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MODELS, COPRODUCTS AND EXCHANGEABILITY: NOTES ON STATES ON BAIRE FUNCTIONS

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ABSTRACT. We discuss exchangeability and independence in the setting of σ-complete Riesz MV-algebras. We define and link to each other the notions of exchangeability and distribution law for a sequence of observables (i.e. non classical random variables), as well as the notion of independence for a sequence of algebras. We obtain two categorical dualities for σ-complete Riesz MV-algebras endowed with states and we define a "canonical" state on the coproduct of a sequence of probability Riesz tribes, giving a weak version of de Finetti's result. Finally, we discuss statistical models.

1. Introduction

The idea of using logic to reason about probability is not new, one can find several approaches in literature, that vary from first order logics in which the atomic formulas are meant to be comparison of probabilities (see for instance [10]) to modal approaches in which one uses a modality P to say that a formula of some logic is probable (see for instance [11] for the case for fuzzy probability logic).

In the setting of Łukasiewicz logic, one of the most studied fuzzy logics, probabilities are very efficiently considered via the notion of a *state*, which is an additive and truth-preserving operator. This notion was introduced by D. Mundici in [19] with the idea of defining a "truth averaging" operator for logical formulas.

This intuition was made more precise subsequently, when T. Kroupa and G. Panti proved independently that a state on any MV algebra A is, in fact, a Borel regular probability measure on the space $Max(A)$ of the maximal ideals of A, see Section 2 for a formal statement. If one considers an MV-subalgebra A of some algebra of continuous functions $C(X)$ over a compact and Hausdorff space X, then to each state s on A it corresponds exactly one regular measure on the σ-algebra of Borel subsets of X. Furthermore, s is given by integration with respect to such measure. In this sense, states are averaging operators: they act as expectations on the functions of A, see also [12: Remark 2.8].

With the additional momentum gained from the Kroupa-Panti integral representation, states were deeply investigated. For instance, they are used in [11] to give an algebraic semantics for their

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logical system and they are characterized as terms in the language of special types of MV-algebras in [12: Section 5]. Nevertheless, the second crucial result for the development of the theory of states on MV-algebras was the generalization of de Finetti's Dutch book theorem, see [20: Chapter 1]. This result, proved by D. Mundici, has among its consequences the fact that states on MV-algebras are Bayesian probabilities.

Thus, probability on MV-algebras can be considered from a subjective point of view and one pivotal theorem in probability theory is again a result by B. de Finetti, the so-called de Finetti's theorem on exchangeable probabilities. This theorem, loosely speaking, states that a probability on a cartesian product does not depend on the order of factors if, and only if, it can be build using independent and identically distributed probability measures, see Section 2 for a more precise statement. One crucial feature of this result is that it shows how a statistical model can appear in a subjectivist framework, and it is a bridge between the frequentist and the Bayesian approaches to probability theory. Whence, having at our disposal a subjective notion of probability on MV-algebras, it is only natural trying to understand how de Finetti's exchangeability can be analyzed from the point of view of logic. To do this, a more expressive language than that given by MV-algebras is needed.

Indeed, despite the theory of states is well-developed for MV-algebras, if one wants to tackle a theorem such as de Finetti's one needs a language that allows to write (at least) convex combinations, as de Finetti's theorem deals with mixtures of independent and identically distributed random variables, which are convex combinations when the variables are finitely many. With this idea in mind, we choose to focus our attention on the class of Riesz MV-algebras, that stand to MV-algebras like real vector spaces stand to groups. More precisely, we will work in the class of those Riesz MV-algebras that are closed under countable suprema and infima, which was proven to be an infinitary variety of algebras in [8].

The reason for this choice lies in the fact that in [6] the authors used σ-complete Riesz MV-algebras to start an investigation on the most suitable way of interpreting random variables in logical terms. They defined *observables* as the algebraic counterpart of random variables in a way that, in certain cases, provides a one-one correspondence between classical random variables and observables, adding the expressive power that allows to model more complicated phenomena, while being able to use the powerful tools of measure theory inside a logical system. The variety of σ-complete Riesz MV-algebras has been further investigated in [7], where the authors characterize all free objects and give two categorical dualities.

In this paper we build on [6–8] and we propose and discuss the main notions that one needs in order to start thinking about de Finetti's result on exchangeable probabilities. In other words, we prepare the ground for a non-classical version of this result. We start with giving all needed preliminary notions in Section 2, while in Section 3 we discuss exchangeability and independence for sequences of observables as well as for states. We first generalize two key results of [6] and then we define and link to each other the notions of exchangeability and distribution law for a sequence of observables, as well as the notion of independence for a sequence of tribes. Section 4 aims at defining a suitable algebraic counterpart for the notion of a product measure. Motivated by the dualities of [7], we define this notion using coproducts of tribes: Proposition 4.1, Definition 4.9 and Theorem 4.13 are the main results of the section, culminating in a weak version of de Finetti's exchangeability. In Section 5 we enrich the objects of the categories considered in [7] with states and probability measures and, upon suitable restrictions on morphisms, we obtain two new enriched categorical dualities. Finally, in Section 6 we define statistical models in logical terms and we give some examples.

2. Preliminaries

2.1. σ-complete Riesz MV-algebras

In this paper we will work with *σ-complete Riesz MV-algebras*.

A *Riesz MV-algebra* is an algebra $(A, \oplus, \neg, 0, 1, \{\alpha\}_{\alpha\in[0,1]})$, where $\oplus$ is a binary operation, $\neg$ is an involution, 0 and 1 are respectively a bottom and a top element, and the unary operations $\{\alpha\}_{\alpha\in[0,1]}$ model a scalar multiplication. The standard example of such an algebra is the real interval $[0,1]$, where $x \oplus y = \min(x+y, 1)$, $\neg x = 1 - x$ and αx is the product of real numbers. This example is standard in a very precise sense: Riesz MV-algebras form a variety and $[0,1]$ is a generator for it. From a different point of view, if (V, u) is a Riesz space (vector lattice) with a distinguished strong order unit u, the interval $[0, u]_V = \{x \in V \mid 0 \le x \le u\}$ is a Riesz MV-algebra when endowed with the following operations: $x \oplus y = (x +_V y) \wedge u$, $\neg x = u -_V x$, αx the same as in V. The map that takes (V, u) and sends it into $[0, u]_V$ is actually a functor, denoted by Γ, that gives a more general categorical equivalence between Riesz MV-algebras and Riesz spaces with strong unit, see [5, 9, 18] for all missing definitions and references.

A Riesz MV-algebra is σ-complete if it is closed under countable suprema (and therefore also infima). It was proved in [8] that σ-complete Riesz MV-algebras form an *infinitary* variety in the sense of [22], and the algebra $[0,1]$ is again a generator for the infinitary variety, subsequently denoted by $\mathbf{RMV}_\sigma$. Furthermore, in [7], the authors prove that for any cardinal κ, the κ-generated free algebra in $\mathbf{RMV}_\sigma$ is the MV-algebra all $[0,1]$-valued and *Baire*-measurable functions defined on $[0,1]^\kappa$.

To be more precise, given a topological space (T, τ), where T is the universe of the topology and τ denotes the open sets of the topology, $C(T)$ will denote the set of $[0,1]$-valued continuous functions defined over T. A *zeroset* $Z \subseteq T$ is a set for which there exists $f \in C(T)$ such that $Z = \{x \in T \mid f(x) = 0\}$. A *cozero* set is a complement of a zeroset, that is a set definable as $\{x \in T \mid f(x) \neq 0\}$ for some continuous function f. A *Baire set* is a subset of T belonging to the σ-algebra generated by the zerosets of continuous functions from T to $\mathbb{R}$, while a *Borel set* is a subset of T belonging to the σ-algebra generated by the closed sets. We shall denote the σ-algebras of Baire and Borel subsets of T respectively by $\mathcal{BA}(T)$ and $\mathcal{BO}(T)$. Whence, a Baire function is a function $f : T \to [0,1]$ measurable with respect to the spaces $(T, \mathcal{BA}(T))$, $([0,1], \mathcal{BA}([0,1]))$. Borel functions are analogously defined. Note that every Baire set is a Borel set, that is $\mathcal{BA}(T) \subseteq \mathcal{BO}(T)$, while the converse inclusion holds for a metrizable space, see [7: Remark 2.1]. Thus, considering the cases of interest for us, $\mathcal{BO}([0,1]^X) = \mathcal{BA}([0,1]^X)$ for $|X| \le \omega$.

It is known that the sets of $[0,1]$-valued Baire and Borel functions defined over $[0,1]^X$ are σ-complete Riesz MV-algebras, and they shall be denoted respectively by $\mathrm{Baire}([0,1]^X)$ and $\mathrm{Borel}([0,1]^X)$. Note that $\mathrm{Baire}([0,1]^X) = \mathrm{Borel}([0,1]^X)$ for $|X| \le \omega$, while they do not coincide for $|X| > \omega$. We also recall that in both algebras countable suprema are taken pointwise, see [7, 8].

To give more uniform notations, in this work we follow [7] and therefore $IRL(X)$ will denote the free X-generated algebra, where X is an arbitrary set. The elements of $IRL(X)$ will be called *IRL-polynomials*. Consequently, $IRL(X) = \mathrm{Borel}([0,1]^X) = \mathrm{Baire}([0,1]^X)$ for $|X| \le \omega$, while $IRL(X) = \mathrm{Baire}([0,1]^X)$ for $|X| > \omega$.

Finally, for any $A \in \mathbf{RMV}_\sigma$, we shall call *$\sigma$-ideals* the ideals of A that are closed under countable suprema, while we will call *MV-maximal σ-ideals* those σ-ideals of A that are also maximal ideals for the Riesz MV-reduct of A. The set of all MV-maximal σ-ideals will be denoted by $\mathcal{M}_\sigma(A)$, or $\mathcal{M}_\sigma$ when A is clear from the context. In [7], the authors defined σ-semisimple algebras as those algebras A such that $\bigcap\{M \mid M \in \mathcal{M}_\sigma\} = \{0\}$. Furthermore, upon defining IRL-algebraic varieties as intersections of Baire sets, in [7] it is proved the existence of a duality between IRL-varieties and σ-semisimple σ-complete Riesz MV-algebras. The duality, on the level of objects, is induced by the following operators. For any subset $S \subseteq [0,1]^X$,

$$\mathbb{I}(S) = \{p \in IRL(X) \mid p(\mathbf{x}) = 0 \text{ for any } \mathbf{x} \in S\}.$$

Given a set J of IRL-polynomials in $IRL(X)$,

$$\mathbb{V}(J) = \{\mathbf{x} \in [0,1]^X \mid p(\mathbf{x}) = 0 \text{ for any } p \in J\} = \bigcap_{p \in J} \mathbb{V}(\{p\}).$$

Formally, the dual categories are the following.

(1) The category $\mathbf{ssRMV}_\sigma$ whose objects are presented σ-semisimple algebras in $\mathbf{RMV}_\sigma$ and whose arrows are σ-homomorphisms of Riesz MV-algebras. More precisely, an object is a pair $(IRL(X), I)$, where I is a σ-ideal in the free algebra $IRL(X)$ such that $I = \mathbb{I}(\mathbb{V}(I))$. Consequently, each morphism $h\colon (IRL(X), I) \to (IRL(Y), J)$ between pairs is induced by a unique homomorphism $h^p\colon IRL(X) \to IRL(Y)$ such that $h^p(I) \subseteq J$.

(2) The category **IRL**, whose objects are IRL-algebraic varieties in hypercubes of type $[0,1]^X$, for an arbitrary X, and arrows are tuples of IRL-polynomials, that is, an arrow in **IRL** is a map $\eta = (\eta_{y|_S})_{y \in Y}\colon S \subseteq [0,1]^X \to T \subseteq [0,1]^Y$, where each η_y belongs to $IRL(X)$.

We also note that, as proved in [7: Theorem 4.12], a σ-complete Riesz MV-algebra $A = IRL(X)/I$ is σ-semisimple if, and only if, $A \in ISP([0,1])$. We remark that the universal algebraic operators I, S and P are considered in the infinitary variety of σ-complete Riesz MV-algebras.

2.2. Tribes and functional representation

For any non-empty set X, a *Riesz tribe* (or simply *tribe*) is a Riesz MV-subalgebra of $[0,1]^X$ closed under pointwise countable suprema, that is, an element of $SP([0,1])$. For the purpose of this paper, we stress the fact that $IRL(X)$ is a Riesz tribe for any X, and any σ-complete Riesz MV-algebra is a σ-homomorphic image of a Riesz tribe. This is the so-called Loomis–Sikorski Theorem, see [8: Theorem 5.2]. If $\mathcal{T} \subseteq [0,1]^X$ is a Riesz tribe, the set $\mathcal{S}(\mathcal{T}) = \{A \subseteq X \mid \chi_A \in \mathcal{T}\}$ is a natural σ-algebra of subsets of X, and it is the smallest σ-algebra of X that makes all functions in $\mathcal{T}$ measurable. We will often refer to $\mathcal{S}(\mathcal{T})$ as the Boolean center of $\mathcal{T}$. Notice that we will always consider the codomain $[0,1]$ of each function in $\mathcal{T}$ endowed with its standard Baire σ-algebra $\mathcal{BO}([0,1]) = \mathcal{BA}([0,1])$, as defined before.

We say that a topological space $X \neq \emptyset$ is *basically disconnected* provided the closure of every open F_σ subset (i.e. a countable union of closed subsets) of X is open. By a well-known result of Nakano (see [20: Proposition 11.10] for the case of MV-algebras) if X is a compact Hausdorff space, the algebra $C(X)$ (of $[0,1]$-valued continuous functions on X) is σ-complete if, and only if, X is basically disconnected. We stress the fact that countable suprema on continuous functions need not to be computed pointwise.

Moreover, by the fundamental Stone-Krein-Kakutani-Yosida Theorem (see [8] for the case of Riesz MV-algebras) any σ-complete Riesz MV-algebra A is isomorphic to the algebra $C(X)$, where $X = Max(A)$ is a basically disconnected compact Hausdorff space. This result is made into a duality in [7], where morphisms on topological spaces are defined as follows: A function $f\colon X \to Y$ between compact Hausdorff and basically disconnected spaces is *cozero-closed* if for every countable union U of clopens of Y, we have $f^{-1}(\overline{U}) = \overline{f^{-1}(U)}$. Then, we have the following.

Proposition 2.1. *The algebraic category* $\mathbf{RMV}_\sigma$ *is dual to the category* **BDKH** *whose objects are basically disconnected, compact, Hausdorff spaces and whose morphisms are continuous and cozero-closed functions.*

2.3. States and subjective probability

A theory of subjective probability has been developed on MV-algebras using the notion of a *state*, introduced by D. Mundici. To define this notion properly, we recall that any MV-algebra A can be endowed with a partial operation, subsequently denoted by $+$, defined when $x \odot y = 0$ as $x + y := x \oplus y$.

The definition of a state is easily generalized to Riesz MV-algebras without additional requirements, see [9]. Thus, a *state* of a Riesz MV-algebra A is a map $s\colon A \to [0,1]$ satisfying the following conditions:

(1) $s(1) = 1$,

(2) for all $x, y \in A$ such that $x \odot y = 0$, $s(x \oplus y) = s(x) + s(y)$.

We shall call a state that preserves countable suprema of increasing sequences a σ-state. States on Riesz MV-algebras and σ-states on tribes are in bijective correspondence with measures on suitable measurable spaces. In the case of tribes with σ-states, the correspondence was proved by Butnariu and Klement and, subsequently, by Barbieri and Weber with different techniques [1, 3]. In the case of arbitrary MV-algebras and states, the result was proved independently by Kroupa and Panti [13: Theorem 4.0.1]. We recall both versions of the result in the way most suited to our framework, as we shall use both of them in Section 5.

Theorem 2.2. *Let X be a compact and Hausdorff topological space. For any state $s_X : C(X) \to [0,1]$ there exists a unique Borel regular measure $\mu_X : \mathcal{BO}(X) \to [0,1]$ such that*

$$s_X(g) = \int_X g \, \mathrm{d}\mu_X,$$

and the map $s_X \mapsto \mu_X$ is a homeomorphism.

Theorem 2.3. *For every Riesz tribe $\mathcal{T} \subseteq [0,1]^X$, for every σ-state s of $\mathcal{T}$ there exists a unique measure μ_s such that, for every $f \in \mathcal{T}$,*

$$s(f) = \int_X f \, \mathrm{d}\mu_s.$$

The measure $\mu_s\colon \mathcal{S}(\mathcal{T}) \to [0,1]$ is given by $\mu_s(A) = s(\chi_A)$. The correspondence is a bijection.

We remark that the correspondence of Theorem 2.3 is actually an isometry. Indeed, in [1: Theorem 3.2.1] it is proved that the set of states on a tribe $\mathcal{T}$ and the set of measures on the corresponding $\mathcal{S}(\mathcal{T})$ can both be endowed with a metric structure (making them Banach spaces) in a way that the assignment above defined becomes an isometry.

Unless otherwise specified by a *probability Riesz tribe* is meant a pair $(\mathcal{T}, s)$ where $\mathcal{T}$ is a Riesz tribe and s is a σ-additive state.

Moreover, we will call *additive* a function that satisfies item (2) of the definition of a state. Similarly, for any $k \in \mathbb{N}$ and MV-algebras $A, A_1, \dots, A_k$, a function $\beta\colon A_1 \times \dots \times A_k \to A$ defined on the cartesian product will be called k-additive if it is additive in each component.

Finally, for sake of completeness we discuss below de Finetti's theorem on exchangeability in the version of Hewitt and Savage, see [15].

Take any set X with a σ-algebra of subsets $\mathcal{X}$ and let $\overline{\mathcal{X}}$ be the smallest σ-algebra on X^ω that contains all sets of type $C(E_{i_1}, \dots, E_{i_k}) = \prod_n E_n$ with $E_n = X$ for $n \notin \{i_1, \dots, i_k\}$. A generic measure σ on $\overline{\mathcal{X}}$ is called *exchangeable* if for any permutation π of $\{i_1, \dots, i_k\}$,

$$\sigma\left(C(E_{i_1}, \dots, E_{i_k})\right) = \sigma\left(C(E_{\pi(i_1)}, \dots, E_{\pi(i_k)})\right).$$

For any measure μ on $\mathcal{X}$ let $\overline{\mu}$ denote the unique product measure defined on $\overline{\mathcal{X}}$ using μ. A generic measure σ on $\overline{\mathcal{X}}$ is called *presentable* if there exists a measure ν on the set $\mathcal{P}$ of all probability measure on $\mathcal{X}$ such that for any $A \in \overline{\mathcal{X}}$,

$$\sigma(A) = \int_{\mathcal{P}} \overline{\mu}(A) \, \mathrm{d}\nu(\mu).$$

Loosely speaking, a measure is presentable if it is given by a probability kernel. Then, de Finetti's theorem (in the version of Hewitt and Savage) gives sufficient conditions for the two notions to coincide. One of its versions is the following.

Theorem 2.4 ([15: Theorem 7.2]). *With the above definitions, the notions of presentability and exchangeability coincide when X is a compact and Hausdorff space and $\mathcal{X} = \mathcal{BA}(X)$.*

If one thinks in terms of random variables, exchangeability says that the joint distribution is invariant with respect to any permutation of a given sequence. That is, given a sequence $\{f_i\}_{i\in I}$ of random variables, the sequence is exchangeable if for any $k \in \mathbb{N}$ and any two sets of indexes $i_1, \dots, i_k$, $j_1, \dots, j_k$, the distribution laws $F_{i,k}$ of $(f_{i_1}, \dots, f_{i_k})$ and $F_{j,k}$ of $(f_{j_1}, \dots, f_{j_k})$ coincide. de Finetti's theorem in this case says that any sequence of exchangeable random variables can be obtained by first choosing a probability ν (on the set of all probabilities on the domain of the f_i's) and then asking for the f_i's to be independent and identically distributed with joint distribution ν.

3. Independence and exchangeability in non-classical processes

In [6], *non-classical* stochastic processes posed in a probability Riesz tribe are defined as sequences of observables $\{\mathcal{X}_n\}_{n\in\mathbb{N}}$ with values in a Riesz tribe, in symbols, $\mathcal{X}_n \colon IRL(Y) \to \mathcal{T}$. The tribe $\mathcal{T} \subseteq [0,1]^X$, for some set X, is endowed with the σ-algebra $\mathcal{S}(\mathcal{T}) = \{A \subseteq X \mid \chi_A \in \mathcal{T}\}$ and Y is a countable set.

In this work we shall consider processes defined over $IRL(Y)$, for an arbitrary $Y \neq \emptyset$, indexed in an arbitrary set of cardinality κ, and we shall usually denote them as $\{\mathcal{X}_i\}_{i\in\kappa}$. Indeed, the definition given in [6: Definition 2.1] can be easily generalized as follows.

Definition 3.1. Let A be a σ-complete Riesz MV-algebra. For any cardinal κ, a κ-dimensional *observable* on A is any σ-homomorphism of Riesz MV-algebras from $IRL(Y)$ to A, with $|Y| = \kappa$. When we need to make explicit reference to the dimension of the observable we shall write $IRL(\kappa)$ instead of $IRL(Y)$.

A crucial point in [6] is Theorem 2.3, which is a representation theorem. Let us prove here the analogous result for observables defined over $IRL(Y)$, with Y an arbitrary set. We give the whole proof for sake of completeness, but for the most part it can be deduced from [6].

Theorem 3.2. *Let X, Y be nonempty sets, $\mathcal{T} \subseteq [0,1]^X$ be a Riesz tribe and $f \colon X \to [0,1]^Y$ a measurable function w.r.t. $\mathcal{S}(\mathcal{T}) = \{A \subseteq X \mid \chi_A \in \mathcal{T}\}$ and $\mathcal{BA}([0,1]^Y)$. Then the function*

$$\mathcal{X}_f \colon IRL(Y) \to \mathcal{T}, \quad \mathcal{X}_f(a) = a \circ f, \quad a \in IRL(Y).$$

is a κ-dimensional observable on $\mathcal{T}$, where $|Y| = \kappa$.
Conversely, for any κ-dimensional observable $\mathcal{X} \colon IRL(Y) \to \mathcal{T}$, there exists a unique $f \colon X \to [0,1]^Y$ such that $\mathcal{X} = \mathcal{X}_f$.

P r o o f. The fact that $\mathcal{X}_f$ is a homomorphism of Riesz MV-algebras is straightforward. Thus, we have to prove that it preserves countable joins. Let $\{g_n\}_{n\in\mathbb{N}}$ be a sequence in $IRL(Y)$. We recall that countable suprema and infima are defined pointwise in $IRL(Y)$ and in any tribe, therefore

$$\mathcal{X}_f\Big(\bigvee_n g_n\Big)(x) = \Big(\bigvee_n g_n\Big)(f(x)) = \sup_n\{g_n(f(x))\} = \sup_n\{(g_n \circ f)(x)\}$$
$$= \sup_n\{(\mathcal{X}_f(g_n))(x)\} = \Big(\bigvee_n \mathcal{X}_f(g_n)\Big)(x).$$

Finally, let us prove that for any $a \in IRL(Y)$, $a \circ f \in \mathcal{T}$. By [20: Lemma 11.8(ii)], $\mathcal{T}$ is the tribe of all $\mathcal{S}(\mathcal{T})$-measurable functions, whence it is enough to prove that $a\circ f$ is $\mathcal{S}(\mathcal{T})$-measurable for any $a \in IRL(Y)$. This fact is easily deduced: For any $E \in \mathcal{BA}([0,1])$, $(a \circ f)^{-1}(E) = f^{-1}(a^{-1}(E)) \in \mathcal{S}(\mathcal{T})$ since f is measurable and $a \in IRL(Y)$.

Conversely, given an observable $\mathcal{X}$, we need to prove that there exists a function f that satisfies the claim. Let $\kappa = |Y|$ and let f be the function defined by $f = (f_i)_{i\in\kappa}\colon X \to [0,1]^Y$ with $f_i = \mathcal{X}(\pi_i) \in \mathcal{T}$. Using the fact that the projections form a generating set for $IRL(Y)$, it is straightforward that $\mathcal{X} = \mathcal{X}_f$ and $\mathcal{X}_f(IRL(Y)) \subseteq \mathcal{T}$. Thus, we only need to prove that f is measurable w.r.t. $\mathcal{S}(\mathcal{T})$. To do so, take $E \in \mathcal{BA}([0,1]^Y)$ and let us prove that $\chi_{f^{-1}(E)} \in \mathcal{T}$: since $\chi_{f^{-1}(E)}(x) = 1$ if, and only if, $f(x) \in E$ if, and only if, $\chi_E(f(x)) = 1$, it follows that $\chi_{f^{-1}(E)} = \mathcal{X}_f(\chi_E) = \mathcal{X}(\chi_E)$, which belongs to $\mathcal{T}$ by definition of $\mathcal{X}$.

Finally, let $g\colon X \to [0,1]^Y$ be another function that satisfies the claim. Then, $\pi_i \circ g = \mathcal{X}(\pi_i) = \pi_i \circ f$, from which we deduce that for any $x \in X$, $g(x) = (\pi_i(g(x)))_{i\in\kappa} = (\pi_i(f(x)))_{i\in\kappa} = f(x)$. □

Note that we have endowed $[0,1]^Y$ with the σ-algebra $\mathcal{BA}([0,1]^Y)$. When $|Y| \le \omega$, $\mathcal{BA}([0,1]^Y) = \mathcal{BO}([0,1]^Y)$ and we recover the countable case of [6].

When $\mathcal{T}$ carries a σ-state s, to the pair $(\mathcal{T}, s)$ is naturally associated the probability space $(X, \mathcal{S}(\mathcal{T}), \mu_s)$ as defined in Section 2. Thus, to each observable $\mathcal{X}$ it is associated the measurable function $f\colon X \to [0,1]^Y$ given by Theorem 3.2, which is a random variable, and the correspondence is one-one. See also the discussion after [6: Theorem 2.3].

Let us now generalize [6: Theorem 2.6] to the case of any arbitrary Y. The observables that arise from the following result will be subsequently called *joint observables*.

Theorem 3.3 (Joint observable theorem). *Given the Riesz tribe $\mathcal{T} \subseteq [0,1]^X$ and κ one-dimensional observables over $\mathcal{T}$, namely $\mathcal{X}_i\colon IRL(X_i) \to \mathcal{T}$, with $i \in \kappa$ and X_i singletons, for $X = \bigcup_i X_i$ the joint function $\mathcal{J}_\kappa\colon IRL(X) \to \mathcal{T}$ defined by $\mathcal{J}_\kappa(a) = a \circ f$, with $f = (\mathcal{X}_i(\mathrm{id}))_{i\in\kappa}$, is a κ-dimensional observable over $\mathcal{T}$, where $\mathrm{id}\colon [0,1]^{X_i} \to [0,1]^{X_i}$ is the identity function. Moreover, for any $i \in \kappa$ and any $a \in IRL(X_i)$, $\mathcal{J}_\kappa(a \circ \pi_i) = \mathcal{X}_i(a)$.*

P r o o f. Recalling that $IRL(X)$ is the free X-generated algebra in $\mathbf{RMV}_\sigma$, it follows that the assignment $\pi_i \mapsto f_i = \mathcal{X}_i(\mathrm{id}) \in \mathcal{T}$ extends to a σ-homomorphism $\mathcal{X}\colon IRL(X) \to \mathcal{T}$. Applying now Theorem 3.2 to such an $\mathcal{X}$, we can say that $\mathcal{X} = \mathcal{X}_g$, with $g = (g_i)_{i\in\kappa}$ given by $g_i = \mathcal{X}(\pi_i)$. By definition of $\mathcal{X}$, $g_i = f_i$ and $\mathcal{X}_g$ is exactly the map $\mathcal{J}_\kappa$, which is, therefore, an observable.

Finally, $\mathcal{J}_\kappa(a \circ \pi_i) = (a \circ \pi_i) \circ f = a \circ f_i = \mathcal{X}_i(a)$. □

Following [6], we call *process* any indexed set $\{\mathcal{X}_i\}_{i\in I}$ of one-dimensional observables, with I being an arbitrary set. Unless otherwise specified, for any observable $\mathcal{X}_i\colon IRL(Y) \to \mathcal{T}$ with $\mathcal{T} \subseteq [0,1]^X$, we will denote by f_i the unique measurable function given in Theorem 3.2, that is, the unique function such that $\mathcal{X}_i(a) = a \circ f_i$ for any $a \in IRL(Y)$.

As mentioned in the Introduction, the main goal of this section is to define and analyze, from the point of view of non-classical logic, the pivotal notions that are involved in de Finetti's theorem on exchangeable probabilities. To do so, let us start with analyzing basic notions of probability theory such as distributions and independence.

Definition 3.4. Let $\mathcal{X}\colon IRL(Y) \to \mathcal{T}$ be an observable on $\mathcal{T} \subseteq [0,1]^X$, given by $\mathcal{X}(a) = a \circ f$, and assume that $\mathcal{T}$ carries the state $s\colon \mathcal{T} \to [0,1]$. Its *distribution* is the state:

$$s_{\mathcal{X}}\colon IRL(Y) \to [0,1], \qquad s_{\mathcal{X}}(a) = s(\mathcal{X}(a)).$$

Remark 3.5. With the same notations as above, the function $f\colon X \to [0,1]^Y$ is a classical random variable between the spaces $(X, \mathcal{S}(\mathcal{T}), \mu_s)$ and $\big([0,1]^Y, \mathcal{BA}([0,1]^Y\big)$.

For any $E \in \mathcal{BA}([0,1]^Y)$, it follows that $\mathcal{X}(\chi_E) = \chi_E \circ f = \chi_{f^{-1}(E)}$ and its distribution state is given by $s_{\mathcal{X}}(\chi_E) = s(\mathcal{X}(\chi_E)) = s(\chi_{f^{-1}}(E))$. Thus, if we consider the measure $\mu_{s_{\mathcal{X}}}$ associated to $s_{\mathcal{X}}$ by Theorem 2.3, $\mu_{s_{\mathcal{X}}}(E) = \mu_s(f^{-1}(E))$.

Consequently, $\mu_{s_{\mathcal{X}}}$ is the pushforward measure of μ_s under f, and $F(E) := \mu_s(f^{-1}(E))$ is the probability of the event $(f \in E) = \{x \in X \mid f(x) \in E\}$, that is, the distribution law of f.

Lemma 3.6. *Let $\{\mathcal{X}_i\}_{i \in I}$ be an indexed set of one-dimensional observables, with I being an arbitrary set. For any natural number k, let F_k be the (classical) joint distribution function of $f_{i_1}, \dots, f_{i_k}$. Then, for any characteristic function χ_E, with $E \in \mathcal{BA}([0,1]^k)$, we have $F_k(E) = s_{\mathcal{J}_k}(\chi_E)$, where $s_{\mathcal{J}_k}$ denotes the distribution state of the joint observable of $\mathcal{X}_{i_1}, \dots, \mathcal{X}_{i_k}$.*

Proof. For each $f_{i_j}\colon X \to [0,1]$, its distribution is the function $F_{i_j}\colon \mathcal{BA}([0,1]) \to [0,1]$ that maps $E \mapsto \mu_s(f_{i_j}^{-1}(E)) = s(\chi_{f_{i_j}^{-1}(E)})$. Let us denote by $f_k\colon X \to [0,1]^k$ the function $x \mapsto (f_{i_1}(x), \dots, f_{i_k}(x))$. Then $\mathcal{J}_k(a) = a \circ f_k$ and the (classical) joint distribution of these random variables is given by $F_k\colon \mathcal{BA}([0,1]^k) \to [0,1]$, $B \mapsto \mu_s(f_k^{-1}(B))$. Hence, for $E \in \mathcal{BA}([0,1]^k)$, $s_{\mathcal{J}_k}(\chi_E) = s(\chi_{f_k^{-1}(E)}) = F_k(E)$, settling the claim. □

Lemma 3.7. *Let $(\mathcal{T}, s)$ be a probability Riesz tribe. Any function $f \in \mathcal{T}$ induces a one-dimensional observable $\mathcal{X}_f\colon IRL(1) \to \mathcal{T}$ whose distribution state corresponds to the distribution law of f on $\mathcal{BO}([0,1])$.*

Proof. It follows from the fact that any $f \in \mathcal{T}$ is $\mathcal{S}(\mathcal{T})$-measurable. □

Using this notion of distribution, we now define exchangeable processes. We also recall that the definition of a sequence of exchangeable random variables can be found at the end of Section 2.

Let $\{\mathcal{X}_i\}_{i \in I}$ be an indexed set of one-dimensional observables, with I being an arbitrary set. For any finite $k \in \mathbb{N}$ and two sets of indexes $i_1, \dots, i_k$, $j_1, \dots, j_k$, let us denote by $\mathcal{J}_k^i$ the joint observable of $\mathcal{X}_{i_1}, \dots \mathcal{X}_{i_k}$ and by $\mathcal{J}_k^j$ the joint observable of $\mathcal{X}_{j_1}, \dots \mathcal{X}_{j_k}$.

Definition 3.8. The process $\{\mathcal{X}_i\}_{i \in I}$, posed in the probability Riesz tribe $(\mathcal{T}, s)$, is called

(i) *weakly exchangeable* if the sequence $\{f_i\}_{i \in I}$ of corresponding random variables (such that $\mathcal{X}_i = \mathcal{X}_{f_i}$) is exchangeable.

(ii) *strongly exchangeable* if for any finite $k \in \mathbb{N}$ and two sets of indexes $i_1, \dots, i_k$, $j_1, \dots, j_k$, $\mathcal{J}_k^i$ and $\mathcal{J}_k^j$ have the same distribution state.

Thus, the process $\{\mathcal{X}_i\}_{i \in I}$ is weakly exchangeable when, for any $k \in \mathbb{N}$ and any two sets of indexes $i_1, \dots, i_k$, $j_1, \dots, j_k$, the distribution laws $F_{i,k}$ of $(f_{i_1}, \dots, f_{i_k})$ and $F_{j,k}$ of $(f_{j_1}, \dots, f_{j_k})$ coincide. Strong exchangeability requires the analogous property directly on the joint observable, rather than the joint random variable associated to it. The next lemma shows that indeed strong exchangeability implies weak exchangeability.

Lemma 3.9. *If $\{\mathcal{X}_i\}_{i\in I}$ is a strongly exchangeable process, then it is weakly exchangeable.*

P r o o f. Recalling the definition of exchangeable random variables from Section 2, by Corollary 3.6, for any $E \in \mathcal{BA}([0,1]^k)$, $F_{i,k}(E) = s_{\mathcal{J}_k^i}(\chi_E)$ and $F_{j,k}(E) = s_{\mathcal{J}_k^j}(\chi_E)$. By hypothesis on $\{\mathcal{X}_i\}_{i\in I}$, it follows that $s_{\mathcal{J}_k^i}(\chi_E) = s_{\mathcal{J}_k^j}(\chi_E)$, settling the claim. $\square$

Moving now on the notion of independence, we generalize the definition given for processes in [6: Section 5], which is itself adapted from the definition given in [17].

Definition 3.10. Let I be a set of indexes, $\{(\mathcal{T}_i, s_i)\}_{i\in I}$ and $(\mathcal{T}, s)$ be probability Riesz tribes. The collection $\{(\mathcal{T}_i, s_i)\}_{i\in I}$ is said to be $(\mathcal{T}, s)$-*independent* if for any $k \in \mathbb{N}$ and any $i_1, \dots, i_k$ there exists an k-additive operator (as defined in Section 2)

$$\beta\colon \mathcal{T}_{i_1} \times \dots \times \mathcal{T}_{i_k} \to \mathcal{T}$$

such that for all $a_{i_1} \in \mathcal{T}_{i_1}, \dots, a_{i_k} \in \mathcal{T}_{i_k}$ we have

$$s(\beta(a_{i_1}, \dots, a_{i_k})) = s_{i_1}(a_{i_1}) \cdot \ldots \cdot s_{i_k}(a_{i_k}).$$

If all β are surjective, the collection is called surjectively independent.

In [6], the definition was given for a stochastic process $\{\mathcal{X}_i\}_{i\in I}$ posed in a probability Riesz tribe $(\mathcal{T}, s)$. The process was called independent if any finite subset $\{\mathrm{Im}(\mathcal{X}_{\mathrm{i}_1}), \dots, \mathrm{Im}(\mathcal{X}_{\mathrm{i}_\mathrm{k}})\}$ of the sequence of tribes $\mathrm{Im}(\mathcal{X}_\mathrm{i})$ is $\mathrm{Im}(\mathcal{J}_\mathrm{k})$-independent, where $\mathcal{J}_k$ is their joint observable.

Note that, if any of the k-additive operators β of the definition of an independent process is surjective, for any $f \in \mathrm{Im}(\mathcal{J}_\mathrm{k})$ there exists $f_{i_1} \in \mathrm{Im}(\mathcal{X}_{\mathrm{i}_1}), \dots, \mathrm{f}_{\mathrm{i}_\mathrm{k}} \in \mathrm{Im}(\mathcal{X}_{\mathrm{i}_\mathrm{k}})$ such that $\beta(f_{i_1}, \dots f_{i_k}) = f$. Consequently, for any $a \in IRL(k)$ there exists $a_{i_1}, \dots, a_{i_k} \in IRL(1)$ such that $\mathcal{J}_k^i(a) = \beta(\mathcal{X}_{i_1}(a_1), \dots, \mathcal{X}_{i_k}(a_k))$.

Lemma 3.11. *Let $\{\mathcal{X}_i\}_{i\in I}$ be a one-dimensional process posed on a Riesz tribe $(\mathcal{T}, s)$, with $\mathcal{T} \subseteq [0,1]^X$. If the process is independent and identically distributed with all β's being the product of functions, then it is weakly exchangeable.*

P r o o f. By [6: Proposition 5.9], the associate sequence $\{f_i\}_{i\in I}$ is independent; by Remark 3.5 for any $i \in I$, the distribution law of f_i is $F_i := \mu_s \circ f_i^{-1}$ and by hypothesis for any $i, j \in I$ we have $s \circ \mathcal{X}_i = s \circ \mathcal{X}_j$. Consequently, again by Remark 3.5, for any $E \in \mathcal{BA}([0,1])$, $F_i(E) = \mu_s(f_i^{-1}(E)) = s(\mathcal{X}_i(\chi_E)) = s(\mathcal{X}_j(\chi_E)) = \mu_s(f_j^{-1}(E)) = F_j(E)$. Hence, the classical process $\{f_i\}_{i\in I}$ is independent and identically distributed, and therefore exchangeable. Indeed, for any $k \in \mathbb{N}$ and any two sets of indexes $i_1, \dots, i_k$, $j_1, \dots, j_k$, let $F_{i,k}$ be the distribution law of $f_i^k = (f_{i_1}, \dots, f_{i_k})$ and let $F_{j,k}$ be the distribution law of $f_j^k = (f_{j_1}, \dots, f_{j_k})$. For any generator $\prod_{n=1}^{k} E_n$ of $\mathcal{BA}([0,1]^k)$, since the f_i's are independent and identically distributed,

$$\begin{aligned}
F_{i,k}(E) &= \mu_s((f_i^k)^{-1}(E)) = \mu_s\left(\bigcap_{n=1}^{k} f_{i_n}^{-1}(E_n)\right) \\
&= \mu_s\left(f_{i_1}^{-1}(E_1)\right) \cdot \ldots \cdot \mu_s\left(f_{i_k}^{-1}(E_k)\right) = \mu_s\left(f_{j_1}^{-1}(E_1)\right) \cdot \ldots \cdot \mu_s\left(f_{j_k}^{-1}(E_k)\right) \\
&= \mu_s\left(\bigcap_{n=1}^{k} f_{j_n}^{-1}(E_n)\right) = \mu_s((f_j^k)^{-1}(E)) = F_{j,k}(E).
\end{aligned}$$

Hence, the $\mathcal{X}_i$'s are weakly exchangeable. $\square$

One open problem is to understand when the conclusion of Lemma 3.11 can be extended to strong exchangeability. Furthermore, for a countable and weakly exchangeable process, the classical version of de Finetti's exchangeability implies that the distribution law of the associated sequence of random variables is *conditionally i.i.d.* (see [16: Chapter 11]). It is still unclear how and if we can give a suitable logical interpretation for this notion, in terms of observables. To tackle this issue, in the next section we define a counterpart for the notion of product measure.

4. States on coproducts in $\mathbf{RMV}_\sigma$

Taking inspiration from the definition of presentable measures from [15], we now define an appropriate counterpart of the product measure for states. Thinking of states as an algebraic dual of probability measures, we will first give a characterization for the coproduct of objects in $\mathbf{RMV}_\sigma$.

We recall that, given an arbitrary set I, for $\{A_i\}_{i\in I} \in \mathbf{RMV}_\sigma$, the coproduct $\bigoplus_{i\in I} A_i$ is defined as a pair $(\bigoplus_{i\in I} A_i, \{\alpha_i\}_{i\in I})$ such that $\alpha_i\colon A_i \to \bigoplus_{i\in I} A_i$ and for any other object C and maps $\eta_i\colon A_i \to C$, $i \in I$, there exists a unique map $\eta\colon \bigoplus_{i\in I} A_i \to C$ such that $\eta_i = \eta \circ \alpha_i$ for all $i \in I$. If, in addition, the maps α_i are one-one, and $\bigcup_{i\in I} \alpha_i(A_i)$ generates $\bigoplus_{i\in I} A_i$, $\bigoplus_{i\in I} A_i$ is called *free product* of the A_i.

We note that the existence of all coproducts in $\mathbf{RMV}_\sigma$ follows from [2: Theorem 9.4.14]. In the following proposition we use the same proof strategy sketched in [2] and given in details in [20: Theorem 7.1], and characterize free products of non-trivial σ-semisimple algebras, as defined in Section 2.

Proposition 4.1. *Let $\{A_i\}_{i\in I}$ be a collection of σ-semisimple algebras in $\mathbf{RMV}_\sigma$, indexed in a set I. For any $i \in I$, let $A_i \simeq IRL(X_i)/I_i$, where we assume the sets X_i to be pairwise disjoint. Then the free product $\bigoplus_i A_i$ exists in $\mathbf{RMV}_\sigma$ and*

$$\bigoplus_i A_i = IRL\Big(\bigcup_i X_i\Big)/J,$$

with $J = \langle \bigcup_i I_i \rangle_\sigma$, the σ-ideal generated by $\bigcup_i I_i$.

Proof. Let us denote by A the algebra $IRL(\bigcup_i X_i)/J \in \mathbf{RMV}_\sigma$. For any $i \in I$, define the map

$$\alpha_i\colon A_i \to A, \quad \alpha_i(f/I_i) = f/J.$$

Each α_i is well defined because $IRL(X_i)$ embeds in $IRL\big(\bigcup_i X_i\big)$ and $I_i \subseteq J$.

Let us divide the proof in three steps.

(1) *The maps α_i are embeddings.*
Let f/I_i be an element such that $f/J = 0$. Let us prove that $f \in I_i$.

By hypothesis, $f/J = 0$ implies $f \in J$. By [7: Proposition 2.6] there exists a countable set of elements $G = \{f_n \mid n \in \mathbb{N}\} \in \bigcup_i I_i$ such that $f \le \tau(f_1, \dots, f_n, \dots)$, where τ is a term build using only $\oplus$ and $\bigvee$. We note that the variables that appear in f belong to X_i. Moreover, any f_n belongs to a proper ideal I_j, $j \in I$.

Let $F_i = \{g \in G \mid g \in I_i\}$ be the subset of the f_n's that belong to I_i. For any $g \in G \setminus F_i$ there exists $j \ne i$ and a point in $\mathbf{y_j} \in [0,1]^{X_j}$, such that $g \in I_j$ and $g(\mathbf{y_j}) = 0$. To see this, assume that for some $g \in G \setminus F_i$, $g \in I_j$, we have $\mathbb{V}(g) = \emptyset$. Hence we deduce that $\mathbb{I}(\mathbb{V}(g)) = IRL(X_j)$. But [7: Lemma 4.15] implies $\mathbb{I}(\mathbb{V}(g)) = \langle g \rangle_\sigma \subseteq I_j$, and I_j is proper ideal, which is a contradiction.

Take now $\mathbf{y} \in [0,1]^{X_i}$ such that $g(\mathbf{y}) = 0$ for any $g \in F_i$. Denote by $\mathbf{z}$ the point of $[0,1]^{\bigcup_i X_i}$ such that $\mathbf{z}$ coincides with $\mathbf{y}$ in the X_i-coordinates and it coincides with $\mathbf{y_j}$ in the X_j-coordinates, for $j \neq i$, and it is arbitrary in the coordinates eventually not taken into account. Note that this is well defined as the sets of variables are disjoint. Then $\tau(f_1, \dots, f_n, \dots)(\mathbf{z}) = 0$ and therefore $f(\mathbf{z}) = 0$.

We have proved that $g(\mathbf{y}) = 0$ for any $g \in F_i$ implies $f(\mathbf{z}) = 0$, and consequently (since the only variables appearing in f belong to X_i) $f(\mathbf{y}) = 0$. Thus, $\mathbb{V}(f) \supseteq \mathbb{V}(F_i)$. By [7: Theorem 4.7 and Lemma 4.15] and the hypothesis on A_i we deduce that $\langle f \rangle_\sigma = \mathbb{I}(\mathbb{V}(f)) \subseteq \mathbb{I}(\mathbb{V}(F_i)) \subseteq \mathbb{I}(\mathbb{V}(I_i)) = I_i$. Thus, we have proved that $f \in I_i$ and α_i is one-one.

(2) $\bigcup_i \alpha_i(A_i)$ *generates* A.

Each $IRL(X_i)$ is generated by $\{\pi_x \mid x \in X_i\}$, the coordinate projections. Analogously, $\{\pi_x \mid x \in \bigcup_i X_i\}$ is a set of generators for $IRL\big(\bigcup_i X_i\big)$. Therefore, the set $P = \{\pi_x/J \mid x \in \bigcup_i X_i\}$ generates A. By the definition of α_i, P coincides with $\bigcup_i \{\alpha_i(\pi_x/I_i) \mid x \in X_i\} \subseteq \bigcup_i \alpha_i(A_i)$, which settles the claim.

(3) $(A, \{\alpha_i\}_{i \in I})$ *satisfies the universal property.*

Let $E \in \mathbf{RMV}_\sigma$ be an algebra such that we have homomorphisms $\eta_i \colon A_i \to E$ for any $i \in I$. We have to prove that there exists $\eta \colon A \to E$ such that $\eta \circ \alpha_i = \eta_i$ for any $i \in I$.

Let $Q = \{\pi_x \mid x \in \bigcup_i X_i\}$ be the set of generators for $IRL\big(\bigcup_i X_i\big)$. Define $\varrho \colon Q \to E$ as $\varrho(\pi_x) = \eta_i(\pi_x/I_i)$, when $x \in X_i$. Note that ϱ is well defined since the X_i's are pairwise disjoint. Let $\overline{\varrho} \colon IRL\big(\bigcup_i X_i\big) \to E$ the unique σ-homomorphism that extends ϱ to the free algebra. Note that, by definition, and because $\{\pi_x \mid x \in X_i\}$ generates $IRL(X_i) \subseteq IRL\big(\bigcup_i X_i\big)$, for any $i \in I$ and any $f \in IRL(X_i)$, $\overline{\varrho}(f) = \eta_i(f/I_i)$. Take now $\eta \colon A \to E$ to be defined as $\eta(f/J) = \overline{\varrho}(f)$.

Let us prove that η is well defined. Take $f, g \in IRL\big(\bigcup_i X_i\big)$ such that $f/J = g/J$. Then, denoted by d Chang's distance, $d(f,g) \in J$, whence there exists a countable set $\{h_n\}_n \in \bigcup_i I_i$ such that $d(f,g) \leq \tau(h_1, \dots, h_n, \dots)$, where in τ only $\oplus$ and $\bigvee$ appear. We have

$$\begin{aligned} d(\overline{\varrho}(f), \overline{\varrho}(g)) &= \overline{\varrho}(d(f,g)) \leq \overline{\varrho}(\tau(h_1, \dots, h_n, \dots)) \\ &= \tau(\overline{\varrho}(h_1), \dots, \overline{\varrho}(h_n), \dots). \end{aligned}$$

Since each h_n belongs to $\bigcup_i I_i$, $\overline{\varrho}(h_n) = 0$ and $\tau(\overline{\varrho}(h_1), \dots, \overline{\varrho}(h_n), \dots) = 0$. Consequently, $d(\overline{\varrho}(f), \overline{\varrho}(g)) = 0$ and $\overline{\varrho}(f) = \overline{\varrho}(g)$.

Finally, for any $i \in I$ and any $f \in IRL(X_i)$, $(\eta \circ \alpha_i)(f/I_i) = \eta(\alpha_i(f/I_i)) = \eta(f/J) = \overline{\varrho}(f) = \eta_i(f/I_i)$, settling the final part of the claim. □

Definition 4.2. We shall call an algebra $A \in \mathbf{RMV}_\sigma$ countably presented if $A \simeq IRL(X)/I$ where X is a *countable* set of generators and I is principal. We shall denote by $\mathbf{RMV}_\sigma^\omega$ the full subcategory of $\mathbf{RMV}_\sigma$ whose objects are countably presented and σ-complete Riesz MV-algebras.

Remark 4.3. For any algebra $A \in \mathbf{RMV}_\sigma$, it is an easy exercise to show that countably generated σ-ideals in A are principal. Indeed, for a sequence $\{a_n\}_{n \in \mathbb{N}}$ in A, we have $\langle \{a_n\}_{n \in \mathbb{N}} \rangle_\sigma = \big\langle \bigvee_{n \in \mathbb{N}} a_n \big\rangle_\sigma$.

Corollary 4.4. *The full subcategory* $\mathbf{RMV}_\sigma^\omega$ *of* $\mathbf{RMV}_\sigma$ *whose objects are countably presented and* σ*-complete Riesz MV-algebras is closed under countable free products. The full subcategory* $\mathbf{RMV}_\sigma^{\mathbf{fp}}$ *of* $\mathbf{RMV}_\sigma$ *whose objects are finitely presented and* σ*-complete Riesz MV-algebras is closed under finite free products.*

Proof. It follows from [7: Lemma 4.15] that both countably presented algebras and finitely presented algebras are σ-semisimple. Moreover, if we deal with a countable set of algebras $A_i \simeq {IRL(X_i)}/{I_i}$, and for some $f_i \in IRL(X_i)$ we have $I_i = \langle f_i \rangle_\sigma$, $J = \left\langle \bigvee_i f_i \right\rangle_\sigma$ by Remark 4.3. Thus, ${IRL(\bigcup_i X_i)}/{J}$ is a countably presented algebra. Analogously, a finite free product will be finitely presented. □

The next proposition, given in [7], is crucial for what follows.

Proposition 4.5 ([7: Proposition 4.12])**.** *Let* A *be a* σ*-semisimple* σ*-complete Riesz MV-algebra and assume that* ${IRL(X)}/{I}$ *is a presentation for* A*. Then* A *is isomorphic to the Riesz tribe* $IRL(X)|_{\mathbb{V}(I)} = \{g \in [0,1]^{\mathbb{V}(I)} \mid g = f_{|\mathbb{V}(I)}$ *for some* $f \in IRL(X)\}$*.*

Given a tribe $\mathcal{T} \subseteq [0,1]^T$, we recall that $\mathcal{S}(\mathcal{T}) = \{A \subseteq T \mid \chi_A \colon [0,1]^T \to \{0,1\},\ \chi_A \in \mathcal{T}\}$ denotes its Boolean center, as defined in Section 2. For any $P \subseteq [0,1]^X$ we can define the tribe of restrictions $IRL(X) \mid_P$. In this case, for brevity and if it is clear from the context, we shall denote the Boolean center $\mathcal{S}(IRL(X)|_P)$ by $\mathcal{S}(P)$.

Lemma 4.6. *For any* $P \subseteq [0,1]^X$*, denoted by* $\mathcal{BA}(P) = \{A \cap P \mid A \in \mathcal{BA}([0,1]^X)\}$*, the algebra of restrictions* $IRL(X)|_P = \{f|_P \mid f \in IRL(X)\}$ *is the Riesz tribe of all* $\mathcal{BA}(P)$*-measurable functions.*

Proof. As remarked before, the algebra of restrictions $IRL(X)|_P = \{f|_P \mid f \in IRL(X)\}$ is a Riesz tribe and by [20: Lemma 11.8] it is the algebra of all $\mathcal{S}(P)$-measurable functions, where $\mathcal{S}(P) = \{B \subseteq P \mid \chi_B \in IRL(X) \mid_P\}$.

Thus, we only need to prove that the σ-algebras $\mathcal{BA}(P)$ and $\mathcal{S}(P)$ coincide.

For any $B \subseteq P$, $B \in \mathcal{S}(P)$ if, and only if, $\chi_B \in IRL(X)|_P$. The latter is equivalent to saying that there exists $p \in IRL(X)$ such that $\chi_B = p|_P$. Take $A \subseteq [0,1]^X$ to be the Baire set $\mathbb{V}(1-p)$. It is easily seen that $B = A \cap P$.

Conversely, for any set of type $A \cap P$, with $A \in \mathcal{BA}([0,1]^X)$, $\chi_{A\cap P} \colon P \to \{0,1\}$ coincide with $\chi_{A|_P}$. Since A is Baire-measurable, it follows that $\chi_A \in IRL(X)$ and $A \cap P \in \mathcal{S}(P)$. □

Building on the previous results, let us describe a somewhat canonical way to define a state on the coproduct starting from algebras endowed with states.

Take a sequence of pairs $\left({IRL(X_i)}/{I_i}, s_i\right)_{i\in I}$. With the same notation as before, let ${IRL(X)}/{J}$ be the free product of the sequence. We have $X = \bigcup_i X_i$, the X_i are chosen to be pairwise disjoint, ${IRL(X)}/{J} \simeq IRL(X) \mid_{\mathbb{V}(J)}$. When it is clear from the context, we denote by π_{x_i} the x_i-coordinate projection in either X-coordinates or X_i-coordinates, that is, $\pi_{x_i} \colon [0,1]^X \to [0,1]$ and $\pi_{x_i} \colon [0,1]^{X_i} \to [0,1]$, with $x_i \in X_i \subseteq X$.

Following the notation set out in [7] and briefly recalled in Section 2, we denote by $\mathbf{Baire}^\omega$ the full subcategory of $\mathbf{IRL}$ whose objects are Baire subsets of hypercubes of type $[0,1]^X$, for a countable X. Note that the duality given in [7] for finitely presented algebras is straightforwardly extended to a duality between $\mathbf{Baire}^\omega$ and $\mathbf{RMV}_\sigma^\omega$. More precisely, the duality is between $\mathbf{Baire}^\omega$ and the category of presentations for algebras in $\mathbf{RMV}_\sigma^\omega$. Nonetheless, this category of presentations is equivalent to $\mathbf{RMV}_\sigma^\omega$, see [4: Remark 4.7]. Thus, the coproduct of a sequence ${IRL(X_i)}/{I_i} \in \mathbf{RMV}_\sigma^\omega$ is reflected in a product in the dual category $\mathbf{Baire}^\omega$.

Hence, to a sequence of algebras ${IRL(X_i)}/{I_i} \in \mathbf{RMV}_\sigma^\omega$ we can naturally associate the sequence of measure spaces $(\mathbb{V}(I_i), \mathcal{S}(\mathbb{V}(I_i))$, and to their coproduct we can associate the space $(\mathbb{V}(J), \mathcal{S}(\mathbb{V}(J)))$.

Lemma 4.7. *Let I be a countable set and let $\{X_i\}_{i\in I}$ be a pairwise disjoint sequence of sets of variables. For any $i \in I$, let $f_i \in IRL(X_i)$ be an IRL-polynomial, let $X = \bigcup_i X_i$ and let J be the σ-ideal generated by the sequence $\{f_i\}_{i\in I}$ in $[0,1]^X$. Then $\mathbb{V}(J)$ is the product in $\mathbf{Baire}^\omega$ of the objects $\mathbb{V}(f_i)$, where the projection from $\mathbb{V}(J)$ to $\mathbb{V}(f_i)$ is induced by the inclusion of X_i in X.*

Proof. By definition, a point $\mathbf{x} \in [0,1]^X$ belongs to $\mathbb{V}(J)$ if, and only if, $f_i(\mathbf{x}) = 0$ for all $i \in I$. Since we have chosen the X_i's to be pairwise disjoint, it follows that $\mathbf{x} \in \mathbb{V}(J)$ if, and only if, $(\pi_{x_i}(\mathbf{x}))_{x_i\in X_i} \in \mathbb{V}(f_i)$. Thus, $\mathbb{V}(J)$ is actually the set of those points in $[0,1]^X$ such that the 'X_i-coordinates' form a point of $\mathbb{V}(f_i)$, that is, their cartesian product.

For brevity, let us denote $\mathbb{V}(J)$ by V and $\mathbb{V}(f_i)$ by V_i. For any $i \in I$, let $\eta_i \colon V \subseteq [0,1]^X \to V_i \subseteq [0,1]^{X_i}$ be defined by $\eta_i(\mathbf{x}) = (\pi_{x_i}(\mathbf{x}))_{x_i\in X_i}$, where $\pi_{x_i} \colon [0,1]^{X_i} \to [0,1]$ are the coordinate projections that generate $IRL(X_i)$. This is well defined because the inclusion $X_i \subseteq X$ allows to think of $IRL(X_i)$ as embedded in $IRL(X)$. Hence, by definition, each η_i is an arrow in $\mathbf{Baire}^\omega$ since it is a tuple of IRL-polynomials.

Take now any other $W \in \mathbf{Baire}^\omega$, $W \subseteq [0,1]^Y$ such that for any $i \in I$ there exists $\nu_i \colon W \to V_i$. Thus, by definition of arrows, $\nu_i(\mathbf{w}) = (\nu_{x_i}(\mathbf{w}))_{x_i\in X_i}$ for some $\nu_{x_i} \in IRL(Y)$.

Define $\nu \colon W \to V$ by $\nu(\mathbf{w}) = (\nu_x(\mathbf{w}))_{x\in \bigcup_i X_i}$. We note that ν is an IRL-map, moreover it is well defined since the X_i-coordinates of the point $\nu(\mathbf{w})$ belong to V_i, making $\nu(\mathbf{w})$ a point in V. Furthermore, for any $i \in I$, $\eta_i(\nu(\mathbf{w})) = (\pi_{x_i}(\nu(\mathbf{w})))_{x_i\in X_i} = (\nu_{x_i}(\mathbf{w}))_{x_i\in X_i} = \nu_i(\mathbf{w})$. Consequently, (V, η_i) is the product of the V_i in $\mathbf{Baire}^\omega$. □

For any collection of σ-algebras $\{\mathcal{B}_i\}_{i\in I}$, we denote by $\bigtimes_i \mathcal{B}_i$ the product σ-algebra generated by $\prod_i A_i$, $A_i \in \mathcal{B}_i$. When I is an infinite countable set, we require that A_i coincides with the whole space for all but a finite number of indexes.

Lemma 4.8. *With the same notations of the previous lemma, the Boolean center $\mathcal{S}(V)$ of $IRL(X)|_V$ is the product σ-algebra of the Boolean centers $\mathcal{S}(V_i)$ of $IRL(X_i)|_{V_i}$.*

Proof. We shall use the characterization of the Boolean center given in Lemma 4.6, as well as the notation fixed there.

Let us first remark that, since each X_i is countable, $[0,1]^{X_i}$ is a compact and metric space, whence it is separable. Thus, we can apply [16: Lemma 1.2] and deduce that $\mathcal{BA}([0,1]^X) = \bigtimes_i \mathcal{BA}([0,1]^{X_i})$. Therefore sets of type $\prod_i B_i$, with $B_i \in \mathcal{BA}([0,1]^{X_i})$ are generators for $\mathcal{BA}([0,1]^X)$. Furthermore, since $V \in \mathcal{BA}([0,1]^X)$ it is easily seen that $\big(\prod_i B_i\big) \cap V$ is a set of generators for $\mathcal{BA}(V)$, where $\prod_i B_i$ is defined as before.

The conclusion is now straightforward since, by Lemma 4.7, $V = \prod_i V_i$. Indeed, take $A_i = B_i \cap V_i$ with $B_i \in \mathcal{BA}([0,1]^{X_i})$. From the previous remarks we have that the product $\prod_i A_i$ is a generator of $\bigtimes_i \mathcal{S}(V_i)$, the product $\prod_i B_i$ is a generator for $\mathcal{BA}([0,1]^X)$ and the intersection $\prod_i B_i \cap V$ is a generator for $\mathcal{BA}(V)$.

Since $\prod_i A_i = \prod_i (B_i \cap V_i) = \prod_i B_i \cap \prod_i V_i = \prod_i B_i \cap V$, we can see that the sets of generators of $\bigtimes_i \mathcal{S}(V_i)$ and $\mathcal{S}(V)$ are the same, and the two σ algebras coincide. $\square$

We can now build on the previous lemmas, using the same notation set out before. Assume that each $IRL(X_i)/_{I_i} \simeq IRL(X_i)|_{V_i}$ carries a state s_i. Then, we can naturally associate the measure space $(V_i, \mathcal{S}(V_i), \mu_i)$, where μ_i is the probability measure defined by $\mu_i(A) = s_i(\chi_A)$. Take now the usual product measure μ of the μ_i and define $s\colon IRL(X)|_{\mathbb{V}(J)} \to [0,1]$ as the integral with respect to μ.

Definition 4.9. We call *coproduct presentable* any tribe $\mathcal{T}$ that can be obtained as a coproduct of a sequence of algebras in $\mathbf{RMV}^{\omega}_{\sigma}$. If, in addition, each algebra of the sequence carries a σ-state, we call *coproduct state* the state obtained via the product measure, as described above.

With the same notations of the previous lemmas, for any finite set of indexes $i_1, \dots, i_k \in I$, define the map

$$\begin{aligned} &\beta\colon IRL(X_{i_1})|_{V_{i_1}} \times \dots \times IRL(X_{i_k})|_{V_{i_k}} \to IRL(X)|_V, \\ &\beta(a_{i_1}, \dots, a_{i_k}) = a_{i_1} \cdot \ \dots \ \cdot a_{i_k}, \end{aligned} \tag{1}$$

where the product $\cdot$ is the usual ring-product of real-valued functions. We note that β is a k-additive function and it is also well defined, since each a_{i_j} can be embedded in $IRL(X)|_V$ and the product of $\mathcal{BA}(V)$-measurable functions is $\mathcal{BA}(V)$-measurable, thus it does belong to $IRL(X)|_V$.

The next theorem shows how, in the finite case, the summands of the coproduct are independent with respect to the coproduct state.

Theorem 4.10. *With the notation fixed above, if $|I| < \omega$, the σ-complete Riesz MV-algebras $\{(IRL(X_i)|_{V_i}, s_i)\}_{i \in I}$ are $(IRL(X)|_V, s)$-independent.*

Proof. Take β as defined in equation (1). Take any subset of $\{(IRL(X_i)|_{V_i}, s_i)\}_{i \in I}$ of cardinality $k \in \mathbb{N}$. Let $h \in \mathbb{N}$ be another integer such that $|I| = k + h$ and assume that $I = \{i_1, \dots, i_k\} \cup \{j_1, \dots, j_h\}$. Furthermore, let $\mathbf{1}_{V_{j_1}}, \dots, \mathbf{1}_{V_{j_h}}$ denote the functions that are identically equal to 1 on $V_{j_1}, \dots, V_{j_h}$. We remark that, by definition of s and β, the measure associated to s is the product measure of the μ_i's, and each μ_i is a finite measure. Thus, we can apply (iteratively) the well known Fubini-Tonelli theorem (see, e.g. [14: Section 36]): for any $a_{i_1} \in IRL(X_{i_1})|_{V_{i_1}}, \dots, a_{i_k} \in IRL(X_{i_k})|_{V_{i_k}}$,

$$\begin{aligned} s(\beta(a_{i_1}, \dots, a_{i_k})) &= \int_V (a_{i_1} \cdot \ \dots \ \cdot a_{i_k}) \, \mathrm{d}\mu = \int_V \Big(a_{i_1} \cdot \ \dots \ \cdot a_{i_k} \cdot \mathbf{1}_{V_{j_1}} \cdot \mathbf{1}_{V_{j_h}} \Big) \, \mathrm{d}\mu \\ &= \Big(\int_{V_{i_1}} a_{i_1} \mathrm{d}\mu_{i_1} \Big) \cdot \ \dots \ \cdot \Big(\int_{V_{i_k}} a_{i_k} \mathrm{d}\mu_{i_k} \Big) \cdot \mu_{j_1}(V_{j_1}) \cdot \ \dots \ \cdot \mu_{j_h}(V_{j_h}) \\ &= s_{i_1}(a_{i_1}) \cdot \ \dots \ \cdot s_{i_k}(a_{i_k}), \end{aligned}$$

settling the claim. $\square$

Definition 4.11. Let $A \in \mathbf{RMV}^{\omega}_{\sigma}$, $A \simeq IRL(X)/_I$. Take the coproduct $\bigoplus_\omega A$ of A with itself countably many times. A σ-state $s\colon \bigoplus_\omega A \to [0,1]$ is called *weakly exchangeable* if the associate measure, on the product of countable copies of $(\mathbb{V}(I), \mathcal{S}(\mathbb{V}(I))$, is exchangeable in the classical sense. Similarly, the σ-state s is called *weakly presentable* if the associated measure is presentable in the classical sense.

As a straightforward remark, the coproduct state is weakly exchangeable.

Lemma 4.12. *For any $n \leq \omega$ and any $A \in \mathbf{RMV}_\sigma^\omega$, endowed with a σ-state $s\colon A \to [0,1]$, the coproduct state on $\bigoplus_{i=1}^{n} A_i$, where $A_i \simeq A$ for any i, is weakly exchangeable.*

Proof. It is a consequence of the fact that a product measure on a countable power V^n is exchangeable, see [15]. □

The next theorem is our weak version of de Finetti's theorem for states, which depends upon the classical result.

Theorem 4.13 (Weak de Finetti's exchangeability)**.** *Let X be a countable set. A state on $IRL(X)$ is weakly exchangeable if, and only if, it is weakly presentable.*

Proof. Since $IRL(X)$ is presented by the trivial ideal, for any state of $\bigoplus_{\omega} IRL(X)$, the associated measure is defined on countable copies of $\mathcal{BA}([0,1]^X)$. Then, since $[0,1]^X$ is a compact and Hausdorff space, we apply Theorem 2.4. □

We close this section with some remarks on these notions. While it is somewhat natural to define independence directly on both stochastic processes and sequences of tribes, and it is also natural to define exchangeability of processes as done in Definition 3.8, it is still unclear how to define "strongly" exchangeable and "strongly" presentable states is in this framework, where the adverb strongly is to be intended as "without reference to the associated measure". Since Definition 4.11 is completely dependent on the associated measure, it will be interesting to understand if and how we can adjust the definition in this sense.

To give more strength to our results, one possibility is to explore states in a categorial setting. With this aim in mind, in the next section we extend the Butnariu-Klement and Kroupa-Panti representations for σ-states and states, into categorical dualities.

5. Dualities for algebras with states

In this section we see how both dualities given in [7] are suitable to be lifted to a setting that incorporates states and measures. We use the results of the previous section.

We start by enriching the duality between the algebraic category $\mathbf{RMV}_\sigma$ of σ-complete Riesz MV-algebras and category $\mathbf{BDKH}$ of compact, Hausdorff and basically disconnected topological spaces endowed with cozero-closed continuous functions, see Proposition 2.1. Consider the following categories:

(i) $\mathbf{pRMV}_\sigma$ is the category whose objects are pairs (A, s), with $A \in \mathbf{RMV}_\sigma$ and s a state, and whose morphisms are *state-preserving* homomorphisms. That is, $\eta\colon (A, s_A) \to (B, s_B)$ is a morphism in $\mathbf{pRMV}_\sigma$ if it a σ-homomorphism of Riesz MV-algebras such that $s_A = s_B \circ \eta$.

(ii) $\mathbf{pBDKH}$ is the category whose objects are pairs (X, μ), with $X \in \mathbf{BDKH}$ and μ Borel regular probability measure on $\mathcal{BO}(X)$, the σ-algebra of Borel subsets of X. Morphisms in $\mathbf{pBDKH}$ are continuous and cozero-closed that are also *measure-preserving*, that is, $f\colon (X, \mu_X) \to (Y, \mu_Y)$ such that $\mu_Y = \mu_X \circ f^{-1}$.

The Kroupa-Panti theorem, together with the restriction of Kakutani's duality given in [7], gives naturally an essentially surjective functor that maps each $(X, \mu_X) \in \mathbf{pBDKH}$ into $(C(X), s_X) \in \mathbf{pRMV}_\sigma$, with s_X being defined by integration as in Theorem 2.2.

Moreover, the duality of [7] allows to associate to each morphism $f\colon X \to Y$ in **BDKH**, a unique $\tilde{f}\colon C(Y) \to C(X)$, with $\tilde{f}(g) = g \circ f$ for any $g \in C(Y)$ σ-homomorphism of Riesz MV-algebras. Note that, since we are dealing with a duality, the assignment $f \mapsto \tilde{f}$ is full and faithful.

Therefore, in order to lift the duality of [7] to $\mathbf{pRMV}_\sigma$ and **pBDKH**, it is enough to show that measure-preserving functions are mapped to state-preserving homomorphisms and viceversa.

Proposition 5.1. *Let $f\colon (X, \mu_X) \to (Y, \mu_Y)$ be morphism in* **BDKH**, *that is, a continuous and cozero closed function. Let $\tilde{f}\colon (C(Y), s_Y) \to (C(X), s_X)$ be its corresponding homomorphism in $\mathbf{RMV}_\sigma$. Then $\mu_Y = \mu_X \circ f^{-1}$ if, and only if, $s_Y = s_X \circ \tilde{f}$.*

P r o o f. By hypothesis $f\colon (X, \mu_X) \to (Y, \mu_Y)$ is continuous, cozero-closed and measure preserving and μ_Y is the pushforward measure of μ_X with respect to f. Hence, by the property of change of variables of pushforward measures,

$$s_Y(g) = \int_Y g \, \mathrm{d}\mu_Y = \int_Y g \, \mathrm{d}(\mu_X \circ f^{-1}) = \int_X (g \circ f) \, \mathrm{d}\mu_X = s_X(\tilde{f}(g)),$$

which settles one direction of the claim.

On the other hand, assume $\tilde{f}$ is a measure preserving σ-homomorphism. That is, for any $g \in C(Y)$, $s_X(\tilde{f}(g)) = s_Y(g)$. Than, as before,

$$\int_Y g \, \mathrm{d}\mu_Y = \int_Y g \, \mathrm{d}(\mu_X \circ f^{-1}).$$

By the Kroupa-Panti theorem, the regular measure associated to s_Y is unique. Thus, if we prove that $\mu_X \circ f^{-1}$ is a regular measure, we can infer that $\mu_Y = \mu_X \circ f^{-1}$.

Following [13], we recall that (since Y is a compact space) it is enough to prove that $\mu_X \circ f^{-1}$ is inner regular, that is, for any E Borel subset of Y,

$$(\mu_X \circ f^{-1})(E) = \sup\{(\mu_X \circ f^{-1})(K) \mid K \subseteq E,\ K\text{compact}\}. \tag{2}$$

Let $\mathcal{K} = \{K \subseteq Y \mid K \subseteq E,\ K\text{compact}\}$. One inequality of equation (2) follows from the fact that for any $K \in \mathcal{K}$, $(\mu_X \circ f^{-1})(K) \le (\mu_X \circ f^{-1})(E)$, whence $(\mu_X \circ f^{-1})(E)$ is greater or equal of their supremum, that is,

$$(\mu_X \circ f^{-1})(E) \ge \sup\{(\mu_X \circ f^{-1})(K) \mid K \subseteq E,\ K\text{compact}\}. \tag{3}$$

Let us prove the other inequality.

To do so, let

$$\mathcal{H} = \{C \subseteq X \mid C \subseteq f^{-1}(E),\ C \text{ compact}\}$$

and

$$\mathcal{D} = \{f^{-1}(D) \subseteq X \mid D \subseteq E,\ D \text{ compact}\}.$$

For any $C \in \mathcal{H}$, $f(C) \subseteq E$ is a compact set. Hence $f^{-1}(f(C)) \in \mathcal{D}$. Moreover, $C \subseteq f^{-1}(f(C))$ implies that $\mu_X(C) \le \mu_X(f^{-1}(f(C)))$. Consequently,

$$\sup\{\mu_X(C) \mid C \in \mathcal{H}\} \le \sup\{\mu_X(f^{-1}(D)) \mid D \subseteq E,\ D \text{ compact}\}. \tag{4}$$

By hypothesis, μ_X is a regular measure on X. Since $f^{-1}(E)$ is a Borelian set of X, it follows that

$$\mu_X(f^{-1}(E)) = \sup\{\mu_X(C) \mid C \subseteq f^{-1}(E),\ C \text{ compact}\} \tag{5}$$

Putting together (4) and (5), we deduce

$$\mu_X(f^{-1}(E)) \leq \sup\{\mu_X(f^{-1}(D)) \mid D \subseteq E,\ D \text{ compact}\},$$

and the latter, together with (3) gives equation (2), settling the claim of regularity for $\mu_X \circ f^{-1}$. □

Using the same notation settled before, we make explicit the duality obtained.

Corollary 5.2. *The functor*

$$\begin{aligned}\mathcal{P}\colon \mathbf{pBDKH} &\to \mathbf{pRMV}_\sigma\\ (X, \mu_X) &\mapsto (C(X), s_X)\\ (f\colon X \to Y) &\mapsto (\tilde{f}\colon C(Y) \to C(X))\end{aligned}$$

yields a duality between $\mathbf{pBDKH}$ *and* $\mathbf{pRMV}_\sigma$.

P r o o f. The restriction of Kakutani's duality proved in [7] and the Kroupa-Panti theorem imply that the functor is well defined and essentially surjective. It is full and faithful by the same duality of [7] and Theorem 5.1, which implies that measure-preserving morphisms in **BDKH** correspond to state-preserving morphisms in $\mathbf{RMV}_\sigma$. □

Remark 5.3. Despite $\mathbf{RMV}_\sigma$ being closed under coproducts, we notice that **BDKH**, the category dual to the whole $\mathbf{RMV}_\sigma$ is not closed under the usual product of topological spaces. Indeed $\mathbf{2} = \{0, 1\}$ with the discrete topology is an object there, while $\mathbf{2}^\omega$ is not, since it is the Stone space of the Lindenbaum-Tarski algebra of the classical propositional calculus, which is not σ-complete. Consequently, products in **BDKH** do not coincide with products in **Top**, the category of topological spaces and continuous maps.

The duality obtained in Corollary 5.2 is built by combining two results: one is based on the classical Kakutani's duality, the other being the Kroupa-Panti integral representation. Both of these references work using the topological space of maximal MV-ideals of a σ-complete Riesz MV-algebra. Following the same ideas, the second duality is built on the integral representation of σ-states given in [1,3], that is, Theorem 2.3. Consider the following categories:

(i) $\mathbf{pRMV}^\omega_\sigma$ is the category whose objects are triplets $(IRL(X), I, s_X)$, where $IRL(X)/_I \simeq IRL(X)|_{\mathbb{V}(I)} \in \mathbf{RMV}^\omega_\sigma$ and s a σ-state on $IRL(X)|_{\mathbb{V}(I)}$, and whose morphisms are *state-preserving* homomorphisms of $\mathbf{RMV}^\omega_\sigma$.
That is, $h\colon (IRL(X), J, s_X) \to (IRL(Y), K, s_Y)$ is induced by a unique $h^*\colon IRL(X) \to IRL(Y)$ such that $h^*(J) \subseteq K$ and $s_X = s_Y \circ h$.

(ii) $\mathbf{pBaire}^\omega$ is the category whose objects are pairs (V, μ), where $V \in \mathbf{Baire}^\omega$ and μ is a probability measure on $\mathcal{BA}(V)$, the σ-algebra of Baire subsets of V. Morphisms in $\mathbf{pBaire}^\omega$ are tuples of IRL-maps that are also *measure-preserving*, that is, $f\colon (V, \mu_V) \to (W, \mu_W)$ such that $\mu_W = \mu_V \circ f^{-1}$.

Consider now the functors $\mathcal{V}^p\colon \mathbf{pRMV}^\omega_\sigma \to \mathbf{pBaire}^\omega$ and $\mathcal{I}^p\colon \mathbf{pBaire}^\omega \to \mathbf{pRMV}^\omega_\sigma$ defined upon the functors $\mathcal{V}$ and $\mathcal{I}$ of [7] as follows:

(i) $\mathcal{V}^p(IRL(X), I, s) = (\mathbb{V}(I), \mu_s)$, where μ_s is given by Theorem 2.3.

(ii) for $V \subseteq [0,1]^X$, $\mathcal{I}^p(V, \mu) = (IRL(X), \mathbb{I}(V), s_\mu)$, with $s_\mu\colon IRL(X)|_V \to [0,1]$ defined by integration with respect to μ. Notice that $\mathbb{V}(\mathbb{I}(V)) = V$.

(iii) on arrows, with the obvious notations, $\mathcal{V}^p(h) = \mathcal{V}(h)$ and $\mathcal{I}^p(\eta) = \mathcal{I}(\eta)$. We remark that $\mathcal{I}(\eta)$ is defined by precomposition with η.

Theorem 5.4. *The functors $\mathcal{V}^p$ and $\mathcal{I}^p$ give a duality.*

Proof. Take $(IRL(X), J, s) \in \mathbf{pRMV}^\omega_\sigma$. By Lemma 4.6, $IRL(X) \mid_{\mathbb{V}(J)}$ is the algebra of all $\mathcal{BA}(\mathbb{V}(J))$-measurable functions. Thus, the measure μ_s given in Theorem 2.3 is a measure on $\mathcal{BA}(\mathbb{V}(J))$ and consequently $\mathcal{V}^p(IRL(X), J, s) \in \mathbf{pBaire}^\omega$. Similarly, for any $(V, \mu) \in \mathbf{pBaire}^\omega$, with $V \subseteq [0,1]^X$, the triplet $(IRL(X), \mathbb{I}(V), s_\mu) = \mathcal{I}^p(V, \mu)$ is an object in $\mathbf{pRMV}^\omega_\sigma$.

Furthermore, by [7: Corollary 4.17] and Theorem 2.3,

$$\mathcal{V}^p(\mathcal{I}^p(V, \mu)) = (V, \mu) \quad \text{and} \quad \mathcal{I}^p(\mathcal{V}^p(IRL(X), I, s)) = (IRL(X), I, s).$$

Whence, in order to prove that we have a duality, it is enough to show that a state-preserving morphism in $\mathbf{RMV}^\omega_\sigma$ is mapped to a measure-preserving map of $\mathbf{Baire}^\omega$, and viceversa.

Let $h\colon (IRL(X), J, s_X) \to (IRL(Y), K, s_Y)$ be a morphism in $\mathbf{RMV}^\omega_\sigma$ induced by some h^*. Then $s_X = s_Y \circ h$ with $s_X : IRL(X) \mid_{\mathbb{V}(J)} \to [0,1]$ and $s_Y : IRL(Y) \mid_{\mathbb{V}(K)} \to [0,1]$.

Take $A \in \mathcal{BA}(\mathbb{V}(J))$. Then $\mu_{s_X}(A) = s_X(\chi_A) = s_Y(h(\chi_A))$. For brevity, let η denote $\mathcal{V}^p(h)$. Note that it follows from [7] that the inverse functor of $\mathcal{V}$ acts as the pre-composition with η. More precisely, for any $p \in IRL(X) \mid_V$, $h(p) = p \circ \eta$. Then, by definition of η, it follows that $h(\chi_A) = \chi_{\eta^{-1}(A)}$. Indeed, $h(\chi_A)$ is again a Boolean member of $IRL(Y)) \mid_{\mathbb{V}(K)}$ and

$$\begin{aligned} \chi_{\eta^{-1}(A)}(\mathbf{y}) = 1 &\iff \mathbf{y} \in \eta^{-1}(A) \\ &\iff \eta(\mathbf{y}) \in A \\ &\iff \chi_A(\eta(\mathbf{y})) = 1. \end{aligned}$$

Then, $\mu_{s_X}(A) = s_Y(\chi_{\eta^{-1}(A)}) = \mu_{s_Y}(\eta^{-1}(A))$, settling the claim.

Finally, we see how $h(p) := p \circ \eta$ is state-preserving for any morphism $\eta\colon (V \subseteq [0,1]^X, \mu_X) \to (W \subseteq [0,1]^Y, \mu_Y)$ in $\mathbf{Baire}^\omega$ that is measure-preserving. By definition of arrows in $\mathbf{pBaire}^\omega$, let $\eta := (\eta_y)_{y \in Y}$ and each $\eta_y \in IRL(X)$. We first remark that η is measurable between $(V, \mathcal{BA}(V))$ and $(W, \mathcal{BA}(W))$. Indeed, for any generator E of $\mathcal{BA}(W)$, $E = W \cap \prod\limits_{y \in Y} A_y$ with $A_y \in \mathcal{BA}([0,1])$, see [16: Lemma 1.2]. Then

$$\begin{aligned} \mathbf{x} \in \eta^{-1}(E) &\iff \eta(\mathbf{x}) \in W \cap \prod_{y \in Y} A_y \\ &\iff \eta(\mathbf{x}) \in W \text{ and } \eta_y(\mathbf{x}) \in A_y \\ &\iff \mathbf{x} \in V \cap \bigcap_{y \in Y} \eta_y^{-1}(A_y). \end{aligned}$$

Since each η_y belongs to $IRL(X)$, it follows that $\bigcap\limits_{y \in Y} \eta_y^{-1}(A_y) \in \mathcal{BA}([0,1]^X)$ and $V \cap \bigcap\limits_{y \in Y} \eta_y^{-1}(A_y) \in \mathcal{BA}(V)$, making η a measurable function.

Now, using the property of change of variables of the pushforward measure, which is μ_Y by hypothesis, we have

$$\begin{aligned} s_Y(p) &= \int_W p \, \mathrm{d}\mu_Y = \int_W p \, \mathrm{d}(\mu_X \circ \eta^{-1}) \\ &= \int_V (p \circ \eta) \, \mathrm{d}\mu_X = \int_V h(p) \, \mathrm{d}\mu_X = s_X(h(p)), \end{aligned}$$

which settles the claim. □

6. Statistical models via IRL-maps

A classical approach in statistics defines a *model* to be a *collection of probability distributions* $\mathcal{M} = \{p_\theta \mid \theta \in \Theta\}$ [21, 23]. The set of parameters Θ is usually taken inside a finite dimensional space, that is, $\Theta \subseteq \mathbb{R}^d$ and often parameters are the values of a density function on a certain dataset. Outcomes are often assumed to be in a finite number and it is possible to identify each $p_\theta \in \mathcal{M}$ as a point of the simplex $\Delta_{k-1} = \{(p_1, \ldots, p_k) \in [0,1]^k \mid \sum_{i=1}^{k} p_i = 1\}$, for some $k \in \mathbb{N}$. This is the starting point for algebraic statistics, which was introduced in [21]. Here we follow a slightly different approach, as described in [23]: a parametric algebraic statistical model is, classically, a model in which the set of parameters Θ is an algebraic variety and $\mathcal{M}$ sits inside a probability simplex Δ_{k-1} for some k.

Thus, inspired by these considerations, let us give the following definition.

Definition 6.1. An *IRL-statistical model* is an IRL-map

$$\eta = (\eta_1, \ldots, \eta_k)\colon P \subseteq [0,1]^X \to \Delta_{k-1}$$

such that $|X| \leq \omega$ and the set P is an IRL-algebraic variety. That is, there exists a set of polynomials $F \subseteq IRL(X)$ such that $P = \mathbb{V}(F)$.

If we denote by Δ_ω the simplex in $[0,1]^\omega$, that is, the set of points $(a_n) \in [0,1]^\omega$ for which the infinite sum $\sum_n a_n$ exists and it is equal to 1, this definition is generalized to the case $\eta\colon [0,1]^d \to \Delta_\omega$, where we take $\eta = (\eta_n)_{n\in\omega}$.

If we look at points of $\eta(P)$ as functions $k \to [0,1]$, every such point can be considered as a discrete probability measure. Hence, we have indeed defined a parametrized statistical model (in the classical sense) with additional conditions on the set of parameters and the assignment. In our case P plays the role of Θ, while $\eta(P) = \mathcal{M}$.

By the very nature of IRL-models, we obtain the following propositions.

Proposition 6.2. *Let P be a Baire set. A model $\eta\colon P \subseteq [0,1]^X \to \Delta_{k-1} \subseteq [0,1]^k$ induces an observable $\mathcal{X}_\eta\colon IRL(Y) \to IRL(X)|_P$, $|Y| = k$, given by $a \in IRL(Y) \mapsto a \circ \eta$. The same holds in the case of a model $\eta\colon P \to \Delta_\omega$.*

Proof. By Lemma 4.6, the algebra $IRL(X)|_P$ is the Riesz tribe of all $\mathcal{BA}(P)$-measurable functions, as defined before. Thus, looking at $IRL(Y)$ and $IRL(X)|_P$ respectively as the free k-generated σ-complete Riesz MV-algebra and a tribe inside $[0,1]^P$, we apply [6: Theorem 2.3] (in both the finite and countable case) getting the desired result, for which we notice that the measurable map η is not required to be surjective into $[0,1]^Y$ or $[0,1]^\omega$. ☐

Furthermore, a consequence of the duality of [7] is the following result.

Proposition 6.3. *Statistical models on Baire subsets of hypercubes $[0,1]^X$, with X countable, are in bijective correspondence with morphisms between countably presented σ-complete Riesz MV-algebras.*

We now give a list of examples, in order to show the applicability of this definition. Note that in every example the IRL-variety taken as domain is a simple Borel subset of $[0,1]$. In particular it is $[0,1]$ itself in the case of a binomial model, it is $(0,1)$ for the geometric model, and it is $(0,1]$ for the Poisson model.

***Example* 6.4** (Binomial model)**.** Let $[k] = \{0, 1, 2, \ldots, k\}$ be a set of data. In this case, our data represent the iteration of an experiment, for example, the tosses of a coin (biased or unbiased, depending on the parameter). Consider, for any $i \in [k]$, the function:

$$\eta_i \colon [0,1] \to [0,1] \quad \eta_i(\alpha) = \binom{k}{i} \alpha^i (1-\alpha)^{k-i},$$

and let $\eta \colon [0,1] \to [0,1]^{k+1}$ be defined as $\eta = (\eta_0, \ldots, \eta_k)$.

Each η_i is continuous, and therefore and element of $IRL(1)$. Moreover, since for a fixed α, $\sum_{i=0}^{k} \eta_i(\alpha)$ gives the cumulative distribution, $\eta([0,1]) \subseteq \Delta_k$. Whence, we obtain a IRL-model.

The binomial distribution is an example of an algebraic model in the definition of [23], since each η_i is actually a polynomial. Nevertheless, our definition allows to capture more complicated distributions, which are not given in polynomial form.

***Example* 6.5** (Geometric distribution)**.** Similarly to the case of a binomial distribution, we take $k \in \mathbb{N}$ and define

$$\eta_i \colon (0,1) \to [0,1], \quad \eta_i(x) = x(1-x)^{i-1},$$

for any $i = 1, \ldots, k$. This distribution gives the probability that the first occurrence of success requires i independent trials, each with success probability x, with a bound of k trials.

The function $\eta = (\eta_i, \ldots, \eta_k)$ is trivially an IRL-map, but in contrast with the previous case, it is not true now that $\sum_{i=1}^{k} \eta_i(x) = 1$ for a fixed $x \in (0,1)$. Indeed, we have

$$\sum_{i=1}^{k} \eta_i(x) = \sum_{i=1}^{k} x(1-x)^{i-1} = 1 - (1-x)^k$$

Since $x \in (0,1)$, the geometric series is convergent, therefore we can consider

$$\eta \colon (0,1) \to [0,1]^\omega, \quad \eta = (\eta_i)_{i \in \omega}.$$

In this case, for any fixed $x \in (0,1)$,

$$\sum_{i \in \omega} \eta_i(x) = x \sum_{i \in \omega} (1-x)^{i-1} = \frac{x}{1-(1-x)} = 1.$$

Thus, $\eta((0,1)) \subseteq \Delta_\omega$ and we have obtained a logico-algebraic representation of a geometric model within $\mathcal{IRL}$.

***Example* 6.6** (Poisson model)**.** Let $[k] = \{0, 1, 2, \ldots, k\}$ be a set of data. In this case, our data represent the number of occurrences of a certain phenomenon, where we put an arbitrary large bound given by $k \in \mathbb{N}$. Consider, for any $i \in [k]$, the function:

$$\eta_i \colon (0,1] \to [0,1] \quad \eta_i(\lambda) = \frac{e^{-\lambda}\lambda^i}{i!},$$

and let $\eta \colon (0,1] \to [0,1]^k$ be defined as $\eta = (\eta_1, \ldots, \eta_k)$.

Each η_i is continuous, and therefore an element of $IRL(1)$. The domain $(0,1]$ is clearly an IRL-algebraic variety, being a Baire set. Moreover, when we consider the case $k = \omega$, the infinite sum $\sum_{i \in \mathbb{N}}^{k} \eta_i(\lambda)$ equals 1, since $\sum_{i \in \mathbb{N}} \frac{\lambda^i}{i!} = e^\lambda$. Thus, $\eta((0,1]) \subseteq \Delta_\omega$. Thus, once again the possibility to deal with countable sequences allows us to obtain a well defined logical model.

The definitions and examples of this short section are meant to hint towards the idea of a "metamathematics of statistics" and they are a first step in this direction. Propositions 6.2 and 6.3 allow to look at algebraic statistical models with a point of view that is completely encoded into logic, since observables are evaluations of formulas in a standard-complete logical system. It can also be interesting to understand if this setting can be generalized using IRL-maps $\eta\colon P \to C \subseteq [0,1]^X$ where X can be uncountable: The dualities of Section 5 and [7], as well as the results of Section 3 seem to imply that the countable case is the best-behaved one. Furthermore, it will be interesting in the future to see how far we can push this approach, for example in the setting of experiments of designs and when we use morphisms of $\mathbf{pBaire}^{\omega}$.

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