



## Research Article

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# Lipschitz estimates for partial trace operators with extremal Hessian eigenvalues

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**Abstract:** We consider the Dirichlet problem for partial trace operators which include the smallest and the largest eigenvalue of the Hessian matrix. It is related to two-player zero-sum differential games. No Lipschitz regularity result is known for the solutions, to our knowledge. If some eigenvalue is missing, such operators are nonlinear, degenerate, non-uniformly elliptic, neither convex nor concave. Here we prove an interior Lipschitz estimate under a non-standard assumption: that the solution exists in a larger, unbounded domain, and vanishes at infinity. In other words, we need a condition coming from far away. We also provide existence results showing that this condition is satisfied for a large class of solutions. On the occasion, we also extend a few qualitative properties of solutions, known for uniformly elliptic operators, to partial trace operators.

**Keywords:** Lipschitz estimates, partial trace operators, elliptic equations, viscosity solutions

**MSC 2020:** 35J60, 35J70, 35J25, 35B50, 35B51, 35B65

## 1 Introduction and main results

A growing attention has been received by the Hessian partial trace operators in the last few decades. Motivated by geometric problems of mean partial curvature [1–3], many works have been devoted to analytical and geometrical aspects of equations involving partial trace operators. For instance, just to mention a few ones, see the papers of Caffarelli et al. [4–6] and of Harvey and Lawson [7–9].

In this article, we are interested in the more general weighted Hessian partial trace operators:

$$\mathcal{P}(D^2u) = \sum_{i=1}^n a_i \lambda_i(D^2u), \quad (1.1)$$

where  $\lambda_i$  is the eigenvalue of the Hessian matrix, in non-decreasing order, and  $a = (a_i)_{i=1}^n$  is an  $n$ -tuple of numbers  $a_i \geq 0$  such that  $\bar{a} = \max_{1 \leq i \leq n} a_i > 0$ .

For a list of symbols used here we refer to the end of this section.

Equations  $\mathcal{P}(D^2u) = 0$  have been recently recognized to govern two-player zero-sum differential games. See the study of Blanc and Rossi [10], in which an existence and uniqueness theorem for solutions of boundary value problems for the equation  $\lambda_i(D^2u) = 0$  is proved. See also [11] for a time-dependent version.

Such operators share a number of qualitative properties with uniformly elliptic operators. See for instance, recent papers [12–18]. We expect that other properties, typical of uniformly elliptic operators, can be extended to the partial trace operators, like blow-up boundary results or Liouville properties for the Lane-Emden equation, see [19] as we plan to investigate further on.

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Regularity properties seem to be a difficult task. Few Hölder estimates are known, see [14], depending on the coefficients  $a_i$ . See also [20] for an effort to estimate Hölder exponents. Lipschitz regularity is only known for viscosity supersolutions of  $\lambda_1(D^2u) = f$  with  $f$  bounded below, and supersolutions of  $\lambda_n(D^2u) = f$  with  $f$  bounded above. See for instance [14,21]. It is worth noting that classical solutions would be semi-convex and semi-concave, respectively.

An interior  $C^{1,\alpha}$  regularity result has been shown by Oberman and Silvestre [32, Theorem 5.1] for solutions of the equation  $\lambda_1(D^2u) = 0$  in a ball  $B$  with boundary values  $g \in C^{1,\alpha}$ . The result is qualified as “unusual” by the same authors because the  $C^{1,\alpha}$ -regularity is transmitted from the boundary to the interior but it cannot be extended up to the boundary, generally.

The main scope of this article is to show an interior Lipschitz regularity result for weighted partial trace operators with positive coefficients of the smallest and the biggest eigenvalue, in dimension  $n \geq 3$ .

Generally, interior regularity, in a compact set, is obtained supposing that the solution exists in a larger bounded open set containing it. In our case, we have to assume that solutions exist in a larger unbounded set, like a slab or a finite union of slabs, and a boundary condition far away, that the solution vanishes at infinity, is needed.

Here are the contributions of this article, where the measure-geometric condition  $\mathbf{G}$  is given by Definition 2.6.

**Theorem 1.1.** *Let  $S_{d,i}^* = \{x \in \mathbb{R}^n : |x_i - x_i^*| < d\}$  and  $S_d^* = \bigcup_{i=1}^n S_{d,i}^*$ , where  $d \in \mathbb{R}_+$  and  $n \geq 3$ . Assume that  $\mathcal{P} = \sum_{i=1}^n a_i \lambda_i$  with  $a_i a_n > 0$ . Let  $\Omega \supset S_d^*$  be an open set of  $\mathbb{R}^n$ , endowed with a uniform exterior cone property and satisfying condition  $\mathbf{G}$ . Suppose that  $g \in C^0(\partial\Omega)$  and*

$$\sup_{x \in \partial\Omega} |g| \leq M_g, \quad (1.2)$$

and

$$\lim_{\substack{|x| \rightarrow \infty \\ x \in \partial\Omega}} g(x) = 0. \quad (1.3)$$

Let  $u \in C^0(\overline{\Omega})$  be the solution of the Dirichlet problem

$$\begin{cases} \mathcal{P}(D^2u) = 0 & \text{in } \Omega \\ u = g & \text{on } \partial\Omega. \end{cases} \quad (1.4)$$

Then  $u$  is locally Lipschitz continuous in  $Q_d^* = \bigcap_{i=1}^n S_{d,i}^* \equiv \prod_{i=1}^n ]x_i^* - d, x_i^* + d[$ , and

$$\max_{1 \leq i \leq n} |D_i u(x)| \leq \frac{M_g}{\delta(x)} \text{ a.e. in } Q_d^*, \quad (1.5)$$

where  $\delta(x) = \text{dist}(x, Q_d^*)$ . In particular, if  $u$  is differentiable at  $x^*$ ,

$$\max_{1 \leq i \leq n} |D_i u(x^*)| \leq \frac{M_g}{d}. \quad (1.6)$$

The set of functions  $u$  considered in the above theorem is not empty. We will establish an existence (and uniqueness) result in Theorem 1.3.

The previous result is applicable to weighted partial trace operators  $\mathcal{P}$  with non-zero coefficients  $a_1$  and  $a_n$  of the extremal eigenvalues  $\lambda_1$  and  $\lambda_n$ , respectively. Among such ones, we recall:

$$\mathcal{M}(D^2u) = \lambda_1(D^2u) + \lambda_n(D^2u), \quad (1.7)$$

introduced in a previous paper [14].

Note that  $\mathcal{M}$  is neither concave nor convex. It is degenerate, non-uniformly elliptic except for  $n = 2$ , when  $\mathcal{M}(D^2u)$  is the Laplace operator  $\Delta u$ .

**Idea of the proof.** First, we perturb  $\mathcal{P}$  to obtain an operator  $\mathcal{P}_\varepsilon$ , with a small amount of uniform ellipticity, such that  $\mathcal{P}_\varepsilon \rightarrow \mathcal{P}$  as  $\varepsilon \rightarrow 0^+$ . Next, we show a uniform estimate for the gradient of the solution  $u_\varepsilon$  of the equation  $\mathcal{P}_\varepsilon(D^2v) = 0$ . Then we show that  $u_\varepsilon \rightarrow u$  as  $\varepsilon \rightarrow 0^+$ .

Pursuing this program, we have the occasion to extend a few qualitative results, which have not yet been established elsewhere, from the uniformly elliptic case to weighted partial trace equations.

First, in [14, Section 3.2] the existence of solutions was proved for general weighted Hessian partial trace operators  $\mathcal{P}$  in a bounded domain with a suitable convexity assumption derived from [10].

Here we show existence and uniqueness for weighted partial trace operators with extremal eigenvalues in domains satisfying a uniform exterior cone property. See Definition 2.21.

**Theorem 1.2.** *Suppose that  $\Omega$  is a bounded domain of  $\mathbb{R}^n$ . Let  $\Omega$  be endowed with a uniform exterior cone property. Assume that  $\mathcal{P} = \sum_{i=1}^n a_i \lambda_i$  with  $a_1 a_n > 0$ , and  $g \in C^0(\partial\Omega)$  such that  $|g| \leq M_g$  on  $\partial\Omega$  as (1.2).*

*Then the Dirichlet problem,*

$$\begin{cases} \mathcal{P}(D^2u) = 0 & \text{in } \Omega \\ u = g & \text{on } \partial\Omega, \end{cases} \quad (1.8)$$

*has a unique solution  $u \in C^0(\bar{\Omega})$ , and*

$$|u| \leq M_g \text{ in } \bar{\Omega}. \quad (1.9)$$

We shall also consider unbounded subsets of slabs  $S_{d,i} = \{x \in \mathbb{R}^n : |x_i| < d\}$ , star-shaped domains  $S_d = \bigcup_{i=1}^n S_{d,i}$ , and their translates  $S_{d,i}^* = x^* + S_{d,i}$ ,  $S_d^* = x^* + S_d$ , in view of Theorem 1.1.

With respect to the bounded case, for unbounded domains the uniqueness will be proved under the additional assumption that the boundary values tend to zero at infinity.

At the same time, we point out that the following results show that solutions in unbounded sets, like slabs or a finite union of slabs, vanishing at infinity, do exist under assumptions that are not so restrictive.

**Theorem 1.3.** *Suppose that  $\Omega$  is an unbounded domain satisfying condition **G**. Let  $\Omega$  be endowed with a uniform exterior cone property. Assume that  $\mathcal{P} = \sum_{i=1}^n a_i \lambda_i$  with  $a_1 a_n > 0$ , and  $g \in C^0(\partial\Omega)$  such that  $|g| \leq M_g$  on  $\partial\Omega$  as (1.2).*

*Then the Dirichlet problem (1.8) has a viscosity solution  $u \in C^0(\bar{\Omega})$ , and (1.9) holds. Suppose in addition that  $g(x) \rightarrow 0$  as  $|x| \rightarrow \infty$  in  $\partial\Omega$  as (1.3). Then*

$$\lim_{\substack{|x| \rightarrow \infty \\ x \in \Omega}} u(x) = 0 \quad (1.10)$$

*and the solution of the Dirichlet problem (1.8) is unique.*

The same assumptions allow us to obtain the so-called improved Alexandroff-Bakelman-Pucci (ABP) estimate.

**Theorem 1.4.** *Let  $\Omega$  be a domain of  $\mathbb{R}^n$  satisfying condition **G** with constants  $R_0 > 0$  and  $0 < \sigma, \tau < 1$ . Suppose also  $f \in C^0(\Omega) \cap L^n(\Omega \cap B)$  for all  $B \in \mathcal{B}_{R_0}(\Omega)$ , a covering of  $\Omega$  with balls of radius  $R_0$ . Let  $u \in \text{usc}(\bar{\Omega})$  be a subsolution of the equation  $\mathcal{P}(D^2u) = f$ . If  $a_1 > 0$ , then*

$$\sup_{\Omega} u \leq \sup_{\partial\Omega} u + CR_0 \sup_{B \in \mathcal{B}_{R_0}} \|f\|_{L^n(\Omega \cap B)}, \quad (1.11)$$

*where the positive constant  $C$  only depends on the dimension  $n$ ,  $\sigma$ ,  $\tau$ ,  $a_1$ , and it is bounded when  $a_1$ ,  $\sigma$  are bounded away from zero,  $\tau$  is bounded away from 1.*

*Let  $u \in \text{lsc}(\bar{\Omega})$  be a supersolution of the equation  $\mathcal{P}(D^2u) = f$ . If  $a_n > 0$ , then*

$$\inf_{\Omega} u \geq \inf_{\partial\Omega} u - CR_0 \sup_{B \in \mathcal{B}_{R_0}} \|f\|_{L^n(\Omega \cap B)}, \quad (1.12)$$

*where the positive constant  $C$  only depends on the dimension  $n$ ,  $\sigma$ ,  $\tau$ ,  $a_n$ , and it is bounded when  $a_n$ ,  $\sigma$  are bounded away from zero,  $\tau$  is bounded away from 1.*

See [23] for linear uniformly elliptic equations in non-divergence form, [24,25] for generalizations, and [26] for the case of fully nonlinear uniformly elliptic equations; see also [12] for further generalizations and [27] for other kinds of degenerate elliptic operators.

This article is organized as follows: in Section 2 we collect and rearrange a few results on elliptic operators, viscosity solutions, and geometric properties of a domain, which will be useful in the sequel, in particular we establish some connections between uniformly elliptic operators and partial trace operators; in Section 3 we prove existence and uniqueness results; in Section 4 we estimate the gradient of approximate solutions; and finally, in Section 5 we prove the convergence of approximate solutions.

#### List of symbols

- $S^n$ : set of the  $n \times n$  real symmetric matrices
- $X \leq Y$ :  $Y - X$  positive semidefinite
- $\lambda_i(X)$ : eigenvalues of  $X \in S^n$ ,  $\lambda_i \leq \lambda_{i+1}$
- $a = (a_i)_{i=1}^n$ ,  $\underline{a} := \min_{1 \leq i \leq n} a_i$ ,  $\bar{a} := \max_{1 \leq i \leq n} a_i$
- $\mathcal{P} : X \in S^n \rightarrow \sum_{i=1}^n a_i \lambda_i(X) \in \mathbb{R}$ ;  $\mathcal{P}_\varepsilon := \mathcal{P} + \varepsilon \Delta$ ,  $\varepsilon > 0$
- $\mathcal{A} := \{\mathcal{P} : \underline{a} \geq 0, \bar{a} > 0\}$ ,  $\mathcal{A}_{ue} := \{\mathcal{P} : \underline{a} > 0\}$
- $x = (x_1, \dots, x_n)$ ,  $|x| = (\sum_{i=1}^n x_i^2)^{1/2}$
- $\hat{x}_i = (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n)$ ,  $|\hat{x}_i| = (\sum_{j \neq i} x_j^2)^{1/2}$
- $S_{d,i} := \{x \in \mathbb{R}^n : |x_i| < d\}$ ,  $d > 0$ ;  $S_d := \bigcup_{i=1}^n S_{d,i}$
- $C_{\rho,d;i} := \{x \in S_{d,i} : |\hat{x}_i| < \rho\}$ ,  $\rho > 0$ ;  $S_{\rho,d} := \bigcup_{i=1}^n C_{\rho,d;i}$
- $Q_d^* = \bigcap_{i=1}^n S_{d,i}^* \equiv \prod_{i=1}^n [x_i^* - d, x_i^* + d[$ .

## 2 Review of known results and preparatory work

Let  $S^n$  be the set of  $n \times n$  real matrices, with the partial ordering  $X \leq Y$  that means  $Y - X$  positive semi-definite. We call  $\lambda_1(X), \dots, \lambda_n(X)$  the eigenvalues of  $X$  in the non-decreasing order.

We will consider in the sequel mappings  $F : S^n \rightarrow \mathbb{R}$ . We say that  $\mathcal{F}$  is a degenerate elliptic operator if  $X \leq Y$  implies  $\mathcal{F}(X) \leq \mathcal{F}(Y)$ .

We assume  $f \in C^0(\Omega)$ . Let  $u$  be an upper semicontinuous function in  $\Omega$ , for short  $u \in \text{usc}(\Omega)$ . Then  $u$  is a viscosity subsolution of the equation  $\mathcal{F}(D^2u) = f$  in  $\Omega$  if we have

$$\mathcal{F}(D^2\varphi(x)) \geq f(x) \quad (2.1)$$

for all  $x \in \Omega$  and all  $\varphi \in C^2(\Omega)$  touching  $u$  at  $x$  from above.

A lower semicontinuous function  $u$  in  $\Omega$ , for short  $u \in \text{lsc}(\Omega)$ , is a supersolution of the same equation if we have

$$\mathcal{F}(D^2\varphi(x)) \leq f(x) \quad (2.2)$$

for all  $x \in \Omega$  and all  $\varphi \in C^2(\Omega)$  touching  $u$  at  $x$  from below.

A function  $u \in C^0(\Omega)$  is a solution of the equation  $\mathcal{F}(D^2u) = f$  if it is a subsolution and a supersolution.

We also use the notation  $\mathcal{F}(D^2u) \geq f$  for subsolutions and  $\mathcal{F}(D^2u) \leq f$  for supersolutions. Such inequalities correspond to pointwise inequalities in the case  $u \in C^2(\Omega)$ . But in the case of lower regularity, when the Hessian matrices do not exist, we can only use the viscosity sense.

Let  $\mathcal{F}$  and  $\mathcal{G}$  be degenerate elliptic operators such that  $\mathcal{F}(X + Y) \leq \mathcal{F}(X) + \mathcal{G}(Y)$ . We use in the sequel the fact that if  $\mathcal{F}(D^2u) \leq f$  and  $\mathcal{G}(D^2v) \leq g$  in the viscosity sense, then we can still say that  $\mathcal{F}(D^2(u + v)) \leq f + g$ , as we would do for classical solutions, provided at least one between  $u$  and  $v$  is  $C^2$ .

This article is focused on the weighted partial trace operators

$$\mathcal{P}(X) = a_1 \lambda_1(X) + \dots + a_n \lambda_n(X), \quad X \in S^n, \quad (2.3)$$

with coefficients  $a_i \geq 0$  for  $i = 1, \dots, n$ , which are degenerate elliptic.

Depending only on the eigenvalues, they are invariant by reflection.

**Lemma 2.1.** *The weighted partial trace operator  $\mathcal{P}$  are invariant by reflection with respect to any hyperplane  $H$  of  $\mathbb{R}^n$ , that is: if  $R$  is the reflection matrix with respect to  $H$ ,*

$$\mathcal{P}(RXR) = \mathcal{P}(X). \quad (2.4)$$

We refer to [28], and more diffusely [29], for major generality.

Here we consider a sample case in order to obtain to the point of interest. Suppose  $H = \{x_n = 0\}$ , so that

$$R = \begin{pmatrix} \hat{I}_n & 0 \\ 0 & -1 \end{pmatrix}, \quad (2.5)$$

where  $\hat{I}_n$  is the  $(n-1) \times (n-1)$  identity matrix.

Let  $u(x) = u(\hat{x}_n, x_n)$  be a  $C^2$ -function, and  $\tilde{u}(x) = u(Rx) \equiv u(\hat{x}_n, -x_n)$ , then

$$\mathcal{P}(D^2\tilde{u}(\hat{x}_n, x_n)) = \mathcal{P}(RD^2u(\hat{x}_n, -x_n)R) = \mathcal{P}(D^2u(\hat{x}_n, -x_n)). \quad (2.6)$$

Let  $\bar{a} = \max_i a_i$  and  $\underline{a} = \min_i a_i$ . We denote by  $\mathcal{A}$  the class of weighted partial trace operators  $\mathcal{P} \rightarrow \mathbb{R}$  such that  $\bar{a} > 0$ , which are degenerate, non-uniformly elliptic if  $\underline{a} = 0$ , and by  $\mathcal{A}_{ue}$  the subset which consists of  $\mathcal{P} \in \mathcal{A}$  such that  $\underline{a} > 0$ , which will be seen uniformly elliptic.

Deferring later the proof of the uniform ellipticity in such cases, we recall that for any  $\mathcal{P} \in \mathcal{A}$ , the comparison principle holds in bounded domains. See [14, Theorem 3.1].

**Lemma 2.2.** (Comparison principle) *Let  $\Omega$  be a bounded domain,  $\mathcal{P} \in \mathcal{A}$  and  $f$  a continuous function in  $\Omega$ . Let  $u \in \text{usc}(\bar{\Omega})$  and  $v \in \text{lsc}(\bar{\Omega})$ . If  $\mathcal{P}(D^2u) \geq f$ , and  $\mathcal{P}(D^2v) \leq f$  in  $\Omega$  in the viscosity sense, then  $u \leq v$  on  $\partial\Omega$  implies  $u \leq v$  in  $\Omega$ .*

The most important elliptic operators, which depend only on the eigenvalues, are the well-known Pucci extremal operators, the maximal and the minimal one, depending on positive constants  $\lambda$  and  $\Lambda \geq \lambda$ , respectively:

$$\begin{aligned} \mathcal{M}_{\lambda, \Lambda}^+(X) &= \sup_{\substack{M \leq A \leq \Lambda \\ A \in S^n}} \text{Tr}(AX) = \Lambda \sum_{i=1}^n \lambda_i^+(X) - \lambda \sum_{i=1}^n \lambda_i^-(X), \\ \mathcal{M}_{\lambda, \Lambda}^-(X) &= \inf_{\substack{M \leq A \leq \Lambda \\ A \in S^n}} \text{Tr}(AX) = \lambda \sum_{i=1}^n \lambda_i^+(X) - \Lambda \sum_{i=1}^n \lambda_i^-(X), \end{aligned} \quad (2.7)$$

where  $\lambda_i^+ = \max(\lambda_i, 0)$  and  $\lambda_i^- = -\min(\lambda_i, 0)$ .

For any degenerate elliptic operator  $\mathcal{F}$  we define the dual operator as

$$\mathcal{F}^*(X) = -\mathcal{F}(-X). \quad (2.8)$$

It is not hard to verify that the maximal and the minimal Pucci operators  $\mathcal{M}_{\lambda, \Lambda}^+$  and  $\mathcal{M}_{\lambda, \Lambda}^-$ , with the same constants  $\lambda$  and  $\Lambda$ , are the dual of each other.

The dual of the weighted partial trace operator  $\mathcal{P} = a_1\lambda_1 + \dots + a_n\lambda_n$  is again a weighted partial trace operator,  $\mathcal{P}^* = a_n\lambda_1 + \dots + a_1\lambda_n$ .

If  $\mathcal{P} \in \mathcal{A}$ , resp.  $\mathcal{P} \in \mathcal{A}_{eu}$ , then  $\mathcal{P}^* \in \mathcal{A}$ , resp.  $\mathcal{P}^* \in \mathcal{A}_{eu}$ . Self-dual operators, that is to say,  $\mathcal{P} = \mathcal{P}^*$ , are  $\mathcal{M}_{1,1}^+ = \mathcal{M}_{1,1}^- = \lambda_1 + \dots + \lambda_n$ , the Laplace operator, and the partial trace operator  $\mathcal{M} = \lambda_1 + \lambda_n$ .

In this respect, note that, if  $\mathcal{F}(D^2u) \geq f$ , that is,  $u$  is a subsolution of the equation  $\mathcal{F}(D^2u) = f$ , then  $v = -u$  is a supersolution of the equation  $\mathcal{F}^*(D^2v) = f$ , namely,  $\mathcal{F}^*(D^2v) \leq f$ .

**Proposition 2.3.** *Let  $\mathcal{P}$  be a weighted partial trace operator with  $a_i \geq 0$ . Then*

$$\mathcal{M}_{\underline{a}, \bar{a}}^-(X) \leq \mathcal{P}(X) \leq \mathcal{M}_{\underline{a}, \bar{a}}^+(X) \text{ for all } X \in S^n. \quad (2.9)$$

**Proof.** We show only the right-hand inequality, since the left-hand one can be obtained by duality. It is enough to note that

$$\mathcal{M}_{\underline{a}, \bar{a}}^+(X) = \sum_{i=1}^n \mu_i^+ \lambda_i(X), \quad (2.10)$$

where

$$\mu_i^+ \lambda_i(X) = \begin{cases} \bar{a} \lambda_i(X) & \text{if } \lambda_i(X) \geq 0 \\ \underline{a} \lambda_i(X) & \text{if } \lambda_i(X) \leq 0. \end{cases} \quad (2.11)$$

On the other hand,

$$a_i \lambda_i(X) \leq \begin{cases} \bar{a} \lambda_i(X) & \text{if } \lambda_i(X) \geq 0 \\ \underline{a} \lambda_i(X) & \text{if } \lambda_i(X) \leq 0. \end{cases} \quad (2.12)$$

Comparing (2.11) and (2.12) we obtain the right-hand inequality of (2.9) under consideration.  $\square$

The above inequalities also hold in the case  $\underline{a} = 0$ , when the extremal Pucci operators are degenerate, non-uniformly elliptic.

But in this case (2.9) are almost useless. Conversely, if we consider weighted partial trace operators  $\mathcal{P}$  with coefficients  $a_1 > 0$  or  $a_n > 0$ , we can dispose of more useful inequalities; see also [14, Section 3.2].

In this case, we can control  $\mathcal{P}$  with Pucci extremal operators  $\mathcal{M}_{\lambda, \Lambda}^\pm$  having  $\lambda > 0$ .

**Proposition 2.4.** *Let  $\mathcal{P}$  be the partial trace operator  $\mathcal{P} = a_1 \lambda_1 + \dots + a_n \lambda_n$ . Suppose that  $a_n > 0$ . Then*

$$\mathcal{P}(X) \geq \mathcal{M}_{\frac{a_n}{n}, \bar{a} + \frac{a_n}{n}}^-(X). \quad (2.13)$$

On the other hand, if  $a_1 > 0$ , then

$$\mathcal{P}(X) \leq \mathcal{M}_{\frac{a_1}{n}, \bar{a} + \frac{a_1}{n}}^+(X). \quad (2.14)$$

**Proof.** For (2.13), it is sufficient to observe that

$$\mathcal{P}(X) = \sum_{i=1}^n a_i \lambda_i(X) \geq \sum_{i=1}^{n-1} \left( a_i + \frac{a_n}{n} \right) \lambda_i(X) + \frac{a_n}{n} \lambda_n(X) \geq \mathcal{M}_{\frac{a_n}{n}, \bar{a} + \frac{a_n}{n}}^-(X), \quad (2.15)$$

where the last inequality is a consequence of Proposition 2.3.

The proof of (2.14) is similar. One can also use duality, observing that, if  $a_1 > 0$  the inequality (2.13) holds for the dual operator  $\mathcal{P}^*$  with  $a_1$  instead of  $a_n$ .  $\square$

Let  $\mathcal{F} : \mathcal{S}^n \rightarrow \mathbb{R}$ . We say that  $\mathcal{F}$  is uniformly elliptic if

$$Y \geq 0 \Rightarrow \lambda \operatorname{Tr}(Y) \leq \mathcal{F}(X + Y) - \mathcal{F}(X) \leq \Lambda \operatorname{Tr}(Y), \quad (2.16)$$

with positive constants  $\lambda$  and  $\Lambda \geq \lambda$ , called ellipticity constants.

The extremal Pucci operators are uniformly elliptic, in particular  $\mathcal{M}_{1,1}^\pm(X) = \operatorname{Tr}(X)$ , which corresponds to the Laplace operator:  $\Delta u = \operatorname{Tr}(D^2 u)$ .

Differently, the partial trace operator  $\mathcal{M} = \lambda_1 + \lambda_n$  is not uniformly elliptic when  $n > 2$ . In fact, let  $X = e_1 \otimes e_1$  and  $Y = e_2 \otimes e_2$ . Then  $(\lambda_1 + \lambda_n)(X + Y) = 1 = (\lambda_1 + \lambda_n)(X)$ , while  $\operatorname{Tr}(Y) = 1$ , violating the left-hand-side inequality (2.16).

A fundamental tool for dealing with uniformly elliptic operators is the well-known ABP estimate. But in view of the inequalities of Proposition 2.4, it can be used for weighted partial trace operators  $\mathcal{P}$  with  $a_1 > 0$  and/or  $a_n > 0$ . See [14, Lemma 4.1].

**Lemma 2.5.** (ABP estimate). Let  $\mathcal{P} = \sum_{i=1}^n a_i \lambda_i \in \mathcal{A}$ ,  $\Omega$  be a bounded domain of  $\mathbb{R}^n$  of diameter  $d$ , and  $f \in C^0(\Omega) \cap L^n(\Omega)$ . Let  $u \in \text{usc}(\overline{\Omega})$  be a subsolution of the equation  $\mathcal{P}(D^2u) = f$ . If  $a_1 > 0$ , then

$$\sup_{\Omega} u \leq \sup_{\partial\Omega} u + Cd \|f^-\|_{L^n(\Omega)}, \quad (2.17)$$

where the positive constant  $C$  depends only on the dimension  $n$  and  $a_1$ , and it is bounded when  $a_1$  is bounded away from zero.

Let  $u \in \text{lsc}(\overline{\Omega})$  be a supersolution of the equation  $\mathcal{P}(D^2u) = f$ . If  $a_n > 0$ , then

$$\inf_{\Omega} u \geq \inf_{\partial\Omega} u - Cd \|f^+\|_{L^n(\Omega)}, \quad (2.18)$$

where the positive constant  $C$  only depends on the dimension  $n$  and  $a_n$ , and it is bounded when  $a_n$  is bounded away from zero.

Note that in the case  $f \equiv 0$ , the above inequalities yield the maximum and the minimum principle.

We also recall that the ABP estimate has been improved by Cabré [30] and extended to domains, possibly unbounded, with the measure-geometric property **G**.

**Definition 2.6.** Let  $\Omega$  be a domain of  $\mathbb{R}^n$ . We say that  $\Omega$  satisfies the geometric condition **G** if there exist constant  $R_0 > 0$  and  $\sigma, \tau \in (0, 1)$  such that: for all  $y \in \Omega$  there exists a ball  $B_R(y^*)$  of radius  $R \leq R_0$ , containing  $y$ , such that

$$|B_R(y^*) \setminus \Omega_{y,\tau}| \geq \sigma |B_R(y^*)|, \quad (2.19)$$

where  $\Omega_{y,\tau}$  is the connected component of  $\Omega \cap B_{R/\tau}(y^*)$  containing  $y$ .

We point out that all the bounded sets satisfy condition **G**. Moreover, any subset of a domain with condition **G** satisfies in turn condition **G**.

The result of Cabré [30, Theorem 1.1] provides the estimate (2.17) with  $R_0$  instead of the diameter  $d$ , and a constant  $C$  which also depends on  $\sigma$  and  $\tau$ . A refinement [26, Theorem 1] allow us to substitute  $\|f\|_{L^n(\Omega)}$  with a local norm  $\sup_{y^* \in \Omega} \|f\|_{L^n(B_{R_0}(y^*) \cap \Omega)}$ .

Here we extend the result to weighted partial trace operators  $\mathcal{P}$ . Theorem 1.4 is a direct consequence of [26, Theorem 1] and Proposition 2.4.

Let  $d, \rho > 0$ ,  $i \in \{1, \dots, n\}$ . Examples of domains satisfying condition **G** are for instance the following domains, which will be used in the sequel:

- Finite cylinders  $C_{\rho,d,i} := \{x = (\hat{x}_i, x) \in \mathbb{R}^n : |\hat{x}_i| < \rho, |x_i| < d\}$ ,
- Bounded star-shaped domains  $S_{\rho,d} := \bigcup_{i=1}^n C_{\rho,d,i}$ ,
- Slabs  $S_{d,i} := \{x \in \mathbb{R}^n : |x_i| < d\}$ ,
- Unbounded star-shaped domains  $S_d := \bigcup_{i=1}^n S_{d,i}$ .

But very different domains with respect to cylinders and slabs or finite unions are possible. For instance, in the plane, the complement of a regular lattice of balls with centers at integer coordinates, and the complement of the spiral  $\rho = e^\theta$ ,  $\theta \in \mathbb{R}$ , in polar coordinates, are domains  $\Omega$  endowed with a condition **G**. Note that in the latter case that  $\mathbb{R}^2 \setminus \Omega$  has empty interior.

An application of the ABP estimate, which will be used in the sequel, is the local Hölder continuity of solutions of uniformly elliptic equations. See [23]. Using Proposition 2.4, this can be extended to weighted partial trace operators. See [14, Theorem 5.3].

**Lemma 2.7.** Let  $B_\delta(x_0)$  be a ball of radius  $\delta > 0$ , and let  $\mathcal{P} \in \mathcal{A}$  be a weighted partial trace operator with  $a_1 a_n > 0$ . Suppose that  $f \in C^0(B_{2\delta}(x_0)) \cap L^n(B_{2\delta}(x_0))$ . Let  $u \in C^0(\overline{B_{2\delta}(x_0)})$  be a solution of the equation  $\mathcal{P}(D^2u) = f$  in  $B_{2\delta}(x_0)$ . Then  $u \in C^\alpha(B_\delta(x_0))$  for some  $\alpha \in (0, 1)$ , and there exists a positive constant  $C$  such that

$$|u(x'') - u(x')| \leq C(M + N_f)|x'' - x'|^\alpha, \quad x', x'' \in B_\delta(x_0), \quad (2.20)$$



where  $M = \sup_{B_{2\delta}(x_0)} u$ ,  $N_f = \|f\|_{L^n(B_{2\delta}(x_0))}$ . The Hölder exponent  $\alpha$  depends on  $n$ ,  $a_1$ , and  $a_n$ . The constant  $C$  also depends on  $\delta$ .

We recall that for Pucci extremal operators with ellipticity constants  $\lambda > 0$  and  $\Lambda \geq \lambda$  the Hölder inequality (2.20) holds with  $\alpha$  and  $C$  depending on the same quantities but  $\lambda$  and  $\Lambda$  in the place of  $a_1$  and  $a_n$ .

The issue of the boundary regularity will be treated using the uniform exterior cone property and the Miller barrier functions.

**Definition 2.8.** (Exterior cone condition) Let  $\Omega$  be a domain of  $\mathbb{R}^n$ . Following [31, Section 3], we say that  $\Omega$  is endowed with a uniform exterior cone property if there exists a circular cone

$$T_\psi = \{x \in \mathbb{R}^n : x_n \geq |x| \cos \psi\}, \quad (2.21)$$

where  $\psi \in (0, \pi)$  is the angle with the negative half-line in the direction  $x_n$ , such that for any point on  $z \in \partial\Omega$  we have

$$\bar{\Omega} \cap B_{r_0}(z) \subset z + RT_\psi \quad (2.22)$$

for a suitable rotation matrix  $R$ .

**Definition 2.9.** (Miller barrier functions) Let  $T_\psi$  be the circular cone of Definition 2.8. A Miller barrier function is, in spherical coordinates  $\rho = |x|$  and  $\theta = \arccos \frac{x_n}{|x|}$ :

$$\varphi(x) = \rho^\gamma f(\theta). \quad (2.23)$$

Let  $\lambda > 0$  and  $\Lambda \geq \lambda$ . According to [32, Theorem 3], for all  $\psi \in (0, \pi)$  there exist  $\gamma \in (0, 1)$  and a function

$$f \in C^2[0, \pi], \quad f'(0) = 0, \quad f > 0 \text{ in } [0, \psi] \quad (2.24)$$

such that

$$\mathcal{M}_{\lambda, \Lambda}^+(D^2\varphi) \leq 0 \text{ in } T_\psi. \quad (2.25)$$

Multiplying by a constant, we may suppose  $f \geq 1$  in  $[0, \psi]$  so that

$$\varphi(x) \geq r^\gamma \text{ on } |x| = r. \quad (2.26)$$

A further application of the ABP estimate, which will be helpful in the uniqueness issue, is the boundary weak Harnack inequality, which follows from [14, Theorem 5.2] suitably extending non-negative supersolutions outside the domain  $\Omega$ .

**Proposition 2.10.** Let  $\mathcal{P}$  be such that  $a_1 a_n > 0$ . Let  $\Omega$  be a domain of  $\mathbb{R}^n$  and  $B = B_{R_0}(\bar{x})$ ,  $\tilde{B} = B_{R_0/\tau}(\bar{x})$  concentric balls of radius  $R_0$ ,  $R_0/\tau$ , respectively, such that  $B \cap \Omega \neq \emptyset$  and  $\tilde{B} \setminus \Omega \neq \emptyset$ , for a constant  $\tau \in (0, 1)$ . Suppose that  $u$  is a non-negative supersolution of the equation  $\mathcal{P}(D^2u) = f$  in  $\Omega$ , where  $f \in C^0(\tilde{B} \cap \Omega) \cap L^n(\tilde{B} \cap \Omega)$ . There exist positive constants  $p_0$  and  $C$  such that

$$\left( \frac{1}{|B|} \int_B \underline{u}_l^{p_0} dx \right)^{1/p_0} \leq C \left( \inf_{B \cap \Omega} u + \|f\|_{L^n(\tilde{B} \cap \Omega)} \right), \quad (2.27)$$

where setting  $l = \liminf_{x \rightarrow \tilde{B} \cap \partial\Omega} u$ ,

$$\underline{u}_l = \begin{cases} \min(u, l) & \text{in } B \cap \Omega \\ l & \text{in } B \setminus \Omega. \end{cases} \quad (2.28)$$

The constants  $p_0$  and  $C$  only depend on  $n$ ,  $a_1$ ,  $a_n$ , and  $\tau$ .

Finally, a useful tool in the approximation argument that will be used in the sequel is the uniform ellipticity of weighted partial trace operators with all positive weights.



**Proposition 2.11.** *Let  $\mathcal{P}$  be the partial trace operator  $\mathcal{P} = a_1\lambda_1 + \cdots + a_n\lambda_n$ . In particular,  $\mathcal{P}$  is uniformly elliptic with ellipticity constants  $\lambda = \underline{a}$  and  $\Lambda = \bar{a}$ , that is, for  $Y \geq 0$ :*

$$\underline{a}\mathrm{Tr}(Y) \leq \mathcal{P}(X + Y) - \mathcal{P}(X) \leq \bar{a}\mathrm{Tr}(Y). \quad (2.29)$$

**Proof.** Let  $Y \geq 0$ . Then

$$\begin{aligned} \mathcal{P}(X + Y) &= \underline{a}\mathrm{Tr}(X + Y) + \sum_{i=1}^n (a_i - \underline{a})\lambda_i(X + Y) \\ &\geq \underline{a}\mathrm{Tr}(X + Y) + \sum_{i=1}^n (a_i - \underline{a})\lambda_i(X) \\ &= \underline{a}\mathrm{Tr}(X) + \underline{a}\mathrm{Tr}(Y) + \sum_{i=1}^n (a_i - \underline{a})\lambda_i(X) \\ &= \mathcal{P}(X) + \underline{a}\mathrm{Tr}(Y). \end{aligned} \quad (2.30)$$

Analogously,

$$\begin{aligned} \mathcal{P}(X + Y) &= \bar{a}\mathrm{Tr}(X + Y) - \sum_{i=1}^n (\bar{a} - a_i)\lambda_i(X + Y) \\ &\leq \bar{a}\mathrm{Tr}(X + Y) - \sum_{i=1}^n (\bar{a} - a_i)\lambda_i(X) \\ &= \bar{a}\mathrm{Tr}(X) + \bar{a}\mathrm{Tr}(Y) + \sum_{i=1}^n (a_i - \bar{a})\lambda_i(X) \\ &= \mathcal{P}(X) + \bar{a}\mathrm{Tr}(Y). \end{aligned} \quad (2.31)$$

Therefore, we have (2.29). □

### 3 Existence and uniqueness

In this section, we show the existence and the uniqueness of solution for the equation  $\mathcal{P}(D^2u) = 0$  in  $\Omega$  with the boundary condition  $u = g$  on  $\partial\Omega$ , where  $\Omega$  is a domain with an exterior cone property, possibly unbounded.

**Proof.** (Theorem 1.2) We use Perron's method, see [33, Theorem 4.1], which in particular provides the existence of a unique solution  $u \in C^0(\bar{\Omega})$  once that:

- (i) The comparison principle holds;
- (ii) There exist a subsolution  $\underline{u} \in C^0(\bar{\Omega})$  and a supersolution  $\bar{u} \in C^0(\bar{\Omega})$  of the equation  $\mathcal{P}(D^2u) = 0$  in  $\Omega$  such that  $\underline{u} = g = \bar{u}$  on  $\partial\Omega$ .

Lemma 2.2 provides (i). Concerning (ii), since  $\Omega$  is endowed with a uniform exterior cone property, we can apply [31, Theorem 1.1]. That is, there exist continuous solutions of the Dirichlet problems

$$\begin{cases} \mathcal{M}_{\frac{a_1}{n}, \bar{a} + \frac{a_1}{n}}^+(D^2\bar{u}) = 0 & \text{in } \Omega \\ \bar{u} = g & \text{on } \partial\Omega \end{cases} \quad (3.1)$$

and

$$\begin{cases} \mathcal{M}_{\frac{a_n}{n}, \bar{a} + \frac{a_n}{n}}^-(D^2\underline{u}) = 0 & \text{in } \Omega \\ \underline{u} = g & \text{on } \partial\Omega. \end{cases} \quad (3.2)$$

Thanks to Proposition 2.4,  $\mathcal{M}_{\frac{a_n}{n}, \bar{a} + \frac{a_n}{n}}^- \leq \mathcal{P} \leq \mathcal{M}_{\frac{a_1}{n}, \bar{a} + \frac{a_1}{n}}^+$ , and therefore  $\bar{u}$  and  $\underline{u}$  are a supersolution and a subsolution of the equation  $\mathcal{P}(D^2u) = 0$  equal to  $g$  on  $\partial\Omega$  that we were searching for.

The pointwise estimate (1.9) is a consequence of the ABP estimate. See Definition 2.6 and [23, Theorem 1.1] for the uniformly elliptic version of Theorem 1.4.  $\square$

The proof for unbounded domains will proceed by approximation.

**Proof.** (Theorem 1.3) First, by means of the Tietze-Urysohn-Brouwer extension theorem we extend  $g$  to a continuous function on  $\mathbb{R}^n$ , called in turn  $g$ , such that

$$\sup_{\mathbb{R}^n} |g| \leq M_g. \quad (3.3)$$

*Existence and interior continuity*

Let  $\Omega_\rho = Q_\rho \cap \Omega$ , where  $\rho > 0$  and  $Q_\rho = \{x \in \mathbb{R}^n : |x_i| < \rho\}$ . We approximate  $\Omega$  with the bounded domains  $\Omega_{\rho_k}$ ,  $k \in \mathbb{N}$ , such that  $\rho_k \rightarrow \infty$  as  $k \rightarrow \infty$ .

Since  $\Omega$  and  $Q_\rho$  satisfy condition **G** and a uniform exterior cone condition, so does their intersection.

From the bounded case, then we can solve the Dirichlet problem in  $\Omega_{\rho_k}$ . That is, we find  $u_k \in C^0(\bar{\Omega}_{\rho_k})$  such that, in the viscosity sense:

$$\begin{cases} \mathcal{P}(D^2u_k) = 0 & \text{in } \Omega_{\rho_k} \\ u_k = g & \text{on } \partial(\Omega_{\rho_k}). \end{cases} \quad (3.4)$$

The pointwise estimate (1.9) of the bounded case, combined with (3.3), yields

$$\sup_{\Omega_{\rho_k}} |u_k| \leq M_g, \quad (3.5)$$

which implies the full inequality (1.9) even in the unbounded case.

We know that the functions  $u_k$  are locally Hölder. See Lemma 2.7. In fact, let  $K_1$  be a compact subset of  $\Omega$ . For  $k \in \mathbb{N}$  large enough we have  $K_1 \subset \Omega_{\rho_k}$ , and

$$\text{dist}(K_1, \Omega_{\rho_k}) \geq \delta_1 \equiv \text{dist}(K_1, \Omega_{\rho_1}). \quad (3.6)$$

Hence, for all  $x_0 \in K_1$  the Hölder inequality (2.20) holds in a ball  $B_{\delta_1/2}(x_0)$ :

$$|v(x'') - v(x')| \leq CM_g |x'' - x'|^\alpha, \quad x', x'' \in B_{\delta_1/2}(x_0). \quad (3.7)$$

Such estimate is uniform with respect to  $k \in \mathbb{N}$ , depending on the Hölder coefficient  $C \in \mathbb{R}_+$  and the Hölder exponent  $\alpha \in (0, 1)$  only on  $n$ ,  $a_1$ ,  $a_n$ , and  $\delta_1$ .

Therefore, the functions  $u_k$  are equi-bounded and equi-continuous on each compact subset of  $\Omega$ .

Let  $\delta_\ell$  sequence of positive number such that  $\delta_\ell \downarrow 0$  as  $\ell \rightarrow \infty$ . We invade  $\Omega$  with the increasing sequence of bounded subsets  $\Omega_{\ell, \delta_\ell} = \{x \in \Omega_k : \text{dist}(x, \partial\Omega) > \delta_\ell\}$ , such that  $\bar{\Omega}_{\ell, \delta_\ell} \subset \Omega$  and  $\Omega = \cup_\ell \bar{\Omega}_{\ell, \delta_\ell}$ .

From the Ascoli-Arzelà theorem it follows that the sequence  $u_k$  uniformly converges over  $\Omega_{\ell, \delta_\ell}$  up to a subsequence for any  $\ell \in \mathbb{N}$ . So by a Cantor diagonalization process, we can extract from  $\{u_k\}_{k \in \mathbb{N}}$  a subsequence (called again  $u_k$ ) that converges, locally uniformly in  $\Omega$ , to a continuous function  $u$ .

By construction, given  $x \in \partial\Omega$ , we have  $u_k(x) = g(x)$  for sufficiently large  $k \in \mathbb{N}$ , so that the limit function

$$u(x) = \lim_{k \rightarrow \infty} u_k(x) \quad (3.8)$$

exists for all points of  $\bar{\Omega}$ , and  $u = g$  on  $\partial\Omega$ .

Recalling (3.5), we also obtain the pointwise estimate (1.9).

By stability theorems on viscosity solutions, for instance [23, Proposition 2.9], we can also conclude that the limit function  $u$  is a viscosity solution of the equation  $\mathcal{P}(D^2u) = 0$  in  $\Omega$ .

So we have a solution  $u \in C^0(\Omega)$  of the Dirichlet problem (1.8) with boundary values  $u = g$ , as required. But we are left with showing the continuity at points of  $\partial\Omega$ , in order that  $u \in C^0(\bar{\Omega})$ .

*Continuity up to the boundary*

In order to show that we need to prove that for every  $y_0 \in \partial S$ :

$$\lim_{\substack{x \rightarrow y_0 \\ x \in \bar{\Omega}}} u(x) = u(y_0) \equiv g(y_0). \quad (3.9)$$

For this purpose, up to translations, we may suppose  $y_0 = 0$ .

Let us fix  $\varepsilon > 0$ . By the continuity of  $g$  we can choose  $\delta_g = \delta_g(\varepsilon)$  such that

$$|u(y) - g(0)| < \frac{\varepsilon}{2} \quad \text{for } y \in \partial\Omega : |y| < \delta_g. \quad (3.10)$$

We want to show (3.9) with  $y_0 = 0$ , that is,

$$\lim_{\substack{x \rightarrow 0 \\ x \in \bar{\Omega}}} u(x) = u(0) \equiv g(0). \quad (3.11)$$

To this end, observe that  $\Omega$  is endowed with a uniform exterior cone property. See Definition 2.8. We may suppose, up to rotations, that the axis of cone, with vertex at the boundary point  $x = 0$ , has direction  $x_n$ . Thus,

$$\bar{\Omega} \cap B_{r_0} \subset T_\psi, \quad (3.12)$$

where  $B_{r_0} = \{x \in \mathbb{R}^n : |x| < r_0\}$ .

Since  $\mathcal{P}(D^2u) = 0$ , Proposition 2.4 implies that

$$\mathcal{M}_{\frac{a_1}{n}, \bar{a} + \frac{a_1}{n}}^+(D^2u) \geq 0. \quad (3.13)$$

Let  $\varphi(x) = \rho^\gamma f(\theta)$  be the corresponding Miller barrier function, in spherical coordinates  $\rho = |x|$  and  $\theta = \arccos \frac{x_n}{|x|}$ . See Definition 2.9. Then

$$\mathcal{M}_{\frac{a_1}{n}, \bar{a} + \frac{a_1}{n}}^+(D^2\varphi) \leq 0. \quad (3.14)$$

Next, let  $\delta_0 > 0$  be such that  $\delta_0 < \min(\delta_g, r_0)$ . We compare the function  $u$  with the function

$$w = g(0) + \frac{\varepsilon}{2} + \frac{2M_g}{\delta_0^\gamma} \varphi \quad (3.15)$$

in  $\Omega \cap B_{\delta_0}$ . On the boundary, we have  $u \leq w$ : in fact,  $u \leq g(0) + \frac{\varepsilon}{2} \leq w$  on  $\partial\Omega \cap B_{\delta_0}$ , by (3.10), and  $u \leq g(0) + 2M_g \leq w$  on  $\Omega \cap \partial B_{\delta_0}$ , by (1.9) and (2.26).

So, by (3.13) and (3.14),  $u$  and  $w$  are a subsolution and a supersolution, respectively, of the same uniformly elliptic equation in  $\Omega \cap B_{\delta_0}$ , and  $u \leq w$  on  $\partial(\Omega \cap B_{\delta_0})$ .

Therefore, by the comparison principle for uniformly elliptic equations (see [23]), we have  $u \leq w$  in  $\Omega \cap B_{\delta_0}$ , that is,

$$u(x) - g(0) \leq \frac{\varepsilon}{2} + \frac{2M_g}{\delta_0^\gamma} \rho^\gamma. \quad (3.16)$$

Choosing  $\delta_1 > 0$  such that  $\frac{2M_g}{\delta_0^\gamma} \delta_1^\gamma < \frac{\varepsilon}{2}$ , we obtain for  $x \in \bar{\Omega} \cap B_{\delta_1}$ :

$$u(x) - g(0) < \varepsilon. \quad (3.17)$$

In order to complete the proof of the continuity at the point  $x = 0 \in \partial\Omega$ , we also need to show that there exists  $\delta_2 > 0$  such that for  $x \in \bar{\Omega} \cap B_{\delta_2}$ :

$$g(0) - u(x) < \varepsilon. \quad (3.18)$$

We use a similar procedure, observing that the function  $u^* = -u$  is the solution of the Dirichlet problem

$$\begin{cases} \mathcal{P}^*(D^2u^*) = 0 & \text{in } \Omega \\ u^* = -g & \text{on } \partial\Omega, \end{cases} \quad (3.19)$$

where  $\mathcal{P}^*$  is the dual operator.

This time we use the fact that by Proposition 2.4

$$\mathcal{M}_{\frac{a_1}{n}, \bar{a} + \frac{a_1}{n}}^+(D^2u^*) \geq 0, \quad (3.20)$$

and we take  $\gamma \in (0, 1)$  such that

$$\mathcal{M}_{\frac{a_1}{n}, \bar{a} + \frac{a_1}{n}}^+(D^2\varphi) \leq 0, \quad (3.21)$$

in the construction of the barrier function  $\varphi(x) = \rho^\gamma f(\theta)$ .

We compare now the function  $u^*$  with

$$w = -g(0) + \frac{\varepsilon}{2} + \frac{2M_g}{\delta_g^\gamma} \varphi \quad (3.22)$$

in  $\Omega \cap B_{\delta_0}$ . On the boundary, we have  $u^* \leq w$ : in fact,  $u^* = -u \leq -g(0) + \frac{\varepsilon}{2} \leq w$  on  $\partial\Omega \cap B_{\delta_0}$ , by (3.10), and  $u^* \leq -g(0) + 2M_g \leq w$  on  $\Omega \cap \partial B_{\delta_0}$ , by (1.9) and (2.26).

Then we apply the comparison principle (Lemma 2.2), and so  $u^* \leq w$  in  $\Omega \cap B_{\delta_0}$ .

Like before in view of inequality (3.17), we obtain  $\delta_2 > 0$  such that for  $x \in \bar{\Omega} \cap B_{\delta_2}$ :

$$u^*(x) + g(0) < \varepsilon, \quad (3.23)$$

thereby proving inequality (3.18), and so (3.11).

Since this can be done for an arbitrary point  $y_0 \in \partial\Omega$ , the above shows that the solution  $u \in C^0(\bar{\Omega})$ , and the proof of the continuity up to the boundary is complete.

*Behavior of solutions at infinity*

Here we show that the solution  $u$  vanishes at infinity, under the assumption that  $g$  vanishes at infinity.

First, by condition **G** of Definition 2.6, with parameters  $R_0 > 0$  and  $0 < \sigma, \tau < 1$  we construct a covering  $\mathcal{B}$  of  $\Omega$ , with balls  $B = B_r(y^*)$ .

Let us fix  $\varepsilon > 0$ , and let

$$E_r := \{x \in \Omega : |x| \geq r\}. \quad (3.24)$$

By assumption there exists  $R_\varepsilon > 0$  such that for all  $r \geq R_\varepsilon$

$$|g(x)| < \varepsilon \quad \text{for } x \in E_r \cap \partial\Omega. \quad (3.25)$$

Let  $R > R_\varepsilon + R_0\left(\frac{1}{\tau} - 1\right)$ , then we introduce the families of balls

$$\mathcal{B}_R := \{B = B_r(y^*) : |y| > R\} \quad (3.26)$$

and

$$\tilde{\mathcal{B}}_R := \{\tilde{B} = B_{r/\tau}(y^*) : B = B_r(y^*) \in \mathcal{B}_R\}. \quad (3.27)$$

Note that

$$E_R \subset \bigcup_{B \in \mathcal{B}_R} (B \cap \Omega) \subset \bigcup_{\tilde{B} \in \tilde{\mathcal{B}}_R} (\tilde{B} \cap \Omega) \subset E_{R-R_0\left(\frac{1}{\tau}-1\right)}. \quad (3.28)$$

According to condition **G**, we have  $y \in B_r(y^*)$  and  $|B_r(y^*) \setminus \Omega_{y,\tau}| \geq \sigma |B_r(y^*)|$ , where  $\Omega_{y,\tau}$  is the connected component of  $B_{r/\tau}(y^*) \cap \Omega$  containing  $y$ . Set also  $M_{r/\tau} = \sup\{u(x) | x \in B_{r/\tau}(y^*) \cap \Omega\}$ .

By assumption  $a_1 a_n > 0$ . Hence, we can apply the boundary weak Harnack inequality (Proposition 2.10) to the non-negative function  $v = M_{r/\tau} - u$ .

This is a solution of the Dirichlet problem

$$\begin{cases} \mathcal{P}^*(D^2v) = 0 & \text{in } \Omega \\ v = M_{r/\tau} - g & \text{on } \partial\Omega, \end{cases} \quad (3.29)$$

where  $\mathcal{P}^*$  is the dual of  $\mathcal{P}$ .

From (3.25) and (3.28) we have:

$$l = \liminf_{x \rightarrow \bar{B} \cap \partial\Omega} (M_{r/\tau} - u) = M_{r/\tau} - \limsup_{x \rightarrow \bar{B} \cap \partial\Omega} u \geq M_{r/\tau} - \varepsilon. \quad (3.30)$$

Then the boundary weak Harnack inequality (2.27), with  $p := p_0$ , yields

$$\left( \frac{|B \setminus \Omega|}{|B|} \right)^{1/p} (M_{r/\tau} - \varepsilon) \leq C \inf_{B \cap \Omega} (M_{r/\tau} - u) \quad (3.31)$$

from which

$$2^{-1/p} (M_{r/\tau} - \varepsilon) \leq C M_{r/\tau} - C \sup_{B \cap \Omega} u. \quad (3.32)$$

It follows that

$$\sup_{B_r(y^*) \cap \Omega} u \leq \mu \sup_{B_{r/\tau}(y^*) \cap \Omega} u + (1 - \mu)\varepsilon \quad (3.33)$$

with  $0 < \mu \equiv 1 - 2^{-1/p}/C < 1$ .

From (3.28) and (3.33) we deduce that

$$\sup_{\substack{|x| \geq R \\ x \in \Omega}} u(x) \leq \mu \sup_{\substack{|x| \geq R - R_0 \left(\frac{1}{\tau} - 1\right) \\ x \in \Omega}} u(x) + (1 - \mu)\varepsilon. \quad (3.34)$$

Letting  $R \rightarrow \infty$ ,

$$\limsup_{\substack{|x| \rightarrow \infty \\ x \in \Omega}} u(x) \leq \mu \limsup_{\substack{|x| \rightarrow \infty \\ x \in \Omega}} u(x) + (1 - \mu)\varepsilon, \quad (3.35)$$

from which

$$\limsup_{\substack{|x| \rightarrow \infty \\ x \in \Omega}} u(x) \leq \varepsilon. \quad (3.36)$$

Since  $\varepsilon > 0$  is arbitrary we can take the limit for  $\varepsilon \rightarrow 0^+$  to conclude that

$$\limsup_{\substack{|x| \rightarrow \infty \\ x \in \Omega}} u(x) \leq 0. \quad (3.37)$$

An analogous argument can be used to show that

$$\liminf_{\substack{|x| \rightarrow \infty \\ x \in \Omega}} u(x) \geq 0. \quad (3.38)$$

In this case, setting  $m_{2R_0} = \inf\{u(x) | x \in B_{2R_0}(\bar{x}) \cap \Omega\}$ , we observe that  $v = u - m_{2R_0}$  is a solution of the Dirichlet problem

$$\begin{cases} \mathcal{P}(D^2v) = 0 & \text{in } \Omega \\ v = g - m_{2R_0} & \text{on } \partial\Omega. \end{cases} \quad (3.39)$$

Finally, (3.37) and (3.38) show that  $u(x) \rightarrow 0$  as  $|x| \rightarrow \infty$  in  $\Omega$ , as claimed.

*Uniqueness*

Let  $u, w$  be two solutions of the Dirichlet problem under consideration. Then both  $u$  and  $w$  vanish at infinity. That is, for all  $\varepsilon > 0$  there exists  $R_\varepsilon > 0$  such that

$$|u(x)| < \varepsilon, \quad |w(x)| < \varepsilon \quad \text{for } x \in \Omega \quad \text{such that } |x - x^*| \geq R_\varepsilon. \quad (3.40)$$

We compare  $u$  and  $w + 2\varepsilon$  in  $B_{R_\varepsilon}(x^*)$  as a subsolution and a supersolution of the equation  $\mathcal{P}(D^2v) = 0$  in  $\Omega \cap B_{R_\varepsilon}(x^*)$ . Since  $u \leq w + 2\varepsilon$  on the boundary, by the comparison principle (Lemma 2.2) we obtain:

$$u \leq w + 2\varepsilon \quad \text{in } \Omega \cap B_{R_\varepsilon}(x^*). \quad (3.41)$$

On the other hand, by (3.40):

$$u - w \leq 2\varepsilon \quad \text{in } \Omega \setminus B_{R_\varepsilon}(x^*). \quad (3.42)$$

Therefore,  $u - w \leq 2\varepsilon$  in  $\Omega$ . On the other hand, we can interchange the role of  $u$  and  $w$ , obtaining the same inequality for  $w - u$ . Therefore,  $|u - w| \leq \varepsilon$  for all  $\varepsilon > 0$ . Letting  $\varepsilon \rightarrow 0^+$ , we obtain  $u = w$ . That is, the solution is unique as claimed.  $\square$

## 4 Approximate solutions and Lipschitz constants

In this section, we consider perturbations of the weighted partial trace operators  $\mathcal{P}$ , which leads to uniformly elliptic operators, namely:  $\mathcal{P}_\varepsilon := \mathcal{P} + \varepsilon\Delta$  for  $\varepsilon > 0$ .

We show that solutions  $u_\varepsilon$  of the equation  $\mathcal{P}_\varepsilon(D^2v) = 0$ , which are bounded in a cylinder of center  $x^*$  and axis of fixed height with sufficiently large section, depending on  $\varepsilon$ , have a partial derivative bound in the axis direction, uniform with respect to  $\varepsilon$ , at the point  $x^*$ .

Note indeed that in the uniformly elliptic case the argument works without any constraint on the diameter of the section of the cylinder. In the degenerate case, instead, a larger and larger radius  $\rho_\varepsilon$  is needed as  $\varepsilon \rightarrow 0^+$ . See (4.2) in the following lemma, when  $\underline{a} \equiv \min_{i=1, \dots, n} a_i = 0$ .

**Lemma 4.1.** *Let  $d, \delta, \varepsilon$  be positive numbers, and  $C_{\rho_\varepsilon, d; i}^* = x^* + C_{\rho_\varepsilon, d; i}$ , the finite cylinder, with center  $x^*$  and axis direction  $x_i$ :*

$$C_{\rho_\varepsilon, d; i}^* = \{x \in \mathbb{R}^n : |\hat{x}_i - \hat{x}_i^*| < \rho_\varepsilon, |x_i - x_i^*| < d\}, \quad (4.1)$$

where  $\hat{x}_i = (x_1, \dots, x_i, \dots, x_n)$  and

$$\rho_\varepsilon^2 \geq (n-1) \frac{\bar{a} + \varepsilon d^2}{\underline{a} + \varepsilon \delta}. \quad (4.2)$$

Let  $u_\varepsilon$  be a bounded continuous viscosity solution, namely,

$$\sup_{C_{\rho_\varepsilon, d; i}^*} |u_\varepsilon| \leq M_i, \quad (4.3)$$

of the equation  $\mathcal{P}_\varepsilon(D^2v) = 0$  for  $\mathcal{P} \in \mathcal{A}$ .

Then

$$\left| \frac{\partial u_\varepsilon}{\partial x_i}(x^*) \right| \leq (1 + \delta) \frac{M_i}{d}. \quad (4.4)$$

**Proof.** We may suppose  $x^* = 0$  and  $i = n$ , so that we are left with the cylinder  $C_{\rho_\varepsilon, d; n} = \{|\hat{x}_n| < \rho_\varepsilon; |x_n| < d\}$ . The operator

$$\mathcal{P}_\varepsilon := \mathcal{P} + \varepsilon\Delta \equiv (a_1 + \varepsilon)\lambda_1 + \dots + (a_n + \varepsilon)\lambda_n \quad (4.5)$$

is uniformly elliptic. More precisely, according to Proposition 2.11,

$$\mathcal{M}_{\underline{a}+\varepsilon, \bar{a}+\varepsilon}^-(Y) \leq \mathcal{P}_\varepsilon(X+Y) - \mathcal{P}_\varepsilon(X) \leq \mathcal{M}_{\underline{a}+\varepsilon, \bar{a}+\varepsilon}^+(Y). \quad (4.6)$$

Hence, by [23, Corollary 5.7],  $u_\varepsilon$  is locally  $C^{1,\alpha}$  in  $C_{\rho_\varepsilon, d; n}$ , and the derivative on the left-hand side of inequality (4.4) exists.

The operator  $\mathcal{P}_\varepsilon$  is also invariant by reflection. As observed in Lemma 2.1 and in the subsequent discussion, leading to (2.6), we have therefore both

$$\mathcal{P}_\varepsilon(D^2u_\varepsilon) = 0 \quad \text{and} \quad \mathcal{P}_\varepsilon(D^2\tilde{u}_\varepsilon) = 0, \quad (4.7)$$

where  $\tilde{u}_\varepsilon(\hat{x}_n, x_n) = u_\varepsilon(\hat{x}_n, -x_n)$ .

Next, we introduce the function

$$\varphi(x) = \varphi(\hat{x}_n, x_n) = \frac{2M_n}{\rho_\varepsilon^2} |\hat{x}_n|^2 + \frac{2M_n}{d^2} x_n((1+\delta)d - \delta x_n). \quad (4.8)$$

We compare the functions  $u_\varepsilon$  and  $v_\varepsilon := \tilde{u}_\varepsilon + \varphi$  on the upper cylinder

$$C_{\rho_\varepsilon, d; n}^+ = \{x \in C_{\rho_\varepsilon, d; n} : 0 \leq x_n \leq d\}, \quad (4.9)$$

using the comparison principle (Lemma 2.2).

By uniform ellipticity (4.6):

$$\mathcal{P}_\varepsilon(D^2v_\varepsilon) = \mathcal{P}_\varepsilon(D^2\tilde{u}_\varepsilon + D^2\varphi) \leq \mathcal{P}_\varepsilon(D^2\tilde{u}_\varepsilon) + \mathcal{M}_{\underline{a}+\varepsilon, \bar{a}+\varepsilon}^+(D^2\varphi). \quad (4.10)$$

Note that  $D^2\varphi$  is diagonal with eigenvalues

$$\lambda_1(D^2\varphi) = -\frac{4M_n\delta}{d^2}, \quad \lambda_i(D^2\varphi) = \frac{4M_n}{\rho_\varepsilon^2}, \quad i = 2, \dots, n \quad (4.11)$$

and so, by (4.2):

$$\mathcal{M}_{\underline{a}+\varepsilon, \bar{a}+\varepsilon}^+(D^2\varphi) = (n-1) \frac{4M_n}{\rho_\varepsilon^2} - (\underline{a}+\varepsilon) \frac{4M_n\delta}{d^2} \leq 0. \quad (4.12)$$

This shows, together with (4.7) and (4.10), that

$$\mathcal{P}_\varepsilon(D^2v_\varepsilon) \leq \mathcal{P}_\varepsilon(D^2\tilde{u}_\varepsilon) + \mathcal{M}_{\underline{a}+\varepsilon, \bar{a}+\varepsilon}^+(D^2\varphi) \leq 0. \quad (4.13)$$

By (4.7) and (4.13), it turns out that  $u_\varepsilon$  and  $v_\varepsilon$  are a subsolution and a supersolution of the uniformly elliptic equation  $\mathcal{P}_\varepsilon(D^2v) = 0$ , respectively.

Next, we compare  $u_\varepsilon$  and  $v_\varepsilon$  on the boundary of  $C_{\rho_\varepsilon, d; n}^+$ :

(base  $x_n = 0$ ) being  $\tilde{u}_\varepsilon = u_\varepsilon$  and  $\varphi \geq 0$ , we have

$$v_\varepsilon(x', 0) = \tilde{u}_\varepsilon(\hat{x}_n, 0) + \varphi(\hat{x}_n, 0) \geq u_\varepsilon(\hat{x}_n, 0); \quad (4.14)$$

(base  $x_n = d$ ) we have  $\varphi(\hat{x}_n, d) \geq 2M_n$  and therefore by (4.3)

$$v_\varepsilon(\hat{x}_n, d) = \tilde{u}_\varepsilon(\hat{x}_n, d) + \varphi(\hat{x}_n, d) \geq -M_n + 2M_n = M_n \geq u_\varepsilon(\hat{x}_n, d); \quad (4.15)$$

(lateral boundary  $|\hat{x}_n| = \rho_\varepsilon$ ) we have  $\varphi(\hat{x}_n, x_n) \geq 2M_n$ , and again by (4.3),

$$v_\varepsilon(x) = \tilde{u}_\varepsilon(x) + \varphi(x) \geq -M_n + 2M_n = M_n \geq u_\varepsilon(x). \quad (4.16)$$

We conclude that  $u_\varepsilon$  and  $v_\varepsilon$  are a subsolution and a supersolution of equation  $\mathcal{P}_\varepsilon(D^2v) = 0$  in  $C_{\rho_\varepsilon, d; n}^+$  such that  $u_\varepsilon \leq v_\varepsilon$  on  $\partial C_{\rho_\varepsilon, d; n}^+$ . Then the comparison principle (Lemma 2.2) yields

$$\begin{aligned} u_\varepsilon(\hat{x}_n, x_n) &\leq v_\varepsilon(\hat{x}_n, x_n) \equiv u_\varepsilon(\hat{x}_n, -x_n) + \varphi(\hat{x}_n, x_n) \\ &\leq u_\varepsilon(\hat{x}_n, -x_n) + \frac{2M_n}{\rho_\varepsilon^2} |\hat{x}_n|^2 + \frac{2M_n}{d^2} x_n((1+\delta)d - \delta x_n) \end{aligned} \quad (4.17)$$



in  $C_{\rho_\varepsilon, d, n}^+$ . That is, for  $\hat{x}_n = 0$

$$\frac{u_\varepsilon(0, x_n) - u_\varepsilon(0, -x_n)}{x_n} \leq \frac{2M_n}{d^2} x_n ((1 + \delta)d - \delta x_n). \quad (4.18)$$

Recalling that  $u_\varepsilon$  is  $C^{1, \alpha}$  and therefore the derivative exists, we take the limit as  $x_n \rightarrow 0$  in the above, and we obtain

$$\frac{\partial u_\varepsilon}{\partial x_n}(0) = \frac{1}{2} \lim_{x_n \rightarrow 0} \frac{u_\varepsilon(0, x_n) - u_\varepsilon(0, 0)}{x_n} + \frac{1}{2} \lim_{x_n \rightarrow 0} \frac{u_\varepsilon(0, 0) - u_\varepsilon(0, -x_n)}{2x_n} \leq (1 + \delta) \frac{M_n}{d}. \quad (4.19)$$

Interchanging the role of  $u_\varepsilon(\hat{x}_n, x_n)$  and  $u_\varepsilon(\hat{x}_n, -x_n)$ , we also obtain the opposite estimate

$$\frac{\partial u_\varepsilon}{\partial x_n}(0) \geq -(1 + \delta) \frac{M_n}{d}. \quad (4.20)$$

We have therefore proved the estimate (4.4) with  $x^* = 0$ , as wanted.  $\square$

In the slab  $S_{d,i}^*$ , which can be regarded as a limiting set of the cylinders  $C_{\rho_\varepsilon, d, i}^*$  as  $\delta \rightarrow 0^+$ , assuming that  $u_\varepsilon$  is bounded on  $S_{d,i}^*$ , the above lemma holds *a fortiori*, and with the sharper upper bound

$$\left| \frac{\partial u_\varepsilon}{\partial x_i}(x^*) \right| \leq \frac{M_i}{d}, \quad (4.21)$$

where

$$\sup_{S_{d,i}^*} |u_\varepsilon| \leq M_i. \quad (4.22)$$

By translational invariance along the directions of  $\hat{x}_i$ , orthogonal to the direction  $x_i$ , the above bound (4.21) also holds for the points, other than  $x^*$ , of the middle hyperplane of the slab

$$H_{d,i}^* = \{x \in S_{d,i} : x_i = x_i^*\}. \quad (4.23)$$

Suppose in addition that  $u$  is a bounded continuous viscosity solution of the equation  $\mathcal{P}_\varepsilon(D^2v) = 0$  in the unbounded star-shaped domain  $S_d^* = x^* + S_d$ .

We obtain the above bound (4.21) for all partial derivatives, and consequently a bound for the gradient of  $u_\varepsilon$  at the point  $x^*$ .

In the next corollary, we observe that an analogous bound can be established for all  $x \in Q_d^* = \bigcap_{i=1}^n S_{d,i}^*$ , with a Lipschitz constant growing faster and faster as  $x \rightarrow x_i^* \pm d$ .

**Corollary 4.2.** *Let  $d, \varepsilon$  be positive numbers, and  $S_d^* = \bigcup_{i=1}^n S_{d,i}^*$ , where  $S_{d,i}^* = x^* + S_{d,i}$ . As before, for  $\mathcal{P} \in \mathcal{A}$  we set  $\mathcal{P}_\varepsilon = \mathcal{P} + \varepsilon \Delta$ .*

*Let  $u_\varepsilon$  be a bounded continuous viscosity solution of the equation  $\mathcal{P}_\varepsilon(D^2v) = 0$  in  $S_{2d}^*$ , such that*

$$\sup_{S_{d,i}^*} |u_\varepsilon| \leq M_i. \quad (4.24)$$

*Then*

$$|D_i u_\varepsilon(x)| \leq \frac{M_i}{\delta_i(x)} \quad \text{for all } x \in Q_d^*, \quad (4.25)$$

*where  $\delta_i(x) = \text{dist}(x, \partial S_{d,i}^*)$ . That is the following gradient bound holds:*

$$\max_i |D_i u_\varepsilon(x)| \leq \frac{\max_i M_i}{\text{dist}(x, \partial Q_d^*)} \quad \text{for all } x \in Q_d^*. \quad (4.26)$$

## 5 Asymptotic convergence

The results of the previous section show existence and uniqueness of regular solutions for Dirichlet problems concerning the approximate equation  $\mathcal{P}_\varepsilon(D^2v) = 0$  of Section 4 in an unbounded star-shaped domain  $S_d^*$ .

Here, we prove that solutions of the approximate equations actually converge to the solution of the limit equation in  $S_d^*$ .

**Lemma 5.1.** *Let  $\Omega$  be a domain of  $\mathbb{R}^n$  as in Theorem 1.3, and  $\mathcal{P} = \sum_{i=1}^n a_i \Lambda_i$  be a weighted partial trace operator such that  $a_1 a_n > 0$ .*

*Suppose  $g \in C^0(\partial\Omega)$  be such that*

$$\lim_{\substack{|x| \rightarrow \infty \\ x \in \partial\Omega}} g(x) = 0. \quad (5.1)$$

*For  $\varepsilon > 0$ , let  $u_\varepsilon$  be the solution of the Dirichlet problem*

$$\begin{cases} \mathcal{P}_\varepsilon(D^2v) = 0 & \text{in } \Omega \\ v = g & \text{on } \partial\Omega. \end{cases} \quad (5.2)$$

*Then the  $u_\varepsilon$ s converge by subsequence to the solution  $u$  of the Dirichlet problem*

$$\begin{cases} \mathcal{P}(D^2u) = 0 & \text{in } \Omega \\ u = g & \text{on } \partial\Omega. \end{cases} \quad (5.3)$$

**Proof.** The existence and the uniqueness of solutions  $u_\varepsilon \in C^0(\bar{\Omega})$  and  $u \in C^0(\bar{\Omega})$  of the Dirichlet problems (5.2) and (5.3) are assured by Theorem 1.3.

*Convergence of  $u_\varepsilon$  (by subsequence) to a solution  $v \in C^0(\Omega)$*

The solutions  $u_\varepsilon$  are uniformly bounded by (1.9). Consequently, the Hölder estimates (3.7) are uniform with respect to  $\varepsilon \in (0, 1)$ , depending on the Hölder coefficient and the Hölder exponent only on  $n$ ,  $a_1$ , and  $a_n$ . See Lemma 2.7.

Hence, the  $u_\varepsilon$ s uniformly converge by subsequence in any compact subset of  $\Omega$ .

Proceeding as in the proof of Theorem 1.3 (*Existence and interior continuity*), we can use the Ascoli-Arzelà theorem and a Cantor diagonalization process to obtain a sequence  $u_{\varepsilon_k}$ , locally uniformly converging to a function  $v \in C^0(\Omega)$ .

Since  $u_{\varepsilon_k} = g$  on  $\partial\Omega$ , then the  $u_{\varepsilon_k}$ s converge pointwise in all points of  $\bar{\Omega}$ :

$$v(x) = \lim_{k \rightarrow \infty} u_{\varepsilon_k}(x), \quad (5.4)$$

and  $v = g$  on  $\partial\Omega$ .

By the locally uniform convergence, using stability results in the viscosity setting [23], the limit function  $v$  is a solution of the limit equation, that is,

$$\mathcal{P}(D^2v) = 0 \quad \text{in } \Omega. \quad (5.5)$$

*Continuity of the limit  $v$  up to the boundary*

Arguing as in the proof of Theorem 1.3, due to the uniform bound (1.9) and the uniform exterior cone property, the  $u_{\varepsilon_k}$ s have the same modulus of continuity at any point  $y_0 \in \partial\Omega$ , that is:

$$\forall \varepsilon > 0 \quad \exists \delta = \delta(\varepsilon) > 0 : |u_{\varepsilon_k}(x) - g(y_0)| < \varepsilon \quad \forall x \in \bar{\Omega} \cap B_\delta(y_0), \quad (5.6)$$

and we are pointing out that  $\delta$  does not depend on  $k \in \mathbb{N}$ .

Taking the limit as  $k \rightarrow \infty$ , we have therefore

$$\forall \varepsilon > 0 \quad \exists \delta > 0 : |v(x) - g(y_0)| \leq \varepsilon \quad \forall x \in \bar{\Omega} \cap B_\delta(y_0). \quad (5.7)$$

So  $v$  is continuous at an arbitrary point  $y_0 \in \partial\Omega$ . We conclude that  $v \in C^0(\bar{\Omega})$ .

*End of the proof*

Actually,  $v \in C^0(\bar{\Omega})$  is a solution of the Dirichlet problem (1.8) as well as  $u$ .

Then by Theorem 1.3 (*Uniqueness*) we have  $v = u$  in  $\Omega$ , thereby proving that  $u_{\varepsilon_k} \rightarrow u$  in  $\bar{\Omega}$  as  $k \rightarrow \infty$ .  $\square$

We are in position to show the main result.

**Proof.** (Theorem 1.1). The viscosity solution  $u \in C^0(\bar{\Omega})$  of the Dirichlet problem (1.8) exists and is unique by Theorem 1.3.

The sequence of viscosity solutions  $u_{\varepsilon_k} \in C^0(\bar{\Omega})$  of the approximate problems (5.2) converge to  $u$  by virtue of Lemma 5.1.

The approximate solutions are locally  $C^{1,\alpha}$ . Since  $|u_{\varepsilon_k}| \leq M_g$  in  $\Omega$  by Theorem 1.3, then Corollary 4.2 yields in  $Q_d^* = \bigcap_{i=1}^n S_{d,i}^*$  the gradient bound

$$|D_i u_{\varepsilon_k}(x_0)| \leq \frac{M_g}{\delta(x_0)}, \quad (5.8)$$

where  $\delta(x_0) = \text{dist}(x_0, \partial Q_d^*)$ , for all  $x_0 \in Q_d^*$ .

In particular, let  $0 < \rho < \delta(x_0)$ ,  $\tilde{Q}_\rho$  be the cube centered at  $x_0$  of side  $2\rho$  and  $x', x'' \in Q_\rho$  such that  $x'_j = x''_j$  for  $j \neq i$ . Then

$$|u_{\varepsilon_k}(x'') - u_{\varepsilon_k}(x')| \leq \max_{\tilde{Q}_\rho} |D_i u_{\varepsilon_k}| |x''_i - x'_i| \leq \frac{M_g}{\delta(x_0) - \rho} |x''_i - x'_i|. \quad (5.9)$$

Letting  $k \rightarrow \infty$ , then we obtain, for  $i = 1, \dots, n$ ,

$$|u(x'') - u(x')| \leq \frac{M_g}{\delta(x_0) - \rho} |x''_i - x'_i|. \quad (5.10)$$

So, by Rademacher's theorem, the function  $u$  is differentiable a.e. in  $\tilde{Q}_\rho$ , and then in  $Q_d^*$ , holding (5.10) for all  $x_0 \in Q_d^*$  provided that  $0 < \rho < \delta(x_0)$ . Therefore,  $u$  is locally Lipschitz continuous in  $Q_d^*$ . Moreover, if  $x_0$  is a differentiability point, from (5.10), letting  $\rho \rightarrow 0^+$  we also obtain

$$|D_i u(x_0)| \leq \frac{M_g}{\delta(x_0)} \quad (5.11)$$

as claimed.  $\square$

**Remark 5.2.** Suppose that in Theorem 1.1 we only have  $\Omega \supset S_{d,i}^*$  for some  $i$  instead of  $\Omega \supset S_d^*$ .

From the proof it is clear that, without changing the other assumptions, then we have a Lipschitz estimate along the direction  $x_i$ :

$$|D_i u(x)| \leq \frac{M_g}{\delta_i(x)} \quad \text{for a.e. } x \in S_{d,i}^*. \quad (5.12)$$

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