# Exact Controllability for Evolutionary Imperfect Transmission Problems

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## Abstract

In this paper we study the asymptotic behaviour of an exact controllability problem for a second order linear evolution equation defined in a two-component composite with $\varepsilon$-periodic disconnected inclusions of size $\varepsilon$. On the interface we prescribe a jump of the solution that varies according to a real parameter $\gamma$. In particular, we suppose that $-1 < \gamma \leq 1$. The case $\gamma = 1$ is the most interesting and delicate one, since the homogenized problem is represented by a coupled system of a P.D.E. and an O.D.E., giving rise to a memory effect. Our approach to exact controllability consists in applying the Hilbert Uniqueness Method, introduced by J. -L. Lions, which leads us to the construction of the exact control as the solution of a transposed problem. Our main result proves that the exact control and the corresponding solution of the $\varepsilon$-problem converge to the exact control of the homogenized problem and to the corresponding solution respectively.

## Résumé

Dans cet article nous étudions le comportement asymptotique d'un problème de contrôlabilité exacte pour une équation d'évolution linéaire du second ordre, dans un milieu composite à deux composantes présentant des inclusions $\varepsilon$-périodiques de taille $\varepsilon$. Sur l'interface entre les deux composantes on prescrit un saut de la solution qui est proportionnel par un facteur $\varepsilon^\gamma$ à la dérivée conormale. On suppose $-1 < \gamma \leq 1$. Le cas $\gamma = 1$, plus délicat, est aussi le plus intéressant, puisque le problème homogénéisé est un système couplé de deux équations, une EDP et une EDO, ce qui génère un effet de mémoire. Notre approche à la contrôlabilité exacte consiste à appliquer la méthode HUM (Hilbert Uniqueness Method), introduite par J. -L. Lions, ce qui nous conduit à la construction du contrôle exact comme solution d'un problème transposé. Notre résultat principal montre que le contrôle exact et la solution correspondante du $\varepsilon$-problème convergent respectivement vers le contrôle exact du problème homogénéisé et la solution correspondante.

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## 1. Introduction

The aim of this paper is to investigate the exact controllability problem related to a linear hyperbolic system of equations with oscillating coefficients defined in an $\varepsilon$-periodic two-component domain $\Omega = \Omega_{1\varepsilon} \cup \overline{\Omega_{2\varepsilon}}$. The first component $\Omega_{1\varepsilon}$ is supposed to be connected, while the second one is the union of $\varepsilon$-periodic disconnected inclusions of size $\varepsilon$. Denoted by $\Gamma^\varepsilon = \partial\Omega_{2\varepsilon}$ the interface separating the two components, we prescribe on it a jump of the solution proportional to the conormal derivatives via a real parameter $\gamma$, meanwhile, a Dirichlet condition is imposed on the exterior boundary $\partial\Omega$ (see Fig.1). This problem models the wave propagation in a medium made up of two components with very different coefficients of propagation, which gives rise to the jump in the boundary condition on the interface. This interface condition is the mathematical interpretation of imperfect interface characterized by the discontinuity of the displacement (see [1, 2], [12]÷[19], [24, 25], [27, 28], [31], [37]÷[41], [43, 44] and references therein). The order of magnitude of the parameter $\gamma$, with respect to the period $\varepsilon$, determines the influence of the contact barrier in the propagation properties of the medium. It is natural to suppose $\gamma \leq 1$, as one cannot expect to have boundedness of the solutions when $\gamma > 1$ (see Hummel in [31], for the homogenization results in the elliptic case). Moreover, it is already known from previous studies (see [14, 15]) that the asymptotic behavior of the $\varepsilon$-problem differs in terms of the homogenized problems in the two cases $\gamma < 1$ and $\gamma = 1$. The second case is more complicated, since the limit problem is a coupled system of a P.D.E. and an O.D.E. and gives rise to what is called a memory effect. In particular, in this paper we want to deal with the exact controllability when $-1 < \gamma \leq 1$. The case $\gamma \leq -1$ will be treated in a forthcoming paper.

The issue of exact controllability can be formulated as follows. Given an evolution system (described by O.D.E/P.D.E.), we are allowed to act on the trajectories (solutions) by means of a suitable control (the right hand side of the system, the boundary conditions, etc.). Then, given a time interval $[0, T]$, is it possible, for all initial data, to find a control (or set of controls) driving the system to a desired state at time $T$? We use a constructive method known as the Hilbert Uniqueness Method introduced by Lions (see [33, 34]). The idea is to build a control as the solution of a transposed problem associated to some suitable initial conditions. These initial conditions are obtained by calculating at zero time the solution of a backward problem. The control obtained by HUM is also an energy minimizing control.

The first question we pose deals with the existence of an exact control of the $\varepsilon$-problem. If such a control exists, the second and more interesting question is: does the exact control of the $\varepsilon$-problem converge, as $\varepsilon \to 0$, to the exact control of the homogenized problem? We are able to answer successfully both these questions.

The plan of the paper is the following one. In Section 2, we give the precise setting of the problem and recall some useful properties of specific functional spaces, introduced in [19] and [39] in the elliptic framework, suitable for the solutions of these kinds of interface problems. Then we recall the homogenization result from [14]. In Section 3, we state Theorem 3.1 which provides the exact controllability of the $\varepsilon$-problem. In order to seek an answer to the second question, we also state the exact controllability of two different problems of the same type of the homogenized ones corresponding to the cases $-1 < \gamma < 1$ (Theorem 3.2) and $\gamma = 1$ (Theorem 3.3). Theorem 3.1 is proved in Section 4. In Section 5, we detail only the noteworthy points of the proof of Theorem 3.2. On the contrary, we prove in details Theorem 3.3, which is more delicate since, when trying to use the same arguments as for the case $-1 < \gamma < 1$, some difficulties arise due to the lack of symmetry of the homogenized coupled system. Indeed, the usual existence and uniqueness results for second order evolution equations and the method developed by Lions in [33, 34] cannot be directly applied. We turn around this difficulty thanks to appropriate properties of the homogenized matrix. More

precisely, concerning the existence and uniqueness of the solution of a problem like the homogenized one in the case $\gamma = 1$, in Theorem 5.1, we recall the result proved in [28] which provides also the same estimate as in the symmetric case. Afterwards, in Theorem 5.2, we establish an existence and uniqueness result for the solution of the related transposed problem, which generalizes to the non-symmetric case the one proved in [35], Theorems 9.3 and 9.4, Chapter 3, Section 9 only for the symmetric case. Moreover, following again the idea contained in [33, 34], in Proposition 5.1, we get a suitable observability estimate that allows us to find the exact control.
Finally, in Section 6, we prove the main result of this paper, namely, we describe the asymptotic behavior of the $\varepsilon$-controllability problem. Again some difficulties arise when treating the case $\gamma = 1$. Indeed, one needs to exploit some homogenization results applied to the transposed problem at $\varepsilon$-level. To do that, it is necessary to study the asymptotic behavior of a stationary $\varepsilon$-problem, with weakly converging data (see Theorem 6.1). This is possible thanks to the homogenization result proved in [25].
Optimal control and exact controllability problems in domains with highly oscillating boundary are considered in [7]÷[11], [21]÷[23] and [42]. Moreover we refer to [33] and [4, 6] for the exact controllability of hyperbolic problems with oscillating coefficients in fixed and in perforated domains respectively, to [27, 28] and [29] for the optimal control of hyperbolic problems in composites with imperfect interface and of rigidity parameters of thin inclusions in composite materials respectively. In [16] and [17, 18] the authors study, respectively, the correctors and the approximate control for a class of parabolic equations with interfacial contact resistance, while in [20] the authors study the approximate controllability of linear parabolic equations in perforated domains. In [47], see also [46], the author studies the approximate controllability of a parabolic problem with highly oscillating coefficients in a fixed domain. The null controllability of semilinear heat equations in a fixed domain was done in [30]. The exact controllability and exact boundary controllability for semilinear wave equations can be found in [32] and [45], respectively.

## 2. Preliminaries

### *2.1. Position of the problem*

Let $\Omega$ be a connected open bounded subset of $\mathbb{R}^n$, $n \geq 2$, and $Y := ]0, l_1[\times \cdots \times]0, l_n[$ be the reference cell, with $l_i$, $i = 1, \dots, n$, positive real numbers.
We denote by $Y_1$ and $Y_2$ two nonempty open and disjoint subsets of $Y$ such that

$$Y := Y_1 \cup \overline{Y_2},$$

with $Y_1$ connected and $\Gamma := \partial Y_2$ Lipschitz continuous.
For any $k \in \mathbb{Z}^n$ we define the translated sets $Y_i^k$ and $\Gamma_k$ as follows:

$$Y_i^k := \; k_l + Y_i, \; i = 1, 2, \qquad \Gamma_k := \; k_l + \Gamma,$$

where $k_l = (k_1 l_1, \dots, k_n l_n)$.
Let $\{\varepsilon\}$ be a sequence of positive real numbers converging to zero and for any given $\varepsilon$ let us set

$$K_\varepsilon := \; \{k \in \mathbb{Z}^n |\; \varepsilon\Gamma_k \cap \Omega \neq \emptyset\}.$$

Then we define the two components of $\Omega$ and the interface respectively as follows:

$$\Omega_{i\varepsilon} := \; \Omega \cap \left\{ \bigcup_{k \in K_\varepsilon} \varepsilon Y_i^k \right\}, \; i = 1, 2 \quad \text{ and } \quad \Gamma^\varepsilon := \partial\Omega_{2\varepsilon}.$$

We assume that

$$\partial\Omega \cap \left( \bigcup_{k \in \mathbb{Z}^n} (\varepsilon \Gamma_k) \right) = \emptyset. \tag{2.1}$$

We explicitly observe that, by construction, the set $\Omega$ is decomposed into two components $\Omega = \Omega_{1\varepsilon} \cup \overline{\Omega_{2\varepsilon}}$ where $\Omega_{1\varepsilon}$ is a connected set, while $\Omega_{2\varepsilon}$ is a disconnected union of $\varepsilon$-periodic disjoint translated sets of $\varepsilon Y_2$. Moreover $\Gamma_\varepsilon$ is the interface separating the two components with $\partial\Omega \cap \Gamma_\varepsilon = \emptyset$ (see Fig. 1).

`figure1.pdf`

Figure 1: The two-component domain $\Omega$

In the sequel, we denote by

- $\tilde{}$ the zero extension to the whole of $\Omega$ of functions defined in $\Omega_{1\varepsilon}$ or $\Omega_{2\varepsilon}$;
- $\chi_E$ the characteristic function of any measurable set $E \subseteq \mathbb{R}^n$;
- $m_E(v) = \dfrac{1}{|E|}\displaystyle\int_E v\,dx$ the average on E of any function $v \in L^1(E)$.

Let us recall (see for istance [5]) that, as $\varepsilon \longrightarrow 0$,

$$\chi_{\Omega_{i\varepsilon}} \rightharpoonup \theta_i := \frac{|Y_i|}{|Y|} \quad \text{weakly in } L^2(\Omega), \text{ for } i = 1, 2, \tag{2.2}$$

$\theta_i$ being the proportion of the material occupying $\Omega_{i\varepsilon}$.

For any $\varepsilon > 0$, let us define the functional space $V^\varepsilon$, introduced in [3], as

$$V^\varepsilon := \{v_1 \in H^1(\Omega_{1\varepsilon}) |\; v_1 = 0 \,\text{on}\, \partial\Omega\},$$

which is a Banach space endowed with the norm

$$\|v_1\|_{V^\varepsilon} := \|\nabla v_1\|_{L^2(\Omega_{1\varepsilon})}.$$

The condition on $\partial\Omega$ in the definition of $V^\varepsilon$ has to be understood in a density sense, since we don't require any regularity on $\partial\Omega$. Namely, $V^\varepsilon$ is the closure, with respect to the $H^1(\Omega_{1\varepsilon})$-norm, of the set of the functions in $C^\infty(\Omega_{1\varepsilon})$ with a compact support contained in $\Omega$. This can be done in view of (2.1).

The first question we deal with concerns the study of the exact controllability of a hyperbolic imperfect transmission problem defined in the domain $\Omega$ previously described. More precisely, let $\zeta_\varepsilon := (\zeta_{1\varepsilon}, \zeta_{1\varepsilon}) \in L^2\left(0, T; L^2(\Omega_{1\varepsilon}) \times L^2(\Omega_{2\varepsilon})\right)$ be a control. For any fixed $T > 0$, let us consider the following problem

$$\begin{cases} u''_{1\varepsilon} - \operatorname{div}\left(A\left(\frac{x}{\varepsilon}\right)\nabla u_{1\varepsilon}\right) = \zeta_{1\varepsilon} & \text{in } \Omega_{1\varepsilon}\times]0,T[, \\ u''_{2\varepsilon} - \operatorname{div}\left(A\left(\frac{x}{\varepsilon}\right)\nabla u_{2\varepsilon}\right) = \zeta_{2\varepsilon} & \text{in } \Omega_{2\varepsilon}\times]0,T[, \\ A\left(\frac{x}{\varepsilon}\right)\nabla u_{1\varepsilon}\cdot n_{1\varepsilon} = -A\left(\frac{x}{\varepsilon}\right)\nabla u_{2\varepsilon}\cdot n_{2\varepsilon} & \text{on } \Gamma^\varepsilon\times]0,T[, \\ A\left(\frac{x}{\varepsilon}\right)\nabla u_{1\varepsilon}\cdot n_{1\varepsilon} = -\varepsilon^\gamma h\left(\frac{x}{\varepsilon}\right)(u_{1\varepsilon} - u_{2\varepsilon}) & \text{on } \Gamma^\varepsilon\times]0,T[, \\ u_{1\varepsilon} = 0 & \text{on } \partial\Omega\times]0,T[, \\ u_{1\varepsilon}(0) = U^0_{1\varepsilon}, \quad u'_{1\varepsilon}(0) = U^1_{1\varepsilon} & \text{in } \Omega_{1\varepsilon}, \\ u_{2\varepsilon}(0) = U^0_{2\varepsilon}, \quad u'_{2\varepsilon}(0) = U^1_{2\varepsilon} & \text{in } \Omega_{2\varepsilon}, \end{cases} \tag{2.3}$$

where $n_{i\varepsilon}$ is the unitary outward normal to $\Omega_{i\varepsilon}$, $i = 1, 2$, $-1 < \gamma \leq 1$ and

$$\begin{cases} \text{(i) } U^0_\varepsilon := \left(U^0_{1\varepsilon}, U^0_{2\varepsilon}\right) \in V^\varepsilon \times H^1(\Omega_{2\varepsilon}), \\ \text{(ii) } U^1_\varepsilon := \left(U^1_{1\varepsilon}, U^1_{2\varepsilon}\right) \in L^2(\Omega_{1\varepsilon}) \times L^2(\Omega_{2\varepsilon}). \end{cases} \tag{2.4}$$

We suppose that $A$ is a symmetric Y-periodic $n \times n$ matrix field in $M(\alpha, \beta, \Omega)$, that is

$$\begin{cases} (i)\ A \in \left(L^\infty(Y)^{n^2}\right) \text{ and } a_{ij} = a_{ji},\ 1 \leq i, j \leq n, \\ (ii)\ (A(x)\lambda, \lambda) \geq \alpha|\lambda|^2,\ |A(x)\lambda| \leq \beta|\lambda|, \end{cases} \tag{2.5}$$

for every $\lambda \in \mathbb{R}^n$ and a.e. in $\Omega$ where $\alpha, \beta \in \mathbb{R}$ with $0 < \alpha < \beta$. Moreover we suppose that $h$ is a Y-periodic function such that

$$h \in L^\infty(\Gamma) \text{ and } \exists\, h_0 \in \mathbb{R} \text{ such that } 0 < h_0 < h(y),\ y \text{ a.e. in } \Gamma. \tag{2.6}$$

In the sequel, for any $\varepsilon > 0$, we set

$$h^\varepsilon(x) := \ h\left(\frac{x}{\varepsilon}\right) \tag{2.7}$$

and

$$A^\varepsilon(x) := \ A\left(\frac{x}{\varepsilon}\right). \tag{2.8}$$

We underline that for clearness sake, throughout the paper, we denote by $u_\varepsilon\left(\zeta_\varepsilon\right) := \left(u_{1\varepsilon}\left(\zeta_\varepsilon\right), u_{2\varepsilon}\left(\zeta_\varepsilon\right)\right)$ the solution of problem (2.3) and where no ambiguity arises, we omit the explicit dependence on the control.

**Definition 2.1.** *System* (2.3) *is exactly controllable at time* $T > 0$, *if for every* $\left(U_\varepsilon^0, U_\varepsilon^1\right), \left(Z_\varepsilon^0, Z_\varepsilon^1\right)$ *in* $\left(V^\varepsilon \times H^1\left(\Omega_{2\varepsilon}\right)\right) \times \left(L^2\left(\Omega_{1\varepsilon}\right) \times L^2\left(\Omega_{2\varepsilon}\right)\right)$, *there exists a control* $\zeta_\varepsilon^{ex} := \left(\zeta_{1\varepsilon}^{ex}, \zeta_{2\varepsilon}^{ex}\right)$ *belonging to* $L^2\left(0, T; L^2\left(\Omega_{1\varepsilon}\right) \times L^2\left(\Omega_{2\varepsilon}\right)\right)$ *such that the corresponding solution* $u_\varepsilon$ *of problem* (2.3) *satisfies*

$$u_\varepsilon(T) = Z_\varepsilon^0, \ \ u_\varepsilon'(T) = Z_\varepsilon^1.$$

**Remark 2.1.** *It is well known that for a linear system, driving it to any state is equivalent to driving it to the null state and this is known as null controllability. Hence, in the sequel we study the null controllability of the considered systems.*

Therefore, (2.3) is *null controllable* if there exists a control $\zeta_\varepsilon^{ex} \in L^2\left(0, T; L^2\left(\Omega_{1\varepsilon}\right) \times L^2\left(\Omega_{2\varepsilon}\right)\right)$ such that $u_\varepsilon(T) = u_\varepsilon'(T) = 0$.

A second interesting question is the following: if the system (2.3) is exactly controllable, do the exact control and its corresponding solution converge, as $\varepsilon$ goes to zero, to the exact control of the homogenized problem and to the corresponding solution, respectively?
In this paper, we give positive answers to both questions.
Concerning the exact controllability, we use a constructive method known as the Hilbert Uniqueness Method introduced by Lions (see [33, 34]). The idea is to build a control as the solution of a transposed problem associated to some suitable initial conditions. These initial conditions are obtained by calculating at zero time the solution of a backward problem. The control obtained by HUM is also the energy minimizing control.
In order to answer the second question, we will use some homogenization results which were studied in [14, 15] and [25].

### *2.2. Recall of the asymptotic behavior of the $\varepsilon$-problem*

Let us define a class of function spaces that are suitable for the solutions of this particular kind of interface problems. They were introduced for the first time in [39] and successively in [19] in the framework of the study of the analogous stationary problem. Clearly, these spaces must take into account the geometry of the domain where the material is confined and both the boundary and interfacial conditions.
For any $\varepsilon > 0$ and $\gamma \in \mathbb{R}$, we set

$$H_\gamma^\varepsilon := \ \left\{v = (v_1, v_2) \mid v_1 \in V^\varepsilon \quad \text{and} \quad v_2 \in H^1(\Omega_{2\varepsilon})\right\}. \tag{2.9}$$

The space $H_\gamma^\varepsilon$ is a Hilbert space when equipped with the norm

$$\|v\|_{H_\gamma^\varepsilon}^2 := \ \|\nabla v_1\|_{L^2(\Omega_{1\varepsilon})}^2 + \|\nabla v_2\|_{L^2(\Omega_{2\varepsilon})}^2 + \varepsilon^\gamma \|v_1 - v_2\|_{L^2(\Gamma^\varepsilon)}^2.$$

Moreover, for every fixed $\varepsilon$ the norms of $H^\varepsilon_\gamma$ and $V^\varepsilon \times H^1(\Omega_{2\varepsilon})$ are equivalent, see [14] for details. It has also been proved that there exist two positive constants $C_1$ and $C_2$, independent of $\varepsilon$, such that

$$C_1\|v\|^2_{H^\varepsilon_\gamma} \le \|v\|^2_{V_\varepsilon \times H^1(\Omega_{2\varepsilon})} \le C_2(1+\varepsilon^{\gamma-1})\|v\|^2_{H^\varepsilon_\gamma} \quad \forall v \in H^\varepsilon_\gamma, \tag{2.10}$$

see [16].

We denote by $(H^\varepsilon_\gamma)'$ the dual of $H^\varepsilon_\gamma$. As proved in [16], for every fixed $\varepsilon$, the norms of $(H^\varepsilon_\gamma)'$ and $(V_\varepsilon)' \times (H^1(\Omega_{2\varepsilon}))'$ are equivalent. Moreover, if $(v_1, v_2) \in (V_\varepsilon)' \times (H^1(\Omega_{2\varepsilon}))'$ and $(u_1, u_2) \in V_\varepsilon \times H^1(\Omega_{2\varepsilon})$, then

$$\langle v, u\rangle_{(H^\varepsilon_\gamma)', H^\varepsilon_\gamma} = \langle v_1, u_1\rangle_{(V_\varepsilon)', V_\varepsilon} + \langle v_2, u_2\rangle_{H^1(\Omega_{2\varepsilon})', H^1(\Omega_{2\varepsilon})}\,.$$

For sake of simplicity, throughout this paper, we denote by $L^2_\varepsilon(\Omega) := L^2(\Omega_{1\varepsilon}) \times L^2(\Omega_{2\varepsilon})$. The space $L^2_\varepsilon(\Omega)$ will be equipped with the usual product norm, that is,

$$\|(w_1, w_2)\|^2_{L^2_\varepsilon(\Omega)} = \|w_1\|^2_{L^2(\Omega_{1\varepsilon})} + \|w_2\|^2_{L^2(\Omega_{2\varepsilon})} \quad \forall (w_1, w_2) \in L^2_\varepsilon(\Omega).$$

**Remark 2.2.** *We point out that $H^\varepsilon_\gamma$ is a separable and reflexive Hilbert space dense in $L^2_\varepsilon(\Omega)$. Furthermore, $H^\varepsilon_\gamma \subseteq L^2_\varepsilon(\Omega)$ with continuous imbedding. On the other hand, one has that $L^2_\varepsilon(\Omega) \subseteq \left(H^\varepsilon_\gamma\right)'$, with $L^2_\varepsilon(\Omega)$ separable Hilbert space. This means that the triple $(H^\varepsilon_\gamma, L^2_\varepsilon(\Omega), \left(H^\varepsilon_\gamma\right)')$ is an evolution triple. We refer the reader to [14, 15] for an in-depth analysis on this aspect.*

For reader's convenience, let us now recall the homogenization result proved in [14], [28].

**Theorem 2.1 ([14, 28]).** *Let $A_\varepsilon$ and $h_\varepsilon$ satisfy* (2.5) $\div$ (2.8). *Let $z_\varepsilon = (z_{1\varepsilon}, z_{2\varepsilon})$ be the solution of the following problem*

$$\begin{cases} z''_{i\varepsilon} - \operatorname{div}(A^\varepsilon \nabla z_{i\varepsilon}) = g_{i\varepsilon} & \text{in } \Omega_{i\varepsilon}\times]0,T[,\ i = 1,2, \\ A^\varepsilon \nabla z_{1\varepsilon}\cdot n_{1\varepsilon} = -A^\varepsilon \nabla z_{2\varepsilon}\cdot n_{2\varepsilon} & \text{on } \Gamma^\varepsilon\times]0,T[, \\ A^\varepsilon \nabla z_{1\varepsilon}\cdot n_{1\varepsilon} = -\varepsilon^\gamma h^\varepsilon (z_{1\varepsilon} - z_{2\varepsilon}) & \text{on } \Gamma^\varepsilon\times]0,T[, \\ z_{1\varepsilon} = 0 & \text{on } \partial\Omega\times]0,T[, \\ z_{1\varepsilon}(0) = Z^0_{1\varepsilon}, \quad z'_{1\varepsilon}(0) = Z^1_{1\varepsilon} & \text{in } \Omega_{1\varepsilon}, \\ z_{2\varepsilon}(0) = Z^0_{2\varepsilon}, \quad z'_{2\varepsilon}(0) = Z^1_{2\varepsilon} & \text{in } \Omega_{2\varepsilon} \end{cases} \tag{2.11}$$

*where $n_{i\varepsilon}$ is the unitary outward normal to $\Omega_{i\varepsilon}$, $i = 1, 2$, $-1 < \gamma \le 1$ and*

$$\begin{cases} \text{(i)} & g_\varepsilon := (g_{1\varepsilon}, g_{2\varepsilon}) \in L^2\left(0,T; L^2_\varepsilon(\Omega)\right), \\ \text{(ii)} & Z^0_\varepsilon := \left(Z^0_{1\varepsilon}, Z^0_{2\varepsilon}\right) \in H^\varepsilon_\gamma, \\ \text{(iii)} & Z^1_\varepsilon := \left(Z^1_{1\varepsilon}, Z^1_{2\varepsilon}\right) \in L^2_\varepsilon(\Omega). \end{cases} \tag{2.12}$$

*If*

$$\begin{cases} \text{(i)} \quad \widetilde{Z^0_\varepsilon} \rightharpoonup Z^0 := (Z^0_1, Z^0_2) \text{ weakly in } \left[L^2(\Omega)\right]^2, \text{ with } Z^0_2 \in H^1_0(\Omega) \text{ if } -1 < \gamma < 1, \\ \\ \text{(ii)} \quad \widetilde{Z^1_\varepsilon} \rightharpoonup Z^1 := (Z^1_1, Z^1_2) \text{ weakly in } \left[L^2(\Omega)\right]^2, \\ \\ \text{(iii)} \quad \|Z^0_\varepsilon\|_{H^\varepsilon_\gamma} \le C, \end{cases} \tag{2.13}$$

*with $C$ positive constant independent of $\varepsilon$, and*

$$(\widetilde{g_{1\varepsilon}}, \widetilde{g_{2\varepsilon}}) \rightharpoonup (g_1, g_2) \text{ weakly in } L^2\left(0,T;\left(L^2(\Omega)\right)^2\right), \tag{2.14}$$

*then there exists an extension operator $P_1^\varepsilon \in \mathcal{L}(L^\infty(0,T;H^k(\Omega_{1\varepsilon})); L^\infty(0,T;H^k(\Omega)))$, for $k=1,2$, such that*

$$\begin{cases} P_1^\varepsilon z_{1\varepsilon} \rightharpoonup z_1 & \text{weakly* in } L^\infty\left(0,T;H_0^1(\Omega)\right), \\ \widetilde{z_{1\varepsilon}} \rightharpoonup \theta_1 z_1 & \text{weakly* in } L^\infty\left(0,T;L^2(\Omega)\right), \\ \widetilde{z_{2\varepsilon}} \rightharpoonup z_2 & \text{weakly* in } L^\infty\left(0,T;L^2(\Omega)\right), \end{cases}$$

$$\begin{cases} P_1^\varepsilon z'_{1\varepsilon} \rightharpoonup z'_1 & \text{weakly* in } L^\infty\left(0,T;L^2(\Omega)\right), \\ \widetilde{z'_{1\varepsilon}} \rightharpoonup \theta_1 z'_1 & \text{weakly* in } L^\infty\left(0,T;L^2(\Omega)\right), \\ \widetilde{z'_{2\varepsilon}} \rightharpoonup z'_2 & \text{weakly* in } L^\infty\left(0,T;L^2(\Omega)\right) \end{cases}$$

*and*

$$A^\varepsilon\widetilde{\nabla z_{1\varepsilon}} + A^\varepsilon\widetilde{\nabla z_{2\varepsilon}} \rightharpoonup A^0\nabla z_1 \quad \text{weakly* in } L^\infty\left(0,T;\left[L^2(\Omega)\right]^n\right),$$

*where $\theta_1$ and $\theta_2$ are given by (2.2).*
*Furthermore*

$$\begin{cases} A^\varepsilon\widetilde{\nabla z_{1\varepsilon}} \rightharpoonup A^0\nabla z_1 & \text{weakly* in } L^\infty\left(0,T;\left[L^2(\Omega)\right]^n\right), \\ A^\varepsilon\widetilde{\nabla z_{2\varepsilon}} \rightharpoonup 0 & \text{weakly* in } L^\infty\left(0,T;\left[L^2(\Omega)\right]^n\right). \end{cases}$$

*where the matrix $A^0$ is defined as follows:*

$$A^0\lambda = \frac{1}{|Y|}\int_{Y_1} A\nabla w_\lambda \, dy \tag{2.15}$$

*with $w_\lambda \in H^1(Y_1)$ solution, for any $\lambda \in \mathbb{R}^n$, of*

$$\begin{cases} -\mathrm{div}\,(A\nabla w_\lambda) = 0 & \text{in } Y_1, \\ (A\nabla w_\lambda)\cdot n_1 = 0 & \text{on } \Gamma, \\ w_\lambda - \lambda\cdot y \qquad Y\text{-periodic}, \\ \dfrac{1}{|Y_1|}\displaystyle\int_{Y_1}(w_\lambda - \lambda\cdot y)\,dy = 0. \end{cases} \tag{2.16}$$

*The homogenized problems satisfied by the couple $(z_1, z_2)$ are different for the two cases $-1<\gamma<1$ and $\gamma=1$.*
***Case** $-1<\gamma<1$: The function $z_2$ is given by $z_2 = \theta_2 z_1$ where $z_1 \in L^2(0,T;H_0^1(\Omega))$ with $z'_1 \in L^2\left(0,T;L^2(\Omega)\right)$ is the unique solution of the following homogenized problem*

$$\begin{cases} z''_1 - \mathrm{div}\,\left(A^0\nabla z_1\right) = g_1 + g_2 & \text{in } \Omega\times]0,T[, \\ z_1 = 0 & \text{on } \partial\Omega\times]0,T[, \\ z_1(0) = Z_1^0 + Z_2^0 & \text{in } \Omega, \\ z'_1(0) = Z_1^1 + Z_2^1 & \text{in } \Omega. \end{cases}$$

***Case*** *$\gamma = 1$: The pair $(z_1, z_2) \in L^2\left(0,T; H_0^1(\Omega) \times L^2(\Omega)\right)$ with $(z_1', z_2') \in L^2\left(0,T; \left(L^2(\Omega)\right)^2\right)$ is the unique solution of the coupled system*

$$\begin{cases} \theta_1 z_1'' - \operatorname{div}\left(A^0 \nabla z_1\right) + c_h\left(\theta_2 z_1 - z_2\right) = g_1 & \text{in } \Omega\times]0,T[, \\ z_2'' - c_h\left(\theta_2 z_1 - z_2\right) = g_2 & \text{in } \Omega\times]0,T[, \\ z_1 = 0 & \text{on } \partial\Omega\times]0,T[, \\ z_1(0) = \dfrac{Z_1^0}{\theta_1},\ z_1'(0) = \dfrac{Z_1^1}{\theta_1} & \text{in } \Omega, \\ z_2(0) = Z_2^0,\ z_2'(0) = Z_2^1 & \text{in } \Omega, \end{cases} \tag{2.17}$$

*where $c_h = \dfrac{1}{|Y_2|}\displaystyle\int_\Gamma h(y)d\sigma_y > 0$ and $\theta_i$, $i = 1,2$ are defined in* (2.2).

**Remark 2.3.** *The matrix field $A^0$ is the same obtained by D. Cioranescu and J. Saint Jean Paulin, in [3], for the homogenization of the elliptic problem in the perforated domain $\Omega_{1\varepsilon}$ with a Neumann condition on the boundary of the holes.*
*Moreover let us observe that (see for instance [5]), $A^0$ is a symmetric constant matrix such that*

$$A^0 \in M\left(\alpha, \beta, \Omega\right), \tag{2.18}$$

*where $\alpha$ and $\beta$ are defined in* (2.5).

**Remark 2.4.** *The boundness (iii) in* (2.13) *is necessary in order to have a priori estimates for the solution of problem* (2.11).

**Remark 2.5.** *Observe that for $\gamma = 1$ the homogenized problem takes into account also the data $h$ in the boundary condition of* (2.11). *Moreover problem* (2.17) *is equivalent to the following one*

$$\begin{cases} \theta_1 z_1'' - \operatorname{div}\left(A^0 \nabla z_1\right) + c_h \theta_2 z_1 - c_h^2 \theta_2 \displaystyle\int_0^t \mathrm{K}(t,s) z_1(s) ds = F & \text{in } \Omega\times]0,T[, \\ z_1 = 0 & \text{on } \partial\Omega, \\ z_1(0) = \dfrac{Z_1^0}{\theta_1} & \text{in } \Omega, \\ z_1'(0) = \dfrac{Z_1^1}{\theta_1} & \text{in } \Omega. \end{cases} \tag{2.19}$$

*with*

$$\mathrm{K}(t,s) := \frac{1}{c_h} \sin\left(\sqrt{c_h}(t-s)\right) \tag{2.20}$$

*and*

$$F(x,t) := g_1 + c_h Z_2^0(x) \cos\left(\sqrt{c_h} t\right) + \sqrt{c_h} Z_2^1(x) \sin\left(\sqrt{c_h} t\right) + c_h \int_0^t \mathrm{K}(t,s) g_2(x,s) ds.$$

*Moreover, $z_2$ is given by*

$$z_2(x,t) = Z_2^0(x) \cos\left(\sqrt{c_h} t\right) + \frac{Z_2^1}{\sqrt{c_h}} \sin\left(\sqrt{c_h} t\right) + \int_0^t \mathrm{K}(t,s)\left(c_h \theta_2 z_1(x,s) + g_2(x,s)\right) ds.$$

*In other words, for* $\gamma = 1$ *the limit function* $z_1$ *is the unique solution of the second order evolution problem* (2.19) *with a linear memory effect, due to the presence of the term* $c_h^2 \theta_2 \int_0^t \mathrm{K}(t,s) z_1(s)\,ds$*,* $\mathrm{K}$ *being a periodic memory kernel explicitly computed in* (2.20)*.*

For any $\varepsilon > 0$, we set

$$W^\varepsilon := \left\{ v = (v_1, v_2) \in L^2\left(0, T; V^\varepsilon \times H^1(\Omega_{2\varepsilon})\right) \text{ such that } v' = (v_1', v_2') \in L^2\left(0, T; L^2_\varepsilon(\Omega)\right) \right\}, \tag{2.21}$$

which is a Hilbert space if equipped with the norm

$$\|v\|_{W^\varepsilon} = \|v_1\|_{L^2(0,T;V^\varepsilon)} + \|v_2\|_{L^2(0,T;H^1(\Omega_{2\varepsilon}))} + \|v_1'\|_{L^2(0,T;L^2(\Omega_{1\varepsilon}))} + \|v_2'\|_{L^2(0,T;L^2(\Omega_{2\varepsilon}))}.$$

Thanks to Remark 2.2, by using an approach to evolutionary problems based on evolution triples, as far as the weak formulation of problem (2.11) is concerned, we assume as precise formulation of the formal problem the following one (see [14]):

$$\begin{cases} \text{Find } z_\varepsilon = (z_{1\varepsilon}, z_{2\varepsilon}) \text{ in } W^\varepsilon \text{ such that} \\ \langle z_{1\varepsilon}'', v_1 \rangle_{(V^\varepsilon)', V^\varepsilon} + \langle z_{2\varepsilon}'', v_2 \rangle_{(H^1(\Omega_{2\varepsilon}))', H^1(\Omega_{2\varepsilon})} + \displaystyle\int_{\Omega_{1\varepsilon}} A^\varepsilon \nabla z_{1\varepsilon} \nabla v_1 \, dx + \int_{\Omega_{2\varepsilon}} A^\varepsilon \nabla z_{2\varepsilon} \nabla v_2 \, dx \\ + \varepsilon^\gamma \displaystyle\int_{\Gamma^\varepsilon} h^\varepsilon (z_{1\varepsilon} - z_{2\varepsilon})(v_1 - v_2)\, d\sigma_x = \int_{\Omega_{1\varepsilon}} g_{1\varepsilon} v_1 \, dx + \int_{\Omega_{2\varepsilon}} g_{2\varepsilon} v_2 \, dx, \\ \forall (v_1, v_2) \in V^\varepsilon \times H^1(\Omega_{2\varepsilon}) \text{ in } \mathcal{D}'(0,T), \\ z_{1\varepsilon}(0) = Z^0_{1\varepsilon}, \quad z_{1\varepsilon}'(0) = Z^1_{1\varepsilon} \quad \text{in } \Omega_{1\varepsilon}, \\ z_{2\varepsilon}(0) = Z^0_{2\varepsilon}, \quad z_{2\varepsilon}'(0) = Z^2_{2\varepsilon} \quad \text{in } \Omega_{2\varepsilon}. \end{cases} \tag{2.22}$$

As observed in [14], an abstract Galerkin's method provides the existence and uniqueness result for the solution of problem (2.22), for any $\varepsilon > 0$, and also some a priori estimates.

**Theorem 2.2 ([14]).** *Let* $T \in ]0, +\infty[$*. Let* $H^\varepsilon_\gamma$ *and* $W_\varepsilon$ *be defined as in* (2.9) *and* (2.21)*,* $h_\varepsilon$ *and* $A_\varepsilon$ *as in* (2.5)$\div$(2.8)*. Under assumptions* (2.12)*,* (2.13) *and* (2.14)*, problem* (2.11) *admits a unique weak solution* $z_\varepsilon \in W_\varepsilon$*. Moreover, there exists a positive constant* $C$*, independent of* $\varepsilon$*, such that*

$$\|z_\varepsilon\|_{L^\infty(0,T;H^\varepsilon_\gamma)} + \|z_\varepsilon'\|_{L^\infty(0,T;L^2_\varepsilon(\Omega))} \leq C \left( \|Z^0_\varepsilon\|_{H^\varepsilon_\gamma} + \|Z^1_\varepsilon\|_{L^2_\varepsilon(\Omega)} + \|g_\varepsilon\|_{L^2(0,T;L^2_\varepsilon(\Omega))} \right).$$

Let us point out that, for any fixed $\varepsilon$, the solution of problem (2.11) has some further properties (see [35], Chapter 3, Theorem 8.2). In fact, under the same hypotheses of Theorem 2.2, the unique solution $z_\varepsilon$ of problem (2.11) is such that

$$z_\varepsilon \in C\left([0,T]; H^\varepsilon_\gamma\right), \; z_\varepsilon' \in C\left([0,T]; L^2_\varepsilon(\Omega)\right).$$

## 3. Statement of the main results

This section is devoted to the statements of our main results. More precisely, in Subsection 3.1, we give exact controllability results for problem (2.3) for any $\varepsilon > 0$, as well as for two evolution problems posed in a fixed domain. In Subsection 3.2, we state that for both cases $-1 < \gamma < 1$ and $\gamma = 1$, the exact control of the $\varepsilon$-problem and its corresponding solution converge, as $\varepsilon$ goes to zero, respectively to the exact control and to the solution of the homogenized problems.

### *3.1. Exact controllability results*

In the following theorem, we give a positive answer to the first question posed in Sections 1 and 2. Indeed we show that for any fixed $\varepsilon > 0$, problem (2.3) is exactly controllable at time $T$ (see Remark 2.1).

**Theorem 3.1.** *Let $-1 < \gamma \leq 1$ and $T > 0$. Suppose* (2.4)÷(2.8) *hold. Then, for given $U^0_\varepsilon \in V^\varepsilon \times H^1(\Omega_{2\varepsilon})$, $U^1_\varepsilon \in L^2_\varepsilon(\Omega)$ there exists a control $\zeta^{ex}_\varepsilon := (\zeta^{ex}_{1\varepsilon}, \zeta^{ex}_{2\varepsilon}) \in L^2\left(0,T;L^2_\varepsilon(\Omega)\right)$ such that the corresponding solution of problem* (2.3) *satisfies*

$$u_\varepsilon(T) = u'_\varepsilon(T) = 0. \tag{3.1}$$

This theorem will be proved in Section 4. We want to show that the exact control (found in Theorem 3.1) and the corresponding solution converge, as $\varepsilon \to 0$, respectively to the exact control and to the solution of the homogenized problem. As there are two different homogenized problems for $-1 < \gamma < 1$ and $\gamma = 1$ (see [14]), we will treat these two cases separately.
To do that, in Section 5, we will prove the following results (cfr. Remark 2.1):

**Theorem 3.2.** *Let $T > 0$ and $A^0$ be the matrix defined in Theorem 2.1 by* (2.15) *and* (2.16). *The system*

$$\begin{cases} v'' - \text{div}\,\left(A^0 \nabla v\right) = \zeta & \text{in } \Omega \times ]0,T[, \\ v = 0 & \text{on } \partial\Omega \times ]0,T[, \\ v(0) = V^0 & \text{in } \Omega, \\ v'(0) = V^1 & \text{in } \Omega \end{cases} \tag{3.2}$$

*is exactly controllable, that is, for given $\left(V^0, V^1\right) \in H^1_0(\Omega) \times L^2(\Omega)$, there exists a control $\bar\zeta^{ex} \in L^2(0,T;L^2(\Omega))$ such that the corresponding solution satisfies $v(T) = v'(T) = 0$.*

**Theorem 3.3.** *Let $T > 0$ and $A^0$ be the matrix defined in Theorem 2.1 by* (2.15) *and* (2.16). *The system*

$$\begin{cases} \theta_1 v_1'' - \text{div}\,\left(A^0 \nabla v_1\right) + c_h\left(\theta_2 v_1 - v_2\right) = \zeta_1 & \text{in } \Omega \times ]0,T[, \\ v_2'' - c_h\left(\theta_2 v_1 - v_2\right) = \zeta_2 & \text{in } \Omega \times ]0,T[, \\ v_1 = 0 & \text{on } \partial\Omega \times ]0,T[, \\ v_1(0) = V^0_1,\ v_1'(0) = V^1_1 & \text{in } \Omega, \\ v_2(0) = V^0_2,\ v_2'(0) = V^1_2 & \text{in } \Omega, \end{cases} \tag{3.3}$$

*where $c_h = \dfrac{1}{|Y_2|}\displaystyle\int_\Gamma h(y)\,d\sigma_y > 0$ and $\theta_i$, $i = 1,2$, are defined in* (2.2), *is exactly controllable, that is, for given $\left(V^0, V^1\right) \in (H^1_0(\Omega) \times L^2(\Omega)) \times (L^2(\Omega))^2$, there exists a control $\bar\zeta^{ex} := (\bar\zeta^{ex}_1, \bar\zeta^{ex}_2) \in L^2(0,T;(L^2(\Omega))^2)$ such that the corresponding solution satisfies*

$$v_i(T) = v_i'(T) = 0, \ i = 1,2. \tag{3.4}$$

### *3.2. Limit behaviour of the exact controllability problem*

The following theorem, which is the main result of our work, gives a positive answer to the second question posed in Sections 1 and 2.

**Theorem 3.4.** *Let $T > 0$ and $(U^0_\varepsilon, U^1_\varepsilon) \in H^\varepsilon_\gamma \times L^2_\varepsilon(\Omega)$ satisfies*

$$\begin{cases} \text{(i)} \quad \widetilde{U^0_\varepsilon} \rightharpoonup U^0 := (U^0_1, U^0_2) \ \textit{weakly in } \big[L^2(\Omega)\big]^2, \ \textit{with } U^0_2 \in H^1_0(\Omega), \ \textit{if } -1 < \gamma < 1, \\[2ex] \text{(ii)} \quad \widetilde{U^1_\varepsilon} \rightharpoonup U^1 := (U^1_1, U^1_2) \ \textit{weakly in } \big[L^2(\Omega)\big]^2, \\[2ex] \text{(iii)} \quad \|U^0_\varepsilon\|_{H^\varepsilon_\gamma} \le C, \end{cases} \tag{3.5}$$

*with $C$ positive constant independent of $\varepsilon$.*
*Further, assume that (2.5) $\div$ (2.8) hold. Let $\theta_i$, $i = 1, 2$, be given in (2.2) and $u_\varepsilon(\zeta^{ex}_\varepsilon) = (u_{1\varepsilon}(\zeta^{ex}_\varepsilon), u_{2\varepsilon}(\zeta^{ex}_\varepsilon))$ be the solution of problem (2.3) where as control $\zeta_\varepsilon$ we take the exact control $\zeta^{ex}_\varepsilon = (\zeta^{ex}_{1\varepsilon}, \zeta^{ex}_{2\varepsilon}) \in L^2(0, T; L^2_\varepsilon(\Omega))$ given in Theorem 3.1.*

***Case*** $-1 < \gamma < 1$*: There exist $\zeta^{ex}_1 \in L^2(0, T; L^2(\Omega))$, a function $u_1(\zeta^{ex}_1)$ and an extension operator $P^\varepsilon_1 \in \mathcal{L}(L^\infty(0, T; H^k(\Omega_{1\varepsilon})); L^\infty(0, T; H^k(\Omega)))$, for $k = 1, 2$, such that*

$$\begin{cases} \widetilde{\zeta^{ex}_{1\varepsilon}} \rightharpoonup \theta_1 \zeta^{ex}_1 & \textit{weakly in } L^2(0, T; L^2(\Omega)), \\ \widetilde{\zeta^{ex}_{2\varepsilon}} \rightharpoonup \theta_2 \zeta^{ex}_1 & \textit{weakly in } L^2(0, T; L^2(\Omega)), \end{cases} \tag{3.6}$$

$$\begin{cases} P^\varepsilon_1 u_{1\varepsilon}(\zeta^{ex}_\varepsilon) \rightharpoonup u_1(\zeta^{ex}_1) & \textit{weakly* in } L^\infty\big(0, T; H^1_0(\Omega)\big), \\ \widetilde{u_{1\varepsilon}(\zeta^{ex}_\varepsilon)} \rightharpoonup \theta_1 u_1(\zeta^{ex}_1) & \textit{weakly* in } L^\infty\big(0, T; L^2(\Omega)\big), \\ \widetilde{u_{2\varepsilon}(\zeta^{ex}_\varepsilon)} \rightharpoonup \theta_2 u_1(\zeta^{ex}_1) & \textit{weakly* in } L^\infty\big(0, T; L^2(\Omega)\big), \end{cases} \tag{3.7}$$

*and*

$$\begin{cases} P^\varepsilon_1 u'_{1\varepsilon}(\zeta^{ex}_\varepsilon) \rightharpoonup u'_1(\zeta^{ex}_1) & \textit{weakly* in } L^\infty\big(0, T; L^2(\Omega)\big), \\ \widetilde{u'_{1\varepsilon}(\zeta^{ex}_\varepsilon)} \rightharpoonup \theta_1 u'_1(\zeta^{ex}_1) & \textit{weakly* in } L^\infty\big(0, T; L^2(\Omega)\big), \\ \widetilde{u'_{2\varepsilon}(\zeta^{ex}_\varepsilon)} \rightharpoonup \theta_2 u'_1(\zeta^{ex}_1) & \textit{weakly* in } L^\infty\big(0, T; L^2(\Omega)\big). \end{cases} \tag{3.8}$$

*Moreover the function $u_1 := u_1(\zeta^{ex}_1) \in L^2(0, T; H^1_0(\Omega))$, with $u'_1 := u'_1(\zeta^{ex}_1) \in L^2\big(0, T; L^2(\Omega)\big)$, is the unique solution of the following homogenized problem*

$$\begin{cases} u''_1 - \operatorname{div}\big(A^0 \nabla u_1\big) = \zeta^{ex}_1 & \textit{in } \Omega \times ]0, T[, \\ u_1 = 0 & \textit{on } \partial\Omega \times ]0, T[, \\ u_1(0) = U^0_1 + U^0_2 & \textit{in } \Omega, \\ u'_1(0) = U^1_1 + U^1_2 & \textit{in } \Omega, \end{cases} \tag{3.9}$$

*where $\zeta^{ex}_1 \in L^2(0, T; L^2(\Omega))$ is the exact control in Theorem 3.2 with $\big(V^0, V^1\big) = \big(U^0_1 + U^0_2, U^1_1 + U^1_2\big)$.*

***Case*** $\gamma = 1$*: There exist $\zeta^{ex} := (\zeta^{ex}_1, \zeta^{ex}_2) \in L^2(0, T; (L^2(\Omega))^2)$, a function $u(\zeta^{ex}) = (u_1(\zeta^{ex}), u_2(\zeta^{ex}))$ and an extension operator $P^\varepsilon_1 \in \mathcal{L}(L^\infty(0, T; H^k(\Omega_{1\varepsilon})); L^\infty(0, T; H^k(\Omega)))$, for $k = 1, 2$, such that*

$$\begin{cases} \widetilde{\zeta^{ex}_{1\varepsilon}} \rightharpoonup \zeta^{ex}_1 & \textit{weakly in } L^2(0, T; L^2(\Omega)), \\ \widetilde{\zeta^{ex}_{2\varepsilon}} \rightharpoonup \zeta^{ex}_2 & \textit{weakly in } L^2(0, T; L^2(\Omega)), \end{cases} \tag{3.10}$$

$$\begin{cases} P^\varepsilon_1 u_{1\varepsilon}(\zeta^{ex}_\varepsilon) \rightharpoonup u_1(\zeta^{ex}) & \textit{weakly* in } L^\infty\big(0, T; H^1_0(\Omega)\big), \\ \widetilde{u_{1\varepsilon}(\zeta^{ex}_\varepsilon)} \rightharpoonup \theta_1 u_1(\zeta^{ex}) & \textit{weakly* in } L^\infty\big(0, T; L^2(\Omega)\big), \\ \widetilde{u_{2\varepsilon}(\zeta^{ex}_\varepsilon)} \rightharpoonup u_2(\zeta^{ex}) & \textit{weakly* in } L^\infty\big(0, T; L^2(\Omega)\big) \end{cases} \tag{3.11}$$

*and*

$$\begin{cases} P_1^\varepsilon u'_{1\varepsilon}(\zeta_\varepsilon^{ex}) \rightharpoonup u'_1(\zeta^{ex}) & \textit{weakly* in } L^\infty\left(0,T;L^2(\Omega)\right), \\ \widetilde{u'_{1\varepsilon}(\zeta_\varepsilon^{ex})} \rightharpoonup \theta_1 u'_1(\zeta^{ex}) & \textit{weakly* in } L^\infty\left(0,T;L^2(\Omega)\right), \\ \widetilde{u'_{2\varepsilon}(\zeta_\varepsilon^{ex})} \rightharpoonup u'_2(\zeta^{ex}) & \textit{weakly* in } L^\infty\left(0,T;L^2(\Omega)\right). \end{cases} \tag{3.12}$$

*Moreover the pair* $(u_1,u_2) := (u_1(\zeta^{ex}), u_2(\zeta^{ex})) \in L^2\left(0,T;H_0^1(\Omega)\times L^2(\Omega)\right)$*, with* $(u'_1(\zeta^{ex}), u'_2(\zeta^{ex})) \in L^2\left(0,T;(L^2(\Omega))^2\right)$*, is the unique solution of the coupled system*

$$\begin{cases} \theta_1 u''_1 - \operatorname{div}\left(A^0\nabla u_1\right) + c_h\left(\theta_2 u_1 - u_2\right) = \zeta_1^{ex} & \textit{in } \Omega\times]0,T[, \\ u''_2 - c_h\left(\theta_2 u_1 - u_2\right) = \zeta_2^{ex} & \textit{in } \Omega\times]0,T[, \\ u_1 = 0 & \textit{on } \partial\Omega\times]0,T[, \\ u_1(0) = \dfrac{U_1^0}{\theta_1},\ u'_1(0) = \dfrac{U_1^1}{\theta_1} & \textit{in } \Omega, \\ u_2(0) = U_2^0,\ u'_2(0) = U_2^1 & \textit{in } \Omega, \end{cases} \tag{3.13}$$

*where* $c_h = \dfrac{1}{|Y_2|}\displaystyle\int_\Gamma h(y)d\sigma_y > 0$ *and* $(\zeta_1^{ex},\zeta_2^{ex}) \in L^2\left(0,T;(L^2(\Omega))^2\right)$ *is the exact control given by Theorem 3.3, with* $\left(V_1^0,V_2^0\right) = \left(\dfrac{U_1^0}{\theta_1}, U_2^0\right)$ *and* $\left(V_1^1,V_2^1\right) = \left(\dfrac{U_1^1}{\theta_1}, U_2^1\right)$*.*

Let us observe that by (3.5)ii), $U_1^0$ is in fact in $H_0^1(\Omega)$ (see [14], Remark 2.7 for details).

## 4. Proof of the exact controllability for the $\varepsilon-$problem

In this section, by using the Hilbert Uniqueness Method introduced by Lions (see [33, 34]), we prove the exact controllability of system (2.3), for fixed $\varepsilon$, stated in Theorem 3.1.
For $T>0$, let $\left(\varphi_\varepsilon^0,\varphi_\varepsilon^1\right) \in L^2_\varepsilon(\Omega)\times\left(H_\gamma^\varepsilon\right)'$ and consider the problem

$$\begin{cases} \varphi''_{i\varepsilon} - \operatorname{div}(A^\varepsilon\nabla\varphi_{i\varepsilon}) = 0 & \text{in } \Omega_{i\varepsilon}\times]0,T[,\ i=1,2, \\ A^\varepsilon\nabla\varphi_{1\varepsilon}\cdot n_{1\varepsilon} = -A^\varepsilon\nabla\varphi_{2\varepsilon}\cdot n_{2\varepsilon} & \text{on } \Gamma^\varepsilon\times]0,T[, \\ A^\varepsilon\nabla\varphi_{1\varepsilon}\cdot n_{1\varepsilon} = -\varepsilon^\gamma h^\varepsilon(\varphi_{1\varepsilon}-\varphi_{2\varepsilon}) & \text{on } \Gamma^\varepsilon\times]0,T[, \\ \varphi_{1\varepsilon} = 0 & \text{on } \partial\Omega\times]0,T[, \\ \varphi_{1\varepsilon}(0) = \varphi_{1\varepsilon}^0, \quad \varphi'_{1\varepsilon}(0) = \varphi_{1\varepsilon}^1 & \text{in } \Omega_{1\varepsilon}, \\ \varphi_{2\varepsilon}(0) = \varphi_{2\varepsilon}^0, \quad \varphi'_{2\varepsilon}(0) = \varphi_{2\varepsilon}^1 & \text{in } \Omega_{2\varepsilon}, \end{cases} \tag{4.1}$$

where $n_{i\varepsilon}$ is the unitary outward normal to $\Omega_{i\varepsilon},\ i=1,2$.
Since the initial data are in a weak space, in order to give an appropriate definition of weak solution of problem (4.1), one needs to apply the so called transposition method (see [35], Chapter 3, Section 9). To this aim for every $f_\varepsilon := (f_{1\varepsilon}, f_{2\varepsilon}) \in L^2(0,T;L^2_\varepsilon(\Omega))$, let us consider the following backward

problem

$$\begin{cases} \psi''_{i\varepsilon} - \operatorname{div}(A^\varepsilon \nabla \psi_{i\varepsilon}) = f_{i\varepsilon} & \text{in } \Omega_i \times ]0,T[,\ i=1,2, \\ A^\varepsilon \nabla \psi_{1\varepsilon} \cdot n_{1\varepsilon} = -A^\varepsilon \nabla \psi_{2\varepsilon} \cdot n_{2\varepsilon} & \text{on } \Gamma^\varepsilon \times ]0,T[, \\ A^\varepsilon \nabla \psi_{1\varepsilon} \cdot n_{1\varepsilon} = -\varepsilon^\gamma h^\varepsilon (\psi_{1\varepsilon} - \psi_{2\varepsilon}) & \text{on } \Gamma^\varepsilon \times ]0,T[, \\ \psi_{1\varepsilon} = 0 & \text{on } \partial\Omega \times ]0,T[, \\ \psi_{1\varepsilon}(T) = \psi'_{1\varepsilon}(T) = 0 & \text{in } \Omega_{1\varepsilon}, \\ \psi_{2\varepsilon}(T) = \psi'_{2\varepsilon}(T) = 0 & \text{in } \Omega_{2\varepsilon}. \end{cases} \tag{4.2}$$

As previously, for clearness sake, throughout the paper, we denote by $\psi_\varepsilon(f_\varepsilon) := (\psi_{1\varepsilon}(f_\varepsilon), \psi_{2\varepsilon}(f_\varepsilon))$ the solution of problem (4.2) and where no ambiguity arises, we omit the explicit dependence on the right hand member. As the initial conditions of problem (4.2) are in right spaces, we can give the following definition

**Definition 4.1.** *For any fixed* $\left(\varphi^0_\varepsilon, \varphi^1_\varepsilon\right) \in L^2_\varepsilon(\Omega) \times \left(H^\varepsilon_\gamma\right)'$, *we say that a function* $\varphi_\varepsilon := (\varphi_{1\varepsilon}, \varphi_{2\varepsilon}) \in L^2(0,T; L^2_\varepsilon(\Omega))$ *is a solution of problem* (4.1), *in the sense of transposition, if it satisfies*

$$\begin{aligned} &\int_0^T \int_{\Omega_{1\varepsilon}} \varphi_{1\varepsilon} \left(\psi''_{1\varepsilon} - \operatorname{div}(A^\varepsilon \nabla \psi_{1\varepsilon})\right) dxdt + \int_0^T \int_{\Omega_{2\varepsilon}} \varphi_{2\varepsilon} \left(\psi''_{2\varepsilon} - \operatorname{div}(A^\varepsilon \nabla \psi_{2\varepsilon})\right) dxdt \\ &\qquad = - \int_{\Omega_{1\varepsilon}} \varphi^0_{1\varepsilon} \psi'_{1\varepsilon}(0) dx + \left\langle \varphi^1_{1\varepsilon}, \psi_{1\varepsilon}(0) \right\rangle_{(V^\varepsilon)', V^\varepsilon} \\ &\qquad\quad - \int_{\Omega_{2\varepsilon}} \varphi^0_{2\varepsilon} \psi'_{2\varepsilon}(0) dx + \left\langle \varphi^1_{2\varepsilon}, \psi_{2\varepsilon}(0) \right\rangle_{(H^1(\Omega_{2\varepsilon}))', H^1(\Omega_{2\varepsilon})} \end{aligned} \tag{4.3}$$

*for every* $\psi_\varepsilon$ *unique solution of problem* (4.2).

By classical results (see [35], Chapter 3, Section 9, Theorems 9.3 and 9.4), problem (4.1) admits a unique solution $\varphi_\varepsilon \in C\left([0,T]; L^2_\varepsilon(\Omega)\right) \cap C^1\left([0,T]; \left(H^\varepsilon_\gamma\right)'\right)$ satisfying the estimate

$$\|\varphi_\varepsilon\|_{L^\infty(0,T;L^2_\varepsilon(\Omega))} + \|\varphi'_\varepsilon\|_{L^\infty(0,T;(H^\varepsilon_\gamma)')} \le C(\|\varphi^0_\varepsilon\|_{L^2_\varepsilon(\Omega)} + \|\varphi^1_\varepsilon\|_{(H^\varepsilon_\gamma)'}), \tag{4.4}$$

with $C$ positive constant independent of $\varepsilon$.

Now, let $\psi_\varepsilon \in C\left([0,T]; H^\varepsilon_\gamma\right) \cap C^1\left([0,T]; L^2_\varepsilon(\Omega)\right)$ be the unique solution of the backward problem

$$\begin{cases} \psi''_{i\varepsilon} - \operatorname{div}(A^\varepsilon \nabla \psi_{i\varepsilon}) = -\varphi_{i\varepsilon} & \text{in } \Omega_{i\varepsilon} \times ]0,T[,\ i=1,2, \\ A^\varepsilon \nabla \psi_{1\varepsilon} \cdot n_{1\varepsilon} = -A^\varepsilon \nabla \psi_{2\varepsilon} \cdot n_{2\varepsilon} & \text{on } \Gamma^\varepsilon \times ]0,T[, \\ A^\varepsilon \nabla \psi_{1\varepsilon} \cdot n_{1\varepsilon} = -\varepsilon^\gamma h^\varepsilon (\psi_{1\varepsilon} - \psi_{2\varepsilon}) & \text{on } \Gamma^\varepsilon \times ]0,T[, \\ \psi_{1\varepsilon} = 0 & \text{on } \partial\Omega \times ]0,T[, \\ \psi_{1\varepsilon}(T) = \psi'_{1\varepsilon}(T) = 0 & \text{in } \Omega_{1\varepsilon}, \\ \psi_{2\varepsilon}(T) = \psi'_{2\varepsilon}(T) = 0 & \text{in } \Omega_{2\varepsilon}, \end{cases} \tag{4.5}$$

where $n_{i\varepsilon}$ is the unitary outward normal to $\Omega_{i\varepsilon},\ i=1,2$, and $\varphi_\varepsilon$ is the unique solution of problem (4.1).

Inspired by HUM method, we introduce the linear operator

$$\Lambda_\varepsilon : L^2_\varepsilon(\Omega) \times \left(H^\varepsilon_\gamma\right)' \to L^2_\varepsilon(\Omega) \times H^\varepsilon_\gamma \tag{4.6}$$

by setting for all $\left(\varphi_\varepsilon^0, \varphi_\varepsilon^1\right) \in L^2_\varepsilon(\Omega) \times \left(H^\varepsilon_\gamma\right)'$,

$$\Lambda_\varepsilon\left(\varphi_\varepsilon^0, \varphi_\varepsilon^1\right) = \left(\psi'_\varepsilon(0), -\psi_\varepsilon(0)\right), \tag{4.7}$$

where $\psi_\varepsilon$ is the unique solution of problem (4.5). Moreover it results

$$\begin{aligned} \left\langle \Lambda_\varepsilon\left(\varphi_\varepsilon^0, \varphi_\varepsilon^1\right), \left(\varphi_\varepsilon^0, \varphi_\varepsilon^1\right)\right\rangle &= \left\langle \left(\psi'_\varepsilon(0), -\psi_\varepsilon(0)\right), \left(\varphi_\varepsilon^0, \varphi_\varepsilon^1\right)\right\rangle = -\left\langle \varphi^1_{1\varepsilon}, \psi_{1\varepsilon}(0)\right\rangle_{(V^\varepsilon)', V^\varepsilon} \\ &- \left\langle \varphi^1_{2\varepsilon}, \psi_{2\varepsilon}(0)\right\rangle_{(H^1(\Omega_{2\varepsilon}))', H^1(\Omega_{2\varepsilon})} + \int_{\Omega_{1\varepsilon}} \varphi^0_{1\varepsilon}\psi'_{1\varepsilon}(0)dx + \int_{\Omega_{2\varepsilon}} \varphi^0_{2\varepsilon}\psi'_{2\varepsilon}(0)dx, \end{aligned} \tag{4.8}$$

for every $\left(\varphi_\varepsilon^0, \varphi_\varepsilon^1\right) \in L^2_\varepsilon(\Omega) \times \left(H^\varepsilon_\gamma\right)'$.

The following lemma provides an explicit formula for the operator $\Lambda_\varepsilon$.

**Lemma 4.1.** *Let us fix $\left(\varphi_\varepsilon^0, \varphi_\varepsilon^1\right) \in L^2_\varepsilon(\Omega) \times \left(H^\varepsilon_\gamma\right)'$. Let $\varphi_\varepsilon$ be the corresponding solution of problem* (4.1)*. Then the following identity holds*

$$\left\langle \Lambda_\varepsilon\left(\varphi_\varepsilon^0, \varphi_\varepsilon^1\right), \left(\varphi_\varepsilon^0, \varphi_\varepsilon^1\right)\right\rangle = \int_0^T \int_{\Omega_{1\varepsilon}} \left|\varphi_{1\varepsilon}\right|^2 dxdt + \int_0^T \int_{\Omega_{2\varepsilon}} \left|\varphi_{2\varepsilon}\right|^2 dxdt. \tag{4.9}$$

**Proof.** Taking into account (4.8), it sufficies to consider in (4.3) the function $\psi_\varepsilon$, unique solution of the backward problem (4.5). ☐

**Remark 4.1.** *For any $\varepsilon > 0$, by* (4.4)*, the operator $\Lambda_\varepsilon$ is linear, continuous and injective. If $\Lambda_\varepsilon$ is an isomorphism we define the control $\zeta^{ex}_\varepsilon \in L^2\left(0, T; L^2_\varepsilon(\Omega)\right)$ by $\zeta^{ex}_\varepsilon := -\varphi_\varepsilon$, where $\varphi_\varepsilon$ is the unique solution of problem* (4.1) *with initial data $\left(\varphi_\varepsilon^0, \varphi_\varepsilon^1\right) = \Lambda_\varepsilon^{-1}\left(U^1_\varepsilon, -U^0_\varepsilon\right)$. Indeed, by uniqueness, the state is given by $u_\varepsilon(\zeta^{ex}_\varepsilon) = \psi_\varepsilon$, where $\psi_\varepsilon$ is the unique solution of the backward problem* (4.5)*. Hence* (3.1) *is satisfied and we obtain the exact controllability at time $T > 0$ for system* (2.3)*.*

In order to prove that, for any fixed $\varepsilon$, the operator $\Lambda_\varepsilon$ is an isomorphism from $L^2_\varepsilon(\Omega) \times \left(H^\varepsilon_\gamma\right)'$ to $L^2_\varepsilon(\Omega) \times H^\varepsilon_\gamma$ and obtain estimates independent of $\varepsilon$, we need to show the following observability estimate (4.10), the converse inequality being an immediate consequence of (4.4).

**Proposition 4.1.** *Let us fix $\left(\varphi_\varepsilon^0, \varphi_\varepsilon^1\right) \in L^2_\varepsilon(\Omega) \times \left(H^\varepsilon_\gamma\right)'$. Let $\varphi_\varepsilon$ be the corresponding solution of problem* (4.1)*. Then, for any $\varepsilon$, we have*

$$\left\|\varphi^0_\varepsilon\right\|^2_{L^2_\varepsilon(\Omega)} + \left\|\varphi^1_\varepsilon\right\|^2_{(H^\varepsilon_\gamma)'} \leq C\left(\int_0^T \int_{\Omega_{1\varepsilon}} \left|\varphi_{1\varepsilon}\right|^2 dxdt + \int_0^T \int_{\Omega_{2\varepsilon}} \left|\varphi_{2\varepsilon}\right|^2 dxdt\right), \tag{4.10}$$

$$\left\|\Lambda_\varepsilon^{-1}\right\|_{\mathcal{L}\left(L^2_\varepsilon(\Omega)\times H^\varepsilon_\gamma; L^2_\varepsilon(\Omega)\times (H^\varepsilon_\gamma)'\right)} \leq C, \tag{4.11}$$

*with $C$ positive constant independent of $\varepsilon$.*

Let us establish a preliminary result with more regular data $\left(\varphi_\varepsilon^0, \varphi_\varepsilon^1\right) \in H^\varepsilon_\gamma \times L^2_\varepsilon(\Omega)$. We can define the solution $\varphi_\varepsilon$ of problem (4.1) by the usual weak formulation. Hence the associated energy is

$$\begin{aligned} E^\varepsilon(t) := \ & \frac{1}{2}\left[\int_{\Omega_{1\varepsilon}} |\varphi'_{1\varepsilon}(t)|^2 dx + \int_{\Omega_{2\varepsilon}} |\varphi'_{2\varepsilon}(t)|^2 dx + \int_{\Omega_{1\varepsilon}} A^\varepsilon \nabla\varphi_{1\varepsilon}(t)\nabla\varphi_{1\varepsilon}(t)dx\right. \\ & \left. + \int_{\Omega_{2\varepsilon}} A^\varepsilon \nabla\varphi_{2\varepsilon}(t)\nabla\varphi_{2\varepsilon}(t)dx + \varepsilon^\gamma \int_{\Gamma_\varepsilon} h^\varepsilon \left|\varphi_{1\varepsilon}(t) - \varphi_{2\varepsilon}(t)\right|^2 d\sigma_x\right]. \end{aligned} \tag{4.12}$$

**Lemma 4.2.** *For any $\varepsilon$, one has*

$$E^\varepsilon(0) \le C\left(\int_0^T \int_{\Omega_{1\varepsilon}} |\varphi'_{1\varepsilon}|^2 dxdt + \int_0^T \int_{\Omega_{2\varepsilon}} |\varphi'_{2\varepsilon}|^2 dxdt\right), \tag{4.13}$$

*with $C$ positive constant independent of $\varepsilon$ and where*

$$\begin{aligned} E^\varepsilon(0) = \ & \frac{1}{2}\left(\int_{\Omega_{1\varepsilon}} \left|\varphi^1_{1\varepsilon}\right|^2 dx + \int_{\Omega_{2\varepsilon}} \left|\varphi^1_{2\varepsilon}\right|^2 dx + \int_{\Omega_{1\varepsilon}} A^\varepsilon \nabla\varphi^0_{1\varepsilon}\nabla\varphi^0_{1\varepsilon} dx + \int_{\Omega_{2\varepsilon}} A^\varepsilon \nabla\varphi^0_{2\varepsilon}\nabla\varphi^0_{2\varepsilon} dx \right. \\ & \left. + \varepsilon^\gamma \int_{\Gamma_\varepsilon} h^\varepsilon \left|\varphi^0_{1\varepsilon} - \varphi^0_{2\varepsilon}\right|^2 d\sigma_x \right). \end{aligned}$$

**Proof.** First note that the energy defined in (4.12) is conserved (see [15], Lemma 4.1), that is

$$E^\varepsilon(t) = E(0), \quad \text{for every } t \in [0, T]. \tag{4.14}$$

Let $\rho(t)$ be the function defined by

$$\rho(t) = t^2(T-t)^2, \text{ for every } t \in [0, T]. \tag{4.15}$$

By choosing $(\rho(t)\varphi_{1\varepsilon}(x,t), \rho(t)\varphi_{2\varepsilon}(x,t))$ as test function in problem (4.1) and integrating by parts, we obtain

$$\begin{aligned} & \int_0^T \int_{\Omega_{1\varepsilon}} \rho|\varphi'_{1\varepsilon}|^2 dxdt + \int_0^T \int_{\Omega_{2\varepsilon}} \rho|\varphi'_{2\varepsilon}|^2 dxdt + \int_0^T \int_{\Omega_{1\varepsilon}} \rho' \varphi_{1\varepsilon}\varphi'_{1\varepsilon} dxdt \\ & + \int_0^T \int_{\Omega_{2\varepsilon}} \rho' \varphi_{2\varepsilon}\varphi'_{2\varepsilon} dxdt - \varepsilon^\gamma \int_0^T \int_{\Gamma_\varepsilon} \rho h^\varepsilon \left|\varphi_{1\varepsilon} - \varphi_{2\varepsilon}\right|^2 d\sigma_x dt \\ & = \int_0^T \int_{\Omega_{1\varepsilon}} \rho A^\varepsilon \nabla\varphi_{1\varepsilon}\nabla\varphi_{1\varepsilon} dxdt + \int_0^T \int_{\Omega_{2\varepsilon}} \rho A^\varepsilon \nabla\varphi_{2\varepsilon}\nabla\varphi_{2\varepsilon} dxdt. \end{aligned} \tag{4.16}$$

Denoting for every $t \in [0, T]$

$$\begin{aligned} E^\varepsilon_\rho(t) := \ & \int_{\Omega_{1\varepsilon}} \rho(t) A^\varepsilon \nabla\varphi_{1\varepsilon}(t)\nabla\varphi_{1\varepsilon}(t) dx + \int_{\Omega_{2\varepsilon}} \rho(t) A^\varepsilon \nabla\varphi_{2\varepsilon}(t)\nabla\varphi_{2\varepsilon}(t) dx \\ & + \varepsilon^\gamma \int_{\Gamma_\varepsilon} \rho(t) h^\varepsilon \left|\varphi_{1\varepsilon}(t) - \varphi_{2\varepsilon}(t)\right|^2 d\sigma_x, \end{aligned} \tag{4.17}$$

(4.16) can be written as

$$\begin{aligned} & \int_0^T \int_{\Omega_{1\varepsilon}} \rho|\varphi'_{1\varepsilon}|^2 dxdt + \int_0^T \int_{\Omega_{2\varepsilon}} \rho|\varphi'_{2\varepsilon}|^2 dxdt + \int_0^T \int_{\Omega_{1\varepsilon}} \rho' \varphi_{1\varepsilon}\varphi'_{1\varepsilon} dxdt \\ & + \int_0^T \int_{\Omega_{2\varepsilon}} \rho' \varphi_{2\varepsilon}\varphi'_{2\varepsilon} dxdt = \int_0^T E^\varepsilon_\rho \, dt. \end{aligned} \tag{4.18}$$

Then, by making use of Young's inequality, for any $\delta > 0$, we get

$$\int_0^T \int_{\Omega_{1\varepsilon}} \rho' \varphi_{1\varepsilon}\varphi'_{1\varepsilon} dxdt + \int_0^T \int_{\Omega_{2\varepsilon}} \rho' \varphi_{2\varepsilon}\varphi'_{2\varepsilon} dxdt$$

$$\leq \delta \left( \int_0^T \rho \left( \int_{\Omega_{1\varepsilon}} |\varphi_{1\varepsilon}|^2 dx + \int_{\Omega_{2\varepsilon}} |\varphi_{2\varepsilon}|^2 dx \right) dt \right) \tag{4.19}$$

$$+C(\delta) \left( \int_0^T \int_{\Omega_{1\varepsilon}} |\varphi'_{1\varepsilon}|^2 dxdt + \int_0^T \int_{\Omega_{2\varepsilon}} |\varphi'_{2\varepsilon}|^2 dxdt \right)$$

where $C(\delta) = \dfrac{1}{4\delta} \left\| \dfrac{(\rho')^2}{\rho} \right\|_{L^\infty(0,T)} = \dfrac{1}{4\delta} \max_{[0,T]} \dfrac{(\rho')^2}{\rho} = \dfrac{T^2}{\delta}$.
By collecting together (4.18) and (4.19), we get

$$\begin{aligned} \int_0^T E^\varepsilon_\rho \, dt \leq{}& \int_0^T \int_{\Omega_{1\varepsilon}} \rho |\varphi'_{1\varepsilon}|^2 dxdt + \int_0^T \int_{\Omega_{2\varepsilon}} \rho |\varphi'_{2\varepsilon}|^2 dxdt \\ &+\delta \int_0^T \rho \left( \int_{\Omega_{1\varepsilon}} |\varphi_{1\varepsilon}|^2 dx + \int_{\Omega_{2\varepsilon}} |\varphi_{2\varepsilon}|^2 dx \right) dt \\ &+C(\delta) \left( \int_0^T \int_{\Omega_{1\varepsilon}} |\varphi'_{1\varepsilon}|^2 dxdt + \int_0^T \int_{\Omega_{2\varepsilon}} |\varphi'_{2\varepsilon}|^2 dxdt \right). \end{aligned} \tag{4.20}$$

Note that by (2.5), (2.8), Remark 2.3 of [39] stating the Poincaré inequality in the space $V_\varepsilon$ with a constant independent of $\varepsilon$ and (2.10), we get

$$\begin{aligned} &\int_{\Omega_{1\varepsilon}} A^\varepsilon \nabla\varphi_{1\varepsilon}(t) \nabla\varphi_{1\varepsilon}(t) dx + \int_{\Omega_{2\varepsilon}} A^\varepsilon \nabla\varphi_{2\varepsilon}(t) \nabla\varphi_{2\varepsilon}(t) dx + \varepsilon^\gamma \int_{\Gamma_\varepsilon} h^\varepsilon |\varphi_{1\varepsilon}(t) - \varphi_{2\varepsilon}(t)|^2 d\sigma_x \\ &\geq C' \left( \int_{\Omega_{1\varepsilon}} |\varphi_{1\varepsilon}(t)|^2 + \int_{\Omega_{2\varepsilon}} |\varphi_{2\varepsilon}(t)|^2 \right) \quad \text{for a.e. } t \in [0,T], \end{aligned} \tag{4.21}$$

with $C'$ positive constant independent of $\varepsilon$.
By (4.20) and (4.21), we obtain

$$\begin{aligned} \left( 1 - \frac{\delta}{C'} \right) \int_0^T E^\varepsilon_\rho \, dt \leq{}& \int_0^T \int_{\Omega_{1\varepsilon}} \rho |\varphi'_{1\varepsilon}|^2 dxdt + \int_0^T \int_{\Omega_{2\varepsilon}} \rho |\varphi'_{2\varepsilon}|^2 dxdt \\ &+C(\delta) \left( \int_0^T \int_{\Omega_{1\varepsilon}} |\varphi'_{1\varepsilon}|^2 dxdt + \int_0^T \int_{\Omega_{2\varepsilon}} |\varphi'_{2\varepsilon}|^2 dxdt \right). \end{aligned} \tag{4.22}$$

Thus if we choose $\delta < C'$, there exists a positive constant $C''$ independent of $\varepsilon$ such that

$$\int_0^T E^\varepsilon_\rho \, dt \leq C'' \left( \int_0^T \int_{\Omega_{1\varepsilon}} |\varphi'_{1\varepsilon}|^2 dxdt + \int_0^T \int_{\Omega_{2\varepsilon}} |\varphi'_{2\varepsilon}|^2 dxdt \right). \tag{4.23}$$

Multiplying equation in (4.14) by $\rho(t)$ and integrating from 0 to $T$, we obtain

$$E^\varepsilon(0)\int_0^T \rho\, dt = \frac{1}{2}\int_0^T E^\varepsilon_\rho\, dt + \frac{1}{2}\int_0^T \rho\left(\int_{\Omega_{1\varepsilon}} |\varphi'_{1\varepsilon}|^2 dx + \int_{\Omega_{2\varepsilon}} |\varphi'_{2\varepsilon}|^2 dx\right) dt. \tag{4.24}$$

By virtue of (4.23) and (4.24) estimate (4.13) holds. $\square$

Now we are able to prove Proposition 4.1.
Let $\pi_\varepsilon \in H^\varepsilon_\gamma$ be the unique solution of the problem

$$\begin{cases} -\mathrm{div}(A^\varepsilon \nabla \pi_{i\varepsilon}) = -\varphi^1_{i\varepsilon} & \text{in } \Omega_{i\varepsilon},\ i = 1,2, \\ A^\varepsilon \nabla \pi_{1\varepsilon}\cdot n_{1\varepsilon} = -A^\varepsilon \nabla \pi_{2\varepsilon}\cdot n_{2\varepsilon} & \text{on } \Gamma^\varepsilon, \\ A^\varepsilon \nabla \pi_{1\varepsilon}\cdot n_{1\varepsilon} = -\varepsilon^\gamma h^\varepsilon(\pi_{1\varepsilon} - \pi_{2\varepsilon}) & \text{on } \Gamma^\varepsilon, \\ \pi_{1\varepsilon} = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.25}$$

with $\varphi^1_\varepsilon \in \left(H^\varepsilon_\gamma\right)'$. By classical arguments concerning symmetric bilinear forms associated to elliptic equations (see [33] in the proof of Proposition 1.2), one has

$$C_1\|\varphi^1_\varepsilon\|_{(H^\varepsilon_\gamma)'} \le \|\pi_\varepsilon\|_{H^\varepsilon_\gamma} \le C_2\|\varphi^1_\varepsilon\|_{(H^\varepsilon_\gamma)'}, \tag{4.26}$$

with $C_1$ and $C_2$ positive constants independent of $\varepsilon$.
Let $\varphi_\varepsilon$ be the transposition solution of problem (4.1) corresponding to the initial data $\left(\varphi^0_\varepsilon, \varphi^1_\varepsilon\right) \in L^2_\varepsilon(\Omega) \times \left(H^\varepsilon_\gamma\right)'$. Then the function $\varpi_\varepsilon = (\varpi_{1\varepsilon}, \varpi_{2\varepsilon})$ defined by

$$\varpi_{i\varepsilon}(x,t) := \int_0^t \varphi_{i\varepsilon}(x,s)ds + \pi_{i\varepsilon}(x),\ \ i = 1,2, \tag{4.27}$$

satisfies problem

$$\begin{cases} \varpi''_{i\varepsilon} - \mathrm{div}(A^\varepsilon \nabla \varpi_{i\varepsilon}) = 0 & \text{in } \Omega_{i\varepsilon}\times]0,T[,\ i = 1,2, \\ A^\varepsilon \nabla \varpi_{1\varepsilon}\cdot n_{1\varepsilon} = -A^\varepsilon \nabla \varpi_{2\varepsilon}\cdot n_{2\varepsilon} & \text{on } \Gamma^\varepsilon\times]0,T[, \\ A^\varepsilon \nabla \varpi_{1\varepsilon}\cdot n_{1\varepsilon} = -\varepsilon^\gamma h^\varepsilon(\varpi_{1\varepsilon} - \varpi_{2\varepsilon}) & \text{on } \Gamma^\varepsilon\times]0,T[, \\ \varpi_{1\varepsilon} = 0 & \text{on } \partial\Omega\times]0,T[, \\ \varpi_{1\varepsilon}(0) = \pi_{1\varepsilon}, \quad \varpi'_{1\varepsilon}(0) = \varphi^0_{1\varepsilon} & \text{in } \Omega_{1\varepsilon}, \\ \varpi_{2\varepsilon}(0) = \pi_{2\varepsilon}, \quad \varpi'_{2\varepsilon}(0) = \varphi^0_{2\varepsilon} & \text{in } \Omega_{2\varepsilon}. \end{cases} \tag{4.28}$$

Observe that, since $\pi_\varepsilon \in H^\varepsilon_\gamma$ and $\varphi^0_\varepsilon \in L^2_\varepsilon(\Omega)$, the solution $\varpi_\varepsilon$ is defined by usual weak formulation. Hence, by applying Lemma 4.2, in view of (4.27) we get

$$\begin{aligned} &\frac{1}{2}\left(\int_{\Omega_{1\varepsilon}} \left|\varphi^0_{1\varepsilon}\right|^2 dx + \int_{\Omega_{2\varepsilon}} \left|\varphi^0_{2\varepsilon}\right|^2 dx + \int_{\Omega_{1\varepsilon}} A^\varepsilon \nabla \pi_{1\varepsilon} \nabla \pi_{1\varepsilon} dx + \int_{\Omega_{2\varepsilon}} A^\varepsilon \nabla \pi_{2\varepsilon} \nabla \pi_{2\varepsilon} dx\right. \\ &\left. + \varepsilon^\gamma \int_{\Gamma_\varepsilon} h^\varepsilon \left|\pi_{1\varepsilon} - \pi_{2\varepsilon}\right|^2 d\sigma_x\right) \le C\left(\int_0^T \int_{\Omega_{1\varepsilon}} |\varpi'_{1\varepsilon}|^2 dxdt + \int_0^T \int_{\Omega_{2\varepsilon}} |\varpi'_{2\varepsilon}|^2 dxdt\right) \\ &\qquad = C\left(\int_0^T \int_{\Omega_{1\varepsilon}} |\varphi_{1\varepsilon}|^2 dxdt + \int_0^T \int_{\Omega_{2\varepsilon}} |\varphi_{2\varepsilon}|^2 dxdt\right) \end{aligned} \tag{4.29}$$

with $C$ positive constant independent of $\varepsilon$. Inequality (4.10) of Proposition 4.1 is then a direct consequence of (2.5), (2.8), (4.26) and (4.29).
Now we prove (4.11). Using Young's inequality, (4.9) and (4.10), one obtains that

$$\begin{aligned}
&\|(\varphi^0_\varepsilon,\varphi^1_\varepsilon)\|^2_{L^2_\varepsilon(\Omega)\times\left(H^\varepsilon_\gamma\right)'} \le 2\left(\|\varphi^0_\varepsilon\|^2_{L^2_\varepsilon(\Omega)}+\|\varphi^1_\varepsilon\|^2_{\left(H^\varepsilon_\gamma\right)'}\right)\\
&\le C\left(\int_0^T\int_{\Omega_{1\varepsilon}}|\varphi_{1\varepsilon}|^2dxdt+\int_0^T\int_{\Omega_{2\varepsilon}}|\varphi_{2\varepsilon}|^2dxdt\right)=C\left\langle\Lambda_\varepsilon\left(\varphi^0_\varepsilon,\varphi^1_\varepsilon\right),\left(\varphi^0_\varepsilon,\varphi^1_\varepsilon\right)\right\rangle\\
&\le C\|\Lambda_\varepsilon\left(\varphi^0_\varepsilon,\varphi^1_\varepsilon\right)\|_{L^2_\varepsilon(\Omega)\times H^\varepsilon_\gamma}\|\left(\varphi^0_\varepsilon,\varphi^1_\varepsilon\right)\|_{L^2_\varepsilon(\Omega)\times(H^\varepsilon_\gamma)'},
\end{aligned}\tag{4.30}$$

with $C$ positive constant independent of $\varepsilon$.
By (4.30) and taking into account that $\Lambda_\varepsilon$ is an isomorphism, we get

$$\begin{aligned}
&\|\Lambda_\varepsilon^{-1}\|_{\mathcal{L}\left(L^2_\varepsilon(\Omega)\times H^\varepsilon_\gamma;L^2_\varepsilon(\Omega)\times(H^\varepsilon_\gamma)'\right)}\\
&=\sup\left(\frac{\|(\varphi^0_\varepsilon,\varphi^1_\varepsilon)\|_{L^2_\varepsilon(\Omega)\times\left(H^\varepsilon_\gamma\right)'}}{\|\Lambda_\varepsilon\left(\varphi^0_\varepsilon,\varphi^1_\varepsilon\right)\|_{L^2_\varepsilon(\Omega)\times H^\varepsilon_\gamma}}:\left(\varphi^0_\varepsilon,\varphi^1_\varepsilon\right)\in L^2_\varepsilon(\Omega)\times(H^\varepsilon_\gamma)'\right)\le C,
\end{aligned}\tag{4.31}$$

that is (4.11). □

## 5. Proof of Theorems 3.2 and 3.3

While the proof of Theorem 3.2 follows more or less the same arguments as in [33] (see also [34]), for Theorem 3.3 the situation is much more delicate. Indeed, we would apply again the Hilbert Uniqueness Method. To this aim we need to construct a functional associated to a coupled system of a partial and an ordinary differential equation, which renders this case more difficult.

### *5.1. Proof of Theorem 3.2*

The proof of Theorem 3.2 can be found in [33]. Here, for convenience sake, we detail only the noteworthy points. More precisely, let $\left(\varphi^0,\varphi^1\right)\in L^2(\Omega)\times H^{-1}(\Omega)$ and consider the problem

$$\begin{cases}\varphi''-\operatorname{div}(A^0\nabla\varphi)=0 & \text{in } \Omega\times]0,T[,\\ \varphi=0 & \text{on } \partial\Omega\times]0,T[,\\ \varphi(0)=\varphi^0,\quad \varphi'(0)=\varphi^1 & \text{in } \Omega.\end{cases}\tag{5.1}$$

Since the initial data are in a weak space, one needs to apply the so called transposition method (see [35], Chapter 3, Section 9) to obtain a unique solution $\varphi\in C\left([0,T];L^2(\Omega)\right)\cap C^1\left([0,T];H^{-1}(\Omega)\right)$ of problem (5.1) satisfying the estimate

$$\|\varphi\|_{L^\infty(0,T;L^2(\Omega))}+\|\varphi'\|_{L^\infty(0,T;H^{-1}(\Omega))}\le C(\|\varphi^0\|_{L^2(\Omega)}+\|\varphi^1\|_{H^{-1}(\Omega)}),\tag{5.2}$$

with $C$ positive constant.
Now, let $\psi\in C\left([0,T];H^1_0(\Omega)\right)\cap C^1\left([0,T];L^2(\Omega)\right)$ be the unique solution of the backward problem

$$\begin{cases}\psi''-\operatorname{div}(A^0\nabla\psi)=-\varphi & \text{in } \Omega\times]0,T[,\\ \psi=0 & \text{on } \partial\Omega\times]0,T[,\\ \psi(T)=\psi'(T)=0 & \text{in } \Omega,\end{cases}\tag{5.3}$$

where $\varphi$ is the unique solution of problem (5.1).
Inspired by HUM method, we introduce the linear operator

$$\Lambda : L^2(\Omega) \times H^{-1}(\Omega) \to L^2(\Omega) \times H_0^1(\Omega) \tag{5.4}$$

by setting for all $\left(\varphi^0, \varphi^1\right) \in L^2(\Omega) \times H^{-1}(\Omega)$,

$$\Lambda\left(\varphi^0, \varphi^1\right) = \left(\psi'(0), -\psi(0)\right), \tag{5.5}$$

where $\psi$ is the unique solution of problem (5.3). Moreover it results

$$\begin{aligned} \left\langle \Lambda\left(\varphi^0, \varphi^1\right), \left(\varphi^0, \varphi^1\right)\right\rangle &= \left\langle \left(\psi'(0), -\psi(0)\right), \left(\varphi^0, \varphi^1\right)\right\rangle \\ &= -\left\langle \varphi^1, \psi(0)\right\rangle_{H^{-1}(\Omega), H_0^1(\Omega)} + \int_\Omega \varphi^0 \psi'(0) dx, \end{aligned} \tag{5.6}$$

for every $\left(\varphi^0, \varphi^1\right) \in L^2(\Omega) \times H^{-1}(\Omega)$.
The operator $\Lambda$ is an isomorphism (see [33]). Then, we define the control $\bar{\zeta}^{ex} \in L^2\left(0, T; L^2(\Omega)\right)$ by $\bar{\zeta}^{ex} := -\varphi$, where $\varphi$ is the unique solution of problem (5.1) with initial data $\left(\varphi^0, \varphi^1\right) = \Lambda^{-1}\left(V^1, -V^0\right)$. Indeed, by uniqueness, the state is given by $v(\bar{\zeta}^{ex}) = \psi$, where $\psi$ is the unique solution of the backward problem (5.3).
Hence we obtain the exact controllability at time $T > 0$ for system (3.2).

### *5.2. Proof of Theorem 3.3*

Due to the lack of symmetry of the associated operator, classical existence, uniqueness and regularity results do not directly apply to problem (3.3). Hence we need to recall the following result proved in [28].

**Theorem 5.1 ([28]).** *Let $T > 0$. Let $A^0$ be the matrix defined in Theorem 2.1 by* (2.15) *and* (2.16). *Then problem* (3.3) *admits a unique solution $v := (v_1, v_2) \in L^2\left(0, T; H_0^1(\Omega) \times L^2(\Omega)\right)$ satisfying the following estimate*

$$\begin{aligned} &\|v\|_{L^\infty\left(0,T;H_0^1(\Omega)\times L^2(\Omega)\right)} + \|v'\|_{L^\infty(0,T;(L^2(\Omega))^2)} \\ &\leq C\left(\left\|V^0\right\|_{H_0^1(\Omega)\times L^2(\Omega)} + \left\|V^1\right\|_{(L^2(\Omega))^2} + \|\zeta\|_{L^2(0,T;(L^2(\Omega))^2)}\right), \end{aligned} \tag{5.7}$$

*with $C$ positive constant.*

Let us point out that, by arguing as in [35], (Chapter 3, Theorem 8.2), the energy equality for problem (3.3) (see [15], Theorem 4.4) provides some further properties. In fact, under the same hypotheses of Theorem 5.1, it holds

$$v \in C\left([0, T]; H_0^1(\Omega) \times L^2(\Omega)\right) \cap C^1\left([0, T]; \left(L^2(\Omega)\right)^2\right) \cap C^1\left([0, T]; H^{-1}(\Omega) \times L^2(\Omega)\right). \tag{5.8}$$

Here, some difficulties arise when one tries to obtain an observability inequality corresponding to the coupled system (3.3).

Fixed $\varphi^0 := (\varphi_1^0, \varphi_2^0) \in (L^2(\Omega))^2$ and $\varphi^1 := (\varphi_1^1, \varphi_2^1) \in H^{-1}(\Omega) \times L^2(\Omega)$, let us consider the following problem

$$\begin{cases} \theta_1 \varphi_1'' - \operatorname{div}(A^0 \nabla \varphi_1) + c_h(\theta_2 \varphi_1 - \varphi_2) = 0 & \text{in } \Omega \times ]0, T[, \\ \varphi_2'' - c_h(\theta_2 \varphi_1 - \varphi_2) = 0 & \text{in } \Omega \times ]0, T[, \\ \varphi_1 = 0 & \text{on } \partial\Omega \times ]0, T[, \\ \varphi_1(0) = \varphi_1^0, \quad \varphi_1'(0) = \varphi_1^1 & \text{in } \Omega, \\ \varphi_2(0) = \varphi_2^0, \quad \varphi_2'(0) = \varphi_2^1 & \text{in } \Omega. \end{cases} \tag{5.9}$$

Since the initial data are in a weak space, we should define the solution of problem (5.9) by the transposition method. Unfortunately, classical results (see [35], Chapter 3, Section 9) do not directly apply to problem (5.9), since the associated operator is not symmetric. Neverthless, we can overcome this difficulty thanks to the symmetry of the matrix $A^0$ and (2.18).
At first, let us give an appropriate definition of what we understand as solution of problem (5.9). To this aim for every $f := (f_1, f_2) \in L^2(0, T; (L^2(\Omega))^2)$, let us consider the following backward problem

$$\begin{cases} \theta_1 \psi_1'' - \operatorname{div}(A^0 \nabla \psi_1) + c_h(\theta_2 \psi_1 - \psi_2) = f_1 & \text{in } \Omega \times ]0, T[, \\ \psi_2'' - c_h(\theta_2 \psi_1 - \psi_2) = f_2 & \text{in } \Omega \times ]0, T[, \\ \psi_1 = 0 & \text{on } \partial\Omega \times ]0, T[, \\ \psi_1(T) = \psi_1'(T) = 0 & \text{in } \Omega, \\ \psi_2(T) = \psi_2'(T) = 0 & \text{in } \Omega. \end{cases} \tag{5.10}$$

As previously, for clearness sake, throughout the paper, we denote by $\psi(f) := (\psi_1(f), \psi_2(f))$ the solution of problem (5.10) and where no ambiguity arises, we omit the explicit dependence on the right hand member. As initial conditions of problem (5.10) are in right spaces, then we can give the following definition

**Definition 5.1.** *For every $\varphi^0 = (\varphi_1^0, \varphi_2^0) \in (L^2(\Omega))^2$ and $\varphi^1 = (\varphi_1^1, \varphi_2^1) \in H^{-1}(\Omega) \times L^2(\Omega)$, we call solution of problem* (5.9), *in the sense of transposition, a function $\varphi := (\varphi_1, \varphi_2) \in L^2(0, T; (L^2(\Omega))^2)$ satisfying the condition*

$$\begin{aligned} &\int_0^T \int_\Omega \varphi_1 \left( \theta_1 \psi_1'' - \operatorname{div}(A^0 \nabla \psi_1) + c_h(\theta_2 \psi_1 - \psi_2) \right) dxdt \\ &+ \int_0^T \int_\Omega \theta_2^{-1} \varphi_2 \left( \psi_2'' - c_h(\theta_2 \psi_1 - \psi_2) \right) dxdt = - \int_\Omega \theta_1 \varphi_1^0 \psi_1'(0) dx \\ &+ \langle \theta_1 \varphi_1^1, \psi_1(0) \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} - \int_\Omega \theta_2^{-1} \varphi_2^0 \psi_2'(0) dx + \int \theta_2^{-1} \varphi_2^1 \psi_2(0) dx \end{aligned} \tag{5.11}$$

*for every function $\psi$ solution of problem* (5.10) *such that $-\operatorname{div}(A^0 \nabla \psi_1)$ is in $L^2(0, T; L^2(\Omega))$.*

Now we are able to establish an existence and uniqueness result which generalizes, to the non-symmetric case, the one proved in [35], Chapter 3, Section 9, Theorem 9.3 and Theorem 9.4 only for the symmetric case.

**Theorem 5.2.** *Let $A^0$ be the matrix defined in Theorem 2.1 by* (2.15) *and* (2.16). *Let $\varphi^0 = (\varphi_1^0, \varphi_2^0) \in (L^2(\Omega))^2$ and $\varphi^1 = (\varphi_1^1, \varphi_2^1) \in H^{-1}(\Omega) \times L^2(\Omega)$. Then there exists a unique $\varphi$ solution* *of problem* (5.9) *such that* $\varphi = (\varphi_1, \varphi_2) \in L^\infty(0,T;(L^2(\Omega))^2)$, $\varphi' = (\varphi_1', \varphi_2') \in L^\infty(0,T;H^{-1}(\Omega) \times L^2(\Omega))$ *and satisfying the estimate*

$$\|\varphi\|_{L^\infty(0,T;(L^2(\Omega))^2)} + \|\varphi'\|_{L^\infty(0,T;H^{-1}(\Omega)\times L^2(\Omega))} \leq C \left( \|\varphi^0\|_{(L^2(\Omega))^2} + \|\varphi^1\|_{H^{-1}(\Omega)\times L^2(\Omega)} \right), \quad (5.12)$$

*with* $C$ *positive constant.*
*Furthermore it holds*

$$\varphi \in C\left([0,T];(L^2(\Omega))^2\right) \cap C^1\left([0,T];H^{-1}(\Omega)\times L^2(\Omega)\right). \quad (5.13)$$

**Proof.** First of all, let us observe that, by arguing as in [35], Chapter 3, Section 9, Theorem 9.4, by a density argument, one can obtain that (5.11) holds in fact for every $\psi$ solution of the backward problem (5.10). Hence, if a solution of problem (5.9) exists, it is unique. Let us now prove the existence.
As a preliminary, we define the differential operator $\mathcal{A} : dom(\mathcal{A}) \subset L^2(\Omega) \mapsto L^2(\Omega)$ by

$$\mathcal{A}v = -\mathrm{div}(A^0 \nabla v) \ \text{ for every } v \in dom(\mathcal{A}),$$

where $dom(\mathcal{A}) = \left\{ v \in H_0^1(\Omega) : -\mathrm{div}(A^0\nabla v) \in L^2(\Omega)\right\}$. Due to (2.18) and the symmetry of the matrix $A^0$, $\mathcal{A}$ is a linear, continuous and selfadjoint operator. Moreover it is easy to observe that it establishes an isomorphism from $H_0^1(\Omega)$ onto $H^{-1}(\Omega)$ and the inverse operator $\mathcal{A}^{-1}$ is linear, continuous and selfadjoint itself. Then one has

$$\|v\|^2_{H^{-1}(\Omega)} = \left\|\mathcal{A}^{-1}v\right\|^2_{H_0^1(\Omega)} = \int_\Omega A^0 \nabla \mathcal{A}^{-1} v \nabla \mathcal{A}^{-1} v dx = \int_\Omega v \mathcal{A}^{-1} v dx \ \ \forall v \in H^{-1}(\Omega). \quad (5.14)$$

We consider the sequences $\varphi_n^0 := (\varphi_{1n}^0, \varphi_{2n}^0) \in H_0^1(\Omega) \times L^2(\Omega)$ and $\varphi_n^1 := (\varphi_{1n}^1, \varphi_{2n}^1) \in (L^2(\Omega))^2$ such that

$$\begin{aligned} &\varphi_n^0 \to \varphi^0 \quad \text{in } (L^2(\Omega))^2 \\ &\varphi_n^1 \to \varphi^1 \quad \text{in } H^{-1}(\Omega)\times L^2(\Omega). \end{aligned} \quad (5.15)$$

Let us consider the problem

$$\begin{cases} \theta_1 \varphi_{1n}'' - \mathrm{div}(A^0 \nabla \varphi_{1n}) + c_h(\theta_2 \varphi_{1n} - \varphi_{2n}) = 0 & \text{in } \Omega \times ]0,T[, \\ \varphi_{2n}'' - c_h(\theta_2\varphi_{1n} - \varphi_{2n}) = 0 & \text{in } \Omega \times ]0,T[, \\ \varphi_{1n} = 0 & \text{on } \partial\Omega \times ]0,T[, \\ \varphi_{1n}(0) = \varphi_{1n}^0, \quad \varphi_{1n}'(0) = \varphi_{1n}^1 & \text{in } \Omega, \\ \varphi_{2n}(0) = \varphi_{2n}^0, \quad \varphi_{2n}'(0) = \varphi_{2n}^1 & \text{in } \Omega, \end{cases} \quad (5.16)$$

whose initial data are more regular. Hence its unique solution satisfies $\varphi_n := (\varphi_{1n}, \varphi_{2n}) \in L^\infty(0,T;H_0^1(\Omega)\times L^2(\Omega))$ and $\varphi_n' := (\varphi_{1n}', \varphi_{2n}') \in L^\infty(0,T;(L^2(\Omega))^2)$. According to usual results, $\varphi_n''(t) \in H^{-1}(\Omega) \times L^2(\Omega)$ and $(\mathcal{A}^{-1}\varphi_{1n}', \varphi_{2n}') \in H_0^1(\Omega)\times L^2(\Omega)$ a. e. in $[0,T]$. Hence, by scalarly multiplying the equations of problem (5.16) by $\left(\mathcal{A}^{-1}\varphi_{1n}', \varphi_{2n}'\right)$ and summing up, we get a. e. in $[0,T]$

$$\left\langle \theta_1 \varphi_{1n}''(t), \mathcal{A}^{-1}\varphi_{1n}'(t)\right\rangle + \left\langle \mathcal{A}\varphi_{1n}(t), \mathcal{A}^{-1}\varphi_{1n}'(t)\right\rangle + c_h \left\langle \theta_2 \varphi_{1n}(t) - \varphi_{2n}(t), \mathcal{A}^{-1}\varphi_{1n}'(t)\right\rangle$$

$$+ \left\langle \varphi_{2n}''(t), \varphi_{2n}'(t)\right\rangle - c_h \left\langle \theta_2\varphi_{1n}(t) - \varphi_{2n}(t), \varphi_{2n}'(t)\right\rangle = 0,$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$.
As both $\mathcal{A}$ and $\mathcal{A}^{-1}$ are selfadjoint, we get a. e. in $[0, T]$,

$$\theta_1 \frac{d}{dt}\left(\varphi_{1n}'(t), \mathcal{A}^{-1}\varphi_{1n}'(t)\right) + \frac{d}{dt}\left(\varphi_{1n}(t), \varphi_{1n}(t)\right) + \frac{d}{dt}\left(\varphi_{2n}'(t), \varphi_{2n}'(t)\right) + c_h \frac{d}{dt}\left(\varphi_{2n}(t), \varphi_{2n}(t)\right)$$
$$= 2c_h\left(\varphi_{2n}(t) - \theta_2\varphi_{1n}(t), \mathcal{A}^{-1}\varphi_{1n}'(t)\right) + 2c_h\left(\theta_2\varphi_{1n}(t), \varphi_{2n}'(t)\right),$$

where $(\cdot, \cdot)$ denotes the scalar product in $L^2(\Omega)$.
By (5.14), we can write a.e. in $[0, T]$,

$$\begin{aligned} &\frac{d}{dt}\left[\theta_1\|\varphi_{1n}'(t)\|_{H^{-1}(\Omega)}^2 + \|\varphi_{1n}(t)\|_{L^2(\Omega)}^2 + \|\varphi_{2n}'(t)\|_{L^2(\Omega)}^2 + c_h\|\varphi_{2n}(t)\|_{L^2(\Omega)}^2\right] \\ &= 2c_h\int_\Omega\left(\varphi_{2n}(t) - \theta_2\varphi_{1n}(t)\right)\mathcal{A}^{-1}\varphi_{1n}'(t)dx + 2c_h\theta_2\int_\Omega\varphi_{1n}(t)\varphi_{2n}'(t)dx. \end{aligned} \tag{5.17}$$

Let us estimate separately the two terms in the right hand side of (5.17).
By Holder inequality, the boundness of $\mathcal{A}^{-1}$ and Young inequality, we get

$$\begin{aligned} \int_\Omega\left(\varphi_{2n}(t) - \theta_2\varphi_{1n}(t)\right)\mathcal{A}^{-1}\varphi_{1n}'(t)dx &\le C_1\|\varphi_{2n}(t) - \theta_2\varphi_{1n}(t)\|_{L^2(\Omega)}\|\mathcal{A}^{-1}\varphi_{1n}'(t)\|_{H_0^1(\Omega)} \\ &\le C_1\|\mathcal{A}^{-1}\|\|\varphi_{2n}(t) - \theta_2\varphi_{1n}(t)\|_{L^2(\Omega)}\|\varphi_{1n}'(t)\|_{H^{-1}(\Omega)} \\ &\le C_2\left(\|\varphi_{2n}(t)\|_{L^2(\Omega)}^2 + \|\varphi_{1n}(t)\|_{L^2(\Omega)}^2 + \|\varphi_{1n}'(t)\|_{H^{-1}(\Omega)}^2\right). \end{aligned} \tag{5.18}$$

On the other hand

$$\int_\Omega\varphi_{1n}(t)\varphi_{2n}'(t)dx \le \|\varphi_{1n}(t)\|_{L^2(\Omega)}\|\varphi_{2n}(t)\|_{L^2(\Omega)} \le \frac{1}{2}\left(\|\varphi_{1n}(t)\|_{L^2(\Omega)}^2 + \|\varphi_{2n}'(t)\|_{L^2(\Omega)}^2\right). \tag{5.19}$$

If we put (5.18) and (5.19) in (5.17) we get

$$\begin{aligned} &\frac{d}{dt}\left[\theta_1\|\varphi_{1n}'(t)\|_{H^{-1}(\Omega)}^2 + \|\varphi_{1n}(t)\|_{L^2(\Omega)}^2 + \|\varphi_{2n}'(t)\|_{L^2(\Omega)}^2 + c_h\|\varphi_{2n}(t)\|_{L^2(\Omega)}^2\right] \\ &\le C\left(\theta_1\|\varphi_{1n}'(t)\|_{H^{-1}(\Omega)}^2 + \|\varphi_{1n}(t)\|_{L^2(\Omega)}^2 + \|\varphi_{2n}'(t)\|_{L^2(\Omega)}^2 + c_h\|\varphi_{2n}(t)\|_{L^2(\Omega)}^2\right), \end{aligned} \tag{5.20}$$

with $C$ positive constant.
By applying Gronwall's inequality we obtain a. e. in $[0, T]$,

$$\begin{aligned} &\|\varphi_{1n}'(t)\|_{H^{-1}(\Omega)}^2 + \|\varphi_{1n}(t)\|_{L^2(\Omega)}^2 + \|\varphi_{2n}'(t)\|_{L^2(\Omega)}^2 + \|\varphi_{2n}(t)\|_{L^2(\Omega)}^2 \\ &\le C\left(\|\varphi_{1n}^1\|_{H^{-1}(\Omega)}^2 + \|\varphi_{2n}^1\|_{L^2(\Omega)}^2 + \|\varphi_{1n}^0\|_{L^2(\Omega)}^2 + \|\varphi_{2n}^0\|_{L^2(\Omega)}^2\right) \end{aligned} \tag{5.21}$$

whit $C$ positive constant.
From (5.15) and (5.21) we deduce that $\varphi_n$ (resp. $\varphi_n'$) remains in a bounded set of $L^\infty(0, T; (L^2(\Omega))^2)$

(resp. $L^\infty(0,T;H^{-1}(\Omega)\times L^2(\Omega)))$ as $n\to+\infty$. Therefore, up to a subsequence still denoted by $n$, we may deduce

$$\begin{aligned}
&\varphi_n=(\varphi_{1n},\varphi_{2n})\rightharpoonup\varphi:=(\varphi_1,\varphi_2) &&\text{weakly * in } L^\infty(0,T;(L^2(\Omega))^2)\\
&\varphi_n'=(\varphi_{1n}',\varphi_{2n}')\rightharpoonup\varphi':=(\varphi_1',\varphi_2') &&\text{weakly * in } L^\infty(0,T;H^{-1}(\Omega)\times L^2(\Omega)).
\end{aligned}\tag{5.22}$$

Let us consider the unique solution $\psi=(\psi_1,\psi_2)$ of problem (5.10) and let us take $\left(\psi_1,\theta_2^{-1}\psi_2\right)$ as test function in the variational formulation of problem (5.16). Then, we get

$$\begin{aligned}
&\int_0^T\int_\Omega\varphi_{1n}\left(\theta_1\psi_1''-\operatorname{div}(A^0\nabla\psi_1)+c_h(\theta_2\psi_1-\psi_2)\right)dxdt\\
&\quad+\int_0^T\int_\Omega\theta_2^{-1}\varphi_{2n}\left(\psi_2''-c_h(\theta_2\psi_1-\psi_2)\right)dxdt\\
&=-\int_\Omega\theta_1\varphi_{1n}^0\psi_1'(0)dx+\left\langle\theta_1\varphi_{1n}^1,\psi_1(0)\right\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}\\
&\quad-\int_\Omega\theta_2^{-1}\varphi_{2n}^0\psi_2'(0)dx+\int\theta_2^{-1}\varphi_{2n}^1\psi_2(0)dx.
\end{aligned}\tag{5.23}$$

By passing to the limit in (5.21) and (5.23), by convergences (5.15) and (5.22), we get that $\varphi=(\varphi_1,\varphi_2)$ satisfies (5.11), and (5.12).
Let us observe that inequality (5.21) gives in fact that $\varphi_n$ (resp. $\varphi_n'$) converges uniformly to $\varphi$ (resp. $\varphi'$) in $(L^2(\Omega))^2$ (resp. $H^{-1}(\Omega)\times L^2(\Omega)$) and therefore we get $\varphi=(\varphi_1,\varphi_2)\in C\left([0,T];(L^2(\Omega))^2\right)\cap C^1\left([0,T];H^{-1}(\Omega)\times L^2(\Omega)\right)$. ☐

Now, let $\psi\in C\left([0,T];(H_0^1(\Omega)\times L^2(\Omega))\right)\cap C^1\left([0,T];(L^2(\Omega))^2\right)$ be the unique solution of the backward problem

$$\begin{cases}
\theta_1\psi_1''-\operatorname{div}(A^0\nabla\psi_1)+c_h(\theta_2\psi_1-\psi_2)=-\varphi_1 & \text{in } \Omega\times]0,T[,\\
\psi_2''-c_h(\theta_2\psi_1-\psi_2)=-\varphi_2 & \text{in } \Omega\times]0,T[,\\
\psi_1=0 & \text{on } \partial\Omega\times]0,T[,\\
\psi_1(T)=\psi_1'(T)=0 & \text{in } \Omega,\\
\psi_2(T)=\psi_2'(T)=0 & \text{in } \Omega,
\end{cases}\tag{5.24}$$

where $\varphi$ is the unique solution of problem (5.9). Inspired by HUM method, we introduce the linear operator

$$\Lambda_1:(L^2(\Omega))^2\times(H^{-1}(\Omega)\times L^2(\Omega))\to(L^2(\Omega))^2\times(H_0^1(\Omega)\times L^2(\Omega))\tag{5.25}$$

by setting for all $\left(\varphi^0,\varphi^1\right)\in(L^2(\Omega))^2\times(H^{-1}(\Omega)\times L^2(\Omega))$,

$$\Lambda_1\left(\varphi^0,\varphi^1\right)=\left(\psi'(0),-\psi(0)\right),\tag{5.26}$$

where $\psi$ is the unique solution of problem (5.24). Moreover we define the pairing

$$\begin{aligned}
&\left\langle\Lambda_1\left(\varphi^0,\varphi^1\right),\left(\varphi^0,\varphi^1\right)\right\rangle=\left\langle\left(\psi'(0),-\psi(0)\right),\left(\varphi^0,\varphi^1\right)\right\rangle:=-\left\langle\theta_1\varphi_1^1,\psi_1(0)\right\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}\\
&\quad+\int_\Omega\theta_1\varphi_1^0\psi_1'(0)dx-\int_\Omega\theta_2^{-1}\varphi_2^1\psi_2(0)dx+\int_\Omega\theta_2^{-1}\varphi_2^0\psi_2'(0)dx,
\end{aligned}\tag{5.27}$$

for every $\left(\varphi^0, \varphi^1\right) \in (L^2(\Omega))^2 \times (H^{-1}(\Omega) \times L^2(\Omega))$.
The following lemma provides an explicit formula for the operator $\Lambda_1$.

**Lemma 5.1.** *Let us fix* $\left(\varphi^0, \varphi^1\right) \in (L^2(\Omega))^2 \times (H^{-1}(\Omega) \times L^2(\Omega))$*. Let* $\varphi$ *be the corresponding solution of problem (5.9). Then the following identity holds*

$$\left\langle \Lambda_1 \left(\varphi^0, \varphi^1\right), \left(\varphi^0, \varphi^1\right) \right\rangle = \int_0^T \int_\Omega |\varphi_1|^2 \, dxdt + \theta_2^{-1} \int_0^T \int_\Omega |\varphi_2|^2 \, dxdt. \tag{5.28}$$

**Proof.** Taking into account (5.27), it sufficies to consider in (5.11) the function $\psi = (\psi_1, \psi_2)$, unique solution of the backward problem (5.24). ☐

**Remark 5.1.** *By* (5.12) *and* (5.28)*, the operator* $\Lambda_1$ *is linear, continuous and injective. If* $\Lambda_1$ *is an isomorphism, we define the control* $\bar{\zeta}^{ex} \in L^2\left(0, T; (L^2(\Omega))^2\right)$ *as* $\bar{\zeta}^{ex} := -\varphi$ *where* $\varphi$ *is the solution of problem* (5.9) *with initial data* $\left(\varphi^0, \varphi^1\right) = \Lambda_1^{-1}\left(V^1, -V^0\right)$*. Indeed, by uniqueness, the state is given by* $v(\bar{\zeta}^{ex}) = \psi$*, where* $\psi$ *is the unique solution of the backward problem* (5.24)*. Hence* (3.4) *is satisfied and we obtain the exact controllability at time* $T > 0$ *for system* (3.3)*.*

In order to prove that the operator $\Lambda_1$ is an isomorphism from $(L^2(\Omega))^2 \times (H^{-1}(\Omega) \times L^2(\Omega))$ to $(L^2(\Omega))^2 \times (H_0^1(\Omega) \times L^2(\Omega))$, we need to show the following observability estimate (5.29), the converse inequality being an immediate consequence of (5.12).

**Proposition 5.1.** *Let us fix* $\left(\varphi^0, \varphi^1\right) \in (L^2(\Omega))^2 \times (H^{-1}(\Omega) \times L^2(\Omega))$*. Let* $\varphi$ *be the corresponding solution of problem* (5.9)*. Then, there exists a positive constant* $C$ *such that*

$$\left\|\varphi^0\right\|^2_{(L^2(\Omega))^2} + \left\|\varphi^1\right\|^2_{H^{-1}(\Omega) \times L^2(\Omega)} \leq C \left( \int_0^T \int_\Omega |\varphi_1|^2 \, dxdt + \theta_2^{-1} \int_0^T \int_\Omega |\varphi_2|^2 \, dxdt \right). \tag{5.29}$$

Let us establish a preliminary result with more regular data $\left(\varphi^0, \varphi^1\right) \in (H_0^1(\Omega) \times L^2(\Omega)) \times (L^2(\Omega))^2$. We can define the solution $\varphi$ of problem (5.9) by the usual weak formulation. Hence the associated energy is:

$$\begin{aligned} E(t) := \; & \frac{1}{2} \left[ \theta_1 \int_\Omega |\varphi_1'(t)|^2 dx + \theta_2^{-1} \int_\Omega |\varphi_2'(t)|^2 dx + \int_\Omega A^0 \nabla \varphi_1(t) \nabla \varphi_1(t) dx \right. \\ & \left. + c_h \theta_2^{-1} \int_\Omega |\theta_2 \varphi_1(t) - \varphi_2(t)|^2 dx \right]. \end{aligned} \tag{5.30}$$

**Lemma 5.2.** *There exists a positive constant* $C$ *such that*

$$E(0) \leq C \left( \int_0^T \int_\Omega |\varphi_1'|^2 \, dxdt + \theta_2^{-1} \int_0^T \int_\Omega |\varphi_2'|^2 \, dxdt \right), \tag{5.31}$$

*where*

$$E(0) = \frac{1}{2} \left[ \theta_1 \int_\Omega |\varphi_1^1|^2 dx + \theta_2^{-1} \int_\Omega |\varphi_2^1|^2 dx + \int_\Omega A^0 \nabla \varphi_1^0 \nabla \varphi_1^0 dx + c_h \theta_2^{-1} \int_\Omega |\theta_2 \varphi_1^0 - \varphi_2^0|^2 dx \right]. \tag{5.32}$$

**Proof.** First note that the energy defined in (5.30) is conserved (see the proof of Lemma 4.4 in [15]), that is

$$E(t) = E(0), \quad \text{for every } t \in [0, T]. \tag{5.33}$$

As in the previous subsection, let $\rho(t)$ be the function defined by

$$\rho(t) = t^2(T-t)^2, \quad \text{for every } t \in [0, T]. \tag{5.34}$$

By choosing $\rho(t)\varphi_1(x,t)$ as test function in the first equation of problem (5.9) and $\rho(t)\theta_2^{-1}\varphi_2(x,t)$ as test function in the second equation of problem (5.9) and integrating by parts, we obtain respectively

$$\begin{gathered}\int_0^T \int_\Omega \theta_1 \rho |\varphi_1'|^2 dxdt + \int_0^T \int_\Omega \theta_1 \rho' \varphi_1 \varphi_1' dxdt = \int_0^T \int_\Omega \rho A^0 \nabla\varphi_1 \nabla\varphi_1 dxdt \\ + c_h \int_0^T \int_\Omega \rho(\theta_2\varphi_1 - \varphi_2)\varphi_1 dxdt\end{gathered} \tag{5.35}$$

and

$$\int_0^T \int_\Omega \theta_2^{-1} \rho |\varphi_2'|^2 dxdt + \int_0^T \int_\Omega \theta_2^{-1} \rho' \varphi_2 \varphi_2' dxdt = -c_h \theta_2^{-1} \int_0^T \int_\Omega \rho(\theta_2\varphi_1 - \varphi_2)\varphi_2 dxdt. \tag{5.36}$$

Summing up (5.35) and (5.36) we get

$$\begin{gathered}\int_0^T \int_\Omega \theta_1 \rho |\varphi_1'|^2 dxdt + \int_0^T \int_\Omega \theta_1 \rho' \varphi_1 \varphi_1' dxdt + \int_0^T \int_\Omega \theta_2^{-1} \rho |\varphi_2'|^2 dxdt + \int_0^T \int_\Omega \theta_2^{-1} \rho' \varphi_2 \varphi_2' dxdt \\ = \int_0^T \int_\Omega \rho A^0 \nabla\varphi_1 \nabla\varphi_1 dxdt + c_h \theta_2^{-1} \int_0^T \int_\Omega \rho |\theta_2\varphi_1 - \varphi_2|^2 dxdt.\end{gathered} \tag{5.37}$$

Denoting for every $t \in [0, T]$

$$E_\rho(t) := \int_\Omega \rho(t) A^0 \nabla\varphi_1(t) \nabla\varphi_1(t) dx + c_h \theta_2^{-1} \int_\Omega \rho(t) |\theta_2\varphi_1(t) - \varphi_2(t)|^2 dx, \tag{5.38}$$

(5.37) can be written as

$$\begin{gathered}\int_0^T \int_\Omega \theta_1 \rho |\varphi_1'|^2 dxdt + \int_0^T \int_\Omega \theta_1 \rho' \varphi_1 \varphi_1' dxdt + \int_0^T \int_\Omega \theta_2^{-1} \rho |\varphi_2'|^2 dxdt \\ + \int_0^T \int_\Omega \theta_2^{-1} \rho' \varphi_2 \varphi_2' dxdt = \int_0^T E_\rho \, dt.\end{gathered} \tag{5.39}$$

Then, by making use of Young's inequality, for any $\delta > 0$, we get

$$\begin{aligned}&\int_0^T \int_\Omega \theta_1 \rho' \varphi_1 \varphi_1' dxdt + \int_0^T \int_\Omega \theta_2^{-1} \rho' \varphi_2 \varphi_2' dxdt \leq \delta \int_0^T \int_\Omega \theta_1 |\varphi_1|^2 \rho dxdt \\ &+ C(\delta) \int_0^T \int_\Omega \theta_1 |\varphi_1'|^2 dxdt + \delta \int_0^T \int_\Omega \theta_2^{-1} |\varphi_2|^2 \rho dxdt + C(\delta) \int_0^T \int_\Omega \theta_2^{-1} |\varphi_2'|^2 dxdt \\ &\leq \delta \int_0^T \rho \int_\Omega \left(\theta_1 |\varphi_1|^2 + \theta_2^{-1} |\varphi_2|^2\right) dxdt + C(\delta) \int_0^T \left(\int_\Omega \theta_1 |\varphi_1'|^2 + \theta_2^{-1} |\varphi_2'|^2\right) dxdt,\end{aligned} \tag{5.40}$$

where, as in the proof of Lemma 4.2, $C(\delta) = \dfrac{T^2}{\delta}$.
By collecting together (5.39) and (5.40) we get

$$\begin{aligned}\int_0^T E_\rho dt \leq \int_0^T \rho \int_\Omega \left(\theta_1|\varphi_1'|^2 + \theta_2^{-1}|\varphi_2'|^2\right) dxdt + \delta \int_0^T \rho \underbrace{\int_\Omega \left(\theta_1|\varphi_1|^2 + \theta_2^{-1}|\varphi_2|^2\right) dx\, dt}_{*} \\ +C(\delta)\int_0^T \left(\int_\Omega \theta_1|\varphi_1'|^2 + \theta_2^{-1}|\varphi_2'|^2\right) dxdt.\end{aligned} \tag{5.41}$$

Now we want to estimate (*) in (5.41) in terms of $E_\rho(t)$. To this aim, let us observe that by applying twice Poincaré inequality and by considering the ellipticity of the matrix $A^0$, we get

$$\begin{aligned}\theta_1\|\varphi_1\|^2_{L^2(\Omega)} + \theta_2^{-1}\|\varphi_2\|^2_{L^2(\Omega)} &\leq \left(C_1\|\nabla\varphi_1\|^2_{L^2(\Omega)} + C_2\theta_2^{-1}\|\varphi_2 - \theta_2\varphi_1\|^2_{L^2(\Omega)}\right) \\ &\leq C\left(\int_\Omega A^0\nabla\varphi_1\nabla\varphi_1 dx + c_h\theta_2^{-1}\int_\Omega |\varphi_2 - \theta_2\varphi_1|^2 dx\right)\end{aligned}$$

with $C$ positive constant.
Hence (5.41) becomes

$$\int_0^T E_\rho(t)dt \leq C'(\delta)\int_0^T\int_\Omega \left(\theta_1|\varphi_1'|^2 + \theta_2^{-1}|\varphi_2'|^2\right) dt + C\delta\int_0^T E_\rho(t)dt.$$

Finally we choose $\delta$ such that $1 - C\delta > 0$ and get

$$\int_0^T E_\rho(t)dt \leq C''(\delta)\int_0^T\int_\Omega \left(\theta_1|\varphi_1'|^2 + \theta_2^{-1}|\varphi_2'|^2\right) dt. \tag{5.42}$$

Multiplying equation in (5.33) by $\rho(t)$ and integrating in $[0,T]$, we obtain

$$\int_0^T \rho E(0)dt = \int_0^T \rho E dt = \frac{1}{2}\int_0^T E_\rho dt + \frac{1}{2}\int_0^T \rho\int_\Omega \left(\theta_1|\varphi_1'|^2 + \theta_2^{-1}|\varphi_2'|^2\right) dxdt$$

which by (5.42) and since $\theta_1 < 1$, yields (5.31). ☐

Now we are able to prove Proposition 5.1. Let us fix $\varphi^1 \in H^{-1}(\Omega) \times L^2(\Omega)$ and let $\pi \in H_0^1(\Omega) \times L^2(\Omega)$ be the unique solution of problem

$$\begin{cases} -\mathrm{div}(A^0\nabla\pi_1) + c_h(\theta_2\pi_1 - \pi_2) = -\theta_1\varphi_1^1 & \text{in } \Omega, \\ -c_h(\theta_2\pi_1 - \pi_2) = -\varphi_2^1 & \text{in } \Omega, \\ \pi_1 = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.43}$$

Problem (5.43) is equivalent to

$$\begin{cases} -\mathrm{div}(A_1^0\nabla\pi_1) = -(\theta_1\varphi_1^1 + \varphi_2^1) & \text{in } \Omega, \\ \pi_1 = 0 & \text{on } \partial\Omega. \end{cases}$$

By classical arguments concerning symmetric bilinear forms associated to elliptic equations (see [33] in the proof of Proposition 1.2), one has

$$C_1(\|\theta_1\varphi_1^1+\varphi_2^1\|_{H^{-1}(\Omega)}) \le \|\pi_1\|_{H_0^1} \le C_2(\|\theta_1\varphi_1^1+\varphi_2^1\|_{H^{-1}(\Omega)}). \tag{5.44}$$

Let $\varphi$ be the transposition solution of problem (5.9) corresponding to the initial data $\left(\varphi^0,\varphi^1\right) \in (L^2(\Omega))^2 \times (H^{-1}(\Omega)\times L^2(\Omega))$. Then the function $\varpi=(\varpi_1,\varpi_2)$ defined by

$$\varpi_i(x,t) := \int_0^t \varphi_i(x,s)ds + \pi_i(x), \quad i=1,2, \tag{5.45}$$

satisfies the problem

$$\begin{cases} \theta_1\varpi_1'' - \operatorname{div}(A^0\nabla\varpi_1) + c_h(\theta_2\varpi_1-\varpi_2) = 0 & \text{in } \Omega\times]0,T[,\\ \varpi_2'' - c_h(\theta_2\varpi_1-\varpi_2)=0 & \text{in } \Omega\times]0,T[,\\ \varpi_1 = 0 & \text{on } \partial\Omega\times]0,T[,\\ \varpi_1(0)=\pi_1, \quad \varpi_1'(0)=\varphi_1^0 & \text{in } \Omega,\\ \varpi_2(0)=\pi_2, \quad \varpi_2'(0)=\varphi_2^0 & \text{in } \Omega. \end{cases} \tag{5.46}$$

Observe that, since $\pi \in H_0^1(\Omega)\times L^2(\Omega)$ and $\varphi^0 \in (L^2(\Omega))^2$, the solution $\varpi$ is defined by usual weak formulation. Hence, by applying Lemma 5.2, in view of (5.45) we get

$$\begin{aligned} &\frac{1}{2}\left[\theta_1\int_\Omega |\varphi_1^0|^2dx + \theta_2^{-1}\int_\Omega|\varphi_2^0|^2dx + \int_\Omega A^0\nabla\pi_1\nabla\pi_1dx + c_h\theta_2^{-1}\int_\Omega|\theta_2\pi_1-\pi_2|^2dx\right]\\ &\qquad \le C\left(\int_0^T\int_\Omega|\varpi_1'|^2dxdt + \theta_2^{-1}\int_0^T\int_\Omega|\varpi_2'|^2dxdt\right)\\ &\qquad = C\left(\int_0^T\int_\Omega|\varphi_1|^2dxdt + \theta_2^{-1}\int_0^T\int_\Omega|\varphi_2|^2dxdt\right) \end{aligned} \tag{5.47}$$

with $C$ positive constant. By (5.43), inequality (5.47) gives

$$\begin{aligned} &\frac{1}{2}\theta_1\|\varphi_1^0\|^2_{L^2(\Omega)} + \frac{1}{2}\theta_2^{-1}\|\varphi_2^0\|^2_{L^2(\Omega)} + \alpha\|\nabla\pi_1\|^2_{L^2(\Omega)} + \|\varphi_2^1\|^2_{L^2(\Omega)}\\ &\qquad \le C\left(\int_0^T\int_\Omega|\varphi_1|^2dxdt + \theta_2^{-1}\int_0^T\int_\Omega|\varphi_2|^2dxdt\right). \end{aligned}$$

Hence, by (5.44), we easily get (5.29). □

## 6. Proof of the main convergence result

### *6.1. Homogenization of the transposed problem*

In order to prove the main convergence result given in Theorem 3.4 we need to study the homogenization of problem (4.1) whose solution is defined by the method of transposition. To this aim, at first we need to deduce some a priori estimates for both the initial conditions $\left(\varphi_\varepsilon^0,\varphi_\varepsilon^1\right)$ and the corresponding solution $\varphi_\varepsilon$ of problem (4.1). We start with the following proposition which is a direct consequence of (4.11) in Proposition 4.1.

**Proposition 6.1.** *Let* $\left(\varphi^0_\varepsilon, \varphi^1_\varepsilon\right) \in L^2_\varepsilon(\Omega) \times \left(H^\varepsilon_\gamma\right)'$ *be the initial conditions of problem* (4.1)*. Then, for any* $\varepsilon$*, we have*

$$\|(\varphi^0_\varepsilon, \varphi^1_\varepsilon)\|_{L^2_\varepsilon(\Omega)\times\left(H^\varepsilon_\gamma\right)'} \leq C, \tag{6.1}$$

*with* $C$ *positive constant independent of* $\varepsilon$*.*

From estimate (6.1), we deduce that there exists $\Phi^0 \in (L^2(\Omega))^2$ such that, up to a subsequence still denoted by $\varepsilon$,

$$\widetilde{\varphi^0_\varepsilon} \rightharpoonup \Phi^0 \text{ weakly in} \, (L^2(\Omega))^2. \tag{6.2}$$

Moreover, in view of (4.4), we also get

$$\|\varphi_\varepsilon\|_{L^2(0,T;L^2_\varepsilon(\Omega))} \leq C, \tag{6.3}$$

$$\|\varphi'_\varepsilon\|_{L^2\left(0,T;\left(H^\varepsilon_\gamma\right)'\right)} \leq C, \tag{6.4}$$

with $C$ positive constant independent of $\varepsilon$.

**Theorem 6.1.** *Let* $\varphi_\varepsilon$ *be the unique solution of problem* (4.1) *corresponding to the initial data* $\left(\varphi^0_\varepsilon, \varphi^1_\varepsilon\right) \in L^2_\varepsilon(\Omega) \times \left(H^\varepsilon_\gamma\right)'$*. Then there exists a subsequence of* $\varphi_\varepsilon$*, still denoted by* $\varepsilon$*, such that, as* $\varepsilon \to 0$

$$\begin{aligned} &\widetilde{\varphi_{1\varepsilon}} \rightharpoonup \theta_1 \Phi_1 \;\; in \, L^2\left(0,T;L^2(\Omega)\right) \\ &\widetilde{\varphi_{2\varepsilon}} \rightharpoonup \Phi_2 \;\; in \, L^2\left(0,T;L^2(\Omega)\right), \end{aligned} \tag{6.5}$$

*where* $\Phi_1$ *and* $\Phi_2$ *are different for the two cases* $-1 < \gamma < 1$ *and* $\gamma = 1$*.*
***Case*** $-1 < \gamma < 1$*: There exists a function* $\Phi^*_1$ *in* $H^{-1}(\Omega)$ *such that* $\Phi_1 \in L^2(0,T;L^2(\Omega))$*, with* $\Phi'_1 \in L^2(0,T;L^2(\Omega))$*, is the unique solution of the following homogenized problem*

$$\begin{cases} \Phi''_1 - \operatorname{div}\left(A^0\nabla\Phi_1\right) = 0 & in \ \Omega\times]0,T[, \\ \Phi_1 = 0 & on \ \partial\Omega\times]0,T[, \\ \Phi_1(0) = \Phi^0_1 + \Phi^0_2 & in \ \Omega, \\ \Phi'_1(0) = \Phi^*_1 & in \ \Omega. \end{cases} \tag{6.6}$$

*Moreover*

$$\Phi_2 = \theta_2 \Phi_1 \tag{6.7}$$

*where* $\theta_2$ *is given in* (2.2)*.*
***Case*** $\gamma = 1$*: There exist two functions* $\Phi^*_1$ *and* $\Phi^*_2$ *in* $H^{-1}(\Omega)$ *with* $\Phi^*_1 - \Phi^*_2 \in L^2(\Omega)$*, such that the pair* $(\Phi_1, \Phi_2) \in L^2\left(0,T;(L^2(\Omega))^2\right)$*, with* $(\Phi'_1, \Phi'_2) \in L^2\left(0,T;L^2(\Omega)\times H^{-1}(\Omega)\right)$*, is the unique solution of the coupled system*

$$\begin{cases} \theta_1\Phi''_1 - \operatorname{div}\left(A^0\nabla\Phi_1\right) + c_h\left(\theta_2\Phi_1 - \Phi_2\right) = 0 & in \ \Omega\times]0,T[, \\ \Phi''_2 - c_h\left(\theta_2\Phi_1 - \Phi_2\right) = 0 & in \ \Omega\times]0,T[, \\ \Phi_1 = 0 & on \ \partial\Omega\times]0,T[, \\ \Phi_1(0) = \dfrac{\Phi^0_1}{\theta_1}, \ \Phi'_1(0) = \dfrac{\Phi^*_2}{\theta_1} & in \ \Omega, \\ \Phi_2(0) = \Phi^0_2, \ \Phi'_2(0) = \Phi^*_1 - \Phi^*_2 & in \ \Omega. \end{cases} \tag{6.8}$$

*where* $c_h = \dfrac{1}{|Y_2|}\displaystyle\int_\Gamma h(y)d\sigma_y > 0$ *and* $\theta_i$*,* $i = 1, 2$*, are defined in* (2.2)*.*

**Proof.** Estimate (6.3) provides the existence of two functions $\bar{\varphi} \in L^2\left(0,T;L^2(\Omega)\right)$ and $\Phi_2 \in L^2\left(0,T;L^2(\Omega)\right)$ such that, up to a subsequence,

$$\begin{aligned} &\widetilde{\varphi_{1\varepsilon}} \rightharpoonup \bar{\varphi} \ \text{ in } L^2\left(0,T;L^2(\Omega)\right), \\ &\widetilde{\varphi_{2\varepsilon}} \rightharpoonup \Phi_2 \ \text{ in } L^2\left(0,T;L^2(\Omega)\right). \end{aligned} \tag{6.9}$$

Let $\xi_\varepsilon := (\xi_{1\varepsilon}, \xi_{2\varepsilon})$ be the unique solution of the following system

$$\begin{cases} -\mathrm{div}(A^\varepsilon \nabla \xi_{i\varepsilon}) = -\varphi^1_{i\varepsilon} & \text{in } \Omega_{i\varepsilon},\ i=1,2, \\ A^\varepsilon \nabla \xi_{1\varepsilon} \cdot n_{1\varepsilon} = -A^\varepsilon \nabla \xi_{2\varepsilon} \cdot n_{2\varepsilon} & \text{on } \Gamma^\varepsilon, \\ A^\varepsilon \nabla \xi_{1\varepsilon} \cdot n_{1\varepsilon} = -\varepsilon^\gamma h^\varepsilon (\xi_{1\varepsilon} - \xi_{2\varepsilon}) & \text{on } \Gamma^\varepsilon, \\ \xi_{1\varepsilon} = 0 & \text{on } \partial\Omega. \end{cases} \tag{6.10}$$

By hypotheses (2.5)-(2.8), the assumption $\varphi^1_\varepsilon \in (H^\varepsilon_\gamma)'$ and estimate (6.1) the results of Theorem 2.1 in [25] apply obtaining that there exist an extension operator $Q^\varepsilon_1 \in \mathcal{L}\left(V^\varepsilon; H^1_0(\Omega)\right)$ and a function $\Phi^*_1 \in H^{-1}(\Omega)$ sucht that

$$\begin{cases} \text{(i) } Q^\varepsilon_1 \xi_{1\varepsilon} \rightharpoonup \xi_1 & \text{weakly in } H^1_0(\Omega), \\ \text{(ii) } \widetilde{\xi_{1\varepsilon}} \rightharpoonup \theta_1 \xi_1 & \text{weakly in } L^2(\Omega), \\ \text{(iii) } \widetilde{\xi_{2\varepsilon}} \rightharpoonup \xi_2 & \text{weakly in } L^2(\Omega), \end{cases} \tag{6.11}$$

with $\theta_1$ given by (2.2) and $\xi_1 \in H^1_0(\Omega)$ unique solution of

$$\begin{cases} -\mathrm{div}\ \left(A^0 \nabla \xi_1\right) = -\Phi^*_1 & \text{in } \Omega, \\ \xi_1 = 0 & \text{on } \partial\Omega, \end{cases} \tag{6.12}$$

where $A^0$ is the matrix defined in (2.15) and (2.16). Let us observe that in (6.12), the function $\Phi^*_1$ is not exactly the sum of the weak limits of $\varphi^1_{1\varepsilon}$ and $\varphi^1_{2\varepsilon}$ as in the case of more regular data, but a more complicated function depending on a subsequence of $\varphi^1_{i\varepsilon}$, $i=1,2$ and on some auxiliary functions (see [25] for details).
Denote

$$\sigma_{i\varepsilon}(x,t) := \int_0^t \varphi_{i\varepsilon}(x,s)ds + \xi_{i\varepsilon}(x), \quad i=1,2. \tag{6.13}$$

We do observe that this transformation leads to a system whose initial data are more regular than $\left(\varphi^0_\varepsilon, \varphi^1_\varepsilon\right)$. Indeed, $\sigma_\varepsilon := (\sigma_{1\varepsilon}, \sigma_{2\varepsilon})$ satisfies

$$\begin{cases} \sigma''_{i\varepsilon} - \mathrm{div}(A^\varepsilon \nabla \sigma_{i\varepsilon}) = 0 & \text{in } \Omega_{i\varepsilon} \times ]0,T[,\ i=1,2, \\ A^\varepsilon \nabla \sigma_{1\varepsilon} \cdot n_{1\varepsilon} = -A^\varepsilon \nabla \sigma_{2\varepsilon} \cdot n_{2\varepsilon} & \text{on } \Gamma^\varepsilon \times ]0,T[, \\ A^\varepsilon \nabla \sigma_{1\varepsilon} \cdot n_{1\varepsilon} = -\varepsilon^\gamma h^\varepsilon (\sigma_{1\varepsilon} - \sigma_{2\varepsilon}) & \text{on } \Gamma^\varepsilon \times ]0,T[, \\ \sigma_{1\varepsilon} = 0 & \text{on } \partial\Omega \times ]0,T[, \\ \sigma_{1\varepsilon}(0) = \xi_{1\varepsilon}, \quad \sigma'_{1\varepsilon}(0) = \varphi^0_{1\varepsilon} & \text{in } \Omega_{1\varepsilon}, \\ \sigma_{2\varepsilon}(0) = \xi_{2\varepsilon}, \quad \sigma'_{2\varepsilon}(0) = \varphi^0_{2\varepsilon} & \text{in } \Omega_{2\varepsilon} \end{cases} \tag{6.14}$$

and since $\varphi^1_\varepsilon \in (H^\varepsilon_\gamma)'$, one has $\xi_\varepsilon \in H^\varepsilon_\gamma$, hence the initial data $\left(\xi_\varepsilon, \varphi^0_\varepsilon\right) \in H^\varepsilon_\gamma \times L^2_\varepsilon(\Omega)$. Moreover, by (6.1) and (6.10) we get, for any $\varepsilon$,

$$\|\xi_\varepsilon\|_{H^\varepsilon_\gamma} \leq C \tag{6.15}$$

with $C$ positive constant independent of $\varepsilon$.
By (6.2), (6.11)ii), (6.11)iii), with $\xi_2 = \theta_2 \xi_1$ for $-1 < \gamma < 1$, and (6.15) we can apply Theorem 2.1 to system (6.14) obtaining the existence of an extension operator

$$P_1^\varepsilon \in \mathcal{L}\left(L^\infty(0,T;H^k(\Omega_{1\varepsilon})); L^\infty(0,T;H^k(\Omega))\right) \text{ for } k = 1,2,$$

such that

$$\begin{cases} \text{(i) } P_1^\varepsilon \sigma_{1\varepsilon} \rightharpoonup \sigma_1 & \text{weakly * in } L^\infty(0,T;H_0^1(\Omega)), \\ \text{(ii) } P_1^\varepsilon \sigma_{1\varepsilon}' \rightharpoonup \sigma_1' & \text{weakly * in } L^\infty(0,T;L^2(\Omega)), \\ \text{(iii) } \widetilde{\sigma_{2\varepsilon}} \rightharpoonup \sigma_2 & \text{weakly * in } L^\infty(0,T;L^2(\Omega)), \\ \text{(iv) } \widetilde{\sigma_{2\varepsilon}'} \rightharpoonup \sigma_2' & \text{weakly * in } L^\infty(0,T;L^2(\Omega)). \end{cases} \tag{6.16}$$

Moreover one gets

$$\begin{cases} \text{(i) } \widetilde{\sigma_{1\varepsilon}} \rightharpoonup \theta_1 \sigma_1 & \text{weakly in } L^2(0,T;L^2(\Omega)), \\ \text{(ii) } \widetilde{\sigma_{1\varepsilon}'} \rightharpoonup \theta_1 \sigma_1' & \text{weakly in } L^2(0,T;L^2(\Omega)). \end{cases} \tag{6.17}$$

Now let us study in a separate way the cases $-1 < \gamma < 1$ and $\gamma = 1$.
**Case** $-1 < \gamma < 1$: Since, as already observed, in this case, $\xi_2 = \theta_2 \xi_1$ where $\xi_1$ is the unique solution of problem (6.12)(see [25]), by applying the results in Theorem 2.1 to problem (6.14), we have that $\sigma_1$ is the unique solution of the homogenized system

$$\begin{cases} \sigma_1'' - \text{div } \left(A^0 \nabla \sigma_1\right) = 0 & \text{in } \Omega \times ]0,T[, \\ \sigma_1 = 0 & \text{on } \partial\Omega \times ]0,T[, \\ \sigma_1(0) = \xi_1 & \text{in } \Omega, \\ \sigma_1'(0) = \Phi_1^0 + \Phi_2^0 & \text{in } \Omega \end{cases} \tag{6.18}$$

and $\sigma_2 = \theta_2 \sigma_1$.
By (6.13) it results

$$\widetilde{\sigma_{i\varepsilon}'} = \widetilde{\varphi_{i\varepsilon}}, \quad i = 1,2. \tag{6.19}$$

Hence (6.9), (6.16)iv) and (6.17)ii), by passing to the limit in (6.19), provide $\bar{\varphi} = \theta_1 \sigma_1'$ and $\varphi_2 = \theta_2 \sigma_1'$. 225

By classical regularity results for hyperbolic equations we have

$$\sigma_1 \in C\left([0,T]; H_0^1(\Omega)\right) \cap C^1\left([0,T]; L^2(\Omega)\right) \cap C^2\left([0,T]; H^{-1}(\Omega)\right).$$

Hence, by (6.12) and (6.14)

$$\sigma_1''(0) = \text{div } \left(A^0 \nabla \sigma_1(0)\right) = \text{div } \left(A^0 \nabla \xi_1\right) = \Phi_1^*.$$

Therefore, the function $\Phi_1 := \sigma_1' = \dfrac{\bar{\Phi}}{\theta_1}$ is the unique solution in the sense of transposition of system (6.6) and $\Phi_2 = \theta_2 \Phi_1$.
**Case** $\gamma = 1$: In this case

$$\xi_2 = \theta_2 \xi_1 - c_h^{-1} \Phi_1^* + c_h^{-1} \Phi_2^*, \tag{6.20}$$

where $\xi_1$ is the unique solution of problem (6.12), $c_h = \dfrac{1}{|Y_2|} \displaystyle\int_\Gamma h(y)\, d\sigma_y > 0$, $\theta_2$ is given in (2.2), $\Phi_1^*$ is the same as in the previous case and $\Phi_2^*$ is again a function in $H^{-1}(\Omega)$ depending on a subsequence of $\varphi^1_{1\varepsilon}$ and on some auxiliary functions (see [25] for details). Hence, by applying the results in Theorem 2.1 to problem (6.14), we have that the couple $(\sigma_1, \sigma_2)$ is the unique solution of the following homogenized system

$$\begin{cases} \theta_1 \sigma_1'' - \operatorname{div}\left(A^0 \nabla \sigma_1\right) + c_h\left(\theta_2 \sigma_1 - \sigma_2\right) = 0 & \text{in } \Omega \times ]0,T[, \\ \sigma_2'' - c_h\left(\theta_2 \sigma_1 - \sigma_2\right) = 0 & \text{in } \Omega \times ]0,T[, \\ \sigma_1 = 0 & \text{on } \partial\Omega \times ]0,T[, \\ \sigma_1(0) = \xi_1, \ \sigma_1'(0) = \dfrac{\Phi_1^0}{\theta_1} & \text{in } \Omega, \\ \sigma_2(0) = \xi_2, \ \sigma_2'(0) = \Phi_2^0 & \text{in } \Omega. \end{cases} \tag{6.21}$$

By (6.13) again, it results

$$\widetilde{\sigma'_{i\varepsilon}} = \widetilde{\varphi_{i\varepsilon}}, \quad i = 1, 2. \tag{6.22}$$

Hence, (6.9), (6.16)iv) and (6.17)ii), by passing to the limit in (6.22), provide $\bar{\varphi} = \theta_1 \sigma_1'$ and $\varphi_2 = \sigma_2'$.

By (5.8) and (6.21) we have

$$\sigma_1 \in C\left([0,T]; H_0^1(\Omega)\right) \cap C^1\left([0,T]; L^2(\Omega)\right) \cap C^2\left([0,T]; H^{-1}(\Omega)\right)$$

and

$$\sigma_2 \in C\left(0,T; L^2(\Omega)\right) \cap C^1\left(0,T; L^2(\Omega)\right) \cap C^2\left(0,T; L^2(\Omega)\right).$$

Hence, by (6.12), (6.20) and (6.21), we have

$$\sigma_2''(0) = c_h\left(\theta_2 \sigma_1(0) - \sigma_2(0)\right) = c_h\left(\theta_2 \xi_1 - \xi_2\right) = \Phi_1^* - \Phi_2^* \tag{6.23}$$

and

$$\theta_1 \sigma_1''(0) = \operatorname{div}\left(A^0 \nabla \sigma_1(0)\right) - c_h\left(\theta_2 \sigma_1(0) - \sigma_2(0)\right) = \Phi_2^*. \tag{6.24}$$

Therefore the pair $(\Phi_1, \Phi_2) := (\sigma_1', \sigma_2')$ is the unique solution in the sense of transposition of system (6.8). Now the proof is complete. □

### 6.2. *Proof of Theorem 3.4*

Let $\zeta^{ex}_\varepsilon$ be the exact control of problem (2.3) given by Theorem 3.1, with respect to the initial conditions $(U^0_\varepsilon, U^1_\varepsilon) \in H^\varepsilon_\gamma \times L^2_\varepsilon(\Omega)$. In Section 4, see also Remark 4.1, it is proved that

$$\zeta^{ex}_\varepsilon = -\varphi_\varepsilon, \tag{6.25}$$

where $\varphi_\varepsilon$ is the unique solution of the transposed problem (4.1) with respect to the initial conditions $(\varphi^0_\varepsilon, \varphi^1_\varepsilon)$ satisfying

$$\Lambda_\varepsilon\left(\varphi^0_\varepsilon, \varphi^1_\varepsilon\right) = \left(U^1_\varepsilon, -U^0_\varepsilon\right).$$

Moreover the corresponding state of system (2.3) is such that

$$u_\varepsilon(\zeta^{ex}_\varepsilon) = \psi_\varepsilon, \tag{6.26}$$

where $\psi_\varepsilon = \psi_\varepsilon(\varphi_\varepsilon)$ is the unique solution of the corresponding backward problem (4.5). Observe that, as a result of (6.5) in Theorem 6.1 and (6.25), we get

$$\begin{cases} \text{(i) } \widetilde{\zeta^{ex}_{1\varepsilon}} \rightharpoonup \widehat{\zeta}_1 & \text{weakly in } L^2(0,T; L^2(\Omega)), \\ \text{(ii) } \widetilde{\zeta^{ex}_{2\varepsilon}} \rightharpoonup \widehat{\zeta}_2 & \text{weakly in } L^2(0,T; L^2(\Omega)) \end{cases} \tag{6.27}$$

up to a subsequence, with

$$\widehat{\zeta_1} = -\theta_1 \Phi_1, \quad \widehat{\zeta_2} = -\Phi_2. \tag{6.28}$$

In view of (3.5) and (6.27), Theorem 2.1 applies to problem (2.3) for the choice $Z^0_\varepsilon = U^0_\varepsilon$, $Z^1_\varepsilon = U^1_\varepsilon$, $Z^0 = U^0$, $Z^1 = U^1$ and $g_\varepsilon = \zeta^{ex}_\varepsilon$. Analogously, by (6.5) in Theorem 6.1, we can apply Theorem 2.1 to problem (4.5).
Let $P^\varepsilon_1 \in \mathcal{L}(L^\infty(0,T;H^k(\Omega_{1\varepsilon})); L^\infty(0,T;H^k(\Omega)))$, for $k = 1, 2$ the extension operator introduced in Theorem 2.1 and $\theta_i$, $i = 1, 2$ be given in (2.2).

**Case** $-1 < \gamma < 1$: By (6.28) and (6.7) in Theorem 6.1, one has

$$\widehat{\zeta_2} = \frac{\theta_2}{\theta_1} \widehat{\zeta_1}. \tag{6.29}$$

Hence, up to a subsequence, we obtain the following convergences

$$\begin{cases} P^\varepsilon_1 u_{1\varepsilon}(\zeta^{ex}_\varepsilon) \rightharpoonup u_1(\widehat{\zeta_1}) & \text{weakly* in } L^\infty\left(0,T;H^1_0(\Omega)\right), \\ \widetilde{u_{1\varepsilon}(\zeta^{ex}_\varepsilon)} \rightharpoonup \theta_1 u_1(\widehat{\zeta_1}) & \text{weakly* in } L^\infty\left(0,T;L^2(\Omega)\right), \\ \widetilde{u_{2\varepsilon}(\zeta^{ex}_\varepsilon)} \rightharpoonup \theta_2 u_1(\widehat{\zeta_1}) & \text{weakly* in } L^\infty\left(0,T;L^2(\Omega)\right), \end{cases} \tag{6.30}$$

$$\begin{cases} P^\varepsilon_1 u'_{1\varepsilon}(\zeta^{ex}_\varepsilon) \rightharpoonup u'_1(\widehat{\zeta_1}) & \text{weakly* in } L^\infty\left(0,T;L^2(\Omega)\right), \\ \widetilde{u'_{1\varepsilon}(\zeta^{ex}_\varepsilon)} \rightharpoonup \theta_1 u'_1(\widehat{\zeta_1}) & \text{weakly* in } L^\infty\left(0,T;L^2(\Omega)\right), \\ \widetilde{u'_{2\varepsilon}(\zeta^{ex}_\varepsilon)} \rightharpoonup \theta_2 u'_1(\widehat{\zeta_1}) & \text{weakly* in } L^\infty\left(0,T;L^2(\Omega)\right), \end{cases} \tag{6.31}$$

where $u_1 := u_1(\widehat{\zeta_1}) \in L^2(0,T;H^1_0(\Omega))$ with $u'_1 := u'_1(\widehat{\zeta_1}) \in L^2\left(0,T;L^2(\Omega)\right)$ is the unique solution of the following homogenized problem

$$\begin{cases} u''_1 - \operatorname{div}\left(A^0 \nabla u_1\right) = \dfrac{\widehat{\zeta_1}}{\theta_1} & \text{in } \Omega \times ]0,T[, \\ u_1 = 0 & \text{on } \partial\Omega \times ]0,T[, \\ u_1(0) = U^0_1 + U^0_2 & \text{in } \Omega, \\ u'_1(0) = U^1_1 + U^1_2 & \text{in } \Omega. \end{cases} \tag{6.32}$$

On the other hand, by(6.7) in Theorem 6.1, for the backward problem (4.5) we obtain, up to a subsequence, the following convergences

$$\begin{cases} P^\varepsilon_1 \psi_{1\varepsilon}(\varphi_\varepsilon) \rightharpoonup \psi_1(\Phi_1) & \text{weakly* in } L^\infty\left(0,T;H^1_0(\Omega)\right), \\ \widetilde{\psi_{1\varepsilon}(\varphi_\varepsilon)} \rightharpoonup \theta_1 \psi_1(\Phi_1) & \text{weakly* in } L^\infty\left(0,T;L^2(\Omega)\right), \\ \widetilde{\psi_{2\varepsilon}(\varphi_\varepsilon)} \rightharpoonup \theta_2 \psi_1(\Phi_1) & \text{weakly* in } L^\infty\left(0,T;L^2(\Omega)\right), \end{cases} \tag{6.33}$$

$$\begin{cases} P^\varepsilon_1 \psi'_{1\varepsilon}(\varphi_\varepsilon) \rightharpoonup \psi'_1(\Phi_1) & \text{weakly* in } L^\infty\left(0,T;L^2(\Omega)\right), \\ \widetilde{\psi'_{1\varepsilon}(\varphi_\varepsilon)} \rightharpoonup \theta_1 \psi'_1(\Phi_1) & \text{weakly* in } L^\infty\left(0,T;L^2(\Omega)\right), \\ \widetilde{\psi'_{2\varepsilon}(\varphi_\varepsilon)} \rightharpoonup \theta_2 \psi'_1(\Phi_1) & \text{weakly* in } L^\infty\left(0,T;L^2(\Omega)\right), \end{cases} \tag{6.34}$$

where $\psi_1 := \psi_1(\Phi_1) \in L^2(0,T;H^1_0(\Omega))$ with $\psi'_1 := \psi'_1(\Phi_1) \in L^2\left(0,T;L^2(\Omega)\right)$, is the unique solution of the following homogenized backward problem

$$\begin{cases} \psi_1'' - \operatorname{div}(A^0 \nabla \psi_1) = -\Phi_1 & \text{in } \Omega \times ]0,T[, \\ \psi_1 = 0 & \text{on } \partial\Omega \times ]0,T[, \\ \psi_1(T) = \psi_1'(T) = 0 & \text{in } \Omega. \end{cases}$$

As both $\psi_1$ and $u_1$ belong to $C([0,T]; H_0^1(\Omega)) \cap C^1([0,T]; L^2(\Omega)))$, by (6.26), (6.30) and (6.33), we get

$$u_1(T) = u_1'(T) = 0. \tag{6.35}$$

Therefore $\zeta_1^{ex} := \dfrac{\widehat{\zeta_1}}{\theta_1} = -\Phi_1$ is an exact control for problem (6.32). On the other hand, if we apply HUM method directly to problem (6.32), in view of Theorem 3.2, by (5.5) and since $\psi_1 = u_1$, we get

$$\Lambda\left(\Phi_1^0 + \Phi_2^0, \Phi_1^*\right) = \left(U_1^1 + U_2^1, -(U_1^0 + U_2^0)\right). \tag{6.36}$$

This identifies $\zeta_1^{ex}$ in a unique way as the energy minimizing control of problem (6.32). Moreover by (6.27)ii) and (6.29), we get

$$\zeta_2^{ex} := \widehat{\zeta_2} = \theta_2 \zeta_1^{ex}. \tag{6.37}$$

This implies that convergences (6.27), (6.30) and (6.31) hold for the whole sequences. Hence, by (6.37), we get (3.6), (3.7) and (3.8).

**Case** $\gamma = 1$: Let $\widehat{\zeta} := \left(\widehat{\zeta_1}, \widehat{\zeta_2}\right)$. We obtain, up to a subsequence, the following convergences

$$\begin{cases} P_1^\varepsilon u_{1\varepsilon}(\zeta_\varepsilon^{ex}) \rightharpoonup u_1(\widehat{\zeta}) & \text{weakly* in } L^\infty\left(0,T; H_0^1(\Omega)\right), \\ \widetilde{u_{1\varepsilon}(\zeta_\varepsilon^{ex})} \rightharpoonup \theta_1 u_1(\widehat{\zeta}) & \text{weakly* in } L^\infty\left(0,T; L^2(\Omega)\right), \\ \widetilde{u_{2\varepsilon}(\zeta_\varepsilon^{ex})} \rightharpoonup u_2(\widehat{\zeta}) & \text{weakly* in } L^\infty\left(0,T; L^2(\Omega)\right), \end{cases} \tag{6.38}$$

$$\begin{cases} P_1^\varepsilon u_{1\varepsilon}'(\zeta_\varepsilon^{ex}) \rightharpoonup u_1'(\widehat{\zeta}) & \text{weakly* in } L^\infty\left(0,T; L^2(\Omega)\right), \\ \widetilde{u_{1\varepsilon}'(\zeta_\varepsilon^{ex})} \rightharpoonup \theta_1 u_1'(\widehat{\zeta}) & \text{weakly* in } L^\infty\left(0,T; L^2(\Omega)\right), \\ \widetilde{u_{2\varepsilon}'(\zeta_\varepsilon^{ex})} \rightharpoonup u_2'(\widehat{\zeta}) & \text{weakly* in } L^\infty\left(0,T; L^2(\Omega)\right), \end{cases} \tag{6.39}$$

where the pair $(u_1, u_2) := \left(u_1(\widehat{\zeta}), u_2(\widehat{\zeta})\right) \in L^2\left(0,T; H_0^1(\Omega) \times L^2(\Omega)\right)$, with $(u_1', u_2') := \left(u_1'(\widehat{\zeta}), u_2'(\widehat{\zeta})\right) \in L^2\left(0,T; (L^2(\Omega))^2\right)$, is the unique solution of the coupled system

$$\begin{cases} \theta_1 u_1'' - \operatorname{div}\left(A^0 \nabla u_1\right) + c_h\left(\theta_2 u_1 - u_2\right) = \widehat{\zeta_1} & \text{in } \Omega \times ]0,T[, \\ u_2'' - c_h\left(\theta_2 u_1 - u_2\right) = \widehat{\zeta_2} & \text{in } \Omega \times ]0,T[, \\ u_1 = 0 & \text{on } \partial\Omega \times ]0,T[, \\ u_1(0) = \dfrac{U_1^0}{\theta_1}, \ u_1'(0) = \dfrac{U_1^1}{\theta_1} & \text{in } \Omega, \\ u_2(0) = U_2^0, \ u_2'(0) = U_2^1 & \text{in } \Omega, \end{cases} \tag{6.40}$$

where $c_h = \dfrac{1}{|Y_2|} \displaystyle\int_\Gamma h(y)\, d\sigma_y > 0$.

Analogously, let $\Phi := (\Phi_1, \Phi_2)$. By (6.5) in Theorem 6.1, we can apply Theorem 2.1 to problem

(4.5) and obtain, up to a subsequence, the following convergences

$$\begin{cases} P_1^\varepsilon \psi_{1\varepsilon}(\varphi_\varepsilon) \rightharpoonup \psi_1(\Phi) & \text{weakly* in } L^\infty\left(0,T;H_0^1(\Omega)\right), \\ \widetilde{\psi_{1\varepsilon}(\varphi_\varepsilon)} \rightharpoonup \theta_1\psi_1(\Phi) & \text{weakly* in } L^\infty\left(0,T;L^2(\Omega)\right), \\ \widetilde{\psi_{2\varepsilon}(\varphi_\varepsilon)} \rightharpoonup \psi_2(\Phi) & \text{weakly* in } L^\infty\left(0,T;L^2(\Omega)\right), \end{cases} \tag{6.41}$$

$$\begin{cases} P_1^\varepsilon \psi'_{1\varepsilon}(\varphi_\varepsilon) \rightharpoonup \psi'_1(\Phi) & \text{weakly* in } L^\infty\left(0,T;L^2(\Omega)\right), \\ \widetilde{\psi'_{1\varepsilon}(\varphi_\varepsilon)} \rightharpoonup \theta_1\psi'_1(\Phi) & \text{weakly* in } L^\infty\left(0,T;L^2(\Omega)\right), \\ \widetilde{\psi'_{2\varepsilon}(\varphi_\varepsilon)} \rightharpoonup \psi'_2(\Phi) & \text{weakly* in } L^\infty\left(0,T;L^2(\Omega)\right), \end{cases}$$

where the pair $(\psi_1,\psi_2) := (\psi_1(\Phi),\psi_2(\Phi)) \in L^2\left(0,T;H_0^1(\Omega)\times L^2(\Omega)\right)$ with $(\psi'_1,\psi'_2) := (\psi'_1(\Phi),\psi'_2(\Phi)) \in L^2\left(0,T;(L^2(\Omega))^2\right)$ is the unique solution of the coupled backward system

$$\begin{cases} \theta_1\psi''_1 - \operatorname{div}\left(A^0\nabla\psi_1\right) + c_h\left(\theta_2\psi_1-\psi_2\right) = -\theta_1\Phi_1 & \text{in } \Omega\times]0,T[, \\ \psi''_2 - c_h\left(\theta_2\psi_1-\psi_2\right) = -\Phi_2 & \text{in } \Omega\times]0,T[, \\ \psi_1 = 0 & \text{on } \partial\Omega\times]0,T[, \\ \psi_1(T) = \psi_2(T) = 0, & \text{in } \Omega, \\ \psi'_1(T) = \psi'_2(T) = 0 & \text{in } \Omega \end{cases}$$

where $c_h$ is as before.
As $\psi_1 \in C([0,T];H_0^1(\Omega)) \cap C^1([0,T];L^2(\Omega)))$ and $\psi_2 \in C([0,T];L^2(\Omega) \cap C^1([0,T];L^2(\Omega)))$, by (6.26), (6.38) and (6.41) we get

$$u_i(T) = u'_i(T) = 0 \quad i=1,2. \tag{6.42}$$

Therefore $\zeta^{ex} := (\zeta_1^{ex},\zeta_2^{ex}) = \widehat{\zeta} = (\widehat{\zeta}_1,\widehat{\zeta}_2) = (-\theta_1\Phi_1,-\Phi_2)$ is an exact control for problem (6.40).
On the other hand, if we apply directly HUM method to problem (6.40), in view of Theorem 3.3, by (5.26) and since $\psi = u$, we get

$$\Lambda_1\left(\bar{\Phi}_0,\bar{\Phi}_1\right) = \left(\bar{U}_0,\bar{U}_1\right),$$

where $\bar{\Phi}_0 = \left(\dfrac{\Phi_1^0}{\theta_1},\Phi_2^0\right)$, $\bar{\Phi}_1 = \left(\dfrac{\Phi_1^*}{\theta_1},\Phi_1^*-\Phi_2^*\right)$, $\bar{U}_0 = \left(\dfrac{U_1^0}{\theta_1},U_2^0\right)$ and $\bar{U}_1 = \left(\dfrac{U_1^1}{\theta_1},U_2^1\right)$. As before, this indentifies $\zeta^{ex}$ in a unique way as the energy minimizing control of problem (6.40). This implies that, also in this case, convergences (6.27), (6.38) and (6.39) hold for the whole sequences and we get (3.10), (3.11) and (3.12).
Theorem 3.4 is now completely proved.

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## Conflicts of interest

None.

## Authors' contributions

The authors conceived and wrote this article in collaboration and with the same responsibility. All of them read and approved the final manuscript.

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