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GROUPS WITH FINITELY MANY ISOMORPHIC CLASSES OF NON-ABELIAN SUBGROUPS

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Abstract. We study groups in which the non-abelian subgroups fall into finitely many isomorphic classes. We prove that a locally generalized radical group with this property is abelian-by-finite and reduced minimax. The same conclusion holds for locally generalized coradical groups. Here a generalized radical group is a group with an ascending series whose factors are either locally nilpotent or locally finite, and a generalized coradical group is a group with a descending series whose factors are either locally nilpotent or locally finite.

1. Introduction.

The concept of isomorphism is one of the basic concept in algebra. It gives a natural equivalence between groups and their subgroups. More precisely, let G be a group, $\mathcal{M}$ a family of subgroups of the group G and $H, K \in \mathcal{M}$. Then the relation "H is isomorphic to K" is an equivalence relation in $\mathcal{M}$. Denote by $\mathbf{Isom}\mathcal{M}$ the set of the equivalence classes defined by this relation. Choose in every equivalence class one representative and denote the set of all these representatives by $\mathbf{Itype}\mathcal{M}$. The set $\mathbf{Itype}\mathcal{M}$ is called the isomorphic type of the family $\mathcal{M}$.

If G is non-trivial and $\mathcal{L}(G)$ denotes the family of all subgroups of G, then $\mathbf{Itype}\mathcal{L}(\mathbf{G})$ contains at least two subgroups: G and $< 1 >$. It is not hard to prove that if $|\mathbf{Itype}\mathcal{L}(G)| = 2$, then either G has prime order p or G is an infinite cyclic group. Denote by $\mathcal{L}_{\mathbf{ab}}(G)$ the family of all abelian subgroups of G. It is possible to prove that if G is some generalized soluble group (for example G is a locally (soluble-by-finite) group), then the equation $|\mathbf{Itype}\mathcal{L}_{\mathbf{ab}}(G)| = 2$ implies that either G has prime order p or G is an infinite cyclic group. However this is

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not true in the general case. A.Yu. Olshankii in the paper [19] constructed an example of a torsion-free group whose proper non-trivial subgroups are infinite cyclic. In another paper [20] A.Yu. Olshanskii constructed an example of an infinite p-group whose proper non-trivial subgroups have order p, where p is a relatively large prime number. It is clear that for both these groups $|\mathbf{Itype}\mathcal{L}_{\mathbf{ab}}(G)| = 2$. The case of the family $\mathcal{L}_{\mathbf{non-ab}}(G)$ of all non-abelian subgroups of G is more complicated. Clearly, if G is a group whose proper subgroups are abelian, then $\mathbf{Itype}\mathcal{L}_{\mathbf{non-ab}}(G) = \{G\}$. The structure of non-abelian groups G in which $\mathbf{Itype}\mathcal{L}_{\mathbf{non-ab}}(G) = \{G\}$ has been described in details by H. Smith and J. Wiegold in [25].

The following question naturally arises: What can be said about the structure of a group G in which the cardinality of the set $\mathbf{Itype}\mathcal{M}$ is finite, for some basic families $\mathcal{M}$ of subgroups of G?

If $\mathcal{M}$ is the family of the commutator subgroups of all subgroups of G, then the problem has been studied by F. de Giovanni and D.J.S. Robinson in [3], as well as by M. Herzog, P. Longobardi, M. Maj and D.J.S. Robinson, H. Smith in a series of papers (see [9], [13], [14] and [15]).

If G is an abelian group, in which the family of all subgroups has finite isomorphic type, then clearly $T = \mathbf{Tor}(G)$, the torsion subgroup of G, is finite, and so $G = T \times A$ where A is a torsion-free subgroup. It is not hard to prove that $r_0(A)$ is finite. Note that if B is a subgroup of A and $B \simeq A$, then A/B is finite, by a theorem due to V.S. Charin (see [2], Theorem 2). Using this Charin's result, it is not difficult to prove that A has to be minimax. It follows that if G is a group, for which $\mathbf{Itype}\mathcal{L}(G)$ or $\mathbf{Itype}\mathcal{L}_{\mathbf{ab}}(G)$ is finite, then every abelian subgroup of G is minimax. The radical groups with such property have been studied by R. Baer (see [1]) and by D.I. Zaitsev (see [28]).

Here we begin the study of groups, in which the family of all non-abelian subgroups has finite isomorphic type. The main results of this paper are the following theorems.

First, however, it is worth recalling some definitions.

If G is an arbitrary group, then we denote by $\mathbf{Tor}(G)$ the maximal normal periodic subgroup of G. We note that if G is locally nilpotent, then $\mathbf{Tor}(G)$ contains all elements of finite order in G.

A group G is said to have $\mathbf{0-rank\ r_0(G) = r}$ if G has a finite subnormal series whose factors are either infinite cyclic or periodic and if the number of infinite cyclic factors is exactly r.

It is not hard to prove that the number of infinite cyclic factors is an invariant of G. In some papers, the 0-rank is also called the **torsion-free rank** of the group G.

If easy to see that if A is an abelian torsion-free group, then $r_0(A)$ is finite if and only if A is isomorphic to a subgroup of a finite dimensional vector space.

A group G is called **minimax**, if G has a finite subnormal series, whose factors satisfy **Min** or **Max**. If G is a soluble-by-finite minimax group, then G has a finite subnormal series, whose factors are Chernikov groups or polycyclic-by-finite groups. We note that if G is a soluble-by-finite minimax group, then $T = \mathbf{Tor}(G)$ is a Chernikov group and the periodic subgroups of G/T are finite. If $\mathbf{Tor}(G)$ is finite (that is G does not include a divisible abelian subgroup), then G is called a **reduced minimax** group.

A group G is called **generalized radical**, if G has an ascending series whose factors are locally nilpotent or locally finite.

It follows easily from the definition that a generalized radical group G has either an ascendant locally nilpotent subgroup or an ascendant locally finite subgroup. In the former case, the locally nilpotent radical is non-trivial. In the latter case, G contains a non-trivial normal locally finite subgroup, so the locally finite radical is non-trivial. Thus every generalized radical group has an ascending series of normal subgroups with factors that are either locally nilpotent or locally finite.

Now we can state the main results of the paper.

Theorem A *Let G be a non-abelian locally generalized radical group in which the family of all non-abelian subgroups has finite isomorphic type. Then G is abelian-by-finite and reduced minimax. In particular, G is residually finite.*

The following class is dual to the class of generalized radical groups. A group G is called **generalized coradical**, if G has a descending series of normal subgroups whose factors are locally nilpotent or locally finite.

Theorem B *Let G be a non-abelian locally generalized coradical group in which the family of all non-abelian subgroups has finite isomorphic type. Then G is abelian-by-finite and reduced minimax.*

The following result plays an important role in the proof of Theorem B.

Theorem C *Let G be a group whose non-abelian finitely generated subgroups can be generated by d elements. If G is residually finite then G is hyperabelian-by-finite and has finite special rank.*

Notice that if G is a finitely generated abelian-by-finite group, then the subgroups of G fall into finitely many isomorphic classes. In fact, G has an abelian finitely generated torsion-free subgroup F of finite index, say t. If F is s-generated, then every subgroup of F is a torsion-free group with $\leq s$ generators, thus F has finite isomorphism type, say v. But if P is a polycyclic group and t is a positive integer, then the groups H which have a subgroup of index at most t isomorphic to P lie in finitely many isomorphic classes (see for example [23], Theorem 6, page 176). Therefore the converse of Theorem A and B holds for finitely generated groups. But in general it is not true that every minimax torsion-free abelian group has finitely many non-isomorphic subgroups (see [11]), hence the converse of Theorem A and Theorem B generally is not true.

2. The normalizers of periodic abelian subgroups.

We start our investigation of groups in which the family of non-abelian subgroups has finite isomorphic type with two useful Lemmas and with a couple of definitions.

Lemma 2.1 *Let G be a group in which the family of all non-abelian subgroups has finite isomorphic type. If K is an infinite locally finite subgroup of G, then K is abelian.*

Proof. Suppose that K is non-abelian. Being locally finite, K includes a finite non-abelian subgroup F. Then G has an ascending chain

$$F = F_0 \leq F_1 \leq \cdots \leq F_n \leq F_{n+1} \leq \cdots$$

of finite subgroups such that $|F_n| < |F_{n+1}|$ for each $n \in \mathbb{N}$. But in this case, the subgroups F_n and F_m cannot be isomorphic for $n, m \in \mathbb{N}$, $n \neq m$, and we obtain a contradiction. This contradiction shows that K has to be abelian.

□

Lemma 2.2 *Let G be a group in which the family of all non-abelian subgroups has finite isomorphic type. If G is locally nilpotent, then G is nilpotent.*

Proof. Without loss of generality we can assume that G is non-abelian. Then G includes a non-abelian finitely generated subgroup K. Suppose that G is not nilpotent. Then G has an ascending chain

$$K = K_0 \leq K_1 \leq \cdots \leq K_n \leq K_{n+1} \leq \cdots$$

of finitely generated subgroups such that $\mathbf{ncl}(K_n) < \mathbf{ncl}(K_{n+1})$ for each $n \in \mathbb{N}$ (here, if K is a nilpotent group, we denote by $\mathbf{ncl}(K)$ the nilpotence class of K). But in this case the subgroups K_n and K_m cannot be isomorphic for $n, m \in \mathbb{N}$, $n \neq m$. This contradiction shows that G must be nilpotent.

□

Let R be an integral domain and A be an R-module. An element $a \in A$ is said to be R**-periodic** if $\mathbf{Ann}_R(a) \neq < 0 >$. It is not hard to see that the subset $\mathbf{Tor}_R(A)$ consisting of all R-periodic elements of a module A is an R-submodule.

As usual, a module A is called R**-periodic** if $A = \mathbf{Tor}_R(A)$.

If $\mathbf{Tor}_R(A) =< 0 >$, then we will say that A is R**-torsion-free**.

If A is an abelian group, then we can consider A as a $\mathbb{Z}$-module. In this case obviously $\mathbf{Tor}_{\mathbb{Z}}(A) = \mathbf{Tor}(A)$.

In the following Lemmas we study the structure of the normalizer of an infinite periodic abelian subgroup of a group in which the family of all non-abelian subgroups has finite isomorphic type.

Lemma 2.3 *Let G be a group in which the family of all non-abelian subgroups has finite isomorphic type. Let p be a prime, A be an elementary abelian p-subgroup of G, and x be an element of infinite order such that $x \in \mathbf{N}_G(A)$. Then A, considered as an $\mathbf{F}_p < x >$-module, is periodic.*

Proof. Suppose the contrary, and let A be not $\mathbf{F}_p < x >$-periodic. Put $J = \mathbf{F}_p < x >$. Then there exists an element $d \in A$ such that $\mathbf{Ann}_J(d) = < 0 >$. Put $D = < d >^{<x>}$, then we can consider D as an J-submodule of A, morever $D \simeq_J J$. It follows that $D = \mathbf{Dr}_{n\in\mathbb{Z}} < d_n >$ and $d_n^x = d_{n+1}$, $n \in \mathbb{Z}$.
In particular, $[< D, x >, < D, x >] =< d_n^{-1} d_{n+1} \mid n \in \mathbb{Z} >$.
In this case, the factor group $< D, x > /[< D, x >, < D, x >]$ is an extension of a subgroup of order p by an infinite cyclic group.

Let k be a positive integer and put $J_k = \mathbf{F}_p < x^k >$. Then we have $J = J_k \oplus J_k x \oplus \cdots \oplus J_k x^{k-1}$. Put $D_k = \mathbf{Dr}_{n\in\mathbb{Z}} < d_{kn} >$ and $D_{k+j} = \mathbf{Dr}_{n\in\mathbb{Z}} < d_{kn+j} >$, $1 \leq j \leq k-1$. Every subgroup D_{k+j} is $< x^k >$-invariant and the factor group $< D_{k+j}, x^k > /[< D_{k+j}, x^k >, < D_{k+j}, x^k >]$ is an extension of a subgroup of order p^k by an infinite cyclic group. Hence the subgroups $< D, x^k >, k \in \mathbb{N}$, cannot be isomorphic, and we obtain a contradiction. This contradiction proves the result.

□

Corollary 2.4 *Let G be a group in which the family of all non-abelian subgroups has finite isomorphic type. Let p be a prime, A be an elementary abelian p-subgroup of G, and x be an element of infinite order such that $x \in \mathbf{N}_G(A)$. Then $< a >^{<x>}$ is finite, for each element $a \in A$.*

Proof. Put $J = \mathbf{F}_p < x >$. By Lemma 2.3, $\mathbf{Ann}_J(a)$ is non-zero for every element $a \in A$. We have $< a >^{<x>} \simeq_J J/\mathbf{Ann}_J(a)$. Now our assertion follows from the fact that every non-zero ideal of J has finite index in J. □

Now let G be a group, x an element of G. Set

$$x^G = \{x^g = g^{-1}xg \mid g \in G\} \text{ and } \mathbf{FC}(G) = \{x \in G \mid x^G \text{ is finite}\}.$$

It is not difficult to prove that $\mathbf{FC}(G)$ is a characteristic subgroup of G. This subgroup is called the **FC-center** of G.

Lemma 2.5 *Let G be a group in which the family of all non-abelian subgroups has finite isomorphic type. Let A be an infinite periodic abelian subgroup of G, $x \in \mathbf{N}_G(A)$. If $A \leq \mathbf{FC}(< A, x >)$, then $x \in \mathbf{C}_G(A)$.*

Proof. If the element x has finite order, the subgroup $< A, x >$ is periodic. Being locally finite, it is abelian by Lemma 2.1. It follows that $x \in \mathbf{C}_G(A)$.

Consider now the case when x has infinite order. Then $< A, x >= A \rtimes < x >$. Suppose that $x \notin \mathbf{C}_G(A)$. In this case, the subgroup A has an element a such that $ax \neq xa$. It follows that the subgroup $< a, x >$ is not abelian. We have $< a, x > = < a >^{<x>} < x >$. Since $A \leq \mathbf{FC}(< a, x >)$, the normal closure $< a >^{<x>}$ is finite. Furthermore, $\mathbf{Tor}(< a, x >) =< a, x > \cap A =< a >^{<x>}$ is finite. Therefore we can choose an element $a_1 \in A$ such that $a_1 \notin < a, x > \cap A$. Again the subgroup $< a, a_1, x >$ is non-abelian and $\mathbf{Tor}(< a, a_1, x >)$ is finite, moreover we have $|\mathbf{Tor}(< a, x >)| < |\mathbf{Tor}(< a, a_1, x >)|$. Using similar arguments we can choose elements $\{a_n \mid n \in \mathbb{N}\}$ satisfying the following conditions:

$$< a, a_1, \cdots, a_n, x > \text{ is non-abelian},$$

$$\mathbf{Tor}(< a, a_1, \cdots, a_n, x >) \text{ is finite},$$

$$|\mathbf{Tor}(< a, a_1, \cdots, a_n, x >)| < |\mathbf{Tor}(< a, a_1, \cdots, a_{n+1}, x > |, \; n \in \mathbb{N}.$$

These conditions show that the subgroups $< a, a_1, \cdots, a_n, x >$ and $< a, a_1, \cdots, a_m, x >$ cannot be isomorphic whenever $n \neq m$, and we obtain a contradiction. This contradiction proves the result.

□

We are now able to prove the following

Corollary 2.6 *Let G be a group in which the family of all non-abelian subgroups has finite isomorphic type. Let p be a prime and A be an infinite elementary abelian p-subgroup of G. Then $\mathbf{N}_G(A) = \mathbf{C}_G(A)$.*

Proof. Indeed for each element $x \in \mathbf{N}_G(A)$ and each element $a \in A$ the normal closure $< a >^{<x>}$ is finite by Corollary 2.4. It follows that $A \leq \mathbf{FC}(< A, x >)$. Then Lemma 2.5 implies that $x \in \mathbf{C}_G(A)$. □

We study now the more general situation in which A is any infinite periodic subgroup.

Lemma 2.7 *Let G be a group in which the family of all non-abelian subgroups has finite isomorphic type. Let A be an infinite periodic abelian subgroup of G, $x \in \mathbf{N}_G(A)$. If the subgroup $< A, x >$ is locally nilpotent, then $x \in \mathbf{C}_G(A)$.*

Proof. If the element x has finite order, then the subgroup $< A, x >$ is periodic. Being locally finite, by Lemma 2.1 it is abelian. It follows $x \in \mathbf{C}_G(A)$.

Consider now the case, when x is of infinite order. Then $< A, x >= A \rtimes < x >$. Let a be an arbitrary element of A. Then the subgroup $< a, x >$ is nilpotent. Being finitely generated, it has a finite periodic part.
We have $< a, x >=< a >^{<x>} < x >$, so that $\mathbf{Tor}(< a, x >) =< a >^{<x>}$. Thus $< a >^{<x>}$ is finite and $a \in \mathbf{FC}(< A, x >)$. Now we can apply Lemma 2.5 and obtain that $x \in \mathbf{C}_G(A)$. □

We are now able to extend the result of Corollary 2.6 to any periodic abelian subgroup A.

Corollary 2.8 *Let G be a group in which the family of all non-abelian subgroups has finite isomorphic type. Let p be a prime and A be an infinite abelian p-subgroup of G. Then $\mathbf{N}_G(A) = \mathbf{C}_G(A)$.*

Proof. Let x be an arbitrary element of $\mathbf{N}_G(A)$. If the element x has finite order, then the subgroup $< A, x >$ is periodic. Being locally finite, by Lemma 2.1, it is abelian. It follows that $x \in \mathbf{C}_G(A)$. Consider now the case, when x is of infinite order. Put $A_j = \Omega_j(A)$. Suppose first that A_1 is infinite. Then Corollary 2.6 implies that $x \in C_G(A_1)$.

It is not difficult to prove that in this case $x \in C_G(A_{j+1}/A_j)$ for each $j \in \mathbf{N}$. In other words, the ascending series

$$< 1 > = A_0 \leq A_1 \leq A_2 \leq \cdots \leq A_j \leq A_{j+1} \leq \cdots$$

is central in the subgroup $< A, x >$. It follows that $< A, x >$ is a hypercentral subgroup. Being hypercentral, it is locally nilpotent abelian (see, for example, [12], § 63). Using now Lemma 2.7 we obtain that $x \in C_G(A)$. Suppose now that A_1 is finite. Then A satisfies the minimal condition (see, for example, [7], Theorem 25.1). Then $A = C_1 \times \cdots \times C_k \times F$ where C_j is a Prüfer p-subgroup, $1 \leq j \leq k$, F is a finite subgroup. But in this case $A \leq \mathbf{FC}(< A, x >)$. An application of Lemma 2.5 ensures that $x \in \mathbf{C}_G(A)$. □

Corollary 2.9 *Let G be a group in which the family of all non-abelian subgroups has finite isomorphic type. Let A be an infinite abelian periodic subgroup of G. Then $\mathbf{N}_G(A) = \mathbf{C}_G(A)$.*

Proof. Let x be an arbitrary element of $N_G(A)$. We have

$$A = \mathbf{Dr}_{p \in \Pi(A)} A_p,$$

where A_p is a Sylow p-subgroup of A, $p \in \mathbf{\Pi}(A)$. Clearly every subgroup A_p is $< x >$-invariant. If A_p is infinite, then, by Corollary 2.6, $x \in C_G(A_p)$. In particular, $A_p \leq \mathbf{FC}(< A, x >)$. If A_p is finite, then again we have the inclusion $A_p \leq \mathbf{FC}(< A, x >)$. It follows that $A \leq \mathbf{FC}(< A, x >)$. Lemma 2.5 implies that in this case $x \in \mathbf{C}_G(A)$. □

We end this section with the following result that will be useful in the sequel.

Lemma 2.10 *Let G be a group in which the family of all non-abelian subgroups has finite isomorphic type. Let A be an infinite abelian periodic subgroup of G. Suppose that H is a subgroup of $\mathbf{N}_G(A)$, such that all periodic subgroups of H are finite and their order are bounded. Then H is abelian.*

Proof. By Corollary 2.9, $[H, A] = < 1 >$. Suppose that H is non-abelian. The intersection $A_0 = H \cap A$ is finite. Let t be the least positive integer such that the orders of all finite subgroups of H are at most t. Being infinite, A has an ascending series

$$A_0 < A_1 < \cdots < A_n < A_{n+1} < \cdots$$

of finite subgroups. Consider the ascending series

$$H = HA_0 \leq HA_1 \leq \cdots \leq HA_n \leq HA_{n+1} \leq \cdots$$

We note that every subgroup HA_n is non-abelian, $n \in \mathbb{N}$. If F is a finite subgroup of HA_n, then

$$F/(F \cap A_n) \simeq FA_n/A_n \leq HA_n/A_n \simeq H/(H \cap A_n) = H/A_0.$$

It follows that $|FA_n| \leq (t|A_n|)/|A_0|$, $n \in \mathbb{N}$. Since $|A_n|/|A_0| < |A_{n+1}|/|A_0|$, we can conclude that HA_{n+1} includes a finite subgroup, whose order is greater than $(t|A_n|)/|A_0|$, which shows that the subgroups HA_n and HA_m cannot be isomorphic for $n, m \in \mathbb{N}$, $n < m$, and we obtain a contradiction. This contradiction shows that H must be abelian. $\square$

3. The normalizers of abelian torsion-free subgroups.

In this section we study the structure of the normalizer of a torsion-free abelian subgroup of a group in which the family of all non-abelian subgroups has finite isomorphic type. We start with three Lemmas.

Lemma 3.1 *Let G be a group and A be a normal abelian torsion-free subgroup of G. Suppose that G has an element x such that $G = A < x >$. If $< x^k > = < x > \cap \mathbf{C}_G(A)$, $k > 0$, then there exists a positive integer d such that k divides d and G^d is torsion-free and abelian.*

Proof. Suppose first that x has finite order. Then $y = x^k \in \mathbf{C}_G(A)$, moreover, $C = \mathbf{C}_G(A)$ is abelian and $< y > = \mathbf{Tor}(C)$. We may assume that K divides $|x|$. Put $m = |x|/k$, then m is the order of y. We have $C = A \times < y >$, so that $C^m = A^m$ is torsion-free. Clearly the subgroup $D = C^m$ is normal in G and $(G/D)^{mk} = < 1 >$. It follows that $G^{mk} \leq D$, so that G^{mk} is torsion-free and abelian.

Suppose now that x has infinite order. Again $y = x^k \in \mathbf{C}_G(A)$ and the subgroup $C = \mathbf{C}_G(A)$ is abelian. Here we have two possibilities: $A \cap < x > = < 1 >$ or $y^t \in A$ for some positive integer t. In the first case, $C = A \times < y >$ is abelian and torsion-free. Since $|G/C| = k$, $G^k \leq C$, so that G^k is abelian and torsion-free.

Finally assume that $y^t \in A$. If C is torsion-free, then again G^k is abelian and torsion-free. If not, then $\mathbf{Tor}(C)$ is finite cyclic. In this case we can repeat the above arguments and find a positive integer d such that G^d is abelian and torsion-free.

$\square$

Lemma 3.2 *Let G be a group in which the family of all non-abelian subgroups has finite isomorphic type. Let A be an abelian torsion-free subgroup of G. If $\mathbf{N}_G(A) \setminus \mathbf{C}_G(A)$ contains an element x such that $x^k \in \mathbf{C}_G(A)$, then A is minimax.*

Proof. Since $x \notin \mathbf{C}_G(A)$, there exists an element $a \in A$ such that the subgroup $C_0 = < a, x >$ is non-abelian.
We have $< a, x > = < a >^{<x>} < x >$. Put $A_0 = < a >^{<x>} < x^k >$. By Lemma 3.1 there exists a positive integer d_0 such that $C_0^{d_0}$ is torsion-free and abelian. Since the indices $|C_0 : A_0|$ and $|C_0 : C_0^{d_0}|$ are finite, $r_0(A_0) = r_0(C_0^{d_0})$.

Suppose that $r_0(A)$ is infinite. Then there exists an element a_1 such that $< a_1 > \cap A_0 = < 1 >$. Put $A_1 = < a_1 >^{<x>} A_0$, then $r_0(A_0) < r_0(A_1)$. The subgroup $C_1 = A_1 < x >$ is non-abelian. Using again Lemma 3.1, we obtain that there exists a positive integer d_1 such that $C_1^{d_1}$ is torsion-free and abelian. As above $r_0(A_1) = r_0(C_1^{d_1})$.
Put $t = d_0 d_1$, then $C_0^t \leq C_0^{d_0}$ and $C_1^t \leq C_1^{d_1}$. Since both subgroups C_0, C_1 are abelian-by-finite and finitely generated, both factor-groups C_0/C_0^t and C_1/C_1^t are finite. It follows that $r_0(A_0) = r_0(C_0^t)$, $r_0(A_1) = r_0(C_1^t)$. Thus we obtain that $r_0(C_0^t) < r_0(C_1^t)$, therefore the subgroups C_0 and C_1 cannot be isomorphic. Using similar arguments we can choose elements $\{a_n \mid n \in \mathbb{N}\}$ satisfying the following conditions:

$$< a, a_1, \cdots, a_n, x > \text{ is non} - \text{abelian;}$$

$$< a, a_1, \cdots, a_n, x > \text{ is abelian} - \text{by} - \text{finite;}$$

$$< a, a_1, \cdots, a_n, x >, < a, a_1, \cdots, a_{n+1}, x > \text{ are not isomorphic}, n \in \mathbb{N}.$$

These conditions show that we have an infinite family of non-isomorphic non-abelian subgroups, and we obtain a contradiction. This contradiction proves that $r_0(A)$ is finite.

Choose a maximal $\mathbb{Z}$-independent subset M of A such that $a \in M$ and put $K = < M >^{<x>}$. Then the subgroup K is $< x >$-invariant, finitely generated, the factor-group A/K is periodic and the subgroup $< K, x >$ is not abelian. Suppose that $\mathbf{\Pi}(A/K)$ is infinite. Then we can choose an infinite family $\{X_n \mid n \in \mathbb{N}\}$ of infinite subsets of $\mathbf{\Pi}(A/K)$ such that $X_n \subseteq X_{n+1}$ and $X_{n+1} \setminus X_n$ is infinite for every $n \in \mathbb{N}$. Denote by K_n/K the Sylow X_n-subgroup of A/K, $n \in \mathbb{N}$. By such a choice, the indices $|K_1 : K|$ and $|K_{n+1} : K_n|$ are infinite. It follows that the subgroups K_m and K_n, $m, n \in \mathbb{N}, n < m$, cannot be isomorphic (see [2], Theorem 2).

Consider first the case $A \cap < x > = < 1 >$.
Then $\mathbf{C}_G(A) \geq A \times < x^k >$. Let $n, m \in \mathbb{N}$, $n < m$. The subgroups $E_n = K_n \rtimes < x >$ and $E_m = K_m \rtimes < x >$ are non-abelian and

include the normal abelian subgroups $K_n \times < x^k >$ and $K_m \times < x^k >$ respectively, having index k. Since K_n and K_m are not isomorphic, then $K_n \times < x^k >$ and $K_m \times < x^k >$ are not isomorphic. By Lemma 3.1, there exists a positive integer s_n (respectively s_m) such that $E_n^{s_n}$ (respectively $E_m^{s_m}$) is torsion-free and abelian. Put $s = s_n s_m$. Then $E_n^s \leq E_n^{s_n}$ and $E_m^s \leq E_m^{s_m}$, in particular, E_n^s and E_m^s are abelian and torsion-free. Since the subgroups E_m and E_n have finite special rank, E_n/E_n^s and E_m/E_m^s are finite. By Lemma 3.1, k divides s, therefore $E_n^s \leq K_n \times < x^k >$ (respectively $E_m^s \leq K_m \times < x^k >$). The finiteness of the index $|K_n \times < x^k > : E_n^s|$ (respectively $|K_m \times < x^k > : E_m^s|$) implies that E_n^s is isomorphic to $K_n \times < x^k >$ (respectively E_m^s is isomorphic to $K_m \times < x^k >$). Since $K_n \times < x^k >$ and $K_m \times < x^k >$ are not isomorphic, E_n^s and E_m^s are not isomorphic. In turn it follows that E_n and E_m are not isomorphic. That is true for every $n, m \in \mathbb{N}$, $n < m$, so we obtain a contradiction.

Consider now the case $A \cap < x > = < v > \neq < 1 >$.
Without loss of generality we may assume that $v \in K$. Then using the above arguments, we again obtain a contradiction. This contradiction shows that $\mathbf{\Pi}(A/K)$ is finite. This means that A is a minimax subgroup. □

Lemma 3.3 *Let G be a group in which the family of all non-abelian subgroups has finite isomorphic type. Let A be an abelian torsion-free subgroup of G. If $x \in \mathbf{N}_G(A)$ and x has infinite order, then A, considered as a $\mathbb{Z} < x >$ module, is periodic.*

Proof. Suppose that A is not $\mathbb{Z} < x >$-periodic. Put $J = \mathbb{Z} < x >$. Then there exists an element $d \in A$ such that $\mathbf{Ann}_J(d) = < 0 >$. Put $D = < d >^{<x>}$. Then we can consider D as a J-submodule of A, moreover $D \simeq_J J$. It follows that $D = \mathbf{Dr}_{n \in \mathbb{Z}} < d_n >$ and $d_n^x = d_{n+1}$, $n \in \mathbb{Z}$. In particular, $[< D, x >, < D, x >] = < d_n^{-1} d_{n+1} \mid n \in \mathbb{Z} >$. Thus the factor-group $< D, x > /[< D, x >, < D, x >]$ is a free abelian group of 0-rank 2.

Let k be a positive integer and put $J_k = \mathbb{Z} < x^k >$. Then we have $J = J_k \oplus J_k x \oplus \cdots \oplus J_k x^{k-1}$. Put $D_k = \mathbf{Dr}_{n \in \mathbb{Z}} < d_{kn} >$ and $D_{k+j} = \mathbf{Dr}_{n \in \mathbb{Z}} < d_{kn+j} >$, $1 \leq j \leq k-1$.
Every subgroup D_{k+j} is $< x^k >$-invariant and the factor-group
$< D_{k+j}, x^k > /[< D_{k+j}, x^k >, < D_{k+j}, x^k >]$ is a free abelian group of 0-rank 2, $0 \leq j \leq k-1$. It follows that the factor-group
$< D, x^k > /[< D, x^k >, < D, x^k >]$ is a free abelian group of 0-rank $k+1$. Hence the subgroups $< D, x^k >$, $k \in \mathbb{Z}$, cannot be isomorphic, and we obtain a contradiction. This contradiction proves the result.

□

The following result may be compared with Corollary 2.4.

Corollary 3.4 *Let G be a group in which the family of all non-abelian subgroups has finite isomorphic type. Let A be an abelian torsion-free subgroup of G. Then for every element $x \in \mathbf{N}_G(A)$ and for every element $a \in A$ the subgroup $< a >^{<x>}$ is minimax.*

Proof. If $x^k \in \mathbf{C}_G(A)$ for some positive integer k, then the subgroup $< a >^{<x>}$ is finitely generated, in particular, it is minimax. Therefore we will assume that x has infinite order and $< x > \cap \mathbf{C}_G(A) = < 1 >$. Let V be the divisible envelope of A. We can extend in a natural way the action of x on A to the action of x on V. Then we can consider V as a $\mathbb{Q} < x >$-module. By Lemma 3.3 $\mathbf{Ann}_{\mathbb{Z}<x>}(a) \neq < 0 >$. It follows that $U = \mathbf{Ann}_{\mathbb{Q}<x>}(a) \neq < 0 >$. Then the factor $\mathbb{Q} < x > /U$ has finite dimension over $\mathbb{Q}$. The isomorphism $a(\mathbb{Q} < x >) \simeq \mathbb{Q} < x > /U$ shows that $a(\mathbb{Q} < x >)$ has finite dimension over $\mathbb{Q}$. It follows from the inclusion $a(\mathbb{Z} < x >) \leq a(\mathbb{Q} < x >)$ that $a(\mathbb{Z} < x >)$ is of finite 0-rank. Furthermore, $a(\mathbb{Z} < x >)$ includes a free abelian subgroup B such that $\mathbf{\Pi}(A/B)$ is finite (see, for example, [10], Corollary 1.8). This fact and the finiteness of $r_0(a(\mathbb{Z} < x >))$ imply that $a(\mathbb{Z} < x >) = < a >^{<x>}$ is minimax.

□

We have proved in Lemma 2.2 that a locally nilpotent group in which the family of non-abelian subgroups has finite isomorphic type is in fact nilpotent. More is true in the torsion-free case, as the following result shows.

Lemma 3.5 *Let G be a torsion-free nilpotent group. If G is non-abelian, then G has infinitely many non-abelian pairwise non-isomorphic subgroups.*

Proof. Since G is non-abelian, $\zeta_2(G) \neq \zeta(G)$. Therefore we can choose an element $a \in \zeta_2(G) \setminus \zeta(G)$. Then there exists an element b such that $[a, b] = c \neq 1$. For each $n \in \mathbb{N}$ put $H_n = < a^n, b, c >$. Clearly $\zeta(H_n) = < c >$, $[H_n, H_n] = < c^n >$, in particular, the subgroup H_n is non-abelian. Let $n, k \in \mathbb{N}$ and $n \neq k$. If we suppose that the subgroups H_n, H_k are isomorphic, then the factors $\zeta(H_n)/[H_n, H_n]$ and $\zeta(H_k)/[H_k, H_k]$ should be isomorphic. But the first one is a cyclic group of order n and the second is a cyclic group of order k. It follows that the subgroups H_n, H_k cannot be isomorphic, so that we obtain an infinite family $\{H_n \mid n \in \mathbb{N}\}$ of subgroups, which are not pairwise isomorphic. $\square$

Let G be an abelian group of finite 0-rank. Choose in G a maximal $\mathbb{Z}$-independent subset M and put $A = < M >$. Then the factor-group G/A is periodic. Denote by $\mathbf{Sp}(G)$ the set of all primes p such that the Sylow p-subgroup of G/A is infinite. If B is another free abelian subgroup of G such that G/B is periodic, then both factors $A/(A \cap B)$ and $B/(A \cap B)$ are finite. This shows that the set $\mathbf{Sp}(G)$ is independent of a choice of the subgroup A, i.e. $\mathbf{Sp}(G)$ is an invariant of the group G. The set $\mathbf{Sp}(G)$ is called the **spectrum** of the group G. If G includes a subgroup H, then it is not hard to see that $\mathbf{Sp}(G) = \mathbf{Sp}(H) \cup \mathbf{Sp}(G/H)$.

Now let G be a group having a finite series of normal subgroups, whose factors are abelian group of finite 0-rank. We define $\mathbf{Sp}(G)$ as the union of the spectrums of the factors of this series.

Lemma 3.6 *Let G be a group, $G = A < x >$ where A is a normal abelian minimax torsion-free subgroup, and x is an element of infinite order. If $< 1 > = \mathbf{C}_G(A) \cap < x >$, then G has infinitely many non-abelian subgroups which are pairwise non-isomorphic.*

Proof. Let L be the locally nilpotent radical of G.
Then $L = A(L \cap < x >)$. If we suppose that $L \neq A$, then $< z > = L \cap < x > \neq < 1 >$. Hence $L == < A, z >$ is not abelian, otherwise $< z > \leq \mathbf{C}_G(A)$, and we obtain a contradiction. Now we can apply Lemma 3.5.

Therefore we can suppose that A is the locally nilpotent radical of the subgroup $< A, y >$ for every element $1 \neq y \in < x >$.

Let B be a finitely generated subgroup of A such that A/B is periodic. Without loss of generality we can assume that $\mathbf{\Pi}(A/B) = \mathbf{Sp}(A)$. Let π be an infinite set of primes such that $\pi \cap \mathbf{Sp}(A) = \emptyset$. Let $p \in \pi$, then B/B^p is the Sylow p-subgroup of A/B^p. Then $A/B^p = B/B^p \times C/B^p$ where C/B^p is the Sylow p'-subgroup of A/B^p. We have $(A/B^p)^p = (C/B^p)^p = C/B^p$. On the other hand, $(A/B^p)^p = A^pB^p/B^p = A^p/B^p$, so that $A^p/B^p = C/B^p$. It follows that $A^p \cap B = B^p$. Furthermore

$$(\bigcap_{p\in\pi} A^p) \cap B = \bigcap_{p\in\pi}(A^p \cap B) = \bigcap_{p\in\pi} B^p.$$

Since B is a free abelian subgroup and the set π is infinite, $\bigcap_{p\in\pi} B^p = < 1 >$. In other words $(\bigcap_{p\in\pi} A^p) \cap B = < 1 >$, so that $\bigcap_{p\in\pi} A^p$ is isomorphic to a subgroup of A/B. But A/B is periodic and A is torsion-free. It follows that $\bigcap_{p\in\pi} A^p = < 1 >$.

Let $\sigma(g) = \{p \mid p \text{ is a prime}, p \notin \mathbf{Sp}(A), g \in \mathbf{C}_G(A/A^p)\}$. Suppose that, for every non-trivial element $y \in < x >$, the set $\sigma(y)$ is infinite. Then $[a, y] \in A^p$ for each element $a \in A$ and every $p \in \sigma(y)$. Then $[a, y] \in \bigcap_{p\in\sigma(g)} A^p = < 1 >$. This shows that for every element x the set $\sigma(x)$ is finite.

Choose a prime $p_1 \in \mathbf{Sp}(A)$, $p_1 \notin \sigma(x)$, then $x \notin \mathbf{C}_G(A/A^{p_1})$. Since the factor A/A^{p_1} is finite, then $< x_1 > = C_{<x>}(A/A^{p_1})$ has finite index in $< x >$, in particular, it is not trivial. By the previous remark, A is the locally nilpotent radical in G and in the subgroup $G_1 = A < x_1 >$. The factor A/A^{p_1} is not central in G, but it is central in G_1. It follows that G and G_1 cannot be isomorphic.

Choose a prime $p_2 \in \mathbf{Sp}(A)$, $p_2 \notin \sigma(x) \cup \sigma(x_1)$, then $x, x_1 \notin \mathbf{C}_G(A/A^{p_2})$. Since the factor A/A^{p_2} is finite, then $< x_2 > = \mathbf{C}_{<x_1>}(A/A^{p_2})$ has finite index in $< x_1 >$, in particular, it is not trivial. By the previous remark, A is the locally nilpotent radical in G, in G_1 and also in $G_2 = A < x_2 >$. The factor A/A^{p_1} is not central in G, but it is central in G_1 and in G_2. The factor A/A^{p_2} is not central in G and G_1, but it is central in G_2. It follows that G, G_1 and G_2 cannot be pairwise isomorphic.

Using similar arguments, we can construct an infinite family of non-abelian subgroups $\{G_n \mid n \in \mathbb{N}\}$ which are pairwise non-isomorphic.

□

Corollary 3.7 *Let G be a group in which the family of all non-abelian subgroups has finite isomorphic type. Let A be an abelian torsion-free subgroup of G. Then the index $| < x > : \mathbf{C}_{<x>}(a)|$ is finite for every element $x \in \mathbf{N}_G(A)$ and for every element $a \in A$.*

Proof. By Corollary 3.4, the subgroup $D = < a >^{<x>}$ is minimax. If we suppose that $< x > \cap \mathbf{C}_G(D) = < 1 >$, then Lemma 3.6 shows that the subgroup $< D, x >$ has infinitely many non-abelian subgroups, which are pairwise non-isomorphic, and we obtain a contradiction. This proves the result.

□

Corollary 2.9 says that if A is an infinite periodic subgroup of a group in which family of all non-abelian subgroups has finite isomorphic type, then $\mathbf{N}_G(A) = \mathbf{C}_G(A)$. If A is torsion-free, the hypothesis $\mathbf{N}_G(A) \neq \mathbf{C}_G(A)$ also gives important information on the structure of A, as next Lemmas show.

Lemma 3.8 *Let G be a group in which the family of all non-abelian subgroups has finite isomorphic type. Let A be an abelian torsion-free subgroup of G. If $\mathbf{N}_G(A) \neq \mathbf{C}_G(A)$, then A has finite 0-rank.*

Proof. Choose an element $x \in \mathbf{N}_G(A) \setminus \mathbf{C}_G(A)$. Then there exists an element $v \in A$ such that the subgroup $V = < v, x >$ is non-abelian. By Corollary 3.4, the subgroup $< a >^{<x>}$ is minimax for each element $a \in A$. In particular, it has finite 0-rank.

Suppose that $r_0(A)$ is infinite. Put $U_0 = < v >^{<x>}$, $D_0/U_0 = \mathbf{Tor}(A/U_0)$. Then D_0 is $< x >$-invariant. Moreover $r_0(D_0) = r_0(U_0)$ is finite, therefore A/D_0 is a torsion-free abelian group having infinite 0-rank. Furthermore, $V_0 = D_0 < x > \geq V$, so that V_0 is non-abelian. Since A/D_0 is non-trivial and torsion-free, there exists an element $v_1 \in A$ such that $< v_1 > \cap D_0 = < 1 >$. Put $U_1 = < v_1 >^{<x>} D_0$, then $r_0(D_0) < r_0(U_1)$. Let $D_1/U_1 = \mathbf{Tor}(A/U_1)$, then D_1 is $< x >$-invariant and $r_0(D_1) = r_0(U_1)$ is finite. The subgroups $V_0 = U_0 < x >$ and $V_1 = U_1 < x >$ are non-abelian. We have

$$r_0(V_1) = r_0(D_1) + 1 > r_0(D_0) + 1 = r_0(V_0).$$

It follows that the subgroups V_0 and V_1 cannot be isomorphic.

Using similar arguments we construct in A a family of subgroups $\{D_n \mid n \in \mathbb{N}\}$, containing v and satisfying the following conditions:

- D_n is $< x >$-invariant;
- D_n is pure in A;
- $r_0(D_n)$ is finite and $r_0(D_n) < r_0(D_{n+1}),\ n \in \mathbb{N}$.

These conditions show that we have an infinite family of non-isomorphic non-abelian subgroups $D_n < x >$ of G, and we obtain a contradiction. This contradiction proves that $r_0(A)$ is finite.

□

Lemma 3.9 *Let G be a group in which the family of all non-abelian subgroups has finite isomorphic type. Let A be an abelian torsion-free subgroup of G. If $\mathbf{N}_G(A) \neq \mathbf{C}_G(A)$, then A is minimax.*

Proof. Since $\mathbf{N}_G(A) \neq \mathbf{C}_G(A)$, we can choose an element $x \in \mathbf{N}_G(A) \setminus \mathbf{C}_G(A)$. By Lemma 3.8 A has finite 0-rank. Let $\{a_1, \cdots, a_n\}$ be a maximal $\mathbb{Z}$-independent subset of A. By Corollary 3.4, the subgroup $A_j = < a_j >^{<x>}$ is minimax. By Lemma 3.6 $x^{k_j} \in \mathbf{C}_G(A_j)$, for some $k_j > 0$, $1 \leq j \leq n$. Put $k = k_1 \cdots k_n$, then $x^k \in \mathbf{C}_G(< a_1, \cdots, a_n >)$.

Since A is torsion-free and A is the pure envelope of $< a_1, \cdots, a_n >$, we have
$\mathbf{C}_{<x>}(< a_1, \cdots, a_n >) = \mathbf{C}_{<x>}(A)$. Using now Lemma 3.2 we obtain that A is minimax.

□

Corollary 3.10 *Let G be a group in which the family of all non-abelian subgroups has finite isomorphic type. Let A be an abelian torsion-free subgroup of G. If A is not minimax, then $\mathbf{N}_G(A) = \mathbf{C}_G(A)$.*

4. The normalizers of abelian subgroups.

In this section we show that the conclusion of Theorem A holds in some particular cases.

Lemma 4.1 *Let G be a non-abelian group in which the family of all non-abelian subgroups has finite isomorphic type. If the derived subgroup $[G, G]$ is locally finite, then G is reduced minimax.*

Proof. Since $[G, G]$ is locally finite, $\mathbf{Tor}(G) = T$ contains all elements having finite order. Furthermore, G/T is a torsion-free abelian group. Suppose that T is infinite. Then Lemma 2.1 shows that T is abelian. By Corollary 2.9, $T \leq \zeta(G)$, so that G is nilpotent. Since G is non-abelian, it contains elements a, b such that $[a, b] = c \neq 1$.
Put $K =< a, b >$, then K is non-abelian. Being finitely generated and nilpotent, K is polycyclic, and we obtain a contradiction with Lemma 2.10. This contradiction shows that T is finite.

Since G is non-abelian, G includes a non-abelian finitely generated subgroup F including T. Suppose that G/T has infinite 0-rank. Denote by F_0/T the pure envelope of F/T. Then $r_0(F) = r_0(F_0)$, in particular, $r_0(F_0)$ is finite. Then we can choose an element $v_1 \notin F_0$. Denote by F_1/T the pure envelope of $< F/T, v_1T >$. The choice of v_1 yields that $r_0(F_1) > r_0(F_0)$. Using similar arguments, we can construct an ascending series

$$F_0 \leq F_1 \leq \cdots \leq F_n \leq F_{n+1} \leq \cdots$$

of subgroups including T such that $r_0(F_n) < r_0(F_{n+1})$ for each $n \in \mathbb{N}$. But in this case the subgroups F_n and F_m cannot be isomorphic for $n, m \in \mathbb{N}$, $n \neq m$, and we obtain a contradiction. This contradiction shows that G has finite 0-rank.

Assume that G is not minimax. Choose in G a finitely generated subgroup $C \geq F$ such that $r_0(C) = r_0(G)$. Then the factor-group

G/C is periodic. Since G is not minimax and C is finitely generated, the set $\mathbf{\Pi}(G/C)$ is infinite. Then we can choose an infinite family $\{X_n \mid n \in \mathbb{N}\}$ of infinite subset of $\mathbf{\Pi}(G/C)$ such that $X_n \subseteq X_{n+1}$ and $X_{n+1} \setminus X_n$ is infinite for every $n \in \mathbb{N}$. Denote by C_n/C the X_n-Sylow subgroup of G/C, $n \in \mathbb{N}$. By such a choice, the indices $|C_1 : C|$ and $|C_{n+1} : C_n|$ are infinite, $n \in \mathbb{N}$. If we suppose now that the subgroups C_{n+1} and C_n are isomorphic, then $C_{n+1}/\mathbf{Tor}(C_{n+1})$ and $C_n/\mathbf{Tor}(C_n)$ must be isomorphic. But $\mathbf{Tor}(C_{n+1}) = \mathbf{Tor}(C_n) = T$. Therefore, C_{n+1}/T and C_n/T must be isomorphic. On the other hand, the index $|C_{n+1}/T : C_n/T|$ is infinite, which means that C_{n+1}/T and C_n/T are not isomorphic (see [2], Theorem 2) for each $n \in \mathbb{N}$. Then all subgroups C_n are not pairwise isomorphic, $n \in \mathbb{N}$. This contradiction shows that G must be minimax.

□

Now we can prove the result of Theorem A in the case G locally nilpotent.

Corollary 4.2 *Let G be a non-abelian group in which the family of all non-abelian subgroups has finite isomorphic type. If G is locally nilpotent and non-abelian, then G is minimax, $\mathbf{Tor}(G)$ is finite, and $G/\mathbf{Tor}(G)$ is abelian.*

Proof. By Lemma 2.2, G is nilpotent. Put $T = \mathbf{Tor}(G)$. Let F be an arbitrary finitely generated subgroup of G. Then $T_1 = \mathbf{Tor}(F)$ is finite. The subgroup F includes a normal torsion-free subgroup V such that F/V is finite. By Lemma 3.5, V is abelian. Then the nilpotent torsion-free group F/T_1 includes a normal abelian subgroup of finite index. We note that the pure envelope of an abelian normal subgroup in a locally nilpotent torsion-free group is abelian (see, for example, [12], §66). It follows that F/T_1 is abelian. In other words, $[F, F] \leq T$ for each finitely generated subgroup F. It follows that $[G, G] \leq T$. Now we can apply Lemma 4.1.

□

In order to prove Theorem A in the general case, we shall be interested in the locally nilpotent radical of the group G. Two preliminary Lemmas must first be proved.

Lemma 4.3 *Let G be a non-abelian group in which the family of all non-abelian subgroups has finite isomorphic type. Suppose that A is an* *abelian subgroup of G. Then for every element $x \in \mathbf{N}_G(A)$ and every element $d \in A$ the subgroup $< d >^{<x>}$ is reduced minimax.*

Proof. Put $D = < d >^{<x>}$. If $x \in \mathbf{C}_G(d)$, then $D = < d >$ and all is proved. Therefore we will suppose that $x \notin \mathbf{C}_G(d)$. If $x^m \in \mathbf{C}_G(d)$ for some positive integer m, then D is finitely generated, and again all is proved. Thus we will suppose that x has infinite order and

$$< x > \cap \mathbf{C}_G(D) = < 1 > .$$

If d has finite order, then D is finite, by Corollary 2.9. Suppose now that d has infinite order. Put $T = \mathbf{Tor}(D)$. The subgroup D includes a free abelian subgroup C such that D/C is periodic and the set $\mathbf{\Pi}(D/C)$ is finite (see, for example, [10], Corollary 1.8). It follows that $\mathbf{\Pi}(T)$ is finite. Suppose that T is infinite. Using again Corollary 2.9, we obtain that $[T, x] = < 1 >$. Since D is abelian, the mapping $a \longmapsto [a, x]$, $a \in D$, is an endomorphism of D. Assume that $< 1 > \neq T \cap [D, x]$ and let $1 \neq u_1 \in T \cap [D, x]$. Then there exists an element $d_1 \in D$ such that $u_1 = [d_1, x]$. By such a choice the subgroup $< d_1, x >$ is non-abelian, so that the subgroup $< T, d_1, x >$ also is non-abelian and its derived subgroup is periodic. Then Lemma 4.1 implies that T is finite. This contradiction shows that $< 1 > = T \cap [D, x]$. If we assume that the subgroup $< [D, x], x >$ is abelian, then x acts trivially on the factors of the series $< 1 > \leq [D, x] \leq D$. In other words the subgroup $< D, x >$ is nilpotent. Corollary 4.2 implies that in this case $< D, x >$ must be abelian. But then $x \in \mathbf{C}_G(D)$, and we come to a contradiction. This contradiction shows that $< [D, x], x > = [D, x] < x >$ is non-abelian. The equality $< 1 > = T \cap [D, x]$ shows that the periodic subgroups in the group $[D, x] < x >$ are finite and their orders are bounded. An application of Lemma 2.10 gives a contradiction. This contradiction shows that T is finite.

Then $D = T \times K$ for some torsion-free subgroup K (see, for example, [7], Theorem 27.5). If $t = |T|$ then $D^t = K^t$ is torsion-free. Since D is abelian, the mapping $a \longmapsto a^t$, $a \in D$, is a $\mathbb{Z} < x >$-endomorphism. It follows that the $\mathbb{Z} < x >$-submodule D^t is generated by the element d^t. Corollary 3.4 shows that D^t is minimax. Since C is abelian and torsion-free, $K \simeq K^t = D^t$. Thus K is minimax, and hence D is reduced minimax. □

Lemma 4.4 *Let G be a non-abelian group in which the family of all non-abelian subgroups has finite isomorphic type. Suppose that A is an abelian subgroup of G. If $\mathbf{N}_G(A) \neq \mathbf{C}_G(A)$, then A is reduced minimax.*

Proof. Since $\mathbf{N}_G(A) \neq \mathbf{C}_G(A)$, we can choose an element $x \in \mathbf{N}_G(A) \setminus \mathbf{C}_G(A)$. Put $T = \mathbf{Tor}(A)$, and suppose first that T is infinite. Then Corollary 2.9 shows that $[T, x] = < 1 >$. Since $x \notin \mathbf{C}_G(A)$, we can find an element $d \in A$ such that $[d, x] \neq 1$. By Lemma 4.3 the normal closure $< d >^{<x>}$ is reduced minimax, so that the subgroup $< d, x >$ is non-abelian, all periodic subgroups of $< d, x >$ are finite and their orders are bounded. Then Lemma 2.10 shows that $< d, x >$ must be abelian. This contradiction proves that T is finite. Then $A = T \times C$ for some torsion-free subgroup C (see, for example, [7], Theorem 27.5). If $t = |T|$, then $A^t = C^t$ is torsion-free. Obviously the subgroup A^t is $< x >$-invariant. If $x \notin \mathbf{C}_G(A^t)$, then Lemma 3.9 shows that A^t is minimax. Since C is torsion-free, the mapping $c \longmapsto c^t$, $c \in C$ is a monomorphism, so that $A^t = C^t \simeq C$ and C is minimax.

Suppose now that $x \in \mathbf{C}_G(A^t)$. The mapping $\xi : a \longmapsto [a, x]$, $a \in A$, is an endomorphism of A. We have

$$[A, x] = Im(\xi) \simeq A/Ker(\xi) = A/\mathbf{C}_A(x).$$

The inclusion $A^t \leq \mathbf{C}_A(x)$ shows that $A/\mathbf{C}_A(x)$ is periodic, so that $[A, x] \leq T$. It follows that the factor-group $< A, x > /T$ is abelian. By the choice of x, the subgroup $< A, x >$ is non-abelian and we can apply Lemma 4.1. By this Lemma, A is reduced minimax.

□

Corollary 4.5 *Let G be a non-abelian group, in which the family of all non-abelian subgroups has finite isomorphic type. Suppose that L is a locally nilpotent subgroup of G. If $\mathbf{N}_G(L) \neq \mathbf{C}_G(L)$, then L is reduced minimax. Moreover, $L/\mathbf{Tor}(L)$ is abelian.*

Proof. Suppose that L is non-abelian. Then L is reduced minimax by Corollary 4.2. Put $T = \mathbf{Tor}(L)$. There exists a positive integer t such that L^t is torsion-free. Lemma 3.5 shows that L^t is abelian. Then the nilpotent torsion-free group L/T includes the normal abelian subgroup L^tT/T such that L/L^tT is finite. It follows that L/T is abelian (see, for example, [12], §66). If L is abelian, then we can apply Lemma 4.4.
□

5. The structure of locally generalized radical groups having finitely many non-isomorphic non-abelian subgroups.

In this section we will prove Theorem A.

We start with an easy consequence of Corollary 4.5.

Lemma 5.1 *Let G be a radical non-abelian group whose family of all non-abelian subgroups has finite isomorphic type. Then the locally nilpotent radical L of G is reduced minimax.*

Proof. We have $\mathbf{C}_G(L) \leq L$ (see, for example, [21], Lemma 4). Since L is normal in G, $G = \mathbf{N}_G(L) \neq \mathbf{C}_G(L)$ and we can apply Corollary 4.5. □

Now we study the structure of G in an interesting particular case.

Lemma 5.2 *Let G be a generalized radical non-abelian group in which the family of all non-abelian subgroups has finite isomorphic type. If the center of G includes the locally nilpotent radical L of G, then G is reduced minimax and abelian-by-finite.*

Proof. If we assume that G/L includes a normal non-trivial locally nilpotent subgroup L_1/L, then L_1 is locally nilpotent, and we obtain a contradiction with the choice of L. This contradiction shows that G/L does not include a non-trivial normal locally nilpotent subgroups. Since G is a generalized radical group, then G/L must have a non-trivial locally finite radical F/L. For every finite subgroup K/L of F/L the subgroup $[K, K]$ is finite (see [18]). It follows that $[F, F]$ is locally finite. We note that the subgroup F cannot be abelian. Then Lemma 4.1 shows that F is reduced minimax. But in this case, L is also reduced minimax. Furthermore, F/L is finite. If we suppose now that $F/L \neq G/L$, then the choice of F shows that G/F includes a non-trivial normal locally nilpotent subgroup V/F. Moreover, V/F is torsion-free. Since F/L is finite, $\mathbf{C}_{V/L}(F/L)$ has finite index in V/L. In particular, $\mathbf{C}_{V/L}(F/L)$ is non-trivial. Since G/L does not include non-trivial normal locally nilpotent subgroups, we have $F/L \cap \mathbf{C}_{G/L}(F/L) = < 1 >$. But in this case $\mathbf{C}_{V/L}(F/L)$ is a non-trivial normal locally nilpotent subgroup, and we obtain a contradiction. This contradiction proves that G/L must be finite.

□

We need now a result on linear groups, which we present in the following general form. If $\mathcal{X}$ is a class of groups, then, as usual, $\mathbf{S}\mathcal{X}$ will denote the class of all subgroups of $\mathcal{X}$-groups, $\mathbf{D}\mathcal{X}$ will denote the class of all direct products of $\mathcal{X}$-groups and $\mathcal{F}$ will denote the class of finite groups.

Lemma 5.3 *Let F be a field, G a group and A be a simple FG-module such that $\mathbf{dim}_F(A)$ is finite. Let $\mathcal{X}$ be a class of groups, satisfying the following conditions: $\mathcal{X} = \mathbf{S}\mathcal{X}$, $\mathcal{X} = \mathbf{D}\mathcal{X}$. Let H be a normal subgroup of G having finite index. If $H/\mathbf{C}_H(B) \in \mathcal{X}$ for each finite dimensional simple FH-module B, then $G/\mathbf{C}_G(A) \in \mathcal{X}\mathcal{F}$.*

Proof. Since $\mathbf{dim}_F(A)$ is finite, then A includes a simple FH-submodule W. Then there are elements $g_1, \cdots, g_n \in G$ such that $A = W \oplus Wg_1 \oplus \cdots \oplus Wg_n$ (see, for example, [10], Lemma 3.5). Since H is normal in G, Wg is a simple FH-module for every element $g \in G$. By our conditions $H/\mathbf{C}_H(Wg_j) \in \mathcal{X}$, $1 \le j \le n$. The obvious equality

$$\mathbf{C}_H(A) = \mathbf{C}_H(W) \cap \ \mathbf{C}_H(Wg_1) \cap \cdots \cap \mathbf{C}_H(Wg_n)$$

together with Remak's theorem imply the embedding

$$H/\mathbf{C}_H(A) \hookrightarrow H/\mathbf{C}_H(W) \times H/\mathbf{C}_H(Wg_1) \times \cdots \times H/\mathbf{C}_H(Wg_n),$$

which implies that $H/\mathbf{C}_H(A) \in \mathcal{X}$. Since G/H is finite, the relation $H\mathbf{C}_G(A)/\mathbf{C}_G(A) \in \mathcal{X}$ implies that $G/\mathbf{C}_G(A) \in \mathcal{X}\mathcal{F}$.

□

Let A be an abelian torsion-free group, G a subgroup of $\mathbf{Aut}(A)$. We say that A is **G-rationally irreducible** or that G **acts on** A **rationally irreducibly**, if for every non-trivial G- invariant subgroup B of A the factor-group A/B is periodic.

Lemma 5.4 *Let A be an abelian torsion-free minimax group and G be a locally generalized radical subgroup of $\mathbf{Aut}(A)$. If A is G-rationally irreducible, then G is a finitely generated abelian-by-finite group.*

Proof. Let V be the divisible envelope of A. We note that $r_0(A) = r_0(V)$, thus we can consider V as a finite dimensional space over $\mathbb{Q}$. We can extend the action of G on A to the action of G on V. In other words, we can consider G as a subgroup of $\mathbf{GL}_n(\mathbb{Q})$, where $n = r_0(A) = \mathbf{dim}_{\mathbb{Q}}(V)$. The fact that G acts on A rationally irreducibly implies that G is an irreducible subgroup of $\mathbf{GL}_n(\mathbb{Q})$. Being locally generalized radical, G does not includes non-abelian free subgroups, thus G includes a normal soluble subgroup D of finite index (see [26]). Let B be an arbitrary simple $\mathbb{Q}D$-module such that $\mathbf{dim}_{\mathbb{Q}}(B)$ is finite. Using Maltsev's theorem (see [16], Theorem 1), we obtain that $D/\mathbf{C}_D(B)$ is abelian-by-finite. Then Lemma 5.3 implies that $G/\mathbf{C}_G(V)$ is abelian-by-finite. By our conditions, $\mathbf{C}_G(V) = \mathbf{C}_G(A) = < 1 >$.

Let K be a normal abelian subgroup of G, having finite index. Since $r_0(A)$ is finite, A includes a non-trivial K-invariant subgroup

E, which is K-rationally irreducible. Let $\{x_1, \cdots, x_m\}$ be a transversal to K in G. Since K is normal in G, E^{x_j} is K-rationally irreducible, $1 \leq j \leq m$. Clearly the subgroup $E^{x_1} \dots E^{x_m}$ is G-invariant. Since it is non-trivial, $r_0(E^{x_1} \dots E^{x_m}) = r_0(A)$. It is not hard to check that $\mathbf{C}_G(E^{x_1} \dots E^{x_m}) = \mathbf{C}_G(A) = < 1 >$. It follows that $< 1 > = \mathbf{C}_K(E^{x_1} \dots E^{x_m}) = \mathbf{C}_K(E^{x_1}) \cap \dots \cap \mathbf{C}_K(E^{x_m})$. Since every subgroup E^{x_j} is K-rationally irreducible, then $K/\mathbf{C}_K(E^{x_j})$ is finitely generated (see [1], Folderung 3.2), $1 \leq j \leq m$. Using now Remak's theorem, we obtain the embedding

$$K \hookrightarrow K/\mathbf{C}_K(E^{x_1}) \times \dots \times K/\mathbf{C}_K(E^{x_m}),$$

which implies that K is finitely generated. This implies that G is finitely generated and abelian-by-finite.

□

We use Lemma 5.4 to investigate how the locally nilpotent radical of G is embedded in G.

Proposition 5.5 *Let G be a generalized radical non-abelian group in which the family of all non-abelian subgroups has finite isomorphic type. If L is the locally nilpotent radical of G, then L is reduced minimax and G/L is finitely generated and abelian-by-finite.*

Proof. Let L be the locally nilpotent radical of G. If $L \leq \zeta(G)$, then by Lemma 5.2 G is abelian-by-finite and reduced minimax. Suppose now that $\mathbf{C}_G(L) \neq G$. Since L is normal in G, $\mathbf{C}_G(L) \neq \mathbf{N}_G(L) = G$. Using Corollary 4.5 we obtain that L is reduced and minimax. Then $T = \mathbf{Tor}(L)$ is finite and L/T is an abelian torsion-free minimax group. In particular, L is nilpotent. Then L has a finite series of G-invariant subgroups

$$T = A_0 \leq A_1 \leq \dots \leq A_n = L,$$

in which the factor A_j/A_{j-1} is torsion-free, central in L and G-rationally irreducible, $1 \leq j \leq n$. Let $C_0 = \mathbf{C}_G(A_0)$, $C_j = \mathbf{C}_G(A_j/A_{j-1})$, $1 \leq j \leq n$. Then $G_0 = G/C_0$ is finite, $G_j = G/C_j$ is abelian-by-finite and finitely generated by Lemma 5.4, $1 \leq j \leq n$.

Let $C = \bigcap_{0 \leq j \leq n} C_j$. The embedding $G/C \hookrightarrow \mathbf{Dr}_{0 \leq j \leq n-1} G/C_j$ implies that G/C includes a finitely generated torsion-free abelian subgroup K/C such that G/K is finite. By the choice, $L \leq C$. Suppose that $C \neq L$. Since G is a generalized radical group, C/L includes a non-trivial normal subgroup D/L, which either is locally nilpotent or locally finite. If we suppose that D/L is locally nilpotent, then inclusion $L \leq C$ implies that D is locally nilpotent. But in this case

$D \leq L$. This contradiction shows that D/L is locally finite. Without loss of generality we can suppose that D/L is a maximal G-invariant locally finite subgroup of C/L. Then D includes a G-invariant locally finite subgroup F such that D/F is nilpotent (it follows, for example, from Theorem 3.3 of the paper [6]). If F is infinite, then Lemma 2.1 shows that F is abelian. But in this case D is soluble, so that D/L includes a G-invariant non-trivial abelian subgroup, and we again obtain a contradiction. This contradiction shows that F is finite. Then $\mathbf{C}_D(F)$ has finite index in D. The factor-group $\mathbf{C}_D(F)/(F \cap \mathbf{C}_D(F))$ is nilpotent, so the inclusion $F \cap \mathbf{C}_D(F) \leq \zeta(\mathbf{C}_D(F))$ implies that $\mathbf{C}_D(F)$ is nilpotent. It follows that $\mathbf{C}_D(F) \leq L$, which implies that D/L is finite. If we suppose now that $C/L \neq D/L$, then the choice of D shows that C/D includes a non-trivial G-invariant locally nilpotent subgroup V/D. Moreover, V/D is torsion-free. Since D/L is finite, $\mathbf{C}_{V/L}(D/L)$ has finite index in V/L. In particular, it is non-trivial. Since C/L does not include non-trivial G-invariant locally nilpotent subgroups, $D/L \cap \mathbf{C}_{V/L}(D/L) = < 1 >$. But in this case $\mathbf{C}_{V/L}(D/L)$ is a non-trivial G-invariant locally nilpotent subgroup, and we obtain a contradiction. This contradiction proves that C/L must be finite. Hence G/L is finite-by-abelian-by-finite. Being finitely generated, it is abelian-by-finite.

□

Next Lemma shows that a locally generalized radical group G satisfying our hypothesis is in fact generalized radical.

Lemma 5.6 *Let G be a non-abelian locally generalized radical group in which the family of all non-abelian subgroups has finite isomorphic type. Then G is a generalized radical group.*

Proof. If G is periodic, then G is locally finite. Using Lemma 2.1 we obtain that G is finite.

Therefore we can suppose that G is not periodic. Clearly $\zeta(G)$ does not contain all the elements of infinite order. Hence there exist elements a, b such that $[a, b] \neq 1$ and a has infinite order. Let $K_0 = < a, b >$. By Proposition 5.5, every non-abelian finitely generated subgroup of G has finite 0-rank, in particular, $r_0(K_0)$ is finite. Suppose that the 0-ranks of all finitely generated non-abelian subgroups are not bounded. Then there is a non-abelian finitely generated subgroup H such that $r_0(H) > r_0(K_0)$. Put $K_1 = < K_0, H >$, then $K_0 \leq K_1$ and $r_0(K_0) < r_0(K_1)$. Using the same arguments, we can construct an ascending chain

$$K_0 \leq K_1 \leq \cdots \leq K_n \leq K_{n+1} \leq \cdots$$

such that $r_0(K_{n-1}) < r_0(K_n)$ for all $n \in \mathbb{N}$. But in this case the subgroups K_{n-1} and K_n cannot be isomorphic, and we obtain an infinite family $\{K_n \mid n \in \mathbb{N}\}$ of subgroups which are pairwise non-isomorphic. This contradiction shows that 0-ranks of all finitely generated non-abelian subgroups are bounded by some positive integer r. If B is a finitely generated abelian subgroup of G, then the subgroup $< K_0, B >$ is finitely generated and non-abelian. Then $r_0(< K_0, B >) \leq r$, thus $r_0(B) \leq r$. Hence the 0-ranks of all finitely generated subgroups are bounded. By Proposition 2 of the paper [5], G is a generalized radical group.

□

We are now in the position to prove Theorem A.

Proof. of **Theorem A**

By Lemma 5.6, G is a generalized radical group. Denote by L its locally nilpotent radical. If $L \leq \zeta(G)$, then the result follows from Lemma 5.2. Suppose now that $\mathbf{C}_G(L) \neq G$. Since L is normal in G, $\mathbf{C}_G(L) \neq \mathbf{N}_G(L) = G$. Using Corollary 4.5 we obtain that $\mathbf{Tor}(L) = T$ is finite and L/T is a torsion-free abelian minimax group. There exists a positive integer t such that $K = L^t$ is torsion-free and L/K is finite. By Proposition 5.5 G/L is abelian-by-finite and finitely generated. It follows that G/K is abelian-by-finite and finitely generated. Let V/K be a normal abelian torsion-free subgroup of G/K having finite index. Suppose that $V/\mathbf{C}_V(K)$ is infinite. Then V contains an element x of infinite order such that $< x > \cap \mathbf{C}_V(K) = < 1 >$. Choose a maximal $\mathbb{Z}$-independent subset M of K. Since $r_0(K)$ is finite, M is also finite. If $d \in M$ and $\mathbf{C}_{<x>}(< d >^{<x>}) = < 1 >$, then by Lemma 3.6 the subgroup $< d, x > = < d >^{<x>} < x >$ has infinitely many non-abelian subgroups which are pairwise non-isomorphic, and we obtain a contradiction. This contradiction shows that $\mathbf{C}_{<x>}(< d >^{<x>})$ is non-trivial for each element $d \in M$. Since M is finite, $\mathbf{C}_{<x>}(< M >^{<x>})$ is non-trivial. The choice of M and the fact that K is torsion-free imply that $\mathbf{C}_{<x>}(K) = \mathbf{C}_{<x>}(< M >^{<x>})$, in particular $\mathbf{C}_{<x>}(K)$ is non-trivial, and we obtain a contradiction. This contradiction shows that $V/\mathbf{C}_V(K)$ is finite. The subgroup $\mathbf{C}_V(K)$ is nilpotent and torsion-free. Using now Lemma 3.5 we obtain that $\mathbf{C}_V(K)$ is abelian. Thus G is abelian-by-finite.

□

6. The structure of locally generalized coradical groups having finitely many non-isomorphic non-abelian subgroups.

In this section we study generalized coradical groups with a finite number of isomorphism classes of non-abelian subgroups

We recall that a finite group G is called **semisimple** if G does not include non-trivial normal abelian subgroups.

A normal subgroup H of a finite group G is called **completely reducible** if H is a direct product of simple groups.

Every finite semisimple group G has a non-trivial maximal normal completely reducible subgroup. This subgroup is called the **completely reducible radical** of G (see, for example, [12], §61).

Let K be a finitely generated group. We denote by $\mathbf{d}(K)$ the minimal number of generators of K.

We start with the following easy remark.

Proposition 6.1 *Let G be a finite group and suppose that every non-abelian subgroup of G can be generated by d elements. Then G has normal subgroups $R \leq S \leq V \leq G$ such that R is the soluble radical of G, S/R is a direct product of at most d finite simple non-abelian groups, V/S is soluble and $|G/V| \leq d!$.*

Proof. Throughout all proof we will consider the factor-group G/R, therefore without loss of generality we may assume that $R = < 1 >$. Let S be the completely reducible radical of G, then $S = S_1 \times \cdots \times S_m$ where S_j is a non-abelian simple group, $1 \leq j \leq m$. Since S is non-abelian, $d(S) \leq d$, in particular, $m \leq d$. For every element g of the group G define the mapping $\rho_g : S \longrightarrow S$ by the following rule $\rho_g(x) = x^g$, for every $x \in S$. By a corollary to Krull-Remak-Schmidt theorem (see, for example [22], 3.3.10) we have

$$\{S_1, \cdots, S_m\} = \{S_1^g, \cdots, S_m^g\},$$

for every element g of the group G. In other words, the mapping $\rho_g : \{S_1, \cdots, S_m\} \longrightarrow \{S_1, \cdots, S_m\}$, defined by the rule $\rho_g(S_j) = S_j^g$, $1 \leq j \leq m$, is a permutation of the set $\{S_1, \cdots, S_m\}$. It is not hard to prove that the mapping $\upsilon : g \longmapsto \rho_g$, $g \in G$, is a homomorphism, so that $\mathbf{Im}(\upsilon)$ is a subgroup of $\mathbf{S}_m$ and

$$\mathbf{Ker}(\upsilon) = \{g \in G \mid S_j^g = S_j, \text{for every } j,\ 1 \leq j \leq m\} = \bigcap_{1 \leq j \leq m} N_G(S_j).$$

Put $K = \mathbf{Ker}(\upsilon)$, $K_j = S_jC_K(S_j) = SC_K(S_j)$. The section K/K_j can be embedded in $\mathbf{Aut}(S_j)/\mathbf{Inn}(S_j)$. Recall that the group of the outer automorphisms of every finite simple group is soluble (see, for example, [22]), hence K/K_j is soluble for each j. It follows that $K/C_K(S_j)$ includes the normal subgroup $K_j/C_K(S_j) \simeq S_j$ such that K/K_j is soluble for each j, $1 \leq j \leq m$. Since G does not include non-identity normal soluble subgroups, $\bigcap_{1\leq j\leq m} C_K(S_j) = C_K(S) = <1>$. By Remak's theorem we obtain an embedding $f : K \hookrightarrow \mathbf{Dr}_{1\leq j\leq m}K/C_K(S_j)$. Since

$$f(S) = \mathbf{Dr}_{1\leq j\leq m}(SC_K(S_j))/C_K(S_j) = \mathbf{Dr}_{1\leq j\leq m}(S_jC_K(S_j))/C_K(S_j) =$$
$$= \mathbf{Dr}_{1\leq j\leq m}K_j/C_K(S_j),$$

K/S is isomorphic to a subgroup of $\mathbf{Dr}_{1\leq j\leq m}K/K_j$, which is soluble. Finally, G/K is isomorphic to a subgroup of $\mathbf{S}_m$, so that $|G/K| \leq m! \leq d!$

□

Corollary 6.2 *Let G be a finite group and suppose that every non-abelian subgroup of G can be generated by d elements. Then the number of the non-abelian composition factors of G is at most $d + d!$*

The following two results are known.

Lemma 6.3 *Let H be a finite group, $G = \mathbf{Cr}_{\lambda\in\Lambda}G_\lambda$, where $G_\lambda \simeq H$ for each $\lambda \in \Lambda$. Then G is a locally finite group.*

Corollary 6.4 *Let $G = \mathbf{Cr}_{\lambda\in\Lambda}G_\lambda$ where G_λ is a finite group for each $\lambda \in \Lambda$. If there is a positive integer t such that $|G_\lambda| \leq t$, then G is a locally finite group.*

We say that a group G has finite non-abelian rank d and we write $\mathbf{r_{nonb}(G) = d}$, if $\mathbf{d}(K) \leq d$ for every non-abelian finitely generated subgroup K of G.

Suppose that the family of non-abelian subgroups of a group G has finite isomorphic type. If H and K are non-abelian finitely generated subgroups of G, then from $H \simeq K$ it follows $d(H) = d(K)$. Therefore there exists an integer d such that every finitely generated non-abelian subgroup of G can be generated by d elements. Thus the group G has finite non-abelian rank d. We shall prove some results on this class of groups.

Corollary 6.5 *Let G be a group which is not abelian-by-finite, and suppose that G has non-abelian rank at most d. Then between the* *proper normal subgroups of G of finite index there is a characteristic subgroup L whose finite factor-groups are soluble.*

Proof. If G does not include a proper normal subgroup of finite index, all is proved. Therefore suppose that the family

$$\mathcal{R} = \{H \mid H \text{ is a proper normal subgroup of finite index}\}$$

is not empty. If all finite factor-groups of G are soluble, then all is proved. Therefore we can assume that G has a normal subgroup K such that G/K is finite and non-soluble. For an arbitrary finite group F we denote by $\mathbf{sc}(F)$ the number of all non-abelian factors in some composition series of F. If $H \in \mathcal{R}$ and X/H is a non-abelian subgroup of G/H, then we can choose in G a finitely generated non-abelian subgroup Y such that $X = HY$. Being non-abelian and finitely generated, Y is d-generated. It follows that $d(X/H) \leq d$. Corollary 6.2 shows that in this case $\mathbf{sc}(G/H) \leq d + d!$. This is true for every $H \in \mathcal{R}$. Therefore there is a member M of $\mathcal{R}$ such that $\mathbf{sc}(G/M)$ is the largest one. Put

$$\mathcal{B} = \{H < G \mid H \text{ is normal in } G \text{ and } |G/H| \leq |G/M|\}.$$

Let $L = \bigcap_{H \in \mathcal{B}} H$. Then L is a characteristic subgroup of G and G/L is locally finite by Corollary 6.4. Since $K \in \mathcal{B}$, then G/L is non-abelian. As above we can show that $d(X/L) \leq d$ for every non-abelian finite subgroup X/L of G/L. It follows that G/L has finite non-abelian rank. Then G/L has finite special rank (see [4], Theorem 3). Clearly G/L is bounded. It follows that G/L must be finite. Since $M \in \mathcal{B}$, $\mathbf{sc}(G/L) = \mathbf{sc}(G/M)$. If D is a normal subgroup of L such that L/D is finite, then put $U = \mathbf{Core}_G(D)$. It is not hard to show that G/U is finite. By the choice of L, $\mathbf{sc}(G/L) = \mathbf{sc}(G/U)$. It follows that the factors of the composition series of L/U are abelian, that is L/U is soluble. In particular, L/D is soluble.

□

We recall that if G is a soluble subgroup of $\mathbf{GL}_n(F)$, where F is a field, then there exists a value $\zeta(n)$, depending only of n, such that G is soluble of solubility length at most $\zeta(n)$ (see, for example, [27], Theorem 3.7).

Lemma 6.6 *Let G be a subgroup of $\mathbf{GL}_n(F)$, where F is a field. Suppose that G has finite non-abelian rank. If every finite factor-group of G is soluble, then G is soluble of solubility length at most $\zeta(n)$.*

Proof. Clearly a free group of free rank 2 has no finite non-abelian rank. It follows that G includes a normal soluble subgroup K such that G/K is locally finite (see, for example, [27], Theorem 10.17). If G/K is abelian, then G is soluble. If G/K is non-abelian, then G/K has finite special rank (see [4], Theorem 3). It follows that G/K is (locally soluble)-by-finite (see [24]). Since every finite factor-group of G is soluble, G/K is locally soluble. Then G is locally soluble. Being linear, G is soluble of solubility length at most $\zeta(n)$.

□

Let G be a group. Recall that the derived series of G is a series

$$G = \delta_0(G) \geq \delta_1(G) \geq \cdots \geq \delta_\alpha(G) \geq \delta_{\alpha+1}(G) \geq \cdots \geq \delta_\gamma(G)$$

of characteristic subgroups, defined by the rule $\delta_1(G) = [G, G]$, $\delta_{\alpha+1}(G) = [\delta_\alpha(G), \delta_\alpha(G)]$ for all ordinals α, $\delta_\beta(G) = \bigcap_{\alpha<\beta} \delta_\alpha(G)$ if β is a limit ordinal, and $\delta_\gamma(G) = [\delta_\gamma(G), \delta_\gamma(G)]$.

Corollary 6.7 *Let G be a group having finite non-abelian rank. Suppose that G is embedded in $\mathbf{Cr}_{\lambda\in\Lambda} G_\lambda$, where G_λ is a subgroup of $\mathbf{GL}_n(F_\lambda)$ and where F_λ is a field for all $\lambda \in \Lambda$. If every finite factor-group of G is soluble, then G is soluble of solubility length at most $\zeta(n)$.*

Proof. Without loss of generality we may assume that $G_\lambda = \mathbf{pr}_\lambda(G)$ for each $\lambda \in \Lambda$. Put $K = \mathbf{Cr}_{\lambda\in\Lambda} G_\lambda$, $K_\mu = \mathbf{Cr}_{\lambda\in\Lambda,\lambda\neq\mu} G_\lambda$, $L_\mu = G \cap K_\mu$, $\mu \in \Lambda$. Then

$$G/L_\mu = G/(G \cap K_\mu) \simeq GK_\mu/K_\mu = K/K_\mu.$$

It follows that G/L_μ is isomorphic to a subgroup of $\mathbf{GL}_n(F_\mu)$. If G/L_μ is abelian, then $[G, G] \leq L_\mu$. If G/L_μ is non-abelian, then using Lemma 6.6 we obtain that $\delta_{\zeta(n)}(G) \leq L_\mu$. Since it is true for each $\mu \in \Lambda$, we get

$$\delta_{\zeta(n)}(G) \leq \bigcap_{\mu\in\Lambda} L_\mu \leq \bigcap_{\mu\in\Lambda} K_\mu = <1> .$$

□

Lemma 6.8 *Let A be a finitely generated module over a commutative ring R, $A_\lambda = a_{\lambda_1} R_\lambda + \cdots + a_{\lambda_n} R_\lambda$ for some elements $a_{\lambda_1}, \cdots, a_{\lambda_n} \in A_\lambda$, $\lambda \in \Lambda$. Let G_λ be the automorphism group of A_λ, $\lambda \in \Lambda$. Then $G = \mathbf{Cr}_{\lambda\in\Lambda} G_\lambda$ is isomorphic to a subgroup of the automorphism group of the n-generator R-module $A = \mathbf{Cr}_{\lambda\in\Lambda} A_\lambda$, where $R = \mathbf{Cr}_{\lambda\in\Lambda} R_\lambda$.*

Proof. Put $A = \mathbf{Cr}_{\lambda\in\Lambda}A_\lambda$, $\mathbf{a}_j^\nabla = (a_{\lambda j})_{\lambda\in\Lambda}$, $1 \le j \le n$. In a standard way A is an R-module. For each element $\mathbf{g} = (g_\lambda)_{\lambda\in\Lambda} \in G$, $\mathbf{a}^\nabla = (a_\lambda)_{\lambda\in\Lambda} \in A$ we define the action

$$(\mathbf{a}^\nabla)^{\mathbf{g}} = (a_\lambda^{g_\lambda})_{\lambda\in\Lambda}.$$

Then G becomes a group of R-operators for the module A. If $\mathbf{a}^\nabla = (a_\lambda)_{\lambda\in\Lambda}$ is an arbitrary element of A, then $a_\lambda = \sum_{1\le j\le n}\alpha_{\lambda j}a_{\lambda j}$ for some $\alpha_{\lambda j} \in R_\lambda$, $1 \le j \le n$. We have now

$$\mathbf{a}^\nabla = (a_\lambda)_{\lambda\in\Lambda} = (\sum_{1\le j\le n}\alpha_{\lambda j}a_{\lambda j})_{\lambda\in\Lambda} =$$

$$= \sum_{1\le j\le n}(\alpha_{\lambda j}a_{\lambda j})_{\lambda\in\Lambda} = \sum_{1\le j\le n}(\alpha_{\lambda j})_{\lambda\in\Lambda}(a_{\lambda j})_{\lambda\in\Lambda}.$$

This shows that $A = \mathbf{a}_1^\nabla R + \cdots + \mathbf{a}_n^\nabla R$. If $x \in \mathbf{C}_G(A)$, then $\mathbf{c}^\nabla\mathbf{x} = \mathbf{c}^\nabla$ for an arbitrary element $\mathbf{c}^\nabla$ of A. In particular, $a_j^\nabla\mathbf{x}^\nabla = \mathbf{a}_j^\nabla$, for all j, $1 \le j \le n$. Put $\mathbf{x} = (x_\lambda)_{\lambda\in\Lambda}$. We have

$$\mathbf{a}_j^\nabla\mathbf{x} = (a_{\lambda j}x_\lambda)_{\lambda\in\Lambda} = \mathbf{a}_j^\nabla = (a_{\lambda j})_{\lambda\in\Lambda}.$$

In other words, $a_{\lambda j}x_\lambda = a_{\lambda j}$. Since that holds for every j, $1 \le j \le n$, we obtain that $x_\lambda \in \mathbf{C}_{G_\lambda}(A_\lambda) =< 1 >$. Since this is valid for each $\lambda \in \Lambda$, $\mathbf{C}_G(A) = < 1 >$. Therefore G is isomorphic to a subgroup of the automorphism group of A.

□

Proposition 6.9 *Let G be a group having finite non-abelian rank d. Suppose that G is residually (finite and soluble). Then G is abelian-by-(locally nilpotent)-by-soluble.*

Proof. If G is abelian-by-finite, then all is proved. Thus we can suppose that G is not abelian-by-finite. Let $\{H_\lambda \mid \lambda \in \Lambda\}$ be a family of normal subgroups of G such that $\bigcap_{\lambda\in\Lambda} H_\lambda = < 1 >$ and G/H_λ is finite and soluble for every $\lambda \in \Lambda$. Then Remak's theorem shows that the mapping

$$\kappa : G \longrightarrow G^\nabla = \mathbf{Cr}_{\lambda\in\Lambda}G/H_\lambda,$$

defined by the rule $\kappa(g) = (gH_\lambda)_{\lambda\in\Lambda}$ is a (canonical) embedding of G in K. In every $\mathbf{Fitt}(G/H_\lambda)$ choose a maximal abelian G-invariant subgroup A_λ/H_λ, $\lambda \in \Lambda$. By this choice, $\mathbf{C}_{G/H_\lambda}(A_\lambda/H_\lambda) = A_\lambda/H_\lambda$, $\lambda \in \Lambda$. Since G is not abelian-by-finite, the subgroup A_λ is non-abelian for every $\lambda \in \Lambda$. It follows that $d(A_\lambda) \le d$ for all $\lambda \in \Lambda$. Then $d(A_\lambda/H_\lambda) \le d$ for all $\lambda \in \Lambda$. We can consider A_λ/H_λ as a d-generated module over $\mathbb{Z}$. Then $A^\nabla = \mathbf{Cr}_{\lambda\in\Lambda} A_\lambda/H_\lambda$ is a d-generated module over the commutative ring $R = \mathbb{Z}^\nabla$. By Lemma 6.8 the group

$$\mathbf{Cr}_{\lambda\in\Lambda}(G/H_\lambda)/(A_\lambda/H_\lambda) \simeq (\mathbf{Cr}_{\lambda\in\Lambda} G/H_\lambda)/(\mathbf{Cr}_{\lambda\in\Lambda} A_\lambda/H_\lambda) = G^\nabla/A^\nabla$$

is isomorphic to a subgroup of $\mathbf{Aut}_R(A^\nabla)$.
In particular, $K = \kappa(G)A^\nabla/A^\nabla$ is isomorphic to a subgroup of $\mathbf{Aut}_R(A^\nabla)$. Then K includes a normal locally nilpotent subgroup L such that K/L is isomorphic to a subgroup $\mathbf{Cr}_{\mu\in M} V_\mu$, where V_μ is isomorphic to a subgroup of $\mathbf{GL}_n(F_\mu)$, F_μ is a field, $\mu \in M$ and $n \leq d^2 + 1$ (see, for example, [27], Theorem 13.5). Since K is an epimorphic image of G, K has finite non-abelian rank at most d (of course, if K is non-abelian). Hence if K/L is non-abelian, it has finite non-abelian rank at most d. By Corollary 6.7, K/L is soluble. Furthermore, we have $\kappa(G)A^\nabla/A^\nabla \simeq \kappa(G)/(\kappa(G) \cap A^\nabla)$. Since A^∇ is abelian, $A^\nabla \cap \kappa(G)$ is likewise abelian, and the proof is complete.

□

Proof. of **Theorem C**

By Corollary 6.5, G includes a characteristic subgroup K of finite index such that every finite factor-group of K is soluble. By Proposition 6.9, K includes normal subgroups A, L such that $A \leq L$, A is abelian, L/A is locally nilpotent and K/L is soluble. Therefore without loss of generality we can assume that L is G-invariant. The normal closure A^G is a product of finitely many normal in K abelian subgroups, so that A^G is nilpotent (see, for example, [22], 5.2.8). Then A^G has a finite series

$$< 1 > = D_0 \leq D_1 \leq \cdots \leq D_k = A^G$$

of G-invariant subgroups, whose factors are abelian. If L/A^G is non-abelian, then it has finite special rank (see [4], Theorem 2). Then every Sylow p-subgroup of $\mathbf{Tor}(L/A^G) = T/A^G$ is Chernikov (see [17], Theorem 1). Hence T has a series

$$< 1 > = D_0 \leq D_1 \leq \cdots \leq D_k \leq D_{k+1} \leq \cdots \leq D_\alpha = T$$

of G-invariant subgroups, whose factors are abelian and $\alpha \leq 2\omega$. The factor L/T is nilpotent (see [17], Theorem 2), so we obtain a series

$$< 1 > = D_0 \leq D_1 \leq \cdots \leq D_k \leq D_{k+1} \leq ... \leq D_\alpha \leq D_{\alpha+1} \leq \cdots \leq D_{\alpha+t} = L$$

of G-invariant subgroups, whose factors are abelian. Finally K/L is soluble, therefore K is hyperabelian.

□

Corollary 6.10 *Let G be a non-abelian group in which the family of all non-abelian subgroups has finite isomorphic type. If G is residually finite, then G is abelian-by-finite and reduced minimax.*

Proof. If K, F are finitely generated subgroups of G, then the isomorphism $K \simeq F$ implies that $d(K) = d(F)$. It follows that there exists a positive integer d such that every non-abelian finitely generated subgroup of G is d-generated. Theorem C implies that G is hyperabelian-by-finite. In particular, G is a generalized radical group, and we can apply Theorem A.

□

Corollary 6.11 *Let G be a non-abelian group in which the family of all non-abelian subgroups has finite isomorphic type. If G is generalized coradical, then G is abelian-by-finite and reduced minimax.*

Proof. Let

$$G = K_0 \geq K_1 \geq \cdots \geq K_n \geq K_{n+1} \geq \cdots \geq K_\delta = <1>$$

be a series of normal subgroups of G whose factors $K_\alpha/K_{\alpha+1}$ are locally nilpotent or locally finite, $\alpha < \delta$. Since G is non-abelian, there exists an ordinal μ such that G/K_μ is non-abelian. Therefore without loss of generality we may suppose that G/K_1 is non-abelian. For each positive integer m the factor-group G/K_m is generalized radical. By Theorem A, the factor-group G/K_m is residually finite. It follows that G/K_ω is residually finite. By Corollary 6.10, G/K_ω is abelian-by-finite and reduced minimax. Let $\alpha > \omega$ and suppose that we have already proved that the factor-groups G/K_β are abelian-by-finite and reduced minimax for all ordinals $\beta < \alpha$. If α is a not limit ordinal, then $\alpha - 1$ exists, so that $G/K_{\alpha-1}$ is abelian-by-finite and reduced minimax. The factor $K_{\alpha-1}/K_\alpha$ either is locally nilpotent or locally finite. It follows that the factor-group G/K_α is generalized radical. By Theorem A, G/K_α is abelian-by-finite and reduced minimax. If α is a limit ordinal, then G/K_β is residually finite for all ordinals $\beta < \alpha$. It follows that G/K_α is also residually finite. Corollary 6.10, G/K_α is abelian-by-finite and reduced minimax.

For $\alpha = \delta$ we obtain the result. □

Proof. **of Theorem B**

Let K be an arbitrary finitely generated subgroup of G. By Corollary 6.11 K is abelian-by-finite and reduced minimax. In particular, K is generalized radical, and we can apply Theorem A.

□

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