Published on European Journal of Combinatorics
https://doi.org/10.1016/j.ejc.2017.07.006

SMALL DOUBLING IN ORDERED NILPOTENT GROUPS OF CLASS 2

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Abstract. The aim of this paper is to present a complete description of the structure of finite subsets S of torsion-free nilpotent groups of class 2 satisfying $|S^2| = 3|S|-2$. In view of results in [12], this gives a complete description of the structure of finite subsets with the above property in any torsion-free nilpotent group.

1. Introduction.

Let α and β denote real numbers, with $\alpha > 1$. A finite subset S of a group G is said to satisfy the *small doubling property* if

$$|S^2| \leq \alpha|S| + \beta \ ,$$

where $S^2 = \{s_1 s_2 \mid s_1, s_2 \in S\}$.

The classical Freiman's inverse theorems describe the structure of finite subsets of abelian groups, which satisfy the small doubling property (see [?], [?], [?], [?], [?] and [?]). Recently, several authors obtained similar results concerning various classes of groups for an arbitrary α (see for example [?], [?], [?], [?], [?], [?], [?], [?], [?], [?], [?] and [?]).

In particular E. Breuillard and B. Green in [?] and M. Tointon in [?] investigated the problem in the case of G being a nilpotent group.

In [?] we started the investigation of finite subsets of *ordered groups* satisfying the small doubling property with $\alpha = 3$ and small $|\beta|$. We proved that if $(G, <)$ is an ordered group and S is a finite subset of G of size $k \geq 2$, such that $|S^2| \leq 3k - 3$, then $\langle S \rangle$ is abelian. Furthermore, if $k \geq 3$ and $|S^2| \leq 3k - 4$, then there exist $x_1, g \in G$ such that $g > 1$, $gx_1 = x_1 g$ and S is a subset of the geometric progression $\{x_1, x_1 g, x_1 g^2, \cdots, x_1 g^{t-k}\}$, where $t = |S^2|$. We also showed that these

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2000 *Mathematics Subject Classification.* Primary: 11P70, 20F60, 20F18; Secondary: 20F99.

Key words and phrases. ordered groups, finite subsets, small doubling, nilpotent groups.

results are best possible, by presenting an example of an ordered group with a subset S of size k with $\langle S\rangle$ non-abelian and $|S^2| = 3k-2$.

Other recent results concerning small doubling properties appear in [?], [?], [?], [?].

In [?] we studied finite subsets S of torsion-free nilpotent groups such that $|S^2| = 3k-2$.

The aim of the present paper is to give a complete description of the structure of subsets S of size k in torsion-free nilpotent groups satisfying $|S^2| = 3k-2$.

It is known that torsion-free nilpotent groups are orderable (see [?] and [?]), so the results in [?] apply. Furthermore, we can assume that the class of G is 2, because of Theorem 5 and Corollary 3 in [?].

Our main result is the following theorem.

Theorem 1.1 *Let G be a torsion-free nilpotent group and let S be a subset of G of size $k \geq 4$ with $\langle S\rangle$ non-abelian. Then $|S^2| = 3k-2$ if and only if there exist $a, b, c \in G$ and non-negative integers i, j such that*

$$S = \{a, ac, \cdots, ac^i, b, bc, \cdots, bc^j\},$$

with $1+i+1+j = k$, $c \neq 1$ and $[a,b] = c^{\pm 1}$.

Here $[a,b] = a^{-1}b^{-1}ab$ is the commutator of the elements a, b.

We refer to [?] and [?] for results concerning ordered groups. In particular, we use at several places the following result proved by B.H. Neumann in [?].

Proposition *If G is an ordered group and $a, b \in G$ are such that a commutes with b^n for some integer $n \geq 1$, then a commutes with b.*

2. Some general results.

We start with the following two very useful lemmas.

Lemma 2.1 *Let $(G,<)$ be an ordered nilpotent group of class 2 and let S be a subset of G of size $k \geq 3$, satisfying*

$$S = \{x_1, \cdots, x_k\}, \quad x_1 < x_2 < \cdots < x_k.$$

Let $T = \{x_1, \cdots, x_{k-1}\}$ and for any positive $j \leq k-1$, let $T_j = \{x_j, \cdots, x_{k-1}\}$. Suppose that for some positive $i \leq k-2$, we have

$$\{x_i x_k, x_k x_i\} \subseteq T^2.$$

Then either $x_i x_k$ or $x_k x_i$ belongs to T_{i+1}^2.

Proof. Write $x_i x_k = x_j x_l$, for suitable $l, j \leq k-1$, then $j \geq i+1$. Similarly, write $x_k x_i = x_s x_t$, for some $s, t \leq k-1$, then $t \geq i+1$. We claim that either $l \geq i+1$ or $s \geq i+1$. We have $x_i x_k = x_k x_i [x_i, x_k]$ and either $[x_i, x_k] \geq 1$ or $[x_i, x_k] \leq 1$. Suppose, for example, that $[x_i, x_k] \geq 1$. Then $[x_k, x_i] \leq 1$, and $x_s x_t = x_k x_i = x_i x_k [x_k, x_i] \leq x_i x_k$, thus, from $t < k$ it follows that $s \geq i+1$. Similarly, if $[x_i, x_k] \leq 1$, we get $l \geq i+1$.

□

Lemma 2.2 *Let $(G, <)$ be an ordered nilpotent group of class 2 and let S be a subset of G of size $k \geq 3$, satisfying:*

$$S = \{x_1, \cdots, x_k\}, \quad x_1 < x_2 < \cdots < x_k$$

and

$$x_k x_{k-1} \neq x_{k-1} x_k.$$

Let $T = \{x_1, \cdots, x_{k-1}\}$. Then:

$$|S^2| \geq |T^2| + 4.$$

In particular, if $\langle T \rangle$ is non-abelian, then

$$|S^2| \geq 3k - 1.$$

Proof. Write

$$D = \{x_k^2, x_k x_{k-1}, x_{k-1} x_k\}.$$

Then $|D| = 3$ and $D \subseteq S^2 \setminus T^2$. If either $x_k x_{k-2}$ or $x_{k-2} x_k$ does not belong to $T^2 \dot{\cup} D$, then

$$|S^2 \setminus T^2| \geq 4,$$

as required. So we may assume, from now on, that

$$\{x_k x_{k-2}, x_{k-2} x_k\} \subseteq T^2 \dot{\cup} D.$$

Our aim is to reach a contradiction.

First, we claim that the case

$$\{x_k x_{k-2}, x_{k-2} x_k\} \subseteq T^2$$

is impossible. Indeed, in this such case,

$$x_k x_{k-2} = x_j x_{k-1} \text{ and } x_{k-2} x_k = x_{k-1} x_i, \text{for some } i, j \leq k-1.$$

Moreover, by Lemma 2.1, either $j = k-1$ or $i = k-1$.

Assume that $j = k-1$, thus $x_k x_{k-2} = x_{k-1}^2$. Then

$$x_{k-2} x_k = x_k x_{k-2} [x_{k-2}, x_k] = x_{k-1}^2 [x_{k-2}, x_k],$$

which implies that x_{k-1} centralizes $x_{k-2}x_k$, since G has class 2. Therefore $x_{k-2}x_k = x_{k-1}x_i = x_i x_{k-1}$, forcing $i = k-1$ and

$$x_k x_{k-2} = x_{k-1}^2 = x_{k-2}x_k.$$

Thus x_k centralizes x_{k-2} and hence x_k centralizes x_{k-1}^2. But then x_k centralizes x_{k-1}, a contradiction. Similarly, we get a contradiction if we assume that $x_{k-2}x_k = x_{k-1}^2$. This completes the proof of our claim.

Thus either $x_{k-2}x_k$ or $x_k x_{k-2}$ is not in T^2 and hence it belongs to D. We claim now that

$$\{x_k x_{k-2}, x_{k-2}x_k\} \subset D.$$

Indeed, assume that $x_k x_{k-2} \in D$, which implies that

$$x_k x_{k-2} = x_{k-1}x_k.$$

Then $x_{k-2} = x_k^{-1}x_{k-1}x_k = x_{k-1}c$, where $c \in Z(G)$. If $x_{k-2}x_k \in T^2$, then $x_{k-2}x_k = x_{k-1}x_i$ for some $i \leq k-1$, which implies that $x_{k-1}cx_k = x_{k-1}x_i$ and $x_k = c^{-1}x_i$. It follows from $x_{k-2}x_k = x_{k-1}x_i$ that $x_k x_{k-2}[x_{k-2}, x_k] = x_i x_{k-1}[x_{k-1}, x_i]$. But $[x_{k-2}, x_k] = [x_{k-1}c, c^{-1}x_i] = [x_{k-1}, x_i]$, so $x_k x_{k-2} = x_i x_{k-1} \in T^2$, a contradiction. Thus $x_k x_{k-2} \in D$ implies that $x_{k-2}x_k \in D$ and similarly $x_{k-2}x_k \in D$ implies that $x_k x_{k-2} \in D$, as claimed.

Finally, we claim that

$$\{x_k x_{k-2}, x_{k-2}x_k\} \subset D.$$

is impossible. Indeed, if that is the case, then $x_k x_{k-2} = x_{k-1}x_k$ and $x_{k-2}x_k = x_k x_{k-1}$, which implies that

$$x_{k-1} = x_{k-2}^{x_k} = x_{k-2}^{x_k^{-1}}.$$

Thus $[x_{k-2}, x_k^2] = 1$ and hence $[x_{k-2}, x_k] = 1$, which implies that $x_{k-2}x_k = (x_{k-2}x_k)^{x_k} = x_{k-1}x_k$, a final contradiction.

Therefore under our assumptions $|S^2| \geq |T^2| + 4$. In particular, if $\langle T \rangle$ is non-abelian, then by Theorem 1.3 of [**?**] we get that $|T^2| \geq 3(k-1) - 2 = 3k-5$ and hence $|S^2| \geq 3k-1$. $\square$

The following observation will be used repeatedly.

Lemma 2.3 *Let G be a torsion-free nilpotent group of class 2. Let $a, b, c \in G$ and consider the subset*

$$S = \{a, ac, \cdots, ac^i, b, bc, \cdots, bc^j\}$$

of G for some non-negative integers i, j satisfying $i + j \geq 1$. Write $A = \{a, ac, \cdots, ac^i\}$ and $B = \{b, bc, \cdots, bc^j\}$.

If $c \neq 1$ and $[a,b] = c^{\pm v}$ for some $v \in \mathbb{N}$ satisfying $v \leq i+j$, then

$$AB \cup BA = \{abc^l, \;\; l \in \{0, \cdots, i+j+v\}\}.$$

In particular, $|AB \cup BA| = i+j+v+1$ and

$$|S^2| = 3|S| + (v-3).$$

Proof. Suppose, first, that $ba = abc^v$. Then $c^v \in Z(G)$ and hence $c \in Z(G)$. Thus $\langle A \rangle$ and $\langle B \rangle$ are abelian and $a \notin C_G(B)$, $b \notin C_G(A)$. We have $AB = \{ab, abc, \cdots, abc^{i+j}\}$ and $BA = \{ba = abc^v, \cdots, abc^{v+i+j}\}$, so

$$AB \cup BA = \{ab, abc, \cdots, abc^{v+i+j}\},$$

since $v \leq i+j$. Therefore $|AB \cup BA| = i+j+v+1$. Furthermore, $|A^2| = 2(i+1) - 1 = 2i+1$, $|B^2| = 2j+1$, and

$$S^2 = A^2 \cup B^2 \;\cup (AB \cup BA),$$

with $\emptyset = A^2 \cap B^2 = A^2 \cap (AB \cup BA) = B^2 \cap (AB \cup BA)$, since $b \notin C_G(A)$ and $a \notin C_G(B)$. Hence:

$$|S^2| = 2i+1+2j+1+i+j+v+1 = 3i+3j+3+v = 3|S| + (v-3),$$

as required. The case $ab = bac^v$ can be dealt with similarly. □

While studying subsets S of size k of ordered nilpotent groups of class 2 with the small doubling property, we shall often try to reduce the hypotheses to those of the following proposition.

Proposition 2.4 *Let $(G,<)$ be an ordered nilpotent group of class 2 and let S be a subset of G of size $k \geq 4$, with $\langle S \rangle$ non-abelian. Write $S = \{x_1, x_2, \cdots, x_{k-1}, x_k\}$, $x_1 < x_2 < \cdots < x_k$, and $T = \{x_1, x_2, \cdots, x_{k-1}\}$. Suppose that $\langle T \rangle$ is abelian and*

$$|S^2| = 3k + i \leq 4k - 6$$

for some integer i. Then $i \geq -2 > -k$ and there exist $a, b, c \in G$ such that

$$S \subseteq \{a, ac, \cdots, ac^{k+i}, b\} \subseteq \{a, ac, \cdots, ac^{2k-6}, b\}$$

with $[a,b] = c^{\pm v}$ for some $v \in \mathbb{N}$ satisfying $v \leq k+i$.

Proof. Since $\langle S\rangle$ is non-abelian, it follows by Theorem 1.3 of [**?**] that $|S^2| \geq 3k-2$, so $i \geq -2 > -k$. Since $\langle T\rangle$ is abelian, it follows that $x_k \notin C_G(T)$ and hence $|x_kT \cup Tx_k| \geq k$ by Proposition 2.4 of [**?**]. Moreover,

$$S^2 = (T^2 \dot\cup \{x_k^2\}) \,\dot\cup\, (x_kT \cup Tx_k)$$

since $x_k \notin \langle T\rangle \subseteq C_G(T)$, and $x_k^2 \notin T^2$ since otherwise $x_k^2 \in C_G(T)$, which implies that $x_k \in C_G(T)$, a contradiction. Thus

$$3k+i = |S^2| = |T^2| + 1 + |x_kT \cup Tx_k| \geq |T^2| + 1 + k, \qquad (1)$$

and consequently

$$|T^2| \leq 2k+i-1 \leq 3k-7 = 3|T|-4.$$

Hence it follows by Proposition 3.1 in [**?**] that

$$T \subseteq \{a, ac, \cdots, ac^{k+i}\} \subseteq \{a, ac, ac^2, \cdots, ac^{2k-6}\},$$

where $a, c \in G$, $c > 1$ and $ac = ca$. Moreover, as $|T^2| \geq 2|T|-1 = 2(k-1)-1$, our assumptions and (1) also imply that

$$4k-6 \geq 3k+i = |S^2| \geq |T^2|+1+|x_kT \cup Tx_k| \geq (2(k-1)-1)+1+|x_kT \cup Tx_k|,$$

so

$$2|T| - |x_kT \cap Tx_k| = |x_kT \cup Tx_k| \leq 2k-4$$

and

$$|x_kT \cap Tx_k| \geq 2(k-1) - (2k-4) = 2. \qquad (2)$$

Write $x_k = b$. Then

$$bT \subseteq \{ba, bac, \cdots, bac^{k+i}\} \text{ and } Tb \subseteq \{ab, cab, c^2ab, \cdots, c^{k+i}ab\},$$

in view of $ac = ca$. As, by (2), $|bT \cap Tb| \geq 2$, there exist $0 \leq l, j, s, t \leq k+i$ such that

$$bac^l = c^jab \quad \text{and} \quad bac^s = c^tab,$$

with $l \neq s$ and $j \neq t$. Now, $bac^l = ac^jb$ implies that $b^{-1}a^{-1}bac^l = b^{-1}c^jb$, yielding

$$[b,a] = b^{-1}c^jbc^{-j}c^{j-l} = [b, c^{-j}]c^{j-l}.$$

Hence $c^{j-l} \in Z(G)$ and similarly $c^{s-t} \in Z(G)$.

Suppose, first, that $j \neq l$. Then $c^{j-l} \in Z(G)$ implies that $c \in Z(G)$. If $l > j$, then

$$ab = bac^{l-j} \quad \text{with} \quad 0 < l-j \leq k+i,$$

and if $j > l$, then

$$ba = abc^{j-l} \quad \text{with} \quad 0 < j-l \leq k+i.$$

Thus the theorem holds. Similarly, the theorem holds if $s \neq t$.

So assume, finally, that $l = j$ and $s = t$. In this case we shall reach a contradiction. We have $bac^l = ac^lb$, $bac^s = ac^sb$ and $l \neq s$. Thus

$$1 = [b, ac^l] = [b, a][b, c^l] \text{ and } 1 = [b, ac^s] = [b, a][b, c^s].$$

Hence $[b, c^l] = [b, c^s]$, implying that $c^{-l}bc^l = c^{-s}bc^s$. Thus c^{l-s} and b commute with each other and since $l \neq s$, it follows that c and b commute with each other. But then $bac^l = abc^l$ and hence b and a commute with each other. So $b \in C_G(T)$, a contradiction. $\square$

Notice that, conversely, if $S = \{a, ac, ac^2, \cdots, ac^{k-2}, b\}$, with $k \geq 4$, $c \neq 1$, $[a, b] = c^{\pm v}$ for some $v \in \mathbb{N}$ satisfying $v \leq k-3$, then, by Lemma 2.3, $|S^2| = 3k + v - 3 \leq 4k - 6$.

As an easy consequence of Lemma 2.2, we get the following result for *completely-non-abelian subsets*, where we say that a subset S of a group G is completely-non-abelian ($S \in CNA$ in short) if $ab \neq ba$ for any $a, b \in S$, $a \neq b$.

Corollary 2.5 *If S is a CNA-subset of size k of an ordered nilpotent group of class 2, then:*

$$|S^2| \geq 4k - 4.$$

Proof. In fact, the result is certainly true if $k = 1$. If $k = 2$ and $S = \{a, b\}$, then $S^2 = \{a^2, ab, ba, b^2\}$ and $|S^2| = 4 = 4 \cdot 2 - 4$. So let $|S| = k \geq 3$, and suppose that the result holds for $k - 1$. Let $S = \{x_1, \cdots, x_k\}$, $x_1 < x_2 < \cdots < x_k$ and let $T = \{x_1, \cdots, x_{k-1}\}$. Then, by Lemma 2.2, $|S^2| \geq |T^2| + 4 \geq 4(k-1) - 4 + 4 = 4k - 4$, as required.

$\square$

3. Subsets S with $|S^2| = 3|S| - 2$.

The aim of this section is to study subsets S of a torsion-free nilpotent group of class 2 satisfying $|S| = k$ and $|S^2| = 3k - 2$.

If $|S| = 2$ and $|S^2| = 3 \cdot 2 - 2 = 4$, then $S = \{a, b\}$, $ab \neq ba$, and the converse is also true. So we shall study subsets S with $|S| = k \geq 3$.

In the case when $k = 3$, we have the following result.

Proposition 3.1 *Let G be a torsion-free nilpotent group of class 2 and let S be a subset of G of size $|S| = 3$, with $\langle S\rangle$ non-abelian. Then $|S^2| = 7$ if and only if one of the following holds:*
(i) $S \cap Z(\langle S\rangle) \neq \emptyset$;
(ii) $S = \{a, ac, b\}$, where $a, b, c \in G$, $c \neq 1$, $[a, b] = c^{\pm 1}$. In particular, $c \in Z(G)$.

Proof. There exists an order $<$ on G such that $(G, <)$ is an ordered group. Write $S = \{x_1, x_2, x_3\}$ with $x_1 < x_2 < x_3$ and $T = \{x_1, x_2\}$.

Suppose that $|S^2| = 7$. If $x_1x_2 = x_2x_1$ and $x_2x_3 = x_3x_2$, then $x_2 \in Z(\langle S\rangle)$ and S satisfies (i).

So assume, first, that $x_2x_3 \neq x_3x_2$. Since $|T^2| \leq |S^2| - 4 = 3$, by Lemma 2.2 it follows that $x_1x_2 = x_2x_1$. If $x_1x_3 = x_3x_1$, then $x_1 \in Z(\langle S\rangle)$ and S satisfies (i). So assume that $x_1x_3 \neq x_3x_1$. Since $x_2x_3 \neq x_3x_2$, it follows that $x_3 \notin \langle x_1, x_2\rangle$ and the elements

$$x_1^2, x_1x_2, x_2^2, x_1x_3, x_2x_3, x_3^2$$

of S^2 are all different. Since $|S^2| = 7$, we have either $x_3x_1 = x_2x_3$ or $x_3x_2 = x_1x_3$. Thus, if we put $x_1 = a, x_2 = ac, x_3 = b$, then $c > 1$, $ac = ca$ and either $ba = cab$ or $bac = ab$. Hence $c \in Z(G)$ and S satisfies (ii).

Similarly, if $x_1x_2 \neq x_2x_1$, then we get the result by considering the order opposite to $<$.

A direct calculation yields the converse. □

In order to prove Theorem 1.1, we study first the following particular case.

Proposition 3.2 *Let G be a torsion-free nilpotent group of class 2 and let S be a subset of G of size $|S| = k \geq 4$, with $\langle S\rangle$ non-abelian. Assume that $|S^2| = 3k - 2$ and $S = T \cup \{x\}$, with $\langle T\rangle$ abelian. Then there exist $a, b, c \in G$ such that*

$$S = \{a, ac, \cdots, ac^{k-2}, b\},$$

with $[a, b] = c^{\pm 1}$. In particular, $c \in Z(G)$.

Proof. By Proposition 2.4, we have

$$S \subseteq \{a, ac, \cdots, ac^{k-2}, b\},$$

with $[a, b] = c^{\pm v}$ for some $v \in \mathbb{N}$ satisfying $v \leq k - 2$. Since $|S| = k$, it follows that $S = \{a, ac, \cdots, ac^{k-2}, b\}$, and by Lemma 2.3 we get $|S^2| = 3k + (v - 3)$. But $|S^2| = 3k - 2$, so $v = 1$, as required. □

Now we can prove Theorem 1.1.

Proof. Let G be a torsion-free nilpotent group and let S be a subset of G of size $k \geq 4$ with $\langle S \rangle$ non-abelian. By Corollary 3 of [**?**] we can assume that G is of class 2.

If $a, b, c \in G$ with $1 \neq [a, b] = c^{\pm 1}$, and if

$$S = \{a, ac, ac^2, \cdots, ac^i, b, bc, bc^2, \cdots, bc^j\}$$

with i, j denoting non-negative integers satisfying $1+i+1+j = k \geq 4$, then $|S^2| = 3k - 2$ by Lemma 2.3, for $v = 1$.

Conversely, assume that $|S^2| = 3k-2$. Our aim is to prove that there exist $a, b, c \in G$ such that $S = \{a, ac, ac^2, \cdots, ac^i, b, bc, bc^2, \cdots, bc^j\}$, where either $[a, b] = c$ or $[b, a] = c$, and where i, j are non-negative integers satisfying $1 + i + 1 + j = k \geq 4$.

There exists an order $<$ on G, such that $(G, <)$ is an ordered group. Write

$$S = \{x_1, x_2, \cdots, x_k\},\ T = \{x_1, \cdots, x_{k-1}\},\ V = \{x_2, \cdots, x_k\},$$

and suppose that $x_1 < x_2 < \cdots < x_k$. If S contains a subset of size $k-1$ which generates an abelian subgroup of G, then our claim follows by Proposition 3.2. Therefore we may assume that S contains no subsets of size $k - 1$ generating an abelian subgroup of G. In particular, the subgroups $\langle T \rangle$ and $\langle V \rangle$ of G are non-abelian.

If $x_{k-1}x_k \neq x_k x_{k-1}$, then $\langle T \rangle$ is abelian by Lemma 2.2, a contradiction. So we may assume that

$$x_{k-1}x_k = x_k x_{k-1}.$$

Similarly, by considering the order opposite to $<$ and the set V, we may assume that

$$x_1 x_2 = x_2 x_1.$$

Obviously $\{x_k^2, x_{k-1}x_k\} \cap T^2 = \emptyset$. Let $\mu+1$ be a minimal integer such that $\langle x_{\mu+1}, \cdots, x_k \rangle$ is abelian. By our assumptions, $0 < \mu \leq k - 2$.

If $x_\mu x_k$ and $x_k x_\mu$ both belong to T^2, then, by Lemma 2.1, either $x_\mu x_k$ or $x_k x_\mu \in \{x_{\mu+1}, \cdots, x_{k-1}\}^2$. Thus $x_\mu \in \langle x_{\mu+1}, \cdots, x_k \rangle$ and $\langle x_\mu, \cdots, x_k \rangle$ is abelian, in contradiction to the minimality of $\mu + 1$. Hence either $x_\mu x_k \notin T^2$, or $x_k x_\mu \notin T^2$.
So either $S^2 \supseteq T^2 \dot{\cup} \{x_k^2, x_{k-1}x_k, x_\mu x_k\}$ or $S^2 \supseteq T^2 \dot{\cup} \{x_k^2, x_k x_{k-1}, x_k x_\mu\}$, which implies that $3k - 2 = |S^2| \geq |T^2| + 3$. Thus

$$|T^2| \leq 3k - 5 = 3(k - 1) - 2 \text{ and similarly } |V^2| \leq 3(k - 1) - 2.$$

If $|T^2| \leq 3(k-1)-3$, then $\langle T\rangle$ is abelian by Theorem 1.3 of [**?**], in contradiction to our assumptions. Hence we may conclude that $|T^2| = 3(k-1)-2$. Similarly, also $|V^2| = 3(k-1)-2$.
Moreover, we may assume that

$$|\{x_1x_k, x_2x_k, \cdots, x_{k-2}x_k\} \setminus T^2| < 2, \qquad (3)$$

since otherwise, in view of $x_{k-1}x_k, x_k^2 \notin T^2$, we obtain $3k-2 = |S^2| \geq |T^2|+4$, yielding $|T^2| \leq 3k-2-4 = 3(k-1)-3$. But then, again by Theorem 1.3 of [**?**], $\langle T\rangle$ is abelian, in contradiction to our assumptions. A similar argument indicates that

$$|\{x_kx_1, x_kx_2, \cdots, x_kx_{k-2}\} \setminus T^2| < 2. \qquad (4)$$

We now argue by induction on k.

If $k = 4$, then $|T| = 3$, $|T^2| = 7$ and we may apply Proposition 3.1.

We will first show that $T \cap Z(\langle T\rangle) = \emptyset$.

Assume that $T \cap Z(\langle T\rangle) \neq \emptyset$. Recall that we have $x_1x_2 = x_2x_1$, $x_3x_4 = x_4x_3$ and by (3) either $x_1x_4 \in T^2$ or $x_2x_4 \in T^2$. In any case $x_4 \in \langle T\rangle$.

Now, if $x_3 \in Z(\langle T\rangle)$, then $\langle x_1, x_2, x_3\rangle$ is abelian, in contradiction to our assumptions. If $x_r \in Z(\langle T\rangle)$ for $r \in \{1, 2\}$, then $x_rx_3 = x_3x_r$ and $x_rx_4 = x_4x_r$, since $x_4 \in \langle T\rangle$, so $\langle x_r, x_3, x_4\rangle$ is abelian, again in contradiction to our assumptions. Thus $T \cap Z(\langle T\rangle) \neq \emptyset$ is impossible.

Next assume that $T = \{a, ac, b\}$, with $c > 1$ and $[a, b] = c^{\pm 1}$. In particular, $c \in Z(G)$. We have $x_3x_4 = x_4x_3$. If either $x_3 = a$ or $x_3 = ac$, then $\langle a, ac, x_4\rangle$ is abelian, in contradiction to our assumptions. So assume that $x_3 = b$. We have $bx_4 = x_4b$. First suppose $ab = bac$. Then we have

$$T^2 = \{a^2, a^2c, a^2c^2, b^2\} \cup \{abc^{-1}, ab, abc\}.$$

By (3) either $ax_4 \in T^2$ or $acx_4 \in T^2$. If either ax_4 or acx_4 belongs to $\{a^2, a^2c, a^2c^2, b^2\}$, then $bx_4 = x_4b$ implies that $ab = ba$, a contradiction. Hence $\{ax_4, acx_4\} \cap \{abc^{-1}, ab, abc\} \neq \emptyset$. Hence $x_4 = bc^l$ for some $l \leq 1$ and $l = 1$ since $b = x_3 < x_4$. If $ba = abc$, then we have

$$T^2 = \{a^2, a^2c, a^2c^2, b^2\} \cup \{bac^{-1}, ba, bac\}.$$

By (4) either $x_4a \in T^2$ or $x_4ac \in T^2$, and using argument similar to those used above, we obtain $x_4 = bc$ and our claim follows.

Now assume that $k > 4$ and, by induction, the result is true for $k-1$. Then there exist $a, b, c \in G$ such that

$$T = \{a, ac, \cdots, ac^i, b, bc, \cdots, bc^j\},$$

where either $[a, b] = c$ or $[b, a] = c$, and where i, j are non-negative integers satisfying $1+i+1+j = k-1 \geq 4$. Relabelling the elements, we can assume that $c > 1$. We have $c \in Z(G)$ and since x_{k-1} is a maximal element of T, we have either $x_{k-1} = ac^i$, $i \geq 0$ or $x_{k-1} = bc^j$, $j \geq 0$. Assume, without loss of generality, that $x_{k-1} = bc^j$. In order to complete the proof, we need to show that $x_k = bc^{j+1}$. If $[a, b] = c$, then we have

$$T^2 = \{a^2, a^2c, \cdots, a^2c^{2i}\} \cup \{b^2, b^2c, \cdots, b^2c^{2j}\} \cup \{abc^{-1}, ab, abc, \cdots, abc^{i+j}\}.$$

We have $i \geq 1$, since otherwise $\{b, bc, \cdots, bc^j, x_k\}$ is a subset of S of size $k - 1$ and generates an abelian group. Now by (3) we have $\{ac^{i-1}x_k, ac^ix_k\} \cap T^2 \neq \emptyset$. Since $c \in Z(G)$, if

$$\{ac^{i-1}x_k, ac^ix_k\} \cap (\{a^2, a^2c, \cdots, a^2c^{2i}\} \cup \{b^2, b^2c, \cdots, b^2c^{2j}\}) \neq \emptyset,$$

then a belongs to the subgroup generated by the elements x_k, b and c. Since x_k, b and c commute with each other, it follows that a commutes with b, a contradiction. Hence we have

$$\{ac^{i-1}x_k, ac^ix_k\} \cap \{abc^{-1}, ab, abc, \cdots, abc^{i+j}\} \neq \emptyset.$$

Hence $x_k = bc^l$, for some integer $l \leq j + 1$. Since $x_k > x_{k-1} = bc^j$, it follows that $l = j + 1$. Hence we obtain the theorem if $[a, b] = c$.

If $[b, a] = c$, then we have

$$T^2 = \{a^2, a^2c, \cdots, a^2c^{2i}\} \cup \{b^2, b^2c, \cdots, b^2c^{2j}\} \cup \{bac^{-1}, ba, bac, \cdots, bac^{i+j}\}.$$

Now, by (4) we have $\{x_kac^{i-1}, x_kac^i\} \cap T^2 \neq \emptyset$. It follows using the arguments similar to those used above that $x_k = bc^{j+1}$. Hence the theorem follows.

$\square$

ACKNOWLEDGEMENTS

This work was supported by the “National Group for Algebraic and Geometric Structures, and their Applications” (GNSAGA - INDAM), Italy.

The second author is grateful to the Department of Mathematics of the University of Salerno for its hospitality and support, while this investigation was carried out.

The authors would like to thank the referees for very valuable suggestions.

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