Stability analysis of spline collocation methods for fractional differential equations

Angelamaria Cardone[a,*], Dajana Conte[a]

[a] *Dipartimento di Matematica, Università di Salerno, Via Giovanni Paolo II n. 132, I-84084 Fisciano (Salerno), Italy*

Abstract

This paper deals with spline collocation methods for fractional differential equations, introduced by Pedas and Tamme in [J. Comput. Appl. Math. **255**, 216-230 (2014)]. Some practical formulas are derived, for the computation of fractional integrals involved in the method, useful for implementation. Linear stability analysis is carried out and stability regions of several methods are provided. Numerical experiments on linear and nonlinear test problems confirm theoretical expectations.

Key words: fractional differential equations, collocation, fractional integrals, stability

1. Introduction

In this paper we investigate theoretical and computational aspects of spline collocation methods introduced by Pedas and Tamme in [35], for the numerical solution of fractional differential equations (FDEs) of the form

$$\begin{cases} D^\alpha y(t) = f(t, y(t)), & 0 \le t \le b, \\ y^{(i)}(0) = \gamma_i, & i = 0, \dots, n-1, \end{cases} \tag{1.1}$$

*Corresponding author
Email addresses: ancardone@unisa.it (Angelamaria Cardone), dajconte@unisa.it (Dajana Conte)

where $n-1<\alpha<n$, $n\in\mathbb{N}$, $\gamma_i\in\mathbb{R}$, $f:[0,b]\times\mathbb{R}\to\mathbb{R}$ is a given continuous function. Here $D^\alpha y$ is the Caputo-type fractional derivative [19, 26, 39]:

$$D^\alpha y(t)=\frac{1}{\Gamma(n-\alpha)}\int_0^t \frac{y^{(n)}(s)}{(t-s)^{\alpha+1-n}}ds.$$

FDEs arise in various applications in life sciences and engineering, such as anomalous transport models [26], viscoelastic models in mechanics [18, 30, 42] and blood flow [38], options pricing models in financial markets [43], brownian motion [26, 31, 32]. Further models may be found in [6, 14, 24, 44] and references therein.

Existence and uniqueness theorems for FDEs may be found in [19, 26, 35, 39]. In [35] authors also provide important results on the regularity of the solution of a nonlinear FDE of the type (1.1). They carry out this analysis in the space $C^{q,\nu}(0,b]$, defined as follows (compare also [5, 33, 34, 36]).

Definition 1. Let $C^{q,\nu}(0,b]$ with $q\in\mathbb{N}$ and $\nu\in(-\infty,1)$, be the space of the functions $y:[0,b]\to\mathbb{R}$, which are q times continuously differentiable in $(0,b]$ such that:

$$|y^{(i)}(t)|\le c\begin{cases}1, & \text{if } i<1-\nu\\ 1+|\log t|, & \text{if } i=1-\nu\\ t^{1-\nu-i}, & \text{if } i>1-\nu\end{cases},\qquad t\in(0,b],\quad i=1,\dots,q.$$

Under suitable hypotheses on $f(t,y)$, the solution y of the problem (1.1) and its fractional derivative $D^\alpha y$ belong to the space $C^{q,\nu}(0,b]$, as stated by the following theorem.

Theorem 1.1. *[35, Th.2.1] Let $\alpha>0$, $\alpha\in\mathbb{R}$. Let $f:[0,b]\times\mathbb{R}\to\mathbb{R}$ be a continuous function which is q times ($\in\mathbb{N}$) continuously differentiable in $\Omega:=(0,b]\times\mathbb{R}$, whereby there exists a real number $\nu\in[1-\alpha,1)$ such that for all nonnegative integers i and j with $i+j\le q$ and for all $(t,y)\in\Omega$ the following estimation holds:*

$$\left|\frac{\partial^{i+j}}{\partial t^i\partial y^j}f(t,y)\right|\le\psi(|y|)\begin{cases}1, & \text{if } i<1-\nu\\ 1+|\log t|, & \text{if } i=1-\nu\\ t^{1-\nu-i}, & \text{if } i>1-\nu\end{cases}.$$

For $\alpha \in (0,1)$ we assume additionally that

$$\left|\frac{\partial^{i+j}}{\partial t^i \partial y^j}[f(t,y_1) - f(t,y_2)]\right| \leq \psi(\max\{|y_1|,|y_1|\})|y_1 - y_2| \begin{cases} 1, & \text{if } i = 0, \\ t^{1-\nu-i}, & \text{if } i > 0, \end{cases}$$

where (t, y_1) and (t, y_2) belong to Ω. The function $\psi : [0,\infty) \to \mathbb{R}$ is assumed to be monotonically increasing. Finally, suppose that the initial value problem (1.1) *possesses a solution $y \in C[0,b]$ such that $D^\alpha y \in C[0,b]$. Then $y \in C^{q,\nu}(0,b]$ and $D^\alpha y \in C^{q,\nu}(0,b]$.*

Collocation methods represent a well-known class of methods for ordinary differential equations and for Volterra integral and integro-differential equations, thanks to their good convergence and stability properties [4, 5, 10–13, 15–17]. For the numerical solution of FDEs, a great variety of collocation methods is available: spectral collocation methods [6, 44], collocation methods based on cubic B-spline wavelets [27], Chebyshev collocation methods [25], collocation methods based on polynomial splines [3, 33–36, 40]. In the present paper, we focus our attention on the last class of methods. Spline collocation methods were applied to FDEs for the first time by Blank [3]. Later on Rawashdeh extended these methods to fractional integro-differential equations [40]. More recently, Pedas and Tamme made an in-depth study of spline collocation methods for FDEs [33–36]. They proved that such methods have high order of convergence when a suitable graded mesh is adopted. Our aim is to complete the theoretical analysis, by studying the stability properties and by deriving some practical formulas useful to apply the method.

There is a vast amount of literature on the stability of numerical methods for FDEs, as for example [22] for predictor-corrector methods, [20, 23] for product integration and for fractional linear multistep methods, [7, 8] for time-splitting schemes. In most cases, the analysis has been inspired by the work of Lubich [28] on the stability of convolution quadrature methods for Abel-Volterra integral equations (Abel-VIEs). The application of a numerical method to the linear test equation gives rise to a difference equation of unbounded order. Lubich analyzed the asymptotic behavior of its solution by formal power series. In our investigation, we consider as a starting point the work by Blank on stability of collocation methods for Abel-VIEs [1, 2], once again inspired by [28].

The paper is structured as follows. Sections 2 and 3 illustrate spline collocation methods introduced in [35] and some practical formulas for their application. In Sec. 4 we recall stability analysis results of spline collocation methods for Abel-VIEs and then analyze linear stability of analogous methods for FDEs. In Sec. 5 we show several stability regions. Some numerical experiments on linear and nonlinear problems are provided in Sec. 6. Last section contains some concluding remarks.

2. Spline collocation methods

In this section we illustrate spline collocation methods proposed and analyzed by Pedas and Tamme in [35]. These methods were already introduced in [5] for linear Volterra integro-differential equations with weakly singular kernels. In the context of FDEs, we will use this equivalent formulation of initial value problem (1.1)

$$z = f(t, J^\alpha z + Q), \tag{2.1}$$

where $z = D^\alpha y$,

$$(J^\alpha z)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} z(s)\, ds, \quad t > 0, \tag{2.2}$$

$$Q(t) = \sum_{i=0}^{\lceil \alpha \rceil - 1} \frac{\gamma_i}{i!} t^i, \tag{2.3}$$

$\lceil \alpha \rceil$ being the least integer not less than α. Then the solution of (1.1) is

$$y = J^\alpha z + Q. \tag{2.4}$$

Let $I_N = \{0 = t_0 < \cdots < t_N = b\}$ be a mesh on $[0, b]$, with

$$h_j = t_j - t_{j-1}, \quad \sigma_j = [t_{j-1}, t_j], \quad j = 1, \ldots, N,$$

Spline collocation methods introduced in [35] approximate the solution of (2.1) by a function v belonging to the space of piecewise polynomial functions $S_k^{(-1)}(I_N)$:

$$S_k^{(-1)}(I_N) = \left\{ v : v|_{\sigma_j} \in \pi_k,\, j = 1, \ldots, N \right\}, \tag{2.5}$$

where π_k is the space of algebraic polynomials of degree not exceeding k. Therefore $v \in S_k^{(-1)}(I_N)$ may have jump discontinuities at the mesh points. Then, the approximated solution of equation (1.1) is:

$$y_N = J^\alpha v + Q. \tag{2.6}$$

Let us consider the set of collocation abscissae

$$0 \le c_1 < c_2 < \cdots < c_m \le 1, \tag{2.7}$$

and the collocation points

$$t_{jk} = t_{j-1} + c_k h_j, \quad k = 1, \ldots, m,\, j = 1, \ldots, N. \tag{2.8}$$

Spline collocation methods approximate the solution of (2.1) as the piecewise polynomial $v \in S_{m-1}^{(-1)}(I_N)$ with

$$v(t) = \sum_{l=1}^{N} \sum_{k=1}^{m} z_{lk} L_{lk}(t), \quad t \in [0, b], \tag{2.9}$$

where $z_{jk} = v(t_{jk})$. For $t \notin [t_{l-1}, t_l]$, $L_{lk}(t) = 0$, while for $t \in [t_{l-1}, t_l]$, $L_{lk}(t)$ is the $k-$th Lagrange fundamental polynomial with respect to the nodes $\{t_{lk} \mid k = 1, \ldots, m\}$. Function $v(t)$ satisfies equation (2.1) at the collocation points (2.8), i.e. $z_{j1}, \ldots, z_{jm}$, satisfy the nonlinear system

$$z_{jk} = f\left(t_{jk}, (J^\alpha v)(t_{jk}) + Q(t_{jk})\right), \quad k = 1, \ldots, m, \tag{2.10}$$

for all $j = 1, \ldots, N$.

The order of convergence of the method depends both on smoothness of the analytical solution of (1.1) and on the mesh of points. In particular, a graded mesh may be adopted, i.e.:

$$t_j = b\left(\frac{j}{N}\right)^r, \qquad j = 0, \ldots, N, \tag{2.11}$$

with a grading exponent $r \in \mathbb{R}$, $r \ge 1$. Theorem 4.1 of [35] states that, if hypotheses of Th. 1.1 are fulfilled with $q = m$ and $\nu \in [1-\alpha, 1)$ and under other suitable assumptions, the spline collocation method with graded mesh (2.11) converges and

$$\|y_N - y\|_\infty \le C E_N(m,\nu,r),$$

where $C > 0$ does not depend on N and

$$E_N(m,\nu,r) = \begin{cases} N^{-r(1-\nu)}, & \text{if } 1 \le r < \frac{m}{1-\nu}, \\ N^{-m}(1+\log N), & \text{if } r = \frac{m}{1-\nu} = 1, \\ N^{-m}, & \text{if } r = \frac{m}{1-\nu} > 1 \text{ or } r > \frac{m}{1-\nu}. \end{cases} \tag{2.12}$$

The accuracy of collocation methods may be further improved when collocation parameters $c_1, \ldots, c_m$ are nodes of the Gaussian quadrature rule. In this case, the error may even decrease as $N^{-m+\alpha}$ if $0 < \alpha < m$, or as N^{-2m} if $\alpha > m$. For details, refer to [35, Lemma 4.1].

Remark 1. We observe that, especially for large values of r, first intervals may be quite small, which may lead to significant round off errors. This phenomenon, well known for Volterra integral equations with weakly singular kernel (compare, e.g. [9, 21, 41] and references therein), it has been pointed out also for fractional integro-differential equations [29]. To overcome this problem, some strategies are available. Here we mention an hybrid collocation approach [9, 29] and some methods based on a regularization of the problem to avoid the lack of smoothness of the solution at the origin [21, 37, 41]. In the numerical tests provided below, we had no problems caused by round off errors and we did not applied any strategy, however we underline that such phenomena may happen in some cases.

3. Derivation of practical formulas

Spline collocation method (2.9)-(2.10) requires the computation of fractional integrals of polynomials and a suitable matrix formulation. Although similar formulas may be found in many books and papers (cfr. for instance [19, 27]), we explicitly derive these formulas to make the paper self-consistent.

Let consider the nonlinear system (2.10). By replacing $v(t)$ with right-hand side of (2.9), we obtain

$$z_{jk} = f\left(t_{jk}, \sum_{l=1}^{N}\sum_{\mu=1}^{m}(J^\alpha L_{l\mu})(t_{jk})z_{l\mu} + Q(t_{jk})\right), \quad k = 1, \dots, m. \tag{3.1}$$

Since $(J^\alpha L_{l\mu})(t_{jk}) = 0$ for $l > j$, the system (3.1) becomes

$$z_{jk} = f\left(t_{jk}, \sum_{\mu=1}^{m}(J^\alpha L_{j\mu})(t_{jk})z_{j\mu} + \sum_{l=1}^{j-1}\sum_{\mu=1}^{m}(J^\alpha L_{l\mu})(t_{jk})z_{l\mu} + Q(t_{jk})\right), \tag{3.2}$$

$k = 1, \dots, m$. Now we show how to compute the fractional integrals appearing in (3.2). We first consider

$$(J^\alpha L_{j\mu})(t_{jk}) = \frac{1}{\Gamma(\alpha)}\int_0^{t_{jk}}(t_{jk} - s)^{\alpha-1}L_{j\mu}(s)\,ds. \tag{3.3}$$

Since $L_{j\mu}(t) = 0$ for $t \notin [t_{j-1}, t_j]$ the integral in (3.3) becomes:

$$(J^\alpha L_{j\mu})(t_{jk}) = \frac{1}{\Gamma(\alpha)}\int_{t_{j-1}}^{t_{jk}}(t_{jk} - s)^{\alpha-1}\prod_{\substack{i=1\\ i\neq\mu}}^{m}\frac{s - t_{ji}}{t_{j\mu} - t_{ji}}\,ds.$$

By change of variable $s = t_{j-1} + \tau h_j$ we obtain:

$$(J^\alpha L_{j\mu})(t_{jk}) = \frac{1}{\Gamma(\alpha)}h_j^\alpha\int_0^{c_k}(c_k - \tau)^{\alpha-1}\varphi_\mu(\tau)\,d\tau,$$

where $\varphi_\mu(\tau)$ are the Lagrange polynomials

$$\varphi_\mu(\tau) = \prod_{\substack{i=1\\ i\neq\mu}}^{m}\frac{\tau - c_i}{c_\mu - c_i}. \tag{3.4}$$

By writing Lagrange polynomials in the form:

$$\varphi_\mu(\tau) = \sum_{\nu=0}^{m-1}a_\nu^{(\mu)}\tau^\nu, \qquad \mu = 1, \dots, m, \tag{3.5}$$

we get:

$$(J^\alpha L_{j\mu})(t_{jk}) = h_j^\alpha\sum_{\mu=0}^{m-1}a_\nu^{(\mu)}\frac{1}{\Gamma(\alpha)}\int_0^{c_k}(c_k - \tau)^{\alpha-1}\tau^\nu\,d\tau.$$

The following equality

$$\frac{1}{\Gamma(\alpha)}\int_0^t(t - s)^{\alpha-1}s^\nu\,ds = t^{\nu+\alpha}\frac{\Gamma(1+\nu)}{\Gamma(1+\nu+\alpha)}$$

leads to

$$(J^{\alpha}L_{j\mu})(t_{jk}) = h_j^{\alpha} \sum_{\nu=0}^{m-1} a_{\nu}^{(\mu)} c_k^{\nu+\alpha} \frac{\Gamma(1+\nu)}{\Gamma(1+\nu+\alpha)}. \tag{3.6}$$

Secondly, we compute the fractional integral

$$(J^{\alpha}L_{l\mu})(t_{jk}) = \frac{1}{\Gamma(\alpha)} \int_0^{t_{jk}} (t_{jk}-s)^{\alpha-1} L_{l\mu}(s) ds, \quad l = 1, \ldots, j-1.$$

Since $L_{l\mu}(s) = 0$ for $s \notin [t_{l-1}, t_l]$, the previous integral reduces to:

$$(J^{\alpha}L_{l\mu})(t_{jk}) = \frac{1}{\Gamma(\alpha)} \int_{t_{l-1}}^{t_l} (t_{jk}-s)^{\alpha-1} L_{l\mu}(s) ds.$$

By change of variable $s = t_{l-1} + \tau h_l$ and by (3.5), it follows

$$\begin{aligned}(J^{\alpha}L_{l\mu})(t_{jk}) &= \frac{1}{\Gamma(\alpha)} h_l^{\alpha} \int_0^1 \left(\frac{t_{j-1} + c_k h_j - t_{l-1}}{h_l} - \tau \right)^{\alpha-1} \varphi_{\mu}(\tau)\, d\tau \\ &= \frac{1}{\Gamma(\alpha)} h_l^{\alpha} \sum_{\nu=0}^{m-1} a_{\nu}^{(\mu)} \int_0^1 \left(\frac{t_{j-1} + c_k h_j - t_{l-1}}{h_l} - \tau \right)^{\alpha-1} \tau^{\nu}\, d\tau.\end{aligned}$$

By employing

$$\int_0^1 (d-\tau)^{\alpha-1} \tau^{\nu} d\tau = d^{\nu+\alpha} \mathrm{B}\left(\frac{1}{d}; 1+\nu, \alpha\right),$$

where $\mathrm{B}(z; a, b)$ stands for the incomplete Beta function:

$$\mathrm{B}(z; a, b) := \int_0^z \sigma^{a-1} (1-\sigma)^{b-1}\, d\sigma,$$

we obtain

$$(J^{\alpha}L_{l\mu})(t_{jk}) =$$
$$\frac{1}{\Gamma(\alpha)} h_l^{\alpha} \sum_{\nu=0}^{m-1} a_{\nu}^{(\mu)} \left(\frac{t_{j-1} + c_k h_j - t_{l-1}}{h_l} \right)^{\nu+\alpha} \mathrm{B}\left(\frac{h_l}{t_{j-1} + c_k h_j - t_{l-1}}; 1+\nu, \alpha \right). \tag{3.7}$$

Expressions (3.6) and (3.7) become practical formulas if an explicit expression of the coefficients $a_{\nu}^{(\mu)}$ is available or an algorithm for their computation is provided. This task can be easily addressed. By evaluating (3.5) in the collocation abscissae, we obtain the following linear system of dimension m:

$$\mathbf{V}\mathbf{a}^{(\mu)} = e_{\mu}, \qquad \mu = 1, \ldots, m,$$

in the unknown $\mathbf{a}^{(\mu)} = [a_0^{(\mu)}, \dots, a_{m-1}^{(\mu)}]$ Here, $\mathbf{V}$ is the Vandermonde matrix corresponding to abscissae $c_1, \dots, c_m$ and e_μ is the μ-th vector of canonical basis. Thus, we could also write

$$\tilde{\mathbf{A}} = \mathbf{V}^{-1}\mathbf{I},$$

where the μ-th column of matrix $\tilde{\mathbf{A}}$ is $\mathbf{a}^{(\mu)}$ and $\mathbf{I}$ is the identity matrix of dimension m. The bad conditioning of the Vandermonde matrix should not dramatically affect the accuracy, since m has usually a small value.

An alternative procedure may be adopted to compute $a_\nu^{(\mu)}$, which does not involve the solution of a Vandermonde system. It is well known that if $\mathbf{u}$ and $\mathbf{v}$ are vectors of polynomial coefficients, convolving them is equivalent to multiplying the two polynomials. Thus, coefficients $a_\nu^{(\mu)}$ can be derived by iteratively computing the convolution products of vectors $\left[\frac{1}{c_\mu - c_i}, \frac{-c_i}{c_\mu - c_i}\right]$, $i = 1, \dots, m$, $i \neq \mu$ (cfr. (3.4)). This procedure may be realized by Matlab routine `conv`.

From (3.6) and (3.7), by defining the matrices

$$\mathbf{A} = (A_{k\mu}) \in \mathbb{R}^{m\times m}, \quad \mathbf{E}^{[l,j]} = (E_{k\mu}^{[l,j]}) \in \mathbb{R}^{m\times m},$$

with

$$A_{k\mu} = \sum_{\nu=0}^{m-1} a_\nu^{(\mu)} c_k^{\nu+\alpha} \frac{\Gamma(1+\nu)}{\Gamma(1+\nu+\alpha)},$$
$$E_{k\mu}^{[l,j]} = \frac{1}{\Gamma(\alpha)} \sum_{\nu=0}^{m-1} a_\nu^{(\mu)} \left(\frac{t_{j-1} + c_k h_j - t_{l-1}}{h_l}\right)^{\nu+\alpha} \mathrm{B}\left(\frac{h_l}{t_{j-1} + c_k h_j - t_{l-1}}; 1+\nu, \alpha\right),$$

spline collocation method (3.2) can be written in matrix form

$$\mathbf{z}_j = f\left(\mathbf{t}_j, h_j^\alpha \mathbf{A}\mathbf{z}_j + \sum_{l=1}^{j-1} h_l^\alpha \mathbf{E}^{[l,j]}\mathbf{z}_l + \mathbf{q}_j\right), \tag{3.8}$$

$j = 1, 2, \dots, N$, where

$$\mathbf{z}_j = \begin{pmatrix} z_{j1} \\ \vdots \\ z_{jm} \end{pmatrix}, \ \mathbf{t}_j = \begin{pmatrix} t_{j1} \\ \vdots \\ t_{jm} \end{pmatrix}, \ \mathbf{q}_j = \begin{pmatrix} Q(t_{j1}) \\ \vdots \\ Q(t_{jm}) \end{pmatrix},$$

and, for $\mathbf{v} \in \mathbb{R}^m$,

$$f(\mathbf{t}_j, \mathbf{v}) = \begin{pmatrix} f(t_{j1}, v_1) \\ \vdots \\ f(t_{jm}, v_m) \end{pmatrix}.$$

4. Numerical stability of spline collocation methods

The study of stability of spline collocation methods for FDEs relies on the stability analysis of analogous methods for Abel-VIEs. Therefore, we prior summarize some important results on the stability of spline collocation methods for Abel-VIEs, obtained by Luise Blank in [2].

The basic linear test equation for Abel-VIEs is

$$\tilde{y}(t) = g(t) + \tilde{\lambda} \int_0^t (t-s)^{-\tilde{\alpha}} \tilde{y}(s) ds, \qquad t \geq 0. \tag{4.1}$$

If g is continuous on $[0, \infty[$ and possesses a finite limit for $t \to \infty$, then $\lim_{t\to\infty} \tilde{y}(t) = 0$ when the parameter λ satisfies the condition

$$|\arg(\tilde{\lambda}) - \pi| < \frac{1}{2}(1+\tilde{\alpha})\pi. \tag{4.2}$$

Let consider the uniform mesh $t_n = nh$, $n \geq 0$, $h > 0$, the set of collocation parameters $0 \leq c_1 \leq \cdots \leq c_m \leq 1$ and the set of collocation points $t_{nl} = t_n + c_l h$, $l = 1, \ldots, m$. The spline collocation method for Abel-VIEs illustrated in [2] approximates the solution of (4.1) by a suitable piecewise polynomial $u(t)$. The sequence $\mathbf{u_j} = (u(t_{j1}), \ldots, u(t_{jm}))^T$, $j \geq 1$, satisfies the recurrence relation:

$$\mathbf{u_j} = h^{1-\tilde{\alpha}} \tilde{\lambda} \sum_{l=1}^{j} \tilde{W}_{j-l} \mathbf{u_l} + \mathbf{g_j}, \qquad j \geq 1, \tag{4.3}$$

where $\mathbf{g_j} = (g(t_{j1}), \ldots, g(t_{jm}))^T$, $\tilde{W}_j = (\tilde{w}_j(c_k, c_\mu))$, $k, \mu = 1, \ldots, m$, and

$$\tilde{w}_0(c_k, c_\mu) = \int_0^{c_k} (c_k - \sigma)^{-\tilde{\alpha}} \varphi_\mu(\sigma) d\sigma,$$
$$\tilde{w}_j(c_k, c_\mu) = \int_0^1 (j + c_k - \sigma)^{-\tilde{\alpha}} \varphi_\mu(\sigma) d\sigma, \quad j \geq 1.$$

φ_μ is the μ-th Lagrange polynomial corresponding to the collocation parameters (compare (3.4)).

The region of stability $\tilde{\mathcal{R}}$ of the spline collocation method for Abel-VIEs consists of all $z = \tilde{\lambda} h^{1-\tilde{\alpha}} \in \mathbb{C}$, for which the numerical solution $\lim_{t\to\infty} u(t) = 0$, whenever $\lim_{t\to\infty} g(t) = const$. In [2] Blank proved that

$$\tilde{\mathcal{R}} = \mathbb{C}\backslash \left\{ \tilde{\lambda} h^{1-\tilde{\alpha}} \,\middle|\, \frac{1}{\tilde{\lambda} h^{1-\tilde{\alpha}}} \text{ is an eigenvalue of } \tilde{W}(\xi) \text{ for a } \xi \in \mathbb{C} \text{ with } |\xi| \leq 1 \right\}, \quad (4.4)$$

where $\tilde{W}(\xi)$ is the formal power series $\tilde{W}(\xi) = \sum_{n=0}^{\infty} \tilde{W}_n \xi^n$.

Now we can proceed with the study of numerical stability of spline collocation methods for FDEs introduced in [35]. The basic linear test equation for FDEs is

$$\begin{cases} D^{\alpha} y(t) = \lambda y(t), & t \geq 0, \\ y(0) = y_0, \end{cases} \quad (4.5)$$

where $0 < \alpha < 1$. The exact solution of (4.5) is $y(t) = E_\alpha(\lambda t^\alpha) y_0$, where $E_\alpha(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(\alpha k + 1)}$ [19, 20, 39]. Thus $\lim_{t\to\infty} y(t) = 0$ if and only if $\lambda \in \Sigma_\alpha$, where [8, 23]

$$\Sigma_\alpha = \left\{ s \in \mathbb{C} : |\arg(s)| > \frac{\alpha}{2}\pi \right\}. \quad (4.6)$$

The study of numerical stability of collocation methods will be carried out in a fixed stepsize framework (compare [2]). Thus, hereinafter, we set $h_j = h$, for all j. For an extension to nonuniform stepsize see Remark 3 at the end of this section.

In the following theorem, we define

$$w_0(c_k, c_\mu) = \int_0^{c_k} (c_k - \sigma)^{\alpha-1} \varphi_\mu(\sigma) d\sigma,$$

$$w_j(c_k, c_\mu) = \int_0^{1} (j + c_k - \sigma)^{\alpha-1} \varphi_\mu(\sigma) d\sigma, \quad j \geq 1,$$

where φ_μ are the Lagrange polynomials (3.4).

Theorem 4.1. *Spline collocation method* (2.9)-(2.10) *applied to the linear test equation* (4.5) *gives rise to the recurrence relation*

$$\mathbf{z}_j = \frac{\lambda h^\alpha}{\Gamma(\alpha)} \sum_{l=1}^{j} W_{j-l} \mathbf{z}_l + y_0 \mathbf{e}, \qquad j \geq 1, \quad (4.7)$$

where $\mathbf{z}_j = \left(\frac{z_{j1}}{\lambda}, \ldots, \frac{z_{jm}}{\lambda}\right)^T$, $\mathbf{e} = (1, \ldots, 1)^T$, $W_j = (w_j(c_k, c_\mu))$, $k, \mu = 1, \ldots, m$.

Proof. By applying method (2.9)-(2.10) to eq. (4.5), with fixed stepsize h, we obtain

$$z_{jk} = \lambda \sum_{\mu=1}^{m} z_{j\mu}(J^\alpha L_{j\mu})(t_{jk}) + \lambda \sum_{l=1}^{j-1} \sum_{\mu=1}^{m} z_{l\mu}(J^\alpha L_{l\mu})(t_{jk}) + \lambda y_0, \tag{4.8}$$

$k = 1, \ldots, m$, $j \geq 1$. From (2.2) we have

$$(J^\alpha L_{j\mu})(t_{jk}) = \frac{1}{\Gamma(\alpha)} \int_0^{t_{jk}} (t_{jk} - s)^{\alpha-1} L_{j\mu}(s)\, ds.$$

By observing that $L_{j\mu}(t) = 0$ outside the interval $[t_{j-1}, t_j]$ and by the change of variable $s = t_{j-1} + \sigma h$, we get

$$(J^\alpha L_{j\mu})(t_{jk}) = \frac{h^\alpha}{\Gamma(\alpha)} \int_0^{c_k} (c_k - \sigma)^{\alpha-1} \varphi_\mu(\sigma)\, d\sigma = \frac{h^\alpha}{\Gamma(\alpha)} w_0(c_k, c_\mu).$$

Analogously, we obtain

$$(J^\alpha L_{l\mu})(t_{jk}) = \frac{h^\alpha}{\Gamma(\alpha)} \int_0^{1} (j - l + c_k - \sigma)^{\alpha-1} \varphi_\mu(\sigma)\, d\sigma = \frac{h^\alpha}{\Gamma(\alpha)} w_{j-l}(c_k, c_\mu).$$

By substituting the last two expressions in (4.8) and by dividing left- and right-hand side by λ, the thesis follows.

Definition 2. The stability region of method (2.9)-(2.10) is defined as the set

$$\mathcal{R} = \left\{ \lambda h^\alpha \in \mathbb{C} \,\middle|\, \lim_{j\to\infty} \mathbf{z}_j = 0 \right\},$$

where $\mathbf{z}_j$ satisfies (4.7).

The method is said $A(\alpha\frac{\pi}{2})$-stable when $\mathcal{R}$ includes Σ_α, defined in (4.6). We will also define an $A(\theta)$-stable method if $\mathcal{R} \supseteq \{s \in \mathbb{C} : |\arg(s)| > \theta\}$.

The following theorem characterizes the stability region $\mathcal{R}$.

Theorem 4.2. *The region of stability of method* (2.9)-(2.10) *is given by*

$$\mathcal{R} = \mathbb{C} \setminus \left\{ \lambda h^\alpha \,\middle|\, \frac{1}{\lambda h^\alpha} \text{ is an eigenvalue of } \frac{W(\xi)}{\Gamma(\alpha)} \text{ for a } \xi \in \mathbb{C} \text{ with } |\xi| \leq 1 \right\}, \tag{4.9}$$

where $W(\xi)$ *is the formal power series* $W(\xi) = \sum_{n=0}^{\infty} W_n \xi^n$.

Proof. By setting $\tilde{\alpha} = 1-\alpha$, matrices W_j in (4.7) coincide with matrices $\tilde{W}_j$ in (4.3). If we further set $\tilde{\lambda} = \lambda/\Gamma(\alpha)$ and $g(t) \equiv y_0$, recurrence relations (4.3) and (4.7) are equal. As a consequence, the region of stability of method (2.9)-(2.10) is

$$\mathcal{R} = \left\{ \Gamma(\alpha) z \mid z \in \tilde{\mathcal{R}} \right\}. \tag{4.10}$$

Therefore, the thesis follows from (4.4).

Remark 2. Since $\Gamma(\alpha) > 1$ for $\alpha \in (0,1)$, if $\tilde{\mathcal{R}}$ is bounded, $\mathcal{R}$ is larger than $\tilde{\mathcal{R}}$. Otherwise, if $\tilde{\mathcal{R}}$ is unbounded, $\mathcal{R}$ is smaller than $\tilde{\mathcal{R}}$, but still unbounded.

We observe that the coefficients of $W(\xi)$ have a radius of convergence 1, but for $\xi = 1$ the series is not absolutely convergent. Thus, in Th. 4.2, the eigenvalues of $\dfrac{W(1)}{\Gamma(\alpha)}$ should be interpreted as the limit for $\xi \to 1$ of the eigenvalues of $\dfrac{W(\xi)}{\Gamma(\alpha)}$. In order to compute such eigenvalues, we consider, as in [2], the similarity transformation $B\dfrac{W(\xi)}{\Gamma(\alpha)}B^{-1}$, where

$$B := \left(\begin{array}{c|c} I_{m-1} & -\mathbf{e} \\ \hline \mathbb{L}^T & \mathbb{L}_m \end{array} \right), \quad B^{-1} = \left(\begin{array}{c|c} I_{m-1} - \mathbf{e}\mathbb{L}^T & \mathbf{e} \\ \hline -\mathbb{L}^T & 1 \end{array} \right), \tag{4.11}$$

with $\mathbb{L} := (\mathbb{L}_1, \ldots, \mathbb{L}_{m-1})^T$, $\mathbf{e} = (1 \ldots 1)^T \in \mathbb{R}^{m-1}$ and

$$\mathbb{L}_\mu = \int_0^1 \varphi_\mu(\tau) d\tau, \quad \mu = 1, \ldots, m.$$

To compute B^{-1}, one has to take into account that $\sum_{\mu=1}^{m} \mathbb{L}_\mu = 1$.

The following result will be useful to construct stability regions.

Theorem 4.3.

$$\frac{1}{\lambda h^\alpha} \textit{ is not an eigenvalue of } \frac{W(1)}{\Gamma(\alpha)} \Longleftrightarrow$$
$$\frac{1}{\lambda h^\alpha} \textit{ is not an eigenvalue of } A(1) \textit{ and } \lambda h^\alpha \neq 0,$$

where $A(\xi)$ is the matrix composed by the first $m-1$ lines and the first $m-1$ columns of the matrix $B\dfrac{W(\xi)}{\Gamma(\alpha)}B^{-1}$ and has coefficient sequences in ℓ^1.

Proof. The proofs follows the lines of analogous results presented in [2].

Remark 3. The analysis of stability has been carried out in a fixed stepsize framework. However, it may deliver guidance on the number of mesh points N to choose when a graded mesh is adopted, too. In particular, we recommend to take N such that $h_{max} \leq \bar{h}$, where $\lambda \bar{h}^{\alpha} \in \mathcal{R}$. Eq. (2.11) leads to

$$h_{max} = h_N = b - b\left(\frac{N-1}{N}\right)^r, \tag{4.12}$$

therefore the following limitation for N holds:

$$N \geq \left(1 - \left(1 - \frac{\bar{h}}{b}\right)^{\frac{1}{r}}\right)^{-1}.$$

The effect of this choice will be shown in the numerical tests.

5. Stability regions

In this section we construct several stability regions of spline collocation methods. They have been numerically derived by exploiting the property, described in [2]:

$$\partial\mathcal{R} \subseteq \left\{\frac{1}{z} \in \mathbb{C} | z \text{ is an eigenvalue of } \frac{W(\xi)}{\Gamma(\alpha)}, \text{ for a } \xi \text{ with } |\xi| = 1\right\}. \tag{5.1}$$

There may be some curves defined in the right-hand side of (5.1) which do not belong to $\partial\mathcal{R}$. Such curves can be excluded by plotting

$$\left\{\frac{1}{z} \in \mathbb{C} | z \text{ is an eigenvalue of } \frac{W(\xi)}{\Gamma(\alpha)}, \text{ for a } \xi \text{ with } |\xi| = 1 - \varepsilon\right\}$$

with small ε.

To construct the boundaries of the stability regions, we approximate the infinite series with $\sum_{j=0}^{N} W_j \xi^j$ for $\xi \neq 1$, and with $\sum_{j=0}^{N} B W_j B^{-1} \xi^j$ for $\xi = 1$, where B has been defined in (4.11). We fix $N = 1000$. Moreover we express ξ as $\xi = e^{i\omega}$, with 500-equally spaced angles ω in $[0, 2\pi]$.

Figures 5.1-5.4 show the stability regions of many spline collocation methods. Unless otherwise specified, we mean collocation methods for FDEs. When present, the dashed lines represent the boundaries of the region Σ_α (4.6).

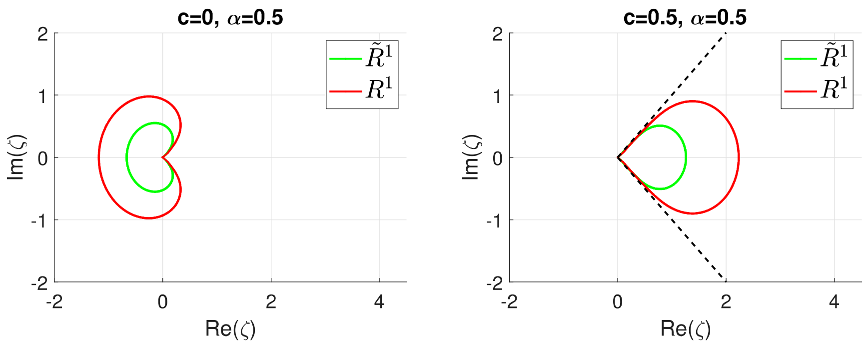

Figure 5.1: Regions of stability of spline collocation methods with $m = 1$ for FDEs (R^1) and for Abel-VIEs with $\tilde{\alpha} = 1 - \alpha$ ($\tilde{R}^1$). On the left, the stability regions lie within the plotted curves, while on the right they lie outside the boundaries. $\zeta = \lambda h^\alpha$ for FDEs, while $\zeta = \tilde{\lambda} h^{1-\tilde{\alpha}}$ for Abel-VIEs.

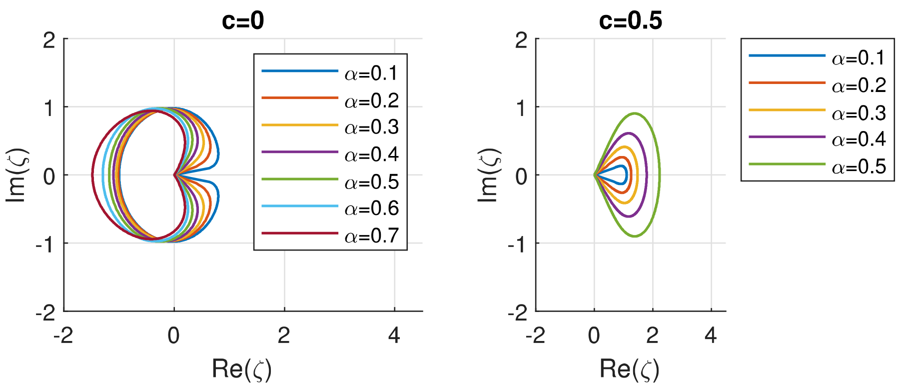

Figure 5.2: Regions of stability of method (2.9)-(2.10), with $m = 1$. $\zeta = \lambda h^\alpha$. On the left, $c = 0$ and $\alpha = 0.1, 0.2, \ldots, 0.7$; the stability regions lie within the plotted curves and the curves intersect the negative axis in decreasing values. On the right, $c = 0.5$ and $\alpha = 0.1, 0.2, \ldots, 0.5$; the stability regions lie outside the plotted curves and the curves intersect the positive axis in increasing values.

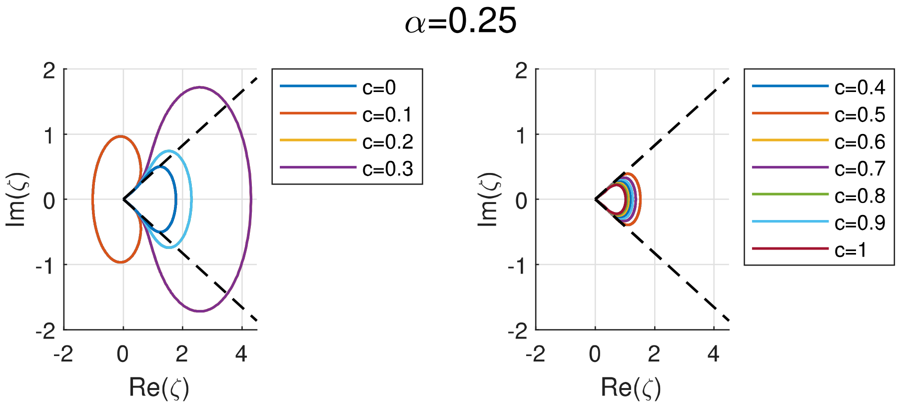

Figure 5.3: Regions of stability of method (2.9)-(2.10), with $m = 1$ and $\alpha = 0.25$. $\zeta = \lambda h^{\alpha}$. On the left $c = 0, 0.1, 0.2, 0.3$, on the right $c = 0.4, 0.5, \dots, 1$.

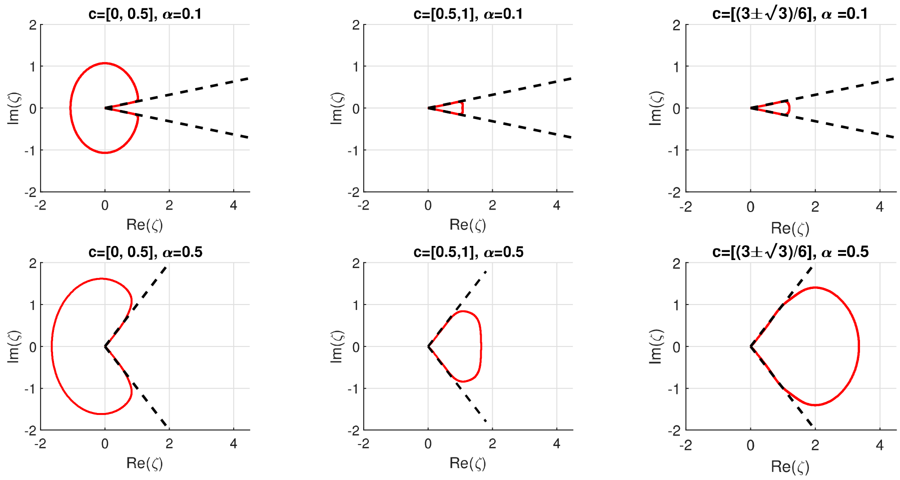

Figure 5.4: Regions of stability of method (2.9)-(2.10), with $m = 2$. $\zeta = \lambda h^{\alpha}$. On the top $\alpha = 0.1$, at the bottom $\alpha = 0.5$. In both lines, on the left $\mathbf{c} = [0, 0.5]$, in the middle $\mathbf{c} = [0.5, 1]$, on the right $\mathbf{c} = \left[\frac{3 \pm \sqrt{3}}{6}\right]$.

In Fig. 5.1 we compare the stability regions of spline collocation methods for Abel-VIEs and for FDEs. This figure confirms observations of Remark 2 and shows as, with suitable choices of c, it is possible to obtain $A(\alpha\frac{\pi}{2})$-stable methods.

In Figures 5.2 we fix the collocation abscissa, namely $c = 0$ on the left and $c = 0.5$ on the right, and let α vary in $[0, 1]$. For $c = 0$ and $\alpha = 0.1, \ldots, 0.9$ collocation methods have all bounded stability regions. For $c = 0.5$, collocation methods are $A\left(\alpha\frac{\pi}{2}\right)$-stable for $\alpha = 0.1, \ldots, 0.8$, while they have a bounded stability region for $\alpha = 0.9$.

In Fig. 5.3 we fix $\alpha = 0.25$ and let c vary in $[0, 1]$. The regions of stability increase monotonically with respect to c. In particular, collocation methods are $A(\alpha\frac{\pi}{2})$-stable for $c = 0.4, \ldots, 1$, $A(\theta)$-stable for $c = 0.1, 0.2, 0.3$, while the stability region is bounded for $c = 0$.

Fig. 5.4 illustrates the regions of stability of some spline collocation methods with $m = 2$, for $\alpha = 0.1$ and $\alpha = 0.5$. In both cases, collocation methods are $A\left(\alpha\frac{\pi}{2}\right)$-stable for $\mathbf{c} = [0.5, 1]$ and for $\mathbf{c} = \left[\frac{3\pm\sqrt{3}}{6}\right]$, while they have a bounded stability region when $\mathbf{c} = [0, 0.5]$.

We notice that the superconvergent method with $m = 1$ has $c = 0.5$ as collocation abscissa, while the superconvergent method with $m = 2$ has $\mathbf{c} = \left[\frac{3\pm\sqrt{3}}{6}\right]$. When $\alpha = 0.1$ and $\alpha = 0.5$, they are both $A\left(\alpha\frac{\pi}{2}\right)$-stable, as showed in Figures 5.2 and 5.4 .

6. Numerical experiments

Here we illustrate numerical experiments carried out on a selection of test equations. We measure the errors as the absolute error

$$error(t) = |y(t) - y_N(t)|, \qquad t \in [0, b],$$

and the relative error

$$E_\infty^{rel} = \frac{\max_{n=0,\ldots,N} |y(t_n) - y_N(t_n)|}{\max_{n=0,\ldots,N} |y(t_n)|},$$

where $y(t)$ is the analytical solution where available, otherwise it is replaced by a reference solution, and $y_N(t)$ is defined in (2.6).

Example 6.1. Basic test equation
The first problem is the basic test equation (4.5) *with* $\alpha = 0.5$, $\lambda = -10$, $y_0 = 1$ *and* $b = 20$. *This problem fulfills hypotheses of Th. 1.1 with* $\nu = 0.5$ *and arbitrary* $q \in \mathbb{N}$. *Therefore method* (2.9)-(2.10) *with* $m = 2$ *and arbitrary abscissae has error* $\|y_N - y\|_\infty \leq O(N^{-m})$ *when the grading exponent* $r \geq 4$ *(cfr.* (2.12)*). If* $m = 2$ *and* $\mathbf{c} = \left[\frac{3\pm\sqrt{3}}{6}\right]$ *the method is superconvergent and* $\|y_N - y\|_\infty \leq O(N^{-m-\alpha})$*, for* $r \geq 2.5$ *(cfr.[35, Th. 4.2]).*

We consider method (2.9)-(2.10) with $m = 2$ and $c = [0, 0.5]$, which has a bounded stability region, as showed in Fig. 5.4. If we apply this method with $h = 0.02$, $\lambda h^\alpha \approx -1.4142$ lies inside the stability region, while if we consider $h = 0.04$, $\lambda h^\alpha = -2$ lies outside the stability region. As expected, the numerical solution exhibits instability for $h = 0.04$, see on the top of Fig. 6.1.

In the case of graded meshes, we show as limitation (4.12) illustrated in Remark 3, is useful to avoid instabilities. As a matter of fact, when $h_{max} \approx 0.02$ the numerical solution behaves as the exact one, while when $h_{max} \approx 0.04$ an unstable behaviour is observed, see at the bottom of Fig. 6.1. In Fig. 6.2 we compare the absolute errors of spline collocation method with fixed stepsize $h = 0.02$ and graded mesh with $h_{max} \approx 0.02$: as expected the error is smaller when the graded mesh is adopted.

In the second set of experiments, we solve the basic test equation (4.5) with $\alpha = 0.5$, $\lambda = -1000$, $y_0 = 1$ and $b = 20$. We consider the superconvergent spline collocation method with $m = 2$, which has abscissae $\mathbf{c} = \left[\frac{3\pm\sqrt{3}}{6}\right]$. This method is $A(\alpha\frac{\pi}{2})$-stable, as showed in Fig. 5.4. In fact, even with large stepsize $h = 0.1$ and $h_{max} \approx 0.1$, no instability arises, see Fig. 6.3.

Example 6.2. Nonlinear fractional differential equation
The second problem is borrowed from [7]

$$D^\alpha y(t) = \lambda y + \rho y(1 - y^2) + g(t), \quad t \in (0, b], \qquad y(0) = y_0, \tag{6.1}$$

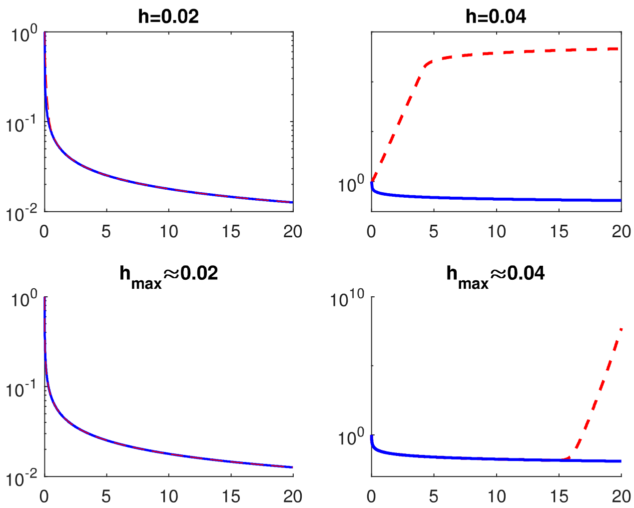

Figure 6.1: Numerical solution (dashed line) and exact solution (solid line) for test problem (4.5) with $\alpha = 0.5$, $\lambda = 10$, $y_0 = 1$ and $b = 20$. $m = 2$ and $\mathbf{c} = [0, 0.5]$. On the top the stepsize is fixed, while at the bottom a graded mesh is adopted, with $r = 4$. There is a logarithmic scale on the y-axis.

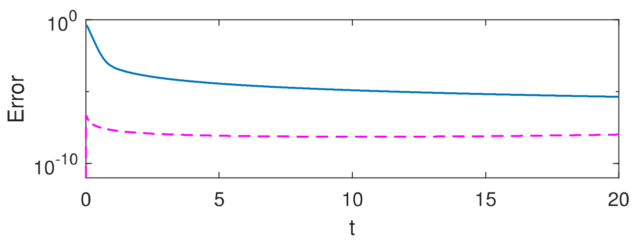

Figure 6.2: Errors of the collocation solution with $m = 2$ and $\mathbf{c} = [0, 0.5]$ for test problem (4.5) with $\alpha = 0.5$, $\lambda = -10$, $y_0 = 1$ and $b = 20$. Solid line: fixed stepsize $h = 0.2$; dashed line: graded mesh with $r = 4$ and $h_{max} \approx 0.02$. There is a logarithmic scale on the y-axis.

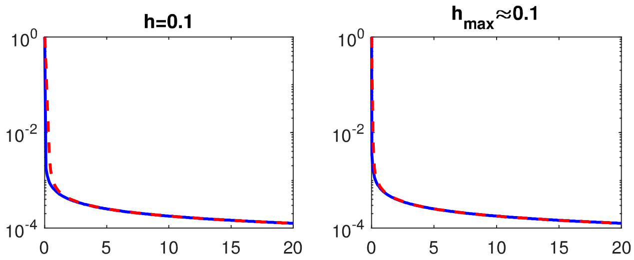

Figure 6.3: Numerical solution (dashed line) and exact solution (solid line) for test problem (4.5) with $\alpha = 0.5$, $\lambda = -1000$, $y_0 = 1$ and $b = 20$. $m = 2$ and $\mathbf{c} = \left[\frac{3\pm\sqrt{3}}{6}\right]$. On the left the stepsize is fixed, on the right a graded mesh is used, with $r = 2.5$. There is a logarithmic scale on the y-axis.

with $\alpha = 0.15$, $\lambda = -3$, $\rho = 0.8$, $b = 8$ *and* $y_0 = 2$.

- *Case I.* $g(t)$ *is chosen such that the solution of* (6.1) *is* $y(t) = y_0 + \sum_{k=1}^{6} t^{\sigma_k}$, *with* $\sigma_k = k\alpha$, $k = 1, \ldots, 5$, *and* $\sigma_6 = 2 + \alpha$.
- *Case II.* $g(t) = 0$. *The analytical solution is not known, thus for comparison we consider as reference solution the one obtained by applying method* (2.9)-(2.10) *with* $N = b/2^{-9}$.

In both cases, $f(t, y)$ fulfills assumptions of Th. 1.1 with $\nu = 1 - \alpha = 0.85$ and arbitrary $q \in \mathbb{N}$. If we apply method (2.9)-(2.10) with $m = 2$ and arbitrary abscissae, the method is convergent of order $O(N^{-m})$ when the grading exponent $r \geq 1.\bar{3}$. If $m = 2$ and collocation abscissae $\mathbf{c} = \left[\frac{3\pm\sqrt{3}}{6}\right]$, the method is superconvergent of order $O(N^{-(m+\alpha)})$, when the grading exponent $r \geq 7.1\bar{6}$.

We solve problem (6.1) by method (2.9)-(2.10) with $m = 2$ and h_{max} such that (4.12) holds. The results obtained with collocation abscissae $\mathbf{c} = \left[\frac{3\pm\sqrt{3}}{6}\right]$ are listed in Table 1. If we choose $\mathbf{c} = [0, 0.5]$, for the same choices of N, the numerical solution blows up.

7. Conclusions

We considered spline collocation methods for FDEs introduced in [35]. We reformulated the method by computing the fractional integrals and deriving a

Table 1: Relative error E_{∞}^{rel} of collocation method with $m = 2$, $\mathbf{c} = \left[\frac{3 \pm \sqrt{3}}{6}\right]$ and $r = 7.17$ on Example 6.2.

N	$h_{\max}$	Case I	Case II
$b/2^{-4}$	0.4375	2.71E-04	2.50E-05
$b/2^{-5}$	0.2214	6.28E-05	5.02E-06
$b/2^{-6}$	0.1114	1.43E-05	1.76E-06
$b/2^{-7}$	0.0558	3.25E-06	4.96E-07

matrix formulation. We analyzed the stability properties in a fixed stepsize framework, by exploiting the results on numerical stability of spline collocation methods for Abel-VIEs. We constructed several stability regions and found $A(\alpha\frac{\pi}{2})$-stable methods. Numerical experiments on significant test problems confirmed theoretical expectations and showed the usefulness of stability results also on nonlinear test problems and on graded meshes.

Acknowledgements

The authors are members of the INdAM Research group GNCS. This work is supported by GNCS-INDAM project and by PRIN2017-MIUR project.

References

[1] L. Blank, Stabilitätsanalyse der Kollokationsmethode für Volterra-Integral-Gleichungen mit schwach singulärem Kern, Ph.D. thesis, Universität Bonn, 1991.

[2] L. Blank, Stability of collocation for weakly singular Volterra equations, IMA J. Numer. Anal. 15 (1995) 357–375.

[3] L. Blank, Numerical treatment of differential equations of fractional order, Technical Report, University of Manchester, Department of Mathematics, 1996. Numerical Analysis Report.

[4] H. Brunner, Collocation methods for Volterra integral and related functional differential equations, volume 15 of *Cambridge Monographs on Applied and Computational Mathematics*, Cambridge University Press, Cambridge, 2004.

[5] H. Brunner, A. Pedas, G. Vainikko, Piecewise polynomial collocation methods for linear Volterra integro-differential equations with weakly singular kernels, SIAM J. Numer. Anal. 39 (2001) 957–982 (electronic).

[6] K. Burrage, A. Cardone, R. D'Ambrosio, B. Paternoster, Numerical solution of time fractional diffusion systems, Appl. Numer. Math. 116 (2017) 82–94.

[7] W. Cao, F. Zeng, Z. Zhang, G.E. Karniadakis, Implicit-explicit difference schemes for nonlinear fractional differential equations with nonsmooth solutions, SIAM J. Sci. Comput. 38 (2016) A3070–A3093.

[8] W. Cao, Z. Zhang, G.E. Karniadakis, Time-splitting schemes for fractional differential equations I: smooth solutions, SIAM J. Sci. Comput. 37 (2015) A1752–A1776.

[9] Y. Cao, T. Herdman, Y. Xu, A hybrid collocation method for Volterra integral equations with weakly singular kernels, SIAM J. Numer. Anal. 41 (2003) 364–381.

[10] G. Capobianco, D. Conte, B. Paternoster, Construction and implementation of two-step continuous methods for Volterra integral equations, Appl. Numer. Math. 119 (2017) 239–247.

[11] A. Cardone, D. Conte, Multistep collocation methods for Volterra integro-differential equations, Appl. Math. Comput. 221 (2013) 770–785.

[12] A. Cardone, D. Conte, R. D'Ambrosio, B. Paternoster, Collocation methods for Volterra integral and integro-differential equations: A review, Axioms 7 (2018).

[13] A. Cardone, D. Conte, B. Paternoster, A family of multistep collocation methods for Volterra integro–differential equations, AIP Conf. Proc. 1168 (2009) 358–361.

[14] A. Cardone, D. Conte, B. Paternoster, Two-step collocation methods for fractional differential equations, Discrete Contin. Dyn. Syst. Ser. B 23 (2018) 2709–2725.

[15] D. Conte, R. D'Ambrosio, Z. Jackiewicz, B. Paternoster, Numerical search for algebraically stable two-step almost collocation methods, J. Comput. Appl. Math. 239 (2013) 304–321.

[16] D. Conte, B. Paternoster, A family of multistep collocation methods for Volterra integral equations, AIP Conf. Proc. 936 (2007) 128–131.

[17] D. Conte, B. Paternoster, Multistep collocation methods for Volterra integral equations, Appl. Numer. Math. 59 (2009) 1721–1736.

[18] M. Di Paola, A. Pirrotta, A. Valenza, Visco-elastic behavior through fractional calculus: an easier method for best fitting experimental results, Mech. Mater. 43 (2011) 799–806.

[19] K. Diethelm, The analysis of fractional differential equations, volume 2004 of *Lecture Notes in Mathematics*, Springer-Verlag, Berlin, 2010. An application-oriented exposition using differential operators of Caputo type.

[20] L. Galeone, R. Garrappa, Explicit methods for fractional differential equations and their stability properties, J. Comput. Appl. Math. 228 (2009) 548–560.

[21] E.A. Galperin, E.J. Kansa, A. Makroglou, S.A. Nelson, Variable transformations in the numerical solution of second kind Volterra integral equations with continuous and weakly singular kernels; extensions to Fredholm integral equations, J. Comput. Appl. Math. 115 (2000) 193–211.

[22] R. Garrappa, On linear stability of predictor-corrector algorithms for fractional differential equations, Int. J. Comput. Math. 87 (2010) 2281–2290.

[23] R. Garrappa, Trapezoidal methods for fractional differential equations: theoretical and computational aspects, Math. Comput. Simulation 110 (2015) 96–112.

[24] R. Hilfer (Ed.), Applications of fractional calculus in physics, World Scientific Publishing Co., Inc., River Edge, NJ, 2000.

[25] M.M. Khader, On the numerical solutions for the fractional diffusion equation, Commun. Nonlinear Sci. Numer. Simul. 16 (2011) 2535–2542.

[26] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, Theory and applications of fractional differential equations, volume 204 of *North-Holland Mathematics Studies*, Elsevier Science B.V., Amsterdam, 2006.

[27] X. Li, Numerical solution of fractional differential equations using cubic B-spline wavelet collocation method, Commun. Nonlinear Sci. Numer. Simul. 17 (2012) 3934–3946.

[28] C. Lubich, A stability analysis of convolution quadratures for Abel-Volterra integral equations, IMA J. Numer. Anal. 6 (1986) 87–101.

[29] J. Ma, H. Liu, Fractional collocation boundary value methods for the second kind Volterra equations with weakly singular kernels, Numer. Algorithms 84 (2020) 743–760.

[30] F. Mainardi, Fractional calculus and waves in linear viscoelasticity, Imperial College Press, London, 2010. An introduction to mathematical models.

[31] B. Mandelbrot, J. Van Ness, Fractional Brownian motions, fractional noises and applications, SIAM Rev. 10 (1968) 422–437.

[32] C. Necula, Option pricing in a fractional brownian motion environment, Available at SSRN 1286833 (2002).

[33] A. Pedas, E. Tamme, On the convergence of spline collocation methods for solving fractional differential equations, J. Comput. Appl. Math. 235 (2011) 3502–3514.

[34] A. Pedas, E. Tamme, Spline collocation methods for linear multi-term fractional differential equations, J. Comput. Appl. Math. 236 (2011) 167–176.

[35] A. Pedas, E. Tamme, Numerical solution of nonlinear fractional differential equations by spline collocation methods, J. Comput. Appl. Math. 255 (2014) 216–230.

[36] A. Pedas, E. Tamme, Spline collocation for nonlinear fractional boundary value problems, Appl. Math. Comput. 244 (2014) 502–513.

[37] A. Pedas, G. Vainikko, Smoothing transformation and piecewise polynomial collocation for weakly singular Volterra integral equations, Computing 73 (2004) 271–293.

[38] P. Perdikaris, G. Karniadakis, Fractional-order viscoelasticity in one-dimensional blood flow models, Annals of Biomedical Engineering 42 (2014) 1012–1023.

[39] I. Podlubny, Fractional differential equations, volume 198 of *Mathematics in Science and Engineering*, Academic Press, Inc., San Diego, CA, 1999. An introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications.

[40] E.A. Rawashdeh, Numerical solution of fractional integro-differential equations by collocation method, Appl. Math. Comput. 176 (2006) 1–6.

[41] L. Tao, H. Yong, Extrapolation method for solving weakly singular nonlinear Volterra integral equations of the second kind, J. Math. Anal. Appl. 324 (2006) 225–237.

[42] P. Torvik, R. Bagley, On the appearance of the fractional derivative in the behavior of real materials, J. Appl. Mech. 51 (1984) 294–298.

[43] W. Wang, X. Chen, D. Ding, S.L. Lei, Circulant preconditioning technique for barrier options pricing under fractional diffusion models, Int. J. Comput. Math. 92 (2015) 2596–2614.

[44] M. Zayernouri, G.E. Karniadakis, Fractional spectral collocation method, SIAM J. Sci. Comput. 36 (2014) A40–A62.