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BANACH FUNCTION NORMS VIA CAUCHY POLYNOMIALS AND APPLICATIONS

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Abstract. Let $X_1, \dots, X_k$ be quasinormed spaces with quasinorms $|\cdot|_j$, $j = 1, \dots, k$, respectively. For any $\boldsymbol{f} = (f_1, \dots, f_k) \in X_1 \times \dots \times X_k$ let $\rho(\boldsymbol{f})$ be the unique nonnegative root of the Cauchy polynomial $p_{\boldsymbol{f}}(x) = x^k - \sum_{j=1}^{k} |f_j|_j^j x^{k-j}$. We prove that $\rho(\cdot)$ (which in general cannot be expressed by radicals when $k \geq 5$) is a quasinorm on $X_1 \times \dots \times X_k$, which we call *root quasinorm*, and we find a characterization of this quasinorm as limit of ratios of consecutive terms of a linear recurrence relation. If $X_1, \dots, X_k$ are normed, Banach or Banach function spaces, then the same construction gives respectively a normed, Banach or a Banach function space. Norms obtained as roots of polynomials are already known in the framework of the variable Lebesgue spaces, in the case of the exponent simple function with values $1, \dots, k$. We investigate the properties of the root quasinorm and we establish a number of inequalities, which come from a rich literature of the past century.

1. Introduction

Let $\Omega \subset \mathbb{R}^n$ be a set of positive Lebesgue measure and let $\mathcal{M}(\Omega)$ be the set of all measurable functions with respect to the Lebesgue measure, defined on Ω, with values in $[-\infty, +\infty]$. Given a mapping $\|\cdot\|_X : \mathcal{M}(\Omega) \to [0, \infty]$, the set

$$X = \{f \in \mathcal{M}(\Omega) : \|f\|_X < \infty\},$$

is a Banach function space over Ω (see [3]; for the following equivalent definition, see [10]) if the pair $(X, \|\cdot\|_X)$ satisfies the following properties for all $f, g \in \mathcal{M}(\Omega)$:

- ***i*** $\|f\|_X = \big\||f|\big\|_X$ and $\|f\|_X = 0$ if and only if $f \equiv 0$;
- ***ii*** $\|af\|_X = |a|\|f\|_X \qquad \forall a \in \mathbb{R}$;
- ***iii*** $\|f + g\|_X \leq \|f\|_X + \|g\|_X$;
- ***iv*** if $|f| \leq |g|$ almost everywhere, then $\|f\|_X \leq \|g\|_X$;
- ***v*** if $\{f_n\} \subset \mathcal{M}(\Omega)$ is a sequence such that $|f_n|$ increases to $|f|$ almost everywhere, then $\|f_n\|_X$ increases to $\|f\|_X$;
- ***vi*** if $E \subset \Omega$ is measurable set and $|E| < \infty$, then $\|\chi_E\|_X < \infty$;
- ***vii*** $\int_E |f(x)|\, dx \leq C_E \|f\|_X$ if $|E| < \infty$, where $C_E < \infty$ depends on E, but not on f.

Observe that all bounded, compactly supported functions on Ω are in every Banach function space X.

A class of Banach function spaces of high interest in recent years is that one of the variable Lebesgue spaces (see [10]). Given a finite exponent function on Ω, i.e. a Lebesgue

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measurable function $p(\cdot) : \Omega \to [1, \infty[$, the space $L^{p(\cdot)}(\Omega)$ is defined as the set of Lebesgue measurable functions f on Ω such that

$$\int_\Omega \left|\frac{f(x)}{\lambda}\right|^{p(x)} dx < \infty$$

for some $\lambda > 0$. The norm in $L^{p(\cdot)}(\Omega)$ is given by

$$\|f\|_{L^{p(\cdot)}(\Omega)} = \inf\left\{\lambda > 0 : \int_\Omega \left|\frac{f(x)}{\lambda}\right|^{p(x)} dx \le 1\right\}.$$

Suppose that the exponent $p(\cdot)$ is a particular simple function:

$$p(x) = \sum_{j=1}^k j\chi_{\Omega_j}(x), \qquad \Omega = \bigcup_{j=1}^k \Omega_j, \quad \text{with } \Omega_j,\ |\Omega_j| > 0\ \forall j,\ \text{pairwise disjoint.}$$

In this case the norm is given by the nonnegative λ such that

$$\int_{\Omega_1} \left|\frac{f(x)}{\lambda}\right| dx + \int_{\Omega_2} \left|\frac{f(x)}{\lambda}\right|^2 dx + \ldots + \int_{\Omega_k} \left|\frac{f(x)}{\lambda}\right|^k dx = 1$$

i.e.

$$\lambda^k - \sum_{j=1}^k \|f\|^j_{L^j(\Omega_j)} \lambda^{k-j} = 0$$

The question now is to understand what happens when in the place of $L^j(\Omega_j)$ one substitutes a generic quasinormed space $(X_j, |\cdot|_j)$. Given $\boldsymbol{f} = (f_1, \ldots, f_k) \in X_1 \times \ldots \times X_k$, is there a unique nonnegative root of the following Cauchy polynomial?

$$p_{\boldsymbol{f}}(x) = x^k - \sum_{j=1}^k |f_j|_j^j x^{k-j}$$

Is this root still a quasinorm? The aim of this paper is to show that the answers are positive and, moreover, if the quasinorm satisfies more properties (i.e. is a norm, or the norm of a Banach space, or a Banach function norm [3]), then also this root has the same properties. Moreover, we find a characterization of this quasinorm as limit of ratios of consecutive terms of a linear recurrence relation having $p_{\boldsymbol{f}}$ as characteristic polynomial.

We remark that even in the simplest case of $X_1 = \ldots = X_k = \mathbb{R}$ (so that $\boldsymbol{f}$ is in $\mathbb{R}^k$) the result is that such root, which in general cannot be expressed by radicals when $k \ge 5$, is a norm which seems to be new in the literature.

2. Background

2.1. **Quasinorms.** Let X be a (real) vector space. A functional $|\cdot| : X \to \mathbb{R}_0^+$ such that for some $C > 0$, for every $f, g \in X$

- *(j)* $|f|_X = 0$ if and only if $f \equiv 0$
- *(jj)* $|af|_X = |a||f|_X \quad \forall a \in \mathbb{R}$
- *(jjj)* $|f + g|_X \le C(|f|_X + |g|_X)$

is said to be a *quasinorm*.

We observe that if $[\cdot] : X \longrightarrow \mathbb{R}_0^+$ is a functional *equivalent* to a quasinorm $|\cdot|_X$, i.e.

$$c_1|f|_X \le [f] \le c_2|f|_X \quad \forall f \in X,$$

for some $c_1, c_2 > 0$, then $[\cdot]$ is not necessarily a quasinorm. It is immediate to check that $[\cdot]$ satisfies ***(j)***; it satifies ***(jjj)***:

$$[f+g] \le c_2|f+g|_X \le c_2 C(|f|_X + |g|_X) \le \frac{c_2}{c_1} C([f]+[g]) \tag{2.1}$$

but it does not necessarily satisfies ***(jj)***: consider for instance $L^1(0,1)$ with the quasinorm

$$|f|_{L^1} = \left(1 + \arctan\left(\int_0^1 |f(x)|dx\right)\right) \int_0^1 |f(x)|dx$$

which evidently is equivalent to the $L^1(0,1)$ norm of f but it does not satisfies ***(jj)***.

If a quasinorm satisfies ***(jjj)*** with $C = 1$, then it is called a *norm* and usually it is denoted by $\|\cdot\|$. It is well known that a quasinorm is not necessarily a norm, but this holds even when a quasinorm is equivalent to a norm: consider, for instance, the following two examples.

Example 2.1. Let us consider the Lebesgue space $L^1(0,1)$ with the quasinorm

$$|f|_{L^1} = (2 - \sin(\pi|\mathrm{supp} f|)) \int_0^1 |f(x)|dx$$

where $\mathrm{supp} f$ denotes the support of f and $|\mathrm{supp} f|$ its Lebesgue measure. We observe that $|\cdot|_{L^1}$ is a quasinorm, because it satisfies ***(jjj)*** with $C = 2$, it is evidently equivalent to the standard L^1 norm, and it is not a norm because it does not satisfy the triangle inequality when one considers $f = \chi_{(0,1/2)}$ and $g = \chi_{(1/2,1)}$.

Example 2.2. Let $p \ge 1$. For w nonnegative measurable function on $(0,\infty)$, $w \not\equiv 0$, $W(t) = \int_0^t w(s)ds < \infty$, consider the Lorentz space $\Lambda^q(w)$ as the set of the measurable functions f such that

$$|f|_{\Lambda^q(w)} = \left(\int_0^\infty (f^*(t))^q w(t)dt\right)^{1/q} < \infty \tag{B_p}$$

where f^* denotes the decreasing rearrangement of f (see [3]). Setting $w(t) = (q/p)t^{(q/p)-1}$ one gets the classical Lorentz space $L^{p,q}(0,\infty)$. When $q > p$ the weight w is not non-increasing, $W \in \Delta_2$ (i.e. $W(2t) \le K_W W(t)$ for all $t > 0$, for some $K_W > 0$) and

$$t^p \int_t^\infty x^{-p} w(x)dx \le K \int_0^t w(x)dx,$$

and therefore (see [2], [11] and references therein; for the importance of this class of weights see e.g. [7]) $|\cdot|_{\Lambda^p(w)}$ is a quasinorm, is not a norm but is equivalent to a norm.

We recall that quasinorms are not necessarily equivalent to norms, but, on the other hand, a classical result by Aoki-Rolewicz states that quasinorms are always equivalent to the so called q-norms (see [29]).

Sums of quasinorms are evidently quasinorms. They appear typically in the following contexts:

(A) Let $X_1, \dots, X_k$ be quasinormed spaces with quasinorms $|\cdot|_j$, $j = 1, \dots, k$, respectively. Then, if for every $\boldsymbol{f} = (f_1, \dots, f_k) \in X_1 \times \ldots \times X_k$ we set

$$|\boldsymbol{f}|_{X_1 \times \ldots \times X_k} = |f_1|_1 + \cdots + |f_k|_k,$$

then $|\cdot|$ is a quasinorm on $X_1 \times \ldots \times X_k$.

(B) Let X be a quasinormed space with quasinorm $|\cdot|$, and let $X_1,\dots,X_k$ be quasinormed spaces with quasinorms $|\cdot|_j$, $j=1,\dots,k$, respectively. Let $T_j : X \to X_j$, $j=1,\dots,k$, be sublinear operators. Then

$$[f] = |f| + |T_1 f|_1 + \cdots + |T_k f|_k$$

is a quasinorm on the vector subspace Y of X

$$Y = \{f \in X : |T_j f|_j < \infty,\ j = 1,\dots,k\}.$$

Let us highlight two special cases of **(A)** (in the sense of isomorphisms). Assume that $X_1,\dots,X_k$ are, as vector spaces, subspaces of a vector space X. If for every $f \in X_1 \cap \cdots \cap X_k$ we set

$$|f| = |(f,\dots,f)|_{X_1\times\ldots\times X_k} = |f|_1 + \cdots + |f|_k,$$

then $|\cdot|$ is a quasinorm on $X_1 \cap \ldots \cap X_k$ (obviously equivalent to the maximum of the $|\cdot|_j$'s). On the other hand, assume that $X_1,\dots,X_k$ are, as quasinormed spaces, subspaces of a quasinormed space X such that for every j it is $X_j \cap \langle X_1,\dots,\widehat{X_j},\dots,X_k\rangle = \{0\}$ (here the symbol $\langle X_1,\dots,\widehat{X_j},\dots,X_k\rangle$ stands for the subspace of X generated by $X_1,\dots,X_k$ except the subspace X_j). If for every $f = f_1 + \cdots + f_k \in X_1 \oplus \cdots \oplus X_k$ we set

$$|f| = |(f_1,\dots,f_k)|_{X_1\times\ldots\times X_k} = |f_1| + \cdots + |f_k|,$$

then $|\cdot|$ is a quasinorm on $X_1 \oplus \ldots \oplus X_k$.

The context **(B)** is natural in Sobolev spaces theory. A classical example is $X = X_j = L^p(\Omega)$, $\Omega \subset \mathbb{R}^k$, $p \geq 1$, $j = 1,\dots,k$ and $T_j f = \partial_j f$, which gives a norm $[\cdot]_{W^{1,p}(\Omega)}$ of the space $W^{1,p}(\Omega)$; for an example of recent interest see [9].

2.2. **Cauchy polynomials.** Let $a_1,\dots,a_k$ be nonnegative real numbers. The polynomial

$$p(x) = x^k - \sum_{j=1}^{k} a_j^j x^{k-j}$$

is said *Cauchy polynomial* (usually the coefficients of x^{k-j} are not written with the jth power, but of course this is an equivalent definition). By the classical Descartes' rule of signs (see e.g. [27, Theorem 2] or [33, Theorem 1.1.3 p.3]; see also [4]), when at least one of the coefficients a_j's is nonzero, $p(x)$ has exactly a unique simple positive root, which we will call λ_+; if all the a_j's are zero, we will set $\lambda_+ = 0$. The importance of the root λ_+ relies on a classical result of Cauchy (1829) (recalled, for instance in [30, Theorem (27,1)p.122]) about the problem, which can be therefore reduced to the Cauchy polynomials, to bound the modulus of the (complex) roots of polynomials.

Some facts about λ_+ will be of crucial importance in the following. Next Lemma is essentially known since 1931, however, we give a proof which is implicit in ([36]).

Lemma 2.1. *Let $a_j \geq 0$, $j = 0,\dots,k-1$, and let λ_+ be the nonnegative root of*

$$p(x) = x^k - \sum_{j=1}^{k} a_j^j x^{k-j}.$$

Then

$$x > 0,\ p(x) > 0 \Rightarrow x > \lambda_+. \tag{2.2}$$

and $p(x)$ is increasing in $[\lambda_+, +\infty[$.

Proof. The assertion is trivial if each $a_i = 0$. Let $\lambda_+ > 0$ be the unique positive root of p. Since $p(0) < 0$, it is $p(x) < 0$ in $]0, \lambda_+[$ and since $p(x) \to +\infty$ as $x \to +\infty$, it is $p(x) > 0$ for all $x > \lambda_+$. Therefore (2.2) holds and $p(x)$ changes sign in λ_+, becoming from negative to positive when x increases. Moreover, since the derivative of p is of the same type as p, it changes its sign in its unique positive root, becoming from negative to positive when x increases. Such root cannot be greater than λ_+, otherwise p would be decreasing in $]0, \lambda_+[$ and therefore positive in $]0, \lambda_+[$. The conclusion is that in $]\lambda_+, +\infty[$ the sign of the derivative of p is positive, hence p is increasing in $]\lambda_+, +\infty[$. □

Remark 2.1. Let $p_1(x) = x^k - \sum_{j=1}^{k} a_j^j x^{k-j}$, $a_j \geq 0\ \forall j \in \{1, \dots, k\}$, be such that there exists $a_{j_0} \neq 0$ and let $\lambda_+(p_1)$ be the its (unique) positive root. Let $(b_1, \dots, b_k) \in \mathbb{R}^k$, $b_j \geq 0$ $\forall j \in \{1, \dots, k\}$, and let $p_2(x) = x^k - \sum_{j=1}^{k} (a_j + b_j)^j x^{k-j}$ and again call $\lambda_+(p_2)$ its (unique) positive root. We observe that

$$p_2(\lambda_+(p_2)) = 0 = p_1(\lambda_+(p_1)) \geq p_2(\lambda_+(p_1))$$

and therefore by Lemma 2.1

$$\lambda_+(p_1) \leq \lambda_+(p_2)$$

Remark 2.2. As a simple consequence of Lemma 2.1 (this is known, too, but with a different proof, see [5, 36]), λ_+ can be majorized by the sum of the a_j's: since $p(a_j) \leq 0$, it is $a_j \leq \lambda_+$ for all $j = 1, \dots, k$ and therefore

$$\lambda_+ = \lambda_+(\lambda_+)^{-k}(\lambda_+)^k = \lambda_+(\lambda_+)^{-k} \sum_{j=1}^{k} a_j^j \lambda_+^{k-j} = \lambda_+(\lambda_+)^{-k} \sum_{j=1}^{k} a_j a_j^{j-1} \lambda_+^{k-j}$$

$$\leq \lambda_+(\lambda_+)^{-k} \sum_{j=1}^{k} a_j \lambda_+^{j-1} \lambda_+^{k-j} = \lambda_+(\lambda_+)^{-k} \sum_{j=1}^{k} a_j \lambda_+^{k-1} = \sum_{j=1}^{k} a_j$$

Remark 2.3. Since $a_j \leq \lambda_+$ for all $j = 1, \dots, k$, summing on j we get immediately that $(1/k) \sum_{j=1}^{k} a_j \leq \lambda_+$

As a consequence of the above Remarks, we get

$$\frac{1}{k} \sum_{j=1}^{k} a_j \leq \lambda_+ \leq \sum_{j=1}^{k} a_j \tag{2.3}$$

Finally, let us recall the following property of λ_+, consequence of a classical result by Ostrovsky (see e.g. [33, Theorem 1.1.4 p.3]):

Proposition 2.2. *Let $a_j > 0$, $j = 1, \dots, k$. If λ_+ is the unique positive root of*

$$p(x) = x^k - \sum_{j=1}^{k} a_j^j x^{k-j},$$

then for every root $\lambda \neq \lambda_+$ of $p(x)$ it is $|\lambda| < \lambda_+$.

2.3. **Linear recurrences.** A special class of sequences is of intrinsic interest and has been a central part of number theory for many years: the linear recurrences. They appear in several areas of mathematics, and in computer science. They are sequences defined through relations of the type

$$F_n = \sum_{j=1}^{k} \alpha_j F_{n-j} \tag{2.4}$$

for all $n \geq k$, with fixed $\alpha_j \in \mathbb{R}$, $j = 1, \dots, k$. The k first elements $F_0, F_1, \dots, F_{k-1}$ of (F_n) are called *initial conditions*. If the initial conditions are given, every element of the sequence is uniquely determined by the law (2.4). The *characteristic polynomial* is the polynomial

$$p(x) = x^k - \sum_{j=1}^{k} \alpha_j x^{k-j}$$

whose (complex) roots $\lambda_1, \dots, \lambda_h$ (pairwise distinct) are called *characteristic roots*. It is well known that (F_n) admits the explicit expression

$$F_n = \sum_{i=1}^{h} c_{i,1}\lambda_i^n + c_{i,2} n \lambda_i^n + \cdots + c_{i,k_i} n^{k_i - 1} \lambda_i^n$$

where $k_1, \dots, k_h$ are respectively the multiplicity of $\lambda_1, \dots, \lambda_h$, $\sum_{i=1}^{h} k_i = k$. For details on recurrence sequences, see e.g. [15].

It is known that the *Kepler limit* of (F_n), i.e. the limit of the ratio of consecutive terms of (F_n) does not always exist. A necessary and sufficient condition for its existence is in [16, 17]. We remark that the nonexistence of the Kepler limit may happen even if $\alpha_j > 0$ for all $j = 1, \dots, k$ (in this case, after Proposition 2.2, we know that there exists a unique root of maximum modulus, called *dominant*): consider, for instance, the sequence

$$\begin{cases} F_1 &= -1 \\ F_2 &= 2 \\ F_3 &= -1 \\ F_n &= F_{n-1} + F_{n-2} + 2F_{n-3} \quad \forall n \in \mathbb{N}, n > 3, \end{cases}$$

which is the periodic sequence $-1, 2, -1, 2, \dots$

It is our purpose to show that if the initial conditions are nonnegative, then this phenomenon does not happen. We begin with the following

Lemma 2.3. *If $\alpha_j > 0$, $j = 1, \dots, k$, $j_0 \in \{1, \dots, k\}$, and (G_n) is the sequence defined by*

$$\begin{cases} G_j &= 0 \quad \forall j \in \{1, \dots, k\}, j \neq j_0 \\ G_{j_0} &= 1 \\ G_n &= \sum_{j=1}^{k} \alpha_j G_{n-j} \quad \forall n \in \mathbb{N}, n > k, \end{cases}$$

then (G_n) admits the representation

$$G_n = c_+ \lambda_+^n + o(\lambda_+^n) \qquad \forall n \in \mathbb{N} \tag{2.5}$$

where $c_+ > 0$ *and* λ_+ *is the unique positive root of* $p(x) = x^k - \sum_{j=1}^{k} \alpha_j x^{k-j}$

Proof. The sequence (G_n) admits the characteristic polynomial $p(x)$ whose pairwise distinct roots $\lambda_1, \ldots, \lambda_h$ (among which there is λ_+), with multiplicity respectively $k_1, \ldots, k_h$, $k_1 + \ldots + k_h = k$, are such that

$$G_n = c_{1,1}\lambda_1^n + c_{1,2}n\lambda_1^n + \ldots + c_{1,k_1}n^{k_1-1}\lambda_1^n + \ldots + c_{h,k_h}n^{k_h-1}\lambda_h^n \tag{2.6}$$

Since the α_j's are all positive, by Proposition 2.2 there exists exactly one positive real dominant root λ_+ of multiplicity 1. We first show that $c_+ \neq 0$, where c_+ is the coefficient in (2.6) of λ_+^n. The number c_+ is a coordinate of the solution of a system whose coefficient matrix A (which has nonzero determinant, see [28, Theorem 21] or the proof of Proposition 2.11 in [19])) is given by

$$\begin{pmatrix} 1 & 0 & \ldots & 0 & \ldots & 1 & 0 & \ldots & 0 \\ \lambda_1 & \lambda_1 & \ldots & \lambda_1 & \ldots & \lambda_h & \lambda_h & \ldots & \lambda_h \\ \lambda_1^2 & 2\lambda_1^2 & \ldots & 2^{k_1-1}\lambda_1^2 & \ldots & \lambda_h^2 & 2\lambda_h^2 & \ldots & 2^{k_h-1}\lambda_h^2 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \lambda_1^{k-1} & (k-1)\lambda_1^{k-1} & \ldots & (k-1)^{k_1-1}\lambda_1^{k-1} & \ldots & \lambda_h^{k-1} & (k-1)\lambda_h^{k-1} & \ldots & (k-1)^{k_h-1}\lambda_h^{k-1} \end{pmatrix}$$

By Cramer's rule, c_+ is equal to a ratio whose numerator is a determinant of a matrix whose rows are of the same type as the rows of A, therefore linearly independent. Hence $c_+ \neq 0$. Since $G_n > 0$ for all $n > k$, it must be $c_+ > 0$ and therefore, by (2.6) and Proposition 2.2, (2.5) holds. □

Of course, if the characteristic polynomial is a Cauchy polynomial and the initial conditions are all positive, the Kepler limit, if it exists, must be exactly λ_+. Here we are going to show that indeed the Kepler limit exists.

Proposition 2.4. *If* $\alpha_j > 0$, $j = 1, \ldots, k$, *and* (F_n) *is any sequence such that* $F_j \geq 0$, $j = 1, \ldots, k$, $(F_1, \ldots, F_k) \neq \mathbf{0}$,

$$F_n = \sum_{j=1}^{k} \alpha_j F_{n-j} \quad \forall n \in \mathbb{N}, n > k,$$

then (F_n) *admits the Kepler limit, i.e. there exists finite the*

$$\lim_{n\to\infty} \frac{F_{n+1}}{F_n}$$

and it is positive.

Proof. For each $j \in \{1, \ldots, k\}$ let $G_n^{(j)}$ be the sequence defined as in Lemma 2.3, where $G_j = 1$, and let $c_+^{(j)}$ be the first coefficient in the representation (2.5). By the linearity of the recursive sequence (F_n), it is

$$F_n = \sum_{j=1}^{k} F_j G_n^{(j)} \quad \forall n \in \mathbb{N}$$

and therefore, by Lemma 2.3,

$$F_n = \sum_{j=1}^{k} F_j c_+^{(j)} \lambda_+^n + o(\lambda_+^n)$$

Hence

$$\lim_{n\to\infty}\frac{F_{n+1}}{F_n}=\sum_{j=1}^{k}F_j c_+^{(j)}\lambda_+$$

which is positive because $F_j \geq 0$, $j=1,\dots,k$, $(F_1,\dots,F_k)\neq \mathbf{0}$, and $c_+^{(j)}>0$. □

We remark that the assumption $F_j \geq 0$, $j=1,\dots,k$, $(F_1,\dots,F_k)\neq \mathbf{0}$, together with $c_+^{(j)}>0$, $j=1,\dots,k$, ensures that $\sum_{j=1}^{k}F_j c_+^{(j)}>0$ therefore $\sum_{j=1}^{k}F_j c_+^{(j)}\neq 0$. If it would be $\sum_{j=1}^{k}F_j c_+^{(j)}=0$, then in $o(\lambda_+^n)$ it may exist more than one dominant term and therefore (see [16, 17]) the Kepler limit may not exist.

Remark 2.4. Given any Cauchy polynomial $p(x)$ with at least one α_j's different from zero, there exists a linear recurrence whose characteristic polynomial coincides with $p(x)$ having as Kepler limit the positive root λ_+ of $p(x)$: in fact it is sufficient to consider, for instance, as linear recurrence, any geometric sequence whose common ratio is exactly λ_+.

3. The root quasinorm

Let $X_1,\dots,X_k$ be quasinormed spaces with quasinorms $|\cdot|_j$, $j=1,\dots,k$, respectively. Then, for every $\boldsymbol{f}=(f_1,\dots,f_k)\in X_1\times\cdots\times X_k$, consider the Cauchy polynomial:

$$p_{\boldsymbol{f}}(x)=x^k-\sum_{j=1}^{k}|f_j|_j^j x^{k-j} \tag{3.1}$$

If $\boldsymbol{f}=\mathbf{0}$, $p_{\boldsymbol{f}}(x)=x^k$ and admits only the root 0; if $\boldsymbol{f}\neq\mathbf{0}$, $p_{\boldsymbol{f}}(x)$ is a Cauchy polynomial and has exactly a unique simple positive root (we will call it $\lambda_+(p_{\boldsymbol{f}})$; when $\boldsymbol{f}=\mathbf{0}$ we set $\lambda_+(p_{\boldsymbol{f}})=0$). The main feature of this root is the following: the function $\boldsymbol{f}$ plays the role of a *parameter* in a set of Cauchy polynomials. We are going to show that the correspondence which associates to $\boldsymbol{f}$ the root $\lambda_+(p_{\boldsymbol{f}})$ has a very interesting property: it is a *quasinorm*, which we call *root quasinorm*. We remark that the functional $\boldsymbol{f}\to\lambda_+(p_{\boldsymbol{f}})$, by (2.3), is equivalent to a quasinorm (which is in fact a sum of quasinorms), but this is, in general, not sufficient to conclude that it is a quasinorm. Therefore it makes sense to prove the following result, where we set

$$[\,\boldsymbol{f}\,]:=\lambda_+(p_{\boldsymbol{f}}) \qquad \forall \boldsymbol{f}\in X_1\times\cdots\times X_k. \tag{3.2}$$

Proposition 3.1. *The map*

$$\boldsymbol{f}\in X_1\times\cdots\times X_k \quad\longrightarrow\quad [\,\boldsymbol{f}\,]\in[0,+\infty[\tag{3.3}$$

defines a quasinorm.

Proof. Property ***(j)*** is immediate from the definition of $[\,\cdot\,]$, while ***(jjj)*** follows from (2.1).

As to property ***(jj)***, since

$$p_{a\boldsymbol{f}}(x)=x^k-\sum_{j=1}^{k}|af_j|_j^j x^{k-j}=x^k-\sum_{j=1}^{k}|a|^j|f_j|_j^j x^{k-j}$$

and

$$p_{a\boldsymbol{f}}(a\lambda_+(p_{\boldsymbol{f}}))=a^k\lambda_+(p_{\boldsymbol{f}})^k-\sum_{j=1}^{k}|a|^j|f_j|_j^j a^{k-j}\lambda_+(p_{\boldsymbol{f}})^{k-j}=a^k p_{\boldsymbol{f}}(\lambda_+(p_{\boldsymbol{f}}))=0$$

we have

$$[\,a\boldsymbol{f}] = \lambda_+(p_{a\boldsymbol{f}}) = a\lambda_+(p_{\boldsymbol{f}}) = a[\,\boldsymbol{f}\,].$$

□

Remark 3.1. In the case $X_1,\dots,X_k$ are normed spaces with norms $\|\cdot\|_j$, $j=1,\dots,k$, respectively, also $[\,\cdot\,]$ is a norm (in this case we will call it *root norm*). We need to prove only the triangle property assuming, without loss of generality, that both $\boldsymbol{f}$ and $\mathbf{g}$ are not $\mathbf{0}$. We have

$$\begin{aligned}
([\,\boldsymbol{f}\,]+[\,\boldsymbol{g}\,])^k &= ([\,\boldsymbol{f}\,]+[\,\boldsymbol{g}\,])^{k-1}([\,\boldsymbol{f}\,]+[\,\boldsymbol{g}\,])\\
&= ([\,\boldsymbol{f}\,]+[\,\boldsymbol{g}\,])^{k-1}\left(\sum_{j=1}^k [\,\boldsymbol{f}\,]\left(\frac{\|f_j\|_j}{[\,\boldsymbol{f}\,]}\right)^j + \sum_{j=1}^k [\,\boldsymbol{g}\,]\left(\frac{\|g_j\|_j}{[\,\boldsymbol{g}\,]}\right)^j\right)\\
&= \sum_{j=1}^k([\,\boldsymbol{f}\,]+[\,\boldsymbol{g}\,])^{k-j}([\,\boldsymbol{f}\,]+[\,\boldsymbol{g}\,])^{j-1}\left([\,\boldsymbol{f}\,]\left(\frac{\|f_j\|_j}{[\,\boldsymbol{f}\,]}\right)^j + [\,\boldsymbol{g}\,]\left(\frac{\|g_j\|_j}{[\,\boldsymbol{g}\,]}\right)^j\right)\\
&= \sum_{j=1}^k([\,\boldsymbol{f}\,]+[\,\boldsymbol{g}\,])^{k-j}\left(([\,\boldsymbol{f}\,]+[\,\boldsymbol{g}\,])^{j-1}[\,\boldsymbol{f}\,]\left(\frac{\|f_j\|_j}{[\,\boldsymbol{f}\,]}\right)^j + ([\,\boldsymbol{f}\,]+[\,\boldsymbol{g}\,])^{j-1}[\,\boldsymbol{g}\,]\left(\frac{\|g_j\|_j}{[\,\boldsymbol{g}\,]}\right)^j\right)\\
&= \sum_{j=1}^k([\,\boldsymbol{f}\,]+[\,\boldsymbol{g}\,])^{k-j}\left(\left(1+\frac{[\,\boldsymbol{g}\,]}{[\,\boldsymbol{f}\,]}\right)^{j-1}\|f_j\|_j^j + \left(\frac{[\,\boldsymbol{f}\,]}{[\,\boldsymbol{g}\,]}+1\right)^{j-1}\|g_j\|_j^j\right)
\end{aligned}$$

We now observe that

$$\min_{t>0}\|f_j\|_j^j\left(1+\frac{1}{t}\right)^{j-1} + \|g_j\|_j^j(1+t)^{j-1} = (\|f_j\|_j+\|g_j\|_j)^j \qquad j=1,\dots,k$$

and therefore we have

$$([\,\boldsymbol{f}\,]+[\,\boldsymbol{g}\,])^k \ge \sum_{j=1}^k(\|f_j\|_j+\|g_j\|_j)^j([\,\boldsymbol{f}\,]+[\,\boldsymbol{g}\,])^{k-j} \ge \sum_{j=1}^k\|f_j+g_j\|_j^j([\,\boldsymbol{f}\,]+[\,\boldsymbol{g}\,])^{k-j}.$$

Hence, by Lemma (2.1),

$$[\,\boldsymbol{f}+\boldsymbol{g}\,] \le [\,\boldsymbol{f}\,]+[\,\boldsymbol{g}\,].$$

Let us now change the point of view. For each $\boldsymbol{f} = (f_1,\dots,f_k) \neq \mathbf{0}$ let us consider the following recurrence relation

$$F_n = \sum_{j=1}^k |f_j|_j^j F_{n-j} \tag{3.4}$$

We already observed (Remark 2.4) that there exist initial conditions such that (3.4) admits the Kepler limit, which coincides with the root λ_+ of the Cauchy polynomial (3.1). Therefore Proposition 3.1 is still true if one defines $[\,\cdot\,]$ as follows:

$$[\,\boldsymbol{f}\,] := \begin{cases}\lim\limits_{n\to\infty}\dfrac{F_{n+1}}{F_n} & \text{if } \boldsymbol{f}\in X_1\times\cdots\times X_k,\quad \boldsymbol{f}\neq\mathbf{0}\\ 0 & \text{if } \boldsymbol{f}=\mathbf{0}\end{cases} \tag{3.5}$$

Example 3.1. If Y is a quasinormed space with quasinorm $|\cdot|_Y$, then for every $f \in Y$ the root quasinorm of $\boldsymbol{f} = (f, f) \in Y \times Y$ is given by

$$[\,(f, f)\,] = \Phi |f|_Y$$

where $\Phi = \dfrac{1+\sqrt{5}}{2}$ denotes the Golden Mean, because $\Phi|f|_Y$ is the nonnegative root of the Cauchy polynomial $x^2 - |f|_Y x - |f|_Y^2$.

Now let X be a quasinormed space with quasinorm $|\cdot|$, and let $X_1, \ldots, X_k$ be quasinormed spaces with quasinorms $|\cdot|_j$, $j = 1, \ldots, k$, respectively. Let $T_j : X \to X_j$, $j = 1, \ldots, k$, be linear operators. For every $f \in X$, consider the Cauchy polynomial:

$$p_f(x) = x^{k+1} - |f|x^k - \sum_{j=1}^{k} |T_j f|_j^{j+1} x^{k-j} \tag{3.6}$$

If $f = 0$, $p_f(x) = x^{k+1}$ and admits only the root 0; if $f \neq 0$, $p_f(x)$ is a Cauchy polynomial and therefore has exactly a unique simple positive root (we will call it $\lambda_+(p_f)$; when $f = 0$ we set $\lambda_+(p_f) = 0$). As before, one can prove that the correspondence which associates to f the root $\lambda_+(p_f)$ is a quasinorm, which is again a Kepler limit in the case $f \neq 0$. We conclude observing that also in this case the functional $f \to \lambda_+(p_f)$ is equivalent to the quasinorm $|\cdot| + \sum_{j=1}^{k} |T_j \cdot|_j$, namely,

$$c_1 \left(|f| + \sum_{j=1}^{k} |T_j f|_j \right) \leq \lambda_+(p_f) \leq c_2 \left(|f| + \sum_{j=1}^{k} |T_j f|_j \right)$$

4. Strict and uniform convexity

4.1. **Strict convexity.** A normed space is strictly convex if every chord of the unit sphere has its midpoint below the surface of the unit sphere, or, equivalently,

$$\|x\| = 1, \ \|y\| = 1, \ \left\| \frac{x+y}{2} \right\| = 1 \Rightarrow x = y \tag{4.1}$$

Let $X_1, \ldots, X_k$ be normed spaces with norms $\|\cdot\|_j$, $j = 1, \ldots, k$, respectively, and let X be the normed space with norm $[\cdot]$ given by (3.2). Our purpose is to establish the strict convexity of the space X from the strict convexity of the spaces $(X_j, \|\cdot\|_j)$. Conditions (also necessary and sufficient) for strict convexity of absolute norms (see [34]) are known (see e.g. [13, Theorem 6], [34, Theorem 6], [14, Theorem 1] and [22, Theorem 3.3]; for the notion in the framework of Banach function lattices see [21]). However, such results cannot be directly applied in our case, because they require that the root norm, obtained starting from k copies of $\mathbb{R}$, is itself strictly convex. The proof that such root norm is strictly convex is in fact the same as the proof of the full result, which is given in next Proposition. Other results do not require the "external" norm strict convex (the central notion here is that one of $\psi-$direct sum of spaces; for recent results see e.g. [32]), but cannot be applied as well ([31]).

Proposition 4.1. *If $(X_j, \|\cdot\|_j)$ is strictly convex for $j = 1, \ldots, k$, the root norm $\|\cdot\|$ on X is strictly convex.*

Proof. We observe that if $\|x\| = 1,\ \|y\| = 1,\ \left\|\frac{x+y}{2}\right\| = 1$ then

$$2^k = \sum_{j=1}^{k} \|x_j + y_j\|_j^j 2^{k-j}, \quad 1 = \sum_{j=1}^{k} \|x_j\|_j^j, \quad 1 = \sum_{j=1}^{k} \|y_j\|_j^j$$

and therefore

$$\sum_{j=1}^{k} \left\|\frac{x_j + y_j}{2}\right\|_j^j = \sum_{j=1}^{k} \frac{\|x_j\|_j^j + \|y_j\|_j^j}{2} = 1 \tag{4.2}$$

by

$$\left\|\frac{x_j + y_j}{2}\right\|_j^j \leq \left(\frac{\|x_j\|_j + \|y\|_j}{2}\right)^j = \frac{(\|x_j\|_j + \|y\|_j)^j}{2^j} \leq \frac{\|x_j\|_j^j + \|y\|_j^j}{2}$$

summing on j, we get $=$ in (4.2) only if

$$\left\|\frac{x_j + y_j}{2}\right\|_j^j = \left(\frac{\|x_j\|_j + \|y\|_j}{2}\right)^j = \frac{(\|x_j\|_j + \|y\|_j)^j}{2^j} = \frac{\|x_j\|_j^j + \|y\|_j^j}{2} \quad j = 1, \dots, k$$

Therefore

$$\|x_j\|_j^j = \|y_j\|_j^j \quad j = 1, \dots, k$$

and

$$\left\|\frac{x_j + y_j}{2}\right\|_j = \frac{\|x_j\|_j + \|y_j\|_j}{2} \quad j = 1, \dots, k$$

hold.

Hence

$$\left\|\frac{\frac{x_j}{\|x_j\|} + \frac{y_j}{\|y_j\|}}{2}\right\|_j = 1 \quad j = 1, \dots, k$$

Then by hypothesis we have $x_j = y_j$, $j = 1, \dots, k$. from which $x = y$. ☐

4.2. **Uniform convexity.** A normed X space is uniformly convex (see [8]) if

$$\forall \epsilon > 0, \exists \delta > 0, \|x\| = 1,\ \|y\| = 1,\ \|x - y\| \geq \epsilon \Rightarrow \|x + y\| \leq 2 - \delta \tag{4.3}$$

Proposition 4.2. *If $(X_j, \|\cdot\|_j)$ is uniformly convex for $j = 1, \dots, k$, then the root norm $\|\cdot\|$ on X is uniformly convex.*

Proof. We apply a classical result by Day ([12]), recalled in ([14, Theorem 1(2)]), which establishes the uniform convexity from that one of the X_j's and that one of the root norm, obtained starting from k copies of $\mathbb{R}$. This latter norm is uniformly convex because in finite dimensional spaces the strict convexity implies the uniform convexity. ☐

We have already seen that by (2.3) the root norm $\|\cdot\|$ is equivalent to a norm given by a sum of norms which does not satisfy the strict convexity property (therefore also does not satisfy the uniform convexity property).

5. The case of Banach function spaces

Let $X_1, \dots, X_k$ be Banach function spaces respectively over $\Omega_1 \subset \mathbb{R}^{n_1}, \dots, \Omega_k \subset \mathbb{R}^{n_k}$ with norms $\|\cdot\|_j$, $j = 1, \dots, k$, respectively. Then, for every $\boldsymbol{f} = (f_1, \dots, f_k) \in X_1 \times \cdots \times X_k$, consider the Cauchy polynomial:

$$p_{\boldsymbol{f}}(x) = x^k - \sum_{j=1}^{k} \|f_j\|_j^j x^{k-j}$$

As before, if $\boldsymbol{f} = \boldsymbol{0}$, $p_{\boldsymbol{f}}(x) = x^k$ and admits only the root 0; if $\boldsymbol{f} \neq \boldsymbol{0}$, $p_{\boldsymbol{f}}(x)$ is a Cauchy polynomial and has exactly a unique simple positive root (we will call it $\lambda_+(p_{\boldsymbol{f}})$; when $\boldsymbol{f} = \boldsymbol{0}$ we set $\lambda_+(p_{\boldsymbol{f}}) = 0$). Setting, for every $\boldsymbol{f} \in \mathcal{M}(\Omega_1) \times \cdots \times \mathcal{M}(\Omega_k)$,

$$[\,\boldsymbol{f}\,] := \begin{cases} 0 & \text{if } \boldsymbol{f} = \boldsymbol{0} \\ \lambda_+(p_{\boldsymbol{f}}) & \text{if } \boldsymbol{f} \in X_1 \times \cdots \times X_k, \quad \boldsymbol{f} \neq \boldsymbol{0} \\ +\infty & \text{otherwise} \end{cases} \tag{5.1}$$

by (2.3) we have

$$c_1 \sum_{j=1}^{k} \|f_j\|_j \leq [\,\boldsymbol{f}\,] \leq c_2 \sum_{j=1}^{k} \|f_j\|_j$$

and by Remark 3.1 we know that $\boldsymbol{f} \to [\,\boldsymbol{f}\,]$ is a norm in $X_1 \times \cdots \times X_k$. The fact that $\|\cdot\|_j$ enjoys for all j's more properties with respect to the norm axioms implies that our construction generates a Banach function norm in the sense that $[\cdot] : \mathcal{M}(\Omega_1) \times \cdots \times \mathcal{M}(\Omega_k) \to [0, \infty]$ satisfies the following properties for all $\boldsymbol{f}, \boldsymbol{g} \in \mathcal{M}(\Omega_1) \times \cdots \times \mathcal{M}(\Omega_k)$:

- ***i'*** $[\boldsymbol{f}] = [\,(|f_1|, \dots, |f_k|)\,]$ and $[\boldsymbol{f}] = 0$ if and only if $\boldsymbol{f} \equiv \boldsymbol{0}$;
- ***ii'*** $[\,a\boldsymbol{f}] = |a|[\boldsymbol{f}] \qquad \forall a \in \mathbb{R}$;
- ***iii'*** $[\boldsymbol{f} + \boldsymbol{g}] \leq [\boldsymbol{f}] + [\,\boldsymbol{g}]$;
- ***iv'*** if $\forall j\ |f_j| \leq |g_j|$ almost everywhere, then $[\boldsymbol{f}] \leq [\,\boldsymbol{g}]$;
- ***v'*** if $\{(f_1^{(n)}, \dots, f_k^{(n)})\} \subset \mathcal{M}(\Omega_1) \times \cdots \times \mathcal{M}(\Omega_k)$ is a sequence such that $\forall j\ |f_j^{(n)}|$ increases to $|f_j|$ almost everywhere, then $[\,(f_1^{(n)}, \dots, f_k^{(n)})\,]$ increases to $[\,(f_1, \dots, f_k)\,]$;
- ***vi'*** if $E_1 \times \cdots \times E_k \subset \Omega_1 \times \cdots \times \Omega_k$ is measurable set and $|E_j| < \infty\ \forall j$, then $[\,(\chi_{E_1}, \dots, \chi_{E_k})\,] < \infty$;
- ***vii'*** $\int_{E_1 \times \cdots \times E_k} |\boldsymbol{f}|\, d\boldsymbol{x} \leq C_{E_1 \times \cdots \times E_k} [\boldsymbol{f}]$ if $|E_j| < \infty\ \forall j$, where $C_{E_1 \times \cdots \times E_k} < \infty$ depends on $E_1, \dots, E_k$ but not on $\boldsymbol{f}$.

Theorem 5.1. *The map*

$$\boldsymbol{f} \in X_1 \times \cdots \times X_k \quad \longrightarrow \quad [\,\boldsymbol{f}\,] \in [0, +\infty] \tag{5.2}$$

defines a Banach function space.

Proof. Properties ***i'***, ***ii'***, ***iii'*** have been proved in Proposition 3.1 and Remark 3.1 in the case $\boldsymbol{f} \in X_1 \times \cdots \times X_k$. If $\boldsymbol{f} \in \mathcal{M}(\Omega_1) \times \cdots \times \mathcal{M}(\Omega_k)$, $\boldsymbol{f} \notin X_1 \times \cdots \times X_k$, these properties are trivial.

As to ***iv'***, we consider first the case $\boldsymbol{f}, \boldsymbol{g} \in X_1 \times \cdots \times X_k$. without loss of generality we may assume $\boldsymbol{f}, \boldsymbol{g} \neq \boldsymbol{0}$. Since $[\,\boldsymbol{f}\,]$ is the unique positive root of

$$p_1(x) = x^k - \sum_{j=1}^{k} \|f_j\|_j^j x^{k-j}$$

and $[\,\boldsymbol{g}\,]$ is the unique positive root of

$$p_2(x) = x^k - \sum_{j=1}^{k} \|g_j\|_j^j x^{k-j}$$

by

$$\|f_j\|_j \le \|g_j\|_j \quad j = 1, \dots, k$$

we have

$$\|f_j\|_j^j \le \|g_j\|_j^j \quad j = 1, \dots, k,$$

and therefore

$$p_2([\,\boldsymbol{g}\,]) = 0 = p_1([\,\boldsymbol{f}\,]) \ge p_2([\,\boldsymbol{f}\,])$$

If by absurd it is $[\,\boldsymbol{f}\,] > [\,\boldsymbol{g}\,]$, then by Lemma 2.1 it would be $p_2([\,\boldsymbol{f}\,]) > 0$, which is a contradiction. If $\boldsymbol{g} \notin X_1 \times \cdots \times X_k$, the assertion is trivial; finally, if $\boldsymbol{f} \notin X_1 \times \cdots \times X_k$, then there exists j_0 such that $f_{j_0} \notin X_{j_0}$, from which $g_{j_0} \notin X_{j_0}$, hence $\boldsymbol{g} \notin X_1 \times \cdots \times X_k$ and $[\,\boldsymbol{f}\,]=[\,\boldsymbol{g}\,]= \infty$.

We prove now $\boldsymbol{v}'$. By $\boldsymbol{iv}'$ the sequence $[\,(f_1^{(n)}, \dots, f_k^{(n)})\,]$ increases to its supremum and, since every X_j satisfies the property $\boldsymbol{v}$, we can pass to the limit as $n \to \infty$ in the equality

$$[\,(f_1^{(n)}, \dots, f_k^{(n)})\,]^k - \sum_{j=1}^{k} \|f_j^{(n)}\|_j^j [\,(f_1^{(n)}, \dots, f_k^{(n)})\,]^{k-j} = 0$$

and we get

$$\left(\sup_n [\,(f_1^{(n)}, \dots, f_k^{(n)})\,]\right)^k - \sum_{j=1}^{k} \|f_j\|_j^j \left(\sup_n [\,(f_1^{(n)}, \dots, f_k^{(n)})\,]\right)^{k-j} = 0$$

from which the assertion follows.

Let us assume now that $\boldsymbol{f} \in \mathcal{M}(\Omega_1) \times \cdots \times \mathcal{M}(\Omega_k)$, $\boldsymbol{f} \notin X_1 \times \cdots \times X_k$. The assertion is immediate if for some $n \in \mathbb{N}$ it is also $f_j^{(n)} \notin X_j$, so that we may assume that $[\,(f_1^{(n)}, \dots, f_k^{(n)})\,]$ satisfies, for all $n \in \mathbb{N}$,

$$[\,(f_1^{(n)}, \dots, f_k^{(n)})\,]^k - \sum_{j=1}^{k} \|f_j^{(n)}\|_j^j [\,(f_1^{(n)}, \dots, f_k^{(n)})\,]^{k-j} = 0 \tag{5.3}$$

Since

$$\left(\max_j \|f_j^{(n)}\|_j\right)^k \le \sum_{j=1}^{k} \|f_j^{(n)}\|_j^j \left(\max_j \|f_j^{(n)}\|_j\right)^{k-j}$$

it is

$$\left(\max_j \|f_j^{(n)}\|_j\right)^k - \sum_{j=1}^{k} \|f_j^{(n)}\|_j^j \left(\max_j \|f_j^{(n)}\|_j\right)^{k-j} \le 0$$

and therefore, by (5.3) and Lemma 2.1,

$$[\,(f_1^{(n)}, \dots, f_k^{(n)})\,] \ge \max_j \|f_j^{(n)}\|_j$$

Since there exists j_0 such that $f_{j_0} \notin X_{j_0}$, it is $\|f_{j_0}^{(n)}\|_{j_0} \to \infty$ and therefore also $[\,(f_1^{(n)}, \dots, f_k^{(n)})\,] \to \infty$.

Finally, for $\boldsymbol{vi'}$ and $\boldsymbol{vii'}$, we observe that such properties are invariant by equivalences, therefore they are true since they are immediate with $[\cdot]= [\,(|f_1|,\dots,|f_k|)\,]$ replaced by $\left(\sum_{j=1}^{k}\|f_j\|_{X_j}^2\right)^{1/2}$. □

Properties $\boldsymbol{i'}$, ..., $\boldsymbol{vii'}$ may be considered as an extension to the vectorial case of properties $\boldsymbol{i}$, ..., $\boldsymbol{vii}$. We observe that another approach is to consider the norm $\||f(\cdot)|_X\|_E$ (see e.g. [9]), for functions f with values in a Banach space X, where E is a Banach function space (see also the approach in [26]). When $X = X_1 \times \cdots \times X_k$, X_j Banach function space for every j, and $E = L^1$, one obtains a Banach function space in the sense of properties $\boldsymbol{i'}$, ..., $\boldsymbol{vii'}$.

6. Norm inequalities

In the first decades of the last century several papers appeared about the bounds of the roots of polynomials in terms of their coefficients (see e.g. [18, 6, 25, 36] and references therein). Mainly the research was made in finding an upper bound for the modulus of the roots; we stress that recently the interest moved to the research of a bound for just the positive roots of polynomials, because this gives also a lower bound for the positive roots (see e.g. [1, 24, 35]). We have seen in Subsection 2.2 that a crucial role is played by Cauchy polynomials. In this section we wish to stress that every estimate of the positive root of the Cauchy polynomial (which is automatically an estimate of the modulus of any root of any polynomial) can be read as norm inequality. Let us consider, for instance, the following result by Cauchy (see e.g. [30, Theorem (27,2)p.123]):

Theorem 6.1. *All the zeroes of $f(z) = a_0 + a_1 z + \cdots + a_k z^k$, $a_k \neq 0$, lie in the circle $|z| < 1 + \max\{|a_j/a_k| : j = 0, 1, \dots, k-1\}$.*

Let $X_1, \dots, X_k$ be normed spaces with norms $\|\cdot\|_j$, $j = 1, \dots, k$, respectively, and let X be the normed space with norm $[\cdot]$ given by $[\,\boldsymbol{f}\,] = \lambda_+(p_{\boldsymbol{f}}) \quad \forall \boldsymbol{f} \in X_1 \times \cdots \times X_k$, where

$$p_{\boldsymbol{f}}(x) = x^k - \sum_{j=1}^{k} \|f_j\|_j^j x^{k-j}.$$

By Theorem 6.1,

$$[\,\boldsymbol{f}\,] < 1 + \max_j\{\|f_j\|_j^j\}. \tag{6.1}$$

We observe that this inequality is not satisfied by the norm given by the sum of the norms of $\|f_j\|_j$, in fact the following inequality in general is *not* true (for instance, consider the case when all the norms are 1):

$$\sum_{j=1}^{k} \|f_j\|_j < 1 + \max_j\{\|f_j\|_j^j\}$$

and also, for any classical $p-$norm, $1 < p < \infty$, the following inequality

$$\left(\sum_{j=1}^{k} \|f_j\|_j^p\right)^{1/p} < 1 + \max_j\{\|f_j\|_j^j\}$$

does not hold for each k sufficiently large.

We remark that property (6.1) is instead satisfied by the so-called *p-H H-norm* introduced in [23], (at least) in the case of Euclidean spaces:

$$\left(\int_0^1 |t\mathbf{x}+(1-t)\mathbf{y}|^p dt\right)^{1/p} < 1+\max\{|\mathbf{x}|,|\mathbf{y}|^2\} \qquad \forall \mathbf{x},\mathbf{y}\in\mathbb{R}^n, \quad 1\le p<\infty,$$

in fact the left hand side can be majorized by

$$\max_{t\in[0,1]}\{|t\mathbf{x}+(1-t)\mathbf{y}|\} = \max\{|\mathbf{x}|,|\mathbf{y}|\} < 1+\max\{|\mathbf{x}|,|\mathbf{y}|^2\} \qquad \forall \mathbf{x},\mathbf{y}\in\mathbb{R}^n.$$

We give now a short selection of norm inequalities which can be deduced from classical results, by an analogous procedure.

From [30, n.1, p.144] we have the following refinement of (6.1), also due to Cauchy:

$$[\,\boldsymbol{f}\,] < 1+\max_j\{1+\|f_1\|_1,\dots,1+\|f_{k-1}\|_{k-1}^{k-1},\|f_k\|_k^k\}.$$

Bounds obtained by Fujiwara ([18]) are

$$[\,\boldsymbol{f}\,] \le \max\left\{\lambda_j^{1/j}\|f_j\|_j : \sum_{j=1}^k \lambda_j^{-1}\le 1\right\}$$

and

$$[\,\boldsymbol{f}\,] \le 2\max_j\{\|f_j\|_j\}. \tag{6.2}$$

An improvement of (6.2) is the following (see [20, (9a)])

$$[\,\boldsymbol{f}\,] \le \max_j\left\{2\|f_1\|_1, \frac{j}{(j-1)^{\frac{j-1}{j}}}\|f_j\|_j\right\}$$

Another estimate is from Carmichael and Mason ([6]):

$$[\,\boldsymbol{f}\,] \le \sqrt{1+\sum_j \|f_j\|_j^{2j}}$$

and more generally ([25])

$$[\,\boldsymbol{f}\,] \le \left\{1+\left(\sum_j \|f_j\|_j^{j(p+1)}\right)^{1/p}\right\}^{p/(p+1)}$$

Finally, from [36, (e), p.32], one gets another improvement of (6.2):

$$[\,\boldsymbol{f}\,] \le \max\{\|f_r\|_r+\|f_s\|_s : r,s=1,\dots,k, r\ne s\}. \tag{6.3}$$

We conclude this section with two remarks. First, inequalities (6.2) and (6.3) can be complemented with a lower bound, in view of Remark 2.2:

$$\max_j\{\|f_j\|_j\} \le [\,\boldsymbol{f}\,] \le 2\max_j\{\|f_j\|_j\}$$

$$\max_j\{\|f_j\|_j\} \le [\,\boldsymbol{f}\,] \le \max\{\|f_r\|_r+\|f_s\|_s : r,s=1,\dots,k, r\ne s\}.$$

Second, all the inequalities above have a corresponding statement in the context **(B)**. We do not give a full list. As an example, we can affirm, after (6.3), that

$$[\,\boldsymbol{f}\,]_{W^{1,p}(\Omega)} \le \max\{\|\partial_r f\|_r+\|\partial_s f\|_s : r,s=0,\dots,k, r\ne s\}.$$

where we set $\partial_0 f = f$ and $[\,\boldsymbol{f}\,]_{W^{1,p}(\Omega)}$, $\Omega\subset\mathbb{R}^k$, is the norm in the Sobolev space $W^{1,p}(\Omega)$ defined in Subsection 2.1.

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