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Gradient estimates in generalized Morrey spaces for parabolic operators

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We obtain global regularity in generalized Morrey spaces for the gradient of the weak solutions to divergence form linear parabolic operators with measurable data. Assuming partial BMO smallness of the coefficients and Reifenberg flatness of the boundary of the underlying domain, we develop a Calderón-Zygmund type theory for such operators. Problems like the considered here arise in the modeling of composite materials and in the mechanics of membranes and films of simple nonhomogeneous materials which form a linear laminated medium.

1 Introduction

The classical Morrey spaces $L^{q,\lambda}$ are originally introduced in Morrey [21] in order to prove local Hölder continuity of solutions to certain systems of partial differential equations. A real valued function f is said to belong to the Morrey space $L^{q,\lambda}$ with $q \in (1,\infty)$, $\lambda \in (0,n)$ provided the following norm is finite

$$\|f\|_{L^{q,\lambda}(\mathbb{R}^n)} = \left(\sup_{(x,r)\in\mathbb{R}^n\times\mathbb{R}_+} \frac{1}{r^\lambda} \int_{\mathcal{B}_r(x)} |f(y)|^q \, dy \right)^{1/q}$$

where the supremum is taken over all balls $\mathcal{B}_r(x)$. The main result connected with these spaces is the following celebrated lemma: let $|Df| \in L^{q,n-\lambda}$ even locally, with $n-\lambda < q$, then u is Hölder continuous of exponent $\alpha = 1 - \frac{n-\lambda}{q}$. This result has found many applications in the study the regularity of the solutions to elliptic and parabolic equations and systems. In [9] Chiarenza and Frasca showed boundedness of the Hardy-Littlewood maximal operator in $L^{q,\lambda}(\mathbb{R}^n)$ that allows them to prove continuity in that spaces of some classical integral operators.

In [20] Mizuhara extended the concept of Morrey of integral average over a ball with a certain growth, taking a weight function $\varphi(x,r) : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}_+$ instead of r^λ. Thus he set the beginning of the study of the generalized Morrey spaces $L^{q,\varphi}$ with φ belonging to suitable classes of weight functions. In [22] Nakai proved boundedness of the maximal operator also in $L^{q,\varphi}$ imposing suitable integral and doubling conditions on φ (see Lemma 2.2).

In the present work we are going to establish Calderón-Zygmund type estimates in generalized Morrey spaces for the weak solutions to divergence structure linear parabolic operators with measurable coefficients. We consider Cauchy-Dirichlet boundary problem in a cylinder with a base having a rough boundary. More precisely, we derive global estimate for the spatial gradient of the solution in $L^{q,\varphi}$, extending this way the recent L^q and $L^{q,\lambda}$-results of Byun [1], [2], Byun, Palagachev, Wang [6], Byun, Wang [7], Dong, Kim [13], Dong [12] and Byun, Palagachev, Softova [5].

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Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, $n \geq 2$, and set $Q = \Omega \times (0, T]$ for the cylinder in $\mathbb{R}^{n+1}$ with base Ω and of height T. We consider the following Cauchy-Dirichlet problem

$$\begin{cases} u_t - D_\alpha(a^{\alpha\beta}(x,t)D_\beta u) = D_\alpha f^\alpha(x,t) & \text{in } Q, \\ u(x,t) = 0 & \text{on } \partial_P Q, \end{cases} \tag{1.1}$$

where $\partial_P Q = (\partial\Omega \times [0, T]) \cup (\Omega \times \{t = 0\})$ stands for the parabolic boundary of Q and the summation convention on the repeated lower and upper indeces, running from 1 to n, is understood.

Suppose that the coefficient matrix $\mathbf{a}(x,t) = \{a^{\alpha\beta}(x,t)\}_{\alpha,\beta=1}^n : Q \to \mathbb{M}^{n\times n}$ is measurable, uniformly bounded and uniformly parabolic, that is, there exist positive constants L and ν such that

$$\begin{cases} \|a^{\alpha\beta}\|_{L^\infty(Q)} \leq L, \\ a^{\alpha\beta}(x,t)\xi_\alpha\xi_\beta \geq \nu|\xi|^2, & \forall\, \xi \in \mathbb{R}^n, \text{ for almost all } (x,t) \in Q. \end{cases} \tag{1.2}$$

Denote the nonhomogeneous term in (1.1) by $\mathbf{F}(x,t) = \big(f^1(x,t), \dots, f^n(x,t)\big)$. As it is known by [2], [7], if $\mathbf{F} \in L^2(Q)$, then the problem (1.1) has a unique weak solution. Recall that under a weak solution of (1.1) we mean a function

$$u \in C^0\left(0, T; L^2(\Omega)\right) \cap L^2\left(0, T; H_0^1(\Omega)\right)$$

that satisfies

$$\int_Q u\varphi_t \, dx\, dt - \int_Q a^{\alpha\beta} D_\beta u D_\alpha \varphi \, dx\, dt = \int_Q f^\alpha D_\alpha \varphi \, dx\, dt$$

for all $\varphi \in C_0^\infty(Q)$ with $\varphi(\cdot, T) \equiv 0$. Moreover, the following L^2-estimate

$$\int_Q |Du(x,t)|^2 \, dx\, dt \leq c \int_Q |\mathbf{F}(x,t)|^2 \, dx\, dt, \tag{1.3}$$

holds where the constant c depends only on n, L, ν and T.

Our goal is to develop an optimal regularity theory regarding Calderón-Zygmund estimates for the problem (1.1) in the settings of generalized Morrey spaces. Namely, taking $\mathbf{F} \in L^{p,\varphi}(Q)$ under suitable assumptions over p and φ, we are going to show that the spatial gradient of the weak solution Du belongs to the same space $L^{p,\varphi}(Q)$ under a minimal regularity assumption on $\mathbf{a}(x,t)$ and a very lower level of geometric assumption on $\partial\Omega$.

Restricting the value of the exponent p in the range $(2, \infty)$, we consider weights φ satisfying certain doubling and integral conditions. This is a reasonable restriction ensuring boundedness of the Hardy-Littlewood maximal operator when acting on $L^{p,\varphi}$, see [22], while the monotonicity assumption on φ permits to estimate the upper level sets of the maximal operator. For what concerns the coefficients $a^{\alpha\beta}(x,t)$ we suppose these are only measurable with respect to one spatial variable and are averaged in the sense of small bounded mean oscillation (BMO) in the remaining space and time variables. This partially BMO assumption on the coefficients is quite general and allows arbitrary discontinuity in one spatial direction which is often related to problems of linear laminates, while the behavior with respect to the other directions, including the time, are controlled in terms of small-BMO, such as small multipliers of the Heaviside step function for instance. It is clear that the cases of continuous, VMO or small-BMO principal coefficients with respect to all variables are particular cases of the situation considered here. Regarding the underlying domain Ω, we suppose that its non-smooth boundary is Reifenberg flat (cf. Reifenberg [23]). It means that $\partial\Omega$ is well approximated by hyperplanes at each point and at each scale. This kind of minimal regularity of the boundary ensures the validity in Ω of some natural properties of geometric and functional analysis such as $W^{1,p}$-extension, non tangential accessibility property, measure density condition, the Poincaré inequality and so on. We refer the reader to the works of Kenig, Toro [17], Toro [28], Lemenant, Milakis, Spinolo [18] and the references therein for further details. In particular, a domain which is sufficiently flat in the sense of Reifenberg is also Jones flat. Moreover, domains with C^1-smooth or Lipschitz continuous boundaries with small Lipschitz constant belong to that category, but the class of Reifenberg flat domains extends beyond these common examples and contains domains with rough fractal boundaries such as the Helge von Koch snowflake with a small angle of the edge.

It is worth noting that the boundary problems and the corresponding regularity theory developed here are related to important variational problems arising in modeling of deformations in composite materials as fiber-reinforced media or, more generally, in the mechanics of membranes and films of simple nonhomogeneous materials which form a linear laminated medium. In particular, a highly twinned elastic or ferroelectric crystal is a typical situation where a laminate appears. The equilibrium equations of such a linear laminate usually have only bounded and measurable coefficients in the direction of the stratification. We refer the reader to the seminal papers by Chipot, Kinderlehrer, Vergara-Caffarelli [10], Li, Vogelius [19] for the general statement of the problem and various issues regarding regularity of solutions in case of piecewise smooth coefficients. The non-smoothness of the underlying Reifenberg flat domain, instead, is related to models of real-world systems over media with fractal geometry such as blood vessels, the internal structure of lungs, bacteria growth, graphs of stock market data, clouds, semiconductor devices, etc.

The paper is organized as follows. In Section 2 we give some definitions and auxiliary results regarding the space $L^{p,\varphi}$, the weight φ and the Hardy-Littlewood maximal operator. In Section 3 we set down the hypotheses on the data of problem (1.1) and state the main result of the paper (Theorem 3.2). The gradient estimate in the generalized Morrey spaces is obtained in Section 4. The main analytic tools employed in that proof rely on the Vitali covering lemma, boundedness properties of the Hardy-Littlewood maximal operator and power decay estimates of the upper level sets of the spatial gradient.

Without essential difficulties, the technique employed in studying the regularity of the solution to (1.1) could be extended to the case of systems and, that is why, in the final Section 5 we restrict ourselves to announce only the $L^{p,\varphi}$-regularity result for the weak solutions to linear, second order parabolic systems with partially BMO coefficients over Reifenberg flat domains. The same results hold also for elliptic divergence form operators.

Let us note also that a global $L^{p,\varphi}$-regularity of strong solutions to linear non divergence form operators have been recently derived in Softova [25] and in Guliyev, Softova [15], [16], [26] in the framework of generalized Morrey spaces under different conditions over the weight functions.

Throughout the paper, the letter c will denote a universal constant that can be explicitly computed in terms of known quantities such as $n,\ L,\ \nu,\ p,\varphi$ and the geometric structure of Q. The exact value of c may vary from one occurrence to another.

2 Generalized parabolic Morrey spaces

Let us start this section with the definitions of the families of domains that we need:

- *parabolic cylinders* centered in a point $(y,\tau)\in\mathbb{R}^{n+1}$ and of radius $r>0$

  $$\mathcal{I}\equiv\mathcal{I}_r(y,\tau)=\left\{(x,t)\in\mathbb{R}^{n+1}:\ |x-y|<r,|t-\tau|<r^2\right\}$$

  with Lebesgue measure $|\mathcal{I}_r|=c(n)r^{n+2}$.
- *parabolic cubes* centered in a point $(y,\tau)=(y_1,y',\tau)$, $y'=(y_2,\dots,y_n)$, such that

  $$\mathcal{C}\equiv\mathcal{C}_r(y,\tau)=\left\{(x_1,x',t)\in\mathbb{R}^{n+1}:|x_1-y_1|<r,\ |x'-y'|<r,|t-\tau|<r^2\right\}$$

  with Lebesgue measure $|\mathcal{C}_r|=c(n)r^{n+2}$.
- *elliptic cubes* in $\mathbb{R}^n$ centered in $y=(y_1,y')$ defined as

  $$\mathcal{C}'_r(y)=\left\{(x_1,x')\in\mathbb{R}^n:\ |x_1-y_1|<r,|x'-y'|<r\right\}$$

  with Lebesgue measure $|\mathcal{C}'_r|=c(n)r^n$.

We call *weight* a positive measurable function $\varphi:\mathbb{R}^{n+1}\times\mathbb{R}_+\to\mathbb{R}_+$ and for any parabolic cylinder $\mathcal{I}_r(y,\tau)$ we use the notation $\varphi(y,\tau;r)=\varphi(\mathcal{I}_r(y,\tau))\equiv\varphi(\mathcal{I})$.

Definition 2.1 Let Q be a cylinder in $\mathbb{R}^{n+1}$. A function $f\in L^q(Q)$,$1<q<\infty$, belongs to the generalized Morrey space $L^{q,\varphi}(Q)$ if the following norm is finite

$$\|f\|_{L^{q,\varphi}(Q)}=\sup_{\substack{(y,\tau)\in Q\\ r>0}}\left(\frac{1}{\varphi(\mathcal{I}_r(y,\tau))}\int_{\mathcal{I}_r(y,\tau)\cap Q}|f(x,t)|^q\,dx\,dt\right)^{\frac{1}{q}}. \tag{2.1}$$

If $\varphi \equiv r^{\lambda}$, $\lambda \in (0, n+2)$, then $L^{q,\varphi}$ coincides with the classical Morrey space $L^{q,\lambda}$. However, there exist examples of weight functions of more general form as $\varphi(r) = r\ln(r+2)$ or $\varphi(\mathcal{I}_r(y,\tau)) = \left(\int_{\mathcal{I}_r(y,\tau)} w(x,t)\,dx\,dt\right)^{\alpha}$, $0 < \alpha < 1$, where $w \in A_q$ is a Muckenhoupt weight with $q \in \left(1, \frac{1}{\alpha}\right)$ (see [22]). One more example is the following.

The function $f(x) = \chi_{[-1,1]}|x|^{-1/2}$ belongs to $L^{1,\varphi}(\mathbb{R})$ with

$$\varphi(\mathcal{I}) = \int_{\mathcal{I}} |x|^{\alpha}\,dx, \quad -1 < \alpha \le -\frac{1}{2},$$

where $\mathcal{I}$ is any interval in $\mathbb{R}$.

Let $\mathcal{M}$ denote the *Hardy-Littlewood maximal operator* on $\mathbb{R}^{n+1}$. For any $f \in L^1_{\mathrm{loc}}\left(\mathbb{R}^{n+1}\right)$ we have

$$\mathcal{M}f(y,\tau) = \sup_{r>0} \frac{1}{|\mathcal{I}_r(y,\tau)|} \int_{\mathcal{I}_r(y,\tau)} |f(x,t)|\,dx\,dt\,.$$

If D is a bounded domain in $\mathbb{R}^{n+1}$ and $f \in L^1(D)$, then $\mathcal{M}f = \mathcal{M}\tilde{f}$, where $\tilde{f}$ is the zero extension of f in the whole space. It is well known that $\mathcal{M}$ is a bounded sub-linear operator from L^q into itself. Precisely, if $f \in L^q\left(\mathbb{R}^{n+1}\right)$, $q \in (1,\infty)$, then

$$\int_{\mathbb{R}^{n+1}} |f(x,t)|^q\,dx\,dt \le \int_{\mathbb{R}^{n+1}} |\mathcal{M}f(x,t)|^q\,dx\,dt \le c\int_{\mathbb{R}^{n+1}} |f(x,t)|^q\,dx\,dt \tag{2.2}$$

for some positive constant $c = c(q,n)$. Moreover, the following weak type estimate

$$\left|\left\{(x,t) \in \mathbb{R}^{n+1} : \mathcal{M}f(x,t) > \lambda\right\}\right| \le \frac{c_q}{\lambda^q}\int_{\mathbb{R}^{n+1}} |f(x,t)|^q\,dx\,dt$$

holds for any $1 \le q < \infty$ and any $\lambda > 0$.

Lemma 2.2 (Maximal inequality, [22]) *Assume that there are positive constants $\kappa_1, \kappa_2, \kappa_3$ such that for any fixed $(y,\tau) \in \mathbb{R}^{n+1}$ and any $r > 0$ it holds*

$$\kappa_1 \le \frac{\varphi(\mathcal{I}_s(y,\tau))}{\varphi(\mathcal{I}_r(y,\tau))} \le \kappa_2 \quad \text{for all } r \le s \le 2r, \tag{2.3}$$

$$\int_r^{\infty} \frac{\varphi(\mathcal{I}_s(y,\tau))}{s^{n+3}}\,ds \le \kappa_3\,\frac{\varphi(\mathcal{I}_r(y,\tau))}{r^{n+2}}\,. \tag{2.4}$$

For $1 < q < \infty$, there is a constant $c_q > 0$ such that

$$\|f\|_{L^{q,\varphi}(\mathbb{R}^{n+1})} \le \|Mf\|_{L^{q,\varphi}(\mathbb{R}^{n+1})} \le c_q\|f\|_{L^{q,\varphi}(\mathbb{R}^{n+1})} \quad \forall f \in L^{q,\varphi}\left(\mathbb{R}^{n+1}\right).$$

Impose in addition a kind of monotonicity condition on φ, precisely

$$\varphi(\mathcal{I}_r(y,\tau)) \le \varphi(\mathcal{I}_s(z,\xi)) \quad \text{for all} \quad \mathcal{I}_r(y,\tau) \subset \mathcal{I}_s(z,\xi)\,. \tag{2.5}$$

This implies that for a given $Q = \Omega \times (0,T] \subset \mathbb{R}^{n+1}$, there holds

$$\sup_{\substack{(y,\tau)\in Q\\ r>0}} \frac{\left|\mathcal{I}_r(y,\tau) \cap Q\right|}{\varphi\left(\mathcal{I}_r(y,\tau)\right)} \le \kappa_4, \tag{2.6}$$

with a positive constant κ_4 depending on n, φ and Q. In fact, since Q is bounded domain there exists $d > 0$ such that $Q \subset \mathcal{I}_d(0,0)$. Then, if $r \ge 2d$ for any $(y,\tau) \in Q$ we have

$$\frac{|\mathcal{I}_r(y,\tau) \cap Q|}{\varphi(\mathcal{I}_r(y,\tau))} \le \frac{|Q|}{\varphi(\mathcal{I}_d(0,0))}.$$

On the other hand, if $0 < r < 2d$, then we see from (2.4) that

$$\kappa_3 \frac{\varphi(\mathcal{I}_r(y,\tau))}{r^{n+2}} \geq \int_{2d}^{\infty} \frac{\varphi(\mathcal{I}_s(y,\tau))}{s^{n+3}}\,ds \geq \varphi(\mathcal{I}_{2d}(y,\tau)) \int_{2d}^{\infty} \frac{1}{s^{n+3}}\,ds \geq \frac{\varphi(\mathcal{I}_d(0,0))}{(n+2)(2d)^{n+2}}\,.$$

It implies that for some positive constant $c = c(n)$ it holds

$$\frac{|\mathcal{I}_r(y,\tau) \cap Q|}{\varphi(\mathcal{I}_r(y,\tau))} \leq \frac{c r^{n+2}}{\varphi\,(\mathcal{I}_r(y,\tau))} \leq \frac{c\kappa_3 (n+2)(2d)^{n+2}}{\varphi(\mathcal{I}_d(0,0))}.$$

3 Assumptions and main result

For each parabolic cube $\mathcal{C}_r(y,\tau)$ and for some fixed $x_1 \in (y_1 - r, y_1 + r)$ we set $\mathcal{C}_r^{x_1}(y,\tau)$ to denote the x_1-slice of $\mathcal{C}_r(y,\tau)$, that is,

$$\mathcal{C}_r^{x_1}(y,\tau) = \left\{(x',t) \in \mathbb{R}^{n-1} \times \mathbb{R} : (x_1, x', t) \in \mathcal{C}_r(y,\tau)\right\}.$$

Then we define the integral average

$$\bar{\mathbf{a}}_{\mathcal{C}_r(y,\tau)}(x_1) = \frac{1}{|\mathcal{C}_r^{x_1}(y,\tau)|} \int_{\mathcal{C}_r^{x_1}(y,\tau)} \mathbf{a}(x_1, x', t)\,dx'dt.$$

Definition 3.1 We say that the couple $(\mathbf{a}, \Omega)$ is (δ, R)-*vanishing of codimension* 1, if the following properties are satisfied:

- For every point $(y,\tau) \in Q$ and for every number $r \in (0, \frac{1}{3}R]$ with

  $$\operatorname{dist}(y, \partial\Omega) > \sqrt{2}r, \tag{3.1}$$

  there exists a coordinate system depending on (y,τ) and r, whose variables we still denote by (x,t) so that in this new coordinate system (y,τ) is the origin and

  $$\frac{1}{|\mathcal{C}_r(0,0)|} \int_{\mathcal{C}_r(0,0)} |\mathbf{a}(x,t) - \bar{\mathbf{a}}_{\mathcal{C}_r(0,0)}(x_1)|^2\,dx\,dt \leq \delta^2. \tag{3.2}$$

- For any point $(y,\tau) \in Q$ and for every number $r \in (0, \frac{1}{3}R]$ such that

  $$\operatorname{dist}(y, \partial\Omega) = \operatorname{dist}(y, x_0) \leq \sqrt{2}r$$

  and for some $x_0 \in \partial\Omega$, there exists a coordinate system depending on (y,τ) and r, whose variables we still denote by (x,t) such that in this new coordinate system (x_0,τ) is the origin,

  $$\Omega \cap \{x \in \mathcal{C}'_{3r}(0) : x_1 > 3r\delta\} \subset \Omega \cap \mathcal{C}'_{3r}(0) \subset \Omega \cap \{x \in \mathcal{C}'_{3r}(0) : x_1 > -3r\delta\} \tag{3.3}$$

  and

  $$\frac{1}{|\mathcal{C}_{3r}(0,0)|} \int_{\mathcal{C}_{3r}(0,0)} |\mathbf{a}(x,t) - \bar{\mathbf{a}}_{\mathcal{C}_{3r}(0,0)}(x_1)|^2\,dx\,dt \leq \delta^2.$$

We add some comments regarding the above definition. Thanks to the scaling invariance property, one can take for simplicity $R = 1$ or any other constant bigger than 1. On the other hand δ is a small positive constant, being invariant under such a scaling argument. If $\mathbf{a}$ is (δ, R)-vanishing of codimension 1, then for each point and for each sufficiently small scale, there is a coordinate system so that the coefficients have small oscillation in (x',t)-variables while these are only measurable in the x_1-variable and therefore may have arbitrary jumps with respect to it. In addition, the boundary of the domain is (δ, R)-Reifenberg flat (see Reifenberg [23]) and the coefficients have a small oscillation along the flat direction x' of the boundary and are only measurable along the normal direction x_1. The number $\sqrt{2}r$ in (3.1) is selected for convenience since we need to take the size of the parabolic cubes in (3.2) such that there is enough room to have the rotation of $\mathcal{C}_r(y,\tau)$ in any spatial direction.

We suppose that $\mathbf{F} \in L^{p,\varphi}(Q)$, $p \in (2,\infty)$, and weight satisfying (2.5) which implies $\mathbf{F} \in L^p(Q)$. Precisely, choose $(y,\tau) \in Q$, then $\sup_{(z,\xi)\in Q}\left\{|y-z| + \sqrt{|\tau-\xi|}\right\} < \operatorname{diam} Q$. Hence there exists $r^* < \operatorname{diam} Q$ and such that $Q \subset \mathcal{I}_{r^*}(y,\tau)$. This gives the relation

$$\|\mathbf{F}\|_{L^p(Q)} \le \varphi(\mathcal{I}_{r^*}(y,\tau))^{\frac{1}{p}}\|\mathbf{F}\|_{L^{p,\varphi}(Q)} \le \varphi(\mathcal{I}_{2d}(0,0))^{\frac{1}{p}}\|\mathbf{F}\|_{L^{p,\varphi}(Q)}\,.$$

Then the Hölder inequality implies

$$\|\mathbf{F}\|^2_{L^2(Q)} \le |Q|^{1-\frac{2}{p}}\left\|\,|\mathbf{F}|^2\right\|_{L^{\frac{p}{2}}(Q)} \le |Q|^{1-\frac{2}{p}}\varphi(\mathcal{I}_{2d}(0,0))^{\frac{2}{p}}\left\|\,|\mathbf{F}|^2\right\|_{L^{\frac{p}{2},\varphi}(Q)} \tag{3.4}$$

that ensures the existence of a unique weak solution u of (1.1) (see [2], [7]). We are going to prove the following regularity result.

Theorem 3.2 *Let $p \in (2,\infty)$ and $\varphi : \mathbb{R}^{n+1}\times\mathbb{R}_+ \to \mathbb{R}_+$ be a weight satisfying* (2.3), (2.4) *and* (2.5). *Then there exists a small positive constant $\delta = \delta(n, L, \nu, p, \varphi, Q)$ such that if the couple $(\mathbf{a},\Omega)$ is (δ, R)-vanishing of codimension* 1 *and $\mathbf{F} \in L^{p,\varphi}(Q)$, then $Du \in L^{p,\varphi}(Q)$ and the following estimate*

$$\|Du\|_{L^{p,\varphi}(Q)} \le c\|\mathbf{F}\|_{L^{p,\varphi}(Q)}$$

holds with a constant c depending on known quantities.

The present work is a natural extension of the previous papers Byun, Palagachev, Wang [6] which deals with the regularity problem for parabolic equations in classical Lebesgue classes and Byun, Palagachev, Softova [5] where the problem (1.1) is studied in the framework of the weighted Lebesgue spaces with a Muckenhoupt weight and the classical Morrey spaces $L^{p,\lambda}(Q)$ with $\lambda \in (0, n+2)$.

In what follows we start with finding a suitable version of the Vitali covering lemma. We then apply the covering lemma to derive a power decay estimate of the upper level sets for the Hardy-Littlewood maximal function of the spatial gradient of the weak solution. The required estimate in the main result follows then by the standard procedure of summation over the level sets.

4 Gradient estimates in $L^{p,\varphi}(Q)$

Because of the scaling invariance property of the Reifenberg domains discussed above (cf. [6, Lemma 5.2]), we can take $R = 1$ hereafter. Fix $(y_0,\tau_0) \in Q$, take a parabolic cylinder $\mathcal{I}_r(y_0,\tau_0)$ and denote $Q_r = \mathcal{I}_r(y_0,\tau_0) \cap Q$. We start with some auxiliary lemmas.

Lemma 4.1 *Suppose Ω is a bounded $(\delta,1)$-Reifenberg flat domain verifying* (3.3). *Let $\mathfrak{C} \subset \mathfrak{D} \subset Q_r$ be measurable subsets of Q_r satisfying the following conditions: there exists $\varepsilon \in (0,1)$ such that*

- *for each $(y,\tau) \in \mathfrak{C}$*

$$|\mathfrak{C} \cap \mathcal{C}_1(y,\tau)| < \varepsilon|\mathcal{C}_1(y,\tau)|; \tag{4.1}$$

- *for each $(y,\tau) \in \mathfrak{C}$ and $\rho > 0$*

$$|\mathfrak{C} \cap \mathcal{C}_\rho(y,\tau)| \ge \varepsilon|\mathcal{C}_\rho(y,\tau)| \quad \textit{implies} \quad Q_r \cap \mathcal{C}_\rho(y,\tau) \subset \mathfrak{D}. \tag{4.2}$$

Then

$$|\mathfrak{C}| \le \varepsilon\left(\frac{10\sqrt{2}}{1-\delta}\right)^{n+2}|\mathfrak{D}|\,. \tag{4.3}$$

Proof. Fix $(y,\tau) \in \mathfrak{C}$ and for each $\rho > 0$ define the function

$$\Theta(\rho) = \frac{|\mathfrak{C} \cap \mathcal{C}_\rho(y,\tau)|}{|\mathcal{C}_\rho(y,\tau)|}.$$

We have $\Theta \in C^0(0,\infty), \Theta(1) < \varepsilon$ according to (4.1) and $\Theta(0) = \lim_{\rho\to 0_+}\Theta(\rho) = 1$ by the Lebesgue Differentiation Theorem. Therefore, for almost all $(y,\tau) \in \mathfrak{C}$, there exists $\rho_{(y,\tau)} \in (0,1)$ such that $\Theta(\rho_{(y,\tau)}) = \varepsilon$ and $\Theta(\rho) < \varepsilon$ for all $\rho > \rho_{(y,\tau)}$.

Define the family of parabolic cubes $\{\mathcal{C}_{\rho_{(y,\tau)}}(y,\tau)\}_{(y,\tau)\in\mathfrak{C}}$ which forms an open covering of $\mathfrak{C}$. By the Vitali lemma (cf. [27, Lemma I.3.1]), there exists a disjoint sub-collection $\{\mathcal{C}_{\rho_i}(y_i,\tau_i)\}_{i\geq 1}$ with $\rho_i = \rho_{(y_i,\tau_i)} \in (0,1)$, $(y_i,\tau_i) \in \mathfrak{C}$ such that $\Theta(\rho_i) = \varepsilon$,

$$\sum_{i\geq 1} |\mathcal{C}_{\rho_i}(y_i,\tau_i)| \geq c|\mathfrak{C}|, \qquad \text{and} \qquad \mathfrak{C} \subset \bigcup_{i\geq 1} \mathcal{C}_{5\rho_i}(y_i,\tau_i),$$

with a positive constant $c = c(n)$. Since $\Theta(5\rho_i) < \varepsilon$, we have

$$|\mathfrak{C} \cap \mathcal{C}_{5\rho_i}(y_i,\tau_i)| < \varepsilon|\mathcal{C}_{5\rho_i}(y_i,\tau_i)| = \varepsilon 5^{n+2}|\mathcal{C}_{\rho_i}(y_i,\tau_i)|\,.$$

Further, making use of the bound obtained in [6] (see also [8]) we get

$$|\mathcal{C}_{\rho_i}(y_i,\tau_i)| \leq \left(\frac{2\sqrt{2}}{1-\delta}\right)^{n+2} |Q_r \cap \mathcal{C}_{\rho_i}(y_i,\tau_i)|.$$

Now we have

$$\begin{aligned} |\mathfrak{C}| &= \left|\bigcup_{i\geq 1}\left(\mathfrak{C} \cap \mathcal{C}_{5\rho_i}(y_i,\tau_i)\right)\right| \leq \sum_{i\geq 1} |\mathfrak{C} \cap \mathcal{C}_{5\rho_i}(y_i,\tau_i)| \\ &< \varepsilon \sum_{i\geq 1} |\mathcal{C}_{5\rho_i}(y_i,\tau_i)| \leq \varepsilon 5^{n+2} \sum_{i\geq 1} |\mathcal{C}_{\rho_i}(y_i,\tau_i)| \\ &\leq \varepsilon \left(\frac{10\sqrt{2}}{1-\delta}\right)^{n+2} \sum_{i\geq 1} |Q_r \cap \mathcal{C}_{\rho_i}(y_i,\tau_i)|. \end{aligned}$$

Having in mind that the $\{\mathcal{C}_{\rho_i}(y_i,\tau_i)\}_{i\geq 1}$ are mutually disjoint, $\Theta(\rho_i) = \varepsilon$ and (4.2), we get

$$|\mathfrak{C}| \leq \varepsilon \left(\frac{10\sqrt{2}}{1-\delta}\right)^{n+2} \left|\bigcup_{i\geq 1} Q_r \cap \mathcal{C}_{\rho_i}(y_i,\tau_i)\right| \leq \varepsilon \left(\frac{10\sqrt{2}}{1-\delta}\right)^{n+2} |\mathfrak{D}|\,.$$

□

The following approximation lemma has been proved for various functional spaces in [13, Corollary 8.4], [6, Lemma 5.3] and [8, Lemma 5.5].

Lemma 4.2 *Assume* (1.2) *and let u be a weak solution of* (1.1). *Then there is a constant $\lambda_1 = \lambda_1(L,\nu,n) > 1$ such that for each $\varepsilon \in (0,1)$ there exists $\delta = \delta(\varepsilon) > 0$ such that if the couple* $(\mathbf{a},\Omega)$ *is* $(\delta,1)$*-vanishing of codimension* 1 *and if $\mathcal{C}_\rho(y,\tau)$ satisfies*

$$\left|\left\{(x,t) \in Q_r : \mathcal{M}(|Du|^2) > \lambda_1^2\right\} \cap \mathcal{C}_\rho(y,\tau)\right| \geq \varepsilon|\mathcal{C}_\rho(y,\tau)|,$$

then we have

$$Q_r \cap \mathcal{C}_\rho(y,\tau) \subset \left\{(x,t) \in Q_r : \mathcal{M}\big(|Du|^2\big) > 1\right\} \cup \left\{\mathcal{M}\big(|\mathbf{F}|^2\big) > \delta^2\right\}.$$

For any weak solution u of (1.1) and for the fixed $Q_r = \mathcal{I}_r(y_0,\tau_0) \cap Q$ we set

$$\mathfrak{C} = \left\{(x,t) \in Q_r : \mathcal{M}\left(|Du|^2\right) > \lambda_1^2\right\} \tag{4.4}$$

and

$$\mathfrak{D} = \left\{(x,t) \in Q_r : \mathcal{M}\left(|Du|^2\right) > 1\right\} \cup \left\{\mathcal{M}\left(|\mathbf{F}|^2\right) > \delta^2\right\} \tag{4.5}$$

with λ_1 and δ as in Lemma 4.2. The next assertion shows that the assumption (4.2) holds for the such defined sets $\mathfrak{C}$ and $\mathfrak{D}$.

Lemma 4.3 *Under the assumptions of Lemma 4.2, we suppose additionally that for each* $(y,\tau)\in Q_r$,

$$\Theta(1)=\frac{|\mathfrak{C}\cap\mathcal{C}_1(y,\tau)|}{|\mathcal{C}_1(y,\tau)|}<\varepsilon \tag{4.6}$$

with $\mathfrak{C}$ *as in* (4.4). *Then for each* $k=1,2,\ldots,$ *we have*

$$\begin{aligned}\left|\left\{(x,t)\in Q_r:\mathcal{M}\left(|Du|^2\right)>\lambda_1^{2k}\right\}\right| &\le \varepsilon_1^k\left|\left\{(x,t)\in Q_r:\mathcal{M}\left(|Du|^2\right)>1\right\}\right|\\ &\quad+\sum_{i=1}^{k}\varepsilon_1^i\left|\left\{(x,t)\in Q_r:\mathcal{M}\left(|\mathbf{F}|^2\right)>\delta^2\lambda_1^{2(k-i)}\right\}\right| \end{aligned}\tag{4.7}$$

where $\varepsilon_1=\varepsilon\left(\frac{10\sqrt{2}}{1-\delta}\right)^{n+2}$.

Proof. Lemma 4.2 and condition (4.6) ensure the validity of the hypotheses of Lemma 4.1 for the sets (4.4) and (4.5). Thus, we get by (4.3)

$$\begin{aligned}\left|\left\{(x,t)\in Q_r:\mathcal{M}\left(|Du|^2\right)>\lambda_1^{2}\right\}\right| &\le \varepsilon_1\left|\left\{(x,t)\in Q_r:\mathcal{M}\left(|Du|^2\right)>1\right\}\right|\\ &\quad+\varepsilon_1\left|\left\{(x,t)\in Q_r:\mathcal{M}\left(|\mathbf{F}|^2\right)>\delta^2\right\}\right|,\end{aligned}$$

where $\varepsilon_1=\varepsilon\left(\frac{10\sqrt{2}}{1-\delta}\right)^{n+2}$.

The last inequality is exactly (4.7) with $k=1$. Further, we proceed with the proof by induction, as it is done in [1, Corollary 4.15]. Suppose that (4.7) holds true for each weak solution of (1.1) and for some $k\ge 1$. Define the functions $u_1=\frac{u}{\lambda_1}$ and $\mathbf{F}_1=\frac{\mathbf{F}}{\lambda_1}$. It is easy to see that u_1 is a weak solution to the problem (1.1) with a right-hand side $\mathbf{F}_1$. Hence, (4.6) and Lemma 4.2 hold with sets $\mathfrak{C}$ and $\mathfrak{D}$ corresponding to u_1 as defined in (4.4) and (4.5). According to (4.7), the inductive assumption holds true for u_1 with the same $k\ge 1$. The definition of u_1 ensures the inductive passage from k to $k+1$ for u. Namely,

$$\begin{aligned}&\left|\left\{(x,t)\in Q_r:\mathcal{M}\left(|Du|^2\right)>\lambda_1^{2(k+1)}\right\}\right|\\ &\quad=\left|\left\{(x,t)\in Q_r:\mathcal{M}\left(|Du_1|^2\right)>\lambda_1^{2k}\right\}\right|\\ &\quad\le\varepsilon_1^k\left|\{(x,t)\in Q_r:\mathcal{M}\left(|Du_1|^2\right)>1\}\right|\\ &\qquad+\sum_{i=1}^{k}\varepsilon_1^i\left|\left\{(x,t)\in Q_r:\mathcal{M}\left(|\mathbf{F}_1|^2\right)>\delta^2\lambda_1^{2(k-i)}\right\}\right|\\ &\quad=\varepsilon_1^k\left|\left\{(x,t)\in Q_r:\mathcal{M}\left(|Du|^2\right)>\lambda_1^2\right\}\right|\\ &\qquad+\sum_{i=1}^{k}\varepsilon_1^i\left|\left\{(x,t)\in Q_r:\mathcal{M}\left(|\mathbf{F}|^2\right)>\delta^2\lambda_1^{2(k-i)}\lambda_1^2\right\}\right|\\ &\quad\le\varepsilon_1^{k+1}\left|\left\{(x,t)\in Q_r:\mathcal{M}\left(|Du|^2\right)>1\right\}\right|\\ &\qquad+\sum_{i=1}^{k+1}\varepsilon_1^i\left|\left\{(x,t)\in Q_r:\mathcal{M}\left(|\mathbf{F}|^2\right)>\delta^2\lambda_1^{2(k+1-i)}\right\}\right|.\end{aligned}$$

□

Let us note that because of the arbitrary choice of the point $(y_0,\tau_0)\in Q$ the above estimates hold locally for any $Q_r=\mathcal{I}_r(y,\tau)\cap Q$. The next result follows from the standard measure theory.

Lemma 4.4 *Let* $h\in L^1(Q)$ *be a nonnegative function,* φ *be a weight function satisfying* (2.3), (2.4) *and* (2.5), $q\in(1,\infty)$ *and* $\lambda>0,\theta>1$ *be constants. Then* $h\in L^{q,\varphi}(Q)$ *if and only if*

$$\mathcal{S}:=\sup_{\substack{(y,\tau)\in Q\\ r>0}}\sum_{k\ge1}\frac{\theta^{kq}\left|\left\{(x,t)\in Q_r:h(x,t)>\lambda\theta^k\right\}\right|}{\varphi(\mathcal{I}_r(y,\tau))}<\infty.$$

Moreover,

$$\frac{1}{c}\mathcal{S} \le \|h\|^q_{L^{q,\varphi}(Q)} \le c(1+\mathcal{S}),$$

where $c = c(\theta, \lambda, q, \varphi, Q)$.

Proof. Choose $(y,\tau) \in Q$ and take $\mathcal{I}_r(y,\tau)$, then

$$\begin{aligned}
&\frac{1}{\varphi(\mathcal{I}_r(y,\tau))}\int_{Q_r} h^q(x,t)\,dx\,dt\\
&\qquad = \frac{1}{\varphi(\mathcal{I}_r(y,\tau))}\int_{\{(x,t)\in Q_r : h\le\lambda\theta\}} h^q(x,t)\,dx\,dt\\
&\qquad\quad + \sum_{k\ge1}\frac{1}{\varphi(\mathcal{I}_r(y,\tau))}\int_{\{(x,t)\in Q_r:\lambda\theta^k<h\le\lambda\theta^{k+1}\}} h^q(x,t)\,dx\,dt\\
&\qquad \le (\lambda\theta)^q\frac{|Q_r|}{\varphi(\mathcal{I}_r(y,\tau))} + \sum_{k\ge1}\frac{\left(\lambda\theta^{k+1}\right)^q}{\varphi(\mathcal{I}_r(y,\tau))}\left|\left\{(x,t)\in Q_r : h(x,t) > \lambda\theta^k\right\}\right|\\
&\qquad = (\lambda\theta)^q\left(\frac{|Q_r|}{\varphi(\mathcal{I}_r(y,\tau))} + \sum_{k\ge1}\frac{\theta^{kq}\left|\left\{(x,t)\in Q_r : h(x,t) > \lambda\theta^k\right\}\right|}{\varphi(\mathcal{I}_r(y,\tau))}\right).
\end{aligned}$$

Taking the supremum over $(y,\tau) \in Q, r > 0$ and making use of (2.6), we get

$$\|h\|^q_{L^{q,\varphi}(Q)} \le c\,(1+\mathcal{S})$$

with a constant depending on $q, n, \varphi, \lambda, \theta$ and Q. On the other hand

$$\begin{aligned}
&\frac{1}{\varphi(\mathcal{I}_r(y,\tau))}\int_{Q_r} h^q(x,t)\,dx\,dt\\
&\qquad = \frac{q}{\varphi(\mathcal{I}_r(y,\tau))}\int_{Q_r}\left(\int_0^{h(x,t)}\xi^{q-1}d\xi\right)dx\,dt\\
&\qquad = \frac{q}{\varphi(\mathcal{I}_r(y,\tau))}\int_0^\infty \left|\{(x,t)\in Q_r : h(x,t) > \xi\}\right|\xi^{q-1}\,d\xi\\
&\qquad \ge \frac{q}{\varphi(\mathcal{I}_r(y,\tau))}\sum_{k\ge1}\left|\left\{(x,t)\in Q_r : h(x,t) > \lambda\theta^k\right\}\right|\int_{\lambda\theta^{k-1}}^{\lambda\theta^k}\xi^{q-1}\,d\xi\\
&\qquad = \lambda^q(1-\theta^{-q})\frac{1}{\varphi(\mathcal{I}_r(y,\tau))}\sum_{k\ge1}\theta^{kq}\left|\left\{(x,t)\in Q_r : h(x,t) > \lambda\theta^k\right\}\right|.
\end{aligned}$$

Taking again the supremum over $(y,\tau) \in Q, r > 0$ we get $\|h\|^q_{L^{q,\varphi}(Q)} \ge \frac{1}{c}\mathcal{S}$ with a positive constant $c = c(\theta, \lambda, q)$. ☐

We are in a position now to prove Theorem 3.2.

Proof. Recall that $\mathbf{F} \in L^{p,\varphi}(Q)$, $p \in (2,\infty)$, with a weight φ satisfying (2.3), (2.4) and (2.5). Because of the scaling invariance property of (1.1) under a normalization, we can assume that the norm of $\mathbf{F}$ is small enough. In fact, taking

$$\bar{u}(x,t) = \frac{\delta u(x,t)}{\sqrt{\||\mathbf{F}|^2\|_{L^{\frac{p}{2},\varphi}(Q)}}} \qquad \text{and} \qquad \bar{\mathbf{F}}(x,t) = \frac{\delta\mathbf{F}(x,t)}{\sqrt{\||\mathbf{F}|^2\|_{L^{\frac{p}{2},\varphi}(Q)}}}$$

instead of u and $\mathbf{F}$ in (1.1) we get $\left\| |\bar{\mathbf{F}}|^2 \right\|_{L^{\frac{p}{2},\varphi}(Q)} = \delta^2$. Then we need to prove boundedness of the norm of the gradient $|D\bar{u}|$. Because of the properties of the maximal function (see Lemma 2.2), it is enough to get

$$\left\| \mathcal{M}\left(|D\bar{u}|^2\right) \right\|_{L^{\frac{p}{2},\varphi}(Q)} \le c.$$

For this goal, we apply Lemma 4.4 with $h = \mathcal{M}\big(|D\bar{u}|^2\big)$,$\theta = \lambda_1^2$, $\lambda = 1$ and $q = \frac{p}{2}$.

Let $\mathfrak{C}$ be the set defined in (4.4) and corresponding to the solution $\bar{u}$. We note that for each $(y,\tau) \in \mathfrak{C}$,

$$\begin{aligned}
\frac{|\mathfrak{C} \cap \mathcal{C}_1(y,\tau)|}{|\mathcal{C}_1(y,\tau)|} &\le c|\mathfrak{C}| = c\left| \left\{ (x,t) \in Q_r : \mathcal{M}\left(|D\bar{u}|^2\right) > \lambda_1^2 \right\} \right| \\
&\le c \int_{Q_r} \mathcal{M}\left(|D\bar{u}|^2\right)(x,t)\,dx\,dt \le c \int_{Q_r} |D\bar{u}(x,t)|^2\,dx\,dt \\
&\le c \int_{Q} |D\bar{u}(x,t)|^2\,dx\,dt \le c \int_{Q} |\bar{\mathbf{F}}(x,t)|^2\,dx\,dt \\
&\le c \left\| |\bar{\mathbf{F}}|^2 \right\|_{L^{\frac{p}{2},\varphi}(Q)} \le c\delta^2,
\end{aligned}$$

according to (3.4) with a constant depending on n, p, φ and Q. Taking δ small enough, we get

$$\Theta(1) = \frac{|\mathfrak{C} \cap \mathcal{C}_1(y,\tau)|}{|\mathcal{C}_1(y,\tau)|} \le c\delta^2 < \varepsilon$$

which ensures (4.6). Therefore Lemma 4.3 gives

$$\begin{aligned}
&\sum_{k\ge 1} \lambda_1^{2k\frac{p}{2}} \frac{\left| \left\{ (x,t) \in Q_r : \mathcal{M}\left(|D\bar{u}|^2\right) > \lambda_1^{2k} \right\} \right|}{\varphi(\mathcal{I}_r(y,\tau))} \\
&\quad \le \sum_{k\ge 1} \lambda_1^{kp} \varepsilon_1^k \frac{\left| \left\{ (x,t) \in Q_r : \mathcal{M}\left(|D\bar{u}|^2\right) > 1 \right\} \right|}{\varphi(\mathcal{I}_r(y,\tau))} \\
&\qquad + \sum_{k\ge 1} \sum_{i=1}^{k} \lambda_1^{kp} \varepsilon_1^i \frac{\left| \left\{ (x,t) \in Q_r : \mathcal{M}\left(|\bar{\mathbf{F}}|^2\right) > \delta^2 \lambda_1^{2(k-i)} \right\} \right|}{\varphi(\mathcal{I}_r(y,\tau))} \\
&\quad \le \sum_{k\ge 1} \left(\lambda_1^p \varepsilon_1\right)^k \frac{|Q_r|}{\varphi(\mathcal{I}_r(y,\tau))} \\
&\qquad + \sum_{i\ge 1} \left(\lambda_1^p \varepsilon_1\right)^i \underbrace{\sum_{k\ge i} \lambda_1^{p(k-i)} \frac{\left| \left\{ (x,t) \in Q_r : \mathcal{M}\left(|\bar{\mathbf{F}}|^2\right) > \delta^2 \lambda_1^{2(k-i)} \right\} \right|}{\varphi(\mathcal{I}_r(y,\tau))}}_{\mathcal{S}'} \\
&\quad \le \kappa_4 \sum_{k\ge 1} \left(\lambda_1^p \varepsilon_1\right)^k + \sum_{i\ge 1} \left(\lambda_1^p \varepsilon_1\right)^i \mathcal{S}',
\end{aligned}$$

where we have used (2.6) for the last inequality. Let us note that

$$\begin{aligned}
\mathcal{S}' &= \sum_{k\ge i} \lambda_1^{p(k-i)} \frac{\left| \left\{ (x,t) \in Q_r : \mathcal{M}\left(|\bar{\mathbf{F}}|^2\right) > \delta^2 \lambda_1^{2(k-i)} \right\} \right|}{\varphi(\mathcal{I}_r(y,\tau))} \\
&= \sum_{k\ge i} \left(\lambda_1^{2(k-i)}\right)^{\frac{p}{2}} \frac{\left| \left\{ (x,t) \in Q_r : \mathcal{M}\left(\frac{|\bar{\mathbf{F}}|^2}{\delta^2}\right) > \lambda_1^{2(k-i)} \right\} \right|}{\varphi(\mathcal{I}_r(y,\tau))}
\end{aligned}$$

$$\leq \frac{c_p}{\varphi(\mathcal{I}_r(y,\tau))}\left(|Q_r| + \int_{Q_r} \mathcal{M}\left(\frac{|\bar{\mathbf{F}}|^2}{\delta^2}\right)^{\frac{p}{2}}(x,t)\,dx\,dt\right)$$

$$\leq \frac{c_p}{\varphi(\mathcal{I}_r(y,\tau))}\left(|Q_r| + \int_{Q_r} \left(\frac{|\bar{\mathbf{F}}|^2}{\delta^2}\right)^{\frac{p}{2}}(x,t)\,dx\,dt\right).$$

Taking again the supremum over $(y,\tau)\in Q$, $r>0$ and making use of (2.6) we get

$$\mathcal{S}' \leq c_p\left(\kappa_4 + \left\|\left|\frac{\bar{\mathbf{F}}}{\delta}\right|^2\right\|^{\frac{p}{2}}_{L^{\frac{p}{2},\varphi}(Q)}\right) \leq c\left(1+\frac{1}{\delta^p}\left\||\bar{\mathbf{F}}|^2\right\|^{\frac{p}{2}}_{L^{\frac{p}{2},\varphi}(Q)}\right) \leq c.$$

Taking ε, and the corresponding δ, small enough such that $0<\lambda_1^p\varepsilon_1<1$ we get

$$\sum_{k\geq 1}\lambda_1^{2k\frac{p}{2}}\frac{\left|\left\{(x,t)\in Q_r : \mathcal{M}\left(|D\bar{u}|^2\right) > \lambda_1^{2k}\right\}\right|}{\varphi(\mathcal{I}_r(y,\tau))} \leq c\sum_{k\geq 1}\left(\lambda_1^p\varepsilon_1\right)^k \leq c\,.$$

Taking again the supremum over $(y,\tau)\in Q$, $r>0$ in the estimates above and making use of Lemma 4.4 we find that

$$\left\|\mathcal{M}\left(|D\bar{u}|^2\right)\right\|_{L^{\frac{p}{2},\varphi}(Q)} \leq c < \infty.$$

This way, Lemma 2.2 and the definition of $\bar{u}$ imply

$$\left\||Du|^2\right\|_{L^{\frac{p}{2},\varphi}(Q)} \leq c\left\||\mathbf{F}|^2\right\|_{L^{\frac{p}{2},\varphi}(Q)}$$

with constant depending on known quantities. □

5 Linear parabolic systems in divergence form

The previous result can be easily extended to the case of nonhomogeneous parabolic systems in divergence form

$$\begin{cases} u^i_t - D_\alpha\big(a^{\alpha\beta}_{ij}(x,t)D_\beta u^j\big) = D_\alpha f^i_\alpha(x,t) & \text{in } Q,\\ u^i(x,t)=0 & \text{on } \partial_P Q,\end{cases} \tag{5.1}$$

for $i=1,\dots,m$.

The tensor matrix of the coefficients

$$\mathbf{A} = \left\{a^{\alpha\beta}_{ij}(x,t)\right\} : Q \longrightarrow \mathbb{R}^{mn\times mn}$$

is assumed to be uniformly bounded and uniformly parabolic, namely, we suppose that there exists positive constants L and ν such that

$$\|\mathbf{A}\|_{L^\infty(Q,\mathbb{R}^{mn\times mn})} \leq L, \qquad a^{\alpha\beta}_{ij}(x,t)\xi^i_\alpha\xi^j_\beta \geq \nu|\xi|^2 \tag{5.2}$$

for all matrices $\xi\in\mathcal{M}^{m\times n}$ and for almost every $(x,t)\in Q$.

When the nonhomogeneous term $\mathbf{F}(x,t)=\left\{f^i_\alpha(x,t)\right\}$ belongs to $L^2(Q,\mathbb{R}^{mn})$, the Cauchy–Dirichlet problem (5.1) has a unique weak solution $\mathbf{u}=(u^1,\dots,u^m)$ with the standard L^2-estimate

$$\|D\mathbf{u}\|_{L^2(Q,\mathbb{R}^{mn})} \leq c\|\mathbf{F}\|_{L^2(Q,\mathbb{R}^{mn})},$$

where c is a positive constant depending only on n, m, L, ν and $|Q|$ (see [2]). In particular, the weak solution of (5.1) belongs to

$$H^{\frac{1}{2}}\big(0,T;L^2(\Omega,\mathbb{R}^m)\big)\cap L^2\left(0,T;H^1_0(\Omega,\mathbb{R}^m)\right),$$

and satisfies the estimate

$$\|\mathbf{u}\|_{H^{\frac{1}{2}}(0,T;L^2(\Omega,\mathbb{R}^m))\cap L^2(0,T;H^1_0(\Omega,\mathbb{R}^m))} + \|D\mathbf{u}\|_{L^2(Q,\mathbb{R}^{mn})} \leq c\|\mathbf{F}\|_{L^2(Q,\mathbb{R}^{mn})},$$

where the constant c is independent of $\mathbf{u}$ and $\mathbf{F}$ (see [8]).

The proofs given in Sections 4 apply also to the weak solutions of the system (5.1). That is why, we shall restrict ourselves only to announce the corresponding regularity result.

Theorem 5.1 *Assume* (5.2) *and let* $p \in (2, \infty)$ *and* $\varphi : \mathbb{R}^{n+1} \times \mathbb{R}^+ \to \mathbb{R}^+$ *be a weight satisfying* (2.3), (2.4) *and* (2.5). *There exists a small positive constant* $\delta = \delta(n, m, L, \nu, p, \varphi, Q)$ *such that if the couple* $(\mathbf{A}, \Omega)$ *is* (δ, R)*-vanishing of codimension 1 and* $\mathbf{F} \in L^{p,\varphi}(Q, \mathbb{R}^{mn})$, *then the spatial gradient* $D\mathbf{u}$ *of the weak solution to* (5.1) *lies in* $L^{p,\varphi}(Q, \mathbb{R}^{mn})$ *and satisfies the estimate*

$$\|D\mathbf{u}\|_{L^{p,\varphi}(Q,\mathbb{R}^{mn})} \leq c\|\mathbf{F}\|_{L^{p,\varphi}(Q,\mathbb{R}^{mn})},$$

with a constant c independent of $\mathbf{u}$ *and* $\mathbf{F}$.

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