SCHRÖDINGER TYPE OPERATORS WITH UNBOUNDED DIFFUSION AND POTENTIAL TERMS

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Abstract. We prove that the realization A_p in $L^p(\mathbb{R}^N)$, $1 < p < \infty$, of the Schrödinger type operator $A = (1 + |x|^\alpha)\Delta - |x|^\beta$ with domain $D(A_p) = \{u \in W^{2,p}(\mathbb{R}^N) : Au \in L^p(\mathbb{R}^N)\}$ generates a strongly continuous analytic semigroup provided that $N > 2$, $\alpha > 2$ and $\beta > \alpha - 2$. Moreover this semigroup is consistent, irreducible, immediately compact and ultracontractive.

1. Introduction

In this paper we study the generation of analytic semigroups in L^p-spaces of Schrödinger type operators of the form

$$Au(x) = a(x)\Delta u(x) - V(x)u(x), \quad x \in \mathbb{R}^N, \tag{1.1}$$

where $a(x) = 1 + |x|^\alpha$ and $V(x) = |x|^\beta$ with $\alpha > 2$ and $\beta > \alpha - 2$. We investigate also spectral properties of such semigroups. In the case when $\alpha \in [0, 2]$ and $\beta \geq 0$, generation results of analytic semigroups for suitable realizations A_p of the operator A in $L^p(\mathbb{R}^N)$ have been proved in [4].

For $\beta = 0$ and $\alpha > 2$, the generation results depend upon N as it is proved in [8]. More specifically, if $N = 1, 2$ no realization of A in $L^p(\mathbb{R}^N)$ generates a strongly continuous (resp. analytic) semigroup. The same happens if $N \geq 3$ and $p \leq N/(N - 2)$. On the other hand, if $N \geq 3$ and $p > N/(N-2)$, then the maximal realization A_p of the operator A in $L^p(\mathbb{R}^N)$ generates a positive analytic semigroup, which is also contractive if $\alpha \geq (p - 1)(N - 2)$.

Generation results concerning the case where $\beta = 0$ and with drift terms of the form $|x|^{\alpha-2}x$ were obtained recently in [9]. The operator with a more general diffusion term was also investigated in [10] and [14].

We quote also the recent paper [5]. Here the authors studied the generation of C_0 and analytic semigroups in $L^p(\mathbb{R}^N)$, $1 < p < \infty$, of operators of the form $\mathcal{A} = |x|^\alpha\Delta + c|x|^{\alpha-2}x \cdot \nabla - b|x|^{\alpha-2}$. They prove for $\alpha \neq 2$, in particular for $c = 0$ and $b = 1$, that a suitable L^p-realization of $\mathcal{A}$ generates a bounded analytic semigroup in $L^p(\mathbb{R}^N)$ if and only if $N/p < (N - 2)/2 + \sqrt{1 + (N - 2)^2/4}$, see [5, Theorem 1.2]. We note here that $\beta = \alpha - 2$

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corresponds to a critical case. The methods used in [5] are completely different from our and lead to results which are not comparable with our case ($\beta > \alpha - 2$).

Here we consider the case where $\alpha > 2$ and assume that $N > 2$. Let us denote by A_p the realization of A in $L^p(\mathbb{R}^N)$ endowed with its maximal domain

$$D_{p,max}(A) = \{u \in L^p(\mathbb{R}^N) \cap W^{2,p}_{loc}(\mathbb{R}^N)\ :\ Au \in L^p(\mathbb{R}^N)\}. \tag{1.2}$$

After proving a priori estimates, we deduce that $D_{p,max}(A)$ coincides with

$$D_p(A) := \{u \in W^{2,p}(\mathbb{R}^N)\ :\ Vu,\ (1+|x|^{\alpha-1})\nabla u,\ (1+|x|^{\alpha})D^2u \in L^p(\mathbb{R}^N)\}.$$

So we show in the main result of this paper that, for any $1 < p < \infty$, the realization A_p of A in $L^p(\mathbb{R}^N)$, with domain $D_p(A)$ generates a positive strongly continuous and analytic semigroup $(T_p(t))_{t\geq 0}$ for any $\beta > \alpha - 2$. This semigroup is also consistent, irreducible, immediately compact and ultracontractive.

The paper is structured as follows. In Section 2 we study the invariance of $C_0(\mathbb{R}^N)$ under the semigroup generated by A in $C_b(\mathbb{R}^N)$ and show its compactness. In Section 3 we use reverse Hölder classes and some results in [13] to study the solvability of the elliptic problem in $L^p(\mathbb{R}^N)$. Then, in Section 4 we prove the generation results.

Notation. For any $k \in \mathbb{N}\cup\{\infty\}$ we denote by $C_c^k(\mathbb{R}^N)$ the set of all functions $f : \mathbb{R}^N \to \mathbb{R}$ that are continuously differentiable in $\mathbb{R}^N$ up to k-th order and have compact support (say supp(f)). The space $C_b(\mathbb{R}^N)$ is the set of all bounded and continuous functions $f : \mathbb{R}^N \to \mathbb{R}$, and we denote by $\|f\|_\infty$ its sup-norm, i.e., $\|f\|_\infty = \sup_{x\in\mathbb{R}^N} |f(x)|$. We use also the space $C_0(\mathbb{R}^N) := \{f \in C_b(\mathbb{R}^N) : \lim_{|x|\to\infty} f(x) = 0\}$. If f is smooth enough we set

$$|\nabla f(x)|^2 = \sum_{i=1}^{N} |D_i f(x)|^2, \qquad |D^2 f(x)|^2 = \sum_{i,j=1}^{N} |D_{ij} f(x)|^2.$$

For any $x_0 \in \mathbb{R}^N$ and any $r > 0$ we denote by $B(x_0, r) \subset \mathbb{R}^N$ the open ball, centered at x_0 with radius r. We simply write $B(r)$ when $x_0 = 0$. The function χ_E denotes the characteristic function of the (measurable) set E, i.e., $\chi_E(x) = 1$ if $x \in E$, $\chi_E(x) = 0$ otherwise.

For any $p \in [1,\infty)$ we denote by $L^p(\mathbb{R}^N)$ the Banach space of all measurable and p-integrable functions in $\mathbb{R}^N$ with respect to the Lebesgue measure endowed with its usual norm $\|\cdot\|_p$. Finally, by $x\cdot y$ we denote the Euclidean scalar product of the vectors $x, y \in \mathbb{R}^N$.

2. Generation of semigroups in $C_0(\mathbb{R}^N)$

In this section we recall some properties of the elliptic and parabolic problems associated with A in $C_b(\mathbb{R}^N)$. We prove the existence of a Lyapunov function for A in the case where $\alpha > 2$ and $\beta > \alpha-2$. This implies the uniqueness of the solution semigroup $(T(t))_{t\geq 0}$ to the associated parabolic problem. Using a domination argument, we show that $T(t)$ is compact and $T(t)C_0(\mathbb{R}^N) \subset C_0(\mathbb{R}^N)$.

First, we endow A with its maximal domain in $C_b(\mathbb{R}^N)$

$$D_{max}(A) = \{u \in C_b(\mathbb{R}^N) \cap W^{2,p}_{loc}(\mathbb{R}^N), \quad 1 \leq p < \infty :\ Au \in C_b(\mathbb{R}^N)\}.$$

Then, we consider for any $\lambda > 0$ and $f \in C_b(\mathbb{R}^N)$ the elliptic equation

$$\lambda u - Au = f. \tag{2.1}$$

It is well-known that equation (2.1) admits at least one solution in $D_{max}(A)$ (see [3, Theorem 2.1.1]). A solution is obtained as follows.
Take the unique solution to the Dirichlet problem associated with $\lambda - A$ into the balls $B(0,n)$ for $n \in \mathbb{N}$. Using Schauder interior estimates one can prove that the sequence of solutions so obtained converges to a solution u of (2.1). It is also known that a solution to (2.1) is in general not unique. The solution u, which we obtained by approximation, is nonnegative whenever $f \geq 0$.

As regards the parabolic problem

$$\begin{cases} u_t(t,x) = Au(t,x) & x \in \mathbb{R}^N,\ t > 0, \\ u(0,x) = f(x) & x \in \mathbb{R}^N \ , \end{cases} \tag{2.2}$$

where $f \in C_b(\mathbb{R}^N)$, it is well-known that one can associate a semigroup $(T(t))_{t\geq 0}$ of bounded operator in $C_b(\mathbb{R}^N)$ such that $u(t,x) = T(t)f(x)$ is a solution of (2.2) in the following sense:

$$u \in C([0,+\infty) \times \mathbb{R}^N) \cap C_{loc}^{1+\frac{\sigma}{2},2+\sigma}((0,+\infty) \times \mathbb{R}^N)$$

and u solves (2.2) for any $f \in C_b(\mathbb{R}^N)$ and some $\sigma \in (0,1)$. Uniqueness of solutions to (2.2) in general is not guaranteed. Moreover the semigroup $(T(t))_{t\geq 0}$ is not strongly continuous in $C_b(\mathbb{R}^N)$ and does not preserve in general the space $C_0(\mathbb{R}^N)$. We note here that the obtained solution u is the minimal solutions among all positive solution of (2.2). For this reason the semigroup $T(t)$ will be called the minimal semigroup. For more details we refer to [3, Chapter 2, Section 2].

Uniqueness is obtained if there exists a positive function $\varphi(x) \in C^2(\mathbb{R}^N)$, called *Lyapunov function*, such that $\lim_{|x|\to\infty} \varphi(x) = +\infty$ and $A\varphi - \lambda\varphi \leq 0$ for some $\lambda > 0$.

Proposition 2.1. *Let $N > 2$, $\alpha > 2$ and $\beta > \alpha - 2$. Let $\varphi = 1 + |x|^\gamma$ where $\gamma > 2$. Then there exists a constant $C > 0$ such that*

$$A\varphi \leq C\varphi.$$

Proof. An easy computation gives

$$A\varphi = \gamma(N+\gamma-2)(1+|x|^\alpha)|x|^{\gamma-2} - (1+|x|^\gamma)|x|^\beta \ .$$

Then, since $\beta > \alpha - 2$, there exists a $C > 0$ such that

$$\gamma(N+\gamma-2)(1+|x|^\alpha)|x|^{\gamma-2} \leq (1+|x|^\gamma)|x|^\beta + C(1+|x|^\gamma).$$

□

Then we can assert that the problem (2.2) admits a unique solution in $C([0,\infty) \times \mathbb{R}^N) \cap C^{1,2}((0,\infty) \times \mathbb{R}^N)$ and problem (2.1) admits a unique solution in $D_{max}(A)$.

In order to investigate the compactness of the semigroup and the invariance of $C_0(\mathbb{R}^N)$ we check the behaviour of $T(t)\mathbb{1}$. We use the following result (see [3, Theorem 5.1.11]).

Theorem 2.2. *Let us fix $t > 0$. Then $T(t)\mathbb{1} \in C_0(\mathbb{R}^N)$ if and only if $T(t)$ is compact and $C_0(\mathbb{R}^N)$ is invariant under $T(t)$.*

Let A_0 be the operator defined by $A_0 := a(x)\Delta$. By [6, Example 7.3] or [8, Proposition 2.2 (iii)], we have that the minimal semigroup $(S(t))$ is generated by $(A_0, D_{max}(A_0)) \cap C_0(\mathbb{R}^N)$. Moreover the resolvent and the semigroup map $C_b(\mathbb{R}^N)$ into $C_0(\mathbb{R}^N)$ and are compact.

Set $v(t,x) = S(t)f(x)$ and $u(t,x) = T(t)f(x)$ for $t > 0$, $x \in \mathbb{R}^N$ and $0 \le f \in C_b(\mathbb{R}^N)$. Then the function $w(t,x) = v(t,x) - u(t,x)$ solves

$$\begin{cases} w_t(t,x) = A_0 w(t,x) + V(x)u(t,x), \quad t > 0, \\ w(0,x) = 0, \quad x \in \mathbb{R}^N. \end{cases}$$

So, applying [3, Theorem 4.1.3], we have $w \ge 0$ and hence $T(t) \le S(t)$. Thus, $T(t)\mathbb{1} \in C_0(\mathbb{R}^N)$, since $S(t)\mathbb{1} \in C_0(\mathbb{R}^N)$ for any $t > 0$ (see [8, Proposition 2.2 (iii)]). Thus, $T(t)$ is compact and $C_0(\mathbb{R}^N)$ is invariant under $T(t)$ (cf. [3, Theorem 5.1.11]). Then we have proved the following proposition:

Proposition 2.3. *The semigroup $(T(t))$ is generated by $(A, D_{max}(A)) \cap C_0(\mathbb{R}^N)$, maps $C_b(\mathbb{R}^N)$ into $C_0(\mathbb{R}^N)$ and is compact.*

3. Solvability of the elliptic problem in $L^p(\mathbb{R}^N)$

In this section we study the existence and uniqueness of solutions of the elliptic problem $\lambda u - A_p u = f$ for a given $f \in L^p(\mathbb{R}^N)$, $1 < p < \infty$ and $\lambda \ge 0$. Let us consider first the case $\lambda = 0$.

We note that the equation $(1+|x|^\alpha)\Delta u - Vu = f$ is equivalent to the equation

$$\Delta u - \frac{V}{1+|x|^\alpha} u = \frac{f}{1+|x|^\alpha} =: \tilde{f}.$$

Therefore we focus our attention to the L^p-realization $\tilde{A}_p$ of the Schrödinger operator

$$\tilde{A} = \Delta - \frac{V}{1+|x|^\alpha} = \Delta - \tilde{V}\ .$$

Let us denote by G the Green function (or the fundamental solution) for $\tilde{A}$. Thus,

$$u(x) = \int_{\mathbb{R}^N} G(x,y)\tilde{f}(y)dy, \tag{3.1}$$

Thus, $u(x) = \int_{\mathbb{R}^N} G(x,y)\frac{f(y)}{1+|y|^\alpha}dy$ and solves $Au = f$ for every $f \in L^p(\mathbb{R}^N)$. So we have to study the operator

$$u(x) = Lf(x) := \int_{\mathbb{R}^N} G(x,y)\frac{f(y)}{1+|y|^\alpha}dy. \tag{3.2}$$

To this purpose, we use the bounds of $G(x,y)$ obtained in [13] when the potential of $\tilde{A}_p$ belongs to the reverse Hölder class B_q for some $q \ge N/2$.

We recall that a nonnegative locally L^q-integrable function V on $\mathbb{R}^N$ is said to be in B_q, $1 < q < \infty$, if there exists $C > 0$ such that the reverse Hölder inequality

$$\left(\frac{1}{|B|}\int_B V^q(x)dx\right)^{1/q} \le C\left(\frac{1}{|B|}\int_B V(x)dx\right)$$

holds for every ball B in $\mathbb{R}^N$. A nonnegative function $V \in L^\infty_{\rm loc}(\mathbb{R}^N)$ is in B_∞ if

$$\|V\|_{L^\infty(B)} \leq C\left(\frac{1}{|B|}\int_B V(x)dx\right)$$

for any ball B in $\mathbb{R}^N$.

One can verify that

$$\tilde{V} \in \begin{cases} B_\infty & \text{if } \beta-\alpha \geq 0 \\ B_q & \text{if } \beta-\alpha > -\frac{N}{q} \\ B_{\frac{N}{2}} & \text{if } \beta-\alpha > -2 \\ B_N & \text{if } \beta-\alpha > -1 \end{cases} \tag{3.3}$$

for some $q > 1$. So, it follows from [13, Theorem 2.7] that, if $\beta-\alpha > -2$ then for any $k > 0$ there is some constant $C_k > 0$ such that for any x, $y \in \mathbb{R}^N$

$$|G(x,y)| \leq \frac{C_k}{(1+m(x)|x-y|)^k}\cdot\frac{1}{|x-y|^{N-2}}, \tag{3.4}$$

where the function m is defined by

$$\frac{1}{m(x)} := \sup_{r>0}\left\{r\ :\ \frac{1}{r^{N-2}}\int_{B(x,r)}\tilde{V}(y)dy \leq 1\right\},\quad x \in R^N. \tag{3.5}$$

Due to the importance of the auxiliary function m we give a lower bound.

Lemma 3.1. *Let $\alpha-2 < \beta < \alpha$. There exists $C = C(\alpha,\beta,N)$ such that*

$$m(x) \geq C\,(1+|x|)^{\frac{\beta-\alpha}{2}}. \tag{3.6}$$

Proof. Fix $x \in \mathbb{R}^N$, and set $f_x(r) = \frac{1}{r^{N-2}}\int_{B(x,r)}\tilde{V}(y)dy$, $r > 0$. Since $\tilde{V} \in B_{N/2}$ implies $V \in B_q$ for some $q > \frac{N}{2}$, by [13, Lemma 1.2], we have

$$\lim_{r\to 0} f_x(r) = 0 \text{ and } \lim_{r\to\infty} f_x(r) = \infty.$$

Thus, $0 < m(x) < \infty$.
In order to estimate $\frac{1}{m(x)}$ we need to find $r_0 = r_0(x)$ such that $r \in [r_0,\infty[$ implies $f_x(r) \geq 1$. In this case we will have $\frac{1}{m(x)} \leq r_0$.

Since $\tilde{V} \in B_{N/2}$, there exists a constant C_1 depending only α,β,N such that

$$\left(\frac{1}{|B|}\int_B \tilde{V}^{N/2}(y)dy\right)^{2/N} \leq C_1\left(\frac{1}{|B|}\int_B \tilde{V}(y)\,dy\right)$$

for any ball B in $\mathbb{R}^N$. Then we have

$$\begin{aligned} f_x(r) &= N^{-1}\sigma_N r^2\frac{1}{|B(x,r)|}\int_{B(x,r)}\tilde{V}(y)dy \\ &\geq \frac{N^{-1}\sigma_N r^2}{C_1}\left(\frac{1}{|B(x,r)|}\int_{B(x,r)}\tilde{V}(y)^{N/2}dy\right)^{2/N} \end{aligned}$$

$$= \frac{(N^{-1}\sigma_N)^{1-2/N}}{C_1}\left(\int_{B(x,r)}\tilde{V}(y)^{N/2}dy\right)^{2/N},$$

where σ_N is the $(N-1)$-dimensional measure of $\partial B(0,1)$. Hence, if

$$\int_{B(x,r)}\tilde{V}(y)^{N/2}dy - C_2 \geq 0\,, \tag{3.7}$$

then $f_x(r) \geq 1$, where $C_2 = C_2(\alpha,\beta,N) = \frac{C_1^{N/2}}{(N^{-1}\sigma_N)^{N/2-1}}$. Note that $\tilde{V} \geq \tilde{V}^*$ in $\mathbb{R}^N \setminus B(0,1)$ with $\tilde{V}^*(x) = \frac{1}{2}|x|^{\beta-\alpha}$. Hence,

$$\begin{aligned}
\int_{B(x,r)}\tilde{V}(y)^{N/2}dy &\geq \int_{B(x,r)\setminus B(0,1)}\tilde{V}(y)^{N/2}dy \geq \int_{B(x,r)\setminus B(0,1)}\tilde{V}^*(y)^{N/2}dy \\
&= \int_{B(x,r)}\tilde{V}^*(y)^{N/2}dy - \int_{B(x,r)\cap B(0,1)}\tilde{V}^*(y)^{N/2}dy \\
&\geq \int_{B(x,r)}\tilde{V}^*(y)^{N/2}dy - \int_{B(0,1)}\tilde{V}^*(y)^{N/2}dy \\
&= \int_{B(x,r)}\tilde{V}^*(y)^{N/2}dy - \frac{2^{1-N/2}\sigma_{N-1}}{N(2-\alpha+\beta)} \\
&\geq N^{-1}\sigma_N r^N \inf_{B(x,r)}(\tilde{V}^*)^{N/2} - C_3(\alpha,\beta,N) &(3.8)\\
&= N^{-1}\sigma_N \frac{2^{-N/2}r^N}{(|x|+r)^{\frac{\alpha-\beta}{2}N}} - C_3. &(3.9)
\end{aligned}$$

Let $\eta = \frac{\alpha-\beta}{2} < 1$ and $\delta > 0$ a parameter to be chosen later, and set

$$r_0 = \delta(1+|x|)^\eta\,.$$

By (3.8) condition (3.7) becomes

$$\begin{aligned}
\int_{B(x,r_0)}\tilde{V}(y)^{N/2}dy - C_2 &\geq N^{-1}\sigma_N \frac{2^{-N/2}r_0^N}{(|x|+r_0)^{\frac{\alpha-\beta}{2}N}} - C_2 - C_3 \\
&= N^{-1}2^{-N/2}\sigma_N \frac{\delta^N(1+|x|)^{\eta N}}{(|x|+\delta(1+|x|)^\eta)^{\frac{\alpha-\beta}{2}N}} - C_4 \\
&\geq N^{-1}2^{-N/2}\sigma_N \frac{\delta^N(1+|x|)^{\eta N}}{(1+|x|+\delta(1+|x|)^\eta)^{\frac{\alpha-\beta}{2}N}} - C_4 \\
&\geq N^{-1}2^{-N/2}\sigma_N \frac{\delta^N(1+|x|)^{\eta N}}{((\delta+1)(1+|x|))^{\frac{\alpha-\beta}{2}N}} - C_4 \\
&= N^{-1}2^{-N/2}\sigma_N \left(\frac{\delta}{(1+\delta)^{\frac{\alpha-\beta}{2}}}\right)^N - C_4\,.
\end{aligned}$$

Since $\frac{\alpha-\beta}{2}<1$ we can choose $\delta>0$ such that $N^{-1}2^{-N/2}\sigma_N\left(\frac{\delta}{(1+\delta)^{\frac{\alpha-\beta}{2}}}\right)^N - C_4 \geq 0\,.$

So, (3.7) is satisfied for $r=r_0$ and hence it is satisfied for any $r>r_0$. Thus, $f_x(r)\geq 1$ for $r>r_0$, and, hence, $\frac{1}{m(x)}\leq r_0=\delta(1+|x|)^{\eta}$. □

The same lower bound holds in the case $\beta\geq\alpha$ as the following lemma shows.

Lemma 3.2. *Let $\beta\geq\alpha$. There exists $C=C(\alpha,\beta,N)$ such that*

$$m(x)\geq C\,(1+|x|)^{\frac{\beta-\alpha}{2}}\,. \tag{3.10}$$

Proof. From [13, Lemma 1.4 (c)], there exist $C_1>0$ and $0<\eta_0<1$ such that, for $x,y\in\mathbb{R}^N$,

$$m(x)\geq\frac{C_1 m(y)}{(1+|x-y|m(y))^{\eta_0}}\,.$$

In particular,

$$m(x)\geq\frac{C_1 m(0)}{(1+|x|m(0))^{\eta_0}},$$

where $\frac{1}{m(0)}=\sup_{r>0}\{r\,:\,f_0(r)\leq 1\}$ with

$$f_0(r)=\frac{1}{r^{N-2}}\int_{B(0,r)}\frac{|z|^\beta}{1+|z|^\alpha}dz=\frac{\sigma_N}{r^{N-2}}\int_0^r\frac{\rho^{\beta+N-1}}{1+\rho^\alpha}d\rho\,.$$

We have $\frac{\sigma_N}{(\beta+N)(1+r^\alpha)}r^{\beta+2}\leq f_0(r)\leq\frac{\sigma_N}{\beta+N}r^{\beta+2}$. Since $\beta>0$ and $\beta-\alpha+2>0$ it follows that $\lim_{r\to 0}f_0(r)=0$ and $\lim_{r\to\infty}f_0(r)=\infty$. Consequently,

$$0<\sup_{r>0}\{r\,:\,f_0(r)\leq 1\}<\infty$$

and, hence, $m(0)=C_2$ for some constant $C_2>0$. Then

$$m(x)\geq\frac{C_1C_2}{(1+C_2|x|)^{\eta_0}}\geq\frac{C_3}{(1+|x|)^{\eta_0}} \tag{3.11}$$

for some constant $C_3>0$.

On the other hand, since $\beta\geq\alpha$, we obtain by (3.3) that $\tilde{V}\in B_\infty$. Then, by [13, Remark 2.9], we have

$$m(x)\geq C_5\tilde{V}^{1/2}(x)=C_5|x|^{\frac{\beta}{2}}\,(1+|x|)^{-\frac{\alpha}{2}}\,. \tag{3.12}$$

The thesis follows taking into account (3.11) and (3.12). □

Applying the estimate (3.4) and the previous lemma we obtain the following upper bounds for the Green function G.

Lemma 3.3. *Let $G(x,y)$ denotes the Green function of the Schrödinger operator $\Delta-\frac{|x|^\beta}{1+|x|^\alpha}$ and assume that $\beta>\alpha-2$. Then,*

$$G(x,y)\leq C_k\,\frac{1}{1+|x-y|^k\,(1+|y|)^{\frac{\beta-\alpha}{2}k}}\,\frac{1}{|x-y|^{N-2}},\qquad x,y\in\mathbb{R}^N \tag{3.13}$$

for any $k>0$ and some constant $C_k>0$ depending on k.

Using the above lemma we have the following estimate.

Lemma 3.4. *Assume that $\alpha > 2$, $N > 2$ and $\beta > \alpha - 2$. Then there exists a positive constant C such that for every $0 \le \gamma \le \beta$ and $f \in L^p(\mathbb{R}^N)$*

$$\||x|^\gamma Lf\|_p \le C\|f\|_p, \tag{3.14}$$

where L is defined in (3.2).

Proof. Let $\Gamma(x,y) = \frac{G(x,y)}{1+|y|^\alpha}$, $f \in L^p(\mathbb{R}^N)$ and

$$u(x) = \int_{\mathbb{R}^N} \Gamma(x,y) f(y)dy.$$

We have to show that

$$\||x|^\gamma u\|_p \le C\|f\|_p.$$

Let us consider the regions $E_1 := \{|x-y| \le (1+|y|)\}$ and $E_2 := \{|x-y| > (1+|y|)\}$ and write

$$u(x) = \int_{E_1} \Gamma(x,y)f(y)dy + \int_{E_2} \Gamma(x,y)f(y)dy =: u_1(x) + u_2(x)\ .$$

In E_1 we have

$$\frac{1+|x|}{1+|y|} \le \frac{1+|x-y|+|y|}{1+|y|} \le 2.$$

So, by Lemma 3.2

$$\begin{aligned} ||x|^\gamma u_1(x)| &\le |x|^\gamma \int_{E_1} \Gamma(x,y)|f(y)|dy \le \frac{1+|x|^\beta}{1+|x|^\alpha} \int_{E_1} \frac{1+|x|^\alpha}{1+|y|^\alpha} G(x,y)|f(y)|dy \\ &\le C(1+|x|)^{\beta-\alpha} \int_{\mathbb{R}^N} G(x,y)|f(y)|dy \le Cm^2(x)\tilde{u}(x), \end{aligned}$$

where $\tilde{u}(x) = \int_{\mathbb{R}^N} G(x,y)|f(y)|dy$. By (3.3) we have $\tilde{V} \in B_{\frac{N}{2}}$. So applying [13, Corollary 2.8], we obtain $\|m^2\tilde{u}\|_p \le C\|f\|_p$ and then $\||x|^\gamma u_1\|_p \le C\|f\|_p$.

In the region E_2, we have, by Hölder's inequality,

$$\begin{aligned} ||x|^\gamma u_2(x)| &\le |x|^\gamma \int_{E_2} \Gamma(x,y)|f(y)|dy = \int_{E_2} \left(|x|^\gamma \Gamma(x,y)\right)^{\frac{1}{p'}} \left(|x|^\gamma \Gamma(x,y)\right)^{\frac{1}{p}} |f(y)|dy \\ &\le \left(\int_{E_2} |x|^\gamma \Gamma(x,y)dy\right)^{\frac{1}{p'}} \left(\int_{E_2} |x|^\gamma \Gamma(x,y)|f(y)|^p dy\right)^{\frac{1}{p}}\ . \end{aligned} \tag{3.15}$$

We propose to estimate first $\int_{E_2} |x|^\gamma \Gamma(x,y)dy$. In E_2 we have $1+|x| \le 1+|y|+|x-y| \le 2|x-y|$, then from (3.13) it follows that

$$\begin{aligned} |x|^\gamma \Gamma(x,y) &\le |x|^\gamma G(x,y) \\ &\le C \frac{1+|x|^\beta}{|x-y|^k (1+|y|)^{k\frac{\beta-\alpha}{2}}} \frac{1}{|x-y|^{N-2}} \\ &\le C \frac{1}{|x-y|^{k-\beta+N-2}} \frac{1}{(1+|y|)^{k\frac{\beta-\alpha}{2}}}\ . \end{aligned}$$

For every $k > \beta - N + 2$, taking into account that $\frac{1}{|x-y|} < \frac{1}{1+|y|}$, we get

$$|x|^\gamma \Gamma(x,y) \le \frac{1}{(1+|y|)^{k\frac{\beta-\alpha+2}{2}+N-2-\beta}} \,.$$

Since $\beta - \alpha + 2 > 0$ we can choose k such that $\frac{k}{2}(\beta-\alpha+2)+N-2-\beta > N$, then

$$\int_{E_2} |x|^\gamma \Gamma(x,y)dy \le \int_{E_2} |x|^\gamma G(x,y)dy \le C\int_{\mathbb{R}^N} \frac{1}{(1+|y|)^{\frac{k}{2}(2+\beta-\alpha)+N-2-\beta}}dy < C\,.$$

Moreover by the symmetry of G we have

$$\begin{aligned} |x|^\gamma \Gamma(x,y) &\le |x|^\gamma G(x,y) \\ &\le C\frac{1+|x|^\beta}{|x-y|^k\,(1+|x|)^{k\frac{\beta-\alpha}{2}}}\frac{1}{|x-y|^{N-2}} \\ &\le C\frac{1}{|x-y|^{k-\beta+N-2}}\frac{1}{(1+|x|)^{k\frac{\beta-\alpha}{2}}}\,. \end{aligned}$$

Taking into account that $\frac{1}{|x-y|} \le 2\frac{1}{1+|x|}$, arguing as above we obtain

$$\int_{E_2} |x|^\gamma \Gamma(x,y)dx \le C. \tag{3.16}$$

Hence (3.15) implies

$$||x|^\gamma u_2(x)|^p \le C\int_{E_2} |x|^\gamma \Gamma(x,y)|f(y)|^p dy. \tag{3.17}$$

Thus, by (3.17) and (3.16), we have

$$\begin{aligned} \||x|^\gamma u_2\|_p^p &\le C\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |x|^\gamma \Gamma(x,y)\chi_{\{|x-y|>1+|y|\}}(x,y)|f(y)|^p dydx \\ &= C\int_{\mathbb{R}^N} |f(y)|^p \left(\int_{E_2} |x|^\gamma \Gamma(x,y)dx\right)dy \le C\|f\|_p^p\,. \end{aligned}$$

□

We are now ready to show the invertibility of A_p and $D_{p,max}(A) \subset D(V)$.

Proposition 3.5. *Assume that $N > 2$, $\alpha > 2$ and $\beta > \alpha - 2$. Then the operator A_p is closed and invertible. Moreover there exists $C > 0$ such that, for every $0 \le \gamma \le \beta$, we have*

$$\||\cdot|^\gamma u\|_p \le C\|A_p u\|_p, \quad \forall u \in D_{p,max}(A)\,. \tag{3.18}$$

Proof. Let us first prove the injectivity of A_p. Let $u \in D_{p,max}(A)$ such that $A_p u = 0$, in particular $\tilde{A}_p u = 0$. It follows that $u \in D_{p,max}(\tilde{A}) = D(\Delta) \cap D\left(\frac{|x|^\beta}{1+|x|^\alpha}\right)$, (see [11]). Then multiplying $A_p u$ by $u|u|^{p-2}$ and integrating over $\mathbb{R}^N$ we obtain, by [7],

$$0 = \int_{\mathbb{R}^N} u|u|^{p-2}\Delta u\,dx - \int_{\mathbb{R}^N} \frac{|x|^\beta}{1+|x|^\alpha}|u|^p dx$$

$$= \quad -(p-1)\int_{\mathbb{R}^N}|u|^{p-2}|\nabla u|^2dx - \int_{\mathbb{R}^N}\frac{|x|^\beta}{1+|x|^\alpha}|u|^p dx,$$

from which we have $u \equiv 0$. On the other hand, we recall that the function given by (3.2) solves $Au = f$ for every $f \in L^p(\mathbb{R}^N)$. Applying Lemma 3.4 with $\gamma = 0$, we deduce that $u \in L^p(\mathbb{R}^N)$ and so by elliptic regularity we have $u \in D_{p,max}(A)$. This together with the injectivity of A_p gives the invertibility of A_p and $A_p^{-1} \in \mathcal{L}(L^p(\mathbb{R}^N))$. This implies in particular that A_p is closed. Finally, the estimate (3.18) follows from (3.14). □

The previous Theorem gives in particular the A_p-boundedness of the potential V and the following regularity result.

Corollary 3.6. *Assume that $N > 2$, $\alpha > 2$ and $\beta > \alpha - 2$. Then*

(i) *there exists $C > 0$ such that for every $u \in D_{p,max}(A)$*

$$\|(1+V)u\|_p \le C\|A_p u\|_p;$$

(ii)

$$D_{p,max}(A) = \left\{u \in W^{2,p}(\mathbb{R}^N) \mid Au \in L^p(\mathbb{R}^N)\right\}.$$

Proof. We have only to prove the inclusion $D_{p,max}(A) \subset \left\{u \in W^{2,p}(\mathbb{R}^N) \mid Au \in L^p(\mathbb{R}^N)\right\}$. Let $u \in D_{p,max}(A)$. Then, by (i), $Vu \in L^p(\mathbb{R}^N)$ and hence

$$\Delta u = \frac{Au + Vu}{1+|x|^\alpha} \in L^p(\mathbb{R}^N).$$

So, the thesis follows from the Calderon-Zygmund inequality. □

We can now state the main result of this section.

Theorem 3.7. *Assume that $N > 2$, $\beta > \alpha - 2$ and $\alpha > 2$. Then, $[0,+\infty) \subset \rho(A_p)$ and $(\lambda - A_p)^{-1}$ is a positive operator on $L^p(\mathbb{R}^N)$ for any $\lambda \ge 0$. Moreover, if $f \in L^p(\mathbb{R}^N) \cap C_0(\mathbb{R}^N)$, then $(\lambda - A_p)^{-1}f = (\lambda - A)^{-1}f$.*

Proof. Let us first prove that if $0 \le \lambda \in \rho(A_p)$, then $(\lambda - A_p)^{-1}$ is a positive operator on $L^p(\mathbb{R}^N)$. To this purpose, take $0 \le f \in L^p(\mathbb{R}^N)$ and set $u = (\lambda - A_p)^{-1}f$. Then, by Corollary 3.6, $u \in D(\tilde{A}_p)$ and

$$-(\tilde{A}_p - \lambda q)u = qf =: \tilde{f},$$

where $q(x) = \frac{1}{1+|x|^\alpha}$. Since $\tilde{A}_p$ generates an exponentially stable and positive C_0-semigroup $(\tilde{T}_p(t))_{t\ge 0}$ on $L^p(\mathbb{R}^N)$ (see [4, Theorem 2.5]), it follows that the semigroup $(e^{-t\lambda q}\tilde{T}_p(t))_{t\ge 0}$ generated by $\tilde{A}_p - \lambda q$ is positive and exponentially stable. Hence,

$$u = (\lambda q - \tilde{A}_p)^{-1}\tilde{f} \ge 0.$$

We show that $E = [0,+\infty) \cap \rho(A_p)$ is an non empty open and closed set in $[0,+\infty)$.
By Proposition 3.5 we have $0 \in \rho(A_p)$ and hence $E \ne \emptyset$. On the other hand, using the above positivity property and the resolvent equation we have $(\lambda - A_p)^{-1} \le (-A_p)^{-1} = L$ for any $\lambda \in E$ and therefore

$$\|(\lambda - A_p)^{-1}\| \le \|L\|\ . \tag{3.19}$$

It follows that the operator norm of $(\lambda - A_p)^{-1}$ is bounded in E and consequently E is closed. Finally, since $\rho(A_p)$ is an open set, it follows that E is open in $[0, +\infty)$. Thus, $E = [0, +\infty)$.

Now in order to show the last statement we may assume $f \in C_c^\infty$, the thesis will follow by density. Setting $u := (\lambda - A_p)^{-1}f$, we obtain, by local elliptic regularity (cf. [2, Theorem 9.19]), that $u \in C_{loc}^{2+\sigma}(\mathbb{R}^N)$ for some $0 < \sigma < 1$. On the other hand, $u \in W^{2,p}(\mathbb{R}^N)$, by Corollary 3.6. If $p \geq \frac{N}{2}$, then by the Sobolev's inequality, $u \in L^q(\mathbb{R}^N)$ for all $q \in [p, +\infty)$. In particular, $u \in L^q(\mathbb{R}^N)$ for some $q > \frac{N}{2}$ and hence $Au = -f + \lambda u \in L^q(\mathbb{R}^N)$. Moreover, since $u \in C_{loc}^{2+\sigma}(\mathbb{R}^N)$, it follows that $u \in W_{loc}^{2,q}(\mathbb{R}^N)$. So, $u \in D_{q,max}(A) \subset W^{2,q}(\mathbb{R}^N) \subset C_b(\mathbb{R}^N)$, by Corollary 3.6 and the Sobolev's embedding theorem, since $q > \frac{N}{2}$.

Let us now suppose that $p < \frac{N}{2}$. Take the sequence (r_n), defined by $r_n = 1/p - 2n/N$ for any $n \in \mathbb{N}$, and set $q_n = 1/r_n$ for any $n \in \mathbb{N}$. Let n_0 be the smallest integer such that $r_{n_0} \leq 2/N$ noting that $r_{n_0} > 0$. Then, $u \in D_{p,max}(A) \subset L^{q_1}(\mathbb{R}^N) \cap L^p(\mathbb{R}^N)$, by the Sobolev embedding theorem. As above we obtain that $u \in D_{q_1,max}(A) \subset L^{q_2}(\mathbb{R}^N)$. Iterating this argument, we deduce that $u \in D_{q_{n_0},max}(A)$. So we can conclude that $u \in C_b(\mathbb{R}^N)$ arguing as in the previous case. Thus, $Au = -f + \lambda u \in C_b(\mathbb{R}^N)$. Again, since $u \in C_{loc}^{2+\sigma}(\mathbb{R}^N)$, it follows that $u \in W_{loc}^{2,q}(\mathbb{R}^N)$ for any $q \in (1, +\infty)$. Hence, $u \in D_{max}(A)$. So, by the uniqueness of the solution of the elliptic problem, we have $(\lambda - A_p)^{-1}f = (\lambda - A)^{-1}f$ for any $f \in C_c^\infty(\mathbb{R}^N)$. □

4. Generation of semigroups

In this section we show that A_p generates an analytic semigroup on $L^p(\mathbb{R}^N)$, $1 < p < \infty$, provided that $N > 2$, $\alpha > 2$ and $\beta > \alpha - 2$.

We start by giving the characterization of the domain of A. More precisely we prove that the maximal domain $D_{p,max}(A)$ coincides with the weighted Sobolev space $D_p(A)$ defined by

$$D_p(A) := \{u \in W^{2,p}(\mathbb{R}^N) \ : \ Vu,\ (1 + |x|^{\alpha-1})\nabla u,\ (1 + |x|^\alpha)D^2u \in L^p(\mathbb{R}^N)\}$$

endowed with its canonical norm.

To this purpose we need the following covering result from, see [1, Proposition 6.1], to prove a weighted gradient estimate.

Proposition 4.1. *For every $0 \leq k < 1/2$ there exists a natural number $\zeta = \zeta(N, k)$ with the following property: given $\mathcal{F} = \{B(x, \rho(x))\}_{x \in \mathbb{R}^N}$, where $\rho : \mathbb{R}^N \to \mathbb{R}_+$ is a Lipschitz continuous function with Lipschitz constant k, there exists a countable subcovering $\{B(x_n, \rho(x_n))\}_{n \in \mathbb{N}}$ of $\mathbb{R}^N$ such that at most ζ among the double balls $\{B(x_n, 2\rho(x_n))\}_{n \in \mathbb{N}}$ overlap.*

We need the following weighted gradient and second derivative estimate.

Lemma 4.2. *Assume that $N > 2$, $\alpha > 2$ and $\beta > \alpha - 2$. Then there exists a constant $C > 0$ such that for every $u \in D_p(A)$ we have*

$$\|(1 + |x|^{\alpha-1})\nabla u\|_p \leq C\|A_p u\|_p \ , \tag{4.1}$$

$$\|(1 + |x|^\alpha)D^2u\|_p \leq C\|A_p u\|_p \ . \tag{4.2}$$

Proof. Let $u \in D_p(A)$. We fix $x_0 \in \mathbb{R}^n$ and choose $\vartheta \in C_c^\infty(\mathbb{R}^N)$ such that $0 \le \vartheta \le 1$, $\vartheta(x) = 1$ for $x \in B(1)$ and $\vartheta(x) = 0$ for $x \in \mathbb{R}^N \setminus B(2)$. Moreover, we set $\vartheta_\rho(x) = \vartheta\left(\frac{x-x_0}{\rho}\right)$, where $\rho = \frac{1}{4}(1+|x_0|)$. We apply the well-known inequality

$$\|\nabla v\|_{L^p(B(R))} \le C\|v\|_{L^p(B(R))}^{1/2}\|\Delta v\|_{L^p(B(R))}^{1/2}, \quad v \in W^{2,p}(B(R)) \cap W_0^{1,p}(B(R)),\ R>0, \tag{4.3}$$

to the function $\vartheta_\rho u$ and obtain for every $\varepsilon > 0$,

$$\begin{aligned}
&\|(1+|x_0|)^{\alpha-1}\nabla u\|_{L^p(B(x_0,\rho))} \le \|(1+|x_0|)^{\alpha-1}\nabla(\vartheta_\rho u)\|_{L^p(B(x_0,2\rho))}\\
&\quad\le C\|(1+|x_0|)^{\alpha}\Delta(\vartheta_\rho u)\|_{L^p(B(x_0,2\rho))}^{\frac{1}{2}}\|(1+|x_0|)^{\alpha-2}\vartheta_\rho u\|_{L^p(B(x_0,2\rho))}^{\frac{1}{2}}\\
&\quad\le C\left(\varepsilon\|(1+|x_0|)^{\alpha}\Delta(\vartheta_\rho u)\|_{L^p(B(x_0,2\rho))} + \frac{1}{4\varepsilon}\|(1+|x_0|)^{\alpha-2}\vartheta_\rho u\|_{L^p(B(x_0,2\rho))}\right)\\
&\quad\le C\left(\varepsilon\|(1+|x_0|)^{\alpha}\Delta u\|_{L^p(B(x_0,2\rho))} + \frac{2M}{\rho}\varepsilon\|(1+|x_0|)^{\alpha}\nabla u\|_{L^p(B(x_0,2\rho))}\right.\\
&\qquad\left.+\frac{\varepsilon M}{\rho^2}\|(1+|x_0|)^{\alpha}u\|_{L^p(B(x_0,2\rho))} + \frac{1}{4\varepsilon}\|(1+|x_0|)^{\alpha-2}u\|_{L^p(B(x_0,2\rho))}\right)\\
&\quad\le C\left(\varepsilon\|(1+|x_0|)^{\alpha}\Delta u\|_{L^p(B(x_0,2\rho))} + 8M\varepsilon\|(1+|x_0|)^{\alpha-1}\nabla u\|_{L^p(B(x_0,2\rho))}\right.\\
&\qquad\left.+\left(16\varepsilon M + \frac{1}{4\varepsilon}\right)\|(1+|x_0|)^{\alpha-2}u\|_{L^p(B(x_0,2\rho))}\right)\\
&\quad\le C(M)\left(\varepsilon\|(1+|x_0|)^{\alpha}\Delta u\|_{L^p(B(x_0,2\rho))} + \varepsilon\|(1+|x_0|)^{\alpha-1}\nabla u\|_{L^p(B(x_0,2\rho))}\right.\\
&\qquad\left.+\frac{1}{\varepsilon}\|(1+|x_0|)^{\alpha-2}u\|_{L^p(B(x_0,2\rho))}\right),
\end{aligned}$$

where $M = \|\nabla\vartheta\|_\infty + \|\Delta\vartheta\|_\infty$. Since $2\rho = \frac{1}{2}(1+|x_0|)$ we get

$$\frac{1}{2}(1+|x_0|) \le 1+|x| \le \frac{3}{2}(1+|x_0|), \qquad x \in B(x_0,2\rho).$$

Thus,

$$\begin{aligned}
&\|(1+|x|)^{\alpha-1}\nabla u\|_{L^p(B(x_0,\rho))} \le \left(\frac{3}{2}\right)^{\alpha-1}\|(1+|x_0|)^{\alpha-1}\nabla u\|_{L^p(B(x_0,\rho))}\\
&\quad\le C\left(\varepsilon\|(1+|x_0|)^{\alpha}\Delta u\|_{L^p(B(x_0,2\rho))} + \varepsilon\|(1+|x_0|)^{\alpha-1}\nabla u\|_{L^p(B(x_0,2\rho))}\right.\\
&\qquad\left.+\frac{1}{\varepsilon}\|(1+|x_0|)^{\alpha-2}u\|_{L^p(B(x_0,2\rho))}\right)\\
&\quad\le C\left(2^{\alpha}\varepsilon\|(1+|x|)^{\alpha}\Delta u\|_{L^p(B(x_0,2\rho))} + 2^{\alpha-1}\varepsilon\|(1+|x|)^{\alpha-1}\nabla u\|_{L^p(B(x_0,2\rho))}\right.\\
&\qquad\left.+\frac{2^{\alpha-2}}{\varepsilon}\|(1+|x|)^{\alpha-2}u\|_{L^p(B(x_0,2\rho))}\right).
\end{aligned}\tag{4.4}$$

Let $\{B(x_n,\rho(x_n))\}$ be a countable covering of $\mathbb{R}^N$ as in Proposition 4.1 such that at most ζ among the double balls $\{B(x_n,2\rho(x_n))\}$ overlap.

We write (4.4) with x_0 replaced by x_n and sum over n. Taking into account the above covering result, we get

$$\|(1+|x|)^{\alpha-1}\nabla u\|_p \le C\big(\varepsilon\|(1+|x|)^{\alpha}\Delta u\|_p + \varepsilon\|(1+|x|)^{\alpha-1}\nabla u\|_p + \frac{1}{\varepsilon}\|(1+|x|)^{\alpha-2}u\|_p\big)\ .$$

Choosing ε such that $\varepsilon C < 1/2$ we have

$$\frac{1}{2}\|(1+|x|)^{\alpha-1}\nabla u\|_p \le \frac{1}{2}\|(1+|x|)^{\alpha}\Delta u\|_p + \frac{C}{\varepsilon}\|(1+|x|)^{\alpha-2}u\|_p\ .$$

Furthermore $\||x|^{\alpha-2}u\|_p \le \|(1+|x|^{\beta})u\|_p \le C\|A_p u\|_p$ for any $u \in D_p(A) \subset D_{p,max}(A)$ and some $C>0$ by Corollary 3.6. Hence,

$$\|(1+|x|)^{\alpha-1}\nabla u\|_p \le C(\|A_p u\|_p + \|u\|_p).$$

As regards the second order derivatives we consider the classical Calderón- Zygmund inequality on $B(1)$

$$\|D^2 v\|_{L^p(B(1))} \le C\|\Delta v\|_{L^p(B(1))},\ \ v \in W^{2,p}(B(1)) \cap W^{1,p}_0(B(1))\ ,$$

by rescaling and translating we get

$$\|D^2 v\|_{L^p(B(x_0,R))} \le C\|\Delta v\|_{L^p(B(x_0,R))} \tag{4.5}$$

for every $x_0 \in \mathbb{R}^N$, $R>0$ and $v \in W^{2,p}(B(x_0,R)) \cap W^{1,p}_0(B(x_0,R))$. We observe that the constant C does not depend on R and x_0.

Then we fix $x_0 \in \mathbb{R}^n$ and choose ρ and $\vartheta_\rho \in C_c^\infty(\mathbb{R}^N)$ as above. Applying (4.5) to the function $\vartheta_\rho u$ in $B(x_0,2\rho)$, we obtain

$$\begin{aligned}\|(1+|x_0|)^{\alpha}D^2u\|_{L^p(B(x_0,\rho))} &\le \|(1+|x_0|)^{\alpha}D^2(\vartheta_\rho u)\|_{L^p(B(x_0,2\rho))}\\ &\le C\|(1+|x_0|)^{\alpha}\Delta(\vartheta_\rho u)\|_{L^p(B(x_0,2\rho))}.\end{aligned}$$

Reasoning as above we obtain

$$\|(1+|x|)^{\alpha}D^2u\|_p \le C\left(\|(1+|x|)^{\alpha}\Delta u\|_p + \|(1+|x|)^{\alpha-1}\nabla u\|_p + \|(1+|x|)^{\alpha-2}u\|_p\right).$$

The lemma follows by Corollary 3.6 and by the gradient estimate (4.1).

□

The following lemma shows that $C_c^\infty(\mathbb{R}^N)$ is a core for $(A, D_p(A))$.

Lemma 4.3. *Assume $N>2$, $\alpha>2$ and $\beta>\alpha-2$. The space $C_c^\infty(\mathbb{R}^N)$ is dense in $D_p(A)$ with respect to the graph norm.*

Proof. Let us first observe that $C_c^\infty(\mathbb{R}^N)$ is dense in $W_c^{2,p}(\mathbb{R}^N)$ with respect to the operator norm. Let $u \in W_c^{2,p}(\mathbb{R}^N)$ and consider $u_n = \rho_n * u$, where ρ_n are standard mollifiers. We have $u_n \in C_c^\infty(\mathbb{R}^N)$, $u_n \to u$ in $L^p(\mathbb{R}^N)$ and $D^2u_n \to D^2u$ in $L^p(\mathbb{R}^N)$. Moreover, $supp\, u_n \subset supp\, u + B(1) := K$ for any $n \in \mathbb{N}$. Then

$$\begin{aligned}\|A_p u - Au_n\|_p &= \|A_p u - Au_n\|_{L^p(K)}\\ &\le \|(1+|x|^{\alpha})\Delta(u-u_n)\|_{L^p(K)} + \||x|^{\beta}(u-u_n)\|_{L^p(K)}\\ &\le \|(1+|x|^{\alpha})\|_{L^\infty(K)}\|\Delta(u-u_n)\|_{L^p(K)} + \||x|^{\beta}\|_{L^\infty(K)}\|(u-u_n)\|_{L^p(K)} \to 0 \text{ as } n\to\infty\ .\end{aligned}$$

Now, let u in $D_{p,max}(A)$ and let η be a smooth function such that $\eta = 1$ in $B(1)$, $\eta = 0$ in $\mathbb{R}^N \setminus B(2)$, $0 \le \eta \le 1$ and set $\eta_n(x) = \eta\left(\frac{x}{n}\right)$. Then consider $u_n = \eta_n u \in W^{2,p}_c(\mathbb{R}^N)$. First we have $u_n \to u$ in $L^p(\mathbb{R}^N)$ by dominated convergence. As regard $A_p u_n$ we have

$$\begin{aligned}
A_p u_n(x) &= (1+|x|^\alpha)\Delta(\eta_n u)(x) - |x|^\beta \eta_n(x)u(x)\\
&= \eta_n(x)A_p u(x) + 2(1+|x|^\alpha)\nabla\eta_n(x)\nabla u(x) + (1+|x|^\alpha)\Delta\eta_n(x)u(x)\\
&= \eta_n(x)A_p u(x) + \frac{2}{n}(1+|x|^\alpha)\nabla\eta\left(\frac{x}{n}\right)\nabla u(x) + \frac{1}{n^2}(1+|x|^\alpha)\Delta\eta\left(\frac{x}{n}\right)u(x)\ ,
\end{aligned}$$

and

$$\eta_n A_p u \to A_p u \qquad \text{in} \qquad L^p(\mathbb{R}^N)$$

by dominated convergence. As regards the last terms we note that $\nabla\eta(x/n)$ and $\Delta\eta(x/n)$ can be different from zero only for $n \le |x| \le 2n$, then we have

$$\frac{1}{n}(1+|x|^\alpha)\left|\nabla\eta\left(\frac{x}{n}\right)\right||\nabla u| \le C(1+|x|^{\alpha-1})|\nabla u|\chi_{\{n\le|x|\le 2n\}},$$

and

$$\frac{1}{n^2}(1+|x|^\alpha)\left|\Delta\eta\left(\frac{x}{n}\right)\right||u| \le C(1+|x|^{\alpha-2})|u|\chi_{\{n\le|x|\le 2n\}}\,.$$

The right hand sides tend to 0 as $n\to\infty$, since by Proposition 3.5 and Lemma 4.2 we have $\|(1+|x|^{\alpha-2})u\|_p \le C\|A_p u\|_p$ and $\|(1+|x|^{\alpha-1})\nabla u\|_p \le C\|A_p u\|_p$. So, applying again the dominated convergence theorem, we obtain $A_p u_n \to A_p u$ in $L^p(\mathbb{R}^N)$. This ends the proof of the lemma. □

We can give now the complete characterization of $D_{p,max}(A)$.

Theorem 4.4. *Assume that $N > 2$, $\alpha > 2$ and $\beta > \alpha - 2$. Then maximal domain $D_{p,max}(A)$ coincides with $D_p(A)$.*

Proof. We have to prove only the inclusion $D_{p,max}(A) \subset D_p(A)$.

Let $\tilde{u} \in D_{p,max}(A)$ and set $f = A\tilde{u}$. The operator A in $B(\rho)$, $\rho > 0$, is an elliptic operator with bounded coefficients, then the problem

$$\begin{cases} Au = f & \text{in } B(\rho)\\ u = 0 & \text{on } \partial B(\rho)\ , \end{cases} \tag{4.6}$$

admits a unique solution u_ρ in $W^{2,p}(B(\rho)) \cap W^{1,p}_0(B(\rho))$ (cf. [2, Theorem 9.15]). Now $u_\rho \in D_p(A)$ and by Lemma 4.2 and Corollary 3.6 (i)

$$\begin{aligned}
&\|(1+|x|^{\alpha-2})u_\rho\|_{L^p(B(\rho))} + \|(1+|x|^{\alpha-1})\nabla u_\rho\|_{L^p(B(\rho))}\\
&\qquad + \|(1+|x|^\alpha)D^2 u_\rho\|_{L^p(B(\rho))} + \|Vu_\rho\|_{L^p(B(\rho))} \le C\|Au_\rho\|_p
\end{aligned}$$

with C independent of ρ. Using a standard weak compactness argument we can construct a sequence u_{ρ_n} which converges to a function u in $W^{2,p}_{loc}$ such that $Au = f$. Since the estimates above are independent of ρ, also $u \in D_p(A)$. Then we have $A\tilde{u} = Au$ and since $D_p(A) \subset D_{p,max}(A)$ and A is invertible on $D_{p,\max}(A)$ by Proposition 3.5, we have $\tilde{u} = u$. □

Let us give now the main result of this section.

Theorem 4.5. *Assume $N > 2$, $\alpha > 2$ and $\beta > \alpha - 2$. Then the operator A_p with domain $D_{p,max}(A)$ generates an analytic semigroup in $L^p(\mathbb{R}^N)$.*

Proof. Let $f \in L^p$, $\rho > 0$. Consider the operator $\widetilde{A_p} := A_p - \omega$ where ω is a constant which will be chosen later. It is known that the elliptic problem in $L^p(B(\rho))$

$$\begin{cases} \lambda u - \widetilde{A_p} u = f & \text{in } B(\rho) \\ u = 0 & \text{on } \partial B(\rho)\ , \end{cases} \tag{4.7}$$

admits a unique solution u_ρ in $W^{2,p}(B(\rho)) \cap W^{1,p}_0(B(\rho))$ for $\lambda > 0$, (cf. [2, Theorem 9.15]).

Let us prove that that $e^{\pm i\theta}\widetilde{A_p}$ is dissipative in $B(\rho)$ for $0 \le \theta \le \theta_\alpha$ with suitable $\theta_\alpha \in (0, \frac{\pi}{2}]$. To this purpose observe that

$$\widetilde{A_p} u_\rho = \operatorname{div}\left((1 + |x|^\alpha)\nabla u_\rho\right) - \alpha |x|^{\alpha-1}\frac{x}{|x|}\nabla u_\rho - |x|^\beta u_\rho - \omega u_\rho\,.$$

Set $u^\star = \overline{u}_\rho |u_\rho|^{p-2}$ and recall that $a(x) = 1 + |x|^\alpha$. Multiplying $\widetilde{A_p} u_\rho$ by $u^\star$ and integrating over $B(\rho)$, we obtain

$$\begin{aligned}
&\int_{B(\rho)} \widetilde{A_p} u_\rho\, u^\star dx = -\int_{B(\rho)} a(x)|u_\rho|^{p-4}|Re(\overline{u}_\rho \nabla u_\rho)|^2 dx \\
&- \int_{B(\rho)} a(x)|u_\rho|^{p-4}|Im(\overline{u}_\rho \nabla u_\rho)|^2 dx - (p-2)\int_{B(\rho)} a(x)|u_\rho|^{p-4}\overline{u}_\rho \nabla u_\rho Re(\overline{u}_\rho \nabla u_\rho) dx \\
&-\alpha \int_{B(\rho)} \overline{u}_\rho |u_\rho|^{p-2}|x|^{\alpha-1}\frac{x}{|x|}\nabla u_\rho\, dx - \int_{B(\rho)} \left(|x|^\beta + \omega\right)|u_\rho|^p dx.
\end{aligned}$$

We note here that the integration by part in the singular case $1 < p < 2$ is allowed thanks to [7]. By taking the real and imaginary part of the left and the right hand side, we have

$$\begin{aligned}
& Re\left(\int_{B(\rho)} \widetilde{A_p} u_\rho\, u^\star dx\right) \\
= \;& -(p-1)\int_{B(\rho)} a(x)|u_\rho|^{p-4}|Re(\overline{u}_\rho \nabla u_\rho)|^2 dx - \int_{B(\rho)} a(x)|u_\rho|^{p-4}|Im(\overline{u}_\rho \nabla u_\rho)|^2 dx \\
& -\alpha \int_{B(\rho)} |u_\rho|^{p-2}|x|^{\alpha-1}\frac{x}{|x|} Re(\overline{u}_\rho \nabla u_\rho)\, dx - \int_{B(\rho)} \left(|x|^\beta + \omega\right)|u_\rho|^p dx \\
= \;& -(p-1)\int_{B(\rho)} a(x)|u_\rho|^{p-4}|Re(\overline{u}_\rho \nabla u_\rho)|^2 dx - \int_{B(\rho)} a(x)|u_\rho|^{p-4}|Im(\overline{u}_\rho \nabla u_\rho)|^2 dx \\
& -\frac{\alpha}{p}\int_{B(\rho)} |x|^{\alpha-1}\frac{x}{|x|}\nabla(|u_\rho|^p)\, dx - \int_{B(\rho)} \left(|x|^\beta + \omega\right)|u_\rho|^p dx \\
= \;& -(p-1)\int_{B(\rho)} a(x)|u_\rho|^{p-4}|Re(\overline{u}_\rho \nabla u_\rho)|^2 dx - \int_{B(\rho)} a(x)|u_\rho|^{p-4}|Im(\overline{u}_\rho \nabla u_\rho)|^2 dx \\
& + \int_{B(\rho)} \left(\frac{\alpha(N-2+\alpha)}{p}|x|^{\alpha-2} - |x|^\beta - \omega\right)|u_\rho|^p dx
\end{aligned}$$

and

$$\begin{aligned} Im\Big(\int_{B(\rho)}\widetilde{A_p}u_\rho\,u^\star dx\Big) &= -(p-2)\int_{B(\rho)}a(x)|u_\rho|^{p-4}Im(\overline{u}_\rho\nabla u_\rho)Re(\overline{u}_\rho\nabla u_\rho)\,dx \\ &\quad -\alpha\int_{B(\rho)}|u_\rho|^{p-2}|x|^{\alpha-1}\frac{x}{|x|}Im(\overline{u}_\rho\nabla u_\rho)\,dx. \end{aligned}$$

We can choose $\tilde{c}>0$ and $\omega>0$ (depending on $\tilde{c}$) such that

$$\frac{\alpha(N-2+\alpha)}{p}|x|^{\alpha-2}-|x|^\beta-\omega\le -\tilde{c}|x|^{\alpha-2}.$$

So, we obtain

$$\begin{aligned} -Re\left(\int_{B(\rho)}\widetilde{A_p}u_\rho\,u^\star dx\right) &\ge (p-1)\int_{B(\rho)}a(x)|u_\rho|^{p-4}|Re(\overline{u}_\rho\nabla u_\rho)|^2dx \\ &\quad +\int_{B(\rho)}a(x)|u_\rho|^{p-4}|Im(\overline{u}_\rho\nabla u_\rho)|^2dx+\tilde{c}\int_{B(\rho)}|u_\rho|^p|x|^{\alpha-2}dx \\ &= (p-1)B^2+C^2+\tilde{c}D^2. \end{aligned}$$

Moreover,

$$\begin{aligned} &\left|Im\left(\int_{B(\rho)}\widetilde{A_p}u_\rho\,u^\star dx\right)\right| \\ \le\ & |p-2|\left(\int_{B(\rho)}|u_\rho|^{p-4}a(x)|Re(\overline{u}_\rho\nabla u_\rho)|^2dx\right)^{\frac12}\left(\int_{B(\rho)}|u_\rho|^{p-4}a(x)|Im(\overline{u}_\rho\nabla u_\rho)|^2dx\right)^{\frac12} \\ &\quad +\alpha\left(\int_{B(\rho)}|u_\rho|^{p-4}|x|^\alpha|Im(\overline{u}_\rho\nabla u_\rho)|^2\,dx\right)^{\frac12}\left(\int_{B(\rho)}|u_\rho|^p|x|^{\alpha-2}\,dx\right)^{\frac12} \\ =\ & |p-2|BC+\alpha CD, \end{aligned}$$

where

$$\begin{aligned} B^2 &= \int_{B(\rho)}|u_\rho|^{p-4}a(x)|Re(\overline{u}_\rho\nabla u_\rho)|^2dx, \\ C^2 &= \int_{B(\rho)}|u_\rho|^{p-4}a(x)|Im(\overline{u}_\rho\nabla u_\rho)|^2dx, \\ D^2 &= \int_{B(\rho)}|u_\rho|^p|x|^{\alpha-2}\,dx. \end{aligned}$$

Let us observe that, choosing $\delta^2=\frac{|p-2|^2}{4(p-1)}+\frac{\alpha^2}{4\tilde{c}}$ (which is independent of ρ), we obtain

$$\left|Im\Big(\int_{B(\rho)}\widetilde{A_p}u_\rho\,u^\star dx\Big)\right|\le\delta\left\{-Re\Big(\int_{B(\rho)}\widetilde{A_p}u_\rho\,u^\star dx\Big)\right\}.$$

If $\tan\theta_\alpha = \delta$, then $e^{\pm i\theta}\widetilde{A_p}$ is dissipative in $B(\rho)$ for $0 \le \theta \le \theta_\alpha$. From [12, Theorem I.3.9] follows that the problem (4.7) has a unique solution u_ρ for every $\lambda \in \Sigma_\theta$, $0 \le \theta < \theta_\alpha$ where

$$\Sigma_\theta = \{\lambda \in \mathbb{C} \setminus \{0\} : |Arg\,\lambda| < \pi/2 + \theta\}.$$

Moreover, there exists a constant C_θ which is independent of ρ, such that

$$\|u_\rho\|_{L^p(B(\rho))} \le \frac{C_\theta}{|\lambda|}\|f\|_{L^p}, \quad \lambda \in \Sigma_\theta. \tag{4.8}$$

Let us now fix $\lambda \in \Sigma_\theta$, with $0 < \theta < \theta_\alpha$ and a radius $r > 0$. We apply the interior L^p estimates (cf. [2, Theorem 9.11]) to the functions u_ρ with $\rho > r + 1$. So, by (4.8) we have

$$\|u_\rho\|_{W^{2,p}(B(r))} \le C_1\left(\|\lambda u_\rho - \widetilde{A_p}u_\rho\|_{L^p(B(r+1))} + \|u_\rho\|_{L^p(B(r+1))}\right) \le C_2\|f\|_{L^p}.$$

Using a weak compactness and a diagonal argument, we can construct a sequence $(\rho_n) \to \infty$ such that the functions (u_{ρ_n}) converge weakly in $W^{2,p}_{loc}$ to a function u which satisfies $\lambda u - \widetilde{A_p}u = f$ and

$$\|u\|_p \le \frac{C_\theta}{|\lambda|}\|f\|_p, \quad \lambda \in \Sigma_\theta. \tag{4.9}$$

Moreover, $u \in D_{p,max}(A_p)$. We have now only to show that $\lambda - \widetilde{A_p}$ is invertible on $D_{p,max}(A_p)$ for $\lambda \in \Sigma_\theta$. Consider the set

$$E = \{r > 0 : \Sigma_\theta \cap C(r) \subset \rho(\widetilde{A_p})\},$$

where $C(r) := \{\lambda \in \mathbb{C} : |\lambda| < r\}$. Since, by Theorem 3.7, 0 is in the resolvent set of $\widetilde{A_p}$, then $R = \sup E > 0$. On the other hand, the norm of the resolvent is bounded by $C_\theta/|\lambda|$ in $C(R) \cap \Sigma_\theta$, consequently it cannot explode on the boundary of $C(R)$, then $R = \infty$ and this ends the proof of the theorem. □

Remark 4.6. *Since A_p generates an analytic semigroup $T_p(\cdot)$ on $L^p(\mathbb{R}^N)$ and the semigroups $T_q(\cdot)$, $q \in (1,\infty)$ are consistent, see Theorem 3.7, one can deduce (as in the proof of* [4, Proposition 2.6]*) using Corollary 3.6 that $T_p(t)L^p(\mathbb{R}^N) \subset C_b^{1+\nu}(\mathbb{R}^N)$ for any $t > 0$, $\nu \in (0,1)$ and for any $p \in (1,\infty)$.*

We end this section by studying the spectrum of A_p. We recall from Proposition 3.5 that

$$\||x|^\beta u\|_p \le C\|A_p u\|_p, \quad \forall u \in D_{p,max}(A).$$

So, arguing as in [4], we obtain the following results.

Proposition 4.7. *Assume $N > 2$, $\alpha > 2$ and $\beta > \alpha - 2$ then*

(i) the resolvent of A_p is compact in L^p;

(ii) the spectrum of A_p consists of a sequence of negative real eigenvalues which accumulates at $-\infty$. Moreover, $\sigma(A_p)$ is independent of p;

(iii) the semigroup $T_p((\cdot)$ is irreducible, the eigenspace corresponding to the largest eigenvalue λ_0 of A is one-dimensional and is spanned by a strictly positive function ψ, which is radial, belongs to $C_b^{1+\nu}(\mathbb{R}^N) \cap C^2(\mathbb{R}^N)$ for any $\nu \in (0,1)$ and tends to 0 when $|x| \to \infty$.

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