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Complexity of conflict-free colorings of graphs

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ABSTRACT

We consider conflict-free colorings of graph neighborhoods: Each vertex of the graph must be assigned a color so that for each vertex v there is at least one color appearing exactly once in the neighborhood of v. The goal is to minimize the number of used colors. We consider both the case of closed neighborhoods, when the neighborhood of a node includes the node itself, and the case of open neighborhoods when a node does not belong to its neighborhood. In this paper, we study complexity aspects of the problem. We show that the problem of conflict-free coloring of closed neighborhoods is NP-complete. Moreover, we give non-approximability results for the conflict-free coloring of open neighborhoods. From a positive point of view, both problems become tractable if parameterized by the vertex cover number or the neighborhood diversity number of the graph. We present simple algorithms which improve on existing results.

1. Introduction

Let $\mathcal{H} = (V, \mathcal{E})$ be a hypergraph with vertex set V and edge set $\mathcal{E}$. A coloring of (the vertices of) $\mathcal{H}$ is a function $C: V \to \mathbb{Z}^+$. A k-coloring of $\mathcal{H}$ is a coloring C with $|C(V)| \le k$, for some $k \ge 1$. A coloring is called proper if no edge $e \in \mathcal{E}$ containing at least two vertices is monochromatic. The smallest number of colors needed to properly color the vertices of $\mathcal{H}$ is called the chromatic number of $\mathcal{H}$ and is denoted by $\chi(\mathcal{H})$. A coloring is said to be conflict-free if every hyperedge contains a vertex whose color is unique among those assigned to the vertices of the hyperedge.

Definition 1 *(CF coloring).* Let C be a coloring of a hypergraph $\mathcal{H} = (V, \mathcal{E})$. C is a *conflict-free coloring* of $\mathcal{H}$ if for each $e \in \mathcal{E}$ there exists a vertex $v \in e$ such that $C(u) \neq C(v)$ for any $u \in e$ with $u \neq v$.

The study of conflict-free colorings was initially motivated by a frequency assignment problem in cellular networks [10]. Such networks consist of fixed-position *base stations*, each assigned a fixed frequency, and roaming *clients*. Roaming clients have a range of communication and come under the influence of different subsets of base stations. This situation can be modeled by means of a hypergraph whose vertices correspond to the base stations. The range of communication of a mobile agent, that is, the set of base stations it can communicate with, is represented by a hyperedge $e \in \mathcal{E}$. A CF-coloring of such a hypergraph implies an assignment of frequencies, to the base stations, which enables clients to connect to a base station holding a unique frequency in the client's range, thus avoiding interferences. The goal is to minimize the number of assigned frequencies.

CF-coloring also finds application in RFID (Radio Frequency Identification) networks. RFID allows a reader device to sense the presence of a nearby object by reading a tag attached to the object itself. To improve coverage, multiple RFID readers

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can be deployed in an area. However, two readers trying to access a tagged device simultaneously might cause mutual interference. It can be shown that CF-coloring of the readers can be used to assure that every possible tag will have a time slot and a single reader trying to access it in that time slot [23].

Due to both its practical motivations and its theoretical interest, conflict-free coloring has been the subject of several papers; a survey of results in the area is given in [23]. The theoretic study of the CF-chromatic number in general graphs and hypergraphs was initiated in [22] and has recently raised much interest due to the novel combinatorial and algorithmic questions it poses, see [2–4,6,15,16,18].

1.1. CF-colorings of neighborhoods

In this paper we study the conflict-free coloring of hypergraphs induced by the neighborhoods of the vertices of a graph [22].

Given a graph $G=(V,E)$ and a vertex $u \in V$, the *open neighborhood* $N_G(u)$ of u is defined as the set consisting of all the vertices in G connected to u. The set $N_G[u]=N_G(u)\cup\{u\}$ is called the *closed neighborhood* of u. We will write $N(u)$ and $N[u]$ whenever G is clear from the context.

A conflict-free coloring with respect to the open (resp. closed) neighborhoods of G is defined as the conflict-free coloring of the hypergraph with vertex set V and edge set $\{N_G(u) \mid u \in V\}$ (resp. $\{N_G[u] \mid u \in V\}$). For the sake of simplicity, we now reformulate Definition 1 in terms of the graph G.

Given a graph $G=(V,E)$ and a coloring C, we say that the set $U \subseteq V$ has a *unique color* under C if there exists a color c such that $|\{v \in U \mid C(v)=c\}|=1$. Equivalently, we say that c is unique for U. All the graphs considered in this paper are supposed to be connected.

Definition 2. Consider a graph $G=(V,E)$.

CF-ON coloring: A coloring C is called conflict-free with respect to the *open neighborhoods* of G if for each $u \in V$ the set $N(u)$ has a *unique color* under C.

CF-CN coloring: A coloring C is called conflict-free with respect to the *closed neighborhoods* of G if for each $u \in V$ the set $N[u]$ has a *unique color* under C.

The smallest number of colors needed by any possible CF-ON (resp. CF-CN) coloring of G is called the CF-ON (resp. CF-CN) chromatic number of G and is denoted by $\chi_{CF}(G)$ (resp. $\chi_{CF}[G]$).

It is possible to show (see also [22]) that, given a graph G, the same greedy upper bound Δ_G+1 (where Δ_G is the maximum degree of a vertex in G) holds for the chromatic number, the CF-ON chromatic number, and the CF-CN chromatic number. However, as also noticed in [22], these values can be quite different and no ordering among them is valid for any graph. Examples are given in Fig. 1. Consider first the complete graph K_n on n vertices, it is not difficult to see that

$$\chi_{CF}[K_n]=2<\chi_{CF}(K_n)=3<\chi(K_n)=n.$$

Moreover, it is easy to show that for any tree T

$$\chi_{CF}[T]=\chi_{CF}(T)=\chi(T)=2.$$

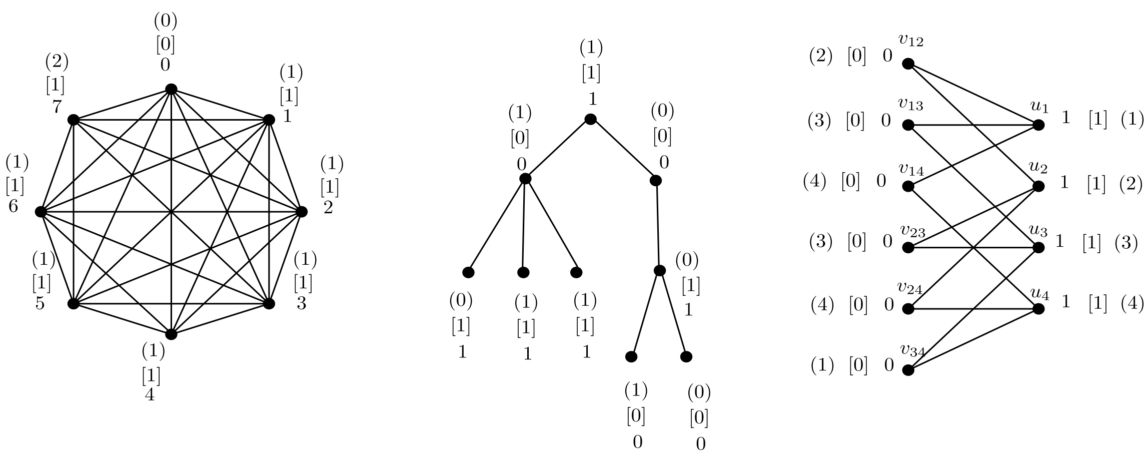

Fig. 1. The graph K_8, a tree T, and the graph B_4 with the corresponding colorings. For each node $c, [c'], (c'')$ represent the colors assigned to the node in a proper, a CF-CN, and a CF-ON coloring, respectively.

Finally, consider the bipartite graph $B_\ell = (U, V, E)$ such that $|U| = \ell$, $V = \{v_{u,w} \mid u, w \in U,\ u \neq w\}$, and E contains the edges $(v_{u,w}, u)$ and $(v_{u,w}, w)$, for each $u, w \in U$. It holds

$$\chi(B_\ell) = 2 = \chi_{\mathrm{CF}}[B_\ell] < \chi_{\mathrm{CF}}(B_\ell) = \ell = \Delta_{B_\ell} + 1 \approx \sqrt{2\bigl(|V| + |U|\bigr)}.$$

In this paper we study complexity aspects of the conflict-free colorings of graph open/closed neighborhoods. In Section 2, we show that the problem of determining the CF-CN chromatic number of a graph is NP-complete and give non-approximability results for the CF-ON coloring problem. On the positive side, in Section 3 we show that both the CF-CN and the CF-ON coloring problems are tractable when restricted to graphs with bounded vertex cover number or with bounded neighborhood diversity [13]; our results improve on the ones implied by the metatheorems in [20].

2. Computational complexity

In this section we study the computational complexity of computing optimal CF-colorings with respect to open and closed neighborhoods. We show that determining whether a graph has a CF-CN coloring of a given size is NP-complete. We prove the hardness of approximating the CF-ON chromatic number.

2.1. Conflict-free closed neighborhood colorings

To prove our NP-completeness result, let us consider the decision version of the CF-CN coloring problem.

`CF-CNC` (CF Closed Neighborhood Coloring)
Instance: A graph $G = (V, E)$ and an integer bound k.
Question: Is there a function $C: V \to \mathbb{Z}^+$ such that $|C(V)| \leq k$, and for each $u \in V$, the set $N[u]$ has a unique color under C?

`CF-CNC` is clearly in NP. We prove its NP-hardness by a reduction from the well known NP-complete `3-CNF SAT` problem. Given any 3-CNF formula $\Phi = K_0 \wedge K_1 \wedge \cdots \wedge K_{\ell-1}$, where $K_i = (x_1^i \vee x_2^i \vee x_3^i)$ for $i = 0, \ldots, \ell-1$, we outline how to construct a graph $G_\Phi = (V, E)$ which has a CF-CN coloring with 2 colors (CF-CN 2-coloring) if and only if Φ is satisfiable.

Let $G_\Phi(K_i)$ be the subgraph, given in Fig. 2(a), associated to the clause K_i, for any $i = 0, \ldots, \ell-1$, where each vertex v_h^i corresponds to the literal x_h^i in K_i, for $h = 1, 2, 3$. Furthermore, let AND_i be the gadget, given in Fig. 2(b), that we use to join vertex u^i in $G_\Phi(K_i)$ with vertex u^{i+1} in $G_\Phi(K_{i+1})$,[1] for $i = 0, \ldots, \ell-1$. The graph G_Φ is then obtained as follows:

- Join $G_\Phi(K_i)$ with $G_\Phi(K_{i+1})$ using AND_i, for $i = 0, \ldots, \ell-1$; in particular, connect u^i and w_1^i with an edge, and connect w_5^i and u_{i+1} with another edge.
- Connect with an edge, each vertex corresponding to a variable x with each vertex corresponding to the variable $\bar{x}$.

The construction of graph G_Φ can obviously be done in polynomial time. An example is given in Fig. 2(c).

Lemma 1. *Consider any CF-CN 2-coloring of G_Φ. If there exists a variable x such that x appears in K_i and $\bar{x}$ appears in K_j, for some $i \neq j$, then the color assigned to the vertex corresponding to x in $G_\Phi(K_i)$ is different from the color assigned to the vertex corresponding to $\bar{x}$ in $G_\Phi(K_j)$.*

Proof. Let C be a CF-CN 2-coloring of G_Φ. Let v_h^i and v_ℓ^j be the vertices corresponding to x and $\bar{x}$ in $G_\Phi(K_i)$ and $G_\Phi(K_j)$, respectively. By contradiction, suppose that $C(v_h^i) = C(v_\ell^j)$. Since the neighbors of v_h^i include v_ℓ^j and two leaves and since $N[v_h^i]$ has a unique color (recall that the coloring uses 2 colors), then at most one neighbor of v_h^i has color different from $C(v_h^i)$. Hence, at least one of the two neighbor leaves of v_h^i has color $C(v_h^i)$. This implies that the closed neighborhood of such a leaf consists of two nodes of the same color. This contradicts the assumption of a CF-CN coloring. □

Lemma 2. *For any CF-CN 2-coloring C of G_Φ, we have $C(u^0) = \cdots = C(u^{\ell-1})$.*

Proof. For each $i = 0, \ldots, \ell-1$, the edge (u^i, w_1^i) connects $G_\Phi(K_i)$ to the AND_i gadget and the edge (w_5^i, u^{i+1}) connects the AND_i gadget to $G_\Phi(K_{i+1})$. We first prove that $C(u^i) = C(w_2^i)$. By contradiction, suppose that $C(u^i) \neq C(w_2^i)$. Recalling that $N[w_1^i] = \{w_1^i, u^i, w_2^i, x\}$, where x is a leaf, must have a unique color under C, we have that $C(w_1^i) = C(x)$. Therefore, we have a contradiction since $N[x] = \{x, w_1^i\}$ would contain two nodes with the same color.

With a proof similar to the above one, we can prove that $C(w_2^i) = C(w_4^i)$ and $C(w_4^i) = C(u^{i+1})$. Hence, the lemma follows. □

[1] The operation + refers to the addition modulo ℓ when applied to the indices of the clauses in G_Φ.

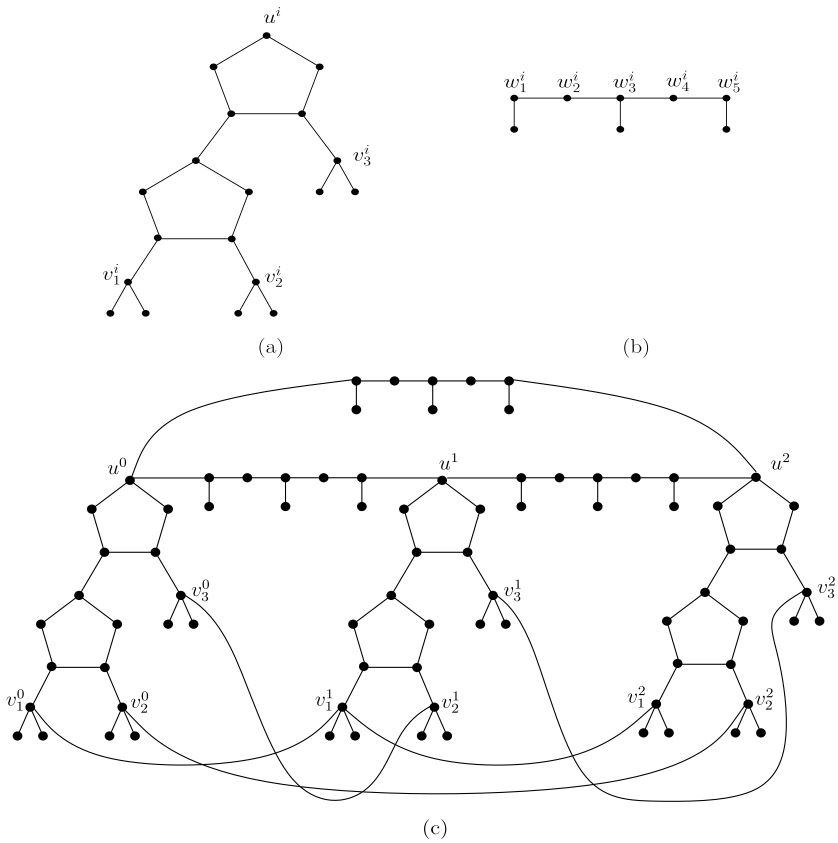

Fig. 2. (a) $G_\Phi(K_i)$. (b) AND_i gadget. (c) G_Φ for $\Phi = (x_1^0 \vee x_2^0 \vee x_3^0) \wedge (x_1^1 \vee x_2^1 \vee x_3^1) \wedge (x_1^2 \vee x_2^2 \vee x_3^2) = (a \vee b \vee \bar{c}) \wedge (\bar{a} \vee c \vee \bar{d}) \wedge (a \vee \bar{b} \vee d)$.

Lemma 3. *For each CF-CN 2-coloring C of G_Φ, and for any $i = 0, \dots, \ell - 1$, if $C(u^i) = 1$ then at least one of the nodes corresponding to the literals in K_i has color 1.*

Proof. We show that any CF-CN 2-coloring C of G_Φ that assigns 0 to the three nodes in $G_\Phi(K_i)$ corresponding to the literals in K_i also assigns color 0 to u^i, that is $C(v_1^i) = C(v_2^i) = C(v_3^i) = 0$ implies $C(u^i) = 0$. We first notice that $N[v_h^i]$, for $h = 1, 2, 3$, includes two leaves whose closed neighborhoods must have a unique color. Hence each of these leaves must have color 1, otherwise the closed neighborhood would consist of two nodes of the same color 0. It follows that all the neighbors of v_h^i must have color 1. This forces any CF-CN 2-coloring of the remaining vertices in $G_\Phi(K_i)$ to be as in Fig. 3, thus proving the lemma. □

Theorem 1. *CF–CNC is NP-complete.*

Proof. We prove that G_Φ has a CF-CN 2-coloring iff Φ is satisfiable. Let C be any CF-CN 2-coloring of G_Φ. By Lemma 2, w.l.o.g. we can assume that

$$C(u^0) = C(u^1) = \dots = C(u^{\ell-1}) = 1. \tag{1}$$

By (1) and Lemma 3, at least one of the vertices corresponding to the literals in K_i has color 1, for $i = 0, \dots, \ell - 1$. Hence, if we set

$$x_h^i = \text{TRUE} \quad \text{iff} \quad C(v_h^i) = 1, \quad \text{for } i = 0, \dots, \ell - 1 \text{ and } h = 1, 2, 3,$$

then by Lemma 1, the values assigned to the literals are consistent, and for each K_i there is at least one literal whose value is TRUE. It follows that Φ is satisfiable.

Assume now that Φ is satisfiable. We use the truth assignment of Φ to get a CF-CN 2-coloring of G_Φ. For each $i = 0, \dots, \ell - 1$ and $h = 1, 2, 3$, we color the vertices corresponding to true literals with color 1 and the vertices corresponding to false literals with color 0, that is, we set

$$C(v_h^i) = \begin{cases} 0 & \text{if } x_h^i = \text{FALSE}, \\ 1 & \text{if } x_h^i = \text{TRUE}. \end{cases}$$

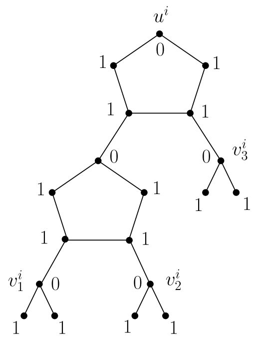

Fig. 3. The CF-CN 2-coloring of $G_\Phi(K_i)$ when v_1^i, v_2^i and v_3^i have color 0.

We then color all the neighbors of v_h^i with the color opposed to that of v_h^i itself, i.e., if v_h^i is colored with 1 then all its neighbors are colored with 0, and vice-verse. This implies that $N[v_h^i]$ has a unique color under C as well as the closed neighborhood of each of the two neighbor leaves of v_h^i. Notice that if v_h^i has a neighbor $v_{h'}^j$, for some $j \neq i$, then v_h^i and $v_{h'}^j$ have properly opposite colors since they correspond to a variable and its negation in two different clauses.

Finally, the coloring of vertices v_h^i, for $i = 0, \ldots, \ell - 1$ and $h = 1, 2, 3$, and the satisfiability of Φ imply that at least one vertex v_h^i, in each $G_\Phi(K_i)$, has color 1, i.e., $C(v_h^i) = 1$. It is then easy to verify that any CF-CN 2-coloring of the remaining vertices of each $G_\Phi(K_i)$ assures that u^i has color 1, for $i = 0, \ldots, \ell - 1$. The coloring of the vertices of $G_\Phi(K_i)$ uniquely specify the color that we can assign to the neighbors w_1^i and w_5^{i-1} of u^i and to the neighbors w_1^{i+1} and w_5^i of u^{i+1}. Now we color w_2^i and w_4^i with 1 and w_3^i with 0. Finally, we complete the CF-CN 2-coloring C of G_Φ by assigning to each leaf connected a vertex w_j^i the color opposite to $C(w_j^i)$, for each $j = 1, \ldots, 5$. □

The NP-completeness of `CF-CNC` is proved for $k = 2$ colors. Hence, Theorem 1 implies the following inapproximability result [14].

Corollary 1. *`CF-CNC` chromatic number is hard to approximate within a factor less that* $3/2$ *unless* $P = NP$.

2.2. Conflict-free open neighborhood colorings

Let us consider the decision version of the CF-ON coloring problem.

> `CF-ONC` (CF Open Neighborhood Coloring)
> *Instance:* A graph $G = (V, E)$ and an integer bound k.
> *Question:* Is there a function $C: V \to \mathbb{Z}^+$ such that $|C(V)| \le k$, and for each $u \in V$, the set $N(u)$ has a unique color under C?

It is easy to see that `CF-ONC` is in NP. We first prove that `CF-ONC` is NP-hard by considering $k = 2$ and showing a reduction from the NP-complete problem `Not-All-Equal-3-SAT` [14]. Moreover, when $k \ge 3$, we show a reduction from `GRAPH COLORING`. Since our reduction is gap-preserving [17], it implies corresponding inapproximability results for the problem of determining the `CF-ONC` chromatic number.

Theorem 2. *`CF-ONC` is NP-complete.*

Proof. We give a reduction from the NP-complete `Not-All-Equal-3-SAT` problem [14].

> `NAE-3SAT` (`Not-All-Equal-3-SAT`)
> *Instance:* A boolean formula Φ that is an instance of `3-SAT`.
> *Question:* Is there a truth-value assignment for the variables of Φ such that each of its clauses has at least one true literal and at least one false literal?

Consider an instance of `NAE-3SAT`, that is, a formula $\Phi = K_1 \wedge K_1 \wedge \cdots \wedge K_\ell$, where $K_i = (x_1^i \vee x_2^i \vee x_3^i)$ for $i = 1, \ldots, \ell$. We outline how to construct a graph $G_\Phi = (V, E)$ which has a CF-ON coloring with 2 colors (CF-ON 2-coloring) if and only if Φ is a *yes* instance for `NAE-3SAT`. The graph G_Φ is obtained as follows:

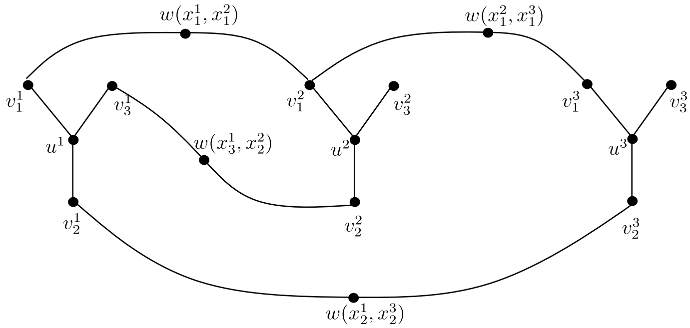

Fig. 4. G_Φ for $\Phi = (x_1^1 \vee x_2^1 \vee x_3^1) \wedge (x_1^2 \vee x_2^2 \vee x_3^2) \wedge (x_1^3 \vee x_2^3 \vee x_3^3) = (a \vee b \vee \bar{c}) \wedge (\bar{a} \vee c \vee d) \wedge (a \vee \bar{b} \vee d)$.

- For $i = 1, \dots, \ell$, the graph G_Φ contains a vertex u_i and the three vertices v_1^i, v_2^i, v_3^i, where v_h^i corresponds to the literal x_h^i in K_i, for $h = 1, 2, 3$. The vertices u_i, v_1^i, v_2^i, v_3^i induce a star with vertex u_i and edges $(u_i, v_1^i), (u_i, v_2^i), (u_i, v_3^i)$.
- For each $1 \le i \le j \le \ell$ and $1 \le h, s \le 3$ if $x_h^i = \bar{x}_s^j$ then the graph G_Φ contains the node $w(x_h^i, x_s^j)$ and the two edges $(v_h^i, w(x_h^i, x_s^j))$ and $(w(x_h^i, x_s^j), v_s^j)$.

The construction of graph G_Φ can obviously be done in polynomial time. An example is given in Fig. 4.

Let C be any CF-ON 2-coloring of G_Φ. Since the open neighborhood of each u_i, for $i = 1, \dots, \ell$, has a unique color then $N(u_i)$ is not monocromatic, that is among the three vertices v_1^i, v_2^i, v_3^i corresponding to the literals in K_i at least one has color 1 and at least one has color 0. If we set

$$x_h^i = \text{TRUE} \quad \text{iff} \quad C(v_h^i) = 1, \quad \text{for } i = 1, \dots, \ell \text{ and } h = 1, 2, 3,$$

then every clause contains at least one true literal and at least one false literal. Moreover, the values of the literals are consistent since the colors of the two vertices in $N(w(x_h^i, x_s^j))$ are different for each $w(x_h^i, x_s^j)$ in G_Φ, thus implying that the literals x_h^i and x_s^j get opposite values.

Assume now that Φ is a *yes* instance for NAE-3SAT. We can use the truth assignment of Φ to get a CF-ON 2-coloring of G_Φ. For each $i = 1, \dots, \ell$ and $h = 1, 2, 3$, set

$$C(v_h^i) = \begin{cases} 0 & \text{if } x_h^i = \text{FALSE}, \\ 1 & \text{if } x_h^i = \text{TRUE}. \end{cases} \tag{2}$$

Furthermore, we set $C(u_i) = 1$ for each $i = 1, \dots, \ell$, and $C(w(\cdot, \cdot)) = 0$ for each $w(\cdot, \cdot)$. The coloring of v_1^i, v_2^i, v_3^i assures that $N(u_i)$ is not monochromatic. Moreover, the neighborhood of each vertex v_h^i consists of the vertex u_i with $C(u_i) = 1$ and, of some $w(x_h^i, \cdot)$, if any; this assures that $N(v_h^i)$ has 1 as a unique color. Finally, any vertex $w(x_h^i, x_s^j)$ has exactly two neighbors corresponding to two literals $x_h^i = \bar{x}_s^j$ and, by (2), $C(v_h^i) \ne C(v_s^j)$. $\square$

Theorem 3. *CF-ONC chromatic number is hard to approximate within a factor $n^{1/2-\epsilon}$, for each $\epsilon > 0$, unless $P = NP$.*

Proof. Consider any graph $G' = (V', E')$ and define the graph $G = (V, E)$ as follows

$$V = V' \cup \{n_{uv} \mid (u, v) \in E'\} \quad \text{and} \quad E = \{(u, n_{uv}), (n_{uv}, v) \mid (u, v) \in E'\},$$

where, n_{uv} and n_{vu}, for $(u, v) \in E'$, denote the same vertex. The construction of graph $G = (V, E)$ can obviously be done in polynomial time. We now show that for any $k \ge 3$ the graph G' is k-colorable (i.e., G' can be properly colored with k colors) if and only if G has a CF-ON k-coloring.

First, assume that G has a CF-ON k-coloring C. By construction, for any $n_{uv} \in V$ we have $N(n_{uv}) = \{u, v\}$; hence, the CF-ON k-coloring C assures that $C(u) \ne C(v)$. This allows to use the coloring C, restricted to the vertices in $V' \subset V$, as a k-coloring of G'.

Now, assume that G' is k-colorable. Let C' be a k-coloring of G'. The algorithm CF-Visit, given in Fig. 5, gives a coloring of the vertices in V that we will show to be a CF-ON k-coloring of G.

The algorithm CF-Visit makes a depth-first graph traversal of G'. It assigns to the vertices in V' the same color as in the coloring C'. Moreover, it assigns colors to the vertices in $V \setminus V'$ so that for each $u \in V'$ exactly one of the neighbors of u in G gets the color of u. The algorithm uses, for each node $v \in V'$, two sets T_v and B_v containing the tree edges and the back edges, respectively, visited starting from v during the depth first traversal of G'. The algorithm starts calling CF-Visit(G', C', r) where r is any node in V'.

1. Set $T_v = B_v = \emptyset$, for each $v \in V'$ *[T_v is the set of tree edges visited starting from v]* *[B_v is the set of back edges visited starting from v]*
2. **CF-Visit**(G', C', u)
3. $\quad C(u) = C'(u)$
4. $\quad$ **while** there exists $(u, v) \in E' \setminus (T_u \cup B_u)$ with v unvisited
5. $\qquad T_u = T_u \cup \{(u, v)\}$
6. $\qquad$ **if** $|T_u| = 1$ **then** $C(n_{uv}) = C'(u)$
7. $\qquad\qquad$ **else** Set $C(n_{uv})$ to be any color in $\{0, \ldots, k-1\} \setminus \{C'(u), C'(v)\}$
8. $\qquad$ **CF-Visit**(G', C', v)
9. $\quad$ **while** there exists $(u, v) \in E' \setminus (T_u \cup B_u)$ with v visited
10. $\qquad B_u = B_u \cup \{(u, v)\}$
11. $\qquad$ **if** $|T_u| = 0$ and $|B_u| = 1$ **then** $C(n_{uv}) = C'(u)$
12. $\qquad\qquad$ **else** Set $C(n_{uv})$ to be any color in $\{0, \ldots, k-1\} \setminus \{C'(u), C'(v)\}$

Fig. 5. The CF-ON k-coloring of G.

We have $C(u) = C'(u) \neq C'(v) = C(v)$ for each $u, v \in V'$ such that $(u, v) \in E'$. Hence, the neighborhood $N(n_{uv}) = \{u, v\}$ has a unique color, for each $n_{uv} \in V \setminus V'$.

To complete the proof we will prove that also the neighborhood $N(u)$ has a unique color, for each $u \in V'$. Let u be any vertex in V'. The algorithm CF-Visit (excepting when $u = r$ is the starting vertex of the algorithm) assures that there exists a vertex x such that $(x, u) \in T_x$ (i.e., (x, u) has been visited starting from x). As effect of this visit we have either $C(n_{ux}) = C'(x)$ or $C(n_{ux})$ in $\{0, \ldots, k-1\} \setminus \{C'(u), C'(x)\}$. In any case

$$C(n_{ux}) \neq C(u), \tag{3}$$

since $C(u) = C'(u) \neq C'(x)$ (recall that u and x are neighbors in G' and C' is a k-coloring of G').

By algorithm CF-Visit, after u has been visited, each edge (u, v), for $v \in N(u) \setminus \{x\}$, is visited and inserted either in T_u or in B_u. Then a color to the vertex n_{uv} is assigned. We now distinguish two cases according to the sizes of T_u and B_u and prove that in each case $N(u)$ has a unique color.

- Let $|T_u| \geq 1$ or $|B_u| \geq 1$. For exactly one vertex $v \in N(u) - \{x\}$ (cf. lines 6 and 11 of the algorithm CF-Visit) we have
$$C(n_{uv}) = C'(u) = C(u). \tag{4}$$
For each $w \in N(u) \setminus \{x, v\}$ we have $C(n_{uw}) \in \{0, \ldots, k-1\} \setminus \{C'(u), C'(w)\}$. Recalling that $C'(u) = C(u)$, we have $C(n_{uw}) \neq C(u)$. By this, (3), and (4) we have
$$C(n_{uv}) = C(u) \quad \text{and} \quad C(n_{uw}) \neq C(u) \quad \text{for each } w \in N(u) \setminus \{v\}$$
and $C(n_{uv})$ is a unique color in $N(u)$.
- Let $|T_u| = |B_u| = 0$. In this case $N(u) = \{n_{ux}\}$ has obviously a unique color.

We conclude the proof, by noticing that the above reduction is a gap-preserving reduction [17] from the MINIMUM GRAPH COLORING, which is known to be NP-Hard to approximate within $n^{1-\epsilon}$, for any $\epsilon > 0$ [24]. □

3. Parameterized complexity

Some NP-hard problems can be solved by algorithms that are exponential only in the size of a fixed parameter while they are polynomial in the size of the input. Such algorithms are called fixed-parameter tractable, because the problem can be solved efficiently for small values of the fixed parameter [9,21]. Formally, a parameterized problem with input size n and parameter t is called *Fixed Parameter Tractable (FPT)* if it can be solved in time $f(t) \cdot n^c$, where f is a function only depending on t and c is a constant.

In this section, we study FPT algorithms for the CF-CNC and CF-ONC problems. Usually, a problem can be parameterized by a parameter that is associated to the problem, such as the number of colors k in CF-CNC and CF-ONC. However, this approach does not make the parameterized complexity theory applicable to CF-CNC and CF-ONC according to Theorems 1, 2, and 3. Courcelle's Theorem [8] shows that problems that can be expressed by an MSO formula are solvable in linear time on graphs of bounded treewidth. Since both CF-CN and CF-ON k-coloring problems can be expressed by MSO formulas for any fixed k, both problems are FPT when parameterized by treewidth.

An important quality of a parameter is its easy computability. Unfortunately there are parameters as clique-width whose computation is very hard; recently, it has been shown that determining clique-width is NP-hard [11], but the parameterized complexity of recognizing graphs of bounded clique-width is still an open problem [7]. Other parameters as treewidth, rankwidth, and vertex cover [5,12] are all computable in FPT time when their respective parameters are bounded. Here, we devote our attention to *neighborhood diversity*, a parameter that was first introduced in [20]. It has recently received attention [13,19] since it is less restrictive than other parameters and, in particular, it has the nice property to be computable in polynomial time (see [20]). We show that both CF-CNC and CF-ONC become tractable if parameterized by the vertex cover or the neighborhood diversity of the graph. We present simple algorithms which improve on existing results.

Definition 3. Given a graph $G = (V, E)$, two vertices $u, v \in V$ have the same *type* iff $N(v) \setminus \{u\} = N(u) \setminus \{v\}$.

The graph G has *neighborhood diversity* t, if there exists a partition of V into at most t sets, $V_1, V_2, \ldots, V_t$, such that all the vertices in V_i have the same type, for $i = 1, \ldots, t$.

The family $\mathcal{V} = \{V_1, V_2, \ldots, V_t\}$ is called the *type partition* of G.

The proof of the following theorem is given in Section 3.1.

Theorem 4. *Let t be the neighborhood diversity of G. It is possible to decide the CF-CNC (resp. CF-ONC) question for a fixed number of colors k in time $2^{O(kt \log k)}$ when $k \le 2t$ and in polynomial time when $k > 2t$.*

Considering that the type partition $\mathcal{V}$ can be obtained in polynomial time, one has that both the CF-CNC problem and the CF-ONC problem are in the class FPT when parameterized by the neighborhood diversity. We stress that the existence of FPT algorithms is also implied by the results presented in [20]. However, it is easy to verify that the running time of the Lampis' results is $2^{O(kt2^k)}$. Furthermore, our algorithm characterizes to be very simple.

The result of Theorem 4 can be used to have FPT linear time algorithms with vertex cover size as parameter for both CF-CNC and CF-ONC. Indeed, if a graph has vertex cover d, it cannot have a type partition with more than $2^d + d$ sets, that is, $t \le 2^d + d$ [13]. We recall that while graphs of bounded vertex cover have bounded neighborhood diversity, the opposite is not true since large cliques have a neighborhood diversity 1 [13].

Theorem 5. *Given a vertex cover of G of size at most d and a fixed number of colors k, it is possible to decide the CF-CNC (resp., CF-ONC) question in time $2^{O(k2^d \log k)}$ when $k \le d$ (resp., $k \le 2d$) and in polynomial time when $k > d$ (resp., $k > 2d$).*

Again Theorem 5 improves on the results implied by the general ones in [20], indeed the time to decide the CF-CNC (resp., CF-ONC) coloring question for a fixed number of colors k, applying the algorithm in [20], is doubly exponential in k and d, namely, $2^{2^{O(k+d)}}$.

We end by noticing that fast coloring exist when the number k of colors is large enough with respect to the vertex cover size d, namely if we are looking either for a CF-CN k-coloring with $k \ge d + 1$ or for a CF-ON k-coloring with $k \ge 2d + 1$.

Lemma 4. *If G has a vertex cover of size d then*

$$\chi_{CF}[G] \le d + 1 \quad \textit{and} \quad \chi_{CF}(G) \le 2d + 1.$$

Proof. Let W be a vertex cover of size d for G and let $I = V(G) \setminus W$. We recall that I is an independent set.

A CF-CN $(d + 1)$-coloring C of G can be obtained as follows:

1. To each $u \in I$ assign the color $C(u) = 0$.
2. To each $w \in W$ assign $C(w) \in \{1, \ldots, d\}$ so that $C(w) \ne C(v)$ whenever $w \ne v$.

According to this coloring, the unique color in $N[v]$ is $C(v)$ if $v \in W$ and 0 if $v \in I$.

A CF-ON $(2d + 1)$-coloring C of G can be obtained as follows:

1. To each $u \in I$ assign the color $C(u) = 0$.
2. To each $w \in W$ assign $C(w) \in \{1, \ldots, d\}$ so that $C(w) \ne C(v)$ whenever $w \ne v$.
3. For each $w \in W$ if $C(N(w)) = \{0\}$[2] choose one node $u \in N(w)$ and recolor it with any $C(u) \in \{d + 1, \ldots, 2d\}$ that is not already used by C.

According to this coloring, each $N(w)$, for $w \in W$, has at least one unique color, that is, either the color of any of its neighbors in W or the color of the selected neighbor in I (cf. 3). Furthermore, since I is an independent set, any $u \in I$ has so many unique colors in $N(u)$ as the number of its neighbors in W. □

[2] Notice that if W is minimal then each vertex in W has at least one neighbor in I.

3.1. Parameterization with neighborhood diversity

Let $G = (V, E)$ be a graph with type partition $\mathcal{V} = \{V_1, V_2, \ldots, V_t\}$. By Definition 3, the vertices of a given type, i.e., the vertices of V_i, have the same neighborhood (excluding the vertices in V_i itself); furthermore, each V_i induces either a clique or an independent set in G. We will denote by $cl(G)$ and $ind(G)$ the number of sets in the type partition $\mathcal{V} = \{V_1, V_2, \ldots, V_t\}$ of G that are cliques and independent sets, respectively. For each $V_i, V_j \in \mathcal{V}$, we get that either each vertex in V_i is a neighbor of each vertex in V_j or no vertex in V_i is a neighbor of any vertex in V_j.

We can then define the *type graph* of G as the graph $H = (\{1, \ldots, t\}, E_H)$ where

$$E_H = \big\{(i, j) \mid 1 \le i < j \le t \text{ and each vertex in } V_i \text{ is a neighbor of each vertex in } V_j\big\}.$$

The vertex size of H is t and the degree of each vertex i in H is $deg_H(i) \le t - 1$. Hence, if the neighborhood diversity t is bounded, then the size and the maximum degree of H have bounded values.

Lemma 5.

(1) $\chi_{\text{CF}}[G] \le \chi_{\text{CF}}[H] + ind(G) + 1$.
(2) $\chi_{\text{CF}}(G) \le \chi_{\text{CF}}(H) + cl(G) + 1$.

Proof. Let us first prove (1). Consider a CF-CN coloring C_H of H that uses $\chi_{\text{CF}}[H]$ colors. Let $B = \{0, b_1, \ldots, b_{ind(G)}\}$ be a set of $ind(G) + 1$ colors not used by C_H. We define a coloring C_G of the vertices of G by using C_H and the colors in B.

- For each $i = 1, \ldots, t$, choose any vertex $u_i \in V_i$ and assign each $v \in V_i$ the color

$$C_G(v) = \begin{cases} C_H(i) & \text{if } v = u_i, \\ 0 & \text{if } v \in V_i \setminus \{u_i\} \text{ and } V_i \text{ is a clique}, \\ \text{a color in } B \setminus (\bigcup_{j=1}^{i-1} C(V_j) \cup \{0\}) & \text{otherwise.} \end{cases}$$

Now we prove that the above is a CF-CN coloring of G, that is, the closed neighborhood of each vertex in any V_i has a unique color. Consider first the vertex $u_i \in V_i$ whose color is $C_G(u_i) = C_H(i)$. Let $x_i \in N_H[i]$ be such that $C_H(x_i)$ is a unique color in $N_H[i]$ (i.e., either $x_i = i$ or, $x_i \ne i$ and $(x_i, i) \in E_H$). By noticing that $C_H(x_i)$ is used exactly once in V_{x_i} (eventually $x_i = i$), and it is not used in any V_ℓ such that $(i, \ell) \in E_H$ and $\ell \ne x_i$, we get that $C_H(x_i)$ is a unique color in $N_G[u_i]$. Consider now $v \in V_i$ with $v \ne u_i$. If $C_H(i)$ is not a unique color in $N_H[i]$, that is $x_i \ne i$, then by using the same argument used above we have that $C_H(x_i)$ is unique in $N_G[v]$. Assume now that $C_H(i)$ is a unique color in $N_H[i]$, that is $x_i = i$. If V_i is an independent set then $C_G(v)$ is a color in $B \setminus \{0\}$ that is not used in any V_ℓ such that $(i, \ell) \in E_H$. This implies that $C_G(v)$ is unique in $N_G[v]$. If V_i is a clique then $C_G(v) = 0$. Since u_i and v are neighbors in the clique V_i and, since $C_G(u_i) = C_H(i)$ is unique in V_i and it is not used in any V_ℓ such that $(i, \ell) \in E_H$ we have that $C_G(u_i)$ is the unique color in $N_G[v]$.

Now, we prove (2). Consider a CF-ON coloring C_H of H that uses $\chi_{\text{CF}}(H)$ colors. Let $B = \{0, b_1, \ldots, b_{cl(G)}\}$ be a set of $cl(G) + 1$ colors not used by C_H. Define a coloring C_G of the vertices of G as follows.

- For each $i = 1, \ldots, t$, choose any vertex $u_i \in V_i$ and assign each $v \in V_i$ the color

$$C_G(v) = \begin{cases} C_H(i) & \text{if } v = u_i \text{ and } V_i \text{ is an independent set}, \\ \text{a color in } B \setminus (\bigcup_{j=1}^{i-1} C(V_j) \cup \{0\}) & \text{if } v = u_i \text{ and } V_i \text{ is a clique}, \\ 0 & \text{if } v \in V_i \setminus \{u_i\}. \end{cases}$$

Now we prove that C_G is a CF-ON coloring of G. In particular, we show that the (open) neighborhood of each vertex in any V_i has a unique color. Recall that by the coloring C_H, there exists $x_i \ne i$ with $(i, x_i) \in E_H$ such that $C_H(x_i)$ is a unique color in $N_H(i)$. First consider $u_i \in V_i$. The coloring implies that $C_G(u_{x_i})$ is unique in $N_G(u_i)$ (recall that u_{x_i} has either color $C_H(x_i)$ or a color never used in any other set of the partition type). Let $v \in V_i$ and $v \ne u_i$. Recall that $C_G(v) = 0$. If V_i is an independent set then $C_G(u_{x_i})$ is unique in $N_G(v)$. If V_i is a clique then at least one between $C_G(u_i)$ and $C_G(u_{x_i})$ is unique in $N_G(v)$. □

In the following we present an FPT-algorithm for the `CF-CNC` coloring problem with parameter t. To this aim we first need some definitions and a preliminary result.

Fix the number k of colors. Let $h_i = \min\{|V_i|, k+1\}$. For each $V_i \in \mathcal{V}$, consider any subset $V'_i \subseteq V_i$ such that $|V'_i| = h_i$ and denote by G' the subgraph of G induced by the set $\bigcup_{i=1}^{t} V'_i$. The sets $V'_1, \ldots, V'_t$ are disjoint and the number of vertices of G' is upper bounded by $t(k+1)$.

Lemma 6. *If there exists a CF-CN k-coloring of G then G' admits a CF-CN k-coloring.*

Proof. We produce a coloring C' of the vertices in each V'_i by using the colors of a CF-CN k-coloring C of G. If $V'_i = V_i$ then $C'(v) = C(v)$ for each vertex $v \in V'_i$. If $V'_i \subset V_i$ then $|V'_i| = k+1$ and V_i contains at least one color that is used twice; hence, if o_i colors are used exactly once in V_i under C, then C' assigns these colors to o_i vertices in V'_i; moreover C' assigns to the remaining vertices of V'_i some other color in $C(V_i)$.

We now prove that C' is a CF-CN coloring of G'. By definition of C', if a color is unique in V_i then it is also unique in V'_i, for $i = 1, \dots, t$. Moreover, for each $v \in V'_i$ there exists $w \in V_i$ (eventually, $v = w$) such that $C'(v) = C(w)$. Recalling that $N_{G'}[v] \subseteq N_G[w]$ and that $N_G[w]$ has a unique color, we have that $N_{G'}[v]$ has a unique color. □

Lemma 5 gives an algorithm that obtains a CF-CN k-coloring of G when $k \geq \chi_{\mathrm{CF}}[H] + ind(G) + 1$. Now we present an FPT-algorithm with parameter t that decides the CF-CNC coloring problem for a fixed number of colors k, where $k \leq \chi_{\mathrm{CF}}[H] + ind(G)$.

Our algorithm considers all the possible k-colorings of vertices of G'. If none of these colorings is a CF-CN k-coloring of G' then we answer NO to the CF-CNC question for G. Otherwise, we answer YES and use any of the CF-CN k-coloring of G' to color the vertices of G as follows: We use the colors of the vertices in V'_i to color any h_i vertices in V_i; if $h_i = k+1$ then we select any color that is used at least twice in V'_i and we assign it to the remaining $|V_i| - (k+1)$ vertices in V_i.

Lemma 7. *Let t be the neighborhood diversity of G. It is possible to decide the CF-CNC question for a fixed number of colors k in time $O(k^{(k+1)t}(k+1)^2 t^2)$ whenever $k \leq \chi_{\mathrm{CF}}[H] + ind(G)$.*

Proof. Lemma 6 proves that, when the above algorithm answers NO to the CF-CN k-coloring question for G, then it is not possible to have a CF-CN k-coloring of G.

We prove now that if the answers is YES then we obtain a CF-CN k-coloring of G. To this aim we show that $N_G[v]$ has a unique color for any $v \in V_j$ and $V_j \in \mathcal{V}$. If $v \in V'_j \subseteq V_j$ then $N_G[v]$ has a unique color since $N_{G'}[v]$ has a unique color. If, otherwise, $v \in V_j \setminus V'_j$ then v has the same color of some $w \in V'_j$ and, since $N_G[v] = N_G[w]$ and $N_{G'}[w]$ has a unique color, we have that also $N_G[v]$ has a unique color.

The number of all possible colorings of the vertices of G' is $O(k^{(k+1)t})$. Moreover, one needs $O(((k+1)t)^2)$ to check whether a given coloring is a CF-CN k-coloring of G'. Hence, the running time for deciding if a CF-CN k-coloring of G exists is

$$O\left(k^{(k+1)t}\left((k+1)t\right)^2\right).$$

Notice that the time to extend the CF-CN k-coloring to G is included in the above bound. □

In Lemma 5 we have presented an algorithm that obtains a CF-ON k-coloring of G when $k \geq \chi_{\mathrm{CF}}(H) + cl(G) + 1$. Using an algorithm similar to that presented above we can obtain an FPT algorithm with parameter t for the CF-ONC problem when the fixed number of colors is $k \leq \chi_{\mathrm{CF}}(H) + cl(G)$. The time required by this algorithm can be proved (as in the proof of Lemma 7) to be $O(k^{(k+1)t}(k+1)^2 t^2)$. Summarizing, we get the desired Theorem 4.

3.1.1. Conclusion and open problems

We have studied some complexity questions concerning colorings of the vertices of a graph that are conflict-free with respect to the open/closed neighborhoods of the graph. We have shown that the closed neighborhoods conflict-free coloring problem is NP-complete when two color are used; it implies that it is not possible to have approximation factor lower that 3/2. In the case of open neighborhoods, we have obtained inapproximability results. On the positive side, both closed and open neighborhood colorings are fixed parameter tractable problems, when parameterized by the vertex cover number or the neighborhood diversity of the graph. With regard to future work, many questions remain open. In particular, it would be interesting extend the NP-hardness of the closed neighborhoods conflict-free coloring problem for any $k \geq 3$ as well as to further study the (in)approximability of CF-CNC chromatic number. It would also be interesting to improve and/or extend our results to other important parameters. Furthermore, it would be intriguing to consider the fault-tolerant version of the problem, where one requires that each neighborhood contains at least t different unique colors, for some fixed t [1,4,23].

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