New decay results in linear thermoelastodynamics

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Abstract

This paper is concerned with the linear theory of thermoelastodynamics for the class of homogeneous and isotropic media with a strongly semi-elliptic elasticity tensor. For this class of materials, some results about the spatial behavior of solutions for the initial-boundary value problem are obtained by using an appropriate family of surface integral measures.

Keywords: linear thermoelsticity; semi-strongly elliptic; spatial decay estimates.

1 Introduction

The spatial behavior of solution and their decay with respect to the distance from the support of the given data is an important topic from the theoretical point of view and for applications. A comprehensive review on the spatial behavior of solutions for statical and dynamical problems of continua is given by Horgan and Knowles [1] updated successively by Horgan [2, 3] (see also Edelstein [4], Knowles [5], Flavin et al. [6, 7]).

In the linear theory of elastodynamics some useful information on the spatial behavior of solutions is provided by the domain of influence theorem when one assume that the elasticity tensor is positive definite, [8]. Moreover, Gurtin reports the class of strongly elliptic materials and shows how the condition of strong ellipticity is applicable in the study of the uniqueness of solutions or in wave propagation. Ericksen and Toupin [9] prove, in the equilibrium case, that there is uniqueness of solutions if and only if the displacement equations are of the strongly elliptic type. The hypothesis of strong ellipticity, in the dynamical case, may be relaxed without loss of uniqueness. Specifically, with limitation to the first boundary-value problem of elastodynamics (surface displacements prescribed) and for bounded domains, the uniqueness of solutions has been proved by Gurtin and Sternberg [10] and Gurtin and Toupin [11] for semi-strongly elliptic elastic bodies. Toupin and Bernstein [12] have shown that an elastic body propagates plane waves wit real speeds if and only if the semi-strong-ellipticity condition is verified. For isotropic materials, the conditions of semi-strongly elliptic are $\mu \geq 0$ and $\sigma \leq 1/2$ or $1 \leq \sigma$, where μ and σ respectively denote the shear modulus and Poisson's ratio of the material. We can see that the concept of semi-strong elliptic materials is not new, has already been object of investigation and has theoretical interest.

In the context of linear thermoelasticity, under the positive definiteness condition on the elasticity tensor, Chiriță [13] studies the spatial decay of solutions, with a decay rate controlled by the factor $\exp(-r/\left(\nu(t)\sqrt{t}\right))$, where r is the distance from the support of given data and the positive function $\nu(t)$ depends also on the thermoelastic coefficients. Furthermore, in [14] Chiriță and Ciarletta obtain a

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spatial decay estimate of exponential type with a factor independent of time using the time-weighted surface power function method.

Important results have been achieved to describe the spatial behavior of the solutions for some classes of materials for which the internal energy density is not necessarily positive definite. These analyses are motivated by the existence of novel foam structures, as auxetic or anti-rubber materials (see, for example, [15, 16, 17, 18] and the references therein); their main feature is that they expand laterally when stretched, in contrast to ordinary materials. Above said structures have interesting mechanical properties, such as high energy absorption and fracture resistance.

In [19, 20], the authors study the spatial behavior for homogeneous and isotropic materials whose elasticity tensor is strongly elliptic, in linear elastodynamics and linear thermoelastodynamics, respectively. In order to obtain a description of the spatial behavior for these materials, the author investigates two classes of isotropic materials, one characterized by $\mu > 0$ and $3\lambda + 4\mu > 0$ and the other by $\mu > 0$, $2\mu > \lambda + 2\mu > 0$. For each class, they introduces a family of appropriate surface integral measures and, finally, they combines the results obtained. Following [14, 19, 20], important results have been obtained also for other types of material (see also [21, 22, 23, 24] and the references therein). In the context of the linear theory of elastodynamics for homogeneous and isotropic media, whose elasticity tensor is semi-strongly elliptic, Passarella [25] obtains results similar to those presented in [14, 20], but under different hypotheses; similar results are obtained by Tibullo [26] in the linear theory of porous elastic materials.

This work is concerned with classical linear thermoelastodynamics under the strong semi-ellipticity condition on the elastic tensor. In Section 2, we state the set of basic equations. In Section 3, a family of appropriate surface integral measures are associated with the displacement-temperature variations and a set of properties is established. In Section 4, we arrive to spatial decay estimates of exponential type for a class of materials that is wider with respect to [20], by using a single family of measures, so simplifying the whole process. Following [13], we obtain a spatial decay controlled by the factor $\exp\left(-r/\left(\nu(t)\sqrt{t}\right)\right)$ using the energy stored in the part B_r of B in the time interval $[0, t]$.

2 Basic equations

Throughout this paper we shall denote by B a bounded regular region of the physical space $\mathbb{R}^3$, with piecewise smooth boundary surface ∂B. Further, $\bar{B}$ represents the closure of B. We consider that the region B is filled with a homogeneous thermoelastic material.

Identified $\mathbb{R}^3$ with the associated vector space, an orthonormal system of reference is introduced, so that vectors and tensors will have components denoted by latin subscripts ranging over $1, 2, 3$. Summation over repeated subscripts and other typical conventions for differential operations are implied, such as a superposed dot or a comma followed by a subscript to denote partial derivative with respect to time or the corresponding Cartesian coordinate. Further, we suppress the dependence upon the spatial variable when no confusion may occur and, occasionally, we shall use bold-face characters and typical notations for vectors and operations upon them. All involved functions are supposed sufficiently regular to ensure analysis to be valid.

Denoting by $\mathbf{u}$ and θ the displacement vector and the temperature variation from the uniform reference temperature T_0, and according to the linear theory of thermoelastodynamics, the fundamental system of field equations consists of [27]:

equations of motion

$$s_{ji,j} + \rho b_i = \rho \ddot{u}_i, \qquad \text{in } B \times (0, \infty), \tag{1}$$

energy equation

$$-q_{i,i} + T_0 M_{ij} \dot{e}_{ij} + \rho h = c\dot{\theta}, \qquad \text{in } B \times (0, \infty), \tag{2}$$

strain-displacement relation

$$e_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right), \qquad \text{in } \bar{B} \times [0, \infty), \tag{3}$$

constitutive equations

$$s_{ij} = t_{ij} + M_{ij}\theta, \qquad q_i = -K_{ij}\theta_{,j}, \qquad \text{in } \bar{B} \times [0, \infty), \tag{4}$$

with

$$t_{ij} = C_{ijkl}e_{kl}. \tag{5}$$

Here, $\mathbf{s}$ and $\mathbf{b}$ are the stress tensor and the body force vector, respectively; $\mathbf{q}$ and h are the heat flux vector and the heat supply, respectively; $\mathbf{e}$ is the strain tensor. Further, ρ and c are the (strictly) positive constant representing the mass density and the specific heat, respectively; $\mathbf{C}$, $\mathbf{M}$ and $\mathbf{K}$ are the elasticity tensor, the stress-temperature tensor and the thermal conductivity tensor, respectively, satisfying the symmetry relations

$$C_{ijkl} = C_{jikl} = C_{klij}, \qquad M_{ij} = M_{ji}, \qquad K_{ij} = K_{ji}. \tag{6}$$

We assume

$$M^2 = \frac{1}{3}M_{ij}M_{ij} \neq 0, \tag{7}$$

otherwise the basic equations (1) and (2) become uncoupled and can be treated separately.

In what follows, we consider a isotropic materials so that the components of $\mathbf{C}$, $\mathbf{M}$ and $\mathbf{K}$ are given by

$$C_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right), \qquad M_{ij} = M\delta_{ij}, \qquad K_{ij} = K\delta_{ij}, \tag{8}$$

and the constitutive equations become

$$s_{ij} = t_{ij} + M\delta_{ij}\theta, \qquad q_i = -K\theta_{,i}, \tag{9}$$

with

$$t_{ij} = \lambda e\delta_{ij} + 2\mu e_{ij}, \qquad e = e_{rr} = u_{r,r}. \tag{10}$$

As it is known [27, 8], the elasticity tensor $\mathbf{C}$ is positive definite if and only if

$$\mu > 0, \quad 3\lambda + 2\mu > 0, \tag{11}$$

while $\mathbf{C}$ is strongly elliptic if and only if

$$\mu > 0, \quad \lambda + 2\mu > 0. \tag{12}$$

The materials satisfying

$$\mu \geq 0, \quad \lambda + 2\mu \geq 0 \tag{13}$$

will be called semi-strongly elliptic.

It is useful for what follows to introduce

$$\mathcal{T} = \frac{1}{2}\rho\dot{u}_i\dot{u}_i, \qquad \mathcal{S} = \frac{c}{2T_0}\,\theta^2, \qquad \mathcal{D} = \frac{K}{T_0}\,\theta_{,i}\theta_{,i}, \tag{14}$$

and

$$\mathcal{W}^* = \frac{1}{2}\left[4\mu\omega_i\omega_i + (\lambda + 2\mu)\,e^2\right] \qquad \text{with } \omega_i = \frac{1}{2}\varepsilon_{ijk}\omega_{jk},\ \omega_{ij} = \frac{1}{2}\left(u_{i,j} - u_{j,i}\right). \tag{15}$$

It is obvious that 4μ and $\lambda + 2\mu$ are the eigenvalues of the quadratic form $\mathcal{W}^*$ in the variables $\{\omega_1,\ \omega_2,\ \omega_3,\ e\}$ and $\mathcal{W}^*$ is positive semi-definite if and only if eqs. (13) are satisfied. Moreover, using the following relation

$$e_{ij}e_{ij} = 2\omega_i\omega_i + e^2 + [u_{i,j}u_j - u_iu_{j,j}]_{,i} \tag{16}$$

we can see that the internal energy density defined by

$$\mathcal{W} = \frac{1}{2}\left[2\mu e_{ij}e_{ij} + \lambda e^2\right] \tag{17}$$

is equal to

$$\mathcal{W} = \mathcal{W}^* + \mu\left[u_{i,j}u_j - u_iu_{j,j}\right]_{,i}. \tag{18}$$

Moreover, we assume that

$$0 < \rho, \qquad 0 < c, \qquad 0 < K, \qquad \kappa = \max\{4\mu, \lambda + 2\mu\} \neq 0. \tag{19}$$

The materials verifying conditions (13) together with eq. (19), are the semi-strongly elliptic materials, excluded the case in which $\lambda = \mu = 0$. Moreover, with these conditions, κ is strictly positive.

3 Time-weighted surface power function

Let $\mathcal{P}$ be the initial–boundary value problem defined by eqs. (1)-(3), (9), (10), the initial conditions

$$u_i(\mathbf{x}, 0) = u_i^0(\mathbf{x}), \quad \dot{u}_i(\mathbf{x}, 0) = v_i^0(\mathbf{x}), \quad \theta(\mathbf{x}, 0) = \theta^0(\mathbf{x}), \quad \text{on } \bar{B}, \tag{20}$$

and the (Dirichlet) boundary conditions

$$u_i(\mathbf{x}, t) = \tilde{u}_i(\mathbf{x}, t), \qquad \theta(\mathbf{x}, t) = \tilde{\theta}(\mathbf{x}, t) \qquad \text{on } \partial B \times [0, \infty), \tag{21}$$

where $\mathbf{u}^0$, $\mathbf{v}^0$, θ^0, $\tilde{\mathbf{u}}$ and $\tilde{\theta}$ are prescribed functions. Further, we denote by $\Gamma = \left\{\mathbf{b},\ h,\ \mathbf{u}^0,\ \mathbf{v}^0,\ \theta^0,\ \tilde{\mathbf{u}},\ \tilde{\theta}\right\}$ the external data of the problem in concern.

We will refer to $\pi = \{\mathbf{u}, \theta\}$ as a solution of problem $\mathcal{P}$ if it fulfills all equations (1)-(3), (9), (10), (20) and (21) for some assignment of external data Γ.

Fixed a time $T > 0$, the support of the external given data Γ on the time interval $[0, T]$ is defined as the set $\hat{D}_T$ of all $\mathbf{x} \in \bar{B}$ such that:

(1) if $\mathbf{x} \in B$, then it is: $b_i(\mathbf{x}, \tau) \neq 0$ or $h(\mathbf{x}, \tau) \neq 0$ for some $\tau \in [0, T]$, or $u_i^0(\mathbf{x}) \neq 0$ or $v_i^0(\mathbf{x}) \neq 0$ or $\theta^0(\mathbf{x}) \neq 0$;

(2) if $\mathbf{x} \in \partial B$, then it is: $\tilde{u}_i(\mathbf{x}, \tau) \neq 0$ or $\tilde{\theta}(\mathbf{x}, \tau) \neq 0$ for some $\tau \in [0, T]$.

For convenience, $\hat{D}_T$ is assumed to be bounded and nonempty

$$D_r = \left\{\mathbf{x} \in \bar{B} :\ \overline{S(\mathbf{x}, r)} \cap \hat{D}_T \neq \emptyset\right\}, \qquad B_r = B \backslash D_r, \qquad B(r_1, r_2) = B_{r_2} \backslash B_{r_1}$$

where $0 < r \leq L$, $r_2 < r_1$, $\overline{S(\mathbf{x}, r)}$ is the closure of the ball with radius r and center at $\mathbf{x}$ and

$$L = \max_{\mathbf{x} \in \bar{B}}\left(\inf_{\mathbf{y} \in \hat{D}_T} |\mathbf{y} - \mathbf{x}|\right).$$

Then, the subsurface of ∂D_r contained inside B is denoted by S_r, and its outward unit normal vector $\mathbf{n}$ is directed to the exterior of D_r. By the definitions given, the external data Γ vanish on B_r, S_r for all $t \in [0, T]$.

Our crucial point in the study of the spatial behavior is to introduce the following time-weighted surface power family of function on $]0,L]\times[0,T]$ associated with $\pi=\{\mathbf{u},\theta\}$ solution of $\mathcal{P}$

$$\Pi_\sigma(r,t)=-\int_0^t\int_{S_r}e^{-\sigma s}\left[s_{ji}(s)\dot{u}_i(s)-\frac{1}{T_0}\,q_j(s)\theta(s)\right]n_j da ds, \tag{22}$$

with $\sigma\geq 0$.

We can prove the following

Lemma 1. *The derivative of Π_σ with respect to r is given by*

$$\begin{aligned}\frac{\partial\Pi_\sigma}{\partial r}(r,t)=&-\int_{S_r}e^{-\sigma t}\left[\mathcal{T}(t)+\mathcal{W}^*(t)+\mathcal{S}(t)\right]da\\&-\int_0^t\int_{S_r}e^{-\sigma s}\left\{\sigma\left[\mathcal{T}(s)+\mathcal{W}^*(s)+\mathcal{S}(s)\right]+\mathcal{D}(s)\right\}da ds.\end{aligned} \tag{23}$$

Moreover, if the conditions (13) and (19) are true, then $\Pi_\sigma(r,t)$ is a non-increasing function with respect to r for all fixed t, i.e.

$$\frac{\partial\Pi_\sigma}{\partial r}(r,t)\leq 0\qquad\forall t\in[0,T]. \tag{24}$$

Proof. To begin, we multiply eq. (1) by $\dot{u}_i$ and eq. (2) by θ, add the resulting equations and use eqs. (9), (10), (14) and (17), to get

$$\frac{\partial}{\partial s}\left[\mathcal{T}(s)+\mathcal{W}(s)+\mathcal{S}(s)\right]+\mathcal{D}(s)=\left[s_{ji}(s)\dot{u}_i(s)-\frac{1}{T_0}q_j(s)\theta(s)\right]_{,j}+\rho b_i(s)\dot{u}_i(s)+\frac{\rho h(s)}{T_0}\,\theta(s). \tag{25}$$

Taking into account eqs.(18), we further obtain

$$\begin{aligned}\frac{\partial}{\partial s}\left[\mathcal{T}(s)+\mathcal{W}^*(s)+\mathcal{S}(s)\right]+\mathcal{D}(s)=&\left[s_{ji}(s)\dot{u}_i(s)-\frac{1}{T_0}q_j(s)\theta(s)\right]_{,j}+\rho b_i(s)\dot{u}_i(s)\\&+\frac{\rho h(s)}{T_0}\theta(s)-\mu\frac{\partial}{\partial s}\left[u_{i,j}(s)u_j(s)-u_i(s)u_{j,j}(s)\right]_{,i}.\end{aligned} \tag{26}$$

On the other hand, by taking into account eq. (22), the divergence theorem and the definitions of S_r, B_r lead to

$$\begin{aligned}\Pi_\sigma(r_1,t)-\Pi_\sigma(r_2,t)=&-\int_0^t\int_{\partial B(r_1,r_2)}e^{-\sigma s}\left[s_{ji}(s)\dot{u}_i(s)-\frac{1}{T_0}\,q_j(s)\theta(s)\right]n_j da ds=\\=&-\int_0^t\int_{B(r_1,r_2)}e^{-\sigma s}\left\{\frac{\partial}{\partial s}\left[\mathcal{T}(s)+\mathcal{W}^*(s)+\mathcal{S}(s)\right]+\mathcal{D}(s)\right\}dv ds,\end{aligned} \tag{27}$$

for $r_2<r_1$, so that, integrating by parts with respect to time,

$$\begin{aligned}\Pi_\sigma(r_1,t)-\Pi_\sigma(r_2,t)=&-\int_{B(r_1,r_2)}e^{-\sigma t}\left[\mathcal{T}(t)+\mathcal{W}^*(t)+\mathcal{S}(t)\right]dv\\&-\int_0^t\int_{B(r_1,r_2)}e^{-\sigma s}\left\{\sigma\left[\mathcal{T}(s)+\mathcal{W}^*(s)+\mathcal{S}(s)\right]+\mathcal{D}(s)\right\}dv ds.\end{aligned} \tag{28}$$

Dividing by r_1-r_2 and passing to the limit for $r_1\to r_2$, we obtain eq. (23). Moreover, under the assumptions (13) and (19), $\mathcal{T}$, $\mathcal{W}^*$, $\mathcal{S}$ and $\mathcal{D}$ are non negative, so that we arrive to eq. (24). □

Eq. (24) assures that

$$0 = \Pi_\sigma(L,t) \leq \Pi_\sigma(r,t) \qquad \forall r \leq L, \tag{29}$$

and, substituting $r_2 = r$ and $r_1 = L$ into (28), we can easily prove the following Lemma

Lemma 2. *Assumed that the conditions (13) and (19) hold, the function $\Pi_\sigma(r,t)$ is a measure associated with the solution π of the problem $\mathcal{P}$ and it is given by*

$$\begin{aligned} 0 \leq \Pi_\sigma(r,t) = &\int_{B_r} e^{-\sigma t}\left[\mathcal{T}(t) + \mathcal{W}^*(t) + \mathcal{S}(t)\right] dv \\ &+ \int_0^t \int_{B_r} e^{-\sigma s}\left\{\sigma\left[\mathcal{T}(s) + \mathcal{W}^*(s) + \mathcal{S}(s)\right] + \mathcal{D}(s)\right\} dvds. \end{aligned} \tag{30}$$

4 Spatial behavior

In this section we present two spatial decay estimates of exponential type for a wider class of materials than that considered in [13, 20], and by using a single family of measures.

To this end, we introduce

$$S_{ji} = T_{ji} + M\delta_{ij}\theta \tag{31}$$

with

$$T_{ji} = (\lambda + 2\mu)e\delta_{ij} + 2\mu\varepsilon_{ijk}\omega_k. \tag{32}$$

It is easy to prove that

$$\Pi_\sigma(r,t) = -\int_0^t \int_{S_r} e^{-\sigma s}\left[S_{ji}(s)\dot{u}_i(s) + \frac{K}{T_0}\theta_{,j}(s)\theta(s)\right] n_j dads, \tag{33}$$

taking into account the constitutive equation $(9)_2$ and the following identities

$$\begin{aligned} T_{ji,j} &= t_{ji,j}, \\ T_{ji}\dot{u}_{i,j} &= t_{ji}\dot{u}_{i,j} - \mu\frac{\partial}{\partial t}\left[u_{i,j}u_j - u_i u_{j,j}\right]_{,i}. \end{aligned} \tag{34}$$

4.1 Spatial behavior for $\sigma > 0$

We have to first establish the following lemma

Lemma 3. *Under the hypotheses of Lemma 2, for $\sigma > 0$ there exists ξ such that*

$$0 \leq \sigma\Pi_\sigma(r,t) \leq \xi \int_0^t \int_{S_r} e^{-\sigma s}\left\{\sigma\left[\mathcal{T}(s) + \mathcal{W}^*(s) + \mathcal{S}(s)\right] + \mathcal{D}(s)\right\} dads \tag{35}$$

and

$$\sigma\Pi_\sigma(r,t) + \xi\frac{\partial \Pi_\sigma}{\partial r}(r,t) \leq 0. \tag{36}$$

Proof. We denote by $\mathcal{A}$ the matrix associated to $2\mathcal{W}^*$, so that

$$\mathbf{\Psi} \cdot \mathcal{A}\mathbf{\Psi} = 2\mathcal{W}^*, \qquad \bar{\mathbf{T}} = \mathcal{A}\mathbf{\Psi}, \tag{37}$$

where

$$\mathbf{\Psi} = \{\omega_1,\ \omega_2,\ \omega_3,\ e\}, \qquad \bar{\mathbf{T}} = \left\{2T_{23},\ 2T_{31},\ 2T_{12},\ \frac{1}{3}T_{ii}\right\}. \tag{38}$$

Remembering the definition of κ, we have

$$0 \leq \mathbf{\Psi} \cdot \mathcal{A}\mathbf{\Psi} \leq \kappa\left(\omega_i\omega_i + e^2\right).$$

Use of Cauchy–Schwarz's inequality, with respect to the positive semi-definite symmetric bilinear form[1] associated to $\mathcal{A}$, and eqs. (37), we are lead to

$$\bar{\mathbf{T}}^4 = \left(\bar{\mathbf{T}} \cdot \mathcal{A}\Psi\right)^2 \leq \left(\mathbf{\Psi} \cdot \mathcal{A}\mathbf{\Psi}\right)\left(\bar{\mathbf{T}} \cdot \mathcal{A}\bar{\mathbf{T}}\right) \leq 2\mathcal{W}^*\bar{\mathbf{T}} \cdot \mathcal{A}\bar{\mathbf{T}} \leq 2\kappa\mathcal{W}^*\bar{\mathbf{T}}^2 \tag{39}$$

so that

$$\bar{\mathbf{T}}^2 \leq 2\kappa\mathcal{W}^*. \tag{40}$$

Consequently, we obtain

$$\begin{aligned} T_{ji}T_{ji} &= 8\mu^2\omega_i\omega_i + 3\left[(\lambda+2\mu)e\right]^2 \leq \\ &\leq 3\left\{16\mu^2\omega_i\omega_i + \left[(\lambda+2\mu)e\right]^2\right\} = 3\bar{\mathbf{T}}^2 \leq 6\kappa\mathcal{W}^*. \end{aligned} \tag{41}$$

Thanks to the arithmetic-geometric mean inequality for second-order tensors $\mathbf{L}$ and $\mathbf{G}$

$$(L_{ij}+G_{ij})(L_{ij}+G_{ij}) \leq (1+\epsilon)L_{ij}L_{ij} + \left(1+\frac{1}{\epsilon}\right)G_{ij}G_{ij}, \qquad \forall\epsilon > 0 \tag{42}$$

and to eqs. (31), (41) we get

$$S_{ji}S_{ji} \leq (1+\epsilon)T_{ij}T_{ij} + \left(1+\frac{1}{\epsilon}\right)3M^2\theta^2 \leq 6(1+\epsilon)\kappa\mathcal{W}^* + \left(1+\frac{1}{\epsilon}\right)3M^2\theta^2, \qquad \forall\epsilon > 0. \tag{43}$$

On the other hand, using Cauchy–Schwarz's inequality and arithmetic-geometric mean inequality, we get by (14) and (43)

$$|S_{ji}\dot{u}_i n_j| \leq \frac{\varepsilon_1\rho}{2}\dot{u}_i\dot{u}_i + \frac{1}{2\rho\varepsilon_1}S_{ji}S_{ji} \leq \varepsilon_1\mathcal{T} + \frac{6\kappa}{\rho\varepsilon_1}(1+\epsilon)\mathcal{W}^* + \left(1+\frac{1}{\epsilon}\right)\frac{3M^2T_0}{\rho c\varepsilon_1}\mathcal{S} \tag{44}$$

and

$$\left|\frac{K}{T_0}\theta_{,j}\theta n_j\right| \leq \frac{1}{2}\frac{\varepsilon_2 K^2}{cT_0}\theta_{,j}\theta_{,j} + \frac{1}{2}\frac{c}{\varepsilon_2 T_0}\theta^2 \leq \frac{1}{2}\frac{\varepsilon_2 K}{c}\mathcal{D} + \frac{1}{\varepsilon_2}\mathcal{S} \tag{45}$$

where ε_1 and ε_2 are the arbitrary positive constants. Since Π_σ is non negative, we have from eqs. (33), (44) and (45)

$$\Pi_\sigma(r,t) \leq \int_0^t\int_{S_r} e^{-\sigma s}\left\{\varepsilon_1\mathcal{T} + \frac{6\kappa}{\rho\varepsilon_1}(1+\epsilon)\mathcal{W}^* + \left[\left(1+\frac{1}{\epsilon}\right)\frac{3M^2T_0}{\rho c\varepsilon_1} + \frac{1}{\varepsilon_2}\right]\mathcal{S} + \frac{\varepsilon_2 k}{2c}\mathcal{D}\right\} dads \tag{46}$$

We choose ϵ, ε_1 and ε_2 such that

$$\varepsilon_1 = \frac{6\kappa}{\rho\varepsilon_1}(1+\epsilon) = \left(1+\frac{1}{\epsilon}\right)\frac{3M^2T_0}{\rho c\varepsilon_1} + \frac{1}{\varepsilon_2}, \qquad \frac{\varepsilon_1}{\sigma} = \frac{\varepsilon_2 k}{2c}, \tag{47}$$

and put

$$\xi \geq \varepsilon_1. \tag{48}$$

Then, thanks to eqs. (23), (46), we arrive to the thesis. □

Remark. If we choose $\xi = \varepsilon_1$, then

$$\varepsilon_1 = \sqrt{\frac{\alpha_0}{\rho}}, \qquad \epsilon = \frac{\alpha_0 - 6\kappa}{6\kappa}, \qquad \varepsilon_2 = \frac{2c}{k\sigma}\sqrt{\frac{\alpha_0}{\rho}}, \tag{49}$$

[1]This bilinear form is defined by $\mathcal{F}(\mathbf{X},\mathbf{Y}) = \mathbf{X}\cdot\mathcal{A}\mathbf{Y}$ for all vectors $\mathbf{X}$ and $\mathbf{Y}$

where α_0 represents the (positive) largest root of the following equation

$$\alpha^2 - \left(\frac{3M^2T_0}{c} + \frac{\rho K}{2c}\sigma + 6\kappa\right)\alpha + 3\frac{\sigma\rho K}{c}\kappa = 0. \tag{50}$$

In particular, we see that α_0 is given by

$$2\alpha_0 = \frac{3M^2T_0}{c} + \frac{\rho K}{2c}\sigma + 6\kappa + \sqrt{\left(\frac{3M^2T_0}{c} + \frac{\rho K}{2c}\sigma + 6\kappa\right)^2 - 12\frac{\rho K\kappa}{2}\sigma}, \tag{51}$$

and we can prove that

$$2\left(\alpha_0 - 6\kappa\right) = \frac{3M^2T_0}{c} + \frac{\rho K}{2c}\sigma - 6\kappa + \sqrt{\left(\frac{3M^2T_0}{c} + \frac{\rho K}{2c}\sigma - 6\kappa\right)^2 + 72\frac{M^2T_0K}{c}} > 0. \tag{52}$$

Eq. (52) assure that $\epsilon > 0$.

An immediate consequence of differential inequality (36) and the positivity of the function Π_σ is

$$\frac{\partial}{\partial r}\left(e^{\frac{\sigma}{\xi}r}\Pi_\sigma(r,t)\right) \leq 0 \tag{53}$$

so that we straightforwardly arrive to

Theorem 1. *Let hypotheses of Lemma 4 be valid. Then, we have for each fixed $t \in [0,T]$ and for all $0 < r < \zeta t$*

$$\Pi_\sigma(r,t) \leq e^{-\frac{\sigma}{\xi}r}\Pi_\sigma(0,t). \tag{54}$$

Following the procedure in [14] we can easily obtain a spatial decay of Phragmen-Lindelof-type for unbounded bodies.

4.2 Spatial behavior for $\sigma = 0$

We introduce the following surface power function

$$\tilde{P}(r,t) = \int_0^t \Pi_0(r,\tau)d\tau = \int_0^t \int_{B_r}\left[\mathcal{T}(\tau) + \mathcal{W}^*(\tau) + \mathcal{S}(\tau) + \int_0^\tau \mathcal{D}(s)ds\right]dvd\tau$$

Since the external data Γ vanish on B_r and eqs. (25) and (26) imply

$$\frac{d}{dt}\int_{B_r}\left[\mathcal{T}(t) + \mathcal{W}(t) + \mathcal{S}(t)\right]dv = \frac{d}{dt}\int_{B_r}\left[\mathcal{T}(t) + \mathcal{W}^*(t) + \mathcal{S}(t)\right] = -\int_{B_r}\mathcal{D}(t)dv, \qquad t \geq 0, \tag{55}$$

we can see that $\tilde{P}(r,t)$ coincides with the energy stored in the part B_r of B in the time interval $[0,t]$, i.e

$$\tilde{P}(r,t) = \int_0^t \int_{B_r}\left[\mathcal{T}(\tau) + \mathcal{W}(\tau) + \mathcal{S}(\tau) + \int_0^\tau \mathcal{D}(s)ds\right]dvd\tau. \tag{56}$$

Assuming valid the hypotheses of Lemma 1, 2 and following the procedure previously developed, it is easy to prove that $\tilde{P}$ is non–increasing, continuously differentiable with respect to r and

$$\frac{\partial}{\partial r}\tilde{P}(r,t) = -\int_0^t \int_{S_r}\left[\mathcal{T}(t) + \mathcal{W}^*(t) + \mathcal{S}(t) + \int_0^t \mathcal{D}(s)ds\right]dad\tau \leq 0.$$

Following [13], we assume that the boundary ∂B includes a plane portion S_0 lying in the plane $x_3 = 0$ and that B lies in the half-space $x_3 > 0$. The body forces and heat supplies are supposed to vanish. Further, we assume that the initial conditions are null and the boundary conditions are null with the exception of the plane end face S_0.

It is obvious that the support of the data in the interval $[0, T]$ is such that $\widehat{D}_T \subseteq S_0$, and the set S_r is the intersection with B of a plane $x_3 = r$ and $r \in]0, L]$ with L maximum value of x_3 on B. In this case, it is

$$D_r = \{\mathbf{x} \in B : x_3 \leq r\} \qquad B_r = \{\mathbf{x} \in B : x_3 > r\}$$
$$B(r_1, r_2) = \{\mathbf{x} \in B : r_2 < x_3 \leq r_1\}$$

We can obtain the same results found in [13] and, in particular, the following spatial decay estimate of Saint-Venant type

$$\tilde{P}(r, t) \leq e^{-\frac{r}{\nu(t)\sqrt{t}}}\tilde{P}(0, t) \qquad (r, t) \in]0, L] \times (0, T]$$

where $\nu(t)$ is a positive function computable in terms of thermoelastic coefficients.

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