

OBSERVABLES ON PERFECT MV-ALGEBRAS

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ABSTRACT. An observable on an MV-algebra is any σ -homomorphism from the Borel σ -algebra $\mathcal{B}(\mathbb{R})$ into the MV-algebra which maps a sequence of disjoint Borel sets onto summable elements of the MV-algebra. We establish that there is a one-to-one correspondence between observables on Rad-Dedekind σ -complete perfect MV-algebras with principal radicals and their spectral resolutions. It means that we show that our partial information on an observable known only on all intervals of the form $(-\infty, t)$ is sufficient to determine the whole information about the observable. In addition, this correspondence allows us to define the Olson order which is a partial order on the set $\mathcal{O}(M)$ of all observables on an MV-algebra M as well as, we can define a sum of observables, so that $\mathcal{O}(M)$ becomes a lattice-ordered semigroup.

1. INTRODUCTION

MV-algebras were introduced by C.C. Chang exactly 60 years ago in [Cha]. They are the algebraic counterpart of the infinite valued Łukasiewicz sentential calculus which generalize Boolean algebras describing the classical propositional logic. Every MV-algebra M contains the Boolean skeleton which is the largest Boolean subalgebra of M . In addition, in contrast to Boolean algebras, MV-algebras may contain infinitesimal non-zero elements. The intersection of all maximal elements of M is a radical, $\text{Rad}(M)$, which consists of all infinitesimal elements of M . On another pole of the realm of MV-algebras, we have perfect MV-algebras which are characterized by the property that every element of M belongs either to $\text{Rad}(M)$ or to $\text{Rad}(M)'$. Such algebras have no analogues among Boolean algebras except the two-valued ones. Perfect MV-algebras were characterized in [DiLe] showing that every perfect MV-algebra is exactly isomorphic to MV-algebras of the form $\Gamma(\mathbb{Z} \overrightarrow{\times} G, (1, 0))$, where \mathbb{Z} is the group of integers, G any Abelian ℓ -group, $\overrightarrow{\times}$ denotes the lexicographic product, and Γ is the Mundici's functor, see [Mun], and the category of perfect MV-algebras is categorically equivalent to the category of Abelian ℓ -groups.

Perfect MV-algebras have an exceptional property because the Lindenbaum algebra of the first order Łukasiewicz logic is not semisimple and the valid but unprovable formulas are precisely the formulas whose negations determine the radical of the Lindenbaum algebra, i.e. the co-infinitesimal elements of such an algebra, see [DiGr].

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Let M be an MV-algebra. The MV-subalgebra of M generated by $\text{Rad}(M)$ is the set $\text{Rad}(M) \cup \text{Rad}(M)'$ which is a perfect skeleton of M . Then the Boolean skeleton and perfect skeleton are important characteristics of MV-algebras which determines the variety of MV-algebras generated by perfect MV-algebras, or equivalently, generated by the Chang MV-algebra $\Gamma(\mathbb{Z} \overrightarrow{\times} \mathbb{Z}, (1, 0))$, see [DFG].

We note that an observable on a classical measurable space (Ω, \mathcal{S}) is any real-valued mapping f on Ω that is measurable, i.e. $f^{-1}(E) \in \mathcal{S}$ for each Borel set $E \in \mathcal{B}(\mathbb{R})$. Then the mapping $x_f : \mathcal{B}(\mathbb{R}) \rightarrow \mathcal{S}$, defined by $x_f(E) = f^{-1}(E)$, $E \in \mathcal{B}(\mathbb{R})$, is a kind of a homomorphism. According to this analogy, an observable on an MV-algebra M is also a kind of a appropriate homomorphism $x : \mathcal{B}(\mathbb{R}) \rightarrow M$. For example, $x(E)$ can be interpreted as an event that measurement takes value in the Borel set E .

This approach is generally used for examples in quantum mechanical measurement process where observables are assumed to be projector valued measures and positive operator valued measures, respectively, see [Var, DvPu]. The first ones describe yes-no measurement and the second ones “multi-valued” measurement. In these models observables are characterized by self-adjoint operators on a Hilbert space, and they are characterized by spectral resolutions. Observable on σ -complete MV-algebras were studied in [JPV].

A spectral resolution corresponding to an observable x is a system $\{x_t : t \in \mathbb{R}\}$ of elements of M , where $x_t = x((-\infty, t))$, $t \in \mathbb{R}$. There is an important question whether does a system $\{x_t : t \in \mathbb{R}\}$ of elements of M that is monotone, left-continuous, $\bigwedge_t x_t = 0$, and $\bigvee_t x_t = 1$ implies existence of an observable x such that $x_t = x((-\infty, t))$, $t \in \mathbb{R}$?

In the positive answer it means that the partial information on an observable known only on all intervals of the form $(-\infty, t)$ is sufficient to derive the whole information about the observable with values in some algebraic structure. A positive answer in the case of quantum logics was done in [Cat]. This question was studied in more details in [DvKu] for quantum structures like σ -MV-algebras, σ -lattice effect algebras, monotone σ -complete algebras with the Riesz Decomposition Property.

In the present paper, we study this question for observables and spectral resolutions with values in perfect MV-algebras M which are not σ -complete and rather their radical is Dedekind σ -complete, that is every sequence $\{a_n\}_n$ of elements of the radical that is bounded by some element $b \in M$, then $\{a_n\}_n$ has supremum $\bigvee_n a_n$ in M and it belongs to the radical. We will call such MV-algebras Rad-Dedekind σ -complete MV-algebras. In addition, we will assume that M is with principal radical, i.e. there is an element $a \in \text{Rad}(M)$ which generates $\text{Rad}(M)$ as an ideal of M .

The main goals are:

- (1) For these perfect MV-algebras, we show that there is a one-to-one correspondence between observables and spectral resolutions which is the basic result of the paper, see Theorem 4.8.
- (2) A partial information contained in the spectral resolution allows us to define a partial order, called the Olson order. First such an order was define in [Ols] for self-adjoint operators and extended also in [DvuOl] to quantum observables. Here we show that the system of observables on our perfect MV-algebras, $\mathcal{O}(M)$, is a distributive lattice with respect to the Olson order, see Theorem 6.3.
- (3) Finally, spectral resolution allows also to define a sum of our observables. Then $\mathcal{O}(M)$ is a commutative semigroup, Theorem 6.8, and a lattice-ordered semigroup with respect to the sum of observables and the Olson order, Theorem 6.9.

The paper is organized as follows. Section 2 gathers basic facts on MV-algebras. Perfect MV-algebras are presented in more details in Section 3. Observables on Rad-Dedekind σ -complete MV-algebras with principal radical are studied in Section 4, where the representation of observables by their spectral resolutions is established. Different kinds of states on perfect MV-algebras are mentioned in Section 5. The last section studies the Olson order of observables and sum of observables which are defined by their spectral resolutions.

2. BASIC FACTS

We note that an Abelian group $(G; +, 0)$ endowed with a partial order \leq is a *partially ordered group* (po-group, in abbreviation), if for all $f, g, h \in G$, $f \leq g$ yields $f + h \leq g + h$. If \leq entails a lattice structure on G , G is said to be a *lattice ordered group* (ℓ -group, in short). An element $u \geq 0$ of an ℓ -group G is a *strong unit* if given $g \in G$, there is an integer $n \geq 1$ such that $g \leq nu$. The couple (G, u) with a fixed strong unit $u \in G$ is said to be a *unital ℓ -group*. For more information on Abelian po-groups, see [Go].

An ℓ -group G is (i) *Dedekind σ -complete* (or *conditionally σ -complete*) provided that every sequence $\{g_n\}_n$ of elements of G which is bounded above in G has a supremum $\bigvee_{n=1}^{\infty} g_n$ in G , (ii) *Archimedean* if, for $f, g \in G^+$ with $ng \leq f$ for each integer $n \geq 1$, we have $g = 0$. We note that if G is Dedekind σ -complete, then it is Archimedean.

An MV-algebra is an algebra $(M; \oplus, ', 0, 1)$ (henceforth written simply as $M = (M; \oplus, ', 0, 1)$) of type $(2, 1, 0, 0)$, where $(M; \oplus, 0)$ is a commutative monoid with the neutral element 0 and for all $a, b \in M$, we have:

- (i) $a'' = a$;
- (ii) $a \oplus 1 = 1$;
- (iii) $a \oplus (a \oplus b)' = b \oplus (b \oplus a)'$.

In any MV-algebra $(M; \oplus, ', 0, 1)$, we can also define the following operations:

$$a \odot b := (a' \oplus b')', \quad a \ominus b := (a' \oplus b)'$$

Moreover, the relation $a \leq b \Leftrightarrow a' \oplus b = 1$ is a partial order on M and $(M; \leq)$ is a distributive lattice, where $a \vee b = (a \ominus b) \oplus b$ and $a \wedge b = a \odot (a' \oplus b)$. Then we write $b - a := b \ominus a$. We note that an element $a \in M$ is *Boolean* if $a \wedge a' = 0$. The set of Boolean elements of M , $B(M)$, is a Boolean algebra.

MV-algebras are in a one-to-one categorical equivalence with unital Abelian ℓ -groups. If (G, u) is an Abelian unital ℓ -group, then the interval $[0, u] := \{g \in G : 0 \leq g \leq u\}$ can be converted into an MV-algebra $\Gamma(G, u) = ([0, u]; \oplus, ', 0, u)$, where $a \oplus b = (a + b) \wedge u$ and $a' = u - a$ for all $a, b \in [0, u]$. Conversely, for every MV-algebra M there is a unique unital ℓ -group (up to isomorphism of unital ℓ -groups) (G, u) such that $M \cong \Gamma(G, u)$. In addition, there is a categorical equivalence between the category of unital ℓ -groups and the category of MV-algebras given by $(G, u) \mapsto \Gamma(G, u)$, see [Mun] or [CDM, Thm 7.12].

On every MV-algebra M , we can define a partial binary operation $+$ as follows: $a + b$ is defined in M iff $a \odot b = 0$ (equivalently, $a \leq b'$) and in such a case, $a + b := a \oplus b$. If we use the ℓ -group representation, we see that this $+$ corresponds to the group addition. We note that the partial $+$ is commutative (i.e. $a + b$ exists iff $b + a$ exists and then $a + b = b + a$) and associative (i.e. $b + c$ and $a + (b + c)$ is defined in M iff $a + b$ and $(a + b) + c$ exist in M and then $a + (b + c) = (a + b) + c$). Moreover, if $a_i \leq b_i$ for $i = 1, 2$ and if $b_1 + b_2$ is defined in M , so is $a_1 + a_2$ and $a_1 + a_2 \leq b_1 + b_2$.

Therefore, given $a_1, \dots, a_n \in M$, we can define their sum $a = a_1 + \dots + a_n$ unambiguously if it exists; in such a case, the elements a_1, \dots, a_n are said to be *summable* with the sum $a = a_1 + \dots + a_n := \sum_{i=1}^n a_i$. Clearly, if a_{i_1}, \dots, a_{i_n} is any re-ordering of the elements a_1, \dots, a_n , then $\sum_{i=1}^n a_i = \sum_{j=1}^n a_{i_j}$. A sequence $\{a_n\}_n$ of elements of an MV-algebra M is *summable* if every finite subsystem of $\{a_n\}_n$ is summable. If the element

$$a = \bigvee \left\{ \sum \{a_i : i \in F\} : F \text{ is a finite subset of } \mathbb{N} \right\}$$

exists in M , it is said to be the *sum* of $\{a_n\}_n$, and we write $a = \sum_{n=1}^{\infty} a_n$. We note that a does not depend on the ordering of $\{a_n\}_n$. For example, if $\{b_n\}_n$ is a monotone increasing sequence of elements of a σ -complete MV-algebra M , define elements $a_1 = b_1$, and $a_n = b_n \ominus b_{n-1}$ for $n \geq 2$. Then $\{a_n\}_n$ is summable and $a = \sum_{n=1}^{\infty} a_n = \bigvee_{n=1}^{\infty} b_n$.

We note that if a sequence $\{a_n\}_n$ of elements of an MV-algebra is summable, then it does not mean that the sequence has a sum in M . Indeed, let M be the Chang MV-algebra defined below. If we put $a_n = (0, n)$ for each integer $n \geq 1$, then $\{a_n\}_n$ is summable but its sum does not exist in M .

A *state* on an MV-algebra M is a mapping $s : M \rightarrow [0, 1]$ such that (i) $s(1) = 1$ and (ii) $s(a + b) = s(a) + s(b)$ whenever $a + b$ is defined in M . We note that every non-trivial MV-algebra M , i.e. $M \neq \{0\}$, has a state, see [Go, Cor 4.3]. Thus a state is a finitely additive normalized mapping on M which resembles a finitely additive measure on Boolean algebras. States on MV-algebras were introduced in [Mun1]. A σ -*additive state* is a mapping $s : M \rightarrow [0, 1]$ such that $s(1) = 1$ and $s(\sum_n a_n) = \sum_n s(a_n)$.

The state space of M is the set $\mathcal{S}(M)$ of all states on M . It is a convex set, i.e. if s_1, s_2 are states and $\lambda \in [0, 1]$, then the convex combinations of s_1 and s_2 , that is $s(a) = \lambda s_1(a) + (1 - \lambda)s_2(a)$, $\lambda \in [0, 1]$, $a \in M$, is a state. Then $\mathcal{S}(M)$ is the empty set (only if $M = \{0\}$), or a singleton or an uncountable set. A state s is *extremal* if from $s(a) = \lambda s_1(a) + (1 - \lambda)s_2(a)$ for $\lambda \in (0, 1)$, we conclude $s = s_1 = s_2$. If s is any MV-homomorphism from M into the MV-algebra of the real interval $[0, 1]$, s is said to be a *state-morphism*; it is always an extremal state and vice-versa. We denote by $\partial\mathcal{S}(M)$ and $\mathcal{SM}(M)$ the set of extremal states and state-morphisms on M , respectively. Then $\partial\mathcal{S}(M) = \mathcal{SM}(M)$. A state s is *faithful* if $s(a) = 0$ implies $a = 0$. For example, $[0, 1] = \Gamma(\mathbb{R}, 1)$ has a unique state $s(t) = t$, $t \in [0, 1]$, which is extremal and faithful as well as a state-morphism. Moreover, there is a one-to-one correspondence between maximal ideals J of M and state-morphisms s on M given by $J \leftrightarrow \text{Ker}(s) := \{a \in M : s(a) = 0\}$.

A state can be defined also on a unital ℓ -group (G, u) as a mapping $s : G \rightarrow \mathbb{R}$ such that (i) $s(g) \geq 0$ for each $g \in G^+$, (ii) $s(g + h) = s(g) + s(h)$, $g, h \in G$, and (iii) $s(u) = 1$. Thanks to Mundici's categorical equivalence, [Mun], the restriction of every state on (G, u) onto the MV-algebra $M = \Gamma(G, u)$ is a state on M and every state on M can be uniquely extended to a state on (G, u) . Extremal states are defined in the same way as on MV-algebras. Then the restriction of an extremal states on (G, u) onto M is an extremal state on M and vice-versa. Any homomorphism of unital ℓ -groups (it preserves the fixed strong units) into the unital ℓ -group $(\mathbb{R}, 1)$ is said to be a *state-morphism* on (G, u) . Also extremal states and state-morphisms on (G, u) coincide, see [Go, Thm 12.18].

The following simple lemma on generalized distributive laws will be useful for us.

Lemma 2.1. *Let $\{x_i : i \in I\}$ be a system of elements of an MV-algebra M .*

(1) *Let $\bigvee_{i \in I} x_i$ exist in M , and let x be any element of M . Then $\bigvee_{i \in I} (x \wedge x_i)$ exists in M and*

$$\bigvee_{i \in I} (x \wedge x_i) = x \wedge \bigvee_{i \in I} x_i. \quad (2.1)$$

(2) *If $\bigwedge_{i \in I} x_i$ exists in M , then for each $x \in M$, the element $\bigwedge_{i \in I} (x \vee x_i)$ exists in M and*

$$\bigwedge_{i \in I} (x \vee x_i) = x \vee \bigwedge_{i \in I} x_i. \quad (2.2)$$

Proof. (1) Let $a = \bigvee_{i \in I} x_i$. First of all, we prove $\bigvee_{i \in I} (x \odot x_i)$ exists in M and

$$x \odot \bigvee_{i \in I} x_i = \bigvee_{i \in I} (x \odot x_i). \quad (2.3)$$

We have $x_i \leq a$ for each $i \in I$, so that $x \odot x_i \leq x \odot a$. Now let for some $z \in M$, $x \odot x_i \leq z$ for each $i \in I$. By [CDM, Lem 1.1.4(iii)], $x_i \leq x' \oplus z$ for each $i \in I$. Hence, $a \leq x' \oplus z$ and $x \odot a \leq z$ which proves (2.3).

Now we prove (2.1)

$$x \wedge a = (a' \oplus x) \odot x = \bigvee_{i \in I} ((a' \oplus x) \odot x_i)$$

Clearly, $x_i \wedge x \leq a \wedge x$ for each $i \in I$. Assume $x_i \wedge x \leq v \in M$ for each $i \in I$. Then for each $i \in I$, we have

$$\begin{aligned} (x'_i \oplus x) \odot x_i &\leq v \\ x_i \leq (x'_i \oplus x)' \oplus v &= (x' \odot x_i) \oplus v \leq (x' \odot a) \oplus v \\ a &\leq (x' \odot a) \oplus v = (x \oplus a')' \oplus v \\ (x \oplus a') \odot a &\leq v \\ x \wedge a &\leq v, \end{aligned}$$

which finishes the proof of (2.1).

(2) Passing to complements, we have

$$x \vee \bigwedge_{i \in I} x_i = (x' \wedge \bigvee_{i \in I} x'_i)' = (\bigvee_{i \in I} (x' \wedge x'_i))' = \bigwedge_{i \in I} (x \vee x_i),$$

which proves the second distributive law. \square

3. PERFECT MV-ALGEBRAS

In the section we present some important properties of perfect MV-algebras and we define observables on σ -complete MV-algebras.

For any integer $n \geq 1$ and any a of an MV-algebra M , we can define

$$0.a = 0, \quad (n+1).a = (n.a) \oplus a, \quad n \geq 1,$$

$$a^0 = 1, \quad a^n = a^{n-1} \odot a, \quad n \geq 1,$$

and

$$0a = 0, \quad (n+1)a = na + a \quad \text{if } na \text{ and } na + a \text{ exist in } M, \quad n \geq 1,$$

An element $a \in M$ is said to be *infinitesimal* if na exists in M for each $n \geq 1$. We denote by $\text{Infin}(M)$ the set of infinitesimal elements of M . Equivalently, an element a is an infinitesimal iff $n.a \leq a'$ for each $n \geq 1$. The set $\text{Infin}(M)$ is a proper ideal of M and

$$\text{Infin}(M) = \text{Rad}(M) := \bigcap \{I : I \text{ is a maximal ideal of } M\},$$

see [CDM, Prop 3.6.4]. The set $\text{Rad}(M)$ is said to be a *radical* of M . We say that an MV-algebra M is with *principal radical* (and $\text{Rad}(M)$ is a principal radical), if there is an element $a \in \text{Rad}(M)$ such that the ideal of M generated by a coincides with $\text{Rad}(M)$.

An MV-algebra M is said to be *perfect* if every element $a \in M$ either belongs to $\text{Rad}(M)$ or $a' \in \text{Rad}(M)$. For example, if G is an Abelian ℓ -group and \mathbb{Z} is the group of integers, then the MV-algebra $M := \Gamma(\mathbb{Z} \overrightarrow{\times} G, (1, 0))$, where $\overrightarrow{\times}$ denotes the lexicographic product of \mathbb{Z} with G , is an example of a perfect MV-algebra; we note $\text{Rad}(M) = \{(0, g) : g \in G^+\}$ and $\text{Rad}(M)' = \{(1, -g) : g \in G^+\}$. Conversely, according to the basic result on perfect MV-algebras, [DiLe], for every perfect MV-algebra M , there is a unique (up to isomorphism of ℓ -groups) Abelian ℓ -group G such that $M \cong \Gamma(\mathbb{Z} \overrightarrow{\times} G, (1, 0))$. Moreover, there is a categorical equivalence between the category of Abelian ℓ -groups and the category of perfect MV-algebras.

From representation of perfect MV-algebras by ℓ -groups, we see that if $a, b' \in \text{Rad}(M)$, then $a \leq b$ and $a \neq b$. If we denote $\text{Rad}(M)' := \{a' : a \in \text{Rad}(M)\}$, then $\text{Rad}(M) \cap \text{Rad}(M)' = \emptyset$ and $M = \text{Rad}(M) \cup \text{Rad}(M)'$. We note that a perfect MV-algebra M is finite iff $M = \{0, 1\}$, i.e. it is a two-element Boolean algebra. Otherwise, every perfect MV-algebra is infinite. In addition, for a perfect MV-algebra M , we have $B(M) = \{0, 1\}$.

We say that an MV-algebra M is

- (i) *Archimedean* if $\text{Infin}(M) = \{0\}$,
- (ii) *σ -complete* if every sequence of elements of M has a supremum in M ,
- (iii) *Rad-Dedekind σ -complete* if, given a sequence $\{a_n\}_n$ of elements of $\text{Rad}(M)$ which is bounded by above by some element $b \in \text{Rad}(M)$, the element $\bigvee_{n=1}^{\infty} a_n$ exists in M and it belongs to $\text{Rad}(M)$, see [DvPu, p. 470].

For example, the Chang MV-algebra $M = \Gamma(\mathbb{Z} \overrightarrow{\times} \mathbb{Z}, (1, 0))$ is an example of an perfect MV-algebra which is Rad-Dedekind σ -complete, but M is not σ -complete.

Lemma 3.1. *Let M be a Rad-Dedekind σ -complete perfect MV-algebra and let $\{a_n\}_n$ be an infinite summable sequence of elements of M with sum $a = \sum_n a_n$.*

(1) *Then either all a_n are infinitesimal and $a \in \text{Rad}(M)$ or there is a unique index n_0 such that $a_n \in \text{Rad}(M)$ for each $n \neq n_0$, $a_{n_0} \in \text{Rad}(M)'$, $\sum_{n \neq n_0} a_n \in \text{Rad}(M)$ and $a \in \text{Rad}(M)'$.*

(2) *In addition, every subsequence $\{a_{n_i}\}_i$ of $\{a_n\}_n$ is summable and it has a sum in M .*

(3) *Let $\{a_n\}_n$ be an infinite sequence, $\mathbb{N} = \bigcup_{i=1}^{\infty} N_i$, where $N_i \cap N_j = \emptyset$ for $i \neq j$, and let $\alpha_i = \sum\{a_k : k \in N_i\}$ for $i = 1, 2, \dots$. Then each α_i is defined in M and $a = \sum_{i=1}^{\infty} \alpha_i$.*

Proof. (1) First we note that if $u, v \in \text{Rad}(M)'$, then $u + v$ is not defined in M . Indeed, $u + v \in M$ implies $u \leq v' \in \text{Rad}(M)$ which is a contradiction.

Now, let $a = \sum_n a_n$. There cannot be two indices n_1 and n_2 such that $a_{n_1}, a_{n_2} \in \text{Rad}(M)'$. So that there are possible only two cases. Case (i): All a_n are from $\text{Rad}(M)$. If $M = \{0, 1\}$, then all $a_n = 0$ and thus $a = 0 \in \text{Rad}(M)$. Suppose $M \neq \{0, 1\}$. We assert that then also their sum a belongs to $\text{Rad}(M)$. This is true if the sequence $\{a_n\}_n$ is finite. Now suppose $\{a_n\}_n$ is an infinite sequence and assume that $a \in \text{Rad}(M)'$. For each finite subset $F \subset \mathbb{N}$, where \mathbb{N} is the set of positive integers, the element $a_F := \sum\{a_i : i \in F\}$ exists in $\text{Rad}(F)$; we set $a_\emptyset := 0$. Clearly $a_F < a$ for each finite F . If $b \in \text{Rad}(M)'$, then $a_F \leq b$ for each finite F and whence $a \leq b$, $\text{Rad}(M)' = [a, 1]$, and $\text{Rad}(M) = [0, a']$ which is absurd because $a' < a' + a' \in \text{Rad}(M)$. Whence, $a \in \text{Rad}(M)$.

Case (ii): There is a unique index n_0 such that $a_{n_0} \in \text{Rad}(M)'$ and, for every $n \neq n_0$, $a_n \in \text{Rad}(M)$. Since the sequence $\{a_n\}_n$ is summable, for each $n \neq n_0$, $a_n \leq a'_{n_0} \in \text{Rad}(M)$. In addition, for each finite subset $F \subset \mathbb{N} \setminus \{n_0\}$, we have $a_F \leq a'_{n_0}$. The countable system $\{a_F : F \subseteq \mathbb{N} \setminus \{n_0\}, |F| < \infty\}$ is thus bounded in above by $a'_{n_0} \in \text{Rad}(M)$, applying the Rad-Dedekind σ -completeness to this countable system, we get that $a_0 = \sum_{n \neq n_0} a_n$ is defined in M and $a_0 \in \text{Rad}(M)$. In addition, $a_0 \leq a'_{n_0}$ and $a = a_0 + a_{n_0} \in \text{Rad}(M)'$.

(2) Now let $\{a_{n_i}\}_i$ be a subsequence of $\{a_n\}_n$. Since $\{a_n\}_n$ is summable, so is its subsequence $\{a_{n_i}\}_i$. If $a = \sum_n a_n \in \text{Rad}(M)$, then a is an upper bound for the countable system

$$\{a_G : G \text{ is a finite subset of } \{n_1, n_2, \dots\}\},$$

where $a_G = \sum\{a_{n_i} : n_i \in G\}$, so the Rad-Dedekind σ -completeness implies $\sum_i a_{n_i}$ exists in $\text{Rad}(M)$.

Now let $a = \sum_n a_n \in \text{Rad}(M)'$. By Case (ii), there is a unique index n_0 such that $a_{n_0} \in \text{Rad}(M)'$ and $a_{n_i} \in \text{Rad}(M)$ for each $n_i \neq n_0$. For every finite subset G of $\{n_1, n_2, \dots\} \setminus \{n_0\}$, we have $a_G + a_{n_0} \in M$, so that $a_G \leq a'_{n_0} \in \text{Rad}(M)$ and the Rad-Dedekind σ -completeness yields $\sum_i a_{n_i}$ exists in M and it belongs to $\text{Rad}(M)$.

(3) By (2), every α_i exists in M ; if $N_i = \emptyset$, then $\alpha_i = 0$. Let F be any finite subset of \mathbb{N} . Applying again (2), we see that $\alpha_F := \sum\{\alpha_i : i \in F\}$ exists in M , and $\alpha_F \leq a$. Assume $a \in \text{Rad}(M)$. Due to the Rad-Dedekind σ -completeness, $\alpha := \sum_i \alpha_i$ exists in M , it belongs to $\text{Rad}(M)$ and $\alpha \leq a$. On the other hand, if H is a finite subset of \mathbb{N} , then we have $a_H := \sum\{a_n : n \in H\} \in M$ and $a_H \leq \alpha$ so that $a \leq \alpha$ and $a = \alpha$.

Now let $a \in \text{Rad}(M)'$. By Case (ii), there is a unique index n_0 such that $a_{n_0} \in \text{Rad}(M)'$ and $a_n \in \text{Rad}(M)$ for $n \neq n_0$. Clearly, there is a unique i_0 such that $n_0 \in N_{i_0}$. Then $\alpha_i \in \text{Rad}(M)$ for $i \neq i_0$ and $\alpha_{i_0} \in \text{Rad}(M)'$. By (2), $\alpha_{i_0} = a_{n_0} + \alpha_{i_0}^0$, where $\alpha_{i_0}^0 = \sum\{a_k : k \in N_{i_0} \setminus \{i_0\}\} \in \text{Rad}(M)$. By the first part of (3), we have $a_0 = \sum_{n \neq n_0} a_n = \sum_{i \neq i_0} \alpha_i + \alpha_{i_0}^0 \in \text{Rad}(M)$. Then $a = a_0 + a_{n_0} = \sum_{i \neq i_0} \alpha_i + \alpha_{i_0}^0 + a_{n_0} = \sum_{i \neq i_0} \alpha_i + \alpha_{i_0}$. \square

On a σ -complete MV-algebra M , we define an observable as follows: An *observable* on M is any mapping $x : \mathcal{B}(\mathbb{R}) \rightarrow M$, where $\mathcal{B}(\mathbb{R})$ is the Borel σ -algebra of the real line \mathbb{R} , such that (i) $x(\mathbb{R}) = 1$, (ii) if $A, B \in \mathcal{B}(\mathbb{R})$, $A \cap B = \emptyset$, then $x(A \cup B) = x(A) + x(B)$, and (iii) if $\{A_i\}_i$ is a sequence of Borel sets such that $A_i \subseteq A_{i+1}$ for each i and $\bigcup_i A_i = A$, then $x(A) = \bigvee_i x(A_i)$.

An example of an observable on a σ -complete MV-algebra M , can be given as follows: Let $\{a_n\}_n$ be a finite or infinite sequence of summable elements of M , $\sum_n a_n = 1$, and let $\{t_n\}_n$ be a sequence of mutually different real numbers. Then the mapping $x : \mathcal{B}(\mathbb{R}) \rightarrow M$ defined by

$$x(A) := \sum \{a_n : t_n \in A\}, \quad A \in \mathcal{B}(\mathbb{R}), \quad (3.1)$$

is an observable on M . In particular, if $t_0 = 0$, $t_1 = 1$ and $a_0 = a'$, $a_1 = a$ for some fixed element $a \in M$, x defined by (3.1) is an observable, called the *question* corresponding to the element a , and we write $x = q_a$.

If $f : \mathbb{R} \rightarrow \mathbb{R}$ is a Borel measurable function and x is an observable on E , the mapping $f \circ x : A \mapsto x(f^{-1}(A))$, $A \in \mathcal{B}(\mathbb{R})$, is also an observable on M . For example if $f(t) = t^3$, we write $f \circ x = x^3$. Sometimes we write also $f(x) := f \circ x$.

If s is a σ -additive state on M and x is an observable, we define $s_x(A) := s(x(A))$, $A \in \mathcal{B}(\mathbb{R})$. Then s_x is a standard σ -additive probability measure on $\mathcal{B}(\mathbb{R})$ and we can define the expectation $Exp_s(x)$ of x in the state s as follows

$$Exp_s(x) := \int t ds_x(t)$$

if the integral exists and is finite. It is possible to show that $Exp_s(f(x)) = \int f(t) ds_x(t)$ if the integral exists and is finite.

The least closed subset C of \mathbb{R} such that $x(C) = 1$ is said to be the *spectrum* of x , and we denote it by $\sigma(x)$; since the natural topology of the real line satisfies the second countability axiom, $\sigma(x)$ exists, and $x(\sigma(x)) = 1$. An observable x is

- (i) *bounded* if $\sigma(x)$ is compact;
- (ii) *positive* if $\sigma(x) \subseteq [0, \infty)$;
- (iii) *negative* if $\sigma(x) \subseteq (-\infty, 0]$;
- (iv) *simple* if $\sigma(x)$ is a finite non-empty subset of \mathbb{R} .

The basic properties of observables are, see e.g. [DvOb]:

- (i) $x(\mathbb{R} \setminus A) = x(A)'$;
- (ii) $x(\emptyset) = 0$;
- (iii) $x(A) \leq x(B)$ whenever $A \subseteq B$, and $x(B \setminus A) = x(B) - x(A)$;
- (iv) if $\{A_i\}_i \searrow A$, then $x(A) = \bigwedge_i x(A_i)$;
- (v) if $x(A) = x(B) = 0$, then $x(A \cup B) = x(A \setminus B) + x(B \setminus A) + x(A \cap B) = 0$;
- (vi) if $x(A_n) = 0$ for each $n \geq 1$, then $x(\bigcup_n A_n) = 0$;
- (vii) $x(A) + x(B)$ exists iff does $x(A \cup B) + x(A \cap B)$, and in such a case, $x(A) + x(B) = x(A \cup B) + x(A \cap B)$;
- (viii) $x(A \cup B) - x(A) = x(B) - x(A \cap B)$;
- (ix) if $x(B) = 0$, then $x(A \cup B) = x(A)$;
- (x) if $x(B) = 1$, then $x(A) = x(A \cap B)$.

Proposition 3.2. *Let M be a σ -complete MV-algebra. A mapping $x : \mathcal{B}(\mathbb{R}) \rightarrow M$ is an observable if and only if (i) $x(\mathbb{R}) = 1$ and (ii) if $\{A_n\}_n$ is a sequence of mutually disjoint subsets from $\mathcal{B}(\mathbb{R})$, then $\{x(A_n)\}_n$ is summable and $x(A) = \sum_n x(A_n)$.*

Proof. Let x be an observable and $\{A_n\}_n$ be a sequence of mutually disjoint subsets from $\mathcal{B}(\mathbb{R})$. For each n , we denote $B_n = A_1 \cup \dots \cup A_n$. Then $x(B_n) = x(A_1) + \dots + x(A_n)$ and $x(A) = \bigvee_n x(B_n) = \sum_n x(A_n)$.

Conversely, let a mapping $x : \mathcal{B}(\mathbb{R}) \rightarrow M$ satisfy the condition of the proposition. Then $x(\emptyset) = \sum_n x(\emptyset)$ which yields $x(\emptyset) = 0$. Let A and B be disjoint Borel subsets of \mathbb{R} . Then $x(A \cup B) = x(A \cup B \cup \emptyset \cup \emptyset \dots \cup \emptyset \cup \dots) = x(A) + x(B) + x(\emptyset) + \dots + x(\emptyset) + \dots = x(A) + x(B)$.

Finally, let $\{A_n\}_n$ be a sequence of Borel sets such that $A_n \subseteq A_{n+1}$ for each integer n and $\bigcup_n A_n = A$. Define $B_1 = A_1$ and $B_n = A_n \setminus A_{n-1}$ for $n \geq 2$. Then $\{B_n\}_n$ is a sequence of mutually disjoint

Borel subsets with union A such that $A_n = B_1 \cup \dots \cup B_n$ for each $n \geq 1$. Then $x(A) = \sum_n x(B_n) = \bigvee_n x(B_1 \cup \dots \cup B_n) = \bigvee_n x(A_n)$ proving that x is an observable. \square

4. OBSERVABLES ON Rad-DEDEKIND σ -COMPLETE PERFECT MV-ALGEBRAS

Observables were defined on σ -complete MV-algebras. In the section, we show that it is possible to define observables also for Rad-Dedekind σ -complete perfect MV-algebras with principal radical. In addition, we show that there is a one-to-one correspondence between observables and their spectral resolutions. Such a correspondence was established in [DvKu, Thm 3.2] for σ -complete MV-algebras. This new result extends the class of MV-algebras where this correspondence holds.

Nevertheless perfect MV-algebras are σ -complete iff $M = \{0, 1\}$, inspired by Lemma 3.1 and the latter characterization of observables, we define the notion of an observable also for Rad-Dedekind σ -complete perfect MV-algebras as follows:

Definition 4.1. Let M be a Rad-Dedekind σ -complete perfect MV-algebra. A mapping $x : \mathcal{B}(\mathbb{R}) \rightarrow M$ is said to be an *observable* if (i) $x(\mathbb{R}) = 1$ and (ii) if $\{A_n\}_n$ is a sequence of mutually disjoint subsets from $\mathcal{B}(\mathbb{R})$, then the sequence $\{x(A_n)\}_n$ is summable and $x(A) = \sum_n x(A_n)$.

In the following proposition, we show that this definition of observables is the same as that for σ -complete MV-algebras.

Proposition 4.2. Let M be a Rad-Dedekind σ -complete MV-algebra. A mapping $x : \mathcal{B}(\mathbb{R}) \rightarrow M$ is an observable if and only if (i) $x(\mathbb{R}) = 1$, (ii) if $A, B \in \mathcal{B}(\mathbb{R})$, $A \cap B = \emptyset$, then $x(A \cup B) = x(A) + x(B)$, and (iii) if $\{A_i\}_i$ is a sequence of Borel sets such that $A_i \subseteq A_{i+1}$ for each i and $\bigcup_i A_i = A$, then $x(A) = \bigvee_i x(A_i)$.

Proof. Let x be an observable. Then (i) holds by definition. In addition, $x(\emptyset) = 0$, so that (ii) holds, too. (iii): Let $\{A_i\}_i$ be a sequence of Borel sets such that $A_i \subseteq A_{i+1}$ for each i and $\bigcup_i A_i = A$. Set $B_1 = A_1$ and $B_n = A_n \setminus A_{n-1}$ for $n \geq 2$. Then $A_n = B_1 \cup \dots \cup B_n$, $A = \bigcup_n B_n$, and $x(A_n) = x(B_1) + \dots + x(B_n)$. Since $\{x(B_n)\}_n$ is summable with sum $x(A) = \sum_n x(B_n) = \bigvee_n x(A_n)$.

Conversely, let a mapping $x : \mathcal{B}(\mathbb{R}) \rightarrow M$ satisfy (i)–(iii) of the Proposition. Then x is monotone, i.e. if $A \subseteq B$, $A, B \in \mathcal{B}(\mathbb{R})$, we have $x(A) \leq x(B)$; indeed, since we have $B = A \cup (B \setminus A)$, so that $x(B) = x(A) + x(B \setminus A)$. Let $\{A_i\}_i$ be a sequence of mutually disjoint Borel sets with $A = \bigcup_i A_i$. We show that $\{x(A_i)\}_i$ is summable with sum $x(A)$. Let F be any finite subset of \mathbb{N} . Then $a_F := \sum \{x(A_i) : i \in F\} = x(\bigcup \{A_i : i \in F\}) \leq x(A)$. Put $B_n = A_1 \cup \dots \cup A_n$. Then $x(A) = \bigvee_n x(B_n) \leq \bigvee_F a_F$ which establishes $x(A) = \sum_i x(A_i)$. \square

Basic properties of observables for Rad-Dedekind σ -complete MV-algebras are similar to ones (i)–(x) of observables on σ -complete MV-algebras:

Proposition 4.3. Let x be an observable on a Rad-Dedekind σ -complete MV-algebra M . Then (i)–(x) holds.

Proof. (i)–(iii), (v) are evident. (iv) Let $\{A_i\}_i \searrow A$. Then $\{\mathbb{R} \setminus A_i\}_i \nearrow \mathbb{R} \setminus A$. By Proposition 4.2, we have $x(\mathbb{R} \setminus A) = \bigvee_i x(\mathbb{R} \setminus A_i)$ so that by (iii), $x(A) = \bigwedge_i x(A_i)$. (vi) follows from (v) and (iv). (vii) Let $x(A) + x(B)$ exist in M . Then $x(A) + x(B) = x(A \cap B) + x(A \setminus B) + x(A \cap B) + x(B \setminus A) = x(A \cap B) + x(A \cup B)$, and conversely. (viii) It follows from $(A \cup B) \setminus A = B \setminus (A \cap B)$. (ix) and (x) follow from (viii). \square

The spectrum $\sigma(x)$ of an observable is defined in the same way as above. Since the natural topology of the real line satisfies the second countability axiom, $\sigma(x)$ exists, and $x(\sigma(x)) = 1$.

Lemma 4.4. Let M be a perfect MV-algebra. If $a_i \in \text{Rad}(M)$ for each $i \in I$ and $a = \bigvee_{i \in I} a_i$ exists in M , then $a \in \text{Rad}(M)$.

Proof. Suppose the converse, that is, $a \in \text{Rad}(M)'$. For each $b \in \text{Rad}(M)'$, we have $a_i \leq b$, $i \in I$, so that $a \leq b$. Then for each $j \in I$, $a_i \leq a'_j$ so that $a \leq a'_j$ and $a_j \leq a'$ which gives $a \leq a'$ and $a' \in \text{Rad}(M)$ which is a contradiction. \square

The following proposition is important because it characterizes observables on a Rad-Dedekind σ -complete perfect MV-algebra.

Proposition 4.5. *Let x be an observable on a Rad-Dedekind σ -complete perfect MV-algebra M . Given a real number $t \in \mathbb{R}$, we put*

$$x_t := x((-\infty, t)). \quad (4.1)$$

Then

$$x_t \leq x_s \quad \text{if } t < s, \quad (4.2)$$

$$\bigwedge_t x_t = 0, \quad \bigvee_t x_t = 1, \quad (4.3)$$

and

$$\bigvee_{t < s} x_t = x_s, \quad s \in \mathbb{R}. \quad (4.4)$$

Moreover, there is a real number t_0 such that $x_t \in \text{Rad}(M)$ for each $t \leq t_0$, $x_t \in \text{Rad}(M)'$ for each $t > t_0$, $x(\{t_0\}) \in \text{Rad}(M)'$,

$$x(\{t_0\}) = \bigwedge_{t \searrow t_0} (x_t - x_{t_0}), \quad (4.5)$$

and $x(E) \in \text{Rad}(M)$ if and only if $t_0 \notin E$, $E \in \mathcal{B}(\mathbb{R})$.

Proof. (4.2) follows from the monotonicity of x . Since the set of rational numbers is countable and dense in the natural topology of \mathbb{R} , we see that infima and suprema in (4.3) and (4.4) exist in M and they equal to the corresponding elements of M .

Clearly, given $t \in \mathbb{R}$, either x_t belongs to $\text{Rad}(M)$ or $x_t \in \text{Rad}(M)'$. By Lemma 4.4, we conclude that not all $x_t \notin \text{Rad}(M)$, otherwise $1 = \bigvee_t x_t \in \text{Rad}(M)$, a contradiction. Similarly, there is x_t belonging to $\text{Rad}(M)$, otherwise, all $x_t \in \text{Rad}(M)'$ and $0 = \bigwedge_t x_t$ and $1 = \bigvee_t x'_t \in \text{Rad}(M)$ which is absurd.

Let $T = \{t \in \mathbb{R} : x_t \in \text{Rad}(M)\}$. There is t' such that $x_{t'} \in \text{Rad}(M)'$. If we set $t_0 = \sup\{t \in T\}$, then $t_0 \leq t'$ and $\bigvee_{t < t_0} x_t = x_{t_0} \in \text{Rad}(M)$ by Lemma 4.4 and $t_0 \in T$.

Let n be the smallest integer such that $t_0 < n$. Then $[t_0, \infty) = \bigcup_{i=0}^{\infty} A_i$, where $A_0 = [t_0, n)$, and $A_i = [n + i - 1, n + i)$. Since $x([t_0, \infty)) \in \text{Rad}(M)'$, then only one $x(A_i) \in \text{Rad}(M)'$, and this is $x(A_0)$, because if $x(A_0) \in \text{Rad}(M)$, then $x_n \in \text{Rad}(M)$ which contradicts the choice of t_0 . In a similar way, $x([t_0, t_0 + 1/n)) \in \text{Rad}(M)'$ for each $n \geq 1$, and $\bigwedge_n x([t_0, t_0 + 1/n)) = x(\{t_0\}) \in M$. Applying Lemma 4.4, we have $x(\{t_0\}) \in \text{Rad}(M)'$.

Equality (4.5) follows from the following $x(\{t\}) = \bigwedge_{s \searrow t} (x_s - x_t)$ which holds for each $t \in \mathbb{R}$.

Let E be a Borel set. If $t_0 \notin E$, then $x(E) \in \text{Rad}(M)$. Indeed, if not since E and $\{t_0\}$ are disjoint and $x(E) + x(\{t_0\}) \in M$ which is impossible. If $t_0 \in E$, then clearly $x(E) \in \text{Rad}(M)'$. \square

The system of elements $\{x_t : t \in \mathbb{R}\}$ defined by (4.1) is said to be a *spectral resolution* of the observable x , and the point t_0 is said to be a *characteristic point* of the observable x and we denote it by $t_x := t_0$.

For example, let $\{a_n\}_n$ be a summable sequence of elements of a Rad-Dedekind σ -complete perfect MV-algebra M such that $a = \sum_n a_n = 1$, and let $\{t_n\}_n$ be a sequence of mutually different real numbers. Then the mapping $x : \mathcal{B}(\mathbb{R}) \rightarrow M$ defined by (3.1) is an observable on M . Indeed, applying Lemma 3.1, we see that x is a correctly defined observable.

Let x be a simple observable with $\sigma(x) = \{t_1, \dots, t_n\}$ for real numbers $t_1 < \dots < t_n$. Put $a_i = x(\{t_i\})$, $i = 1, \dots, n$, then $a_1 + \dots + a_n = 1$, and there is a unique a_{n_0} from a_1, \dots, a_n belonging to $\text{Rad}(M)'$, and for the spectral resolution of x , we have

$$x((-\infty, t)) = \begin{cases} 0 & \text{if } t \leq t_1, \\ a_1 + \dots + a_i & \text{if } t_i < t \leq t_{i+1}, \quad i = 1, \dots, n-1, \\ 1 & \text{if } t_n < t, \end{cases} \quad (4.6)$$

for $t \in \mathbb{R}$. In particular, if q_a is a question observable corresponding to an element $a \in M$, then the spectral resolution of q_a is as follows

$$q_a((-\infty, t)) = \begin{cases} 0 & \text{if } t \leq 0, \\ a' & \text{if } 0 < t \leq 1, \\ 1 & \text{if } 1 < t, \end{cases} \quad (4.7)$$

for $t \in \mathbb{R}$. We note that if x is simple, then $t_x = t_i$, where t_i is a unique point from $\sigma(x) = \{t_1, \dots, t_n\}$ such that $x(\{t_i\}) \in \text{Rad}(M)'$.

We note that $x(A \cap B) \neq x(A) \wedge x(B)$. It is possible to show that the equality $x(A \cap B) = x(A) \wedge x(B)$ holds for all $A, B \in \mathcal{B}(\mathbb{R})$ iff $x(E) \in B(M)$ for each Borel set E .

To establish the converse to Proposition 4.5, we introduce the following notions. A *g-tribe* (g stands for group) is a non-void system \mathcal{T} of bounded real-valued functions on a set $\Omega \neq \emptyset$ such that

- (i) $0_\Omega, 1_\Omega \in \mathcal{T}$;
- (ii) $f \pm g \in \mathcal{T}$ whenever $f, g \in \mathcal{T}$;
- (iii) if $\{f_n\}_n$ is a sequence of elements of \mathcal{T} for which there exists $f \in \mathcal{T}$ with $f_n(\omega) \leq f(\omega)$, $\omega \in \Omega$, $n \geq 1$, then $\sup_n f_n \in \mathcal{T}$.

Clearly a g -tribe is a Dedekind σ -complete ℓ -group of bounded functions with respect to the pointwise ordering of functions; the unit function $u = 1_\Omega$ is a strong unit for \mathcal{T} .

We note that a perfect MV-algebra with principal radical was introduced in [BeDi]. For example, if $M = \Gamma(\mathbb{Z} \overrightarrow{\times} G, (1, 0))$, then the ideal of M generated by an element $a = (0, u) \in \text{Rad}(M)$ is equal to $\text{Rad}(M)$ iff u is a strong unit for the ℓ -group G . We note that there are even Rad-Dedekind σ -complete perfect pseudo MV-algebras without principal radical. Such an example can be obtained for $M = \Gamma(\mathbb{Z} \overrightarrow{\times} G, (1, 0))$, when we use a Dedekind σ -complete ℓ -group G from [Go, Ex 9.6] which has no strong unit.

For such MV-algebras, we have the following kind of the Loomis–Sikorski theorem, see [DvPu, Thm 7.2.4].

Theorem 4.6. [Loomis–Sikorski Theorem] *Let M be a Rad-Dedekind σ -complete perfect MV-algebra with principal radical. There exist a g -tribe \mathcal{T} of bounded functions on a space $\Omega \neq \emptyset$ and an MV-homomorphism h from the Rad-Dedekind σ -complete perfect MV-algebra $M(\mathcal{T}) := \Gamma(\mathbb{Z} \overrightarrow{\times} \mathcal{T}, (1, 0_\Omega))$ with principal radical onto M such that h preserves all existing countable joins in $M(\mathcal{T})$. In addition, h maps $\text{Rad}(M(\mathcal{T}))$ onto $\text{Rad}(M)$ and $h(0, 1_\Omega) = a$, where a is a fixed element of $\text{Rad}(M)$ generating it as an ideal.*

Lemma 4.7. *Let h be an MV-homomorphism from an MV-algebra M_1 onto an MV-algebra M_2 . If there are two elements $A, B \in M_1$, $A \leq B$, and an element $c \in M_2$ such that $h(A) \leq c \leq h(B)$, then there is an element $C \in M_1$ such that $A \leq C \leq B$ and $h(C) = c$.*

Proof. Since h is onto, there is an element $C_1 \in M_1$ such that $h(C_1) = c$. If we set $C = A \vee (B \wedge C_1)$, then C is the element in question. \square

Let G be a Dedekind σ -complete ℓ -group. A mapping $x : \mathcal{B}(\mathbb{R}) \rightarrow G^+$ is said to be a *G-observable* if (i) $x(A \cup B) = x(A) + x(B)$ whenever A and B are disjoint Borel sets, and (ii) if $\{A_n\}_n$ is a sequence of

mutually disjoint Borel sets, $A = \bigcup_n A_n$, then $\{x(A_n)\}_n$ is bounded above and $x(A) = \bigvee_n (x(A_1) + \cdots + x(A_n))$. Clearly, (i) $x(\emptyset) = 0$, (ii) $x(B \setminus A) = x(B) - x(A)$ whenever $A \subseteq B$.

Now we present the following representation of observables which is a converse to Proposition 4.5.

Theorem 4.8. *Let M be a Rad-Dedekind σ -complete perfect MV-algebra with principal radical. If a system $\{x_t : t \in \mathbb{R}\}$ of elements of M satisfies (4.2)–(4.4), then there is a unique observable x on M for which (4.1) holds for any $t \in \mathbb{R}$.*

Proof. We start with the following two claims:

Claim 1: *There is a number $t_0 \in \mathbb{R}$ such that $x_t \in \text{Rad}(M)$ for $t \leq t_0$ and $x_t \in \text{Rad}(M)'$ for $t > t_0$.*

Clearly, given $t \in \mathbb{R}$, either x_t belongs to $\text{Rad}(M)$ or $x_t \in \text{Rad}(M)'$. By Lemma 4.4, we conclude that not all $x_t \notin \text{Rad}(M)$, otherwise $1 = \bigvee_t x_t \in \text{Rad}(M)$, a contradiction. Similarly, there is x_t belonging to $\text{Rad}(M)$, otherwise, all $x_t \in \text{Rad}(M)'$ and $0 = \bigwedge_t x_t$ and $1 = \bigvee_t x_t' \in \text{Rad}(M)$ which is absurd.

Let $T = \{t \in \mathbb{R} : x_t \in \text{Rad}(M)\}$. There is t' such that $x_{t'} \in \text{Rad}(M)'$. Let $t_0 = \sup\{t \in T\}$ and then $\bigvee_{t < t_0} x_t = x_{t_0} \in \text{Rad}(M)$ by Lemma 4.4.

Claim 2: *For the number t_0 from Claim 1, the element $x(t_0) := \bigwedge_{t \searrow t_0} (x_t - x_{t_0})$ exists in M and it belongs to $\text{Rad}(M)'$. Moreover, $x_t \geq x_{t_0} + x(t_0)$ for each $t > t_0$.*

We note that if $t_1, t_2 > t_0$, then $x_{t_2} - x_{t_1} \in \text{Rad}(M)$. Indeed, if not, then the elements $x_{t_1}, x_{t_2} - x_{t_1} \in \text{Rad}(M)'$ and they are summable, which is a contradiction. To prove the claim, it is enough to verify that $\bigwedge_n (x_{t_0+u_n} - x_{t_0})$ exists in M and it belongs to $\text{Rad}(M)'$ for every $\{u_n\}_n \searrow t_0$, $u_n > u_{n+1} > t_0$ for each $n \geq 1$. We have $x_{t_0+u_n} - x_{t_0} \in \text{Rad}(M)'$ for every $n \geq 1$. Then $x_{t_0+u_1} - (x_{t_0+u_n} - x_{t_0}) \leq (x_{t_0+u_n} - x_{t_0})' \in \text{Rad}(M)$ for $n \geq 1$. Therefore, $\bigvee_n (x_{t_0+u_1} - (x_{t_0+u_n} - x_{t_0})) \in \text{Rad}(M)$ and $x_{t_0+u_1} - \bigvee_n (x_{t_0+u_1} - (x_{t_0+u_n} - x_{t_0})) = \bigwedge_n (x_{t_0+u_n} - x_{t_0}) \in \text{Rad}(M)'$.

If $t > t_0$, then $x_t = x_{t_0} + (x_t - x_{t_0})$, so that $x(t_0) \leq x_t - x_{t_0} \leq x_t'$ which proves $x_t \geq x_{t_0} + x(t_0)$.

Now, let $\{x_t : t \in \mathbb{R}\}$ be a system of elements of M satisfying (4.2)–(4.4). Due to the Loomis-Sikorski Theorem for Rad-Dedekind σ -complete perfect MV-algebras with principal radical, Theorem 4.6, there are a g -tribe \mathcal{T} of bounded real-valued functions $\Omega \neq \emptyset$ and an MV-homomorphism h from $M(\mathcal{T}) = \Gamma(\mathbb{Z} \overrightarrow{\times} \mathcal{T}, (1, 0_\Omega))$ onto M which preserves all existing countable joins in $M(\mathcal{T})$. Given $t \in \mathbb{R}$, there is a function $a_t \in \mathcal{T}^+$ such that $h(0, a_t) = x_t$ if $x_t \in \text{Rad}(M)$ and $h(1, -a_t) = x_t$ if $x_t \in \text{Rad}(M)'$. Since M is perfect, due to the Di Nola and Lettieri representation, [DiLe], there is an Abelian ℓ -group G such that $M \cong \Gamma(\mathbb{Z} \overrightarrow{\times} G, (1, 0))$ and without loss of generality, we can assume that $M = \Gamma(\mathbb{Z} \overrightarrow{\times} G, (1, 0))$.

(I) Let r_1, r_2, \dots be any enumeration of the set of rational numbers in the interval $(-\infty, t_0]$.

We are claiming that it is possible to find such a sequence of functions $\{b_n\}_n$ from $\mathcal{T}^+ \subseteq \mathbb{R}^\Omega$ such that $h(0, b_n) = x_{r_n}$ for any $n \geq 1$ and $b_n \leq b_m$ whenever $r_n < r_m$. Indeed, if $n = 1$, we set $b_1 = a_1$. By mathematical induction suppose that we have found $b_1, \dots, b_n \in \mathcal{T}^+$ such that $h(0, b_i) = x_{r_i}$, and $b_i \leq b_j$ whenever $r_i < r_j$ for $i, j = 1, \dots, n$. Let j_1, \dots, j_n be a permutation of $1, \dots, n$ such that $r_{j_1} < \dots < r_{j_n}$. For r_{n+1} we have three possibilities (i) $r_{n+1} < r_{j_1}$, (ii) there exists $k = 1, \dots, n-1$ such that $r_{j_k} < r_{n+1} < r_{j_{k+1}}$, or (iii) $r_{j_n} < r_{n+1}$. Applying Lemma 4.7, we can find $b_{n+1} \in \mathcal{T}^+$, $h(0, b_{n+1}) = x_{r_{n+1}}$, such that for all $i, j = 1, \dots, n+1$, $b_i \leq b_j$ whenever $r_i < r_j$. Set $b = \bigwedge_n b_n$. Then $h(0, b) = \bigwedge_n h(0, b_n) = 0$ and $h(0, b_n - b) = x_{r_n}$ for each n . We may replace b_n by $b_n - b$, if necessary, to assume that $\bigwedge_n b_n = 0_\Omega$.

Thus, we can assume that the sequence of functions $\{b_{r_n}\}_n$, where $b_{r_n} := b_n$ for $n \geq 1$, is linearly ordered. Due to the density of rational numbers in \mathbb{R} , for any real number $t \in (-\infty, t_0]$, we can find a function $b_t \in \mathcal{T}^+$ such that $h(0, b_t) = x_t$. Indeed, if $\{p_n\}_n \nearrow t$ and $\{q_n\}_n \nearrow t$ for two sequences of rational numbers, $\{p_n\}_n$ and $\{q_n\}_n$, we can show that $\bigvee_n b_{p_n} = \bigvee_n b_{q_n}$. Hence, $b_t := \bigvee_n b_{r_n}$ is a well-defined element of \mathcal{T}^+ satisfying $h(0, b_t) = x_t$. For each $t > t_0$, we set $b_t = b_{t_0}$. Then the system of functions $\{b_t : t \in \mathbb{R}\}$ is also linearly ordered such that $b_t \leq b_s$ if $t < s$ and with (4.4) and $\bigwedge_t b_t = 0_\Omega$.

Let $\omega \in \Omega$ be a fixed element. We define $F_\omega(t) := b_t(\omega)$, $t \in \mathbb{R}$. Due to the above proved arguments, we see that F_ω is a non-decreasing, left continuous function, such that $\lim_{t \rightarrow -\infty} F_\omega(t) = 0$ and $\lim_{t \rightarrow \infty} F_\omega(t) = b_{t_0}(\omega)$. By [Hal, Thm 43.B], F_ω is a distribution function on \mathbb{R} corresponding to a unique finite σ -additive measure P_ω on $\mathcal{B}(\mathbb{R})$, that is $P_\omega((-\infty, t)) = F_\omega(t)$ for every $t \in \mathbb{R}$. Define now a mapping $\xi : \mathcal{B}(\mathbb{R}) \rightarrow \mathcal{T}^+$ by $\xi(E)(\omega) = P_\omega(E)$, $E \in \mathcal{B}(\mathbb{R})$, $\omega \in \Omega$. In particular, we have $\xi((-\infty, t)) = b_t \in \mathcal{T}^+$ for any $t \in \mathbb{R}$. To prove that every $\xi(E) \in \mathcal{T}^+$ for any $E \in \mathcal{B}(\mathbb{R})$, let \mathcal{K} be the system of all $E \in \mathcal{B}(\mathbb{R})$ such that $\xi(E) \in \mathcal{T}^+$. Then \mathcal{K} is a Dynkin system, i.e. a system of subsets containing its universe which is closed under the set theoretical complements and countable unions of disjoint subsets, [Kal]. The system \mathcal{K} contains all intervals $(-\infty, t)$ for $t \in \mathbb{R}$. These intervals form a π -system, i.e. intersection of any two sets from the π -system is from the π -system. Hence, by the Sierpiński Theorem, [Kal, Thm 1.1], \mathcal{K} is a σ -algebra, which proves $\mathcal{K} = \mathcal{B}(\mathbb{R})$.

Therefore, ξ is a \mathcal{T} -observable and $h(0, \xi(E)) \in \text{Rad}(M)$ for each Borel set $E \in \mathcal{B}(\mathbb{R})$. Then

$$y_1(E) := h(0, \xi(E)), \quad E \in \mathcal{B}(\mathbb{R}),$$

defines a $\{0\} \times G$ -observable such that $y_1(\mathbb{R}) = x_{t_0}$, $y_1(\{t_0\}) = 0$, and if $E \cap (-\infty, t_0) = \emptyset$, then $y_1(E) = 0$. Indeed, $y_1(\{t_0\}) = \bigwedge_{t \searrow t_0} (x_t - x_{t_0}) = (\bigwedge_{t \searrow t_0} x_t) - x_{t_0} = x_{t_0} - x_{t_0} = 0$.

(II) Now let s_1, s_2, \dots be any enumeration of rational numbers in (t_0, ∞) . Using Claim 2, we set

$$\hat{x}_t = \begin{cases} 0 & \text{if } t \leq t_0, \\ x_t - (x_{t_0} + x(t_0)) & \text{if } t > t_0. \end{cases}$$

Since $x(t_0) = \bigwedge_{t \searrow t_0} (x_t - x_{t_0}) \leq x'_{t_0}$, we see that $x_{t_0} + x(t_0)$ exists in M and it belongs to $\text{Rad}(M)'$. Moreover, it is possible to show that $\bigwedge_{t \searrow t_0} x_t$ exists in M and $x_{t_0} + x(t_0) = x_{t_0} + \bigwedge_{t \searrow t_0} (x_t - x_{t_0}) = \bigwedge_{t \searrow t_0} x_t \leq x_t$ for each $t > t_0$. Hence, the element \hat{x}_t is correctly defined and $\hat{x}_t \in \text{Rad}(M)$ for each $t \in \mathbb{R}$. Then also for $\{\hat{x}_t : t \in \mathbb{R}\}$ equations (4.2)–(4.3) hold and $\lim_{t \rightarrow \infty} \hat{x}_t = (x_{t_0} + x(t_0))' \in \text{Rad}(M)$. Similarly as in part (I) of the present proof, we can find for each s_n a function $c_n \in \mathcal{T}^+$ such that $h(0, c_n) = \hat{x}_{s_n}$ for each $n \geq 1$ and $s_n < s_m$ entails $c_n \leq c_m$. Similarly, using density of rational numbers in $(-\infty, t_0]$, for each $t \leq t_0$, we find $c_t \in \mathcal{T}^+$ such that $h(0, c_t) = x_t$ and if $t < s \leq t_0$ then $c_t \leq c_s$. In addition, or each $t > t_0$, we put $c_t = c_{t_0} \in \mathcal{T}^+$ so that $h(0, c_t) = \hat{x}_t$.

Let $\omega \in \Omega$ be a fixed element. We define $G_\omega(t) := c_t(\omega)$, $t \in \mathbb{R}$. Due to the above proved arguments, we see that G_ω is a non-decreasing, left continuous function such that $\lim_{t \rightarrow -\infty} G_\omega(t) = 0$ and $\lim_{t \rightarrow \infty} G_\omega(t) = b_{t_0}(\omega)$. By [Hal, Thm 43.B], G_ω is a distribution function on \mathbb{R} corresponding to a unique finite σ -additive measure M_ω on $\mathcal{B}(\mathbb{R})$, that is $M_\omega((-\infty, t)) = G_\omega(t)$ for every $t \in \mathbb{R}$. Define now a mapping $\hat{\xi} : \mathcal{B}(\mathbb{R}) \rightarrow \mathcal{T}^+$ by $\hat{\xi}(E)(\omega) = M_\omega(E)$, $E \in \mathcal{B}(\mathbb{R})$, $\omega \in \Omega$. In particular, we have $\hat{\xi}((-\infty, t)) = c_t \in \mathcal{T}^+$ for any $t \in \mathbb{R}$. To prove that every $\hat{\xi}(E) \in \mathcal{T}^+$ for any $E \in \mathcal{B}(\mathbb{R})$, let \mathcal{L} be the system of all $E \in \mathcal{B}(\mathbb{R})$ such that $\hat{\xi}(E) \in \mathcal{T}^+$. Then \mathcal{L} is a Dynkin system. The system \mathcal{L} contains all intervals $(-\infty, t)$ for $t \in \mathbb{R}$. These intervals form a π -system, i.e. intersection of any two sets from the π -system is from the π -system. Hence, by the Sierpiński Theorem, [Kal, Thm 1.1], \mathcal{L} is a σ -algebra, which proves $\mathcal{L} = \mathcal{B}(\mathbb{R})$.

Therefore, $\hat{\xi}$ is a \mathcal{T} -observable and $h(0, \hat{\xi}(E)) \in \text{Rad}(M)$ for each Borel set E . Then

$$y_2(E) := h(0, \hat{\xi}(E)), \quad E \in \mathcal{B}(\mathbb{R}),$$

defines also a $\{0\} \times G$ -observable such that $y_2(\mathbb{R}) = (x_{t_0} + x(t_0))'$ and if $E \cap (t_0, \infty) = \emptyset$, then $y_2(E) = 0$, in particular, $y_2(\{t_0\}) = 0$. Indeed, $\bigwedge_{t \searrow t_0} \hat{x}_t = \bigwedge_{t \searrow t_0} (x_t - (x_{t_0} + x(t_0))) = (\bigwedge_{t \searrow t_0} (x_t - x_{t_0})) - x(t_0) = x(t_0) - x(t_0) = 0$ and $y_2(\{t_0\}) = \bigwedge_{t \searrow t_0} (\hat{x}_t - \hat{x}_{t_0}) = (\bigwedge_{t \searrow t_0} \hat{x}_t) - \hat{x}_{t_0} = 0 - 0 = 0$.

Now we define a mapping $x : \mathcal{B}(\mathbb{R}) \rightarrow M$ as follows

$$x(E) = \begin{cases} y_1(E) & \text{if } E \subseteq (-\infty, t_0), \\ y_2(E) & \text{if } E \subseteq (t_0, \infty), \\ y_1(E) + y_2(E) & \text{if } t_0 \notin E, E \cap (-\infty, t_0) \neq \emptyset \text{ \& } E \cap (t_0, \infty) \neq \emptyset, \\ y_1(E) + x(t_0) + y_2(E) & \text{if } t_0 \in E, \end{cases} \quad (4.8)$$

for each $E \in \mathcal{B}(\mathbb{R})$. First we note that x is defined correctly. We assert that x is an observable on M such that $x((-\infty, t)) = x_t$ for each $t \in \mathbb{R}$. Indeed, let $t \leq t_0$, then $x((-\infty, t)) = y_1((-\infty, t)) = (0, \xi((-\infty, t))) = x_t$. If $t > t_0$, then $x((-\infty, t)) = y_1((-\infty, t)) + x(t_0) + y_2((-\infty, t)) = x_{t_0} + x(t_0) + \hat{x}_t = x_{t_0} + x(t_0) + x_t - (x_{t_0} + x(t_0)) = x_t$.

We have $x(\mathbb{R}) = y_1(\mathbb{R}) + x(t_0) + y_2(\mathbb{R}) = x_{t_0} + x(t_0) + (x_{t_0} + x(t_0))' = 1$. From the prescription, we see that x is monotone.

Let A, B be disjoint Borel sets. Then $x(A \cup B) = x(A) + x(B)$. Indeed, it is necessary to verify couple simple cases, e.g. let A satisfy the third condition and B the fourth one in (4.8): $x(A \cup B) = y_1(A \cup B) + x(t_0) + y_2(A \cup B) = y_1(A) + y_1(B) + x(t_0) + y_2(A) + y_2(B) = x(A) + x(B)$. Now let $A \subseteq (-\infty, t_0)$ and B satisfy the third condition. Then $B = B_1 \cup B_2$, where $B_1 = B \cap (-\infty, t_0)$ and $B_2 = B \cap (t_0, \infty)$. Then $x(A \cup B) = y_1(A \cup B) + y_2(A \cup B) = y_1(A) + y_1(B) + y_2(A) + y_2(B) = y_1(A) + y_1(B_1) + y_1(B_2) + y_2(A) + y_2(B_1) + y_2(B_2)$. But $y_1(B_2) = y_2(A) = y_2(B_1) = 0$. On the other hand, $x(A) + x(B) = y_1(A) + y_1(B) + y_2(B) = y_1(A) + y_1(B_1) + y_1(B_2) + y_2(B_1) + y_2(B_2) = y_1(A) + y_1(B_1) + y_1(B_2) = x(A \cup B)$.

Now let $A = \bigcup_n A_n$, where $A_n \cap A_m = \emptyset$ for $n \neq m$. If $B_n = A_1 \cup \dots \cup A_n$, then $x(B_n) = x(A_1) + \dots + x(A_n)$ so that the sequence $\{x(A_n)\}_n$ is summable. If $A \subseteq (-\infty, t_0)$ or $A \subseteq (t_0, \infty)$, we have trivially $x(A) = \sum_n x(A_n)$. Let A satisfy the third condition. Then $t_0 \notin B_n$ for each n and define $A_1 = A \cap (-\infty, t_0)$, $A_2 = A \cap (t_0, \infty)$, $B_1^n = B_n \cap (-\infty, t_0)$, $B_2^n = B_n \cap (t_0, \infty)$. Then

$$x(B_n) = x(B_1^n \cup B_2^n) = x(B_1^n) + x(B_2^n) = y_1(B_1^n) + y_2(B_2^n) \nearrow y_1(A_1) + y_2(A_2) = x(A).$$

Finally, let $t_0 \in A$. There is a unique integer n_0 such that $t_0 \in A_{n_0}$. Therefore,

$$\begin{aligned} x(A) &= y_1(A) + x(t_0) + y_2(A) = \sum_n y_1(A_n) + x(t_0) + \sum_n y_2(A_n) \\ &= \sum_{n \neq n_0} y_1(A_n) + y_1(A_{n_0}) + x(t_0) + y_2(A_{n_0}) + \sum_{n \neq n_0} y_2(A_n) \\ &= \sum_{n \neq n_0} x(A_n) + x(A_{n_0}) = \sum_n x(A_n) \end{aligned}$$

which proves that x is an observable. In addition, $x(\{t_0\}) = x(t_0)$.

Finally, we prove the uniqueness of x . Assume, that y is any observable on M such that $y((-\infty, t)) = x_t$, $t \in \mathbb{R}$. Let \mathcal{H} be the system of Borel sets $E \in \mathcal{B}(\mathbb{R})$ such that $x(E) = y(E)$. Hence, \mathcal{H} is a Dynkin system containing all intervals of the form $(-\infty, t)$, $t \in \mathbb{R}$. Similarly as above for \mathcal{L} , the Sierpiński Theorem implies \mathcal{H} is a σ -algebra, which proves $\mathcal{H} = \mathcal{B}(\mathbb{R})$ and $x = y$. \square

Now we characterize all observables on Rad-Dedekind σ -complete perfect MV-algebras.

Theorem 4.9. *Let $M = \Gamma(\mathbb{Z} \overrightarrow{\times} G, (1, 0))$ be a Rad-Dedekind σ -complete perfect MV-algebra. Let y_1, y_2 be $\{0\} \times G$ -observables such that there are a real number $t_0 \in \mathbb{R}$, an element $x(t_0) \in \text{Rad}(M)'$ satisfying $y_1(\mathbb{R}) + x(t_0) + y_2(\mathbb{R}) = 1$ and if $E \cap (-\infty, t_0)$, then $y_1(E)$ and if $E \cap (t_0, \infty)$, then $y_2(E) = 0$. Then the mapping $x : \mathcal{B}(\mathbb{R}) \rightarrow M$ defined by (4.8) is an observable, where we have $x(\{t_0\}) = x(t_0)$.*

Conversely, if M is, in addition, with principal radical, and x is an observable on M , there are unique $\{0\} \times G$ -observables y_1 and y_2 , a real number $t_0 \in \mathbb{R}$, and a unique element $x(t_0) \in \text{Rad}(M)'$ such that for x (4.8) holds.

Proof. To show that x is an observable, we proceed in the same way as in the proof of Theorem 4.8.

For the converse, the existence of $y_1, y_2, t_0, x(t_0)$ follows from the proof of Theorem 4.8. The uniqueness of t_0 and $x(t_0)$ follows from Claim 1 and Claim 2 of the latter proof. Consequently, we have also uniqueness of y_1 and y_2 . \square

We note that in the last two theorems it can happen that $y_1(E) = 0$ for all $E \in \mathcal{B}(\mathbb{R})$ or $y_2(E) = 0$ for all $E \in \mathcal{B}(\mathbb{R})$ or $x(t_0) = 1$.

Example 4.10. Let $M = \Gamma(\mathbb{Z} \overrightarrow{\times} \mathbb{Z}, (1, 0))$ be the Chang MV-algebra. It is a Rad-Dedekind σ -complete perfect MV-algebra with principal radical. Every observable on M is simple.

Proof. We show that if x is an observable on M , its spectrum $\sigma(x)$ is a finite set. Let $t_0 \in \mathbb{R}$ be determined by x according to Proposition 4.5. Then by the same Proposition, $x_{t_0} = x((-\infty, t_0)) \in \text{Rad}(M)$, similarly $x((t_0, \infty)) \in \text{Rad}(M)$ and $x(\{t_0\}) \in \text{Rad}(M)'$. We have $x_{t_0} = (0, n)$ and $x((t_0, \infty)) = (0, m)$ and $x(\{t_0\}) = (1, -n - m)$ which show that $\sigma(x)$ has to be a finite set with at most $n + m + 1$ elements. \square

Example 4.11. Let $M = \Gamma(\mathbb{Z} \overrightarrow{\times} \mathbb{R}, (1, 0))$. It is a Rad-Dedekind σ -complete perfect MV-algebra with principal radical and it possesses unbounded observables.

Proof. Indeed, let $\{s_n : n = \pm 1, \pm 2, \dots\}$ be a convergent series of strictly positive real numbers with sum s and let us define $a_n = (0, s_n)$, $n = \pm 1, \pm 2, \dots$, and $a_0 = (1, -s)$. If we put $t_n = n$, $n = 0, \pm 1, \pm 2, \dots$, then (3.1) defines an unbounded observable. \square

Theorem 4.8 can be extended also for Rad-Dedekind σ -complete perfect MV-algebras with not necessarily principal radical. For this reason, we extend the notion of a g -tribe to a *weak g -tribe* \mathcal{T} of real-valued functions (not necessarily bounded) on a set $\Omega \neq \emptyset$ such that

- (i) $0_\Omega \in \mathcal{T}$;
- (ii) $f \pm g \in \mathcal{T}$ whenever $f, g \in \mathcal{T}$;
- (iii) if $\{f_n\}_n$ is a sequence of elements of \mathcal{T} for which there exists $f \in \mathcal{T}$ with $f_n(\omega) \leq f(\omega)$, $\omega \in \Omega$, $n \geq 1$, then $\sup_n f_n \in \mathcal{T}$.

Clearly every g -tribe is a weak g -tribe and a weak g -tribe is also a Dedekind σ -complete ℓ -group of finite functions with respect to the pointwise ordering of functions. It can happen that a weak g -tribe has no strong unit. We say that a Rad-Dedekind σ -complete perfect MV-algebra M has the *Loomis–Sikorski property* if there are a weak g -tribe \mathcal{T} and an MV-homomorphism h from the Rad-Dedekind σ -complete perfect MV-algebra $M(\mathcal{T}) := \Gamma(\mathbb{Z} \overrightarrow{\times} \mathcal{T}, (1, 0_\Omega))$ onto M such that h preserves all existing countable joins in $M(\mathcal{T})$, and h maps $\text{Rad}(M(\mathcal{T}))$ onto $\text{Rad}(M)$ and $\text{Rad}(M(\mathcal{T}))'$ maps onto $\text{Rad}(M)'$.

By Theorem 4.6, every Rad-Dedekind σ -complete perfect MV-algebra with principal radical has the Loomis–Sikorski property. If we take a Dedekind σ -complete ℓ -group G from [Go, Ex 9.6], then the Rad-Dedekind σ -complete perfect MV-algebra $M = \Gamma(\mathbb{Z} \overrightarrow{\times} G, (1, 0))$ has the radical which is not principal but M has the Loomis–Sikorski property, see the construction of [Go, Ex 9.6].

Theorem 4.12. Let M be a Rad-Dedekind σ -complete MV-algebra with the Loomis–Sikorski property. If a system $\{x_t : t \in \mathbb{R}\}$ of elements of M satisfies (4.2)–(4.4), then there is a unique observable x on M for which (4.1) holds for any $t \in \mathbb{R}$.

Proof. It follows the same ideas as the proof of Theorem 4.8. \square

An easy corollary is the following result:

Corollary 4.13. Let \mathcal{T} be a weak g -tribe of finite functions on $\Omega \neq \emptyset$ and let $M = \Gamma(\mathbb{Z} \overrightarrow{\times} \mathcal{T}, (1, 0))$. If a system $\{x_t : t \in \mathbb{R}\}$ of elements of M satisfies (4.2)–(4.4), then there is a unique observable x on M for which (4.1) holds for any $t \in \mathbb{R}$.

Proof. The MV-algebra M is a Rad-Dedekind σ -complete perfect MV-algebra which trivially has the Loomis–Sikorski property. Applying Theorem 4.12, we obtain the desired result. \square

We note that in [DvKu], a one-to-one relation between observables and spectral resolutions was established for σ -complete MV-algebras and Theorem 4.12 extends this relation also for the class of Rad-Dedekind σ -complete perfect MV-algebras with principal radical.

At the end of this section, we present some remarks on observables on MV-algebras where no “ σ -structure” is assumed a priori. We do it in the same way as a σ -additive state can be defined in the equivalent form as a state s such that $\{a_n\}_n \nearrow a$ implies $\{s(a_n)\}_n \nearrow s(a)$. Thus an observable on an arbitrary MV-algebra M is any mapping $x : \mathcal{B}(\mathbb{R}) \rightarrow M$ such that (i) $x(\mathbb{R}) = 1$, (ii) if $E, F \in \mathcal{B}(\mathbb{R})$, $E \cap F = \emptyset$, then $x(E)$ and $x(F)$ are summable and $x(E \cup F) = x(E) + x(F)$, and if $\{E_n\}_n$ is a sequence of non-decreasing Borel sets with $E = \bigcup_n E_n$ and $\{x(E_n)\}_n \nearrow x(E)$. We can define a spectral resolution $\{x_t : t \in \mathbb{R}\}$ of x , where $x_t = x((-\infty, t))$, $t \in \mathbb{R}$; it satisfies (4.2)–(4.4). However, having a system $\{x_t : t \in \mathbb{R}\}$ of elements of M satisfying (4.2)–(4.4), to find a corresponding observable, we need some σ -structure on M .

It is well known that the Chang MV-algebra $C = \Gamma(\mathbb{Z} \overrightarrow{\times} \mathbb{Z}, (1, 0))$ generates a variety $\mathbf{V}(C)$ whose equational base is $(2.x)^2 = 2.x^2$, see [DiLe, Thm 5.11]. Moreover, every non-trivial MV-algebra $M \in \mathbf{V}(C)$ is due to [DiLe, Thm 5.1] a subdirect product of linearly ordered perfect MV-algebras. If $B(M)$ is the set of Boolean elements of M and $P(M) = \text{Rad}(M) \cup \text{Rad}(M)'$, they are a Boolean skeleton and a perfect skeleton, respectively, of M , and due to [DiLe, Cor 5.9], $M = \langle B(M) \cup P(M) \rangle$, where $\langle A \rangle$ denotes an MV-subalgebra of M generated by $A \subseteq M$. In addition, due to [DiLe, Thm 5.12], $B(M) \cong M/\text{Rad}(M)$.

Choose an MV-algebra $M \in \mathbf{V}(C)$ such that M is a subdirect product of a family $\{M_i : i \in I\}$ of linearly ordered perfect MV-algebras which are Rad-Dedekind σ -complete perfect MV-algebras with principal radical and let φ be a subdirect embedding of M into $\prod_{i \in I} M_i$ such that φ preserves existing countable suprema in M and if $\bigvee_n \varphi(a_n)$ exists in $\prod_{i \in I} M_i$, then $\bigvee_n a_n$ exists in M . Let $\{x_t : t \in \mathbb{R}\}$ be a system of elements of M satisfying (4.2)–(4.4). Given $i \in I$, let $x_t^i = \pi_i(\varphi(x_t))$ for each $t \in \mathbb{R}$, where $\pi_i : \prod_{i \in I} M_i \rightarrow M_i$ is the i -th projection. Then every system $\{x_t^i : t \in \mathbb{R}\}$ satisfies conditions (4.2)–(4.4) for each $i \in I$, so by Theorem 4.8, there is an observable x^i on M_i such that $x^i((-\infty, t)) = x_t^i$, $t \in \mathbb{R}$. Let $\mathcal{K} = \{E \in \mathcal{B}(\mathbb{R}) : \varphi^{-1}((x^i(E))_i) \in M\}$. Then \mathcal{K} is a Dynkin system containing all intervals of the form $(-\infty, t)$, so by the Sierpiński Theorem, [Kal, Thm 1.1], $\mathcal{K} = \mathcal{B}(\mathbb{R})$, so that $x(E) = \varphi^{-1}((x^i(E))_i)$, $E \in \mathcal{B}(\mathbb{R})$, is an observable whose spectral resolution is $\{x_t : t \in \mathbb{R}\}$. In other words, for such a kind of MV-algebras from the variety $\mathbf{V}(C)$, the statement of Theorem 4.8 holds, too.

There is a natural question, if $M \in \mathbf{V}(C)$ such that its Boolean skeleton $B(M)$ is σ -complete and the perfect skeleton $P(M)$ of M is a Rad-Dedekind σ -complete perfect MV-algebra with principal radical, does Theorem 4.8 hold? May be, if $P(M)$ is Dedekind σ -complete, is then $B(M)$ σ -complete?

We present some short observations which give a partial answer to the latter question.

Lemma 4.14. *Let M be an MV-algebra from $\mathbf{V}(C)$ and let M be a subdirect product of a system $\{M_i : i \in I\}$ of linearly ordered perfect MV-algebras. If M is with principal radical, then so is every M_i for each $i \in I$.*

Proof. Let M be subdirectly embedded into $\prod_{i \in I} M_i$. For each $i \in I$, there is a prime ideal $P = P_i$ on M such that $M/P_i = M_i$. There is a unique maximal ideal J of M containing P .

By [DiLe, Cor 5.2], $M = J \cup J'$. Due to a one-to-one correspondence between state-morphisms and maximal ideals, there is a unique state-morphism s_J on M such that $\text{Ker}(s_J) = J$; this state is a 0–1-valued one. If we define a mapping $\hat{s}_J : M_i \rightarrow [0, 1]$ by $\hat{s}_J(x/P) = s_J(x)$, $x/P \in M_i = M/P$, then it is a well-defined mapping; indeed, if $x/P = y/P$, then $x \odot y', y \odot x' \in P \subseteq J$, so that $s_J(x \odot y') = 0 = s_J(y \odot x')$ and $s_J(x) = s_J(y)$. In addition, $\hat{s}_J(1/P) = 1$. Now let $x/J \leq (y/J)' = (y')/J$. There is $x_0 \in x/J$ such that $x_0 \leq y'$. Hence, $\hat{s}_J(x/P + y/P) = \hat{s}_J((x_0 + y)/P) = s_J(x_0 + y) = s_J(x_0) + s_J(y) = s_J(x) + s_J(y) =$

$\hat{s}_J(x/P) + \hat{s}_J(y/P)$ which establishes that \hat{s}_J is a state on M/P . On the other hand, M/P is linearly ordered, it has a unique maximal ideal which is $\text{Rad}(M_i) = \text{Ker}(\hat{s}_J)$. Then $J/P = \text{Rad}(M_i)$.

If $x \in \text{Rad}(M)$, then $x/P \in \text{Rad}(M/P) = \text{Rad}(M_i)$, i.e. $\text{Rad}(M)/P \subseteq \text{Rad}(M/P)$. We show that $\text{Rad}(M)/P = \text{Rad}(M/P)$. If $x \in M$, due to [DFG, Prop 2.1], $x \wedge x' \in \text{Rad}(M)$. Hence, if $x \in J$, then $x \wedge x' \in \text{Rad}(M)$ and $(x \wedge x')/P = x/P \wedge x'/P = x/P$ due to linearity of M/P which shows that for each $x/P \in \text{Rad}(M/P)$, there is $x_0 = x \wedge x' \in \text{Rad}(M)$ such that $x_0/P = x/P$.

Now, let a be an element of $\text{Rad}(M)$ that generates $\text{Rad}(M)$ as an ideal. We assert that a/P generates $\text{Rad}(M/P)$ as an ideal. Indeed, for each $x/P \in \text{Rad}(M/P)$, there is an $x_0 \in \text{Rad}(M)$ such that $x_0/P = x/P$ and there is an integer n such that $x_0 \leq n.a$. Then $x/P = x_0/P \leq (n.a)/P = n.a/P$. \square

Proposition 4.15. *Let M be an MV-algebra that is a subdirect product of a system of MV-algebras, $\{M_i : i \in I\}$. For simplicity, let M be a subalgebra of $\prod_{i \in I} M_i$.*

(1) *Then $\text{Rad}(M) \subseteq \prod_{i \in I} \text{Rad}(M_i)$, and if $M = \prod_{i \in I} M_i$, then $\text{Rad}(M) = \prod_{i \in I} \text{Rad}(M_i)$.*

(2) *If $\text{Rad}(M)$ is a principal radical and $M = \prod_{i \in I} M_i$, so every $\text{Rad}(M_i)$ is a principal radical for each $i \in I$.*

(3) *Let $M = \prod_{i \in I} M_i$. Then M is Rad-Dedekind σ -complete if and only if every M_i is Rad-Dedekind σ -complete for each $i \in I$.*

Proof. (1) Let $a = (a_i)_i \in \text{Rad}(M)$. Then for each integer $n \geq 1$, we have $n.a \leq a'$, i.e. $n.a_i \leq a'_i$ for each $i \in I$ and $a_i \in \text{Rad}(M_i)$ and $a \in \prod_{i \in I} \text{Rad}(M_i)$. Now let $M = \prod_{i \in I} M_i$ and let $a = (a_i)_i \in \prod_{i \in I} \text{Rad}(M_i)$. Since $n.a_i \leq a'_i$, $n \geq 1$, $i \in I$, clearly $a \in \text{Rad}(M)$.

(2) Let $a = (a_i)_i \in \text{Rad}(M)$ be an element which generates $\text{Rad}(M)$ as an ideal. We assert that $a_i \in \text{Rad}(M_i)$ generates $\text{Rad}(M_i)$ as an ideal for each $i \in I$. Let b^i be an arbitrary element from $\text{Rad}(M_i)$ and define an element $c = (c_i)_i$ such that $c_j = b^i$ if $j = i$ and $c_j = 0$ if $j \neq i$, $j \in I$. Then $c \in \text{Rad}(M)$ and there is an integer n such that $c \leq n.a$, consequently, $b^i \leq n.a_i$ and a_i generates $\text{Rad}(M_i)$ as an ideal.

(3) Let $\text{Rad}(M)$ be Dedekind σ -complete. Assume that given $i_0 \in I$, we have a sequence $\{y_n^0\}_n$ of elements of $\text{Rad}(M_{i_0})$ which is bounded by some element $b \in \text{Rad}(M_{i_0})$. Let $y_n^i = y_n^0$ if $i = i_0$ and $y_n^i = 0$ if $i \neq i_0$, $i \in I$. Similarly, let $y^i = b$ if $i = i_0$ and $y^i = 0$ if $i \neq i_0$. Then $y_n = (y_n^i)_i \in \text{Rad}(M)$ for every $n \geq 1$, $y = (y^i)_i \in \text{Rad}(M)$ and $y_n \leq y$ for each $n \geq 1$. The assumption entails there is $z = (z_i)_i = \bigvee_n y_n \in M$ and z belongs to $\text{Rad}(M)$. Clearly $y_n^0 \leq z_{i_0} \in \text{Rad}(M_{i_0})$ for $n \geq 1$. Choose $v \in M_{i_0}$ such that $y_n^0 \leq v$ and define an element $w = (w_i)_i$, where $w_{i_0} = z$ and $w_i = z_i$ if $i \neq i_0$. Then $y_n \leq w$ for each $n \geq 1$, so that $z \leq w$ and $z_{i_0} \leq v$ which yields $z_{i_0} = \bigvee_n y_n^0 \in \text{Rad}(M_{i_0})$.

The converse implication is straightforward. \square

Corollary 4.16. *Let $M \in \mathcal{V}(C)$ be an MV-algebra that is Rad-Dedekind σ -complete with principal radical such that there is a family of (linearly ordered) perfect MV-algebras $\{M_i : i \in I\}$ satisfying $M \cong \prod_{i \in I} M_i$. If $\{x_t : t \in \mathbb{R}\}$ is a system of elements of M satisfying (4.2)–(4.4), then there is a unique observable x on M such that $x((-\infty, t)) = x_t$ for each $t \in \mathbb{R}$.*

Proof. Without loss of generality, we can assume that $M = \prod_{i \in I} M_i$. Due to Proposition 4.15, every M_i is a Rad-Dedekind σ -complete perfect MV-algebras with principal radical. Then $x_t = (x_t^i)_i$ and every $\{x_t^i : t \in \mathbb{R}\}$ satisfies of (4.2)–(4.4). Applying Theorem 4.8, there is an observable x^i on M_i such that $x^i((-\infty, t)) = x_t^i$, $t \in \mathbb{R}$, $i \in I$. Then the mapping $x : E \mapsto (x^i(E))_i$, $E \in \mathcal{B}(\mathbb{R})$, is an observable in question. If there is another observable y on M such that $x((-\infty, t)) = y((-\infty, t))$, $t \in \mathbb{R}$, we put $\mathcal{K} = \{E \in \mathcal{B}(\mathbb{R}) : x(E) = y(E)\}$. Then \mathcal{K} is a Dynkin system containing all intervals of the form $(-\infty, t)$, so by the Sierpiński Theorem, $\mathcal{K} = \mathcal{B}(\mathbb{R})$. \square

We note that Theorem 4.8 holds trivially also for an MV-algebra $\prod_{i \in I} M_i$, where each M_i is either a Rad-Dedekind σ -complete perfect MV-algebra with principal radical or a σ -complete MV-algebra, see [DvKu, Thm 3.2].

5. STATES ON PERFECT MV-ALGEBRAS

We present some kinds of states on perfect MV-algebras which are analogues of finitely additive measures.

The state space of any perfect MV-algebras is only a singleton consisting of a two-valued state:

Proposition 5.1. (1) *Any perfect MV-algebra M possesses only one state s , namely $s(a) = 0$ for each $a \in \text{Rad}(M)$ and $s(b) = 1$ for each $b \in \text{Rad}(M)'$.*

(2) *If M is a Rad-Dedekind σ -complete perfect MV-algebra M , the unique state on M is σ -additive.*

(3) *If x is an observable on a Rad-Dedekind σ -complete perfect MV-algebra with principal radical, and s a state on M , then $\text{Exp}_s(x) = t_x$, where the point t_x is a characteristic point of x .*

Proof. (1) If we define a mapping $s : M \rightarrow \{0, 1\}$ by $s(a) = 0$ iff $a \in \text{Rad}(M)$ otherwise, $s(a) = 1$, s is a two-valued state on M .

If s is an arbitrary state and $a \in \text{Rad}(M)$, then $na \in \text{Rad}(M)$ for each $n \geq 1$ so that $s(na) = n(s(a)) \leq 1$, and $s(a) \leq 1/n$ which gives the result.

(2) Let $a = \sum_n a_n$. If $a \in \text{Rad}(M)$, then each $a_n \in \text{Rad}(M)$ so that $s(a) = 0 = \sum_n s(a_n)$. If $a \in \text{Rad}(M)'$, by Lemma 3.1, there is a unique a_{n_0} such that $a_{n_0} \in \text{Rad}(M)$ and $a_n \in \text{Rad}(M)$ for $a_n \neq a_{n_0}$. Then $s(1) = 1 = \sum_{n \neq n_0} s(a_n) + s(a_{n_0})$ proving s is σ -additive.

(3) Let x be an observable on M and let t_0 be the point determined by Lemma 3.1. Then

$$\begin{aligned} \text{Exp}_s(x) &= \int t \, ds_x(t) = \int_{(-\infty, t_x)} t \, ds_x(t) + \int_{\{t_x\}} t \, ds_x(t) + \int_{(t_x, \infty)} t \, ds_x(t) \\ &= \int_{\{t_x\}} t \, ds_x(t) = t_x s(x(\{t_x\})) = t_x. \end{aligned}$$

□

Since the state space of a perfect MV-algebra is very poor, it contains only a two-valued state, we extend the notion of a state on a perfect MV-algebra in a similar way as it was extended a state on MV-algebra to its representing unital ℓ -group (G, u) . The idea is based on the fact that the radical, $\text{Rad}(M)$, of a perfect MV-algebra $M = \Gamma(\mathbb{Z} \overrightarrow{\times} G, (1, 0))$ corresponds to the positive cone of the ℓ -group $\{0\} \times G$.

The category of perfect MV-algebras with principal radical is a category of whose objects are pairs (M, a) , where M is a perfect MV-algebra with a fixed element $a \in \text{Rad}(M)$ which generates $\text{Rad}(M)$ as an ideal, and morphisms are homomorphisms of MV-algebras preserving fixed elements in radicals. This category is categorically equivalent to the category of Abelian unital ℓ -groups, see [BeDi, Thm 3.5].

Let (M, a) be a perfect MV-algebra with principal element, where a is a fixed element generating $\text{Rad}(M)$ as an ideal. A mapping $s : \text{Rad}(M) \rightarrow \mathbb{R}^+$ such that (i) $s(a + b) = s(a) + s(b)$, (ii) $s(a) = 1$ is said to be a Rad-state on (M, a) . Clearly $s(0) = 0$. An extremal Rad-state on (M, a) is defined in the same way as for states on general MV-algebras, and a Rad-state-morphism on (M, a) is a Rad-state on (M, a) which preserves lattice operations on $\text{Rad}(M)$. A Rad-state s is σ -additive if $a = \bigvee_n a_n \in \text{Rad}(M)$ a sequence $\{a_n\}$ from $\text{Rad}(M)$ with $a_n \leq a_{n+1}$ for each $n \geq 1$, then $s(a) = \lim_n s(a_n)$. We note that Rad-states are called in [DGL] local states.

For example, for the Chang MV-algebra $M = \Gamma(\mathbb{Z} \overrightarrow{\times} \mathbb{Z}, (1, 0))$, every element of the form $a = (0, n)$ generates $\text{Rad}(M)$ as an ideal, so that on (M, a) we have a unique Rad-state s_n with $s_n((0, n)) = 1$; s_n is a σ -additive Rad-state. Similarly, $\Gamma(\mathbb{Z} \overrightarrow{\times} \mathbb{R}, (1, 0))$ with the fixed element $a = (0, 1)$ has a unique Rad-state $s((0, t)) = t$, $t \geq 0$. Each Rad-state on the Chang MV-algebra is faithful, i.e. $s(b) = 0$ iff $b = 0$ but every perfect MV-algebra $M \neq \{0, 1\}$ has only a two-valued state which is not faithful. A non-empty set \mathcal{S} of Rad-states on a perfect MV-algebra M is said to be *faithful* if $s(a) = 0$ for each $s \in \mathcal{S}$ implies $a = 0$.

We note that if $a > 0$, (M, a) possesses at least one Rad-state as it follows from the next proposition.

Proposition 5.2. *Let $M = \Gamma(\mathbb{Z} \overrightarrow{\times} G, (1, 0))$ and let $a = (0, u)$, where u is a strong unit on an ℓ -group $G \neq \{0\}$. Then there is a one-to-one correspondence between Rad-states on (M, a) , states on (G, u) , and states on $\Gamma(G, u)$, respectively, preserving extremal states. In particular, (M, a) possesses at least one Rad-state.*

The ℓ -group G is Archimedean if and only if (M, a) possesses a faithful system of Rad-states.

Proof. Let s be a state on (G, u) . Thanks to the categorical equivalence of the category of MV-algebras with the category of unital ℓ -groups and the categorical equivalence of the category of perfect MV-algebras with principal radical, every the state s on $\Gamma(G, u)$ can be extended to a state \hat{s} on (G, u) . The mapping $\tilde{s} : \text{Rad}(M) \rightarrow \mathbb{R}$ given by $\tilde{s}(0, g) = \hat{s}(g)$, $(0, g) \in \text{Rad}(M)$, gives a Rad-state on (M, a) . Conversely, every Rad-state s on (M, a) can be extended to a state on the unital ℓ -group $(\{0\} \times G, (0, u))$ and consequently on (G, u) and it generates a state \hat{s} on $\Gamma(G, u)$. These correspondences preserve extremal states.

Since every unital ℓ -group (G, u) possesses at least one state, see [Go, Cor 4.3], by the first part of the present proof we conclude that (M, a) possesses at least one Rad-state.

Using [Go, Thm 4.14] and the mentioned equivalences, we have that G is Archimedean iff (M, a) possesses a faithful system of Rad-states. \square

Now let $M = \Gamma(\mathbb{Z} \overrightarrow{\times} G, (1, 0))$ be a Rad-Dedekind σ -complete perfect MV-algebra with principal radical. Let $x : \mathcal{B}(\mathbb{R}) \rightarrow \{0\} \times G^+$ be a $\{0\} \times G$ -observable. If s is a Rad-state on M , then $s_x(E) := s(x(E))$, $E \in \mathcal{B}(\mathbb{R})$, is a finite σ -additive measure on $\mathcal{B}(\mathbb{R})$. Then we can define the expectation $Exp_s(x)$ by

$$Exp_s(x) := \int t ds_x(t)$$

if the integral exists and is finite.

For example, let $M = \Gamma(\mathbb{Z} \overrightarrow{\times} \mathbb{Z}, (1, 0))$ be the Chang MV-algebra, and s be a unique Rad-state on $(M, (0, 1))$ such that $s((0, 1)) = 1$. If x is any $\{0\} \times \mathbb{Z}$ -observable, then there is a least finite set consisting of finitely many mutually different numbers $\lambda_1, \dots, \lambda_n$ such that $x(\{\lambda_1, \dots, \lambda_n\}) = x(\mathbb{R})$. Therefore, $Exp_s(x) = \sum_{i=1}^n \lambda_i s(\{\lambda_i\})$.

As we have seen, Proposition 5.1, every perfect MV-algebra M possesses a unique state and it is a two-valued one. The two-element Boolean algebra $\{0, 1\}$ is also a perfect MV-algebra. For example if O is the zero-group, then $\{0, 1\} \cong \Gamma(\mathbb{Z} \overrightarrow{\times} O, (1, 0))$. Therefore, we can see that every perfect MV-algebra has a state with values in the Chang MV-algebra C which is a perfect MV-algebra which generates the variety $V(C)$. Motivated by these ideas, we show some remarks on states on perfect MV-algebras with values in the perfect algebras $C = \Gamma(\mathbb{Z} \overrightarrow{\times} \mathbb{Z}, (1, 0))$ and $R := \Gamma(\mathbb{Z} \overrightarrow{\times} \mathbb{R}, (1, 0))$, respectively.

Let M be a perfect MV-algebra. We say that a mapping (I) $s : M \rightarrow C$ is a \mathbb{Z} -state if (i) $s(a + b) = s(a) + s(b)$ whenever $a + b$ is defined in M and (ii) $s(1) = 1 := (1, 0)$; (II) $s : M \rightarrow R$ is an \mathbb{R} -state if (i) $s(a + b) = s(a) + s(b)$ whenever $a + b$ is defined in M and (ii) $s(1) = 1 := (1, 0)$. Clearly, every \mathbb{Z} -state is an \mathbb{R} -state, $s(0) = 0$ and s is monotone. Then any \mathbb{Z} -state s maps $\text{Rad}(M)$ into the radical of C and $\text{Rad}(M)'$ maps into $\text{Rad}(C)'$; similarly for any \mathbb{R} -state. Without loss of generality, we can assume that $M = \Gamma(\mathbb{Z} \overrightarrow{\times} G, (1, 0))$ for some Abelian ℓ -group. Therefore, there is a unique additive mapping $\phi_s : G^+ \rightarrow \mathbb{Z}^+$ such that we have a mapping $s(0, g) = (0, \phi_s(g))$, $g \in G^+$. The mapping $\phi : G^+ \rightarrow \mathbb{Z}^+$ can be uniquely extended to a positive homomorphism of po-groups from G into \mathbb{Z} . Conversely, if $\phi : G \rightarrow \mathbb{Z}$ is a positive homomorphism, then $s_\phi : M \rightarrow C$ given by $s_\phi(0, g) = (0, \phi(g))$ and $s_\phi(1, -g) = (1, 0) - (0, \phi(g))$, $g \in G^+$, is a \mathbb{Z} -state. Of course, there is at least one positive homomorphism $\phi : G \rightarrow \mathbb{Z}$, the zero-mapping. Whence, if ϕ is non-zero, given an integer $k \geq 0$, the mapping $\phi_k := k\phi$ gives a new \mathbb{Z} -state s_k , so we have countably many different \mathbb{Z} -states $s_{\phi_0}, s_{\phi_1}, \dots$. We note that s_{ϕ_0} is a two-valued \mathbb{Z} -state corresponding to a unique two-valued state on M by Proposition 4.11.

Similar arguments can be applied to \mathbb{R} -states. Then there is also a one-to-one correspondence between positive homomorphisms $\psi : G \rightarrow \mathbb{R}$, and for each $t \in [0, \infty)$, we define $\psi_t := t\psi$. So if ψ is non-zero, we have uncountably many \mathbb{R} -states $\{s_{\psi_t} : t \in (0, \infty)\}$ generated by some \mathbb{R} -state. We note that s_{ψ_0} corresponds to a unique two-valued state on M .

Due to [Go, Ex 9.6], there is a Dedekind σ -complete ℓ -group G without strong unit such that there is no non-zero positive homomorphism. Therefore, the MV-algebra $M = \Gamma(\mathbb{Z} \overrightarrow{\times} G, (1, 0))$ is a Rad-Dedekind σ -complete perfect MV-algebra whose radical is not principal and M possesses only a two-valued \mathbb{R} -state.

Some limitation on the number of \mathbb{Z} -states or \mathbb{R} -states can be done for MV-algebras (M, a) with principal radical assuming that e.g. a \mathbb{Z} -state s maps a to $b = (0, n)$, where $n \geq 1$ is fixed; similarly for \mathbb{R} -states s mapping a to $b = (0, t)$, where $t \in (0, \infty)$ is fixed. Such states are said to be (\mathbb{Z}, a, n) -states and (\mathbb{R}, a, t) -states, respectively.

If (G, u) is a unital ℓ -group, there is a state h on (G, u) which is in fact a homomorphism of unital po-groups from (G, u) into (\mathbb{R}, t) where $t \in (0, \infty)$ is fixed. Therefore, the mapping $s_h : M \rightarrow R$ given by $s_h(0, g) = (0, h(g))$, $s_h(1, -g) = (1, 0) - (0, h(g))$, $g \in G^+$, is an (\mathbb{R}, a, t) -state mapping $a = (0, u)$ onto $(0, 1)$. In addition, (i) M possesses at least one (\mathbb{R}, a, t) -state and (ii) if (G, u) is Archimedean, then the system of all such (\mathbb{R}, a, t) -states is faithful.

The situation with (\mathbb{Z}, a, n) -states is similar and their existence depends on the existence of $(n + 1)$ -valued states on (G, u) . For example, in the case (M, a) , where $M = C$ is the Chang MV-algebra, $a = (0, 3)$ and $n = 1$, there is no $(\mathbb{Z}, a, 1)$ -state on (M, a) . If $a = (0, m) \in C$, then there is a (\mathbb{Z}, a, n) -state on (M, a) iff m divides n , and in such a case, there is a only one (\mathbb{Z}, a, n) -state on (M, a) .

6. OLSON ORDER AND SUM OF OBSERVABLES

In analogy with [Ols], using spectral resolutions, we introduce a partial order, the Olson order, of observables and we define a sum of observables. Then the system of observables is a distributive lattice which is also a lattice-ordered semigroup.

Let $\mathcal{O}(M)$ and $\mathcal{O}_b(M)$ be the set of all observables and of bounded observables, respectively, on a Rad-Dedekind σ -complete perfect MV-algebra M with principal radical. Given observables x, y, z, \dots on M , let $\{x_t : t \in \mathbb{R}\}$, $\{y_t : t \in \mathbb{R}\}$, $\{z_t : t \in \mathbb{R}\}, \dots$ be their spectral resolutions defined by Proposition 4.5. According to [DvuOl], we introduce on the set of observables of a Rad-Dedekind σ -complete MV-algebra with principal radical a partial ordering \preceq_s , called the *Olson order* (the subscript s stands spectral), which was motivated by an ordering of Hermitian operators introduced originally by Olson [Ols]. Thus, we write

$$x \preceq_s y \quad \text{iff} \quad y_t \leq x_t \quad \text{for each } t \in \mathbb{R}.$$

According to Theorem 4.8, we see that \preceq_s is a partial order on $\mathcal{O}(M)$.

Proposition 6.1. *Let M be a Rad-Dedekind σ -complete perfect MV-algebra with principal radical, and let x, y be observables on M . Define*

$$z_t := x_t \vee y_t, \quad t \in \mathbb{R}. \tag{6.1}$$

Then the system $\{z_t : t \in \mathbb{R}\}$ satisfies (4.2)–(4.4) and it determines a unique observable z on M that is the greatest lower bound of x and y under the Olson order, and we write $z = x \wedge y$.

Proof. Clearly $z_t \leq z_s$ whenever $s < t$. Let $z_t \leq w$ for each $t \in \mathbb{R}$. Then $x_t, y_t \leq w$ which gives $1 = \bigvee_t x_t \leq w$, whence $\bigvee_t z_t = 1$. Now, let $u \leq z_t$ for each $t \in \mathbb{R}$. Let t_0 be a fixed arbitrary real number. Then, for each $t < t_0$, we have $u \leq x_t \vee y_t \leq x_t \vee y_{t_0}$. Applying (2.2), we have $u \leq \bigwedge_{t < t_0} (x_t \vee y_{t_0}) = (\bigwedge_{t < t_0} x_t) \vee y_{t_0} = y_{t_0}$ which yields $u \leq \bigwedge_{t_0} y_{t_0} = 0$. Finally, let $s \in \mathbb{R}$ be given and let $z_t \leq v$ for each $t < s$. Then $x_t, y_t \leq z_t \leq v$ so that $x_s, y_s \leq v$ which proves $\bigvee_{t \nearrow s} z_t = z_s$.

Applying Theorem 4.8, we see that there is an observable z on M such that $\{z_t : t \in \mathbb{R}\}$ is a spectral resolution of z . Since $x_t, y_t \leq z_t$, we have $z \preceq_s x, y$. Now let u be any observable on M such that

$u \preceq_s x, y$. Then $x_t, y_t \leq u_t$ for each $t \in \mathbb{R}$, so that $z_t \leq u_t$, $t \in \mathbb{R}$ which entails $u \preceq_s z$ and $z = x \wedge y$ under the Olson order \preceq_s . \square

Clearly, Proposition 6.1 shows also that if x and y are bounded observables, then $z = x \wedge y$ is also a bounded observable. Similarly, (6.1) shows that the infimum of two simple observables is also a simple observable.

Proposition 6.2. *Let M be a Rad-Dedekind σ -complete perfect MV-algebra with principal radical, and let x, y be observables on M . Define*

$$w_t := x_t \wedge y_t, \quad t \in \mathbb{R}. \quad (6.2)$$

Then the system $\{w_t : t \in \mathbb{R}\}$ satisfies (4.2)–(4.4) and it determines a unique observable w on M that is the least upper bound of x and y under the Olson order, and we write $w = x \vee y$.

Proof. Clearly $w_t \leq w_s$ if $t < s$. Since $w_t \leq x_t$, we have $\bigwedge_t w_t = 0$. Let $w_t \leq a \in M$ for each $t \in \mathbb{R}$. Let t_0 be an arbitrary but fixed real number. Then for each $t > t_0$, we have $x_t \wedge y_{t_0} \leq x_t \wedge y_t = w_t \leq a$. Applying (2.1), we have $y_{t_0} = 1 \wedge y_{t_0} \leq a$ and $1 = \bigvee_{t_0} y_{t_0} \leq a$. Now let $t < s$ and let $x_t \leq b \in M$ for each $t < s$. Choose $t_0 < s$, then for each t with $t_0 < t < s$, we have $x_t \wedge y_{t_0} \leq x_t \wedge y_t \leq b$. Applying again (2.1), we have $x_s \wedge y_{t_0} \leq b$ and $\bigvee_{t_0 < s} (x_s \wedge y_{t_0}) = x_s \wedge y_s \leq b$.

Due to Theorem 4.8, we see that there is an observable w on M such that $\{w_t : t \in \mathbb{R}\}$ is a spectral resolution of w . In addition, $x, y \preceq_s w$. If there is an observable u such that $x, y \preceq_s u$, then $u_t \leq x_t, y_t$, so that $u_t \leq w_t$ for every $t \in \mathbb{R}$ which yields $w \preceq_s u$ and w is the least upper bound of x and y with respect to the Olson order. \square

We note that if x, y are bounded or simple observables, so is w . Let o be the zero observable, i.e. $o(\{0\}) = 1$. Then, for the spectral resolution $\{o_t : t \in \mathbb{R}\}$ of o , we have $o_t = 0$ for each $t \leq 0$ and $o_t = 1$ for each $t > 0$.

Theorem 6.3. *Let M be a Rad-Dedekind σ -complete perfect MV-algebra with principal radical. The system $\mathcal{O}(M)$ of observables on M is a distributive lattice under the Olson order. The same is for the class of simple observables. The null observable o is the least element in the set of positive observables and the top element in the set of negative observables.*

Proof. The fact that $\mathcal{O}(M)$ is a lattice follows from Propositions 6.1–6.2. The definition of the null observable entails that it is the least element of the set of positive observables and the top element in the set of negative observables.

We show the following form of the distributive law $(x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z)$ for all $x, y, z \in \mathcal{O}(E)$. By (6.1)–(6.2), we have for each $t \in \mathbb{R}$

$$\begin{aligned} ((x \vee y) \wedge z)_t &= (x_t \wedge y_t) \vee z_t = (x_t \vee z_t) \wedge (y_t \vee z_t) \\ &= ((x \wedge z) \vee (y \wedge z))_t. \end{aligned}$$

Analogously, we can get the second distributivity law $(x \wedge y) \vee z = (x \vee z) \wedge (y \vee z)$. \square

In the second half of the section, we introduce sum of observables. This notion was defined for bounded observables on σ -complete MV-algebra in [DvSum] and now we generalize it for Rad-Dedekind σ -complete perfect MV-algebras with principal radical and for any couple of observables not necessarily bounded.

Motivation for the sum is as follows:

Let (Ω, \mathcal{S}) be a measurable space, i.e. Ω is a non-void set and \mathcal{S} is a σ -algebra of subsets of Ω . Let $f, g : \Omega \rightarrow \mathbb{R}$ be \mathcal{S} -measurable functions. It is well-known that $f + g$ is also \mathcal{S} -measurable. The proof of this fact is based on the property

$$\{\omega \in \Omega : f(\omega) + g(\omega) < t\} = \bigcup_{r \in \mathbb{Q}} (\{\omega \in \Omega : f(\omega) < r\} \cap \{\omega \in \Omega : g(\omega) < t - r\})$$

for each $t \in \mathbb{R}$, where \mathbb{Q} is the set of rational numbers, see e.g. [Hal, Thm 19.A]. Inspired by this equality, in [DvSum], we have introduced a sum of bounded observables on σ -complete MV-algebras. In what follows, we extend the sum of observables also on Rad-Dedekind σ -complete perfect MV-algebras with principal radical.

The point t_0 from (4.5) can be characterized also as a (unique) point t_0 such that $x(\{t_0\}) \in \text{Rad}(M)'$ and $x(\{t\}) \in \text{Rad}(M)$ for each $t \neq t_0$. Indeed, it is sufficient to show that $t > t_0$ implies $x(\{t\}) \in \text{Rad}(M)$. If not, then for $t' \in \mathbb{R}$ such that $t_0 < t' < t$, we have $x_{t'}, x_{t'} + x(\{t\}) \in \text{Rad}(M)'$ which is impossible. Conversely, if $t \leq t_0$, then $(-\infty, t)$ and $\{t_0\}$ are disjoint sets which implies $x_t \in \text{Rad}(M)$. For $t > t_0$, we have $\{t_0\} \subseteq (-\infty, t)$, so that $x_t \in \text{Rad}(M)'$.

Properties of characteristic points:

- (i) If $x = o$ the null observable, then $t_0 = 0$.
- (ii) If f is a bijective mapping from \mathbb{R} onto (α, β) , where $\alpha \in \mathbb{R} \cup \{-\infty\}$ and $\beta \in \mathbb{R} \cup \{\infty\}$, then $t_{f(x)} = f(t_x)$.
- (iii) If $\alpha \in \mathbb{R}$ and $\alpha(t) = \alpha t$, $t \in \mathbb{R}$, then $t_{\alpha(x)} = \alpha t_x$.

Proposition 6.4. *Let x and y be observables on a Rad-Dedekind σ -complete perfect MV-algebra M with principal radical. Then, for each $t \in \mathbb{R}$, the element*

$$(x + y)_t = \bigvee_{r \in \mathbb{Q}} (x_r \wedge y_{t-r}) \quad (6.3)$$

exists in M .

Moreover, if $t \leq t_x + t_y$, then $(x + y)_t \in \text{Rad}(M)$, and $(x + y)_t \in \text{Rad}(M)'$ if $t > t_x + t_y$.

Proof. First we establish the following Claim.

Claim (i) Let $\{a_n\}_n$ be a sequence of elements from $\text{Rad}(M)$, then $\bigwedge_n a_n$ exists in M and it belongs to $\text{Rad}(M)$.

(ii) Let $\{c_n\}_n$ be a sequence of elements from $\text{Rad}(M)'$. Then $\bigvee_n c_n$ exists in M and it belongs to $\text{Rad}(M)'$.

Indeed: (i) We set $b_n = a_1 \wedge \dots \wedge a_n$ for each n . The elements $e_n = a_1 - b_n$ belong to $\text{Rad}(M)$ and they are bounded by $a_1 \in \text{Rad}(M)$. The Rad-Dedekind property of $\text{Rad}(M)$ entails $\bigvee_n e_n$ exists in M and it belongs to $\text{Rad}(M)$. Then $a_1 - \bigvee_n (a_1 - b_n) = \bigwedge_n b_n = \bigwedge_n a_n \in \text{Rad}(M)$.

(ii) Let $\{c_n\}_n$ be a sequence of elements from $\text{Rad}(M)'$. Then $\{c'_n\}_n$ is a sequence of elements from $\text{Rad}(M)$ and by (i), $\bigwedge_n c'_n \in \text{Rad}(M)$, so that $\bigvee_n c_n$ exists and it belongs to $\text{Rad}(M)'$.

Now, let t_x and t_y be characteristic points corresponding to x and y . Assume $t_x \leq t_y$. If $r \leq t_x$, then $x_r \wedge y_{t-r} \leq x_{t_x} \wedge y_{t-r} \leq x_{t_x} \in \text{Rad}(M)$, see Proposition 4.5, and Rad-Dedekind σ -completeness yields

$$A_1 := \bigvee_{r \leq t_x} (x_r \wedge y_{t-r})$$

exists in M and it belongs to $\text{Rad}(M)$. In the same way, we can prove that if $t - r \leq t_y$, i.e. $t - t_y \leq r$, then $x_r \wedge y_{t-r} \leq x_r \wedge y_{t_y} \leq y_{t_y} \in \text{Rad}(M)$ and

$$A_2 := \bigvee_{t-t_y \leq r} (x_r \wedge y_{t-r}) \in \text{Rad}(M).$$

Finally, either the set $\mathbb{A}_1 := \{r \in \mathbb{Q} : t_x < r < t - t_y\}$ is empty or non-empty. In the first case, the element $\bigvee_{r \in \mathbb{A}_1} (x_r \wedge y_{t-r}) = 0$, in the second case, $x_r \wedge y_{t-r} \in \text{Rad}(M)'$ and by Claim, $\bigvee_{r \in \mathbb{A}_1} (x_r \wedge y_{t-r})$ exists in M and it belongs to $\text{Rad}(M)'$. Consequently, the element

$$(x + y)_t = \bigvee_{r \leq t_x} (x_r \wedge y_{t-r}) \vee \bigvee_{r \in \mathbb{A}_1} (x_r \wedge y_{t-r}) \vee \bigvee_{t-t_y \leq r} (x_r \wedge y_{t-r}) \quad (6.4)$$

exists in M .

Now let $t_y < t_x$. Then for all $r \leq t_x$, we have also that

$$B_1 := \bigvee_{r \leq t_x} (x_r \wedge y_{t-r})$$

exists in M and it belongs to $\text{Rad}(M)$. Similarly, if $t - r \leq t_y$, i.e. $t - t_y \leq r$, the element

$$B_2 := \bigvee_{t-t_r \leq r} (x_r \wedge y_{t-r}) \in \text{Rad}(M).$$

Let $\mathbb{A}_2 = \{r \in \mathbb{Q} : t_x < r < t - t_y\}$. If $\mathbb{A}_2 = \emptyset$, then $\bigvee_{r \in \mathbb{A}_2} (x_r \wedge y_{t-r}) = 0$, if \mathbb{A}_2 is non-empty, then for each $r \in \mathbb{A}_2$, $x_r \wedge y_{t-r} \in \text{Rad}(M)'$ so that by Claim, $\bigvee_{r \in \mathbb{A}_2} (x_r \wedge y_{t-r}) \in \text{Rad}(M)'$ which proves that $(x + y)_t$ exists in M for each $t \in \mathbb{R}$.

Now let $t \leq t_x + t_y$. If $t_x \leq t_y$, the elements $A_1, A_2 \in \text{Rad}(M)$ and the set $\mathbb{A}_1 = \emptyset$, because for $r \in \mathbb{A}_1$, we have $t_x < r < t - t_y \leq t_x + t_y - t_y = t_x$. In view of (6.4), $(x + y)_t \in \text{Rad}(M)$. In the similar way, if $t_y < t_x$, the set $\mathbb{A}_2 = \emptyset$, $B_1, B_2 \in \text{Rad}(M)$ and $(x + y)_t \in \text{Rad}(M)$.

Finally, let $t > t_x + t_y$. If $t_x \leq t_y$, then $t_x < t - t_y$ and due to the density of rational numbers, there is a rational number r such that $t_x < r < t - t_y$, so that $\mathbb{A}_1 \neq \emptyset$ and (6.4) shows that $(x + y)_t \in \text{Rad}(M)'$. Similarly, if $t_y < t_x$, the set \mathbb{A}_2 is non-empty and hence, $(x + y)_t \in \text{Rad}(M)'$. \square

Theorem 6.5. *Let x and y be observables on a Rad-Dedekind σ -complete perfect MV-algebra M with principal radical. Then the system $\{(x + y)_t : t \in \mathbb{R}\}$, where $(x + y)_t$ is defined by (6.3) satisfies the conditions (4.2)–(4.4), and there is a unique observable z on E such that $z_t = (x + y)_t$ for each $t \in \mathbb{R}$.*

In addition, $(x + y)_t = (y + x)_t$ for each $t \in \mathbb{R}$, and consequently $x + y = y + x$. Moreover, $t_{x+y} = t_x + t_y$.

Proof. Clearly, $(x + y)_t \leq (x + y)_s$ for $t < s$.

(4.3): By Proposition 6.4, if $t \leq t_x + t_y$, then $(x + y)_t \in \text{Rad}(M)$, so that by Claim, $\bigwedge_t (x + y)_t$ exists in $\text{Rad}(M)$ and dually, $\bigvee_t (x + y)_t$ exists and is in $\text{Rad}(M)'$. Let r_0 be an arbitrary but fixed rational such that $r_0 < t_x + t_y$. Then

$$\begin{aligned} (x + y)_t &= \bigvee_{r \in \mathbb{Q}} (x_r \wedge y_{t-r}) = \bigvee_{r < r_0} (x_r \wedge y_{t-r}) \vee \bigvee_{r \geq r_0} (x_r \wedge y_{t-r}) \\ &\leq \bigvee_{r < r_0} x_r \vee \bigvee_{r \geq r_0} y_{t-r_0} \\ &= x_{r_0} \vee y_{t-r_0}. \end{aligned}$$

Hence,

$$\begin{aligned} \bigwedge_t (x + y)_t &\leq \bigwedge_t (x_{r_0} \vee y_{t-r_0}) = x_{r_0} \\ \bigwedge_t (x + y)_t &\leq \bigwedge_{r_0} x_{r_0} = 0. \end{aligned}$$

Now we show that $\bigvee_{t \in \mathbb{R}} (x + y)_t = 1$. Take $r_0 > t_x + t_y$. Then

$$\begin{aligned} (x + y)_t &= \bigvee_r (x_r \wedge y_{t-r}) \\ &= \bigvee_{r \leq r_0} (x_r \wedge y_{t-r}) \vee \bigvee_{r > r_0} (x_r \wedge y_{t-r}) \\ &\geq \bigvee_{r \leq r_0} (x_r \wedge y_{t-r_0}) \vee \bigvee_{r > r_0} (x_{r_0} \wedge y_{t-r}) \\ &= x_{r_0} \wedge y_{t-r_0} \vee x_{r_0} \wedge y_{t-r_0} = x_{r_0} \wedge y_{t-r_0}. \end{aligned}$$

Then

$$\begin{aligned} \bigvee_t (x + y)_t &\geq \bigvee_t (x_{r_0} \wedge y_{t-r_0}) = x_{r_0} \\ \bigvee_t (x + y)_t &\geq \bigvee_{r_0} x_{r_0} = 1. \end{aligned}$$

(4.4): Let $t < s$. If $s \leq t_x + t_y$, by Proposition 6.4, $(x + y)_s \in \text{Rad}(M)$, so that due to linearity of the set $\{(x + y)_t : t \in \mathbb{R}\}$, we see that $\bigvee_{t < s} (x + y)_t$ exists and is in $\text{Rad}(M)$. If $s > t_x + t_y$, there is t' with $t_x + t_y < t' < s$, so that by Claim in the proof of Proposition 6.4, we see that $\bigvee_{t < s} (x + y)_t$ exists and is in $\text{Rad}(M)'$.

$$\begin{aligned} \bigvee_{t < s} (x + y)_t &= \bigvee_{t < s} \bigvee_{r \in \mathbb{Q}} (x_r \wedge y_{t-r}) \\ &= \bigvee_{r \in \mathbb{Q}} \bigvee_{t < s} (x_r \wedge y_{t-r}) \\ &= \bigvee_{r \in \mathbb{Q}} (x_r \wedge \bigvee_{t < s} y_{t-r}) \\ &= \bigvee_{r \in \mathbb{Q}} (x_r \wedge y_{s-r}) = (x + y)_s. \end{aligned}$$

Applying Theorem 4.8, we see that $\{(x + y)_t : t \in \mathbb{R}\}$ is a spectral resolution of a unique observable z such that $z_t = (x + y)_t$ for each $t \in \mathbb{R}$.

Now we show that $(x + y)_t = (y + x)_t$ for each $t \in \mathbb{R}$. (i) Let t be a rational number and set $s = t - r$. Then $\bigvee_{r \in \mathbb{Q}} (x_r \wedge y_{t-r}) = \bigvee_{s \in \mathbb{Q}} (x_{t-s} \wedge y_s) = (y + x)_t$. (ii) If t is an arbitrary real number, let $\{t_n\}$ be a sequence of rational numbers such that $\{t_n\} \nearrow t$. Then using (4.4), we have $(x + y)_t = \bigvee_{t_n \nearrow t} (x + y)_{t_n} = \bigvee_{t_n \nearrow t} (y + x)_{t_n} = (y + x)_t$.

Hence, $x + y = y + x$.

Finally, the property $t_{x+y} = t_x + t_y$ follows from Proposition 6.4. \square

Definition 6.6. The observable $x + y$ from Theorem 6.5 is said to be a *sum* of observables x and y on a Rad-Dedekind σ -complete perfect MV-algebra M with principal radical.

Now we present a simple corollary which says that expectation is additive and homogeneous on the set of observables.

Corollary 6.7. *If x and y are observables on a Rad-Dedekind σ -complete perfect MV-algebra M with principal radical, $\alpha, \beta \in \mathbb{R}$, and s is a state on M , then*

$$t_{\alpha(x) + \beta(y)} = \text{Exp}_s(\alpha(x) + \beta(y)) = \text{Exp}_s(\alpha(x)) + \text{Exp}_s(\beta(y)) = \alpha t_x + \beta t_y.$$

Proof. According to Proposition 5.1(3), $\text{Exp}_s(x) = t_x$, $\text{Exp}_s(y) = t_y$, and by Theorem 6.5, we have $t_{x+y} = t_x + t_y$ which gives the result. \square

Theorem 6.8. *If x, y, z are observables on a Rad-Dedekind σ -complete perfect MV-algebra M with principal radical, then $x + y = y + x$ and $(x + y) + z = x + (y + z)$.*

In addition, $x + o = o + x$, where o is the null observable on M . The sum of bounded observables is a bounded observable and the sum of simple ones is a simple observable.

Moreover, the set $\mathcal{O}(M)$ is a commutative semigroup with respect to sum of observables and with a neutral element o .

Proof. Using commutativity of the sum of observables, we get

$$\begin{aligned}
((x + y) + z)_t &= (z + (x + y))_t = \bigvee_{r \in \mathbb{Q}} (z_r \wedge (x + y)_{t-r}) \\
&= \bigvee_{r \in \mathbb{Q}} (z_r \wedge \bigvee_{s \in \mathbb{Q}} (x_s \wedge y_{t-r-s})) \\
&= \bigvee_{r \in \mathbb{Q}} \bigvee_{s \in \mathbb{Q}} (z_r \wedge x_s \wedge y_{t-r-s}) \\
&= \bigvee_{s \in \mathbb{Q}} \bigvee_{r \in \mathbb{Q}} (z_r \wedge x_s \wedge y_{t-r-s}) \\
&= \bigvee_{s \in \mathbb{Q}} (x_s \wedge \bigvee_{r \in \mathbb{Q}} (z_r \wedge y_{t-r-s})) \\
&= \bigvee_{s \in \mathbb{Q}} (x_s \wedge (z + y)_{t-s}) \\
&= \bigvee_{s \in \mathbb{Q}} (x_s \wedge (y + z)_{t-s}) \\
&= (x + (y + z))_t.
\end{aligned}$$

Finally, $(o + x)_t = \bigvee_{r \leq 0} (o_r \wedge x_{t-r}) \vee \bigvee_{r > 0} (o_r \wedge x_{t-r}) = 0 \vee \bigvee_{r > 0} (1 \wedge x_{t-r}) = x_t$.

Let x and y be bounded observables. Then there is a real number $K > 0$ such that $x([-K, K]) = 1 = y([-K, K])$. It is possible to show from (6.3) that $(x + y)_t = 0$ for $t < -2K$ and $(x + y)_t = 1$ for $t > 2K$.

If x and y are simple, the spectral resolutions of x and y have only finitely many elements, so that the spectral resolution of $x + y$ has also only finitely many elements.

Consequently, $\mathcal{O}(M)$ is a semigroup with respect to $+$ and with the null observable o as a neutral element. \square

In what follows, we show that the set $\mathcal{O}(M)$ is a lattice ordered semigroup with respect to the Olson order and the sum of observables. An analogous result was established in [DvSum, Thm 4.4] for the set of bounded observables on a σ -complete MV-algebra.

We note that according to [Fuc], a semigroup E with a partial ordering \preceq is (i) a *partially ordered semigroup* if $x \preceq y$ implies $x + z \preceq y + z$ for every $x, y, z \in E$, see [Fuc, p. 153], and (ii) a *lattice-ordered semigroup* if E is a lattice such that $(x \vee y) + z = (x + z) \vee (y + z)$, [Fuc, p. 191].

Theorem 6.9. *The set $\mathcal{O}(M)$ of observables on a Rad-Dedekind σ -complete perfect MV-algebra M with principal radical is a distributive lattice and a lattice-ordered semigroup with respect to sum of observables and the Olson order.*

Proof. (i) According to Theorem 6.3, the set $\mathcal{O}(M)$ is a distributive lattice under the Olson order \preceq_s , and due to Theorem 6.8, we see that the set $\mathcal{O}(M)$ of observables on M is a semigroup with respect to sum $+$ and with a neutral element o .

(ii) Now we show that if, for two observables x and y on M , we have $x \preceq_s y$, then $x + z \preceq_s y + z$ for each observable z on M . Indeed, if $x \preceq_s y$, then $y_t \leq x_t$ for each $t \in \mathbb{R}$. We have $(y + z)_t = \bigvee_r (y_r \wedge z_{t-r}) \leq \bigvee_r (x_r \wedge z_{t-r}) = (x + z)_t$, which implies $x + z \preceq_s y + z$. Hence, $\mathcal{O}(M)$ is a partially ordered semigroup.

(iii) To establish that $\mathcal{O}(M)$ is a lattice-ordered semigroup, we have to show that $(x \vee y) + z = (x + z) \vee (y + z)$. To show that, we will follow steps from the proof of [DvSum, Thm 4.4]. Calculate

$$((x \vee y) + z)_t = \bigvee_r ((x \vee y)_r \wedge z_{t-r}) = \bigvee_r ((x_r \wedge y_r) \wedge z_{t-r})$$

and

$$\begin{aligned}
((x+z) \vee (y+z))_t &= (x+z)_t \wedge (y+z)_t \\
&= \left(\bigvee_r (x_r \wedge z_{t-r}) \right) \wedge \left(\bigvee_s (y_s \wedge z_{t-s}) \right) \\
&= \bigvee_r \bigvee_s (x_r \wedge z_{t-r} \wedge y_s \wedge z_{t-s}) \\
&= \bigvee_r \bigvee_{s \leq r} (x_r \wedge z_{t-r} \wedge y_s \wedge z_{t-s}) \\
&\vee \bigvee_r \bigvee_{s > r} (x_r \wedge z_{t-r} \wedge y_s \wedge z_{t-s}).
\end{aligned}$$

Denote by

$$\begin{aligned}
B_1 &:= \bigvee_r \bigvee_{s \leq r} (x_r \wedge z_{t-r} \wedge y_s \wedge z_{t-s}), \\
B_2 &:= \bigvee_r \bigvee_{s > r} (x_r \wedge z_{t-r} \wedge y_s \wedge z_{t-s}).
\end{aligned}$$

Then

$$\begin{aligned}
B_1 &= \bigvee_r \bigvee_{s \leq r} (x_r \wedge z_{t-r} \wedge y_s) = \bigvee_r (x_r \wedge z_{t-r} \wedge y_r) \in M, \\
B_2 &= \bigvee_s \bigvee_{r < s} (x_r \wedge y_s \wedge z_{t-s}) = \bigvee_s (x_s \wedge y_s \wedge z_{t-s}) \in M.
\end{aligned}$$

Since $B_1 = B_2$, we see that $((x \vee y) + z)_t = ((x+z) \vee (y+z))_t$, that is, $\mathcal{O}(M)$ is a lattice-ordered semigroup. \square

We note that in the same way it is possible to show that the set of bounded observables and the set of simple observables are lattice ordered semigroups.

Finally we do a short remark on Markov kernels which are associated with a g-tribe \mathcal{T} of bounded functions on a set $\Omega \neq \emptyset$ because they are closely connected with \mathcal{T} -observables, where \mathcal{T} is a g-tribe.

We say that a mapping $K : \Omega \times \mathcal{B}(\mathbb{R}) \rightarrow [0, \infty)$ is a *Markov kernel associated with a g-tribe \mathcal{T}* (simply *Markov kernel*) if

- (i) for any fixed $E \in \mathcal{B}(\mathbb{R})$, the mapping $K(\cdot, E) \in \mathcal{T}^+$;
- (ii) for any fixed $\omega \in \Omega$, the mapping $K(\omega, \cdot)$ is a σ -additive measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$.

We note that the mapping $x_K : \mathcal{B}(\mathbb{R}) \rightarrow \mathcal{T}^+$ defined by $x_K(E) := K(\cdot, E)$, $E \in \mathcal{B}(\mathbb{R})$, is a \mathcal{T} -observable. Conversely, let x be a \mathcal{T} -observable. Then there is a unique Markov kernel K associated with the g-tribe \mathcal{T} such that $x = x_K$.

7. CONCLUSION

In the paper we studied a situation when a partial information on an observable known only on all intervals of the form $(-\infty, t)$ gives us a complete information about the whole observable. That means when a system $\{x_t : t \in \mathbb{R}\}$ of elements of an MV-algebra which is monotone, left continuous, $\bigwedge_t x_t = 0$ and $\bigvee_t x_t = 1$ (called also a spectral resolution) determines an observable x such that $x_t = x((-\infty, t))$, $t \in \mathbb{R}$. This question is studied for Rad-Dedekind σ -complete perfect MV-algebras with principal radical. The main result is Theorem 4.8 which generalizes analogous results from [DvKu] where this problem was solved for σ -complete MV-algebras and Dedekind monotone σ -complete effect algebras with the Riesz Dedekind Property.

Using this one-to-one correspondence between observables and spectral resolutions, we define the Olson order under which the set of all observables $\mathcal{O}(M)$ on M is a distributive lattice, Theorem 6.3. In addition, spectral resolutions enable us to define a sum of observables so that $\mathcal{O}(M)$ is a commutative semigroup, Theorem 6.8, and a lattice-ordered semigroup with respect to the sum of observables and the Olson order, see Theorem 6.9.

We have also an open problem: How big is a class of MV-algebras where every spectral resolution determines an observable? We know that it contains all σ -complete MV-algebras, see [DvKu], and now by Theorem 4.8 also all Rad-Dedekind σ -complete perfect MV-algebras with principal radical. This class is closed under direct product.

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