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Analysis of reflected diffusions via an exponential time-based transformation

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Abstract Let $X(t)$ be a time-homogeneous diffusion process with state-space $[0, +\infty)$, where 0 is a reflecting or entrance endpoint, and let Z denote a random variable that describes the process $X(t)$ evaluated at an exponentially distributed random time.

We propose a method to obtain closed-form expressions for the conditional density and the mean of a new diffusion process $Y(t)$, with the same state-space and with the same infinitesimal variance, whose drift depends on the infinitesimal moments of $X(t)$ and on the hazard rate function of Z. This method also allows us to obtain the Laplace transform of the first-passage-time density of $Y(t)$ through a lower constant boundary. We then discuss the relation between $Y(t)$ and the process $X(t)$ subject to catastrophes, as well as the interpretation of $Y(t)$ as a diffusion in a decreasing potential.

We study in detail some special cases concerning diffusion processes obtained when $X(t)$ is the Wiener, Ornstein-Uhlenbeck, Bessel and Rayleigh process.

Keywords Conditional density · First passage time · Wiener process · Ornstein-Uhlenbeck process · Bessel process · Rayleigh process

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1 Introduction

Determining probability density functions (pdf's) of diffusion processes is a relevant problem both from the theoretical and practical point of view. Nevertheless in many cases one is forced to adopt numerical approximations or simulation techniques since exact solutions are not feasible. Fruitful approaches leading to closed-form results for transition pdf's are based on suitable transformations of diffusion processes. We recall the space-time transformations proposed by Cherkasov [10], Ricciardi [41], Bluman [4], Sacerdote and Ricciardi [43], and Kwok [37], for the mapping of diffusion processes into the Wiener process, and by Capocelli and Ricciardi [9] for transformations into Feller processes. More general criteria are given in Bluman and Shtelen [5], involving nonlocal transformations, and in Borodin [6], for transformations of jump-diffusion processes. Other methods finalized to the determination of quantities of interest have been exploited in the past (among others, see Wong [48], Araujo and Drigo Filho [3]). We point out that many efforts have been devoted specifically to the determination of pdf's and other relevant quantities for reflected diffusion processes (see Linetski [38], Wonho [49], Ricciardi and Sacerdote [42], Giorno *et al.* [25], Inoue *et al.* [34]). Indeed, reflected diffusion processes have attracted a large interest over the last decades. They stem from several applications to physics, mathematical biology, queueing, electrical engineering, economic and finance. Specifically, reflecting diffusions often arise as approximations to stochastic models of interest in applied domains, such as communication networks, manufacturing systems, dam theory, where the content process take only nonnegative real values. In these cases the setting of a reflecting boundary at 0 is necessary. The availability of treatable expressions for the pdf's is essential for the related diffusion processes. Our investigation is thus stimulated by the need of discovering new methods leading to closed-form results. The latter are very relevant in practical applications, since they show the functional importance of the parameters and therefore are useful in design decisions.

Following on the previous studies, we propose a method able to determine the pdf's and means of certain one-dimensional time-homogeneous diffusion processes. The method is based on a transformation involving two diffusion processes with state-space $[0, +\infty)$, having the same infinitesimal variance, and where the drift of the new process, say $Y(t)$, is a suitable modification of the drift of the former process, say $X(t)$. The nature of the endpoints of such processes is specified as follows:

- 0 is a reflecting regular or an entrance endpoint for both processes $X(t)$ and $Y(t)$;
- $+\infty$ is an attracting natural endpoint for $Y(t)$, whereas it is either attracting or non attracting for $X(t)$.

The considered transformation involves the hazard rate function of $X(t)$ evaluated at a random exponential time and conditioned to zero initial state. The proposed method allows us to obtain the pdf and mean of $Y(t)$ conditional on $Y(0) = 0$, as well as for the density of $Y(t)$ evaluated at 0 conditional on $Y(0) = y$ (when 0 is a reflecting endpoint). The proposed method leads to diffusion processes that are suitable for describing populations subject to rapid growth, such as bacterial growth (cf. Golding *et al.* [29], for instance) and buffer occupation of stochastic networks in heavy traffic (see, e.g. Kushner [36]). For the new process $Y(t)$, we get the expressions of the Laplace transform of the first-passage-time density through a lower constant boundary.

In addition, we are able to express the pdf of $Y(t)$ in terms of the density of process $X(t)$ in the presence of catastrophes. We recall that there is some interest in the recent statistical physics literature on stochastic processes subject to catastrophes, also named as 'processes with stochastic reset'. See, for instance, Evans and Majumdar [17], [18] for some results on the Brownian motion with reset, Pal [40] for general diffusions with resetting, Giorno *et al.* [24] for the reflected Ornstein-Uhlenbeck process with catastrophes, and Di Crescenzo *et al.* [12], [13] for certain diffusion processes with catastrophes obtained as approximations of queuing systems. We also point out that the method exploited in this paper has been adopted recently in the context of birth-death processes (see [14]).

The paper is organized as follows. In Section 2, we provide the main results. Specifically, starting from a diffusion processes $X(t)$ with state-space $[0, +\infty)$ we define a new diffusion process $Y(t)$, with the same infinitesimal variance, and express the pdf and the mean of $Y(t)$ conditional on $Y(0) = 0$ in terms of the pdf of $X(t)$ and in terms of a new density $\psi(x)$. The latter is the density of $X(t)$ conditional on $X(0) = 0$ and evaluated at a random exponential time with mean $\xi^{-1} > 0$. The special case $\xi \downarrow 0$ is also investigated. We also point out that the process $Y(t)$ can be described as a diffusion in a decreasing potential depending on the infinitesimal moments of $X(t)$ and on $\psi(x)$. In Section 3, we provide an alternative form for the pdf of $Y(t)$, involving the density of process $X(t)$ in the presence of catastrophes. In Section 4, we obtain certain expressions for the Laplace transforms of the pdf of $Y(t)$, and for the first passage time density of $Y(t)$ through a constant boundary. The theoretical results are then applied to some special cases in Section 5, where we perform a thorough analysis of the diffusion processes obtained as transformations of the Wiener process, Ornstein-Uhlenbeck process, Bessel process, and Rayleigh process. In the last 3 cases the proposed transformation leads to processes that do not seem to have been studied in the past, whose drifts involve ratios of parabolic cylinder functions, modified Bessel functions of the second kind, and Kummer's functions of the second kind. In all cases we effectively compute the relevant closed-form expressions. In Section 6, we provide some concluding remarks, and perform a comparison of the potentials for the studied processes.

Throughout this paper, as usual, $[X|B]$ denotes a random variable having the same distribution of X conditional on B, and $\stackrel{d}{=}$ denotes equality in distribution.

2 Main results

Let $\{X(t), t \geq 0\}$ be a regular one-dimensional time-homogeneous diffusion process with drift $A_1(x)$ and infinitesimal variance $A_2(x)$ satisfying the typical regularity conditions (cf. Feller [19] and [20]). The state-space is $I = [0, +\infty)$, where 0 is a reflecting regular endpoint or an entrance endpoint, and $+\infty$ is a natural endpoint. If 0 is an entrance endpoint, it is not included in I. We recall (cf. Karlin and Taylor [35]) that a reflecting regular endpoint is such that a diffusion process can both enter and leave from it; and entrance endpoint cannot be reached from the interior of the state space, but a diffusion process can begin there; a natural endpoint cannot be reached in finite mean time and cannot be the starting point of a diffusion process. The nature of the endpoints is essential to establish the proper conditions, if any, to be associated with the diffusion equation in order to determine the transition density of the process (for more details see [19]). We denote the transition pdf of $X(t)$ by

$$f_X(x,t\,|\,y) = \frac{d}{dx} P[X(t) \leq x\,|\,X(0) = y], \qquad x, y \in I, \;\; t \geq 0.$$

It satisfies the Fokker-Planck equation

$$\frac{\partial}{\partial t} f_X(x,t\,|\,y) = -\frac{\partial}{\partial x}\left[A_1(x) f_X(x,t\,|\,y)\right] + \frac{1}{2}\frac{\partial^2}{\partial x^2}\left[A_2(x) f_X(x,t\,|\,y)\right], \qquad (1)$$

the Dirac delta initial condition

$$\lim_{t \downarrow 0} f_X(x,t\,|\,y) = \delta(x-y), \qquad (2)$$

and the reflecting or zero-flux boundary condition

$$\lim_{x \downarrow 0} \left\{ A_1(x) f_X(x,t\,|\,y) - \frac{1}{2}\frac{\partial}{\partial x}\left[A_2(x) f_X(x,t\,|\,y)\right] \right\} = \begin{cases} \delta(t), & y = 0, \\ 0, & y > 0. \end{cases} \qquad (3)$$

In several applications one is forced to consider a conditioned version of the process $X(t)$, leading to a new regular diffusion process whose drift, say $B_1(x)$, is given by the following suitable modification of $A_1(x)$:

$$B_1(x) = A_1(x) + A_2(x)\,\frac{\pi'(x)}{\pi(x)}, \qquad (4)$$

with prime denoting derivative, whereas the infinitesimal variance remains unchanged. The transformation shown in (4) has been successfully exploited by several authors:

(i) The case when $\pi(x)$ is equal to the speed function of $X(t)$ is treated in detail in Section 15.9 of [35];

(ii) The case when $\pi(x)$ is the first-exit probability of $X(t)$ from an interval has been investigated by Abundo [2] and by Giorno *et al.* [26], where the ratio of the transition densities of the new process and of the original process $X(t)$ is time-independent;

(iii) The case when $\pi(x)$ is equal to a suitable transition density of $X(t)$, or an its integral, has been studied in Buonocore *et al.* [7] and Giraudo *et al.* [28], in connection with the construction of certain diffusion bridges;
(iv) The case when $\pi(x)$ is a conditional survival probability of a certain killing time of the diffusion process is the object of Frydman [22].

The main aim of this paper is to develop a transformation of $X(t)$ based on a relation similar to (4), and with a new choice of the function appearing in the last term. We are motivated by the need of investigating the further potentiality of relations of such type, which are well suited to obtain new closed-form results for regular diffusion processes. We point out that the cases considered in the above mentioned papers, with reference to (4), do not deal with reflecting processes, whereas the approach exploited below focuses on processes restricted by a reflecting endpoint in 0.

In order to define a new diffusion process suitably related to $X(t)$, we introduce the following mixture density with exponential mixing pdf:

$$\psi(x) = \int_0^{+\infty} \xi\, e^{-\xi t} f_X(x, t \,|\, 0)\, dt, \qquad x \in I, \tag{5}$$

for $\xi > 0$. Hence, (5) is the density of a compound random variable, say

$$Z \stackrel{d}{=} [X(T) \,|\, X(0) = 0], \tag{6}$$

that describes the conditional process evaluated at a random time T exponentially distributed with mean ξ^{-1}, with T independent from $X(t)$.

Remark 1 For a given probability measure ν on I, let

$$L_\nu(t, x) = \lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \frac{\int_0^t \mathbf{1}\{|X(\tau) - x| \le \varepsilon\} \nu(d\tau)}{\int_0^t \nu(d\tau)}, \qquad t > 0, \quad x \in I$$

be a *ν-weighted local time* of $X(t)$. Roughly speaking, $L_\nu(t, x)$ denotes a normalized measure of how much time the process $X(t)$ has spent in the neighborhood of x up to time t, where time is sampled according to measure ν. Hence, if $\nu(d\tau) = \xi e^{-\xi\tau} d\tau$, from (5) we have the following asymptotic mean:

$$\lim_{t \to +\infty} E\left[L_\nu(t, x) \,|\, X(0) = 0\right] = \psi(x), \qquad x \in I.$$

We point out that $\psi(x)$ can also be expressed in terms of a Laplace transform. Indeed, denoting by

$$\widehat{f}_X(s; x \,|\, y) = \int_0^{+\infty} e^{-st} f_X(x, t \,|\, y)\, dt, \qquad s > 0, \quad x, y \in I \tag{7}$$

the Laplace transform of the transition pdf $f_X(x, t \,|\, y)$, from (5) we have

$$\psi(x) = \xi\, \widehat{f}_X(\xi; x \,|\, 0), \qquad x \in I. \tag{8}$$

In addition, in Section 3, we show that $\psi(x)$ can also be viewed as the steady-state pdf of the process $X(t)$ subject to catastrophes governed by a Poisson process with intensity ξ.

We point out that the density $f_X(x,t\,|\,0)$ may be divergent when $x \downarrow 0$. Therefore, in the following, we restrict our investigation to the case when $\lim_{x\downarrow 0} f_X(x,t\,|\,0)$ is finite for all $t>0$, and also $\psi(0)$ is finite.

Let $\{Y(t), t\geq 0\}$ be a regular one-dimensional time-homogeneous diffusion process with state-space I, and described by the stochastic differential equation

$$dY(t) = B_1[Y(t)]\,dt + \sqrt{B_2[Y(t)]}\,d\mathcal{B}(t), \qquad Y(0)=y_0, \tag{9}$$

where $\mathcal{B}(t)$ is a normalized Wiener process (standard Brownian motion) and $y_0 \in I$. Similarly to the condition specified in (4), we assume that the infinitesimal moments $B_i(x)$, $i=1,2$, are related to the drift and the infinitesimal variance of $X(t)$ by

$$B_1(x) = A_1(x) + A_2(x)\,\gamma(x), \qquad B_2(x) = A_2(x), \tag{10}$$

where $\gamma(x)$ is the hazard rate function of the random variable Z given in (6), i.e.

$$\gamma(x) = -\frac{d}{dx}\ln P(Z>x) = \frac{\psi(x)}{\int_x^{+\infty}\psi(z)\,dz}, \qquad x\in I. \tag{11}$$

Briefly, the above assumptions show that one can construct a new diffusion process $Y(t)$ by incorporating a sampling mechanism into a diffusion process $X(t)$ via the hazard rate function (11). Due to (6), the sampling refers to the evaluation of the process $X(t)$ at an independent random time T exponentially distributed with mean ξ^{-1}. Hence, parameter ξ may be viewed as the (constant) sampling rate of the process $X(t)$. In particular, for $X(0)=0$, the distribution of Z tends to the initial distribution of $X(t)$ when ξ grows. Formally, from (5) and (8) we have

$$\lim_{\xi\to+\infty}\psi(x) = \delta(x) \qquad (x\in I).$$

A thorough analysis of the case $\xi \downarrow 0$ is performed in Section 2.2.

For $\xi>0$, the nature of the endpoints for $Y(t)$ can be specified noting that, due to (10) and (11), the scale function and speed density of $Y(t)$ are respectively related to the corresponding functions of $X(t)$ as follows:

$$h_Y(x) = h_X(x)\left[\int_x^{+\infty}\psi(z)\,dz\right]^2, \qquad s_Y(x) = s_X(x)\left[\int_x^{+\infty}\psi(z)\,dz\right]^{-2}, \tag{12}$$

for $x\in I$, where (see, for instance, Karlin and Taylor [35])

$$h_X(x) = \exp\left\{-2\int^x \frac{A_1(z)}{A_2(z)}\,dz\right\}, \qquad s_X(x) = \frac{2}{A_2(x)\,h_X(x)}. \tag{13}$$

Hence, recalling the conditions specified in Feller [19], [20], we have that the endpoint 0 for $Y(t)$ has the same nature as for $X(t)$. Specifically, if 0 is a reflecting regular (entrance) endpoint for $X(t)$ then it is a reflecting regular (entrance) endpoint for $Y(t)$ too. Moreover, $+\infty$ is an attracting natural endpoint for $Y(t)$ (i.e., the sample-paths of the process are attracted to $+\infty$

although their never reach the endpoint), whereas it is either attracting or non attracting for $X(t)$. Intuitively, this is due to the fact that the drift of $Y(t)$ is larger than that of $X(t)$. Indeed, from (10), we have $B_1(x) - A_1(x) = A_2(x)\gamma(x) > 0$ for all $x \in I$. Hence, if we interpret x as a displacement, then the drift is seen as a kind of average velocity. We, finally, recall that if $+\infty$ is attractive, then the diffusion process has no steady-state density in I.

2.1 Conditional pdf

Hereafter we see that, under suitable assumptions, the pdf of $[Y(t) \,|\, Y(0) = 0]$ can be expressed in terms of the pdf of $[X(t) \,|\, X(0) = 0]$.

Proposition 1 *Let* $\lim_{x\downarrow 0} f_X(x, t \,|\, 0)$ *be finite for all* $t > 0$ *and let* $\lim_{x\downarrow 0} \psi(x)$ *be finite. If*

$$\lim_{x\to+\infty} \frac{f_X(x, t \,|\, 0)}{\psi(x)} = 0, \qquad \forall t > 0, \tag{14}$$

then for the diffusion process $Y(t)$ *having infinitesimal moments (10), with a reflecting or zero-flux condition at* 0*, we have:*

$$f_Y(x, t \,|\, 0) = e^{-\xi t} V(x, t) + \int_0^t \xi\, e^{-\xi\tau} V(x, \tau)\, d\tau, \qquad x \in I, \quad t > 0, \tag{15}$$

where

$$\begin{aligned} V(x, t) &= -\frac{\partial}{\partial x}\left[\frac{P[X(t) > x \,|\, X(0) = 0]}{P(Z > x)}\right] \\ &= \gamma(x)\left[\frac{f_X(x, t \,|\, 0)}{\psi(x)} - \frac{\int_x^{+\infty} f_X(z, t \,|\, 0)\, dz}{\int_x^{+\infty} \psi(z)\, dz}\right], \end{aligned} \tag{16}$$

with $\gamma(x)$ *defined in (11).*

Proof The proof consists of 4 steps.
(i) First we show the initial condition for $f_Y(x, t \,|\, 0)$. From Eqs. (2) and (16) we have $\lim_{t\downarrow 0} V(x, t) = \delta(x)$, and thus from (15) we obtain

$$\lim_{t\downarrow 0} f_Y(x, t \,|\, 0) = \delta(x).$$

(ii) From (14) and (16), we get

$$\int_0^{+\infty} V(x, t)\; dx = 1 - \lim_{x\to+\infty} \frac{\int_x^{+\infty} f_X(z, t \,|\, 0)\, dz}{\int_x^{+\infty} \psi(z)\, dz} = 1.$$

Hence, due to (15), we have the following normalization condition:

$$\int_0^{+\infty} f_Y(x, t \,|\, 0) dx = 1.$$

(iii) In order to prove the reflection condition at $x=0$, from (15) we note that for $x \in I$ and $t>0$

$$\frac{\partial}{\partial x} f_Y(x,t\,|\,0) = e^{-\xi t}\frac{\partial}{\partial x}V(x,t) + \int_0^t \xi\, e^{-\xi\tau}\frac{\partial}{\partial x}V(x,\tau)\,d\tau, \tag{17}$$

where (16) yields

$$\begin{aligned}\frac{\partial}{\partial x}V(x,t) = {}& \frac{1}{\int_x^{+\infty}\psi(z)\,dz}\,\frac{\partial}{\partial x} f_X(x,t\,|\,0) + \frac{2\,\psi(x)}{\left[\int_x^{+\infty}\psi(z)\,dz\right]^2}\, f_X(x,t\,|\,0) \\ & - \left\{\frac{\psi'(x)}{\left[\int_x^{+\infty}\psi(z)\,dz\right]^2} + \frac{2\,\psi^2(x)}{\left[\int_x^{+\infty}\psi(z)\,dz\right]^3}\right\}\int_x^{+\infty} f_X(z,t\,|\,0)\,dz.\end{aligned} \tag{18}$$

Making use of Eqs. (10), (17) and (18), we have:

$$B_1(x) f_Y(x,t\,|\,0) - \frac{1}{2}\frac{\partial}{\partial x}\left[B_2(x) f_Y(x,t\,|\,0)\right] = e^{-\xi t}\widetilde{V}(x,t) + \int_0^t \xi\, e^{-\xi\tau}\widetilde{V}(x,\tau)\,d\tau, \tag{19}$$

where

$$\begin{aligned}\widetilde{V}(x,t) = {}& \frac{1}{\int_x^{+\infty}\psi(z)\,dz}\left\{A_1(x) f_X(x,t\,|\,0) - \frac{1}{2}\frac{\partial}{\partial x}\left[A_2(x) f_X(x,t\,|\,0)\right]\right\} \\ & - \frac{\int_x^{+\infty} f_X(z,t\,|\,0)\,dz}{\left[\int_x^{+\infty}\psi(z)\,dz\right]^2}\left\{A_1(x)\psi(x) - \frac{1}{2}\frac{d}{dx}\left[A_2(x)\psi(x)\right]\right\}.\end{aligned} \tag{20}$$

Let us now consider the Fokker-Planck equation (1) for $y=0$; multiplying both sides for $\xi\, e^{-\xi t}$ and integrating over $t \in (0,+\infty)$, we obtain

$$\frac{d}{dx}\left[A_1(x)\psi(x)\right] - \frac{1}{2}\frac{d^2}{dx^2}\left[A_2(x)\psi(x)\right] = -\xi\,\psi(x), \qquad x \in I, \tag{21}$$

and thus

$$A_1(x)\psi(x) - \frac{1}{2}\frac{d}{dx}\left[A_2(x)\psi(x)\right] = \xi\int_x^{+\infty}\psi(z)\,dz, \qquad x \in I. \tag{22}$$

This implies that

$$\lim_{x\downarrow 0}\left\{A_1(x)\psi(x) - \frac{1}{2}\frac{d}{dx}\left[A_2(x)\psi(x)\right]\right\} = \xi. \tag{23}$$

Making use of (3) and (23), from (20) we have

$$\lim_{x\downarrow 0}\widetilde{V}(x,t) = \delta(t) - \xi.$$

By employing this limit in Eq. (19) we finally obtain the reflection condition

$$\lim_{x\downarrow 0}\left\{B_1(x)f_Y(x,t\,|\,0)-\frac{1}{2}\frac{\partial}{\partial x}\left[B_2(x)f_Y(x,t\,|\,0)\right]\right\}=\delta(t).$$

(iv) Let us now prove that $f_Y(x,t\,|\,0)$, given in (15), satisfies the Fokker-Planck equation. Eq. (19) yields

$$\begin{aligned}\frac{\partial}{\partial x}\left[B_1(x)f_Y(x,t\,|\,0)\right]-\frac{1}{2}\frac{\partial^2}{\partial x^2}\left[B_2(x)f_Y(x,t\,|\,0)\right]&\\ =e^{-\xi t}\frac{\partial}{\partial x}\widetilde{V}(x,t)+\int_0^t\xi\,e^{-\xi\tau}\frac{\partial}{\partial x}\widetilde{V}(x,\tau)\,d\tau.&\quad(24)\end{aligned}$$

Making use of (20), (21) and (22), after some calculations, (24) becomes

$$\begin{aligned}\frac{\partial}{\partial x}\left[B_1(x)f_Y(x,t\,|\,0)\right]-\frac{1}{2}\frac{\partial^2}{\partial x^2}\left[B_2(x)f_Y(x,t\,|\,0)\right]=-\frac{e^{-\xi t}}{\int_x^{+\infty}\psi(z)\,dz}\frac{\partial}{\partial t}f_X(x,t\,|\,0)&\\ +\frac{e^{-\xi t}\psi(x)}{\left[\int_x^{+\infty}\psi(z)\,dz\right]^2}\left\{A_1(x)f_X(x,t\,|\,0)-\frac{1}{2}\frac{\partial}{\partial x}\left[A_2(x)f_X(x,t\,|\,0)\right]\right\}.&\quad(25)\end{aligned}$$

Moreover, from (15) we have

$$\begin{aligned}\frac{\partial}{\partial t}f_Y(x,t\,|\,0)&=e^{-\xi t}\frac{\partial}{\partial t}V(x,t)\\ &=\frac{e^{-\xi t}\psi(x)}{\left[\int_x^{+\infty}\psi(z)\,dz\right]^2}\left\{-A_1(x)f_X(x,t\,|\,0)+\frac{1}{2}\frac{\partial}{\partial x}\left[A_2(x)f_X(x,t\,|\,0)\right]\right\}\\ &\quad+\frac{e^{-\xi t}}{\int_x^{+\infty}\psi(z)\,dz}\frac{\partial}{\partial t}f_X(x,t\,|\,0).\qquad(26)\end{aligned}$$

By comparing Eqs. (25) and (26), we finally obtain:

$$\frac{\partial}{\partial t}f_Y(x,t\,|\,0)=-\frac{\partial}{\partial x}\left[B_1(x)f_Y(x,t\,|\,0)\right]+\frac{1}{2}\frac{\partial^2}{\partial x^2}\left[B_2(x)f_Y(x,t\,|\,0)\right],$$

which completes the proof.

Let us now give a remark concerning the function introduced in (16).

Remark 2 Given two random variables X and Y, we recall that X is said to be smaller than Y in the hazard rate order (denoted by $X\le_{hr}Y$) if and only if the hazard rate of X is greater than or equal to the hazard rate of Y (e.g. see Shaked and Shanthikumar [44]). Hence, from Eq. (16) it follows that $V(x,t)$ is a probability density in x if and only if $[X(t)|X(0)=0]\le_{hr}Z$.

Under the assumptions of Proposition 1, from Eq. (16) we have $V(0,t) = f_X(0,t\,|\,0) - \psi(0)$, $t>0$, and thus recalling (5) and (15) it follows:

$$f_Y(0,t\,|\,0) = e^{-\xi t} f_X(0,t\,|\,0) - \int_t^{+\infty} \xi\, e^{-\xi\tau} f_X(0,\tau\,|\,0)\, d\tau, \qquad t>0. \quad (27)$$

The results given in Proposition 1 can be used to obtain the conditional mean of $Y(t)$.

Proposition 2 *Under the assumptions of Proposition 1, if*

$$\lim_{x\to+\infty} x\, \frac{\int_x^{+\infty} f_X(z,t\,|\,0)\, dz}{\int_x^{+\infty} \psi(z)\, dz} = 0, \qquad \forall t>0, \quad (28)$$

then for the diffusion process $Y(t)$ having infinitesimal moments (10), with a reflecting or zero-flux condition at 0*, for $t>0$ we have:*

$$M_1(t|0) := E[Y(t)\,|\,Y(0)=0] = e^{-\xi t} R(t) + \int_0^t \xi\, e^{-\xi\tau} R(\tau)\, d\tau, \quad (29)$$

where

$$R(t) = \int_0^{+\infty} x\, V(x,t)\, dx = \int_0^{+\infty} \left[\frac{\int_x^{+\infty} f_X(z,t\,|\,0)\, dz}{\int_x^{+\infty} \psi(z)\, dz} \right] dx. \quad (30)$$

Proof The expression (29) can be obtained by direct evaluation of the conditional mean of $Y(t)$, making use of (15) and (16). Assumption (28) ensures that the function defined in (30) is finite for all $t>0$.

Let us now determine an expression for the pdf $f_Y(0,t\,|\,y)$, for $y\in I$, under suitable conditions.

Proposition 3 *Under the assumptions of Proposition 1, if the endpoint 0 is a reflecting regular boundary for the process $X(t)$ and if $s_X(0)$ is finite then for $y\in I$ and $t>0$ we have*

$$\begin{aligned} f_Y(0,t\,|\,y) &= \left[e^{-\xi t} f_X(0,t\,|\,y) + \int_0^t \xi\, e^{-\xi\tau} f_X(0,\tau\,|\,y)\, d\tau \right] \int_y^{+\infty} \psi(z)\, dz \\ &\quad - \xi\, \widehat{f}_X(\xi;0\,|\,y) \int_y^{+\infty} \left[e^{-\xi t} f_X(z,t\,|\,0) + \int_0^t \xi\, e^{-\xi\tau} f_X(z,\tau\,|\,0)\, d\tau \right] dz, \quad (31) \end{aligned}$$

where $\widehat{f}_X$ is defined in (7).

Proof We recall that the transition pdf of $Y(t)$ satisfies the following detailed balance equation (also known as time-reversibility relation), for all $t>0$:

$$f_Y(x,t\,|\,y) = \frac{s_Y(x)}{s_Y(y)}\, f_Y(y,t\,|\,x), \qquad x,y\in I.$$

Hence, making use of (10), (12), (15) and (16), after some calculations, we obtain

$$f_Y(0,t\,|\,y) = \left[e^{-\xi t} f_X(0,t\,|\,y) + \int_0^t \xi\, e^{-\xi\tau} f_X(0,\tau\,|\,y)\, d\tau\right] \int_y^{+\infty} \psi(z)\, dz$$
$$-\frac{s_X(0)}{s_X(y)}\, \psi(y) \int_y^{+\infty} \left[e^{-\xi t} f_X(z,t\,|\,0) + \int_0^t \xi\, e^{-\xi\tau} f_X(z,\tau\,|\,0)\, d\tau\right] dz.$$

Eq. (31) thus follows by noting that

$$\frac{s_X(0)}{s_X(y)}\, \psi(y) = \xi\, \widehat{f}_X(\xi; 0\,|\,y), \qquad y \in I,$$

due to the time-reversibility relation for $X(t)$.

2.2 The special case $\xi \downarrow 0$

Note that the previous results refer to the case $\xi > 0$. Hereafter we investigate the process $Y(t)$ in the special case when $\xi \downarrow 0$.

If the process $X(t)$ admits a steady-state behavior, we shall denote by X the corresponding random variable, and by

$$W_X(x) = \lim_{t\to+\infty} f_X(x,t\,|\,y) = \frac{s_X(x)}{\int_0^{+\infty} s_X(z)\; dz}, \qquad x \in I, \tag{32}$$

the steady-state pdf, for $y \in I$. Moreover, the corresponding hazard rate function will be indicated as

$$\gamma_W(x) = -\frac{d}{dx} \ln P(X > x) = \frac{W_X(x)}{\int_x^{+\infty} W_X(z)\, dz}, \qquad x \in I. \tag{33}$$

If $X(t)$ admits the steady-state pdf (32), then from (8) and from a Tauberian theorem for Laplace transform we immediately have

$$\lim_{\xi\downarrow 0} \psi(x) = W_X(x), \qquad x \in I, \tag{34}$$

and thus, from (11) and (33), one has

$$\lim_{\xi\downarrow 0} \gamma(x) = \gamma_W(x), \qquad x \in I. \tag{35}$$

In the present case the diffusion process $Y(t)$ rather than (10) has infinitesimal moments

$$B_1(x) = A_1(x) + A_2(x)\, \gamma_W(x), \qquad B_2(x) = A_2(x), \tag{36}$$

with $\gamma_W(x)$ specified in (33). Again, 0 is a regular endpoint, and $+\infty$ is an attracting natural endpoint for $Y(t)$.

The following result is a consequence of Proposition 1 and Eqs. (34) and (35).

Proposition 4 *Let $X(t)$ admit a steady-state behavior; if* $\lim_{x\downarrow 0} f_X(x,t\,|\,0)$ *is finite for all $t>0$, if $W_X(0)$ is finite, and*

$$\lim_{x\to+\infty} \frac{f_X(x,t\,|\,0)}{W_X(x)} = 0, \qquad \forall t>0, \tag{37}$$

then for the diffusion process $Y(t)$ having infinitesimal moments (36), with a reflecting or zero-flux condition at 0, we have:

$$\begin{aligned} f_Y(x,t\,|\,0) &= -\frac{\partial}{\partial x}\left[\frac{P[X(t)>x\,|\,X(0)=0]}{P(X>x)}\right] \\ &= \gamma_W(x)\left[\frac{f_X(x,t\,|\,0)}{W_X(x)} - \frac{\int_x^{+\infty} f_X(z,t\,|\,0)\,dz}{\int_x^{+\infty} W_X(z)\,dz}\right], \end{aligned} \tag{38}$$

with $\gamma_W(x)$ defined in (33).

We note that, recalling Remark 2, the right-hand-side of Eq. (38) is a probability density in x, so that $[X(t)|X(0)=0] \le_{hr} X$.

The next result follows from Proposition 3 and Eqs. (34) and (35).

Proposition 5 *Under the assumptions of Proposition 4, if the endpoint 0 is a reflecting regular boundary for the process $X(t)$, then for $y\in I$ and $t>0$ we have*

$$f_Y(0,t\,|\,y) = f_X(0,t\,|\,y)\int_y^{+\infty} W_X(z)\,dz - W_X(0)\int_y^{+\infty} f_X(z,t\,|\,0)\,dz.$$

Finally, we point out that when the process $X(t)$ does not admit a steady-state behaviour, if $\xi\downarrow 0$ then $B_1(x)$ tends to $A_1(x)$, for all $x\in I$.

2.3 Diffusion in a potential

Consider the one-to-one transformation

$$\widetilde{y} = \zeta(y) := \int_{y'}^{y} \frac{1}{\sqrt{B_2(z)}}\,dz, \qquad y\in I, \tag{39}$$

where $y'\in I$ is an arbitrary fixed point, with $y'<y$, and let

$$\widetilde{Y}(t) = \zeta[Y(t)], \qquad t\ge 0, \tag{40}$$

be also a diffusion process. Hence, Eq. (39) allows us to transform the stochastic differential equation (9) into (see, for instance, Karlin and Taylor [35])

$$d\widetilde{Y}(t) = \widetilde{B}_1[\widetilde{Y}(t)]\,dt + d\mathcal{B}(t), \qquad \widetilde{Y}(0) = \zeta(y_0). \tag{41}$$

Since the process (40) has drift

$$\widetilde{B}_1(\widetilde{y}) = \left\{\frac{1}{\sqrt{B_2(x)}}\left[B_1(x) - \frac{B_2'(x)}{4}\right]\right\}_{x=\zeta^{-1}(\widetilde{y})}$$

and unity infinitesimal variance, the stochastic differential equation of the type (41) is employed usually to describe a diffusion in a potential (see, for instance, Hongler and Zheng [32], [33], Forman and Sørensen [21], Gardiner [23]). According to (41), we define the potential for the diffusion process $Y(t)$ as a function $U(x)$ such that

$$\frac{d}{dx}U(x) = -\widetilde{B}_1[\zeta(x)] = -\frac{1}{\sqrt{B_2(x)}}\left[B_1(x) - \frac{B_2'(x)}{4}\right], \qquad x \in I.$$

Hence, taking into account Eq. (10), for $\xi > 0$ we have

$$\frac{d}{dx}U(x) = \sqrt{A_2(x)}\,\frac{d}{dx}\left[-\int^x \frac{A_1(z)}{A_2(z)}\,dz + \ln\int_x^{+\infty}\psi(z)\,dz + \frac{1}{4}\ln A_2(x)\right].$$

In the special case when the infinitesimal variance is constant, say $A_2(x) = \sigma^2$, the potential is thus given by

$$U(x) = \sigma\left[-\int^x \frac{A_1(z)}{\sigma^2}\,dz + \ln\int_x^{+\infty}\psi(z)\,dz\right] + c, \tag{42}$$

with c an arbitrary real constant.

3 Connection to processes subject to catastrophes

In this section, we show a result that allows us to express density (15) in a different form.

Let us consider a jump-diffusion process obtained by superimposing to $X(t)$ the occurrence of catastrophes. Formally, let $\{X^c(t), t \geq 0\}$ be a regular one-dimensional time-homogeneous diffusion process with state-space I, having drift $A_1(x)$ and infinitesimal variance $A_2(x)$, and subject to catastrophes occurring according to an independent Poisson process with intensity ξ. A catastrophe is an instantaneous reset of $X^c(t)$ to the zero state.

The pdf of $[X^c(t)\,|\,X^c(0) = y]$ is denoted by $f_X^c(x,t\,|\,y)$, for $x, y \in I$, and satisfies the forward differential equation

$$\begin{aligned}\frac{\partial}{\partial t}f_X^c(x,t\,|\,y) = &-\frac{\partial}{\partial x}\left[A_1(x)f_X^c(x,t\,|\,y)\right] + \frac{1}{2}\frac{\partial^2}{\partial x^2}\left[A_2(x)f_X^c(x,t\,|\,y)\right]\\ &-\xi f_X^c(x,t\,|\,y).\end{aligned} \tag{43}$$

Moreover, $f_X^c(x,t\,|\,y)$ satisfies the Dirac delta initial condition

$$\lim_{t\downarrow 0} f_X^c(x,t\,|\,y) = \delta(x-y) \tag{44}$$

and the reflecting condition

$$\lim_{x\downarrow 0}\left\{A_1(x)f_X^c(x,t\,|\,y) - \frac{1}{2}\frac{\partial}{\partial x}\left[A_2(x)f_X^c(x,t\,|\,y)\right] - \xi\right\} = \begin{cases}\delta(t), & y = 0,\\ 0, & y > 0.\end{cases} \tag{45}$$

Clearly, when $\xi \downarrow 0$ the equations (43) and (45) identify with (1) and (3), respectively.

We recall (see, for instance, Giorno *et al.* [24]) that the pdf's of $X^c(t)$ and $X(t)$ are related by the following relation, for all $x, y \in I$ and $t > 0$:

$$f_X^c(x, t \,|\, y) = e^{-\xi t} f_X(x, t \,|\, y) + \int_0^t \xi e^{-\xi \tau} f_X(x, \tau \,|\, 0)\, d\tau. \tag{46}$$

Hence, from (5) it immediately follows that

$$\lim_{t \to +\infty} f_X^c(x, t \,|\, y) = \psi(x), \qquad x \in I,$$

so that $X^c(t)$ admits a steady-state pdf, which identifies with $\psi(x)$.

Proposition 6 *Under the assumptions of Proposition 1, for $t > 0$ the pdf of $[Y(t) \,|\, Y(0) = 0]$ is given by*

$$\begin{aligned} f_Y(x, t \,|\, 0) &= -\frac{\partial}{\partial x} \left[\frac{P[X^c(t) > x \,|\, X^c(0) = 0]}{P(Z > x)} \right] \\ &= \gamma(x) \left[\frac{f_X^c(x, t \,|\, 0)}{\psi(x)} - \frac{\int_x^{+\infty} f_X^c(z, t \,|\, 0)\, dz}{\int_x^{+\infty} \psi(z)\, dz} \right], \qquad x \in I, \end{aligned} \tag{47}$$

with $\gamma(x)$ defined in (11).

Proof From (15) and (16), for $x \in I$ and $t > 0$ we have

$$\begin{aligned} f_Y(x, t \,|\, 0) = & \frac{1}{\int_x^{+\infty} \psi(z)\, dz} \left[e^{-\xi t} f_X(x, t \,|\, 0) + \int_0^t \xi\, e^{-\xi \tau} f_X(x, \tau \,|\, 0)\, d\tau \right] \\ & - \frac{\gamma(x)}{\left[\int_x^{+\infty} \psi(z)\, dz\right]^2} \int_x^{+\infty} \left[e^{-\xi t} f_X(z, t \,|\, 0) + \int_0^t \xi\, e^{-\xi \tau} f_X(z, \tau \,|\, 0)\, d\tau \right] dz. \end{aligned}$$

Hence, Eq. (47) follows from (46).

We note that, from Eq. (47) and Proposition 6, for $x \in I$ and $t > 0$ we have

$$P[Y(t) > x \,|\, Y(0) = 0] = \frac{P[X^c(t) > x \,|\, X^c(0) = 0]}{P(Z > x)} = \frac{\int_x^{+\infty} f_X^c(z, t \,|\, 0)\, dz}{\int_x^{+\infty} \psi(z)\, dz}, \tag{48}$$

and thus $[X^c(t)|X^c(0) = 0] \leq_{hr} X$, due to Remark 2. Finally, Eq. (48) implies that if Z and $[Y(t) \,|\, Y(0) = 0]$ are independent, then

$$[X^c(t)|X^c(0) = 0] \stackrel{d}{=} \min\{Z, [Y(t) \,|\, Y(0) = 0]\}.$$

4 Laplace transforms and first-passage time

In this section we intend to analyze the downward first-passage-time problem for process $Y(t)$ by adopting a Laplace-transform based approach. To this aim, we first determine the Laplace transform of the densities $f_Y(x,t\,|\,0)$ and $f_Y(0,t\,|\,y)$. Then, by making use of the continuity of the sample paths of $Y(t)$ we obtain the Laplace transform of the first-passage-time density and of the probability of ultimate first-passage occurrence.

Similarly to the notation used in (7), the Laplace transform of the transition pdf $f_Y(x,t\,|\,y)$ will be denoted $\widehat{f}_Y(s;x\,|\,y)$. Hereafter it will be determined in some instances.

Proposition 7 *Under the assumptions of Proposition 1, for all $s>0$ and $x\in I$ we have*

$$\widehat{f}_Y(s;x\,|\,0)=\frac{s+\xi}{s}\,\gamma(x)\left[\frac{\widehat{f}_X(s+\xi;x\,|\,0)}{\psi(x)}-\frac{\int_x^{+\infty}\widehat{f}_X(s+\xi;z\,|\,0)\,dz}{\int_x^{+\infty}\psi(z)\,dz}\right]. \quad (49)$$

Proof Taking the Laplace transform of (15), from (7) we have

$$\widehat{f}_Y(s;x\,|\,0)=\frac{s+\xi}{s}\int_0^{+\infty}e^{-(s+\xi)t}V(x,t)\,dt,\qquad s>0,\ \ x\in I,$$

this giving (49) due to (16).

Proposition 8 *Under the assumptions of Proposition 3, for all $s>0$ and $y\in I$ we have*

$$\begin{aligned}\widehat{f}_Y(s;0\,|\,y)=\frac{s+\xi}{s}\;\Bigg[\;&\widehat{f}_X(s+\xi;0\,|\,y)\int_y^{+\infty}\psi(z)\,dz\\&-\xi\,\widehat{f}_X(\xi;0\,|\,y)\int_y^{+\infty}\widehat{f}_X(s+\xi;z\,|\,0)\,dz\Bigg].\end{aligned} \quad (50)$$

Proof By taking the Laplace transform in (31), we obtain Eq. (50).

We remark that $\psi(0)=\xi\,\widehat{f}_X(\xi;0\,|\,0)$, and thus from (27) we get

$$\widehat{f}_Y(s;0\,|\,0)=\frac{s+\xi}{s}\left[\widehat{f}_X(s+\xi;0\,|\,0)-\frac{\psi(0)}{s+\xi}\right],\qquad s>0.$$

Let us now introduce the first-passage time through $x\in I$ for the process $Y(t)$,

$$T_{y,x}=\inf\{t>0:Y(t)=x\},\qquad Y(0)=y\in I\setminus\{x\},$$

and let $g_Y(x,t\,|\,y)$ denote the pdf of $T_{y,x}$. We remark that for $0\le x<y$ the first passage $T_{y,x}$ is not certain, since $+\infty$ is an attracting natural endpoint for $Y(t)$.

Even for simple diffusion processes, the determination of $g_Y(x,t\,|\,y)$ is not an easy task. In few cases first-passage-time densities can be obtained analytically (see, e.g. Molini *et al.* [39]). One is often forced to adopt simulation or numerical methods (see, for instance, the recent progress obtained by Buonocore *et al.* [8] and Taillefumier and Magnasco [45]). In certain cases it is useful to obtain closed forms for functionals of $g_Y(x,t\,|\,y)$. Hereafter, under suitable assumptions, we evaluate the Laplace transform $\widehat{g}_Y(s;x\,|\,y)$ in the case of a passage from above. This Laplace transform is useful to obtain quantities of interest in applied fields (see, for instance, Ditlevsen [15] and Wonho [49]).

Proposition 9 *Under the assumptions of Proposition 3, let $x, y \in I$, with $0 \le x < y$. For all $s > 0$ we have*

$$\widehat{g}_Y(s;x\,|\,y) = \frac{\widehat{f}_X(s+\xi;0\,|\,y)\int_y^{+\infty}\psi(z)\,dz - \xi\,\widehat{f}_X(\xi;0\,|\,y)\int_y^{+\infty}\widehat{f}_X(s+\xi;z\,|\,0)\,dz}{\widehat{f}_X(s+\xi;0\,|\,x)\int_x^{+\infty}\psi(z)\,dz - \xi\,\widehat{f}_X(\xi;0\,|\,x)\int_x^{+\infty}\widehat{f}_X(s+\xi;z\,|\,0)\,dz}. \tag{51}$$

Proof Due to the continuity of the sample paths of $Y(t)$, for $t > 0$ we have

$$f_Y(0,t\,|\,y) = \int_0^t g_Y(x,\tau\,|\,y)\,f_Y(0,t-\tau\,|\,x)\,d\tau, \qquad 0 \le x < y.$$

Therefore, taking the Laplace transform, for $s > 0$ we have

$$\widehat{g}_Y(s;x\,|\,y) = \frac{\widehat{f}_Y(s;0\,|\,y)}{\widehat{f}_Y(s;0\,|\,x)}, \qquad 0 \le x < y \tag{52}$$

this finally giving Eq. (51) due to (50).

Under the assumptions of Proposition 9, the probability of ultimate first-passage occurrence (with passage from above) can be expressed as

$$\begin{aligned} P_Y(x\,|\,y) := P(T_{y,x} < +\infty) &= \frac{\int_y^{+\infty} h_Y(z)\,dz}{\int_x^{+\infty} h_Y(z)\,dz} \\ &= \frac{\int_y^{+\infty} h_X(z)\left[\int_z^{+\infty}\psi(u)\,du\right]^2 dz}{\int_x^{+\infty} h_X(z)\left[\int_z^{+\infty}\psi(u)\,du\right]^2 dz}, \qquad 0 \le x < y, \end{aligned} \tag{53}$$

with $h_Y(z)$ specified in (12).

5 Analysis of special cases

In this section, making use of the previous results, we study certain diffusion processes obtained via transformation of some classical diffusion processes. In particular, we evaluate various closed-form expressions and perform suitable numerical computations by MATHEMATICA®.

5.1 Transformation of the Wiener process

Let $X(t)$ be a Wiener process having infinitesimal moments

$$A_1(x) = \mu, \qquad A_2(x) = \sigma^2 \qquad (\mu \in \mathbb{R}, \sigma \in \mathbb{R}^+),$$

with state-space I, and 0 a reflecting endpoint. The transition pdf is given by (cf. Cox and Miller [11], for instance)

$$f_X(x,t\,|\,y) = \frac{1}{\sigma\sqrt{2\pi t}}\left[\exp\left\{-\frac{(x-y-\mu t)^2}{2\sigma^2 t}\right\} + \exp\left\{-\frac{4\mu t y + (x+y-\mu t)^2}{2\sigma^2 t}\right\}\right] - \frac{\mu}{\sigma^2}\exp\left\{\frac{2\mu x}{\sigma^2}\right\}\mathrm{Erfc}\left(\frac{x+y+\mu t}{\sigma\sqrt{2t}}\right), \tag{54}$$

for $x, y \in I$ and $t > 0$, where $\mathrm{Erfc}(z) = 2\pi^{-1/2}\int_z^{+\infty} e^{-x^2}\,dx$ is the complementary error function. The reflected Wiener process has long played a key role in various contexts, such as queueing (see, for instance, Harrison [31]). The process $X(t)$ admits a steady-state pdf $W_X(x) = (|\mu|/\sigma^2)\exp\{-2|\mu|x/\sigma^2\}$, $x \in I$, if and only if $\mu < 0$. Moreover, for $x, y \in I$ and $s > 0$ one has

$$\widehat{f}_X(s;x\,|\,y) = \frac{1}{\sqrt{\mu^2+2\sigma^2 s}}\exp\left\{\frac{\mu(x-y)}{\sigma^2}\right\}\left[\exp\left\{-\frac{|x-y|}{\sigma^2}\sqrt{\mu^2+2\sigma^2 s}\right\} + \frac{\mu^2+\sigma^2 s-\mu\sqrt{\mu^2+2\sigma^2 s}}{\sigma^2 s}\exp\left\{-\frac{(x+y)}{\sigma^2}\sqrt{\mu^2+2\sigma^2 s}\right\}\right].$$

Hence, from (8) we get

$$\psi(x) = \frac{\sqrt{\mu^2+2\sigma^2\xi}-\mu}{\sigma^2}\exp\left\{-\frac{\sqrt{\mu^2+2\sigma^2\xi}-\mu}{\sigma^2}\,x\right\}, \qquad x \in I,$$

and, due to (11), we have

$$\gamma(x) = \frac{\sqrt{\mu^2+2\sigma^2\xi}-\mu}{\sigma^2} \equiv \psi(0). \qquad x \in I.$$

Therefore, from (10) we have that $Y(t)$ is a diffusion process with state-space I, infinitesimal moments

$$B_1(x) = \sqrt{\mu^2+2\sigma^2\xi}, \qquad B_2(x) = \sigma^2, \tag{55}$$

and a reflecting endpoint at 0. It follows that $Y(t)$ is again a Wiener process, with $+\infty$ a natural attracting endpoint for $\xi > 0$.

The transition pdf of $Y(t)$ can be easily obtained from (54) by changing μ with $\sqrt{\mu^2+2\sigma^2\xi}$. Moreover, the downward first-passage-time density for $Y(t)$, $t > 0$, is:

$$g_Y(x,t\,|\,y) = \frac{y-x}{\sigma\sqrt{2\pi t^3}}\exp\left\{-\frac{(x-y-t\sqrt{\mu^2+2\sigma^2\xi})^2}{2\sigma^2 t}\right\}, \qquad 0 \le x < y.$$

From (53) one obtains the ultimate first-passage probability:

$$P_Y(x\,|\,y) = \exp\left\{\frac{2\sqrt{\mu^2+2\sigma^2\xi}}{\sigma^2}(x-y)\right\}, \qquad 0 \le x < y.$$

We note that $dP_Y(x\,|\,y)/d\xi < 0$, so that $P_Y(x\,|\,y)$ is a decreasing function of ξ.

In conclusion, we analyze the behavior of the process $Y(t)$ as $\xi \downarrow 0$. In this case, from (55) we obtain $B_1(x) = |\mu|$ and $B_2(x) = \sigma^2$, so that the boundary $+\infty$ is a natural attracting endpoint for $\mu \neq 0$, and a non-attracting endpoint for $\mu = 0$.

5.2 Transformation of the Ornstein-Uhlenbeck process

Let $X(t)$ be an Ornstein-Uhlenbeck process having infinitesimal moments

$$A_1(x) = -\alpha x, \qquad A_2(x) = \sigma^2 \qquad (\alpha \in \mathbb{R}^+, \sigma \in \mathbb{R}^+),$$

with state-space I and 0 a reflecting endpoint. The conditional pdf is given by

$$f_X(x,t\,|\,0) = 2\sqrt{\frac{\alpha}{\pi\sigma^2(1-e^{-2\alpha t})}}\exp\left\{-\frac{\alpha x^2}{\sigma^2(1-e^{-2\alpha t})}\right\}, \tag{56}$$

for $x \in I$ and $t > 0$. The reflected Ornstein-Uhlenbeck process arises in several applied fields. For instance, in queueing theory it is used for approximating queues with reneging or bulking customers (see Ward and Glynn [46] and [47]). The process $X(t)$ admits a steady-state density $W_X(x) = 2\,[\alpha/(\pi\sigma^2)]^{1/2}\exp\{-\alpha\, x^2/\sigma^2\}$, $x \in I$. Due to (56) one immediately has

$$\int_x^{+\infty} f_X(z,t\,|\,0)\,dz = \mathrm{Erfc}\left(x\sqrt{\frac{\alpha}{\sigma^2(1-e^{-2\alpha t})}}\right), \qquad x \in I. \tag{57}$$

From (8), it follows

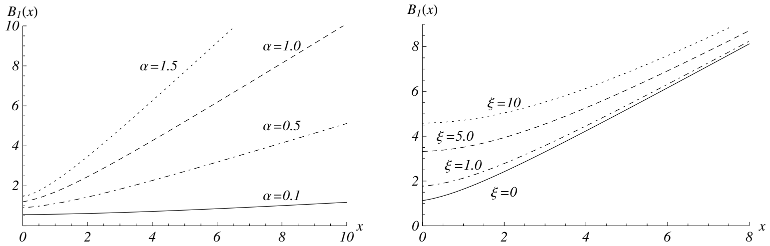
$$\psi(x) = \frac{\xi\, 2^{\xi/(2\alpha)}}{\sigma\sqrt{\pi\alpha}}\,\Gamma\left(\frac{\xi}{2\alpha}\right)\exp\left\{-\frac{\alpha x^2}{2\sigma^2}\right\}D_{-\xi/\alpha}\left(x\frac{\sqrt{2\alpha}}{\sigma}\right), \qquad x \in I, \tag{58}$$

where (cf. Gradshteyn and Ryzhik [30], p. 1028, n. 9.240)

$$D_\nu(z) = 2^{\nu/2}e^{-z^2/4}\left[\frac{\sqrt{\pi}}{\Gamma\left(\frac{1-\nu}{2}\right)}\phi\Big(-\frac{\nu}{2},\frac{1}{2};\frac{z^2}{2}\Big) - \frac{z\sqrt{2\pi}}{\Gamma\left(-\frac{\nu}{2}\right)}\phi\Big(\frac{1-\nu}{2},\frac{3}{2};\frac{z^2}{2}\Big)\right]$$

denotes the parabolic cylinder function, and ϕ is the Kummer's function. We remark that $D_\nu(0) = 2^{\nu/2}\sqrt{\pi}\left[\Gamma\left(\frac{1-\nu}{2}\right)\right]^{-1}$ and thus $\psi(0)$ is finite, being

$$\psi(0) = \frac{\xi}{\sigma\sqrt{\alpha}}\frac{\Gamma\left(\frac{\xi}{2\alpha}\right)}{\Gamma\left(\frac{1}{2}+\frac{\xi}{2\alpha}\right)}.$$



(a) $\xi = 0.1, \sigma = 1.0$ (b) $\alpha = 1.0, \sigma = 1.0$

Fig. 1 Drift of $Y(t)$, given in (61), for different choices of α in (a) and of ξ in (b).

From (58), for $x \in I$ we have

$$\int_x^{+\infty} \psi(z)\, dz = \sqrt{\frac{2}{\pi}}\, 2^{\xi/(2\alpha)}\, \Gamma\left(1 + \frac{\xi}{2\alpha}\right) \exp\left\{-\frac{\alpha x^2}{2\sigma^2}\right\} D_{-1-\xi/\alpha}\left(x\frac{\sqrt{2\alpha}}{\sigma}\right), \tag{59}$$

so that Eq. (11) yields

$$\gamma(x) = \frac{\sqrt{2\alpha}}{\sigma} \frac{D_{-\xi/\alpha}\left(x\frac{\sqrt{2\alpha}}{\sigma}\right)}{D_{-1-\xi/\alpha}\left(x\frac{\sqrt{2\alpha}}{\sigma}\right)}, \qquad x \in I. \tag{60}$$

Consequently, due to (10), $Y(t)$ is a diffusion process with state-space I and infinitesimal moments

$$B_1(x) = -\alpha x + \sigma\sqrt{2\alpha}\, \frac{D_{-\xi/\alpha}\left(x\frac{\sqrt{2\alpha}}{\sigma}\right)}{D_{-1-\xi/\alpha}\left(x\frac{\sqrt{2\alpha}}{\sigma}\right)}, \qquad B_2(x) = \sigma^2, \tag{61}$$

where 0 is a reflecting endpoint and $+\infty$ is a natural attracting endpoint. Some plots of $B_1(x)$ are given in Fig. 1. From Eq. 19.8.1, p. 689, and Eq. 19.3.7, p. 687, of Abramowitz and Stegun [1], we have

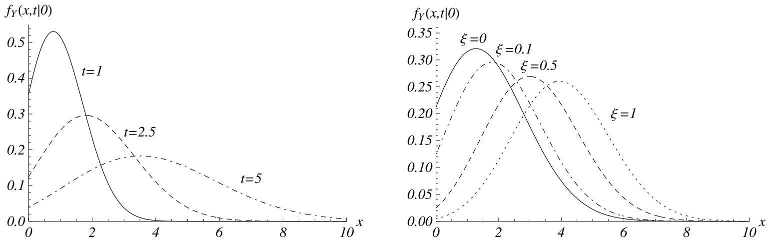
$$D_\nu(z) \sim z^\nu e^{-z^2/4}, \qquad z \to +\infty \tag{62}$$

and thus, due to (61),

$$B_1(x) \sim \alpha x, \qquad x \to +\infty.$$

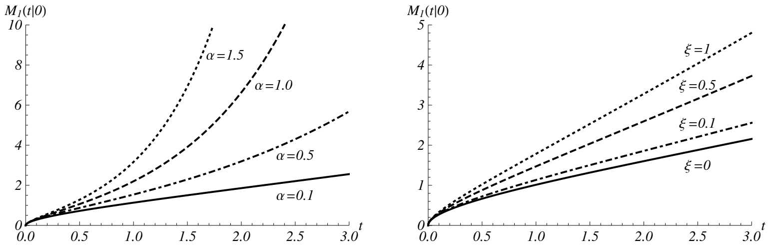
Making use of (56), (58) and (62), we can see that the limit (14) is satisfied. The assumptions of Proposition 1 thus hold. By virtue of (16) and recalling Eqs. (56), (57), (58), (59) and (60), we obtain, for $x \in I$ and $t > 0$,

$$V(x,t) = \frac{2^{-\xi/(2\alpha)}}{\Gamma\left(1 + \frac{\xi}{2\alpha}\right) D_{-1-\xi/\alpha}\left(x\frac{\sqrt{2\alpha}}{\sigma}\right)} \exp\left\{\frac{\alpha x^2}{2\sigma^2}\right\}$$



(a) $\alpha = 0.1, \xi = 0.1, \sigma = 1.0$ (b) $\alpha = 0.1, \sigma = 1.0, t = 2.5$

Fig. 2 For the process with infinitesimal moments (61), the conditional pdf $f_Y(x, t\,|\,0)$ is plotted for some choices of t in (a) and for some choices of ξ in (b).



(a) $\xi = 0.1, \sigma = 1.0$ (b) $\alpha = 0.1, \sigma = 1.0$

Fig. 3 For the process with infinitesimal moments (61), the conditional mean $M_1(t|0)$ is plotted for some choices of α in (a) and for some choices of ξ in (b).

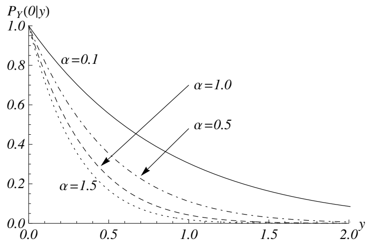
$$\times \left[\sqrt{\frac{2\alpha}{\sigma^2(1-e^{-2\alpha t})}} \exp\left\{ -\frac{\alpha x^2}{\sigma^2(1-e^{-2\alpha t})} \right\} \right.$$
$$\left. - \frac{\sqrt{\pi\alpha}}{\sigma} \, \frac{D_{-\xi/\alpha}\left(x\frac{\sqrt{2\alpha}}{\sigma}\right)}{D_{-1-\xi/\alpha}\left(x\frac{\sqrt{2\alpha}}{\sigma}\right)} \mathrm{Erfc}\left(x\sqrt{\frac{\alpha}{\sigma^2(1-e^{-2\alpha t})}} \right) \right]. \quad (63)$$

Making use of (27), when $\xi > 0$ one has:

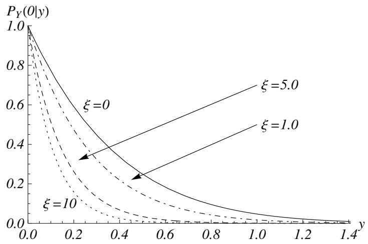
$$f_Y(0, t\,|\,0) = \frac{2}{\sigma}\sqrt{\frac{\alpha}{\pi}}\, e^{-\xi\, t} \left[\frac{1}{\sqrt{1-e^{-2\alpha\, t}}} - F\Big(\frac{\xi}{2\alpha}, \frac{1}{2}; \frac{\xi}{2\alpha} + 1; e^{-2\alpha\, t}\Big) \right], \quad (64)$$

for $t > 0$, where $F(a, b; c; z)$ denotes the hypergeometric series (cf. Gradshteyn and Ryzhik [30], p. 1005, n. 9.100). We remark that, making use of (63), density $f_Y(x, t\,|\,0)$ can be evaluated from (15) via numerical integration. Some plots are given in Fig. 2.

Moreover, (28) is satisfied by virtue of (57) and (59); hence, the conditional mean of $Y(t)$ can be numerically evaluated via (29) and (30). Some plots of $M_1(t|0)$ are shown in Fig. 3.



(a) $\xi = 0.1, \sigma = 1.0$



(b) $\alpha = 1.0, \sigma = 1.0$

Fig. 4 The first passage time probability $P_Y(0\,|\,y)$ is given for the process with infinitesimal moments (61).

We note that for $x \in I$ the Laplace transform $\widehat{f}_Y(s; x\,|\,0)$ is given by

$$\widehat{f}_Y(s; x\,|\,0) = \frac{(s+\xi)\sqrt{2\alpha}}{s\,\sigma\,\xi}\,\frac{2^{s/(2\alpha)}\,\Gamma\left(\frac{s+\xi}{2\alpha}\right)}{\Gamma\left(\frac{\xi}{2\alpha}\right) D_{-1-\xi/\alpha}\left(x\frac{\sqrt{2\alpha}}{\sigma}\right)}$$
$$\times\left\{D_{-(s+\xi)/\alpha}\left(x\frac{\sqrt{2\alpha}}{\sigma}\right) - D_{-1-(s+\xi)/\alpha}\left(x\frac{\sqrt{2\alpha}}{\sigma}\right)\frac{D_{-\xi/\alpha}\left(x\frac{\sqrt{2\alpha}}{\sigma}\right)}{D_{-1-\xi/\alpha}\left(x\frac{\sqrt{2\alpha}}{\sigma}\right)}\right\}.$$

Consider the first passage time problem for $Y(t)$. The Laplace transform $\widehat{g}_Y(s; x\,|\,y)$ can be determined by mean of (52), where for $y \in I$ results

$$\widehat{f}_Y(s; 0\,|\,y) = \frac{s+\xi}{s\,\sigma\pi\sqrt{\alpha}}\,\Gamma\left(\frac{\xi}{2\alpha}+1\right)\Gamma\left(\frac{s+\xi}{2\alpha}\right)2^{1/2+\xi/\alpha+s/(2\alpha)}$$
$$\times\left[D_{-1-\xi/\alpha}\left(y\frac{\sqrt{2\alpha}}{\sigma}\right) D_{-(s+\xi)/\alpha}\left(y\frac{\sqrt{2\alpha}}{\sigma}\right)\right.$$
$$\left. - D_{-\xi/\alpha}\left(y\frac{\sqrt{2\alpha}}{\sigma}\right) D_{-1-(s+\xi)/\alpha}\left(y\frac{\sqrt{2\alpha}}{\sigma}\right)\right].$$

Finally, due to (53), the probability of the ultimate first-passage occurrence is given by

$$P_Y(x\,|\,y) = \frac{\int_{y\sqrt{2\alpha}/\sigma}^{+\infty}\left[D_{-1-\xi/\alpha}(z)\right]^2 dz}{\int_{x\sqrt{2\alpha}/\sigma}^{+\infty}\left[D_{-1-\xi/\alpha}(z)\right]^2 dz}, \qquad 0 \le x < y.$$

Some plots of this probability are given in Fig. 4 for $x = 0$, showing that it is decreasing in α and in ξ.

Let us now discuss the case $\xi \downarrow 0$. We note that (cf. Gradshteyn and Ryzhik [30], p. 1030, n. 9.251 and n. 9.254.1)

$$D_0(x) = e^{-x^2/4}, \qquad D_{-1}(x) = \sqrt{\frac{\pi}{2}}\,e^{x^2/4}\,\mathrm{Erfc}\left(\frac{x}{\sqrt{2}}\right),$$

and thus, recalling (36), we can see that when $\xi \downarrow 0$ the infinitesimal moments given in (61) become

$$B_1(x) = -\alpha x + 2\sigma\sqrt{\frac{\alpha}{\pi}}\,\frac{e^{-\alpha\, x^2/\sigma^2}}{\mathrm{Erfc}\left(x\frac{\sqrt{\alpha}}{\sigma}\right)}, \qquad B_2(x) = \sigma^2, \tag{65}$$

where 0 is a reflecting endpoint and $+\infty$ is a natural attracting endpoint. For the diffusion process (65), from Proposition 4 we have that the conditional pdf, for $x \in I$ and $t > 0$ is given by

$$\begin{aligned} f_Y(x, t\,|\,0) = {} & \frac{2}{\sqrt{\pi}}\,\frac{1}{\mathrm{Erfc}\left(x\frac{\sqrt{\alpha}}{\sigma}\right)}\left[\sqrt{\frac{\alpha}{\sigma^2(1-e^{-2\alpha t})}}\exp\left\{-\frac{\alpha x^2}{\sigma^2(1-e^{-2\alpha t})}\right\}\right. \\ & \left. - \frac{\sqrt{\alpha}}{\sigma}\,\frac{e^{-\alpha\, x^2/\sigma^2}}{\mathrm{Erfc}\left(x\frac{\sqrt{\alpha}}{\sigma}\right)}\,\mathrm{Erfc}\left(x\sqrt{\frac{\alpha}{\sigma^2(1-e^{-2\alpha t})}}\right)\right]. \end{aligned} \tag{66}$$

In particular, from (66) one has:

$$f_Y(0, t\,|\,0) = \frac{2}{\sigma}\sqrt{\frac{\alpha}{\pi}}\left(\frac{1}{\sqrt{1-e^{-2\alpha\, t}}} - 1\right), \qquad t > 0,$$

where 0 is a regular endpoint.

5.3 Transformation of the Bessel process

Let $X(t)$ be a Bessel process (of order n) having infinitesimal moments

$$A_1(x) = (n-1)\frac{\sigma^2}{2x}, \qquad A_2(x) = \sigma^2 \qquad (n \in \mathbb{N}, \sigma \in \mathbb{R}^+), \tag{67}$$

and with state-space I. The endpoint 0 is regular if $n = 1$ and entrance if $n = 2, 3, \ldots$; the endpoint $+\infty$ is natural non-attracting. We assume that a reflecting or zero-flux condition is imposed at $x = 0$. This process can be expressed as

$$X(t) = ||\mathbf{W}(t)|| = \left(\sum_{i=1}^{n} W_i^2(t)\right)^{1/2}, \qquad t \geq 0,$$

where $||w||$ denotes the Euclidean norm, and $\mathbf{W}(t) = [W_1(t), \ldots, W_n(t)]$, $t \geq 0$, is an n-dimensional diffusion process, the components being independent Wiener processes started from the origin, with 0 drift, infinitesimal variance σ^2, and restricted to $[0, +\infty)$ by a reflecting condition at 0. As well known, the conditional pdf is given by (cf. Karlin and Taylor [35], Giorno *et al.* [27], for instance)

$$f_X(x, t\,|\,0) = \frac{2x^{n-1}}{\Gamma(n/2)}\left(\frac{1}{2\sigma^2 t}\right)^{n/2}\exp\left(-\frac{x^2}{2\sigma^2 t}\right), \tag{68}$$

for $x \in I$ and $t > 0$. From (68), we have

$$\int_x^{+\infty} f_X(z,t\,|\,0)\,dz = \frac{1}{\Gamma(n/2)}\Gamma\left(\frac{n}{2},\frac{x^2}{2\sigma^2 t}\right), \qquad x \in I, \quad t > 0, \tag{69}$$

where $\Gamma(\cdot,\cdot)$ denotes the incomplete gamma function. Due to (8), we have

$$\psi(x) = \frac{2\sqrt{2}}{\sigma\Gamma(n/2)}\xi^{1/2+n/4}\left(\frac{x}{\sigma\sqrt{2}}\right)^{n/2} K_{-1+n/2}\left(x\frac{\sqrt{2\xi}}{\sigma}\right), \qquad x \in I, \tag{70}$$

where (see [30], p. 928, n. 8.485)

$$K_\nu(z) = \frac{\pi}{2}\,\frac{I_{-\nu}(z) - I_\nu(z)}{\sin(\nu\pi)}$$

is the modified Bessel functions of the second kind. Recalling that (cf. [1], p. 375, n. 9.6.9)

$$K_\nu(z) \sim \frac{1}{2}\Gamma(\nu)\left(\frac{z}{2}\right)^{-\nu}, \qquad \nu > 0, \quad z \downarrow 0, \tag{71}$$

from (70) we have $\psi(0) = \sqrt{2\xi}/\sigma$. Furthermore, since (cf. [1], p. 378, n. 9.7.2)

$$K_\nu(z) \sim \sqrt{\frac{\pi}{2z}}\,e^{-z}, \qquad z \to +\infty, \tag{72}$$

from (68) and (70) we see that Eq. (14) is satisfied. Making use of (70), for $x \in I$ we get

$$\int_x^{+\infty} \psi(z)\,dz = \frac{2}{\Gamma(n/2)}\left(\frac{x}{\sigma}\sqrt{\frac{\xi}{2}}\right)^{n/2} K_{n/2}\left(x\frac{\sqrt{2\xi}}{\sigma}\right). \tag{73}$$

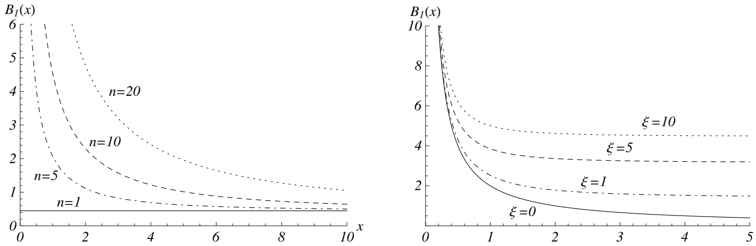
Hence, recalling (11), from (70) and (73) it follows

$$\gamma(x) = \frac{\sqrt{2\xi}}{\sigma}\,\frac{K_{-1+n/2}\left(x\frac{\sqrt{2\xi}}{\sigma}\right)}{K_{n/2}\left(x\frac{\sqrt{2\xi}}{\sigma}\right)}, \qquad x \in I. \tag{74}$$

As a consequence, recalling (10), we have that $Y(t)$ is a diffusion process having state-space I, with infinitesimal moments

$$B_1(x) = (n-1)\frac{\sigma^2}{2x} + \sigma\sqrt{2\xi}\,\frac{K_{-1+n/2}\left(x\frac{\sqrt{2\xi}}{\sigma}\right)}{K_{n/2}\left(x\frac{\sqrt{2\xi}}{\sigma}\right)}, \qquad B_2(x) = \sigma^2, \tag{75}$$

where $n \in \mathbb{N}$, $\sigma \in \mathbb{R}^+$ and $\xi > 0$. From (75) we see that, for $n = 1$, $Y(t)$ becomes a Wiener process with drift $B_1(x) = \sigma\sqrt{2\xi}$. For $Y(t)$ we have that 0 is a regular endpoint for $n = 1$ and an entrance endpoint for $n = 2, 3, \ldots$;



(a) $\xi = 0.1, \sigma = 1.0$ (b) $\sigma = 1.0, n = 5$

Fig. 5 Drift of $Y(t)$, given in (75), for different choices of n in (a), and ξ in (b).

furthermore, $+\infty$ is a natural attracting endpoint. By virtue of (72), from (75) one has:

$$B_1(x) \sim \sigma\sqrt{2\xi}, \qquad x \to +\infty.$$

Some plots of $B_1(t)$ are provided in Fig. 5.

Making use of (68) and (70), by virtue of (72), one can prove that condition (14) is satisfied. Hence, the assumptions of Proposition 1 hold. From (16), making use of Eqs. (68), (69), (70), (73) and (74), after some calculations for $x \in I$ and $t > 0$ we obtain:

$$V(x,t) = \left[K_{n/2}\left(x\frac{\sqrt{2\xi}}{\sigma}\right)\right]^{-1}\left\{\left(\frac{1}{\sigma\, t\sqrt{2\xi}}\right)^{n/2} x^{-1+n/2}\exp\left\{-\frac{x^2}{2\sigma^2 t}\right\}\right.$$
$$\left.-\left(\sigma\sqrt{\frac{2}{\xi}}\right)^{-1+n/2} x^{-n/2}\,\frac{K_{-1+n/2}\left(x\frac{\sqrt{2\xi}}{\sigma}\right)}{K_{n/2}\left(x\frac{\sqrt{2\xi}}{\sigma}\right)}\,\Gamma\left(\frac{n}{2},\frac{x^2}{2\sigma^2 t}\right)\right\}. \tag{76}$$

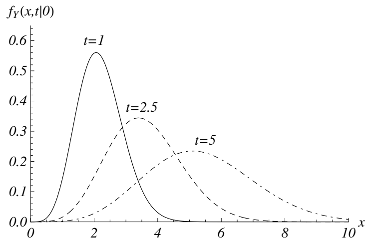
Making use of (27), when $\xi > 0$, for $n = 1$ one obtains

$$f_Y(0,t\,|\,0) = \frac{\sqrt{2}}{\sigma}\left[\frac{e^{-\xi\, t}}{\sqrt{\pi\, t}} + \sqrt{\frac{\pi}{\xi}}\,\mathrm{Erf}(\sqrt{\xi\, t}) - \sqrt{\xi}\right], \qquad t > 0,$$

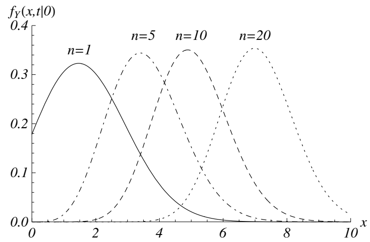
whereas $f_Y(0,t\,|\,0) = 0$, $t > 0$, for $n = 2, 3, \ldots$, due to the fact that 0 is an entrance endpoint. By virtue of (71), we also have that when $\xi \downarrow 0$ the function $V(x,t)$ given in (76) tends to the conditional density (68). For $\xi > 0$, $f_Y(x,t\,|\,0)$ can be determined making use of (15) and (76), since the integral in the right-hand-side can be evaluated numerically. Some plots are given in Fig. 6.

The Laplace transform of density (68) is, for $x \in I$,

$$\widehat{f}_Y(s;x\,|\,0) = \frac{\sqrt{2}\,(s+\xi)^{1/2+n/4}}{s\,\sigma\,\xi^{n/4}}\left[K_{n/2}\left(x\frac{\sqrt{2\xi}}{\sigma}\right)\right]^{-1}\left\{K_{-1+n/2}\left(x\frac{\sqrt{2(s+\xi)}}{\sigma}\right)\right.$$



(a) $\xi = 0.1, \sigma = 1.0, n = 5$



(b) $\xi = 0.1, \sigma = 1.0, t = 2.5$

Fig. 6 For the process with infinitesimal moments (75), the conditional pdf $f_Y(x,t\,|\,0)$ is plotted for some choices of t in (a) and for some choices of n in (b).

$$
-\sqrt{\frac{\xi}{s+\xi}}\,\frac{K_{-1+n/2}\left(x\dfrac{\sqrt{2\xi}}{\sigma}\right)}{K_{n/2}\left(x\dfrac{\sqrt{2\xi}}{\sigma}\right)}K_{n/2}\left(x\frac{\sqrt{2(s+\xi)}}{\sigma}\right)\Bigg\}.
$$

We conclude this section by remarking that, due to (71), when $\xi \downarrow 0$ the infinitesimal moments (75) tend to those of $X(t)$, specified in (67).

5.4 Transformation of the Rayleigh process

Let $X(t)$ be a Rayleigh process (of order n) having infinitesimal moments

$$
A_1(x) = (n-1)\frac{\sigma^2}{2x} - \alpha x, \qquad A_2(x) = \sigma^2 \qquad (n \in \mathbb{N}, \alpha \in \mathbb{R}^+, \sigma \in \mathbb{R}^+),
$$

and with state-space I. The endpoint 0 is regular if $n = 1$ and entrance if $n = 2, 3, \ldots$, whereas the endpoint $+\infty$ is natural non-attracting. We assume that a reflecting or zero-flux condition is imposed at $x = 0$. This process can be expressed as

$$
X(t) = ||\boldsymbol{\Upsilon}(t)|| = \left(\sum_{i=1}^{n} \Upsilon_i^2(t)\right)^{1/2}, \qquad t \geq 0,
$$

where $\boldsymbol{\Upsilon}(t)$, $t \geq 0$, is an n-dimensional diffusion process, whose components are independent Ornstein-Uhlenbeck processes started from the origin, with drift $-\alpha x$, infinitesimal variance σ^2, and restricted to $[0, +\infty)$ by a reflecting condition at 0.

The conditional pdf of $X(t)$ is (cf. Giorno *et al.* [27], for instance)

$$
f_X(x,t\,|\,0) = \frac{2\,x^{n-1}}{\Gamma(n/2)}\left[\frac{\alpha}{\sigma^2\,(1-e^{-2\alpha t})}\right]^{n/2} \exp\left\{-\frac{\alpha x^2}{\sigma^2\,(1-e^{-2\alpha t})}\right\}, \quad (77)
$$

for $x \in I$ and $t > 0$. The process $X(t)$ admits a steady-state density $W_X(x) = 2\,(\alpha/\sigma^2)^{n/2}[x^{n-1}/\Gamma(n/2)]\exp\{-\alpha\,x^2/\sigma^2\}$, $x \in I$. From (77) we have:

$$\int_x^{+\infty} f_X(z,t\,|\,0)\,dz = \frac{1}{\Gamma(n/2)}\Gamma\left(\frac{n}{2},\frac{\alpha\,x^2}{\sigma^2\,(1-e^{-2\alpha\,t})}\right), \qquad x \in I, \quad t>0. \tag{78}$$

Due to (8), for $x \in I$ we have

$$\psi(x) = \frac{\xi x^{n-1}}{\alpha\Gamma(n/2)}\left(\frac{\alpha}{\sigma^2}\right)^{n/2}\exp\left\{-\frac{\alpha x^2}{\sigma^2}\right\}\Gamma\left(\frac{\xi}{2\alpha}\right)\Psi\left(\frac{\xi}{2\alpha},\frac{n}{2};\frac{\alpha x^2}{\sigma^2}\right), \tag{79}$$

where

$$\Psi(a,c;z) = \frac{\Gamma(1-c)}{\Gamma(a-c+1)}\,\Phi(a,c;z) + \frac{\Gamma(c-1)}{\Gamma(a)}\,z^{1-c}\,\Phi(a-c+1,2-c;z)$$

denotes the Kummer's function of the second kind (cf. [30], p. 1023, n. 9.210.2). Taking into account the asymptotic behaviour of $\Psi(a,c;z)$ for $z \downarrow 0$ (see Abramowitz and Stegun [1], p. 508), we obtain that $\psi(0)$ is finite, i.e.

$$\psi(0) = \mathbf{1}_{\{n=1\}}\,\frac{\xi}{\sigma\sqrt{\alpha}}\,\frac{\Gamma\left(\dfrac{\xi}{2\alpha}\right)}{\Gamma\left(\dfrac{\xi}{2\alpha}+\dfrac{1}{2}\right)}, \qquad \alpha>0.$$

Making use of (77) and (79), and recalling that (see [1], p. 504, n. 13.1.8)

$$\Psi(a,c;z) \sim z^{-a}, \qquad z \to +\infty, \tag{80}$$

one can check that condition (14) is fulfilled. Hence, the assumptions of Proposition 1 hold. After some calculations from (79) we obtain, for $x \in I$,

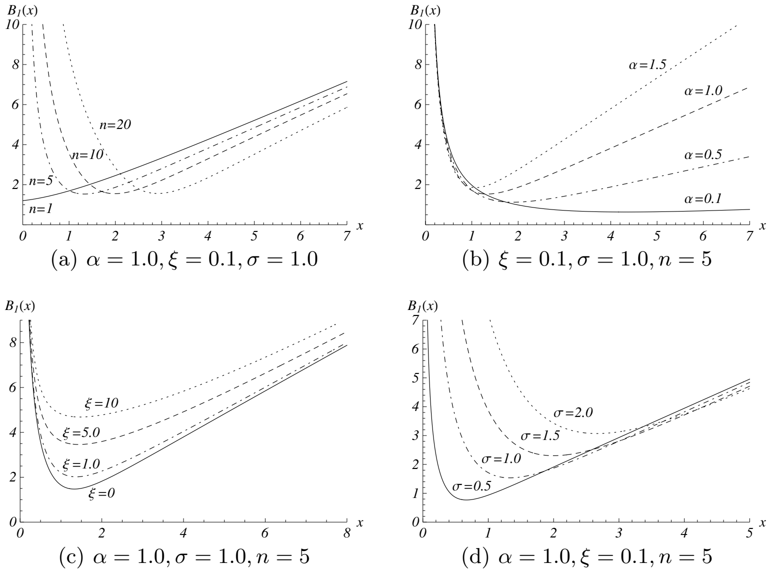
$$\int_x^{+\infty}\psi(z)\,dz = \frac{e^{-\alpha x^2/\sigma^2}}{\Gamma(n/2)}\,\Gamma\left(\frac{\xi}{2\alpha}+1\right)\Psi\left(1-\frac{n}{2}+\frac{\xi}{2\alpha},1-\frac{n}{2};\frac{\alpha x^2}{\sigma^2}\right). \tag{81}$$

Due to (11), from (79) and (81) we have

$$\gamma(x) = \frac{2}{x}\,\frac{\Psi\left(\dfrac{\xi}{2\alpha},\dfrac{n}{2};\dfrac{\alpha x^2}{\sigma^2}\right)}{\Psi\left(\dfrac{\xi}{2\alpha}+1,\dfrac{n}{2}+1;\dfrac{\alpha x^2}{\sigma^2}\right)}, \qquad x \in I. \tag{82}$$

Thus, by virtue of (10), $Y(t)$ is a diffusion process with state-space I, and infinitesimal moments

$$B_1(x) = (n-1)\frac{\sigma^2}{2x} - \alpha x + \frac{2\sigma^2}{x}\,\frac{\Psi\left(\dfrac{\xi}{2\alpha},\dfrac{n}{2};\dfrac{\alpha x^2}{\sigma^2}\right)}{\Psi\left(\dfrac{\xi}{2\alpha}+1,\dfrac{n}{2}+1;\dfrac{\alpha x^2}{\sigma^2}\right)}, \qquad B_2(x) = \sigma^2, \tag{83}$$



(a) $\alpha = 1.0, \xi = 0.1, \sigma = 1.0$ (b) $\xi = 0.1, \sigma = 1.0, n = 5$

(c) $\alpha = 1.0, \sigma = 1.0, n = 5$ (d) $\alpha = 1.0, \xi = 0.1, n = 5$

Fig. 7 Drift of $Y(t)$, given in (83), for different choices of n in (a), of α in (b), of ξ in (c) and of σ in (d).

with $n \in \mathbb{N}$, $\alpha \in \mathbb{R}^+$ and $\sigma \in \mathbb{R}^+$. Since (cf. Eq. (31), p. 267, of Erdélyi *et al.* [16])

$$\Psi\Big(-\frac{\nu}{2}, \frac{1}{2}; \frac{x^2}{2}\Big) = 2^{-\nu/2} e^{x^2/4} D_\nu(x), \quad \Psi\Big(\frac{1-\nu}{2}, \frac{3}{2}; \frac{x^2}{2}\Big) = \frac{2^{(1-\nu)/2}\, e^{x^2/4}}{x} D_\nu(x),$$

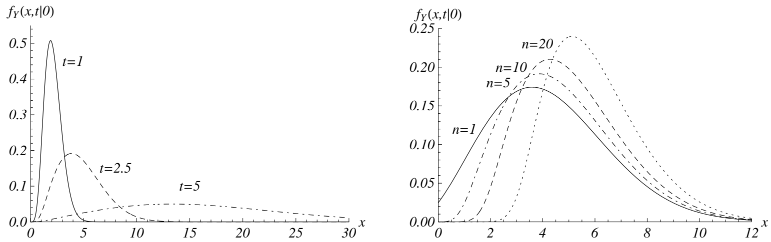
it is not hard to prove that when $n = 1$, the infinitesimal moments (83) identify with (61). This is to be expected, since for $n = 1$ the Rayleigh process $X(t)$ identifies with the Ornstein-Uhlenbeck process. For $Y(t)$, we have that 0 is a regular endpoint for $n = 1$ and an entrance endpoint for $n = 2, 3, \ldots$; furthermore, $+\infty$ is a natural attracting endpoint. From (80) and (83), we note that

$$B_1(x) \sim \alpha x \qquad \text{as } x \to +\infty.$$

Some plots of $B_1(x)$ are given in Fig. 7.

Let us now determine $V(x,t)$, as specified in (16). Making use of (77), (79), (81) and (82), after some calculations we obtain, for $x \in I$ and $t > 0$,

$$V(x,t) = \frac{2}{x} \exp\left\{\frac{\alpha x^2}{\sigma^2}\right\} \frac{\sigma^n}{\alpha^{n/2}} \left[\Gamma\left(\frac{\xi}{2\alpha}+1\right) \Psi\left(\frac{\xi}{2\alpha}+1, \frac{n}{2}+1; \frac{\alpha x^2}{\sigma^2}\right)\right]^{-1}$$
$$\times \left\{ \left[\frac{\alpha}{\sigma^2(1-e^{-2\alpha t})}\right]^{n/2} \exp\left\{-\frac{\alpha x^2}{\sigma^2(1-e^{-2\alpha t})}\right\}\right.$$



(a) $\alpha = 0.5, \xi = 0.1, \sigma = 1.0, n = 5$ (b) $\alpha = 0.5, \xi = 0.1, \sigma = 1.0, t = 2.5$

Fig. 8 For the process with infinitesimal moments (83), the conditional pdf $f_Y(x,t\,|\,0)$ is plotted for some choices of t in (a) and for some choices of n in (b).

$$-\frac{1}{x^n}\,\frac{\Psi\left(\frac{\xi}{2\alpha},\frac{n}{2};\frac{\alpha x^2}{\sigma^2}\right)}{\Psi\left(\frac{\xi}{2\alpha}+1,\frac{n}{2}+1;\frac{\alpha x^2}{\sigma^2}\right)}\,\Gamma\left(\frac{n}{2},\frac{\alpha x^2}{\sigma^2(1-e^{-2\alpha t})}\right)\Bigg\}. \tag{84}$$

By virtue of (15) and (84), for $n=1$ Eq. (64) holds, whereas for $n=2,3,\ldots$ we have $f_Y(0,t\,|\,0)=0$, $t>0$, since 0 is an entrance endpoint. Moreover, the density $f_Y(x,t\,|\,0)$ can be determined making use of (84) in (15), by evaluating numerically the integral in the right-hand-side. Some plots are provided in Fig. 8.

For $\xi>0$, we finally evaluate the Laplace transform

$$\begin{aligned}\widehat{f}_Y(s;x\,|\,0)=&\,\frac{s+\xi}{s\,\alpha x}\Gamma\left(\frac{s+\xi}{2\alpha}\right)\left[\Gamma\left(\frac{\xi}{2\alpha}+1\right)\Psi\left(\frac{\xi}{2\alpha}+1,\frac{n}{2}+1;\frac{\alpha x^2}{\sigma^2}\right)\right]^{-1}\\ &\times\Bigg\{\Gamma\left(\frac{s+\xi}{2\alpha}\right)\Psi\left(\frac{s+\xi}{2\alpha},\frac{n}{2};\frac{\alpha x^2}{\sigma^2}\right)\\ &-\frac{\Psi\left(\frac{\xi}{2\alpha},\frac{n}{2};\frac{\alpha x^2}{\sigma^2}\right)}{\Psi\left(\frac{\xi}{2\alpha}+1,\frac{n}{2}+1;\frac{\alpha x^2}{\sigma^2}\right)}\Psi\left(\frac{s+\xi}{2\alpha}+1,\frac{n}{2}+1;\frac{\alpha x^2}{\sigma^2}\right)\Bigg\}.\end{aligned}$$

Let us now discuss the case $\xi\downarrow 0$. Since $\Psi(0,c;z)=1$ and $\Psi(1,c;z)=e^z z^{1-c}\Gamma(c-1,z)$, by virtue of (36) we can see that when $\xi\downarrow 0$ the infinitesimal moments in (83) become

$$B_1(x)=(n-1)\frac{\sigma^2}{2x}-\alpha x+2\,\sigma^2\left(\frac{\sqrt{\alpha}}{\sigma}\right)^n\frac{x^{n-1}e^{-\alpha\,x^2/\sigma^2}}{\Gamma\left(\frac{n}{2},\frac{\alpha x^2}{\sigma^2}\right)},\qquad B_2(x)=\sigma^2. \tag{85}$$

For the diffusion process (85), from Proposition 4 we have that the conditional pdf for $x\in I$ and $t>0$ is given by

$$f_Y(x,t\,|\,0)=\frac{2\,x^{n-1}}{\Gamma\left(\frac{n}{2},\frac{\alpha x^2}{\sigma^2}\right)}\left\{\left[\frac{\alpha}{\sigma^2(1-e^{-2\alpha t})}\right]^{n/2}\exp\left\{-\frac{\alpha x^2}{\sigma^2(1-e^{-2\alpha t})}\right\}\right.$$

	$A_1(x)$	$B_1(x)$
(a)	μ	$\sqrt{\mu^2+2\sigma^2\xi}$
(b)	$-\alpha x$	$-\alpha x+\sigma\sqrt{2\alpha}\,\frac{D_{-\xi/\alpha}\left(x\frac{\sqrt{2\alpha}}{\sigma}\right)}{D_{-1-\xi/\alpha}\left(x\frac{\sqrt{2\alpha}}{\sigma}\right)}$
(c)	$(n-1)\frac{\sigma^2}{2x}$	$(n-1)\frac{\sigma^2}{2x}+\sigma\sqrt{2\xi}\,\frac{K_{-1+n/2}\left(x\frac{\sqrt{2\xi}}{\sigma}\right)}{K_{n/2}\left(x\frac{\sqrt{2\xi}}{\sigma}\right)}$
(d)	$(n-1)\frac{\sigma^2}{2x}-\alpha x$	$(n-1)\frac{\sigma^2}{2x}-\alpha x+\frac{2\sigma^2}{x}\,\frac{\Psi\left(\frac{\xi}{2\alpha},\frac{n}{2};\frac{\alpha x^2}{\sigma^2}\right)}{\Psi\left(\frac{\xi}{2\alpha}+1,\frac{n}{2}+1;\frac{\alpha x^2}{\sigma^2}\right)}$
		$U(x)$
(a)		$-x\,\frac{\sqrt{\mu^2+2\sigma^2\xi}}{\sigma}+c$
(b)		$\sigma\ln\left[D_{-1-\xi/\alpha}\left(x\frac{\sqrt{2\alpha}}{\sigma}\right)\right]+c$
(c)		$\sigma\left\{\frac{1}{2}\ln x+\ln\left[K_{n/2}\left(x\frac{\sqrt{2\xi}}{\sigma}\right)\right]\right\}+c$
(d)		$\sigma\left\{\frac{n+1}{2}\ln x+\ln\left[\Psi\left(\frac{\xi}{2\alpha}+1,1+\frac{n}{2};\frac{\alpha x^2}{\sigma^2}\right)\right]-\frac{\alpha x^2}{2\sigma^2}\right\}+c$

Table 1 Drift $A_1(x)$ of the diffusion process $X(t)$, drift $B_1(x)$ of the related transformed process $Y(t)$, and potential $U(x)$ of $Y(t)$, for the cases treated in Section 5. For all processes the infinitesimal variance is σ^2.

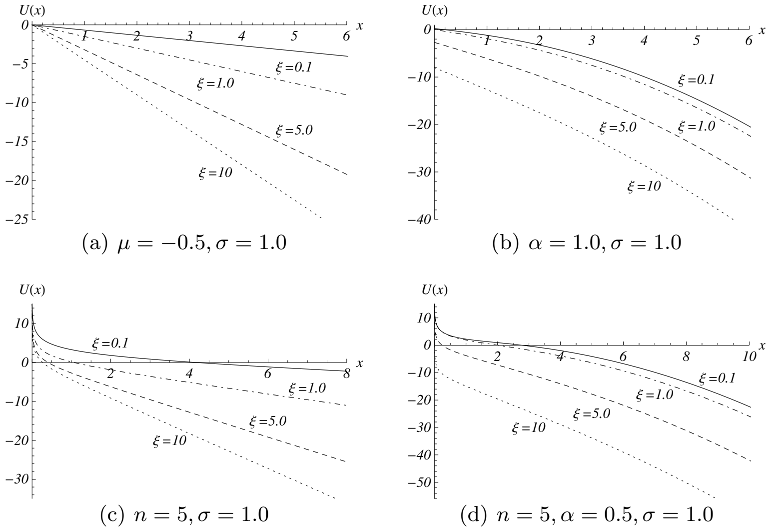
$$-\left(\frac{\sqrt{\alpha}}{\sigma}\right)^n\frac{e^{-\alpha\,x^2/\sigma^2}}{\Gamma\left(\frac{n}{2},\frac{\alpha x^2}{\sigma^2}\right)}\,\Gamma\left(\frac{n}{2},\frac{\alpha x^2}{\sigma^2(1-e^{-2\alpha t})}\right)\Biggr\}. \tag{86}$$

We note that if $n=1$, (85) and (86) identify with (65) and (66), respectively.

6 Concluding remarks

This paper has been devoted to the analysis of certain reflected diffusion processes obtained via suitable transformations. The infinitesimal moments of the new process $Y(t)$ are expressed in terms of those of a former process $X(t)$ by mean of Eq. (10), where $\gamma(x)$ is the hazard rate function of the compound random variable that describes the former process evaluated at an exponentially distributed random time. Thus, the effect of the considered transformation is that the drift of the new diffusion process is larger than the previous one. As a consequence, the process $Y(t)$ can be described as a diffusion in a decreasing potential $U(x)$, according to the notions exploited in Section 2.3.

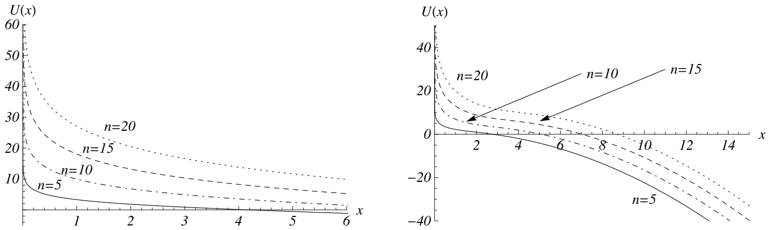
The analysis has been centered on the determination of various expression of interest, such as the pdf and the mean of $Y(t)$ conditional on zero initial state. In Section 5 we investigated the diffusion processes obtained starting from Wiener, Ornstein-Uhlenbeck, Bessel and Rayleigh process, with all processes having constant infinitesimal variance. In Table 1, we report the drifts of the original diffusion process and the corresponding drifts of the



(a) $\mu = -0.5, \sigma = 1.0$ (b) $\alpha = 1.0, \sigma = 1.0$

(c) $n = 5, \sigma = 1.0$ (d) $n = 5, \alpha = 0.5, \sigma = 1.0$

Fig. 9 For the diffusion processes considered in Table 1, the potential $U(x)$ is plotted for $c = 0$ and some choices of the parameter ξ.



$\xi = 0.1, \sigma = 1.0$ $\xi = 0.1, \alpha = 0.5, \sigma = 1.0$

Fig. 10 For the diffusion processes with drift $B_1(x)$ given in (c) and (d) of Table 1, the potential $U(x)$ is plotted for $c = 0$ and some choices of n.

new processes. Furthermore, in Table 1, we also show the potentials $U(x)$ of the new processes, obtained by mean of Eq. (42). We recall that in the context of energy diagrams one can refer to the potentials $U(x)$ by thinking of a ball rolling around on the top of the function. The ball experiences a force whose magnitude is proportional to the slope of the $U(x)$ curve, and whose direction is oriented toward decreasing potential energy. The diffusion processes $Y(t)$ investigated in Section 5 describe diffusive motions directed toward $+\infty$. This is confirmed by the fact that $+\infty$ is an attractive natural endpoint for the processes $Y(t)$ having infinitesimal moments (10), as pointed out in Section 2. Taking such remarks in mind, Figure 9 shows the behavior of the potential $U(x)$ for the processes having drift $B_1(x)$, given in Table 1, for different choices of ξ. In all cases, if ξ increases then the potential decreases more rapidly when x increases. Hence, the effect of the parameter ξ involves a reduction of the potential, this implying a greater growth velocity of the individual processes. Finally, Figure 10 shows the potential $U(x)$ given in cases (c) and (d) of Table 1 for different choices of parameter n. We note that the potential increases with n. The variety of shapes exhibited by the potentials $U(x)$ given in Figures 9 and 10 shows that the method proposed in this paper allows to construct several types of diffusions suitable to describe phenomena characterized by rapid random growth, such as bacterial growth, and buffer occupation processes of stochastic networks in heavy traffic.

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