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Higher order elliptic equations in weighted Banach spaces

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Abstract

We consider m-th order linear, uniformly elliptic equations $\mathcal{L}u = f$ with non-smooth coefficients in Banach–Sobolev spaces $W^m_{X_w}(\Omega)$ generated by weighted Banach Function Spaces (BFS) $X_w(\Omega)$ on a bounded domain $\Omega \subset \mathbb{R}^n$. Supposing boundedness of the Hardy–Littlewood Maximal operator and the Calderón–Zygmund singular integrals in $X_w(\Omega)$ we obtain solvability in the small in $W^m_{X_w}(\Omega)$ and establish interior Schauder type a priori estimates. These results will be used in order to obtain Fredholmness of the operator $\mathcal{L}$ in $X_w(\Omega)$.

Keywords Elliptic equations · Banach Function Spaces · Schauder type estimates · Fredholmness

Mathematics Subject Classification Primary: 35K05; Secondary: 35A01 · 35J05

1 Introduction

The question of solvability in the small and Schauder type estimates play an essential role in the theory of the Fredholmness of boundary value problems for elliptic equations in appropriate BFS. There is a vast number of papers and monographs dedicated to the Fredholmness of Partial Differential Equations (PDEs) in the frame of classical function classes as the Hölder and Sobolev spaces. The appearance of new function spaces and their intensive study had a very big impact on the regularity and existence theory for PDEs. The so-called non standard function spaces turn to be interesting not

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only from theoretical point of view but also for the applied mathematics and mathematical physics. Among the most studied spaces we can find the Morrey $L^{p,\lambda}$ spaces, the weighted Lebesgue spaces and various generalizations, the variable Lebesgue spaces $L^{p(\cdot)}$, the grand Lebesgue spaces $L^{p)}$, the Orlicz spaces L_Φ and many others (see [1, 4, 5, 7, 15, 16, 19–22] and the references therein).

In the present paper we are interested on elliptic differential operators of higher order in weighted general BFS $X_w(\Omega)$ with Muckenhoupt type weight w in a bounded domain $\Omega \subset \mathbb{R}^n$ with a smooth enough boundary. In order to obtain the desired a priori estimates we need that the Hardy–Littlewood maximal and the Calderón–Zygmund singular operators are bounded in $X_w(\Omega)$. Then under certain conditions on the coefficients we obtain solvability in the small, that is local solvability, in Sobolev spaces $W^m_{X_w}(\Omega)$ generated by weighted BFS $X_w(\Omega)$.

More over, we obtain interior Schauder type inequalities in $X_w(\Omega)$ for the solutions of the linear uniformly elliptic equations under consideration. Such estimates play an exceptional role in the establishing of the Fredholmness of the corresponding elliptic operators. At the end, some examples of BFS are given.

In what follows we use the standard notation:

- $\Omega \subset \mathbb{R}^n$ is a bounded domain with Lebesgue measure $|\Omega|$, $d_\Omega = \operatorname{diam}\Omega$, $\mathcal{B}_r(x) = \{y \in \mathbb{R}^n : |x-y| < r\}$, $\mathcal{B}_r \equiv \mathcal{B}_r(0)$, $\Omega_r(x_0) = \Omega \cap \mathcal{B}_r(x_0)$, $\Omega_r = \Omega_r(0)$;
- $[X; Y]$ is the Banach space of bounded operators acting from X to Y endowed by the operator norm $\|\cdot\|_{[X;Y]}$, $[X] = [X; X]$;
- χ_E stands for the characteristic function of the set $E \subset \mathbb{R}^n$;
- $\alpha = (\alpha_1, \ldots, \alpha_n)$, is a multi-index, $\alpha_i \geq 0$ and $|\alpha| = \sum_{i=1}^n \alpha_i$;

$$D_{x_i}u = D_i u = \partial u/\partial x_i, \quad D^\alpha u = D^{\alpha_1}_{x_1} D^{\alpha_2}_{x_2} \ldots D^{\alpha_n}_{x_n} u;$$

- for any $u : \Omega \to \mathbb{R}$ denote $D^k u = \sum_{|\alpha|=k} D^\alpha u$;
- For any normed space X we accept the notion

$$\|D^k u\|_{X(\Omega)} = \sum_{|\alpha|=k} \|D^\alpha u\|_{X(\Omega)};$$

- $F(\Omega)$ is the set of Lebesgue measurable functions in Ω;
- $\overline{\overline{o}}(1)$ stands for a quantity that tends to 0 as $r \to 0$;

2 Definitions and auxiliary results

The question of existence and regularity of the solutions of linear elliptic equations is strongly related to the question of continuity of certain integral operators in the corresponding function spaces. For this goal we recall the definitions of these operators and introduce the function spaces that we are going to use.

Let $f \in L^1(\mathbb{R}^n)$ and $\mathcal{M}f$ be the Hardy–Littlewood maximal operator

$$\mathcal{M}f(x) = \sup_{r>0} \frac{1}{|\mathcal{B}_r(x)|} \int_{\mathcal{B}_r(x)} |f(y)|\,dy, \tag{2.1}$$

and $\mathcal{K}f$ be the Calderón–Zygmund integral operator

$$\mathcal{K}f(x) = p.v. \int_{\mathbb{R}^n} \frac{\omega(y)}{|x-y|^n} f(y)\,dy. \tag{2.2}$$

For each $\gamma \in (0, n)$ we consider the Riesz potential

$$\mathcal{R}_\gamma f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\gamma}}\,dy. \tag{2.3}$$

Simple calculations show that we can estimate the Riesz potential via the maximal function (see [6])

$$\int_{|y-x|\le\delta} \frac{|f(y)|\,dy}{|y-x|^{n-\gamma}} \le c\delta^\gamma \mathcal{M}f(x), \qquad \forall\, x \in \mathbb{R}^n. \tag{2.4}$$

Following [2], we define the BFS and give some of its properties.

Definition 2.1 Let $F(\mathbb{R}^n)$ be the set of Lebesgue measurable functions on $\mathbb{R}^n$. A mapping $\|\cdot\|_X : F(\mathbb{R}^n) \to [0, \infty]$ is called *Banach function norm* if for all $f, g, f_n \in F(\mathbb{R}^n)$, $n = 1, 2, \ldots$, for any constant $a \in \mathbb{R}$ and for any Lebesgue measurable set $E \subset \mathbb{R}^n$ the following properties hold:

(P1) $\|f\|_X = 0 \iff f = 0$ a.e. in $\mathbb{R}^n$,
$\|af\|_X = |a|\|f\|_X, \quad \|f+g\|_X \le \|f\|_X + \|g\|_X;$
(P2) if $0 \le g \le f$ then $\|g\|_X \le \|f\|_X$;
(P3) if $0 \le f_n \uparrow f$ a.e. in $\mathbb{R}^n$, then $\|f_n\|_X \uparrow \|f\|_X$;
(P4) if $|E| < \infty$ then $\|\chi_E\|_X < \infty$;
(P5) if $|E| < \infty$ then $\int_E |f(x)|\,dx \le c\|f\|_X$, with a constant independent of f.

The collection of all functions $f \in F(\mathbb{R}^n)$ for which $\|f\|_X < \infty$ is called Banach Function Space $X(\mathbb{R}^n)$.

We can extend the Definition 2.1 to BFSs $X(\Omega)$ builded on a bounded domain Ω with $|\Omega| > 0$ and regular boundary $\partial\Omega$ by taking $f \in F(\Omega)$ and extending it as zero out of Ω.

It follows immediately, by $(P5)$ that $X(\Omega) \subset L^1(\Omega)$. The associated space $X'(\Omega)$, endowed with the associated norm $\|\cdot\|_{X'}$, consists of all $g \in F(\Omega)$, such that (cf. [2])

$$\|g\|_{X'(\Omega)} = \sup\left\{ \int_\Omega |f(x)g(x)|\,dx : \|f\|_{X(\Omega)} \le 1 \right\} < \infty.$$

At this point we can extend the Hölder inequality in the case of BFSs.

Proposition 2.2 *Let $X(\Omega)$ and $X'(\Omega)$ be associated BFSs. If $f \in X(\Omega)$ and $g \in X'(\Omega)$, then $fg \in L^1(\Omega)$ and*

$$\int_\Omega |f(x)g(x)|\,dx \leq \|f\|_{X(\Omega)}\|g\|_{X'(\Omega)}. \tag{2.5}$$

Let w be a *positive weight, that is a Lebesgue measurable function on Ω for which* $0 < w(x) < \infty$. Then we can define $X_w(\Omega)$ as the space of all $f \in F(\Omega)$ for which $fw \in X(\Omega)$, that is

$$\|f\|_{X_w(\Omega)} = \|fw\|_{X(\Omega)} < \infty.$$

In what follows we assume that $w \in X(\Omega)$ and $w^{-1} \in X'(\Omega)$.

The definition of general Sobolev BFS follows naturally from the definition of the classical Sobolev spaces. By $W^m_{X_w}(\Omega)$ we denote the Sobolev space of all functions $f \in X_w(\Omega)$ differentiable in distributional sense up to order m, that is

$$W^m_{X_w}(\Omega) = \{D^\alpha f \in X_w(\Omega), \quad \forall \alpha : 0 \leq |\alpha| \leq m\},$$

endowed by the norm

$$\|f\|_{W^m_{X_w}(\Omega)} = \sum_{|\alpha|\leq m} \|D^\alpha f\|_{X_w(\Omega)}. \tag{2.6}$$

We denote by $\overset{\circ}{W}{}^m_{X_w}(\Omega)$ the space of $W^m_{X_w}(\Omega)$-functions compactly supported in Ω, endowed with the same norm (2.6).

In the case when $\Omega \equiv \mathcal{B}_r(0)$ we simplify the notion, writing $X_w(r)$ and $W^m_{X_w}(r)$ instead of $X_w(\mathcal{B}_r)$ and $W^m_{X_w}(\mathcal{B}_r)$.

In our further considerations, we are going to use the following norm

$$\|f\|_{W^m_{X_w;d_\Omega}(\Omega)} = \sum_{|\alpha|\leq m} d_\Omega^{|\alpha|}\|D^\alpha f\|_{X_w(\Omega)}, \tag{2.7}$$

which is equivalent of (2.6).

To be able to adapt the classical techniques from the L^p-theory to nonstandard function spaces we assume that that space $X_w(\Omega)$ and the weight function w possess the following properties.

Property 1 Let $X_w(\Omega)$ be a weighted BFS, suppose that w is such that the following inclusions hold:

(*A*) The operators (2.1) and (2.2) are bounded in $X_w(\Omega)$, i.e. $\mathcal{M}, \mathcal{K} \in [X_w(\Omega)]$ and the estimates hold

$$\|\mathcal{M}f\|_{X_w(\Omega)} \leq c\|f\|_{X_w(\Omega)}, \quad \|\mathcal{K}f\|_{X_w(\Omega)} \leq c\|f\|_{X_w(\Omega)} \tag{2.8}$$

with constants independent of f.

(B) There exists $p_0 \in (1, +\infty)$ such that

$$X_w(\Omega) \subset L^p(\Omega), \qquad \forall\, p \in [1, p_0]. \tag{2.9}$$

Property 2 For any bounded domain Ω with $\partial\Omega \in C^m$, the Sobolev–Banach space $W^m_{X_w}(\Omega)$ has the *extension property*. This means that for each domain Ω' such that $\Omega \Subset \Omega'$, there exists a linear bounded operator, called *extension operator*, such that

$$\begin{cases} \theta : W^k_{X_w}(\Omega) \to W^k_{X_w}(\Omega'), & \theta f\big|_{\Omega} = f, \\ \|\theta f\|_{W^k_{X_w}(\Omega')} \le c\|f\|_{W^k_{X_w}(\Omega)}, & \forall\, k = 0, \dots, m. \end{cases} \tag{E}$$

Property 3 For a given space $X_w(\Omega)$ we suppose that one of the following conditions hold.

Condition 1 There exist $p_1 \in (1, \infty)$ and a constant $c > 0$, such that for all disjoint partition of Ω: $\Omega = \bigcup_k \Omega_k$, $\Omega_k \cap \Omega_j = \emptyset$, if $k \ne j$, and for all $1 \le p \le p_1$, it holds

$$\|f\|_{X(\Omega)} \le c \left(\sum_k \|f\|^p_{X(\Omega_k)} \right)^{1/p} \qquad \forall\, f \in X(\Omega).$$

Condition 2 There exists $p_2 \in (1, \infty)$ such that

$$L^p(\Omega) \subset X_w(\Omega), \qquad \forall\, p \in [p_2, \infty).$$

Remark 2.3 Let us note that the Condition 1 is verified if $X(\Omega) \equiv L^p(\Omega)$.

Concerning the Condition 2 we can construct the following example. Let w be a positive weight belonging to $L^{1+0}(\Omega) = \cup_{\delta>0} L^{1+\delta}(\Omega)$, and consider the space $X_w(\Omega) \equiv L^{p(\cdot)}_w(\Omega)$ with $p(\cdot)$ measurable, verifying

$$1 < p_+ \le p_+ < \infty, \qquad p_- = \operatorname*{essinf}_{\Omega} p(x), \quad p_+ = \operatorname*{esssup}_{\Omega} p(x). \tag{2.10}$$

There exists $\alpha > 1$ such that $w \in L^\alpha(\Omega)$ and if we denote $p_1(\cdot) = \alpha' p(\cdot)$, then

$$p_1^- = \alpha' p^-, \qquad p_1^+ = \alpha' p^+$$

and the following continuous embeddings hold

$$L^{p_1^+}(\Omega) \hookrightarrow L^{p_1(\cdot)}(\Omega) \hookrightarrow L^{p_1^-}(\Omega). \tag{2.11}$$

Applying the Hölder inequality we obtain

$$\int_\Omega |f(x)|^{p(x)} w(x)\, dx \le \|w\|_{L^\alpha(\Omega)} \left(\int_\Omega |f(x)|^{p_1(x)}\, dx \right)^{\frac{1}{\alpha'}}$$

that implies $L^{p_1^+}(\Omega) \hookrightarrow L_w^{p(\cdot)}(\Omega)$ and hence

$$L^p(\Omega) \hookrightarrow L_w^{p(\cdot)}(\Omega) \qquad \forall\, p \in [p_1^+, \infty).$$

In addition, the classical weighted Lebesgue spaces $L_w^p(\Omega)$, $p \in (1, \infty)$, $w \in A_p(\Omega)$ verify the Properties 1, 2, and 3 (cf. [8].

It is easy to see that the (E) holds for any weighted space $X_w(\Omega)$, just extending the functions as zero out of Ω, while for the Sobolev spaces it is not so obvious. This property is well-known in the case of classical Sobolev spaces $W^{p,m}(\Omega)$, $p \in (1, +\infty)$ (see [3]) and weighted Sobolev spaces $W_w^{p,m}(\Omega)$ with a Muckenhoupt weight (see [8, 10, 11]). In the our case, we assert that the extension property holds for all $k = 0, \ldots, m$,

Concerning the Conditions 1 and 2 they are not mutually exclusive and we suppose that for a given BFS at least one of them holds true.

3 Statement of the problem

We consider the m-order linear differential operator

$$\mathcal{L}(x, D) = \sum_{|\alpha| \le m} a_\alpha(x) D^\alpha, \tag{3.1}$$

with m being *even number*.

The operator $\mathcal{L}(x, D)$ is *uniformly elliptic* that is, there exist positive constants λ and Λ, such that

$$\lambda |\xi|^m \le \sum_{|\alpha| = m} a_\alpha(x) \xi^\alpha \le \Lambda |\xi|^m, \quad \text{for a.a. } x \in \Omega,\ \forall \xi \in \mathbb{R}^n. \tag{3.2}$$

We say that the operator $\mathcal{L}$ satisfies in $x_0 \in \Omega$ the (P_{x_0})-property if there exists a ball $\mathcal{B}_r(x_0) \Subset \Omega$ and functions $g_\alpha \in L^\infty(\mathcal{B}_r(x_0))$, $|\alpha| = m$, continuous in x_0, such that

$$\begin{cases} a_\alpha(x) = g_\alpha(x) & \text{for a.a. } x \in \mathcal{B}_r(x_0) \setminus \{x_0\}, \\ \|g_\alpha - g_\alpha(x_0)\|_{L^\infty(\mathcal{B}_r(x_0))} \to 0 & \text{as } r \to 0,\ \forall |\alpha| = m, \\ a_\alpha \in L^\infty(\mathcal{B}_r(x_0)) & \forall \alpha : |\alpha| < m. \end{cases} \tag{P_{x_0}}$$

Let $x_0 \in \Omega$ be a point in which (3.2) and (P_{x_0}) hold. Consider the *tangential operator*

$$L_{x_0} = \sum_{|\alpha| = m} a_\alpha(x_0) D^\alpha \tag{3.3}$$

where under the value $a_\alpha(x_0)$ we understand the value of $g_\alpha(x_0)$ defined by (P_{x_0}). The *fundamental solution* J_{x_0} of the equation $L_{x_0}\varphi = 0$ is called a *parametrix* for the equation $\mathcal{L}\varphi = 0$, having a singularity at the point x_0. By the properties of the fundamental solution, for any multiindex β we have

$$\begin{aligned}
& i)\ D^\beta J_{x_0}(x) \in C^\infty(\mathbb{R}^n \setminus \{0\}), \quad \forall\, \beta,\\
& ii)\ D^\beta J_{x_0}(\mu x) = \mu^{-n} D^\beta J_0(x), \quad \forall\, \beta : |\beta| = m,\\
& iii)\ \int_{\mathbb{S}^{n-1}} D^\beta J_{x_0}(\xi)\, d\sigma_\xi = 0, \quad \forall\, \beta : |\beta| = m,\\
& iv)\ |D^\beta J_{x_0}(x)| \le C|x|^{m-n-|\beta|}, \qquad |\beta| = 0, 1, 2, \ldots, m.
\end{aligned} \tag{3.4}$$

These properties ensure that the m-order derivatives of J_{x_0} are *Calderón–Zygmund kernels* (see [18, 19] and the references therein for more details). It is well know, from the classical theory, that for any function $\varphi \in C_0^m(\Omega)$ the following representation holds

$$\varphi(x) = \int_\Omega J_{x_0}(x-y)(L_{x_0} - \mathcal{L})\varphi(y)\, dy + \int_\Omega J_{x_0}(x-y)\mathcal{L}\varphi(y)\, dy. \tag{3.5}$$

Introducing the operators

$$\mathcal{S}_{x_0}\varphi(x) = \int_\Omega J_{x_0}(x-y)\varphi(y)\, dy \tag{3.6}$$

$$\begin{aligned}
\mathcal{T}_{x_0}\varphi(x) &= \mathcal{S}_{x_0}(L_{x_0} - \mathcal{L})\varphi(x)\\
&= \int_\Omega J_{x_0}(x-y) \sum_{|\alpha|=m} (a_\alpha(x_0) - a_\alpha(y)) D^\alpha \varphi(y)\, dy\\
&\quad - \int_\Omega J_{x_0}(x-y) \sum_{|\alpha|<m} a_\alpha(y) D^\alpha \varphi(y)\, dy
\end{aligned} \tag{3.7}$$

we can rewrite (3.5) in the form

$$\varphi(x) = \mathcal{T}_{x_0}\varphi(x) + \mathcal{S}_{x_0}\mathcal{L}\varphi(x).$$

Our goal is to show that $\mathcal{T}_{x_0} + \mathcal{S}_{x_0}\mathcal{L}$ is identity operator in $W^m_{X_w}(\Omega)$. Let us calculate the m-order derivatives of $\mathcal{S}_{x_0}\varphi$ (cf. [3, 12, 21])

$$D^\beta \mathcal{S}_{x_0}\varphi(x) = \int_\Omega D_x^\beta J_{x_0}(x-y)\varphi(y)\, dy + c\varphi(x) = \mathcal{K}_\beta\varphi(x) + c\varphi(x) \tag{3.8}$$

with $|\beta| = m$. Hence $\mathcal{K}_\beta\varphi$ are Calderón–Zygmund integrals satisfying (2.8). Calculating the lower order derivatives we observe that they have weak singularity and can be treated as Riesz potential (2.3).

Lemma 3.1 (Main Lemma) *Let the Property 1 and condition (P_{x_0}) hold at some point $x_0 \in \Omega$. If $\varphi \in W^m_{X_w;r}(\mathcal{B}_r(x_0))$ has a compact support in $\mathcal{B}_r(x_0) \Subset \Omega$, then*

$$\|\mathcal{T}_{x_0}\varphi\|_{W^m_{X_w;r}(\mathcal{B}_r(x_0))} \le \sigma(r)\|\varphi\|_{W^m_{X_w;r}(\mathcal{B}_r(x_0))},$$

where $\sigma(r) \to 0$ as $r \to 0$ and it depends on the coefficients of $\mathcal{L}$, but not on φ.

Proof Following [4] we assume that $x_0 = 0$ and simplify the notation writing $\mathcal{S}_0$, L_0 and $\mathcal{T}_0$. Let $n \geq 3$ be an odd number, in case of even dimension it can be introduced a fictitious new variable and extend all functions as constants along the new variable. Take an arbitrary $\varphi \in W^m_{X_w}(r)$ with a compact support in $\mathcal{B}_r$, then

$$\begin{aligned}(L_0 - \mathcal{L})\varphi(x) = &\sum_{|\alpha|=m} (a_\alpha(0) - a_\alpha(x))D^\alpha \varphi(x) \\ &- \sum_{|\alpha|<m} a_\alpha(x) D^\alpha \varphi(x) =: \psi_1(x) - \psi_2(x).\end{aligned} \tag{3.9}$$

Hence

$$\begin{aligned}\mathcal{T}_0\varphi(x) &= \mathcal{S}_0\psi_1(x) + \mathcal{S}_0\psi_2(x) \\ &= \sum_{|\alpha|=m} \int_\Omega J_0(x-y)\big(a_\alpha(0) - a_\alpha(y)\big) D^\alpha\varphi(y)\, dy \\ &- \sum_{|\alpha|<m} \int_\Omega J_0(x-y) a_\alpha(y) D^\alpha \varphi(y)\, dy.\end{aligned} \tag{3.10}$$

Since $\|a_\alpha(0) - a_\alpha\|_{L^\infty(r)} \to 0$ as $r \to 0$, we have

$$\|\psi_1\|_{X_w(r)} < \bar{\bar{o}}(1)\|D^m\varphi\|_{X_w(r)}, \quad \|\psi_2\|_{X_w(r)} < c\sum_{k<m} \|D^k \varphi\|_{X_w(r)}. \tag{3.11}$$

In order to estimate the Sobolev–Banach norm of (3.10) we need to calculate the derivatives up to order m. Let β be a multi-index and for $|\beta| < m$ we have

$$D^\beta \mathcal{T}_0 \varphi(x) = \int_{\mathcal{B}_r} D^\beta_x J_0(x-y)\psi(y)\, dy.$$

By (3.4) we have

$$|D^\beta \mathcal{T}_0\varphi(x)| \leq c \int_{\mathcal{B}_r} |x-y|^{m-n-|\beta|} |\psi(y)|\, dy = c\mathcal{R}_\gamma|\psi(x)|,$$

where $\mathcal{R}_\gamma$ is defined by (2.3) with $\gamma = m - |\beta|$. Applying (2.4) and Property (1) we obtain

$$\|D^\beta \mathcal{T}_0\varphi\|_{X_w(r)} \leq cr^\gamma \|\mathcal{M}\psi\|_{X_w(r)} \leq cr^\gamma\|\psi\|_{X_w(r)}.$$

Making use of (3.11) we get

$$\begin{aligned}
r^{|\beta|}\|D^\beta \mathcal{T}_0\phi\|_{X_w(r)} &\le cr^m\|\psi\|_{X_w(r)} \le cr^m\big(\|\psi_1\|_{X_w(r)} + \|\psi_2\|_{X_w(r)}\big)\\
&\le c\Big(\overline{\overline{o}}(1)r^m\|D^m\varphi\|_{X_w(r)} + \sum_{k\le m-1} r^{m-k}r^k\|D^k\varphi\|_{X_w(r)}\Big)\\
&\le c\Big(\overline{\overline{o}}(1)r^m\|D^m\varphi\|_{X_w(r)} + r\sum_{k\le m-1} r^k\|D^k\varphi\|_{X_w(r)}\Big)\\
&\le \sigma(r)\|\varphi\|_{W^m_{X_w;r}(r)},
\end{aligned} \tag{3.12}$$

for r small enough and $\sigma(r)$ vanishing function as $r \to 0$.

Consider now the case $|\beta| = m$. By (2.9) it follows that $W^m_{X_w}(r) \subset W^{1,m}(r)$ and therefore, it holds (cf. [3])

$$D^\beta \mathcal{T}_0\varphi(x) = \int_{\mathcal{B}_r} D^\beta J_0(x-y)\psi(y)\,dy + c\psi(x) \quad \text{for a.a. } x \in \mathcal{B}_r \tag{3.13}$$

where c depends on known quantities but not on r and ψ. By the properties of the kernel it follows that $D^\beta J_0$ is singular for each $|\beta| = m$ and therefore by (2.8) and from (3.13) we have

$$\|D^\beta \mathcal{T}_0\varphi\|_{X_w(r)} \le c\|\psi\|_{X_w(r)}.$$

Hence the following estimate holds

$$\begin{aligned}
r^m\|D^\beta \mathcal{T}_0\varphi\|_{X_w(r)} &\le cr^m\|\psi\|_{X_w(r)} \le cr^m\big(\|\psi_1\|_{X_w(r)} + \|\psi_2\|_{X_w(r)}\big)\\
&\le c\left(\overline{\overline{o}}(1)\sum_{|\alpha|=m} r^m\|D^\alpha\varphi\|_{X_w(r)} + \sum_{|\alpha|<m} r^{m-|\alpha|}r^{|\alpha|}\|D^\alpha\varphi\|_{X_w(r)}\right)\\
&\le c\left(\overline{\overline{o}}(1)\sum_{|\alpha|=m} r^m\|D^\alpha\varphi\|_{X_w(r)} + r\sum_{|\alpha|<m} r^{|\alpha|}\|D^\alpha\varphi\|_{X_w(r)}\right)\\
&\le \sigma(r)\|\varphi\|_{W^m_{X_w;r}(r)}.
\end{aligned}$$

Taking into account (3.12) we obtain

$$\|\mathcal{T}_0\varphi\|_{W^m_{X_w;r}(r)} \le \sigma(r)\|\varphi\|_{W^m_{X_w;r}(r)}, \qquad \sigma(r) \to 0 \text{ as } r \to 0.$$

□

4 Local existence of strong solutions

In the next section we are going to obtain some local results concerning solvability in weighted Sobolev–Banach space builded upon X_w.

Lemma 4.1 *Let* (3.2), *(*P_{x_0}*), the Property* 1 *and the condition (E) hold true in a ball* $\mathcal{B}_r(x_0) \Subset \Omega$. *Then*

(1) *If* $\varphi \in W^m_{X_w}(\mathcal{B}_r(x_0))$, *then* $\varphi = \mathcal{T}_{x_0}\varphi + \mathcal{S}_{x_0}\mathcal{L}\varphi$.
(2) *If for some* $f \in X_w(\mathcal{B}_r(x_0))$ *it holds* $\varphi = \mathcal{T}_{x_0}\varphi + \mathcal{S}_{x_0} f$, *then* φ *is a strong solution of the equation* $\mathcal{L}\varphi(x) = f(x)$ *for almost all* $x \in \mathcal{B}_r(x_0)$.

Proof (1) Without loss of generality we may assume that $x_0 = 0 \in \Omega$. Because of (2.9) there exists some $p_0 > 1$ such that $W^m_{X_w}(\Omega) \subset W^{p_0,m}(\Omega)$ continuously.

Since $\varphi \in W^{p_0,m}(r)$ than we can consider $\varphi \in \overset{\circ}{W}{}^{m,p_0}(r+\delta)$ for some $\delta > 0$ small enough and $\varphi = \mathcal{T}_0\varphi + \mathcal{S}_0\mathcal{L}\varphi$, (cf. [3, Lemma A]), i.e. $\mathcal{I}d = \mathcal{T}_0 + \mathcal{S}_0\mathcal{L}$, where $\mathcal{I}d$ is the *identity operator* in $W^{p_0,m}(r)$. Then

$$\mathcal{T}_0\varphi = \mathcal{S}_0(\mathcal{L} - L_0)\varphi = \mathcal{S}_0\psi = \int_{\mathcal{B}(r+\delta)} J_0(x-y)\psi(y)\,dy, \tag{4.1}$$

where $\psi = (\mathcal{L} - L_0)\varphi$ and $x \in \mathcal{B}_{r+\delta}$. It is clear that $\psi \in L^{p_0}(r+\delta)$ and from the classical theory (cf. [3, 14]) we have the representation formula

$$D^\beta \mathcal{T}_0\varphi(x) = \int_{\mathcal{B}_{r+\delta}} D^\beta_x J_0(x-y)\psi(y)\,dy + C\psi(x), \tag{4.2}$$

for each $|\beta| = m$ and C being a positive constant independent of ψ.

Since $D^m_x J_0$ is a Calderón–Zygmund type kernel, then from the boundedness of the singular operators in $L^{p_0}(r+\delta)$ it follows that the formula (4.2) holds also for $\psi \in L^{p_0}(r+\delta)$. By the Property 1 and (2.8) it follows that (4.2) is valid also in $X_w(r+\delta)$ and

$$\begin{aligned} \|D^\beta \mathcal{T}_0\varphi\|_{X_w(r)} &\le \|D^\beta \mathcal{T}_0\varphi\|_{X_w(r+\delta)} \le c\|\psi\|_{X_w(r+\delta)} \\ &\le c\Big(\|L_0\varphi\|_{X_w(r+\delta)} + \|\mathcal{L}\varphi\|_{X_w(r+\delta)}\Big). \end{aligned} \tag{4.3}$$

By the assumptions on the coefficients we have $|a_\alpha(0)| \le \|a_\alpha\|_{L^\infty(r+\delta)}$ for all $|\alpha| = m$. Then from (4.3) it follows

$$\|D^m \mathcal{T}_0\varphi\|_{X_w(r)} \le c\|\varphi\|_{W^m_{X_w}(r+\delta)}.$$

Since $W^m_{X_w}(\Omega)$ verifies the Property 2 we have $\|D^m \mathcal{T}_0\varphi\|_{X_w(r)} \le c\|\varphi\|_{W^m_{X_w}(r)}$ and

$$\|\mathcal{T}_0\varphi\|_{W^m_{X_w}(r)} \le c\|\varphi\|_{W^m_{X_w}(r)}$$

that implies $\mathcal{T}_0 \in [W^m_{X_w}(r)]$.

Analogously $\mathcal{S}_0 \in [X_w(r); W^m_{X_w}(r)]$ and hence $\mathcal{S}_0\mathcal{L} \in [W^m_{X_w}(r)]$. Then

$$\varphi(x) = \mathcal{T}_0\varphi(x) + \mathcal{S}_0\mathcal{L}\varphi(x) \quad \text{for a.a. } x \in \mathcal{B}_r,$$

in $L^{p_0}(r)$. The estimate (3.5) holds also in $W^m_{X_w}(r)$, i.e.

$$\mathcal{I}d = \mathcal{T}_0 + \mathcal{S}_0\mathcal{L} \in [W^m_{X_w}(r)]. \tag{4.4}$$

(2) Let $f \in X_w(r)$ such that $\varphi = \mathcal{T}_0\varphi + \mathcal{S}_0 f$. Hence

$$\mathcal{L}\varphi(x) = L_0\mathcal{S}_0\mathcal{L}\varphi(x) = L_0\mathcal{S}_0 f(x) = f(x) \quad \text{for a.a. } x \in \mathcal{B}_r,$$

where we have used that $L_0\mathcal{S}_0 = \mathcal{I}d$ in $X_w(r)$. Then $\mathcal{L}\varphi(x) = f(x)$. □

Because of the equivalence of the norms (2.6) and (2.7) the Lemma 3.1 holds also in $W^m_{X_{w;r}}(r)$.

Corollary 4.2 *Let $\varphi \in W^m_{X_{w;r}}(\mathcal{B}_r(x_0))$. Under the conditions of Lemma 4.1 it holds*

(1) *$\varphi = \mathcal{T}_{x_0}\varphi + \mathcal{S}_{x_0}\mathcal{L}\varphi$, a.e. in $\mathcal{B}_r(x_0)$;*
(2) *if for some $f \in X_w(\mathcal{B}_r(x_0))$ it holds $\varphi = \mathcal{T}_{x_0}\varphi + \mathcal{S}_{x_0} f$, then φ is a solution of the equation $\mathcal{L}\varphi = f$.*

The previous lemma allows us to proof the following local existence result.

Theorem 4.3 *Let (3.2), (P_{x_0}), and the Properties 1 and 2 hold. Then the equation $\mathcal{L}u = f$ admits a solution in $W^m_{X_w}(\mathcal{B}_r(x_0))$, for all $f \in X_w(\Omega)$, and $r > 0$ small enough.*

Proof By the Corollary 4.2 we can give the proof in the space $W^m_{X_{w;r}}(r)$, taking $x_0 = 0 \in \Omega$. Since $L_0\mathcal{S}_0 = \mathcal{I}d$ in $X_w(r)$, and keeping in mind (3.7) we have

$$L_0\mathcal{T}_0 = L_0\mathcal{S}_0(L_0 - \mathcal{L}) = \mathcal{I}d(L_0 - \mathcal{L}) = L_0 - \mathcal{L}. \tag{4.5}$$

For any $u \in W^m_{X_{w;r}}(r)$ we consider the equation $\mathcal{L}u = f$. By (4.5) we can rewrite it as follows

$$f = \mathcal{L}u = (L_0 - L_0\mathcal{T}_0)u = L_0(\mathcal{I}d - \mathcal{T}_0)u,$$

where the identity operator here acts in $W^m_{X_{w;r}}(r)$. Applying $\mathcal{S}_0$ we obtain

$$\mathcal{S}_0 f = \mathcal{S}_0 L_0(\mathcal{I}d - \mathcal{T}_0)u = (\mathcal{I}d - \mathcal{T}_0)u.$$

By Lemma 3.1 we have

$$\|\mathcal{T}_0\|_{[W^m_{X_{w;r}}(r)]} = \sigma(r) \to 0, \qquad \text{as } r \to 0.$$

Taking r small enough such that $\|\mathcal{T}_0\|_{[W^m_{X_{w;r}}(r)]} < 1$, we obtain that the operator $\mathcal{I}d - \mathcal{T}_0$ is boundedly invertible in $W^m_{X_{w;r}}(r)$ and by Lemma 4.1 the function

$$u = (\mathcal{I}d - \mathcal{T}_0)^{-1}\mathcal{S}_0 f,$$

is a solution of the equation $\mathcal{L}u = f$ in $W^m_{X_{w;r}}(r)$. □

5 Interior Schauder type estimates

Our goal now is to obtain local interior Schauder type estimates for the solutions of $\mathcal{L}\varphi = f$ in $W^m_{X_w}(\Omega)$. For this purpose we need some auxiliary lemma.

Let $\omega \in C_0^\infty([0,1]$ be such that

$$\omega(t) = \begin{cases} 1 & 0 \le t < \frac{1}{3}, \\ 0 & \frac{2}{3} < t \le 1. \end{cases}$$

Then we can define a cut-off function $\xi \in C_0^\infty(\mathcal{B}_{r_2})$ as

$$\xi(x) = \begin{cases} 1 & |x| \le r_1, \\ \omega\Big(\frac{|x|-r_1}{r_2-r_1}\Big) & r_1 < |x| \le r_2, \end{cases} \tag{5.1}$$

for any $0 < r_1 < r_2 \le 1$. The norm of ξ is bounded, as it is proved in [5]

$$\|\xi\|_{C^m(r_2)} \le c\Big(1 - \frac{r_1}{r_2}\Big)^{-m}$$

with a constant independent of r_1 and r_2.

Lemma 5.1 *Let the conditions of Theorem 4.3 be fulfilled in $\mathcal{B}_{r_2} \Subset \Omega$. Then for any $0 < r_1 < r_2$ as in (5.1) and $u \in W^m_{X_w;r_2}(\Omega)$ the following estimate holds*

$$\|u\|_{W^m_{X_w;r_1}(r_1)} \le c\Big(1 - \frac{r_1}{r_2}\Big)^{-m}\Big(\|\mathcal{L}u\|_{X_w(r_2)} + \|u\|_{W^{m-1}_{X_w;r_2}(r_2)}\Big)$$

with a constant independent of r_1, r_2, and u.

Proof Take $\varphi = \xi u \in W^m_{X_w;r_2}(r_2)$ with a compact support in $\mathcal{B}_{r_2}$. Then by Corollary 4.2 we have

$$\varphi = \mathcal{T}_0\varphi + \mathcal{S}_0\mathcal{L}\varphi \tag{5.2}$$

and by Lemma 3.1 there exists $r > 0$ such small that

$$\|\mathcal{T}_0\varphi\|_{W^m_{X_w;r_2}(r_2)} \le \frac{1}{2}\|\varphi\|_{W^m_{X_w;r_2}(r_2)}$$

holds for all $r_2 \in (0, r)$. Then by (5.2) we obtain

$$\|\varphi\|_{W^m_{X_w;r_2}(r_2)} \le 2\|\mathcal{S}_0\mathcal{L}\varphi\|_{W^m_{X_w;r_2}(r_2)} \tag{5.3}$$

where

$$\mathcal{S}_0\mathcal{L}\varphi(x) = \int_{\mathcal{B}_{r_2}} J_0(x-y)\mathcal{L}\varphi(y)\,dy. \tag{5.4}$$

Calculating the higher order derivatives we obtain

$$D^\beta \mathcal{S}_0 \mathcal{L}\varphi(x) = \int_{\mathcal{B}_{r_2}} D_x^\beta J_0(x-y)\mathcal{L}\varphi(y)\,dy + c\mathcal{L}\varphi(x), \quad |\beta| = m$$

with a constant c independent of φ. The integral operator is of Calderón–Zygmund type and, by Property 1, the following estimate holds

$$\|D^m \mathcal{S}_0 \mathcal{L}\varphi\|_{X_w(r_2)} \le c\|\mathcal{L}\varphi\|_{X_w(r_2)}. \tag{5.5}$$

For the lower order derivatives of (5.4) we have the following expression

$$D^\beta \mathcal{S}_0 \mathcal{L}\varphi(x) = \int_{\mathcal{B}_{r_2}} D_x^\beta J_0(x-y)\mathcal{L}\varphi(y)\,dy, \quad |\beta| < m.$$

In this case the kernel has a weak singularity and the integral operator is a Riesz type integral (2.3) that we can estimate as

$$r_2^{|\beta|}\|D^\beta \mathcal{S}_0 \mathcal{L}\varphi\|_{X_w(r_2)} \le c r_2^m \|\mathcal{L}\varphi\|_{X_w(r_2)} \tag{5.6}$$

with a constant independent of r_2 and φ. Unifying (5.5) and (5.6) we obtain

$$\|\mathcal{S}_0 \mathcal{L}\varphi\|_{W^m_{X_w;r_2}(r_2)} \le c r_2^m \|\mathcal{L}\varphi\|_{X_w(r_2)}. \tag{5.7}$$

On the other hand, it is easy to see that $\mathcal{L}\varphi$ can be represented in the form

$$\mathcal{L}\varphi(x) = \xi(x)\mathcal{L}u(x) + M(u;\xi), \tag{5.8}$$

where $M(u;\xi)$ is a linear combination of derivatives $D^\alpha u$, the order of which does not exceed $(m-1)$, multiplied by the derivatives of ξ of order at most m. Precisely

$$M(u;\xi) = \sum_{\substack{|\alpha|+|\alpha'|\le m \\ |\alpha|<m}} c_\alpha(x) D^{\alpha'}\xi(x) D^\alpha u(x).$$

Following [5] we obtain analogously

$$r_2^m \|M(u;\xi)\|_{X_w(r_2)} \le c\|\xi\|_{C^m(r_2)}\|u\|_{W^{m-1}_{X_w;r_2}(r_2)}. \tag{5.9}$$

Unifying (5.9) and (5.7), we obtain

$$\begin{aligned} \|\mathcal{S}_0\mathcal{L}u\|_{W^m_{X_w;r_1}(r_1)} &\leq \|\mathcal{S}_0\mathcal{L}\varphi\|_{W^m_{X_w;r_2}(r_2)} \\ &\leq cr_2^m\Big(\|\xi\|_{C^m(r_2)}\|\mathcal{L}u\|_{X_w(r_2)} + \|M(u,\xi)\|_{X_w(r_2)}\Big) \\ &\leq c\|\xi\|_{C^m(r_2)}\Big(\|\mathcal{L}u\|_{X_w(r_2)} + \|u\|_{W^{m-1}_{X_w;r_2}(r_2)}\Big) \\ &\leq c\Big(1-\frac{r_1}{r_2}\Big)^{-m}\Big(\|\mathcal{L}u\|_{X_w(r_2)} + \|u\|_{W^{m-1}_{X_w;r_2}(r_2)}\Big). \end{aligned} \tag{5.10}$$

Then making use of (5.3) we obtain the desired estimate

$$\begin{aligned} \|u\|_{W^m_{X_w;r_1}(r_1)} &= \|\varphi\|_{W^m_{X_w;r_1}(r_1)} \leq 2\|\mathcal{S}_0\mathcal{L}\varphi\|_{W^m_{X_w;r_1}(r_1)} \\ [6pt] &\leq c\Big(1-\frac{r_1}{r_2}\Big)^{-m}\Big(\|\mathcal{L}u\|_{X_w(r_2)} + \|u\|_{W^{m-1}_{X_w;r_2}(r_2)}\Big). \end{aligned}$$

□

In order to establish interior Schauder's estimate we need the following result.

Lemma 5.2 *Suppose that the space $X_w(\Omega)$ possesses the Properties 1, 2 and 3. Then*

$$\|u\|_{W^k_{X_w}(\Omega)} \leq \varepsilon\|u\|_{W^{k+1}_{X_w}(\Omega)} + \frac{c}{\varepsilon^{p^k}}\|u\|_{X_w(\Omega)}$$

for some $p > 1$ and for all $k = 1, \ldots, m-1$.

Proof Without loss of generality we may assume that $d_\Omega = 1$ (see [5]). Let $\Omega \Subset \Omega'$, where Ω' is a bounded domain with sufficiently smooth boundary.

In order to simplify the calculus, we begin assuming $n = 1, m = 2$ and then we extend the result via induction to more general situation. Let $\Omega = (a, b)$ be an interval of length $b - a = \varepsilon$ for a fixed ε and suppose that $u \in C^2_0(\Omega')$, then by [14, Theorem 7.27] we have

$$|u'(x)| \leq \int_a^b |u''(t)|\,dt + \frac{18}{\varepsilon^2}\int_a^b |u(t)|\,dt, \quad \forall\, x \in (a,b). \tag{5.11}$$

By density arguments it follows that (5.11) holds true also for any $u \in \overset{\circ}{W}{}^{2,1}(\Omega')$. Since $\overset{\circ}{W}{}^2_{X_w}(\Omega') \subset \overset{\circ}{W}{}^{2,1}(\Omega')$, then it is evident that this inequality is true for all $u \in \overset{\circ}{W}{}^2_{X_w}(\Omega')$.

We can estimate the norm of $u'w$ by (5.11):

$$\|u'w\|_{X(a,b)} \leq \|w\|_{X(a,b)}\left(\int_a^b |u''(t)|\,dt + \frac{18}{\varepsilon^2}\int_a^b |u(t)|\,dt\right). \tag{5.12}$$

Then for an arbitrary interval $\Omega = (a, b)$ we construct covering with disjoint intervals I_l with length ε, such that $\Omega = \bigcup_l I_l$. Since (5.12) holds for all I_l, the Hölder inequality for $p > 1$ gives

$$\begin{aligned} \|u'w\|^p_{X(I_l)} &\le \|w\|^p_{X(I_l)} \left[\left(\int_{I_l} |u''(t)|\, dt \right)^p + \frac{c}{\varepsilon^{2p}} \left(\int_{I_l} |u(t)|\, dt \right)^p \right] \\ &\le c\|w\|^p_{X(\Omega)} \left(\varepsilon^{p-1} \int_{I_l} |u''(t)|^p\, dt + \frac{c}{\varepsilon^{p+1}} \int_{I_l} |u(t)|^p\, dt \right), \end{aligned} \tag{5.13}$$

where we have used that for each $I_l \subset \Omega$ we have $\|w\|_{X(I_l)} \le \|w\|_{X(\Omega)}$. Then

$$\sum_l \|u'w\|^p_{X(I_l)} \le c\|w\|^p_{X(\Omega)} \Big(\varepsilon^{p-1} \int_\Omega |u''(x)|^p\, dx + \frac{c}{\varepsilon^{p+1}} \int_\Omega |u(x)|^p\, dx \Big).$$

Let the Condition 1 holds. Then for each $p \in [1, p_1]$ we have

$$\|u'w\|^p_{X(\Omega)} \le c \sum_l \|u'w\|^p_{X(I_l)}.$$

For any $p \le \min\{p_0, p_1\}$, the Property 1 implies $X_w(\Omega) \subset L^p(\Omega)$ and hence

$$\begin{aligned} \|u'w\|_{X(\Omega)} &\le c\|w\|_{X(\Omega)} \Big(\varepsilon^{p-1} \int_\Omega |u''(x)|^p\, dx + \frac{c}{\varepsilon^{p+1}} \int_\Omega |u(x)|^p\, dx \Big)^{\frac{1}{p}} \\ &\le c\|w\|_{X(\Omega)} \Big(\varepsilon \|u''\|_{L^p(\Omega)} + \frac{c}{\varepsilon^{\frac{p+1}{p-1}}} \|u\|_{L^p(\Omega)} \Big). \end{aligned} \tag{5.14}$$

If $x \in \mathbb{R}^n$ then (5.14) holds for any partial derivative $D_i u$ for $i = 1, \ldots, n$. Moreover it holds also for the higher order derivatives, that is

$$\|D^k u\|_{X_w(\Omega)} \le \|w\|_{X(\Omega)} \Big(\varepsilon^{\frac{1}{p'}} \|D^{k+1} u\|_{X_w(\Omega)} + \frac{c}{\varepsilon^{2-\frac{1}{p'}}} \|D^{k-1} u\|_{X_w(\Omega)} \Big),$$

with $1 \le k \le m - 1$. Summing up along k we obtain

$$\|u\|_{W^s_{X_w(\Omega)}} \le \|w\|_{X_w} \Big(\epsilon \|u\|_{W^{s+1}_{X_w(\Omega)}} + \frac{c}{\epsilon^{2-\frac{1}{p'}}} \|u\|_{W^{s-1}_{X_w}(\Omega)} \Big)$$

for all $s = 1, \ldots, m - 1$.

Consider now the case when Condition 2 holds, hence there exists $p_2 \in (1, \infty)$ such that

$$\|u'\|_{X_w(\Omega)} \le c\|u'\|_{L^p(\Omega)} \qquad \forall\, p \in [p_2, \infty).$$

As before, it is sufficient to consider the case of $n = 1$, $m = 2$ and $b - a = \varepsilon$. Taking into account (5.11) and [14, Lemma 7.27] (see also [1, 21]) we have

$$\|u'\|_{X_w(a,b)} \le c \left(\int_a^b |u'(x)|^p \, dx \right)^{\frac{1}{p}} \le c\varepsilon \int_a^b |u''(t)| \, dt + \frac{c}{\varepsilon^{2p}} \int_a^b |u(t)| \, dt \tag{5.15}$$

Taking an arbitrary interval $\Omega = (a, b)$ we construct a covering with a family of disjoint intervals of length ε, that is, $\Omega = \bigcup_l I_l$, where $\Omega \bigcap I_l \ne \emptyset$, $I_i \bigcap I_j = \emptyset$, $i \ne j$. Since (5.15) holds for all I_l, summing up these estimates with respect to l we obtain

$$\begin{aligned} \|u'\|_{X_w(\Omega)} &\le \varepsilon \|u''\|_{L^p(\Omega)} + \frac{c}{\varepsilon} \|u\|_{L^p(\Omega)} \\ &\le \epsilon \int_\Omega |u''(x)| w(x) w^{-1}(x) \, dx + \frac{c}{\epsilon^{2p_2}} \int_\Omega |u(x)| w(x) w^{-1}(x) \, dx \\ &\le \|w^{-1}\|_{X'(\Omega)} \Big(\epsilon \|u''\|_{X_w(\Omega)} + \frac{c}{\epsilon^{2p_2}} \|u\|_{X_w(\Omega)} \int_a^b |u''(t)| \, dt \\ &\quad + \frac{c}{\varepsilon^2} \int_a^b |u(t)| \, dt \Big) \end{aligned}$$

Extending this to functions of n variables and taking $\varepsilon = \|w^{-1}\|_{X'(\Omega)} \epsilon$, we obtain

$$\|u\|_{W^k_{X_w(\Omega)}} \le \varepsilon \|u\|_{W^{k+1}_{X_w(\Omega)}} + \frac{c}{\varepsilon^p} \|u\|_{W^{k-1}_{X_w}(\Omega)}, \quad \forall\, k = 1, \ldots, m-1, \tag{5.16}$$

for any $p \ge 2p_2$ and a constant c depending on known quantities and on the norms $\|w^{-1}\|_{X'(\Omega)}$ and $\|w\|_{X_w(\Omega)}$.

Introducing the notion $A_k := \|u\|_{W^k_{X_w(\Omega)}}$, for all $k = 0, \ldots, m$, we rewrite (5.16)

$$\begin{aligned} A_1 &\le \varepsilon_1 A_2 + \frac{c_1}{\varepsilon_1^p} A_0 \\ A_2 &\le \varepsilon_2 A_3 + \frac{c_2}{\varepsilon_2^p} A_1 \le \varepsilon_2 A_3 + \frac{c_1 c_2}{(\varepsilon_2 \varepsilon_1)^p} A_0 + \frac{c_2 \varepsilon_1}{\varepsilon_2^p} A_2. \end{aligned}$$

Taking $\varepsilon_2 = \varepsilon$, $\varepsilon_1 = \varepsilon^{p+1}$ with $\varepsilon c_2 < 1$, we get

$$A_2 \le \varepsilon A_3 + \frac{1}{\varepsilon^{p^2}} A_0.$$

Repeating the same procedure we obtain a kind of interpolation inequality

$$A_k \le \varepsilon A_{k+1} + \frac{1}{\varepsilon^{p^k}} A_0, \qquad \forall\, k = 1, \ldots, m-1$$

that is the assertion of the lemma. □

Theorem 5.3 (Interior Schauder type estimate) *Consider the uniformly elliptic equation $\mathcal{L}u = f$ in a bounded domain Ω with principal coefficients $a_\alpha \in C(\overline{\Omega})$, for all $|\alpha| = m$ and $a_\alpha \in L^\infty(\Omega)$ for $|\alpha| < m$. Suppose that the Properties 1 and 2 hold. Then for any domain $\Omega_0 \Subset \Omega$ the following a priori estimate holds*

$$\|u\|_{W^m_{X_w}(\Omega_0)} \le C(\|\mathcal{L}u\|_{X_w(\Omega)} + \|u\|_{X_w(\Omega)}). \tag{5.17}$$

Proof We can construct a finite cover of Ω_0 with balls whose closure belong to Ω. That is way, without loss of generality we can consider Ω and Ω_0 to be concentric balls of small radius centered at zero. Let $R > 0$ be sufficiently small, then we need to prove the following estimate

$$\|u\|_{W^m_{X_w;r}(r)} \le C\Big(1 - \frac{r}{R}\Big)^{-m^2} (\|\mathcal{L}u\|_{X_w(R)} + \|u\|_{X_w(R)}).$$

Set

$$A = \sup_{0 \le r \le R} \Big(1 - \frac{r}{R}\Big)^{m^2} \|u\|_{W^m_{X_w;r}(r)}.$$

Suppose that $\|u\|_{W^m_{X_w;R}(R)} \ne 0$, otherwise, there is nothing to prove. It is easy to see that there exists $r_1 \in (0, R)$, such that

$$A \le 2\Big(1 - \frac{r_1}{R}\Big)^{m^2} \|u\|_{W^m_{X_w;r_1}(r_1)}.$$

For a fixed $r_2 \in (r_1, R)$ and Lemmata 5.1 and 5.2 we obtain

$$\begin{aligned} A \le\, & c\Big(1 - \frac{r_1}{R}\Big)^{m^2} \Big(1 - \frac{r_1}{r_2}\Big)^{-m} (\|\mathcal{L}u\|_{X_w(r_2)} + \|u\|_{W^{m-1}_{X_w;r_2}(r_2)}) \\ \le\, & c\Big(1 - \frac{r_1}{R}\Big)^{m^2} \Big(1 - \frac{r_1}{r_2}\Big)^{-m} \Big(\|\mathcal{L}u\|_{X_w(r_2)} + \varepsilon\|u\|_{W^m_{X_w;r_2}(r_2)} + \frac{c}{\varepsilon^{p^{m-1}}}\|u\|_{X_w(r_2)}\Big). \end{aligned}$$

Keeping in mind that $\Big(1 - \frac{r_2}{R}\Big)^{m^2} \|u\|_{W^m_{X_w;r_2}(r_2)} \le A$, we obtain

$$\begin{aligned} A \le\, & c\Big(1 - \frac{r_1}{R}\Big)^{m^2} \Big(1 - \frac{r_1}{r_2}\Big)^{-m} \|\mathcal{L}u\|_{X_w(r_2)} \\ & + \varepsilon c\Big(1 - \frac{r_1}{R}\Big)^{m^2} \Big(1 - \frac{r_1}{r_2}\Big)^{-m} \Big(1 - \frac{r_2}{R}\Big)^{-m^2} A \\ & + \frac{c}{\varepsilon^{p^{m-1}}}\Big(1 - \frac{r_1}{R}\Big)^{m^2} \Big(1 - \frac{r_1}{r_2}\Big)^{-m} \|u\|_{X_w(r_2)}. \end{aligned} \tag{5.18}$$

Set $\delta = 1 - \frac{r_1}{R}$ and choose r_2 and ε in the following way

$$1 - \frac{r_2}{R} = \frac{\delta}{2}, \quad \varepsilon = 2^{-1-4m^2}\delta^{2m}c^{-1}.$$

This implies

$$0 < \delta < 1, \quad \frac{\delta}{2} < 1 - \frac{r_1}{r_2} < \delta, \quad 0 < \varepsilon < 1$$

and, hence

$$\varepsilon c\Big(1 - \frac{r_1}{R}\Big)^{m^2}\Big(1 - \frac{r_1}{r_2}\Big)^{-m}\Big(1 - \frac{r_2}{R}\Big)^{-m^2} < \frac{1}{2}.$$

Making use of (5.18) we obtain the estimate

$$A < c(\|\mathcal{L}u\|_{X_w(r_2)} + \|u\|_{X_w(r_2)}) \le c\Big(\|\mathcal{L}u\|_{X_w(R)} + \|u\|_{X_w(R)}\Big),$$

that gives (5.17). □

6 Some examples of Banach function spaces

Let $X(\Omega)$ be correctly defined for any domain $\Omega \subset \mathbb{R}^n$ with a sufficiently smooth boundary $\partial\Omega$, and suppose that Ω_0 is a bounded domain such that $\Omega \subset \Omega_0 \subset \mathbb{R}^n$.

Lemma 6.1 *Let $\theta_k \in [W^k_{X_w}(\Omega); W^k_{X_w}(\Omega_0)]$ be such that $\theta_k f\big|_\Omega = f$ for all $k = 0, \ldots, m$. Then there exists $\theta \in [W^k_{X_w}(\Omega); W^k_{X_w}(\Omega_0)]$, such that $\theta f\big|_\Omega = f$, for all $k = 0, \ldots, m$.*

Proof Consider the following operator

$$\theta f = \begin{cases} \theta_m f & f \in W^m_{X_w}(\Omega), \\ \theta_{m-1} f & f \in W^{m-1}_{X_w}(\Omega)\backslash W^m_{X_w}(\Omega), \\ \ldots & \ldots\ldots\ldots\ldots \\ \theta_1 f & f \in W^1_{X_w}(\Omega)\backslash W^2_{X_w}(\Omega), \\ \theta_0 f & f \in X_w(\Omega)\backslash W^1_{X_w}(\Omega). \end{cases}$$

From the chain of embedding

$$X_w(\Omega) \supset W^1_{X_w}(\Omega) \supset \ldots \supset W^m_{X_w}(\Omega),$$

it follows that the linear operator $\theta \in [X_w(\Omega)]$ is well defined.

Consider the case $m = 1$, and extend the proof by induction to $m > 1$. For simplicity we carry out the proof in the one dimensional case. Supposing

$$\theta \in [W^1_{X_w}(\Omega); W^1_{X_w}(\Omega_0)] \quad \text{and} \quad \theta f\big|_{\Omega} = f,$$

we have to show that $\theta \in [X_w(\Omega); X_w(\Omega_0)]$. Indeed, if $f \in X_w(\Omega) \setminus W^1_{X_w}(\Omega)$, then $\theta f = \theta_0 f$ and therefore

$$\|\theta f\|_{X_w(\Omega_0)} \le \|\theta_0\| \|f\|_{X_w(\Omega)}.$$

Let $f \in W^1_{X_w}(\Omega)$, hence $\theta f = \theta_1 f$. Consider the function

$$F(x) = \int_a^x \theta_1 f(t)\, dt \qquad \text{for a.a. } x \in \Omega_0,$$

where $a \in \Omega$ is some fixed point. Then $F \in W^1_{X_w}(\Omega_0)$, $\frac{dF(x)}{dx} = \theta_1 f(x)$, for a.a. $x \in \Omega_0$, and $\theta F = \theta_1 F$ that implies $F \in W^1_{X_w}(\Omega)$ and

$$\|\theta_1 F\|_{W^1_{X_w}(\Omega_0)} \le C_1 \|F\|_{W^1_{X_w}(\Omega)}, \tag{6.1}$$

where C_1 is a constant, independent of F. It is obvious that $\frac{dF(x)}{dx} = f(x)$, a.e. in Ω. Moreover

$$\begin{aligned}\|\theta_1 f\|_{X_w(\Omega_0)} &\le \|\theta_1 F\|_{X_w(\Omega_0)} + \|\theta_1 f\|_{X_w(\Omega_0)} \\ &\le C_1\Big(\|f\|_{X_w(\Omega)} + \Big\| \int_a^x f(t)\, dt \Big\|_{X_w(\Omega)}\Big) \le C\|f\|_{X_w(\Omega)}.\end{aligned}$$

Considering functions of n variables, fixing all variables except one and applying the above procedure we can extend this result in the n-dimensional case. □

6.1 The Marcinkiewicz spaces

The Banach space $M^{p,\lambda}(\Omega)$, $p \in (1, \infty)$, $\lambda \in (0, 1)$, consisting of all functions $f \in L^p_{\text{loc}}(\Omega)$ for which

$$\|f\|_{M^{p,\lambda}(\Omega)} = \sup_{E \subset \Omega} \left(\frac{1}{|E|^\lambda} \int_E |f(x)|^p\, dx \right)^{\frac{1}{p}} < \infty$$

is called *the Marcinkiewicz space* (see for instance [13]).

Let us note that in the definition of *the Morrey spaces* $L^{p,\lambda}(\Omega)$ the supremum is taken over all intersections $E = \mathcal{B} \cap \Omega$ where $\mathcal{B}$ are balls in $\mathbb{R}^n$. This gives the continuous embedding $M^{p,\lambda}(\Omega) \subset L^{p,\lambda}(\Omega)$ for all $\lambda \in (0, 1)$.

It is easy to see that $M^{p,\lambda}(\Omega)$ is nonseparable, moreover, it is a rearrangement invariant space (cf. [6]).

Let T be a quasi-linear operator of $(p, p; q, q)$-compatible weak type, (see [2] for the definition). Then we have the following results (cf. [2, 6]).

Lemma 6.2 *The Boyd indices of the Marcinkiewicz space* $M^{p,\lambda}(\Omega)$ *are*

$$\alpha_X = \beta_X = \frac{1-\lambda}{p}, \qquad p \in (1, \infty),\ \lambda \in (0, 1).$$

Theorem 6.3 *Let* X *be a rearrangement invariant space on an nonatomic complete* σ*-finite space with infinite measure. Then any linear (quasilinear) operator* T *of* $(p, p; q, q)$*-compatible weak type is bounded in* X*; i.e.,* $T \in X$ *if and only if the Boyd indices* α_X *and* β_X *satisfy the inequality*

$$\frac{1}{q} < \alpha_X \le \beta_X < \frac{1}{p}, \qquad 1 \le p < q \le +\infty.$$

The Theorem 6.3 (cf. [2]) implies the validity of the Property1 for $M^{q,\lambda}(\Omega)$ for any $q \in (1, \infty)$, $\lambda \in (0, 1)$. The validity of Property 2 can be proved analogously as in the case of the Morrey spaces. Hence, by Theorem 4.3 we have the following result.

Corollary 6.4 *Under the conditions of Theorem 6.3 there exists* $u \in W^m_{M^{q,\lambda}}(\Omega)$*, a solution of* $\mathcal{L}u = f$*, for all* $f \in M^{q,\lambda}(\Omega)$*,* $q \in (1, \infty)$*,* $\lambda \in (0, 1)$*.*

In addition, by Theorem 5.3 we obtain.

Corollary 6.5 *Under the conditions of Theorem 6.3 the following a priori estimate holds*

$$\|u\|_{W^m_{M^{q,\lambda}}(\Omega_0)} \le c \left(\|\mathcal{L}u\|_{M^{q,\lambda}(\Omega)} + \|u\|_{M^{q,\lambda}(\Omega)} \right).$$

6.2 Grand Lebesgue spaces

The Grand Lebesgue spaces $L_{q)}(\Omega)$, $q \in (1, \infty)$, are non separable Banach spaces endowed by the norm

$$\|f\|_{q)} = \sup_{0<\varepsilon<q-1} \left(\varepsilon \int_\Omega |f(x)|^{q-\varepsilon}\, dx \right)^{\frac{1}{q-\varepsilon}}.$$

The Property 1 holds for $L_{q)}(\Omega)$ as it is proved in [17] and the following continuous embedding is valid

$$L^q \subset L^{q)} \subset L^{q-\varepsilon}, \qquad \forall\, \varepsilon \in (0, q-1).$$

Consider the Grand Sobolev spaces $W^m_{L^{q)}}(\Omega)$ builded on the spaces $L^{q)}(\Omega)$.

Corollary 6.6 *Suppose that the conditions of Theorem 5.3 hold true and* $x_0 \in \Omega$ *verifies the property* (P_{x_0})*. Then there exists a solution* $u \in W^m_{L^{q)}}(\mathcal{B}_r(x_0))$ *of the equation* $\mathcal{L}u = f$*, for* r *small enough and for all* $f \in L^{q)}(\mathcal{B}_r(x_0))$*.*

Corollary 6.7 *Under the conditions of Theorem 5.3 the following a priori estimate holds*

$$\|u\|_{W^m_{L^{q)}}(\Omega_0)} \le c\big(\|\mathcal{L}u\|_{L^{q)}(\Omega)} + \|u\|_{L^{q)}(\Omega)}\big).$$

6.3 Variable Lebesgue spaces

Given a bounded domain Ω and a Lebesgue measurable function $p(\cdot) : \Omega \to [1, +\infty)$. We say that $p(\cdot)$ is *locally log-Hölder continuous function* if there exists a constant c_0 such that for all $x, y \in \Omega$ verifying $|x - y| < 1/2$, it holds

$$|p(x) - p(y)| \le \frac{c_0}{-\log(|x-y|)}.$$

We denote this function set as $LH_0(\Omega)$. It follows immediately (see [9]) that if $p(\cdot) \in LH_0(\Omega)$ than it is uniformly continuous and $p(\cdot) \in L^\infty(\Omega)$.

For a Lebesgue measurable function $f \in F(\Omega)$ we define the *modular* associated with $p(\cdot)$ by

$$I_{p(\cdot),\Omega}(f) = \int_\Omega |f(x)|^{p(x)}\,dx.$$

Then the *Variable Lebesgue spaces* $L^{p(\cdot)}(\Omega)$, $p(\cdot) \in LH_0(\Omega)$ consist of all $f \in F(\Omega)$ such that $I_{p(\cdot),\Omega}(f) < +\infty$, for which the following norm is finite

$$\|f\|_{L^{p(\cdot)}(\Omega)} = \inf\left\{\lambda > 0 : \; I_{p(\cdot),\Omega}\Big(\frac{f}{\lambda}\Big) \le 1\right\}. \tag{6.2}$$

The spaces $L^{p(\cdot)}(\Omega)$, $p(\cdot) \in LH_0(\Omega)$, endowed with the norm (6.2) are BFSs.

The corresponding Sobolev spaces are denoted by $W^{p(\cdot),m}(\Omega)$ and for a given weight w we can define also the weighted spaces $L^{p(\cdot)}_w(\Omega)$ and $W^{p(\cdot),m}_w(\Omega)$.

The weighted $L^{p(\cdot)}$ spaces defined by the class of variable Muckenhoupt weights $A_{p(\cdot)}$ are of particular interest.

Definition 6.8 We say that $w \in A_{p(\cdot)}$ if

$$[w]_{A_{p(\cdot)}} = \sup_Q |Q|^{-1}\|w\chi_Q\|_{p(\cdot)}\|w^{-1}\chi_Q\|_{p'(\cdot)} < +\infty,$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$ with sides parallel to coordinate axes and $p'(\cdot)$ is the conjugate function, $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$ for all $x \in \mathbb{R}$.

The properties of the weighted Variable Lebesgue spaces (see [9, 17]) ensure the validity of the Property 1. For a bounded domain Ω we have the embedding $L^{p_+}(\Omega) \subset L^{p(\cdot)}(\Omega) \subset L^{p_-}(\Omega)$ with a measurable function $p(\cdot)$ verifying (2.10).

Corollary 6.9 *Let the conditions of Theorem 4.3 hold and suppose that* $w \in LH_0(\Omega) \cap A_{p(\cdot)}(\mathbb{R}^n)$, $p_- > 1$, *then there exists a solution* $u \in W_w^{p(\cdot),m}(\mathcal{B}_r(x_0))$ *of the equation* $\mathcal{L}u = f$ *for r small enough and for all* $f \in L_w^{p(\cdot)}(\Omega)$.

The validity of the Property 2 in this case follows by Lemma 6.1 and [9].

Theorem 6.10 *Let* $\Omega \subset \mathbb{R}^n$ *be a bounded domain with* C^k *smooth boundary and* $p(\cdot) \in LH_0(\Omega)$ *such that* $1 < p_- \leq p_+ < +\infty$. *Then for every* $k \geq 1$ *there exists a bounded linear extension operator*

$$\theta_k \in \left[W^{p(\cdot),k}(\Omega);\, W^{p(\cdot),k}(\mathbb{R}^n)\right].$$

Applying the Theorem 5.3 to the case of weighted Variable Lebesgue spaces we are able to obtain a local a priori estimate for the solution.

Corollary 6.11 *Let* $\Omega_0 \Subset \Omega$. *Under the conditions of Theorem 5.3 and Corollary 6.9 the following a priori estimate holds*

$$\|u\|_{W^{p(\cdot),m}(\Omega_0)} \leq c(\|\mathcal{L}u\|_{L^{p(\cdot)}(\Omega)} + \|u\|_{L^{p(\cdot)}(\Omega)}). \tag{6.3}$$

In the case of Variable Lebesgue spaces without any weight, the estimate (6.3) is obtained in

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