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Lubomira G. Softova

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Boundedness of the solutions to non-linear systems with Morrey data

Lubomira G. Softova

Department of Mathematics, University of Salerno, Fisciano, Italy

ABSTRACT

We consider non-linear elliptic systems satisfying componentwise coercivity condition. The non-linear terms have controlled growths with respect to the solution and its gradient, while the behaviour in x is governed by functions in Morrey spaces. We firstly prove essential boundedness of the weak solution and then we obtain Morrey regularity of its gradient.

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1. Introduction

Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$ be a bounded domain satisfying the (A)-condition. We are interested in boundedness and Morrey regularity of the weak solutions to non-linear elliptic systems of the type

$$-\operatorname{div} \mathbf{A}(x, \mathbf{u}, D\mathbf{u}) + \mathbf{b}(x, \mathbf{u}, D\mathbf{u}) = \mathbf{f}(x), \quad x \in \Omega \quad (1)$$

where the nonlinear terms are Carathéodory maps

$$\begin{aligned} \mathbf{A}(x, \mathbf{u}, \mathbf{z}) &: \Omega \times \mathbb{R}^N \times \mathbb{M}^{N \times n} \rightarrow \mathbb{R}^{N \times n}, \\ \mathbf{b}(x, \mathbf{u}, \mathbf{z}) &: \Omega \times \mathbb{R}^N \times \mathbb{M}^{N \times n} \rightarrow \mathbb{R}^N. \end{aligned}$$

It is known from the classical theory [1,2] that if we take linear elliptic equation in nondivergence form, that is $\sum_{ij=1}^n A_{ij}(x)D_{ij}u = f(x)$ then for any strong solution $u \in C^0(\bar{\Omega}) \cap W_{\text{loc}}^{2,n}(\Omega)$ we dispose with a maximum principle. As it concerns the divergence form equations, the celebrated result of De Giorgi [3] and Nash [4] implies that any weak solution $u \in W_0^{1,2}(\Omega)$ of the linear elliptic equation $D_i(A_{ij}(x)D_ju + g_i(x)) = f(x)$ is locally Hölder continuous when $g_i \in L^p$ with $p > n$ and $f \in L^q$ with $q > n/2$ even if the coefficients are only L^∞ . Unfortunately the De Giorgi-Nash result does not hold anymore if we consider system of uniformly elliptic equations because of the lack of *Maximum principle*. This was

shown by De Giorgi himself almost 10 years later, constructing a counterexample [5]. Precisely, the function $\mathbf{u} = x/|x|^\gamma \in W^{1,2}(\mathcal{B}_1(0); \mathbb{R}^n)$ is a solution to

$$D_i(A_{ij}^{\alpha\beta}(x)D_j u^\beta(x)) = 0 \quad \text{in } \mathcal{B}_1(0), \quad \mathbf{u} = x \quad \text{on } \partial\mathcal{B}_1(0)$$

with suitably chosen coefficients $A_{ij}^{\alpha\beta}(x)$.

Moreover, the result of De Giorgi-Nash cannot be extended to quasilinear systems even if the coefficients are analytic functions, as it was shown by Giusti and Miranda [6]. In order to get a maximum principle for elliptic systems we need to impose some quite restrictive structural conditions. The simplest one requires the system to be in diagonal form, or *decoupled*.

Example 1: Consider the operator $\operatorname{div}(\mathbf{A}(x, D\mathbf{u})) = 0$ in Ω with coefficients

$$A_i^\alpha(x, D\mathbf{u}) = \sum_{j=1}^n \sum_{\beta=1}^N \delta_{\alpha\beta} A_{ij}^{\alpha\beta}(x) D_j u^\beta$$

where $\delta_{\alpha\beta}$ is the Kronecker delta. Then u^α solves a single elliptic equation and $\sup_\Omega u^\alpha \leq \sup_{\partial\Omega} u^\alpha$, for each $\alpha = 1, \dots, N$.

Such a result is obtained also in [7,8] where the author considers linear uniformly elliptic systems imposing a quite restrictive condition on the ellipticity constants.

One more example was given by Nečas and Stará [9].

Example 2: Consider the system $\operatorname{div} \mathbf{A}(x, \mathbf{u}, D\mathbf{u}) = 0$ in Ω that is diagonal for large values of u^α , that is,

$$0 < \theta_\alpha \leq u^\alpha \implies A_i^\alpha(x, \mathbf{u}, D\mathbf{u}) = \sum_{j=1}^n \sum_{\beta=1}^N \delta_{\alpha\beta} A_{ij}^{\alpha\beta}(x, \mathbf{u}) D_j u^\beta \quad (2)$$

with bounded and elliptic $A_{ij}^{\alpha\beta}$. It turns out that

$$\sup_\Omega u^\alpha \leq \max \left\{ \theta_\alpha; \sup_{\partial\Omega} u^\alpha \right\}$$

also in this case.

The situation becomes more complicated if we consider the *general non-linear system*

$$\operatorname{div} \mathbf{A}(x, \mathbf{u}, D\mathbf{u}) = \mathbf{b}(x, \mathbf{u}, D\mathbf{u}). \quad (3)$$

In [10] Campanato obtained a maximum principle for such kind of operators supposing that the map \mathbf{A} is C^1 and possesses quadratic growth with respect to the gradient. This result covers also the case of linear elliptic systems with constant coefficients studied in [11].

In our case, along with the Carathéodory conditions on the maps $\mathbf{A}(x, \mathbf{u}, \mathbf{z})$ and $\mathbf{b}(x, \mathbf{u}, \mathbf{z})$ we need to control also the growths of \mathbf{A} and \mathbf{b} with respect to \mathbf{u} and \mathbf{z} . These additional *controlled growth conditions* ensure the convergence of the integrals in the definition of *weak solution* to (3) (see (13)).

In [12] Leonetti and Petricca assume *componentwise coercivity condition* on \mathbf{A} and positivity of \mathbf{b} for large values of u^α , that is, there exist positive constants θ_α such that

$$\theta_\alpha \leq u^\alpha \implies \begin{cases} v|z^\alpha|^p - M_\alpha \leq \sum_{i=1}^n A_i^\alpha(x, \mathbf{u}, \mathbf{z})z_i^\alpha \\ 0 \leq b^\alpha(x, \mathbf{u}, \mathbf{z}). \end{cases} \tag{4}$$

Combining the *Sobolev inequality* with the *Stampacchia Lemma* [13] they get a componentwise bound of the solution, covering this way also the systems studied in [9] since (2) is a special case of (4). Let us note that getting essential boundedness of the weak solution to (1) is a starting point for a further study of its regularity in various function spaces. In [14–16] the authors obtain better integrability and Hölder regularity of the bounded solutions to quasilinear elliptic equations ($N = 1$) under controlled growth conditions on the non-linear terms. Further this result has been extended in [17] to semilinear uniformly elliptic systems of the form

$$\operatorname{div}(\mathbf{A}(x)D\mathbf{u} + \mathbf{a}(x, \mathbf{u})) = \mathbf{b}(x, \mathbf{u}, D\mathbf{u}) \quad \text{in } \Omega \tag{5}$$

with minimal regular assumptions on the coefficients and the underlying domain. Precisely, it was shown that if the non-linear terms satisfy the controlled growth conditions (10) with $\varphi \in L^p(\Omega)$, $p > 2$ and $\psi \in L^q(\Omega)$, $q > \frac{2n}{n+2}$ then any bounded weak solution to (5) belongs to $W_0^{1,r}$ with $r = \min\{p, q^*\}$.

The natural question that arises is what kind of regularity of the solution to (1) we could expect, assuming that the given functions φ and ψ belong to some Morrey or weighted Lebesgue spaces. In case of simple equation, we dispose with the results of Byun and Palagachev [18–21]. Combining the Gehring-Giaquinta-Modica lemma, the Adams trace inequality and the Hartmann-Stampacchia maximum principle they obtain L^∞ estimate of the solution. Further, the Morrey-type estimate of the gradient permits the authors to show also Hölder regularity of the solution. In case of quasilinear systems, we can refer to the recent result [22] in which we obtain an optimal Morrey regularity of the gradient $D\mathbf{u}$.

Our goal is to obtain a componentwise maximum principle for any solution of (3) supposing that the operators \mathbf{A} and \mathbf{b} satisfy structural conditions expressed in terms of Morrey functions. As a consequence, we obtain also Morrey regularity of the gradient of \mathbf{u} , extending such a way the regularity results obtained in [9,12,14,17,18,23,24] to non-linear systems with Morrey data.

Recall that a real-valued function $f \in L^p(\Omega)$ belongs to the Morrey space $L^{p,\lambda}(\Omega)$ with $p \in [1, \infty)$, $\lambda \in (0, n)$, if

$$\|f\|_{p,\lambda;\Omega} = \left(\sup_{\mathcal{B}_r(x)} \frac{1}{r^\lambda} \int_{\Omega \cap \mathcal{B}_r(x)} |f(y)|^p dy \right)^{1/p} < \infty \tag{6}$$

where the supremum is taken over all balls $\mathcal{B}_r(x)$, $r \in (0, \operatorname{diam} \Omega]$ and $x \in \overline{\Omega}$. Working in the framework of the Morrey spaces we note that the Sobolev trace inequality is not enough anymore. For this goal, we will use the following result due to Adams.

Lemma 3 (Adams Trace Inequality, [25–27]): *Let m be a positive Radon measure with support in Ω and such that for each ball \mathcal{B}_ρ it holds*

$$m(\mathcal{B}_\rho) \leq K\rho^{\tau_0}, \quad \tau_0 = \frac{s}{r}(n-r), \quad 1 < r < s < \infty, \quad r < n \tag{7}$$

with an absolute constant $K > 0$. Then

$$\left(\int_{\Omega} |v(x)|^s dm \right)^{\frac{1}{s}} \leq C(n, s, r) K^{\frac{1}{s}} \left(\int_{\Omega} |Dv(x)|^r dx \right)^{\frac{1}{r}} \tag{8}$$

for each function $v \in W_0^{1,r}(\Omega)$.

In what follows we suppose that $\Omega \subset \mathbb{R}^n, n \geq 2$, is a bounded domain satisfying the (A)-condition, that is, there exists a constant $A_\Omega > 0$ such that

$$|\Omega_r(x)| \geq A_\Omega r^n \quad \forall x \in \overline{\Omega}, r \in (0, \text{diam } \Omega] \tag{A}$$

where $\Omega_r(x) = \Omega \cap \mathcal{B}_r(x)$. It is worth noting that the (A)-condition excludes interior cusps at each point of the boundary and guarantees the validity of the Sobolev embedding theorem in $W^{1,p}(\Omega)$. This geometric property is surely satisfied when $\partial\Omega$ has the uniform interior cone property (e.g. C^1 -smooth or Lipschitz continuous boundaries), but it holds also for Reifenberg flat domains boundaries (cf. [16]).

Throughout the text the standard summation convention on repeated indexes is adopted. The letter $C > 0$ is used for various constants and may change from one occurrence to another.

2. Maximum principle

Consider the non-linear system

$$-D_i(A_i^\alpha(x, \mathbf{u}, D\mathbf{u})) + b^\alpha(x, \mathbf{u}, D\mathbf{u}) = f^\alpha(x) \text{ in } \Omega \tag{9}$$

where $\mathbf{A} = \{A_i^\alpha(x, \mathbf{u}, \mathbf{z})\}_{i \leq n}^{\alpha \leq N}$ and $\mathbf{b} = (b^1(x, \mathbf{u}, \mathbf{z}), \dots, b^N(x, \mathbf{u}, \mathbf{z}))$ are measurable in $x \in \Omega$ and continuous in (\mathbf{u}, \mathbf{z}) for almost all (a.a.) $x \in \Omega$. Suppose that for each $(x, \mathbf{u}, \mathbf{z}) \in \Omega \times \mathbb{R}^N \times \mathbb{M}^{N \times n}$ the following *controlled growth conditions* hold. Namely,

$$\begin{cases} |\mathbf{A}(x, \mathbf{u}, \mathbf{z})| \leq \Lambda (\varphi(x) + |\mathbf{u}|^{\frac{\nu}{2}} + |\mathbf{z}|) \\ |\mathbf{b}(x, \mathbf{u}, \mathbf{z})| \leq \Lambda (\psi(x) + |\mathbf{u}|^{\nu-1} + |\mathbf{z}|^2)^{\frac{\nu-1}{\nu}} \end{cases} \tag{10}$$

as $|\mathbf{u}|, |\mathbf{z}| \rightarrow \infty$, with some positive constant Λ . Here ν is the Sobolev conjugate of 2, that is,

$$\nu = \begin{cases} \frac{2n}{n-2} & \text{if } n \geq 3 \\ \text{any number } > 2 & \text{if } n = 2, \end{cases} \tag{11}$$

and the given functions φ , ψ and f^α satisfy

$$\begin{cases} \varphi \in L^{p,\lambda}(\Omega), & p > 2, \lambda \in (0, n), p + \lambda > n \\ \psi, f^\alpha \in L^{q,\mu}(\Omega), & q > \frac{\nu}{\nu-1}, \mu \in (0, n), 2q + \mu > n. \end{cases} \quad (12)$$

In the particular case $n = 2$, the powers of $|\mathbf{u}|$ could be arbitrary positive numbers greater than 1, while the growth of $|\mathbf{z}|$ is strictly sub-quadratic (cf. [2,28]).

Under *weak solution* of (9) we mean a function $\mathbf{u} \in W^{1,2}(\Omega; \mathbb{R}^N) \cap L^\nu(\Omega; \mathbb{R}^N)$, satisfying

$$\begin{aligned} \sum_{i=1}^n \int_{\Omega} A_i^\alpha(x, \mathbf{u}, D\mathbf{u}) D_i \phi^\alpha(x) \, dx + \int_{\Omega} b^\alpha(x, \mathbf{u}(x), D\mathbf{u}(x)) \phi^\alpha(x) \, dx \\ = \int_{\Omega} f^\alpha(x) \phi^\alpha(x) \, dx \end{aligned} \quad (13)$$

for all $\phi = (\phi^1, \dots, \phi^N) \in W_0^{1,2}(\Omega; \mathbb{R}^N)$. The conditions (10)-(12) are the natural ones that ensure convergence of the integrals in (13). Moreover, they are optimal as it is seen from the following example in the case of single equation (cf. [2,15]).

Example 4: The function $u(x) = |x|^{\frac{r-2}{r-1}} \in W^{1,2}(\mathcal{B}_1(0))$, with $n \geq 3$ and $\frac{n+2}{n} < r < 2$ is a solution to the equation $\Delta u = C|Du|^r$ in $\mathcal{B}_1(0)$. Note that $u \notin L^\infty(\mathcal{B}_1(0))$.

In order to obtain boundedness of the solutions to (9) we add some restrictions on the structure of the operator (see for example [12,29]). Precisely, we impose componentwise coercivity on A_i^α and a sign condition on b^α .

For every $\alpha \in \{1, \dots, N\}$ there exist positive constants θ_α, γ and a function φ such that for each $u^\alpha \geq \theta_\alpha$ we have

$$\begin{cases} \gamma |\mathbf{z}^\alpha|^2 - \Lambda \varphi(x)^2 \leq \sum_{i=1}^n A_i^\alpha(x, \mathbf{u}, \mathbf{z}) z_i^\alpha \\ \varphi \in L^{p,\lambda}(\Omega), p > 2, p + \lambda > n, \lambda \in (0, n) \\ 0 \leq b^\alpha(x, \mathbf{u}, \mathbf{z}) \quad \text{for a.a. } x \in \Omega, \forall \mathbf{z} \in \mathbb{M}^{N \times n}. \end{cases} \quad (14)$$

Theorem 5 (Maximum principle): Let Ω be an (A)-type domain and $\mathbf{u} \in W^{1,2}(\Omega; \mathbb{R}^N) \cap L^\nu(\Omega; \mathbb{R}^N)$ be a weak solution to (9) under the conditions (10), (12) and (14) and such that $\sup_{\partial\Omega} u^\alpha < \infty$. Then

$$\sup_{\Omega} u^\alpha \leq \max\{\theta_\alpha, \sup_{\partial\Omega} u^\alpha\} + M_\alpha \quad \alpha \in \{1, \dots, N\}$$

where M_α depends on $n, p, q, \lambda, \mu, \Lambda, \gamma, \|\varphi\|_{p,\lambda;\Omega}, \|f^\alpha\|_{q,\mu;\Omega}$ and $\text{diam } \Omega$.

Proof: Fix $\alpha \in \{1, \dots, N\}$. According to our assumptions

$$\sup_{\partial\Omega} u^\alpha = \inf\{k^\alpha : u^\alpha \leq k^\alpha \text{ on } \partial\Omega, k^\alpha \in \mathbb{R}\}$$

(see [1]). Choose a constant $L > 0$ such that $L \geq \max\{\theta_\alpha; k^\alpha\}$ and define a vector-function \mathbf{v} as follows

$$v^\beta = \begin{cases} \max\{u^\alpha - L; 0\} & \text{if } \beta = \alpha \\ 0 & \text{if } \beta \neq \alpha \end{cases}, \quad Dv^\beta = \begin{cases} Du^\alpha & \text{if } \beta = \alpha \\ 0 & \text{if } \beta \neq \alpha \end{cases}.$$

It is clear that $\mathbf{v} \in W_0^{1,2}(\Omega; \mathbb{R}^N)$ and hence $\mathbf{v} \in L^\nu(\Omega; \mathbb{R}^N)$ by the Sobolev embedding theorem.

Defining the set $\mathcal{A}_L^\alpha = \{x \in \Omega : u^\alpha(x) - L > 0\}$, and choosing $\phi = \mathbf{v}$ as a test function we obtain

$$\begin{aligned} \int_{\mathcal{A}_L^\alpha} A_i^\alpha(x, \mathbf{u}, D\mathbf{u}) D_i u^\alpha(x) \, dx + \int_{\mathcal{A}_L^\alpha} b^\alpha(x, \mathbf{u}, D\mathbf{u})(u^\alpha(x) - L) \, dx \\ = \int_{\mathcal{A}_L^\alpha} f^\alpha(x)(u^\alpha(x) - L) \, dx. \end{aligned}$$

We will study firstly the case $n \geq 3$ when $\nu = 2n/(n-2)$. Define the Radon measure dm supported in Ω by

$$dm := (\chi_\Omega(x) + \varphi(x)^2 + |f^\alpha(x)|) \, dx,$$

where χ_Ω is the characteristic function of Ω . Then by (14) we get the estimate

$$\begin{aligned} \int_{\mathcal{A}_L^\alpha} |Du^\alpha(x)|^2 \, dx &\leq \frac{\Lambda}{\gamma} \int_{\mathcal{A}_L^\alpha} \varphi(x)^2 \, dx + \frac{1}{\gamma} \int_{\mathcal{A}_L^\alpha} |f^\alpha(x)|(u^\alpha(x) - L) \, dx \\ &\leq \frac{\Lambda}{\gamma} \int_{\mathcal{A}_L^\alpha} (\chi_\Omega(x) + \varphi(x)^2 + |f^\alpha(x)|) \, dx \\ &\quad + \frac{1}{\gamma} \int_{\mathcal{A}_L^\alpha} |f^\alpha(x)|(u^\alpha(x) - L) \, dx \\ &\leq C(\Lambda, \gamma)(m(\mathcal{A}_L^\alpha) + J). \end{aligned} \tag{15}$$

In order to estimate the integral $J = \int_{\mathcal{A}_L^\alpha} |f^\alpha(x)|(u^\alpha(x) - L) \, dx$ we make use of Lemma 3 applied to the Radon measure $dm' = |f^\alpha(x)| \, dx$. Hence

$$J = \int_{\mathcal{A}_L^\alpha} (u^\alpha(x) - L) dm' \leq \left(\int_{\mathcal{A}_L^\alpha} |u^\alpha(x) - L|^{s'} dm' \right)^{\frac{1}{s}} m'(\mathcal{A}_L^\alpha)^{1-\frac{1}{s}}.$$

Evaluating the measure m' over a ball \mathcal{B}_ρ we get

$$\begin{aligned} m'(\mathcal{B}_\rho) &= \int_{\mathcal{B}_\rho} |f^\alpha(x)| \, dx \leq c(n) \rho^{n-\frac{n-\mu}{q}} \left(\frac{1}{\rho^\mu} \int_{\mathcal{B}_\rho} |f^\alpha(x)|^q \, dx \right)^{1/q} \\ &\leq C(n) \rho^{n-\frac{n-\mu}{q}} \|f^\alpha\|_{q,\mu;\Omega}. \end{aligned}$$

We apply now Lemma 3 with $r' = 2$, $\tau'_0 = n - \frac{n-\mu}{q}$ and $s' = \frac{2}{n-2} \left(n - \frac{n-\mu}{q} \right) > 2$, calculated via (7). Hence

$$\begin{aligned} J &\leq CK^{\frac{1}{s'}} \left(\int_{\mathcal{A}_L^\alpha} |Du^\alpha(x)|^2 dx \right)^{\frac{1}{2}} m'(\mathcal{A}_L^\alpha)^{1-\frac{1}{s'}} \\ &\leq C \left[\varepsilon \int_{\mathcal{A}_L^\alpha} |Du^\alpha(x)|^2 dx + \frac{1}{\varepsilon} m'(\mathcal{A}_L^\alpha)^{2(1-\frac{1}{s'})} \right]. \end{aligned} \tag{16}$$

Combining (15) and (16), taking ε small enough and moving the integral of the gradient on the left-hand side, and keeping in mind that $2(1 - \frac{1}{s'}) > 1$ and $m'(\mathcal{A}_L^\alpha) \leq m(\mathcal{A}_L^\alpha)$ we obtain

$$\int_{\mathcal{A}_L^\alpha} |Du^\alpha(x)|^2 dx \leq C(m(\mathcal{A}_L^\alpha) + m(\mathcal{A}_L^\alpha)^{2(1-\frac{1}{s'})}) \leq Cm(\mathcal{A}_L^\alpha) \tag{17}$$

where the constant C depends on known quantities.

To complete the estimate (15) we will use once again Lemma 3. It is immediate that $m(\mathcal{B}_\rho)$ of a ball $\mathcal{B}_\rho \Subset \Omega$ is

$$\begin{aligned} m(\mathcal{B}_\rho) &= \int_{\mathcal{B}_\rho} (\chi_\Omega(x) + \varphi(x)^2 + |f^\alpha(x)|) dx \\ &\leq C(n)\rho^n + \rho^{n-\frac{2(n-\lambda)}{p}} \|\varphi\|_{p,\lambda;\Omega}^2 + \rho^{n-\frac{n-\mu}{q}} \|f^\alpha\|_{q,\mu;\Omega} \leq K\rho^{\tau_0} \end{aligned} \tag{18}$$

with $K = K(n, p, q, \lambda, \mu, \text{diam } \Omega, \|\varphi\|_{p,\lambda;\Omega}, \|f^\alpha\|_{q,\mu;\Omega})$ and

$$\tau_0 = \min \left\{ n - \frac{2(n-\lambda)}{p}; n - \frac{n-\mu}{q} \right\} > n - 2.$$

Applying (8) with $r = 2 < n$ and calculating s from (7) we get

$$\begin{aligned} \int_{\mathcal{A}_L^\alpha} (u^\alpha(x) - L) dm &\leq \left(\int_{\mathcal{A}_L^\alpha} |u^\alpha(x) - L|^s dm \right)^{\frac{1}{s}} m(\mathcal{A}_L^\alpha)^{1-\frac{1}{s}} \\ &\leq CK^{\frac{1}{s}} \left(\int_{\mathcal{A}_L^\alpha} |Du^\alpha(x)|^2 dx \right)^{\frac{1}{2}} m(\mathcal{A}_L^\alpha)^{1-\frac{1}{s}} \\ &\leq C(n, p, K, \gamma, \Delta) m(\mathcal{A}_L^\alpha)^{1+\frac{1}{2}-\frac{1}{s}} \end{aligned} \tag{19}$$

with $s = \min \left\{ \frac{2np-4(n-\lambda)}{p(n-2)}; \frac{2nq-2(n-\mu)}{q(n-2)} \right\} > 2$.

A similar bound holds also in the case $n = 2$. In fact, for any ball $\mathcal{B}_\rho \subset \mathbb{R}^2$ we have

$$m(\mathcal{B}_\rho) \leq C\rho^2 + \rho^{2-\frac{2(2-\lambda)}{p}} \|\varphi\|_{p,\lambda;\Omega}^2 + \rho^{2-\frac{2-\mu}{q}} \|f^\alpha\|_{q,\mu;\Omega} \leq K\rho^{\tau_0}$$

with $\tau_0 = \min \left\{ 2 - \frac{2(2-\lambda)}{p}; 2 - \frac{2-\mu}{q} \right\} > 0$. Choosing $s = 2$ we calculate r from (7)

$$r = \max \left\{ \frac{2p}{2p-2+\lambda}; \frac{4q}{4q-2+\mu} \right\} \in (1, 2).$$

Then by the Hölder and the Adams trace inequalities we obtain

$$\begin{aligned}
 \int_{\mathcal{A}_L^\alpha} (u^\alpha(x) - L) \, dm &\leq \left(\int_{\mathcal{A}_L^\alpha} (u^\alpha(x) - L)^2 \, dm \right)^{\frac{1}{2}} m(\mathcal{A}_L^\alpha)^{\frac{1}{2}} \\
 &\leq CK^{\frac{1}{2}} \left(\int_{\mathcal{A}_L^\alpha} |Du^\alpha(x)|^r \, dx \right)^{\frac{1}{r}} m(\mathcal{A}_L^\alpha)^{\frac{1}{2}} \\
 &\leq CK^{\frac{1}{2}} \left(\int_{\mathcal{A}_L^\alpha} |Du^\alpha(x)|^2 \, dx \right)^{\frac{1}{2}} \left(\int_{\mathcal{A}_L^\alpha} \chi_\Omega(x) \, dx \right)^{\frac{1}{r} - \frac{1}{2}} m(\mathcal{A}_L^\alpha)^{\frac{1}{2}} \\
 &\leq CK^{\frac{1}{2}} \left(\int_{\mathcal{A}_L^\alpha} |Du^\alpha(x)|^2 \, dx \right)^{\frac{1}{2}} m(\mathcal{A}_L^\alpha)^{\frac{1}{r}}. \tag{20}
 \end{aligned}$$

In order to estimate the integral in the last term, we go back to (15). Consider again the Radon measure $dm' = |f^\alpha(x)|dx$ and calculate $m'(\mathcal{B}_\rho) \leq K\rho^{2-\frac{2-\mu}{q}}$. Then choosing $s' = 2$ we get $r' = \frac{4q}{4q-(2-\mu)} \in (1, 2)$ from (7). This way, Lemma 3 and the Hölder inequality give

$$\begin{aligned}
 J &\leq \left(\int_{\mathcal{A}_L^\alpha} |u^\alpha(x) - L|^2 \, dm' \right)^{\frac{1}{2}} m'(\mathcal{A}_L^\alpha)^{\frac{1}{2}} \\
 &\leq CK^{\frac{1}{2}} \left(\int_{\mathcal{A}_L^\alpha} |Du^\alpha(x)|^{r'} \, dx \right)^{\frac{1}{r'}} m'(\mathcal{A}_L^\alpha)^{\frac{1}{2}} \\
 &\leq CK^{\frac{1}{2}} \left(\int_{\mathcal{A}_L^\alpha} |Du^\alpha(x)|^2 \, dx \right)^{\frac{1}{2}} m(\mathcal{A}_L^\alpha)^{\frac{1}{r'} - \frac{1}{2}} m(\mathcal{A}_L^\alpha)^{\frac{1}{2}} \\
 &\leq C \left[\varepsilon \int_{\mathcal{A}_L^\alpha} |Du^\alpha(x)|^2 \, dx + \frac{1}{\varepsilon} m(\mathcal{A}_L^\alpha)^{\frac{2}{r'}} \right]. \tag{21}
 \end{aligned}$$

Unifying (15) and (21), taking ε small enough and keeping in mind that $\frac{2}{r'} > 1$, we get

$$\int_{\mathcal{A}_L^\alpha} |Du^\alpha(x)|^2 \, dx \leq Cm(\mathcal{A}_L^\alpha)$$

where the constant depends on the same quantities as in (17). Then the estimate (20) becomes

$$\int_{\mathcal{A}_L^\alpha} (u^\alpha(x) - L) \, dm \leq Cm(\mathcal{A}_L^\alpha)^{1+\frac{1}{r}-\frac{1}{2}}. \tag{22}$$

Unifying the estimates (19) and (22) we obtain

$$\int_{\mathcal{A}_L^\alpha} (u^\alpha(x) - L) \, dm \leq Cm(\mathcal{A}_L^\alpha)^{1+\sigma_0} \tag{23}$$

where

$$\sigma_0 = \begin{cases} \frac{1}{2} - \frac{1}{s} = \min \left\{ \frac{p + \lambda - n}{np - 2(n - \lambda)}; \frac{\mu + 2q - n}{2nq - 2(n - \mu)} \right\} & \text{if } n > 2 \\ \frac{1}{r} - \frac{1}{2} = \min \left\{ \frac{p + \lambda - 2}{2p}; \frac{2q + \mu - 2}{4q} \right\} & \text{if } n = 2. \end{cases}$$

Suppose now that $m(\mathcal{A}_L^\alpha) > 0$, otherwise $\sup_\Omega u^\alpha(x) \leq L$. For any $L_1 > L$ we have $\mathcal{A}_{L_1}^\alpha \subset \mathcal{A}_L^\alpha$ and therefore (23) yields

$$\begin{aligned} (L_1 - L)m(\mathcal{A}_{L_1}^\alpha) &\leq \int_{\mathcal{A}_{L_1}^\alpha} (u^\alpha(x) - L) \, dm \\ &\leq \int_{\mathcal{A}_L^\alpha} (u^\alpha(x) - L) \, dm \leq Cm(\mathcal{A}_L^\alpha)^{1+\sigma_0}. \end{aligned}$$

Hence

$$m(\mathcal{A}_{L_1}^\alpha) \leq \frac{C}{L_1 - L} m(\mathcal{A}_L^\alpha)^{1+\sigma_0}.$$

In order to estimate the measure of the set \mathcal{A}_L^α we will apply the following Maximum Principle due to Stampacchia [13, Lemma 4.1].

Lemma 6: *Let $\chi : [L_0, \infty) \rightarrow [0, \infty)$ be a decreasing function. Assume that there exist $c, a \in (0, \infty)$ and $b \in (1, \infty)$ such that*

$$L_1 > L \geq L_0 \implies \chi(L_1) \leq \frac{c}{(L_1 - L)^\alpha} (\chi(L))^b.$$

Then

$$\chi(L_0 + d) = 0 \quad \text{where} \quad d = [c\chi(L_0)^{b-1} 2^{\frac{ab}{b-1}}]^{\frac{1}{a}}.$$

An application of Lemma 6 to the function $\chi(L) = m(\mathcal{A}_L^\alpha)$ with $a = 1, b = 1 + \sigma_0$ and $L_0 = \max\{\theta_\alpha, k^\alpha\}$ yields

$$m(\mathcal{A}_{L_0+d_\alpha}^\alpha) = 0 \quad \text{where} \quad d_\alpha \leq Cm(\Omega)^{\sigma_0} 2^{1+\frac{1}{\sigma_0}}. \tag{24}$$

The last assertion means that for each $\alpha = 1, \dots, N$ there exists a constant M_α depending on $n, p, \lambda, q, \mu, \gamma, \Lambda, \|\varphi\|_{p,\lambda;\Omega}, \|f^\alpha\|_{q,\mu;\Omega}$ and $\text{diam } \Omega$ such that

$$\sup_\Omega u^\alpha \leq \max\{\theta_\alpha; \sup_{\partial\Omega} u^\alpha\} + M_\alpha \tag{25}$$

and this completes the proof of Theorem 5. □

3. The Dirichlet problem

We study the boundedness and the Morrey regularity of the weak solutions to the Dirichlet problem

$$\begin{cases} -\operatorname{div} \mathbf{A}(x, \mathbf{u}(x)D\mathbf{u}(x)) + \mathbf{b}(x, \mathbf{u}, D\mathbf{u}) = \mathbf{f}(x) & x \in \Omega \\ \mathbf{u}(x) = 0 & x \in \partial\Omega \end{cases} \quad (26)$$

in a bounded domain $\Omega \subset \mathbb{R}^n$.

Theorem 7 (Essential Boundedness of the Solution): Let $\mathbf{u} \in W_0^{1,2}(\Omega, \mathbb{R}^N)$ be a solution to (26) and assume (A), (10), and (12). Suppose in addition that

$$\begin{cases} \gamma|\mathbf{z}^\alpha|^2 - \Lambda\varphi(x)^2 \leq \sum_{i=1}^n A_i^\alpha(x, \mathbf{u}, \mathbf{z})z_i^\alpha \\ \varphi(x) \in L^{p,\lambda}(\Omega), p > 2, \lambda \in (0, n), \lambda + p > n, \\ 0 \leq b^\alpha(x, \mathbf{u}, \mathbf{z}) \operatorname{sign} u^\alpha(x) \end{cases} \quad (27)$$

for $|u^\alpha| \geq \theta_\alpha > 0$. Then there exists a constant M depending on known quantities such that

$$\|\mathbf{u}\|_{\infty;\Omega} \leq M.$$

Proof: Take a positive constant L such that $L \geq \theta_\alpha$ and consider the set $\bar{\mathcal{A}}_L^\alpha = \{x \in \Omega : u^\alpha(x) + L < 0\}$. Then Theorem 5 applied to $-u^\alpha$ gives

$$\inf_{\Omega} u^\alpha \geq -\theta_\alpha - M_\alpha. \quad (28)$$

Unifying (25) and (28) we get boundedness of $\|u^\alpha\|_{\infty;\Omega}$ for each $\alpha = 1, \dots, N$. Then

$$\|\mathbf{u}\|_{\infty;\Omega} = \max_{1 \leq \alpha \leq N} \|u^\alpha\|_{\infty;\Omega} =: M < \infty.$$

□

Theorem 8 (Morrey Regularity of the Gradient): Let Ω be a bounded (A)-type domain in $\mathbb{R}^n, n \geq 3$, and $\mathbf{u} \in W_0^{1,2}(\Omega, \mathbb{R}^N)$ be a weak solution to (26) under the assumptions (10), (12), and (27). Then $D\mathbf{u} \in L^{2,n-2}(\Omega, \mathbb{R}^N)$ and

$$\int_{\Omega_\rho(x)} |Du(y)|^2 dy \leq C\rho^{n-2} \quad \forall x \in \Omega, \rho \in (0, \operatorname{diam} \Omega] \quad (29)$$

with a constant depending on known quantities.

Proof: Fix $x_0 \in \Omega$ and $\rho > 0$ be such that $\mathcal{B}_\rho(x_0) \subset \mathcal{B}_{2\rho}(x_0) \subset \Omega, \rho < 1$. Define a cut-off function $\zeta(x) \in C^1(\mathbb{R}^n)$

$$\zeta(x) = \begin{cases} 1 & x \in \mathcal{B}_\rho(x_0), \\ 0 & x \notin \mathcal{B}_{2\rho}(x_0), \end{cases} \quad |D\zeta| \leq \frac{C}{\rho}.$$

For any fixed α take $\phi^\alpha(x) = e^{u^\alpha(x)}\zeta(x)^2$ and $\phi^\beta = 0$ for $\beta \neq \alpha$ as a test function in (13) to get

$$\begin{aligned} & \sum_{i=1}^n \int_{\Omega} A_i^\alpha(x, \mathbf{u}, D\mathbf{u}) e^{u^\alpha(x)} D_i u^\alpha(x) \zeta(x)^2 dx \\ &= \int_{\Omega} f^\alpha(x) e^{u^\alpha(x)} \zeta(x)^2 dx - \int_{\Omega} b^\alpha(x, \mathbf{u}, D\mathbf{u}) e^{u^\alpha(x)} \zeta(x)^2 dx \\ & \quad - \sum_{i=1}^n \int_{\Omega} A_i^\alpha(x, \mathbf{u}, D\mathbf{u}) e^{u^\alpha(x)} 2\zeta(x) D_i \zeta(x) dx. \end{aligned}$$

The left-hand side can be estimated by (27) while for the right-hand side we use (10) and (12)

$$\begin{aligned} & e^{-M} \int_{\Omega} (\gamma |Du^\alpha(x)|^2 - \Lambda \varphi(x)^2) \zeta(x)^2 dx \\ & \leq e^M \int_{\Omega} |f^\alpha(x)| \zeta(x)^2 dx \\ & \quad + 2n\Lambda e^M \int_{\Omega} (\varphi(x) + |\mathbf{u}|^{\frac{n}{n-2}} + |D\mathbf{u}|) \zeta(x) |D\zeta| dx \\ & \quad + \Lambda e^M \int_{\Omega} (\psi(x) + |\mathbf{u}|^{\frac{n+2}{n-2}} + |D\mathbf{u}|^{\frac{n+2}{n}}) \zeta(x)^2 dx. \end{aligned}$$

To proceed further, we use the Young inequality $ab \leq \varepsilon a^p + \frac{b^{p/(p-1)}}{\varepsilon^{1/(p-1)}}$, whence

$$\begin{aligned} & \int_{\Omega} \varphi(x) \zeta(x) |D\zeta(x)| dx \leq \frac{1}{2} \int_{\Omega} \varphi(x)^2 \zeta(x)^2 dx + \frac{1}{2} \int_{\Omega} |D\zeta(x)|^2 dx \\ & \int_{\Omega} |\mathbf{u}|^{\frac{n}{n-2}} \zeta(x) |D\zeta(x)| dx \leq \frac{1}{2} M^{\frac{n}{n-2}} \left(\int_{\Omega} \zeta(x)^2 dx + \int_{\Omega} |D\zeta(x)|^2 dx \right) \\ & \int_{\Omega} |D\mathbf{u}| \zeta(x) |D\zeta| dx \leq \varepsilon \int_{\Omega} |D\mathbf{u}|^2 \zeta(x)^2 dx + \frac{1}{\varepsilon} \int_{\Omega} |D\zeta(x)|^2 dx \\ & \int_{\Omega} |D\mathbf{u}|^{\frac{n+2}{n}} \zeta(x)^2 dx \leq \varepsilon \int_{\Omega} |D\mathbf{u}|^2 \zeta(x)^2 dx + \varepsilon^{-\frac{n+2}{n-2}} \int_{\Omega} \zeta(x)^2 dx. \end{aligned}$$

Unifying the above estimates we get

$$\begin{aligned} & \int_{\Omega} |Du^\alpha(x)|^2 \zeta(x)^2 dx \\ & \leq C \left(1 + \varepsilon^{-\frac{n+2}{n-2}} \right) \int_{\Omega} (1 + |f^\alpha(x)| + \psi(x) + \varphi(x)^2) \zeta(x)^2 dx \\ & \quad + C \left(1 + \frac{1}{\varepsilon} \right) \int_{\Omega} |D\zeta(x)|^2 dx + C\varepsilon \int_{\Omega} |D\mathbf{u}(x)|^2 \zeta(x)^2 dx \end{aligned} \tag{30}$$

with constants depending on n, Λ, γ , and M . Summing up (30) over α from 1 to N , fixing ε small enough and moving the last term to the left-hand side we obtain

$$\begin{aligned} \int_{\Omega} |D\mathbf{u}(x)|^2 \zeta(x)^2 dx &\leq C \int_{\Omega} (1 + \psi(x) + \varphi(x)^2) \zeta(x)^2 dx \\ &\quad + C \sum_{\alpha=1}^N \int_{\Omega} |f^\alpha(x)| \zeta(x)^2 dx + C \int_{\Omega} |D\zeta(x)|^2 dx. \end{aligned} \quad (31)$$

Then, by the definition of ζ and by (12) we have

$$\begin{aligned} \int_{\mathcal{B}_{2\rho}} (1 + \psi(x) + \varphi(x)^2) dx &\leq C[\rho^n + \rho^{n-\frac{n-\mu}{q}} \|\psi\|_{q,\mu;\Omega} \\ &\quad + \rho^{n-\frac{2(n-\lambda)}{p}} \|\varphi\|_{p,\lambda;\Omega}^2] \\ \sum_{\alpha=1}^N \int_{\mathcal{B}_{2\rho}} |f^\alpha(x)| dx &\leq C\rho^{n-\frac{n-\mu}{q}} \|\mathbf{f}\|_{q,\mu;\Omega} \\ \int_{\mathcal{B}_{2\rho}} |D\zeta(x)|^2 dx &\leq C\rho^{n-2}. \end{aligned}$$

Hence

$$\int_{\mathcal{B}_\rho} |D\mathbf{u}|^2 dx \leq C\rho^{\lambda_0} \quad (32)$$

with $\lambda_0 = \min\{n-2, n-\frac{2(n-\lambda)}{p}, n-\frac{n-\mu}{q}\} = n-2$ and the constant depends on known quantities.

Let $\mathcal{B}_{2\rho}(x_0) \cap \partial\Omega \neq \emptyset$. Then we extend u^α and the given functions f^α, φ , and ψ as zero in Ω^c and consider the test functions

$$\phi^\alpha(x) = (e^{|u^\alpha(x)|} - 1)\zeta(x)^2 \operatorname{sign} u^\alpha(x) \text{ and } \phi^\beta(x) = 0 \text{ for } \beta \neq \alpha$$

where $\zeta(x)$ is the cut-off function defined above. Thus, (13) gives

$$\begin{aligned} &\sum_{i=1}^n \int_{\Omega} A_i^\alpha(x, \mathbf{u}, D\mathbf{u}) e^{|u^\alpha(x)|} D_i u^\alpha(x) \zeta(x)^2 dx \\ &= \int_{\Omega} f^\alpha(x) (e^{|u^\alpha(x)|} - 1) \zeta(x)^2 \operatorname{sign} u^\alpha(x) dx \\ &\quad - \sum_{i=1}^n \int_{\Omega} A_i^\alpha(x, \mathbf{u}, D\mathbf{u}) (e^{|u^\alpha(x)|} - 1) 2\zeta(x) D_i \zeta(x) \operatorname{sign} u^\alpha(x) dx \\ &\quad - \int_{\Omega} b^\alpha(x, \mathbf{u}, D\mathbf{u}) (e^{|u^\alpha(x)|} - 1) \zeta(x)^2 \operatorname{sign} u^\alpha(x) dx. \end{aligned}$$

Hence the conditions (10) and (27) give

$$\begin{aligned} \gamma \int_{\Omega} |Du^\alpha(x)|^2 \zeta(x)^2 dx &\leq \Lambda e^M \int_{\Omega} \varphi(x)^2 \zeta(x)^2 dx \\ &+ e^M \int_{\Omega} |f^\alpha(x)| \zeta(x)^2 dx \\ &+ 2n\Lambda e^M \int_{\Omega} (\varphi(x) + |\mathbf{u}|^{\frac{n}{n-2}} + |D\mathbf{u}|) \zeta(x) |D\zeta(x)| dx \end{aligned}$$

and to get the desired estimate (29) we argue as above. □

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References

- [1] Gilbarg G, Trudinger NS. Elliptic partial differential equations of second order. 2nd ed. Berlin: Springer-Verlag; 1983.
- [2] Ladyzhenskaya OA, Ural'tseva NN. Linear and quasilinear equations of elliptic type. Moscow: Nauka; 1964. Russian.
- [3] De Giorgi E. Sulla differenziabilità e l'analiticità delle estremali degli integrali multipli regolari. Mem Accad Sci Torino Cl Sci Fis Mat Natur. 1957;3(3):25–43.
- [4] Nash J. Continuity of solutions of parabolic and elliptic equations. Amer J Math. 1958;80:931–954.
- [5] De Giorgi E. Un esempio di estremali discontinue per un problema variazionale di tipo ellittico. Bull Unione Mat It. 1968;4:135–137.
- [6] Giusti E, Miranda M. Sulla regolarità delle soluzioni deboli di una classe di sistemi ellittici quasi-lineari. Arch Rat Mech Anal. 1968;31:173–184.
- [7] Leonardi S. A maximum principle for linear elliptic systems with discontinuous coefficients. Comment Math Univ Carolin. 2004;45(3):457–474.
- [8] Leonardi S. On constants of some regularity theorems. De Giorgi's type counterexample. Math Nachr. 1998;192:191–204.
- [9] Nečas J, Stará J. Principio di massimo per i sistemi ellittici quasi-lineari non diagonali. Boll Unione Mat Ital. 1972;6(4):1–10.
- [10] Campanato S. A maximum principle for nonlinear elliptic systems: boundary fundamental estimates. Adv Math. 1987;66:291–317.
- [11] Cannarsa P. On a maximum principle for elliptic systems with constant coefficients. Rend Sem Univ Padova. 1981;64:77–84.
- [12] Leonetti F, Petricca PV. Regularity for solutions to some nonlinear elliptic systems. Complex Var Elliptic Equ. 2011;56(12):1099–1113.

- [13] Stampacchia G. Equations elliptiques du second ordre a coefficients discontinuous. In: Séminaire de Mathématiques Supérieures. Vol. 16; Université de Montréal; 1966. 326 p.
- [14] Dong H, Kim D. Global regularity of weak solutions to quasilinear elliptic and parabolic equations with controlled growth. *Commun Partial Differ Equ.* 2011;36:1750–1777.
- [15] Palagachev DK. Global Hölder continuity of weak solutions to quasilinear divergence form elliptic equations. *J Math Anal Appl.* 2009;359:159–167.
- [16] Palagachev DK, Softova LG. The Calderón-Zygmund property for quasilinear divergence form equations over Reifenberg flat domains. *Nonlinear Anal.* 2011;74:1721–1730.
- [17] Softova LG. L^p -integrability of the gradient of solutions to quasilinear systems with discontinuous coefficients. *Differ Int Equ.* 2013;26(9–10):1091–1104.
- [18] Byun S-S, Palagachev D. Boundedness of the weak solutions to quasilinear elliptic equations with Morrey data. *Indiana Univ Math J.* 2013;62(5):1565–1585.
- [19] Byun S-S, Palagachev D. Morrey regularity of solutions to quasilinear elliptic equations over Reifenberg flat domains. *Calc Var Partial Differ Equ.* 2014;49(1–2):37–76.
- [20] Byun S-S, Palagachev D. Weighted L^p -estimates for elliptic equations with measurable coefficients in nonsmooth domains. *Potent l Anal.* 2014;41(1):51–79.
- [21] Byun S-S, Palagachev D, Shin P. Sobolev-Morrey regularity of solutions to general quasilinear elliptic equations. *Nonlinear Anal Theory Methods Appl Ser A.* 2016;147:176–190.
- [22] Fattorusso L, Softova LG. Precise Morrey regularity of the weak solutions to a kind of quasilinear systems with discontinuous data. **Forthcoming.**
- [23] Byun S-S, Softova LG. Gradient estimates in generalized Morrey spaces for parabolic operators. *Math Nachr.* 2015;288(14–15):1602–1614.
- [24] Palagachev DK, Softova LG. Fine regularity for elliptic systems with discontinuous ingredients. *Arch Math.* 2006;86:145–153.
- [25] Adams D. Traces of potentials II. *Indiana Univ Math J.* 1973;22:907–918.
- [26] Chiarenza F. Regularity for solutions of quasilinear elliptic equations under minimal assumptions. *Potential Anal.* 1995;4(4):325–334.
- [27] Rokotson M. Equivalence between the growth of $\int_{B(x,r)} |\nabla u|^p dy$ and T in the equation $P[u] = T$. *J Differ Equ.* 1990;86:102–122.
- [28] Giaquinta M. Multiple integrals in the calculus of variations and nonlinear elliptic systems. Vol. 105, *Annals of mathematics studies*. Princeton (NJ): Princeton University Press; 1983.
- [29] John O, Stará J. On the regularity and nonregularity of elliptic and parabolic systems. In: Kurzweil J, editor. *Equadiff-7, Proceedings of the 7th Czechoslovak Conference on Differential Equations and Their Applications held in Prague, 1989*. Vol. 118, *Teubner-Texte zur Mathematik; Leipzig: Teubner; 1990*. p. 28–36.