

Embedding and compactness results for multiplication operators in Sobolev spaces

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Abstract: The paper deals with the operator $u \rightarrow gu$ defined in the Sobolev space $W^{r,p}(\Omega)$ and which takes values in $L^p(\Omega)$ when Ω is an unbounded open subset in R^n . The functions g belong to Morrey type spaces which provide an intermediate space between $L^\infty(\Omega)$ and $L^p_{loc}(\Omega)$. The main result is an embedding result from which we can deduce a Fefferman type inequality. L^p estimates and a compactness result are also stated.

Keywords: multiplication operator, L^p estimates, compactness results.

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1. Introduction

Let Ω be an unbounded open subset in R^n .

In literature there are different results about the study of *multiplication operator* for a suitable function $g : \Omega \rightarrow C$

$$u \rightarrow gu, \tag{1.1}$$

as an operator defined in a Sobolev space (with or without weight) and which takes values in a $L^p(\Omega)$ space.

In $W_0^{1,p}(\Omega)$ or in $W^{1,p}(\Omega)$ with Ω regular enough, reference results are some well-known inequalities which state the boundedness of (1.1): Hardy type inequalities (see H. Brezis, *Sobolev spaces*, A. Kufner, *Weighted*, J. Nečas, *Les methods*) when $g(x)$ is an appropriate power of the distance of x from a subset of $\partial\Omega$, Fefferman inequality (C. Fefferman, *The uncertainty principle*, see, e.g. F. Chiarenza - M. Franciosi, *A generalization of*, F. Chiarenza - M. Frasca, *A remark on*) obtained when g belongs to suitable Morrey spaces.

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Our aim is to study the *multiplication operator* in the Sobolev space $W^{r,p}(\Omega)$, $r \in \mathbb{N}$, $1 \leq p < +\infty$ when g belongs to suitable Morrey type spaces $\mathbf{M}^p(\Omega)$ which include $L^\infty(\Omega)$ and which are wider than classical Morrey space $L^{p,n}$ when $\Omega = \mathbb{R}^n$. We observe that the space $\mathbf{M}^p(\Omega)$ provides an intermediate space between $L^\infty(\Omega)$ and $L_{loc}^p(\overline{\Omega})$.

The main results we stated concern embedding and compactness results for multiplication operators.

One of the aspects of our interest in these estimates lies in the fact that this type of inequalities are useful tools to prove a priori bounds when studying elliptic equations. Furthermore a priori bounds enable us to state existence and uniqueness results. For some applications in the study of the a priori bounds see A. Canale, *A priori* 2003, *On some results* 2006, *On some results* 2007, *Bounds in spaces* 2008, *Bounds in space* 2009.

In Section 2 and Section 3 we introduce the spaces \mathbf{M}^p and some suitable subspaces analyzing their inclusion properties and the relations between such spaces and the classical Morrey spaces $L^{p,n}$. In Section 4 we state the embedding result and L^p estimates when the functions g belong to the subspaces $\overline{\mathbf{M}}^p$ and \mathbf{M}_0^p defined in Section 2. Moreover we deduce from embedding result a Fefferman type result.

The compactness result for the multiplication operator is stated in Section 5.

2. Notations and Morrey type spaces

Let \mathbb{R}^n be the n -dimensional real euclidean space, and $B_r(x)$ the set

$$B_r(x) = \{y \in \mathbb{R}^n : |y - x| < r\}, \quad B_r = B_r(0) \quad \forall x \in \mathbb{R}^n, \quad \forall r \in \mathbb{R}_+,$$

For any $x \in \mathbb{R}^n$, we call *open infinite cone* having vertex at x every set of the type

$$\{x + \lambda(y - x) : \lambda \in \mathbb{R}_+, |y - z| < r\},$$

where $r \in \mathbb{R}_+$ and $z \in \mathbb{R}^n$ are such that $|z - x| > r$.

For all $\theta \in]0, \pi/2[$ and for all $x \in R^n$ we denote by $C_\theta(x)$ an open infinite cone having vertex at x and opening θ .

For a fixed $C_\theta(x)$, we set

$$C_\theta(x, h) = C_\theta(x) \cap B_h(x), \quad \forall h \in R_+.$$

Let Ω be an open set in R^n . We denote by $\Gamma(\Omega, \theta, h)$ the family of open cones $C \subset\subset \Omega$ of opening θ and height h .

We assume that the following hypothesis is satisfied:

$h_1)$ There exists $\theta \in]0, \pi/2[$ such that

$$\forall x \in \Omega \quad \exists C_\theta(x) \text{ such that } \overline{C_\theta(x, \rho)} \subset \Omega.$$

Let $(\Omega_\rho(x))_{x \in \Omega}$ be the family of open sets in R^n defined as

$$\Omega_\rho(x) = B_\rho(x) \cap \Omega, \quad x \in \Omega, \quad \rho > 0.$$

If $1 \leq p < +\infty$ we define by $\mathbf{M}^p(\Omega)$ as the space of functions $g \in L^p_{loc}(\overline{\Omega})$ such that

$$\|g\|_{\mathbf{M}^p(\Omega)} = \sup_{\substack{x \in \Omega \\ \rho \in]0, d]}} \left(\rho^{-n/p} \|g\|_{L^p(\Omega_\rho(x))} \right) < +\infty, \quad d > 0, \quad (2.1)$$

equipped with the norm defined by (2.1). Furthermore let $\overline{\mathbf{M}^p}(\Omega)$ be the closure of $L^\infty(\Omega)$ in $\mathbf{M}^p(\Omega)$ and $\mathbf{M}_0^p(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $\mathbf{M}^p(\Omega)$.

Remark 2.1 We observe that:

1. If Ω is a bounded open set and $d = \text{diam}\Omega$, then $\mathbf{M}^p(\Omega)$ coincides with the space $L^{p,n}(\Omega)$, the classical Morrey space.
2. If $\Omega = R^n$ then $L^{p,n}(\Omega) \subset \mathbf{M}^p(\Omega)$.
3. The norm (2.1) is a sort of average integral on $B_\rho(x)$. To this end we note that if the family $\{\Omega_\rho(x)\}$ shrinks nicely to x , that is if $\Omega_\rho(x) \subset B_\rho(x)$ for any $\rho > 0$ and there is a constant $\alpha > 0$, independent of ρ , such that $|\Omega_\rho(x)| > \alpha |B_\rho(x)|$, the measure of $\Omega_\rho(x)$ is equivalent to the measure of $B_\rho(x)$.

For functions that belong to the spaces $\overline{\mathbf{M}}^p(\Omega)$ we can state the following alternative definition. Let $\Sigma(\Omega)$ the σ -algebra of Lebesgue measurable subsets of Ω .

Lemma 2.2 *A function $g \in \overline{\mathbf{M}}^p(\Omega)$ if and only if $g \in \mathbf{M}^p(\Omega)$ and the function*

$$\sigma_g^p : t \in [0,1] \rightarrow \sup_{\substack{E \in \Sigma(\Omega) \\ \sup_{\substack{x \\ \rho \in]0,d]} \rho^{-n} |E \cap B_\rho(x)| \leq t}} \left(\|g \chi_E\|_{\mathbf{M}^p(\Omega)} \right) \quad (2.2)$$

is continuous in zero, where χ_E denotes the characteristic function of E .

Proof. Let g be a function belonging to the space $\mathbf{M}^p(\Omega)$ and let σ_g^p be continuous in zero. We denote by δ_ε a positive number such that

$$E \in \Sigma(\Omega), \quad \sup_{\substack{x \in \Omega \\ \rho \in]0,d]} \left(\rho^{-n} |E \cap B_\rho(x)| \right) \leq \delta_\varepsilon \Rightarrow \|g \chi_E\|_{\mathbf{M}^p(\Omega)} \leq \varepsilon$$

and by Ω_r the set

$$\Omega_r(g) = \{x \in \Omega : |g(x)| \geq r\} \quad \forall r \in R_+. \quad (2.3)$$

Then there exists a positive constant $c \in R_+$, independent of r and g , such that

$$\sup_{\substack{x \in \Omega \\ \rho \in]0,d]} \left(\rho^{-n} |\Omega_r \cap B_\rho(x)| \right) \leq c \frac{\|g\|_{\mathbf{M}^p(\Omega)}^p}{r^p}.$$

If we set

$$r_\varepsilon = c \left(\frac{\|g\|_{\mathbf{M}^p(\Omega)}^p}{\delta_\varepsilon} \right)^{1/p},$$

we get

$$\sup_{\substack{x \in \Omega \\ \rho \in]0,d]} \left(\rho^{-n} |\Omega_{r_\varepsilon} \cap B_\rho(x)| \right) \leq \delta_\varepsilon$$

and, as a consequence,

$$\|g\chi_{\Omega_{k_\varepsilon}}\|_{\mathbf{M}^p(\Omega)} \leq \varepsilon.$$

Let us introduce now the function

$$g_\varepsilon = g - g\chi_{\Omega_{k_\varepsilon}}.$$

Evidently $g_\varepsilon \in L^\infty(\Omega)$ and

$$\|g - g_\varepsilon\|_{\mathbf{M}^p(\Omega)} = \|g\chi_{\Omega_{k_\varepsilon}}\|_{\mathbf{M}^p(\Omega)} \leq \varepsilon.$$

Conversely, let us assume $g \in \overline{\mathbf{M}^p}(\Omega)$. Then there exists $g_\varepsilon \in L^\infty(\Omega)$ such that

$$\|g - g_\varepsilon\|_{\mathbf{M}^p(\Omega)} \leq \varepsilon/2.$$

So we get

$$\begin{aligned} \|g\chi_E\|_{\mathbf{M}^p(\Omega)} &\leq \|(g - g_\varepsilon)\chi_E\|_{\mathbf{M}^p(\Omega)} + \|g_\varepsilon\chi_E\|_{\mathbf{M}^p(\Omega)} \leq \\ &\leq \varepsilon/2 + \|g_\varepsilon\|_{L^\infty(\Omega)} \sup_{\substack{x \in \Omega \\ \rho \in]0, d]}} \left(\rho^{-n} |E \cap B_\rho(x)| \right)^{1/p} \end{aligned} \quad (2.4)$$

from which we deduce that

$$\|g\chi_E\|_{\mathbf{M}^p(\Omega)} \leq \varepsilon.$$

for any $E \in \Sigma(\Omega)$ such that

$$\sup_{\substack{x \in \Omega \\ \rho \in]0, d]}} \rho^{-n} |\Omega_\rho(x) \cap E| \leq \left(\frac{\varepsilon}{2\|g_\varepsilon\|_{L^\infty(\Omega)}} \right)^p$$

and the lemma is proved.

3. Inclusion properties

The following Theorem states inclusion properties of the spaces $\mathbf{M}^p(\Omega)$.

Theorem 3.1 *The following inclusions hold:*

- a) $L^\infty(\Omega) \rightarrow \mathbf{M}^p(\Omega) \quad \forall p \in [1, +\infty[$.
- b) $\mathbf{M}^q(\Omega) \rightarrow \mathbf{M}^p(\Omega)$, $1 \leq p \leq q < +\infty$.
- c) $\mathbf{M}^q(\Omega) \subset \widetilde{\mathbf{M}}^p(\Omega)$, $1 \leq p < q < +\infty$.

Proof.

- a) If $g \in L^\infty(\Omega)$ we get

$$\begin{aligned} \|g\|_{\mathbf{M}^p(\Omega)} &= \sup_{\substack{x \in \Omega \\ \rho \in]0, d]}} \left(\rho^{-n/p} \|g\|_{L^p(\Omega_\rho(x))} \right) \leq \\ &\leq \|g\|_{L^\infty(\Omega)} \sup_{\rho \in]0, d]} \left(\rho^{-n/p} |\Omega_\rho(x)|^{1/p} \right) \leq c \|g\|_{L^\infty(\Omega)} \end{aligned} \quad (3.1)$$

where, if ω_n denotes the volume of the unit ball $B_1(0)$, we get $c = \omega_n^{\frac{1}{p}}$.

- b) By Hölder inequality it follows

$$\begin{aligned} \|g\|_{\mathbf{M}^p(\Omega)} &\leq \sup_{\substack{x \in \Omega \\ \rho \in]0, d]}} \left(\rho^{-\frac{n}{p}} \|g\|_{L^q(\Omega_\rho(x))} |\Omega_\rho(x)|^{\frac{1}{p}-\frac{1}{q}} \right) \leq \\ &\leq c_1 \sup_{\substack{x \in \Omega \\ \rho \in]0, d]}} \rho^{-\frac{n}{q}} \|g\|_{L^q(\Omega_\rho(x))}, \end{aligned} \quad (3.2)$$

with $c_1 = \omega_n^{\frac{1}{p}-\frac{1}{q}}$.

- c) We observe that, if $g \in \mathbf{M}^q(\Omega)$, from b) we have $g \in \mathbf{M}^p(\Omega)$. Furthermore

$$\begin{aligned}
 \|g\chi_E\|_{\mathbf{M}^p(\Omega)} &= \sup_{\substack{x \in \Omega \\ \rho \in]0, d]}} \rho^{-n/p} \|g\|_{L^p(\Omega_\rho(x) \cap E)} \leq \\
 &\leq \sup_{\substack{x \in \Omega \\ \rho \in]0, d]}} \rho^{-n/p} |\Omega_\rho(x) \cap E|^{\frac{1}{p} - \frac{1}{q}} \|g\|_{L^q(\Omega_\rho(x) \cap E)} \leq \\
 &\leq \|g\|_{M^q(\Omega)} \sup_{\substack{x \in \Omega \\ \rho \in]0, d]}} \left(\rho^{-n} |\Omega_\rho(x) \cap E| \right)^{\frac{1}{p} - \frac{1}{q}},
 \end{aligned} \tag{3.3}$$

and we deduce that the function σ_g^p , defined by (sigma), is continuous in zero.

From Lemma 2.2 it follows that $g \in \overline{\mathbf{M}^p}(\Omega)$.

4. Embedding result

Let us consider the function

$$\phi : (x, y) \in \Omega \times \Omega \rightarrow \begin{cases} 1 & \text{if } y \in \Omega_\rho(x) \\ 0 & \text{if } y \notin \Omega_\rho(x). \end{cases} \tag{4.1}$$

and, for any $x \in \Omega$, we set

$$E(x) = \{y \in \Omega : x \in \Omega_\rho(y)\}. \tag{4.2}$$

Lemma 4.1 *For any $x \in \Omega$, $E(x)$ is a measurable set and there exist $c_1, c_2 \in \mathbb{R}_+$ such that*

$$c_1 \rho^n \leq |E(x)| \leq c_2 \rho^n \quad \forall x \in \Omega. \tag{4.3}$$

Proof. Clearly the function ϕ is a measurable function. Then, for any fixed $y \in \Omega$, the function $\phi^y : x \in \Omega \rightarrow \phi(x, y)$ is measurable. Since ϕ^y is the characteristic function of $E(y)$, we have that $E(y)$ is measurable.

Now we prove that (4.3) holds. The inequality on the right is easily proved.

We will prove the inequality on the left.

Let us consider $C_\theta(x, \rho)$ such that $\overline{C_\theta(x, \rho)} \subset \Omega$. We get

$$C_\theta(x, \rho) \subset E(x).$$

In fact let $y \in C_\theta(x, \rho)$. Then there exists a cone $C \in \Gamma(\Omega, \theta, h)$ such that $x, y \in C$. So

$$x \in B(y, \rho) \cap \Omega \Rightarrow y \in E(x).$$

Thus the inequality (4.3) is stated.

Now we state a Lemma which we will use in the proof of the embedding result.

Lemma 4.2 *If h_1 holds, then we have $v \in L^1(\Omega)$ if and only if the map $x \in \Omega \rightarrow \rho^{-n} \|v\|_{L^1(\Omega_\rho(x))}$ belongs to $L^1(\Omega)$. Therefore there exist $c_1, c_2 \in R_+$ such that*

$$c_1 \|v\|_{L^1(\Omega)} \leq \int_\Omega \rho^{-n} \|v\|_{L^1(\Omega_\rho(x))} dx \leq c_2 \|v\|_{L^1(\Omega)} \quad \forall v \in L^1(\Omega). \quad (4.4)$$

Proof. The result is a consequence of the relation

$$\begin{aligned} \int_\Omega \rho^{-n} \|v\|_{L^1(\Omega_\rho(x))} dx &= \int_\Omega \rho^{-n} \int_\Omega |v(y)| \phi(x, y) dx dy = \\ &= \int_\Omega |v(y)| dy \int_{E(y)} \rho^{-n} dx \end{aligned} \quad (4.5)$$

and of the Lemma 4.1.

Let r, p, q be real number with the condition

$$h_2) \quad r \in N, \quad 1 \leq p \leq q < +\infty, \quad q \geq \frac{n}{r}, \quad q > \frac{n}{r} \text{ if } \frac{n}{r} = p > 1.$$

Let $u \in W^{r,p}(\Omega)$. For any $x \in \Omega$ we set

$$\Psi^x : y \in \Omega \rightarrow x + \frac{y-x}{\rho}, \quad \Omega_\rho^*(x) = \Psi^x(\Omega_\rho(x)).$$

$$u^* = (u^x)^* : z \in \Omega_\rho^*(x) \rightarrow u(x + \rho(z-x)).$$

We note that

$$u^* \in W^{r,p}(\Omega_\rho^*(x)).$$

We also note that, in consequence of h_1) $\Omega_\rho^*(x)$ has the cone property, with the characteristic cone having height and opening independent of x .

On the other hand, if $\tau = q/p$, from h_2) we get

$$\tau \geq 1, \quad \tau > 1 \text{ if } \frac{n}{r} = p > 1, \quad \frac{\tau-1}{p\tau} \geq \frac{1}{p} - \frac{r}{n}.$$

From well-known embedding theorems of Sobolev spaces (see, e.g., R.A. Adams, *Sobolev*), we deduce that

$$u^* \in L^{\frac{p\tau}{\tau-1}}(\Omega_\rho^*(x))$$

and the following bound holds

$$\|u^*\|_{L^{\frac{p\tau}{\tau-1}}(\Omega_\rho^*(x))} \leq c_0 \|u^*\|_{W^{r,p}(\Omega_\rho^*(x))}, \quad (4.6)$$

where $c_0 = c_0(p, q, r, n)$ is a constant independent of x and u^* .

From (4.6) easily it follows that

$$\rho^{-n \frac{(\tau-1)}{p\tau}} \|u\|_{L^{\frac{p\tau}{\tau-1}}(\Omega_\rho(x))} \leq c_0 \sum_{|\alpha| \leq r} \rho^{|\alpha| - \frac{n}{p}} \|\partial^\alpha u\|_{L^p(\Omega_\rho(x))}. \quad (4.7)$$

Theorem 4.3 *If h_1) and h_2) hold, then for any $g \in \mathbf{M}^q(\Omega)$ and for any $u \in W^{r,p}(\Omega)$ we get $gu \in L^p(\Omega)$ and*

$$\|gu\|_{L^p(\Omega)} \leq c \|g\|_{\mathbf{M}^q(\Omega)} \|u\|_{W^{r,p}(\Omega)}, \quad (4.8)$$

where the constant $c = c(p, q, r, n)$ is independent of g and u .

Proof. Let $u \in W^{r,p}(\Omega)$ and $g \in \mathbf{M}^q(\Omega)$. By (4.4) and by Hölder inequality it follows that

$$\begin{aligned} \int_{\Omega} |gu|^p dx &\leq c_1 \int_{\Omega} \rho^{-n} \int_{\Omega_{\rho}(x)} |gu|^p dy dx \leq c_1 \int_{\Omega} \rho^{-n} \|g\|_{L^{p\tau}(\Omega_{\rho}(x))}^p \|u\|_{L^{\frac{p\tau}{\tau-1}}(\Omega_{\rho}(x))}^p dx \leq \\ &\leq c_1 \|g\|_{\mathbf{M}^q(\Omega)}^p \rho^{-\left(\frac{n\tau-1}{\tau}\right)} \int_{\Omega} \|u\|_{L^{\frac{p\tau}{\tau-1}}(\Omega(x))}^p dx. \end{aligned} \quad (4.9)$$

On the other hand from (4.7) and Lemma (4.2) we obtain

$$\begin{aligned} \rho^{-n\left(\frac{\tau-1}{\tau}\right)} \int_{\Omega} \|u\|_{L^{\frac{p\tau}{\tau-1}}(\Omega_{\rho}(x))}^p dx &\leq c_0 \sum_{|\alpha| \leq r} \rho^{|\alpha|p-n} \int_{\Omega} \|\partial^{\alpha} u\|_{L^p(\Omega_{\rho}(x))}^p dx \leq \\ &\leq c_1 \sum_{|\alpha| \leq r} \rho^{|\alpha|p} \|\partial^{\alpha} u\|_{L^p(\Omega)}^p. \end{aligned} \quad (4.10)$$

From (4.9) and (4.10) the inequality (4.8) follows.

The following theorem is a consequence of the embedding result stated in the Theorem 4.3 (see result of C. Fefferman, *The uncertainty*, and also F. Chiarenza - M. Frasca, *A remark on*, for a simplified proof).

Theorem 4.4 *If $h_1)$ and $h_2)$ hold then for any $g \in \mathbf{M}^q(\Omega)$, we get*

$$\|gu\|_{L^p(\Omega)} \leq c \|g\|_{\mathbf{M}^q(\Omega)} \|\nabla u\|_{L^p(\Omega)} \quad \forall u \in W_0^{1,p}(\Omega).$$

where the constant $c = c(p, q, n)$ is independent of g and u .

Proof. Taking in mind the Poincaré inequality, the proof is a direct consequence of the Theorem 4.3 when $r = 1$.

The L^p estimates below are obtained when the function g belongs to the closure of $L^{\infty}(\Omega)$ in $\mathbf{M}^q(\Omega)$ or to the closure of $C_0^{\infty}(\Omega)$ in $\mathbf{M}^q(\Omega)$. In the

applications to the elliptic differential equations we deal with multiplication operators belonging to these spaces.

Lemma 4.5 *If h_1 , h_2) hold and $g \in \overline{\mathbf{M}}^q(\Omega)$, then for any $\varepsilon \in R_+$ there exists $c(\varepsilon) \in R_+$ such that*

$$\|gu\|_{L^p(\Omega)} \leq \varepsilon \|u\|_{W^{r,p}(\Omega)} + c(\varepsilon) \|u\|_{L^p(\Omega)} \quad \forall u \in W^{r,p}(\Omega).$$

Proof. Let $\phi_\varepsilon \in L^\infty(\Omega)$ such that

$$\|g - \phi_\varepsilon\|_{\mathbf{M}^q(\Omega)} \leq \varepsilon / c, \quad (4.11)$$

where the constant c is the constant in the bound (4.8). Then by Theorem 4.3

$$\begin{aligned} \|gu\|_{L^p(\Omega)} &\leq \|(g - \phi_\varepsilon)u\|_{L^p(\Omega)} + \|\phi_\varepsilon u\|_{L^p(\Omega)} \leq \\ &\leq c \|(g - \phi_\varepsilon)u\|_{\mathbf{M}^q(\Omega)} \|u\|_{W^{r,p}(\Omega)} + \|\phi_\varepsilon\|_{L^\infty(\Omega)} \|u\|_{L^p(\Omega)} \leq \\ &\leq \varepsilon \|u\|_{W^{r,p}(\Omega)} + c(\varepsilon) \|u\|_{L^p(\Omega)} \end{aligned} \quad (4.12)$$

with $c(\varepsilon) = \|\phi_\varepsilon\|_{L^\infty(\Omega)}$.

Lemma 4.6 *If h_1 , h_2) hold and $g \in \mathbf{M}_0^q(\Omega)$, then for any $\varepsilon \in R_+$ there exist $c(\varepsilon) \in R_+$ and an open set $\Omega_\varepsilon \subset\subset \Omega$ with cone property such that*

$$\|gu\|_{L^p(\Omega)} \leq \varepsilon \|u\|_{W^{r,p}(\Omega)} + c(\varepsilon) \|u\|_{L^p(\Omega_\varepsilon)} \quad \forall u \in W^{r,p}(\Omega). \quad (4.13)$$

Proof. Let $\phi_\varepsilon \in C_0^\infty(\Omega)$ be such that (4.11) holds.

Reasoning as in the proof of the Lemma 4.5 we get

$$\|gu\|_{L^p(\Omega)} \leq \varepsilon \|u\|_{W^{r,p}(\Omega)} + c(\varepsilon) \|u\|_{p, \sup p \phi_\varepsilon}, \quad (4.14)$$

with $c(\varepsilon)$ as in the Lemma 4.5.

Then let us fix $\theta \in]0, \pi/2[$ and $h_\varepsilon \in]0, \text{dist}(\partial\Omega, \text{supp } \phi_\varepsilon) / 2[$. If we denote by Ω_ε the open set of R^n union of the cones $C \in \Gamma(\Omega, \theta, h_\varepsilon)$ such that $C \cap \text{supp } g_\varepsilon$ is not empty, then (4.13) follows from (4.14).

5. Compactness result

Now we state the compactness result.

Theorem 5.1 *If h_1 , h_2) hold and $g \in \mathbf{M}_0^q(\Omega)$, then the operator*

$$u \in W^{r,p}(\Omega) \rightarrow gu \in L^p(\Omega)$$

is compact.

Proof. We remark that for any open set $\Omega' \subset\subset \Omega$ the operator

$$u \in W^{r,p}(\Omega) \rightarrow u|_{\Omega'} \in W^{r,p}(\Omega')$$

is linear and bounded.

On the other hand, if Ω' verifies the cone property too, by Rellich-Kondrachov's theorem the operator

$$u \in W^{r,p}(\Omega') \rightarrow u \in L^p(\Omega')$$

is compact. Then also the operator

$$u \in W^{r,p}(\Omega) \rightarrow u|_{\Omega'} \in L^p(\Omega')$$

is compact. Therefore if $\{u_n\} \in W^{r,p}(\Omega')$ is a bounded sequence there exists a subsequence $\{u_{n_k}\}$ converging to u in $L^p(\Omega')$. By Lemma 4.6 we get

$$\|g(u_{n_k} - u)\|_{L^p(\Omega)} \leq \varepsilon \|u_{n_k} - u\|_{W^{r,p}(\Omega)} + c(\varepsilon) \|u_{n_k} - u\|_{L^p(\Omega)} \quad \forall u \in W^{r,p}(\Omega)$$

from which we can deduce the result.

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