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Department of Mathematics

Doctoral dissertation

The Stroh Formalism and Its Applications to Continuum Systems

(A Mathematical Approach to Coupled Field Problems)

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In the name of Allah, the Most Merciful and the Most Magnificent, who granted me the strength to reach where I stand today, who stood by me through every hardship and difficulty, and whose boundless love made all things possible. Blessings and peace be upon my Prophet Muhammad (PBUH), the changer of worlds, whose gentle mercy and timeless wisdom have taught me how to walk through life with faith, patience, and grace.

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Contents

Abstract

The Stroh formalism provides a strict and structured framework of investigating anisotropic continuum systems, discovering natural symmetries and simplifying compact first-order models that are needed to study wave propagation and coupled-field interactions. This makes it a good starting point of generalizing classical elasticity to more multiphysics models due to its versatility. This thesis constructs a unified Stroh-Hamiltonian framework for modeling reversible multiphysics processes in anisotropic porous solids, with a focus on poroelastic and thermoporoelastic interactions. Inspired by structural similarities between classical Stroh sextic formalism of anisotropic elasticity and canonical Hamiltonian systems, the work generalizes this analogy to a family of coupled solid-fluid-thermal systems based on the Biot theories. The first part of the thesis defines a Hamilton representation of incompressible poroelasticity under perfectly drained conditions, and thus explains how incompressibility and the lack of the pressure gradients reorganize the canonical variables and put constraints on the dynamics that are reversible. This formulation sheds new light on the energy conjugate relationships determining the solid deformation and fluid induced effects.

Based on this foundation, the second part introduces a generalized reversible framework for anisotropic thermoporoelasticity, developed from the Duhamel–Neumann constitutive relations, and expressed through a Lagrangian density formulated to reflect both mechanical, hydraulic, and thermal variations. By recasting the governing equations in canonical form, the thesis highlights intrinsic symmetries, clarifies the coupling mechanisms among stress, temperature, pore pressure, and seepage displacement, and provides a systematic platform for analyzing wave propagation and boundary-value problems in multiphysics media.

The last contribution generalizes the methodology of Stroh-Hamiltonian to the special case of thermoporoelasticity when the solid constituents are incompressible. The expression narrows the set of acceptable pairs of dual variables and provides a brevity of first-order form that takes on reversible thermo-fluid-structure interactions with incompressible restrictions, thus offering a serious path of analysis to the investigation

of strongly coupled processes in anisotropic fluid-saturated materials.

Collectively, the thesis not only advances the theoretical basis of reversible coupled field modeling, but also introduces a systematic canonical formalism that can be used throughout the poroelastic, thermoporoelastic regimes and provides a basis on which dissipative processes, heat conduction and nonlinear material behaviour can be incorporated.

Chapter 1

Introduction

1.1 Background and Motivation

Coupled solid-fluid behavior in anisotropic porous materials is a subject of study that is both central to and essential to numerous scientific and engineering fields, such as geomechanics, biomechanics, and materials engineering. Such materials are composed of a skeletal structure of deformable solid material that is interwoven with interdependent pore spaces that are usually filled by fluids, usually liquids or gases. The deformation of the solid matrix and fluid flow in the pores in such systems is complicated and therefore gives rise to their mechanical behavior. Such a description cannot be done without complex analytical systems that are able to deal with anisotropic elasticity and multiphysics coupling. The Stroh formalism is one of the influential approaches that have been predominant to model problems of anisotropic elasticity. This mathematical method re-expresses the governing equations in first-order matrix form, that is, it is a first-order system of equations describing wave propagation, boundary conditions, and directional material properties. The extension of this formalism to poroelastic and thermoporoelastic media is likely to have the effect of bringing multi-physics couplings under the same, consistent, framework.

The key to modeling these systems lies in the fact that external forces, pore pressures, and material anisotropy have to be described in order to obtain a description of the overall response. Direction-dependent material properties, known as the anisotropy, further increase the complexity by making deformation and flow properties nonuniform and therefore cannot be modeled using isotropic models. Besides, the solid-fluid interaction implies that the deformation of the solid and movement of the fluid are coupled, and the result is a system of coupled field equations modeling simultaneous solid deformation and fluid motion. This fundamental interaction renders the task of mathematical formulation of the governing equations and their subsequent solution very

difficult.

Classical poroelasticity theory, first developed by Biot in the 1940s, is a phenomenological basis of such coupled processes. The theory by Biot is effective in modeling the influence of fluid pressure in the pores on the solid deformation and the opposite, which can predict wave propagation and consolidation of the porous media among other behaviour. However, regardless of its achievement, the Biot framework is characterized by the complicated partial differential equations that combine mechanical equilibrium with the mass conservation of fluids. This natural complication becomes particularly significant in cases where the solid matrix can be characterized by anisotropic elastic behavior or where other physical processes are at play such as temperature changes.

The problem is further complicated when thermal effects are implemented in the poroelastic framework to give rise to the field of thermoporoelasticity. Besides the mechanical and hydraulic interactions, it is also important to take into account the heat conduction and consequent thermal expansion or contraction of the solid and fluid phases. Pore pressures and fluid flow can be affected by temperature gradients, and fluid flow can carry heat to form a completely coupled thermal-hydro-mechanical system. The significance of accounting such thermal effects is in geothermal reservoir engineering, energy storage materials and biological tissue modeling where temperature variations play an important role.

The introduction of the thermal field adds some dimensionality and complexity to the governing equations and thus more difficult to analyze and solve. The equations are reduced to a system of fully coupled partial differential equations with several dependent variables displacement, pore pressure, and temperature all dependent on anisotropic material properties and boundary conditions. There are seldom analytical solutions and numerical methods demand sturdy, effective algorithms that can accommodate such couplings without loss of accuracy or stability.

This has led to the desire to come up with analytical and computational tools that can represent these coupled phenomena in a compact, mathematically concise and organized way. These formulations do not only allow more theoretical understanding of the underlying physics but also allow the development of more efficient numerical methods to be used to simulate them. The Stroh formalism, which was first applied to anisotropic elasticity, has the benefit of presenting the equations governing a problem in a matrix form that reduces the equations to a first-order system, and is thus especially well-adapted to the study of wave propagation and boundary value problems in materials with complicated directional responses. The extension of this formalism to poroelastic and thermoporoelastic media is likely to have the effect of bringing multi-physics

couplings under the same, consistent, framework.

Furthermore the Stroh approach is also analogous to the theory of Hamiltonian mechanics, which is an influential theory that underscores the conservation of energy and symplectic forms. This relation gives more analytical instruments to grasp dynamic stability, resonance phenomena and energy transfer mechanisms in coupled solid fluid systems. By developing poroelasticity in a Stroh-Hamiltonian framework, it is possible to maintain the underlying physical symmetry and conservation laws, which can result in stronger and more accurate solution formulas. This is particularly important when dealing with incompressibility constraints, anisotropy, and thermal coupling, all of which add layers of complexity that challenge conventional approaches.

Such a unified formulation of Stroh Hamiltonian of anisotropic thermoporoelastic materials does not only contribute to the theoretical knowledge, but also provides new avenues towards practical application. These encompass better modeling of seismic wave propagation in fluid saturated earth materials, better design of bio-engineered tissues that have thermal and fluid transport issues of importance, and the optimization of smart porous materials that dynamically respond to mechanical and thermal stimuli. Ultimately, this research strives to provide a rigorous yet flexible mathematical toolkit that bridges the gap between complex physical behavior and efficient computational implementation, fostering innovations across geophysics, biomechanics, and materials science.

To address these challenges, this research develops a systematic extension of the classical Biot's poroelastic theory. Beginning with the incorporation of incompressibility constraints into anisotropic poroelastic materials, the study advances the mathematical framework to more accurately reflect the behavior of nearly incompressible solids in fluid-saturated media. Building on this foundation, thermal effects are then integrated, resulting in a coupled thermoporoelastic model that captures the coupled interaction of heat, fluid, and solid deformation. Finally, the framework is unified to treat incompressible coupled thermoporoelastic systems, providing a comprehensive tool set for analyzing complex multiphysics interactions in anisotropic porous materials. The gradual advancement of this process will guarantee both the rigor of theory and its practical application, making important contributions to the gaps in the existing models.

1.2 The Stroh-Hamiltonian Formalism in Mechanics

The Stroh formalism is one of such analytic techniques, and was proposed by Stroh in the context of anisotropic elasticity. This approach (see Chapter 2 for details) represents the governing equations in a first-order matrix form, enabling a systematic treatment

of problems such as wave propagation, boundary-value analysis, and the behavior of layered or constrained media. The principle of its matrix structure vaguely facilitates the analysis of anisotropic materials, in which the directional dependence on stiffness triggers the nontrivial interaction between displacement and stress measures.

The elegance of the Stroh formalism is that it structurally resembles Hamiltonian mechanics (See Chapter 2 to find out more). It maintains a symplectic form, which is directly related to the conservation of energy and reciprocity by putting the governing equations in canonical form. The property does not just permit a natural physical interpretation, but such that physical interpretation also forms the basis of some computational tools, in particular those based on transfer matrices and eigenvalue problems.

Although the Stroh-Hamiltonian formalism should be the standard method adopted in the study of elasticity, and indeed extensions have been made to model a wide range of physical constraints and pre-stress effects, it still remains unexploited in the context of multiphysics continua, including poroelastic or thermoporoelastic materials.

1.3 Poroelasticity and Its Challenges

Poroelastic materials are solids which have pore spaces interconnected together and occur both in nature and through engineering (see Chapter 3 expounded in further detail).

The theory of Biot as canonical is a linear elastic theory of the solid matrix in which the pore fluid is transported in a Darcy-like manner. The theory has since been expanded in the next few decades to include double porosity and multi porosity structures, piezoelectric effects and thermoelastic coupling. These extensions have significantly extended poroelasticity to allow the phenomenon of acoustic attenuation in foams to be explained, and the phenomenon of wave propagation in fluid-saturated rocks to be characterized.

Most formulations that remain, however, are second-order equations in the spatial derivatives and in anisotropic or thermally interacting systems their forms often become cumbersome. This complexity in nature makes analytical derivations as well as numerical implementations difficult, especially in the case of layered or geometrically constrained configurations.

1.4 Literature review

The Stroh formalism, first introduced by Eshelby et al. (1953) and subsequently developed by Stroh (1958, 1962), has become a cornerstone in the analysis of anisotropic elasticity. The formalism was initially developed to deal with two-dimensional linear anisotropic problems, especially in the field of dislocation and crack mechanics, and is understood to be a method of operating with the stress functions and displacements as coupled quantities, so that the equations of the system could be reduced to a single first-order matrix equation. This form of representation does not only make the mathematical treatment of anisotropic solids straightforward but also offers a systematic approach to the analysis of the boundary value problems. Ting (1996) also summarized the theory and its uses, highlighting the technical flexibility and mathematical beauty of the method. Until the 1990s, most studies concentrated on two dimensional deformations, where the state of the solid does not depend on a single Cartesian coordinate, and complex-variable techniques were often used to find solutions, such as the papers of Hwu (1991); Chyanbin (1993); Hwu and Yen (1993); Chyanbin (1992); Hwu (2003, 2010a, 2021). These pioneering works laid a strong theoretical basis of systematic treatment of anisotropic elasticity and established the usefulness of the Stroh formalism in the work on classical problems of materials with a direction-dependent behavior.

The Stroh formalism, over and above its historical beginnings, has since developed into an effective instrument of modern elasticity and wave-propagation studies Hwu and Becker (2022). By reducing the governing equations to a structured first-order system, often expressed as a six-dimensional eigenvalue problem, it preserves the essential physical symmetries of anisotropic materials while enabling systematic investigation of stress fields, wave propagation, and boundary-value problems. The relations of reciprocity, energy-conservation properties, and modal orthogonality are naturally revealed by its mathematical form, and have been used in the study of constrained or pre-stressed materials Edmondson and Fu (2009); Chadwick (1997). The flexibility of the framework has contributed to its effectiveness specifically in the study of layered media Qu et al. (2024), laminated plates Hwu (2010b), and elastic systems Fan (2009) with defects or discontinuities where other formulations can be complicated. Recently, it has also been applied to anisotropic continua with a microstructure and therefore allows direct and rigorous treatment of more complicated material behaviours without losing the underlying symplectic structure Wang et al. (2024); ENSAM (2023); Pan et al. (2023); Shuvalov (2024). Collectively, these studies demonstrate that the Stroh formalism not only provides a mathematically elegant and technically robust framework

for classical elasticity problems but also serves as a unifying approach for contemporary research in wave propagation, material design, and advanced continuum mechanics. It is remarkable that its timeless applicability represents not only its initial significance in the theory of anisotropic elasticity but also its versatility to a broad variety of modern scientific and engineering uses.

Despite the fact that the original development of the Stroh formalism was restricted to the consideration of both the static and dynamic problems of anisotropic elasticity Stroh (1958, 1962), further evolutionary work has shown that the Stroh system has an intrinsic Hamiltonian structure Fu (2007); Barnett (2000); Nobili and Radi (2022). By treating one spatial coordinate analogously to time in Hamiltonian mechanics, the Stroh formulation naturally exposes conservation laws, reciprocity relations, and canonical variable pairings. This interpretation has made possible systematic extensions to a range of more complicated systems. An example is that in constrained elasticity the Hamiltonian structure can be extended to allow the addition of extra constraints on the displacement or stress fields without losing the symplectic properties of the system Edmondson and Fu (2009), and thus can still have energy conservation and reciprocity relations. In the case of plates with laminations, this approach can be described as the Hamiltonian viewpoint where the layers have different anisotropic materials, thereby making it easy to examine interfaces and mode interactions Fu and Ezzinbi (2003). Similarly, when applied to acoustic waves, the first-order Hamiltonian formulation naturally separates propagating and evanescent modes, maintaining consistent energy flux across boundaries and supporting stable modal decomposition Shuvalov (2024). Besides, the extended Hamiltonian framework Nobili and Radi (2022) supports the addition of more internal degrees of freedom, allowing the precise study of the wave propagation and maintaining the inalienable symplectic invariants in microstructures.

Concurrently, poroelasticity has been extensively developed over the years as a fundamental model to characterize the two mechanical and fluid behaviour of porous bodies. The classical work of Biot Biot (1956b) developed a general theory where a porous elastic skeleton, filled with fluid is modeled using coupled stress strain relations as well as motion relations based on the Lagrange equations. Subsequent studies have built upon this foundation to analyze wave propagation Fellah et al. (2004); Tong et al. (2017), attenuation Prasad and Meissner (1992); Williams et al. (2002); Gist (1994); Diallo and Appel (2000) , and dispersion Wang (1990); King et al. (2000) in porous materials, demonstrating how the elastic and viscous properties of both the solid matrix and the pore fluid govern the overall dynamic response of the medium. Predictive properties of the Biot theory have established it as a benchmark in theoretical study and practical modeling of porous media in various situations. Poroelasticity has numerous

applications, such as the study of seismic wave propagation in fluids-saturated rocks and sediments Theodorakopoulos and Beskos (2006)(i.e. as part of subsurface imaging and earthquake modeling). These models are applied in petroleum engineering in reservoir characterization, compaction studies and hydraulic fracturing analysis Wang (2000). In the biomechanics field, poroelastic models are used to model the behaviour of soft tissue (e.g. cartilage and brain matter), in which fluidsolid interactions are essential to sound mechanical reproduction Zhou et al. (2016). Additionally, Poroelastic materials are also widely studied in acoustics and materials engineering Färm et al. (2016); Wang et al. (2024); Yang and Sheng (2017); Benjamin et al. (2024) to emerge as a solution to noise, vibration, and harshness (NVH) in structural engineering; sound absorption; and vibration damping. Recent developments have focused on extending Biot's classical theory to capture more complex material behaviors. These include the incorporation of anisotropy, heterogeneous microstructures Selvadurai (2019), and nonlinear effects Biot (1973), allowing the modeling of a wider range of porous media under various loading conditions. Furthermore, advances in computational and analytical techniques have allowed for systematic analysis of dynamic and transient responses, bridging gap between theory and practical applications.

Over decades, Biot's theory has been extended to address multi-scale and multi-field phenomena. Other additions have involved hierarchical pore structures of dual and multi porosity Berryman and Wang (2000); Pramanik et al. (2024), piezoelectric coupling of smart materials Sharma (2010) adaptations for wave-based characterization of cancellous bone Chen et al. (2018). In rock mechanics, the theory has been fundamental for modeling fractured and fluid-saturated formations, where fluid–solid interactions critically influence deformation and transport Shao (1998); Yang et al. (2024); Abdollahipour et al. (2016).

Integration of thermal behavior has led to thermoporoelasticity, which deals with relationship between temperature, deformation and pore pressure. The theoretical basis was laid by early studies of Biot himself Biot (1956d), of a coupled thermoelastic formulation, which was generalized to body-wave propagation in isotropic Chadwick and Snedon (1958); Chadwick (1962) and anisotropic materials Green (1962); Chad wick (1979); Scott (1989); Shuvalov and Chadwick (1997). Subsequent studies incorporated anisotropy, pre-stress, and coupled field effects Schiffman et al. (1969); Selvadurai and Suvorov (2016); Qiu et al. (2024). Such developments have expanded the applicability of the theory into other areas like geophysics, energy storage, environmental engineering, and biomechanics Giot et al. (2018); Altay and DökmecİĖ (2006).

With these developments, a Stroh-like formalism of poroelastic and thermoporoelastic systems still is pretty much conspicuously missing in the literature,

presumably due to the complexity of multi-field coupling. Initial progress has been made by Nobili Nobili (2024), who developed a Stroh-Hamiltonian framework for reversible poroelasticity under drained conditions, extending it to thermoelastic analogues Nobili and Pichugin (2021).

The limit of the incompressible solid skeleton relevant for saturated clays, certain rocks, and soft biological tissues Lampe et al. (2018); Nespoli et al. (2021); Komijani (2025) introduces non-trivial kinematic constraints that must be incorporated to ensure physical consistency and numerical stability. These limitations have a strong effect on the propagation of waves, stresses transfer, and fluid solid interaction, which makes their incorporation in the Stroh-Hamiltonian formalism an important development.

1.5 Gap in the Literature

Despite the versatility of Biot-type models and the elegance of the Stroh formalism in elasticity, there is currently **no widely adopted Stroh-like representation for poroelastic or thermoporoelastic systems**, especially under incompressibility constraints.

Several factors contribute to this gap:

- **Multifield coupling**: In poroelasticity, displacement and pore pressure fields are intrinsically linked, making it nontrivial to cast the equations in first-order canonical form.
- **Anisotropy**: Direction-dependent material properties further complicate the separation and pairing of variables required for a Hamiltonian structure.
- **Thermal effects**: In thermoporoelasticity, the additional coupling between temperature, pore pressure, and displacement leads to enlarged state vectors and more intricate constitutive matrices.

1.6 Research Aims and Contributions

This dissertation develops and applies a **Stroh-Hamiltonian framework** for the analysis of poroelastic and thermoporoelastic materials, with particular emphasis on incompressibility constraints in the solid skeleton. The work proceeds by first establishing the theoretical background and then advancing new canonical formulations. The main aims and contributions are:

1. **Foundational Framework**
   Provide introductory expositions of the Stroh formalism, poroelasticity, and

Hamiltonian mechanics, setting the stage for their integration into a unified analytical framework.

2. **Incompressible Poroelasticity**
   Develop a Stroh-Hamiltonian representation for anisotropic poroelastic media with incompressible solid skeleton, preserving symplectic structure and enabling energy-based analysis.

3. **Extension to Thermoporoelasticity**
   Incorporate thermal fields into the canonical framework to capture fully coupled thermo-hydro-mechanical interactions in anisotropic porous solids.

4. **Thermoporoelasticity with an incompressible skeleton**
   Extend the formulation to include incompressibility constraints within thermoporoelastic systems, providing a rigorous mathematical method for constrained multiphysics continua.

5. **Applications and Impact**
   Although the present work is primarily theoretical, the canonical framework developed here establishes a foundation for future applications. The formalism offers analytical clarity and computational advantages for addressing problems such as wave propagation in anisotropic media, stability of layered porous systems, and coupled boundary-value problems. More broadly, the methodology provides a versatile mathematical toolset that can inform experimental studies and numerical modeling in diverse areas, including seismic exploration, geotechnical engineering, and biomechanics of soft and hard tissues. By unifying poroelasticity and thermoporoelasticity under a Stroh-Hamiltonian structure, this research enhances both the theoretical understanding of constrained multiphysics continua and their potential application across scientific and engineering domains.

1.7 Structure of the Thesis

The thesis is structured in the following way: Chapter 2 gives an in-depth summary of the relevant literature on the Stroh formalism including the Hamiltonian formulation of anisotropic elasticity and the clarification of the connection between Stroh formalism and it and thus setting the necessary mathematical tools to be used in the successive developments. Chapter 3 further elaborates on classical and extended theories on poroelasticity and thermoporoelasticity, including the principles and latest developments. The fourth chapter 4 derives the Stroh-Hamiltonian formalism

of reversible anisotropic poroelasticity under incompressibility constraints provision of a rigorous mathematical formulation together with a detailed physical interpretation. The Chapter 5 is a continuation of this formalism to the thermoporoelastic systems, which incorporate the effects of thermal effects into the canonical equations and demonstrate the general applicability of the approach. The framework is further extended to the incompressible solid case of thermoporoelastic systems in chapter 6 to consider the extra complexities of the scenario. Chapter 7 summarizes the key findings, addresses the implications of the findings in general and possible extensions, such as inclusion of dissipative mechanisms and nonlinear effects.

Chapter 2

Review of Stroh formalism

The Stroh formalism Stroh (1958) represents a sophisticated and highly effective mathematical framework to address problems in anisotropic elasticity. The equations of elasticity are inherently complex, as they form a coupled system; in the case of anisotropic materials, this complexity is further amplified by the numerous independent parameters arising from the elasticity tensor. The Stroh formalism simplifies the structural complexity hidden in the equations of anisotropic elasticity and deploys a structured methodology for these equations. By leveraging its inherent mathematical elegance, the Stroh method has become a foundational tool in the study of dislocations, fracture mechanics, and wave propagation in anisotropic solids. In the following sections we proceed with the basic introduction of Stroh formalism, with the brief discussion of fundamental solution following Tanuma (2007); Ting (1996).

2.1 Basic elasticity

Consider a Cartesian coordinate system in the three-dimensional space $\mathbb{R}^3$. Let $\mathbf{B} \subset \mathbb{R}^3$ be a region with smooth boundary S, occupied by an elastic solid. At a point $\mathbf{x} \in S$, let $\mathbf{n} = (n_1, n_2, n_3)$ denote the unit outward normal to S.

The *stress tensor* $\boldsymbol{\sigma} = (\sigma_{ij})_{i,j=1}^3$ is defined so that

$$\left(\sum_{j=1}^{3} \sigma_{ij} n_j \right)_{i=1,2,3}$$

represents the traction vector, i.e., the force per unit surface area, acting at $\mathbf{x}$ on the material lying opposite to the direction of $\mathbf{n}$.

2.1.1 Infinitesimal Strain Tensor

The infinitesimal strain tensor $\boldsymbol{\varepsilon}$ is defined in terms of the displacement field $\mathbf{u}(\mathbf{x}) = (u_1, u_2, u_3)$ as

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \quad i, j = 1, 2, 3. \tag{2.1}$$

which quantifies changes in length and angle of an infinitesimal material element.

2.1.2 Hooke's Law

In the linear elastic regime, the relationship between stresses and strains is given by

$$\sigma_{ij} = \sum_{k,l=1}^{3} C_{ijkl}\,\varepsilon_{kl}, \quad i, j = 1, 2, 3, \tag{2.2}$$

where $\mathbf{C} = (C_{ijkl})$ is the *elasticity tensor*.

From the balance of angular momentum, we obtain $\sigma_{ij} = \sigma_{ji}$. Similarly, (2.1) gives $\varepsilon_{ij} = \varepsilon_{ji}$. Thus $\mathbf{C}$ satisfies the *minor symmetries*:

$$C_{ijkl} = C_{jikl}, \quad C_{ijkl} = C_{ijlk}, \quad i, j, k, l = 1, 2, 3. \tag{2.3}$$

Substituting (2.1) into (2.2) leads to:

$$\sigma_{ij} = \sum_{k,l=1}^{3} C_{ijkl}\frac{\partial u_k}{\partial x_l}, \quad i, j = 1, 2, 3. \tag{2.4}$$

2.1.3 Strain Energy Density

Assume there exists a strain energy density function $W(\boldsymbol{\varepsilon})$ such that:

$$W(\mathbf{0}) = 0, \quad \frac{\partial W}{\partial \varepsilon_{ij}} = \sigma_{ij}.$$

Combining with (2.2), we find:

$$\frac{\partial}{\partial \varepsilon_{kl}}\left(\frac{\partial W}{\partial \varepsilon_{ij}}\right) = C_{ijkl}, \quad \frac{\partial}{\partial \varepsilon_{ij}}\left(\frac{\partial W}{\partial \varepsilon_{kl}}\right) = C_{klij}.$$

Therefore $\mathbf{C}$ has the *major symmetry*:

$$C_{ijkl} = C_{klij}, \quad i, j, k, l = 1, 2, 3. \tag{2.5}$$

If $W(\boldsymbol{\varepsilon}) > 0$ for all nonzero $\boldsymbol{\varepsilon}$, then $\mathbf{C}$ is positive definite, which is equivalent to the *strong convexity condition*:

$$\sum_{i,j,k,l=1}^{3} C_{ijkl}\,\varepsilon_{ij}\varepsilon_{kl} > 0, \quad \text{for all nonzero symmetric matrices } (\varepsilon_{ij}). \tag{2.6}$$

2.1.4 Static Equilibrium without Body Forces

When there are no body forces, the equilibrium equations are:

$$\sum_{j=1}^{3} \frac{\partial \sigma_{ij}}{\partial x_j} = 0, \quad i = 1, 2, 3. \tag{2.7}$$

Using (2.4), these become:

$$\sum_{j,k,l=1}^{3} \frac{\partial}{\partial x_j}\left(C_{ijkl}\frac{\partial u_k}{\partial x_l}\right) = 0, \quad i = 1, 2, 3. \tag{2.8}$$

From (2.3) and (2.5), the elasticity tensor has 21 independent components at each point $\mathbf{x}$.

2.1.5 Material Symmetry

An elastic medium is *isotropic* at $\mathbf{x}$ if for every orthogonal matrix $\mathbf{Q} = (Q_{ij})$,

$$C_{ijkl} = \sum_{p,q,r,s=1}^{3} Q_{ip}Q_{jq}Q_{kr}Q_{ls}\,C_{pqrs}, \quad i, j, k, l = 1, 2, 3. \tag{2.9}$$

For isotropic materials, $\mathbf{C}$ is determined by two scalar parameters, the *Lamé moduli* $\lambda(\mathbf{x})$ and $\mu(\mathbf{x})$, and has the form:

$$C_{ijkl} = \lambda\,\delta_{ij}\delta_{kl} + \mu\,(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}), \quad i, j, k, l = 1, 2, 3, \tag{2.10}$$

where δ_{ij} is the Kronecker delta.

If the isotropy condition is not satisfied, the material is *anisotropic*. The *symmetry group* $\mathcal{G}$ of the material at $\mathbf{x}$ is defined as the largest subgroup of the orthogonal group for which the above invariance relation holds for all $\mathbf{Q} \in \mathcal{G}$.

2.1.6 Anisotropic Elasticity and Voigt Notation

The number of independent components of the elasticity tensor $\mathbf{C}(\mathbf{x}) = (C_{ijkl})_{i,j,k,l=1}^{3}$ depends on the symmetry group $\mathcal{G}$ of the material at the point $\mathbf{x}$. This group is generated by a set of orthogonal transformations that leave the constitutive relation invariant Gurtin (1973). Different generators lead to different classifications of anisotropic materials.

For convenience in representing the components C_{ijkl}, we employ the *Voigt notation*, which maps the four-index tensor to a symmetric 6×6 matrix. With this notation, the indices (ij) (and similarly (kl)) are replaced by single indices α (or β) according to the rule:

α or β	(ij)
1	11
2	22
3	33
4	23 or 32
5	13 or 31
6	12 or 21

Table 2.1: Mapping of tensor indices (ij) to single Voigt indices α or β.

With this convention, $\mathbf{C}$ is represented as:

$$(C_{\alpha\beta}) = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ & & C_{33} & C_{34} & C_{35} & C_{36} \\ & & & C_{44} & C_{45} & C_{46} \\ & & & & C_{55} & C_{56} \\ \text{Sym.} & & & & & C_{66} \end{bmatrix}. \tag{2.11}$$

The minor symmetries $C_{ijkl} = C_{jikl} = C_{ijlk}$ guarantee that this mapping is well-defined, while the major symmetry $C_{ijkl} = C_{klij}$ ensures that (2.11) is symmetric. Moreover, the strong convexity condition is equivalent to the requirement that the matrix $(C_{\alpha\beta})$ is positive definite.

Examples of Material Symmetry Classes

Let $\mathbf{R}_{x_i}^{\phi}$ denote the orthogonal transformation corresponding to a right-handed rotation by an angle ϕ about the i-axis in the Cartesian coordinate system. Several common symmetry classes of anisotropic elastic materials, along with their generating rotations and Voigt-representation forms, are listed below.

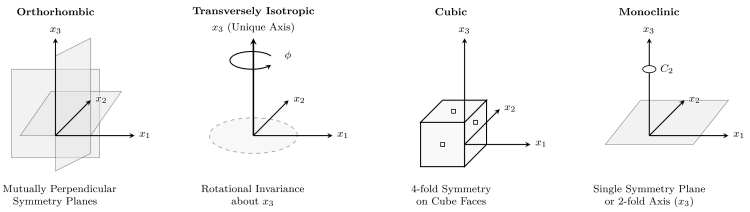

Figure 2.1: Examples of Material Symmetry Classes

1. **Orthorhombic materials**
   Generators: $\mathbf{R}^{\pi}_{x_2}$, $\mathbf{R}^{\pi}_{x_3}$. The Voigt form is:

$$(C_{\alpha\beta}) = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ & C_{22} & C_{23} & 0 & 0 & 0 \\ & & C_{33} & 0 & 0 & 0 \\ & & & C_{44} & 0 & 0 \\ & & & & C_{55} & 0 \\ \text{Sym.} & & & & & C_{66} \end{bmatrix}. \tag{2.12}$$

2. **Transversely isotropic materials**
   Generators: $\mathbf{R}^{\phi}_{x_3}$, $0 < \phi < 2\pi$. The Voigt form is:

$$(C_{\alpha\beta}) = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ & C_{11} & C_{13} & 0 & 0 & 0 \\ & & C_{33} & 0 & 0 & 0 \\ & & & C_{44} & 0 & 0 \\ & & & & C_{44} & 0 \\ \text{Sym.} & & & & & \frac{C_{11}-C_{12}}{2} \end{bmatrix}. \tag{2.13}$$

3. **Cubic materials**
   Generators: $\mathbf{R}^{\pi/2}_{x_1}$, $\mathbf{R}^{\pi/2}_{x_2}$, $\mathbf{R}^{\pi/2}_{x_3}$. The Voigt form is:

$$(C_{\alpha\beta}) = \begin{bmatrix} C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\ & C_{11} & C_{12} & 0 & 0 & 0 \\ & & C_{11} & 0 & 0 & 0 \\ & & & C_{44} & 0 & 0 \\ & & & & C_{44} & 0 \\ \text{Sym.} & & & & & C_{44} \end{bmatrix}. \tag{2.14}$$

4. **Monoclinic materials**

Generators: $\mathbf{R}^{\pi}_{x_3}$. The Voigt form is:

$$(C_{\alpha\beta}) = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & C_{16} \\ & C_{22} & C_{23} & 0 & 0 & C_{26} \\ & & C_{33} & 0 & 0 & C_{36} \\ & & & C_{44} & C_{45} & 0 \\ & & & & C_{55} & 0 \\ \text{Sym.} & & & & & C_{66} \end{bmatrix}. \tag{2.15}$$

Material symmetry is defined pointwise; the generators of $\mathcal{G}$ can, in principle, vary with $\mathbf{x}$. If a rotation $R^{\theta}_{x_i}$ is a member of $\mathcal{G}$ at $\mathbf{x}$, the i-axis is said to be an n-fold symmetry axis, where $n = \frac{2\pi}{\theta}$. For instance, in cubic materials, the 3-axis is a 4-fold symmetry axis.

If the elasticity tensor $\mathbf{C}$ is spatially constant, the material is *homogeneous*. In this case, the equilibrium equations for the displacement field $\mathbf{u}$ reduce to:

$$\sum_{j,k,l=1}^{3} C_{ijkl} \frac{\partial^2 u_k}{\partial x_j \partial x_l} = 0, \quad i = 1, 2, 3. \tag{2.16}$$

In the subsequent analysis, we shall develop the Stroh formalism under the assumption of homogeneity.

2.2 Stroh's Eigenvalue Problem

Let $\mathbf{x} = (x_1, x_2, x_3)$ denote the position vector in $\mathbb{R}^3$. We introduce two orthogonal unit vectors $\mathbf{m} = (m_1, m_2, m_3)$ and $\mathbf{n} = (n_1, n_2, n_3)$. Throughout, $\mathbb{C}$ denotes the set of complex numbers.

We study the equilibrium equations governing a linearly elastic, homogeneous solid occupying the half-space

$$\mathbf{n} \cdot \mathbf{x} = n_1 x_1 + n_2 x_2 + n_3 x_3 \leq 0,$$

under static deformations. The governing system takes the form

$$\sum_{j,k,l=1}^{3} C_{ijkl} \frac{\partial^2 u_k}{\partial x_j \partial x_l} = 0, \quad i = 1, 2, 3, \tag{2.17}$$

where $\mathbf{u} = (u_1, u_2, u_3)$ represents the displacement vector field and $\mathbf{C} = (C_{ijkl})_{i,j,k,l=1}^{3}$ is the fourth-order elasticity tensor. The tensor satisfies the major and minor symmetries,

together with the strong ellipticity (convexity) condition.

We now seek solutions to (2.8) of the exponential form

$$\mathbf{u} = \mathbf{a}\,\exp\Big(-i\xi\big(\mathbf{m}\cdot\mathbf{x} + p\,\mathbf{n}\cdot\mathbf{x}\big)\Big), \quad \mathbf{a}\in\mathbb{C}^3,\ p\in\mathbb{C},\ \xi>0, \tag{2.18}$$

valid within the half-space $\mathbf{n}\cdot\mathbf{x}\le 0$. Such solutions correspond to two-dimensional deformations, with displacement depending only on the projection of $\mathbf{x}$ onto the plane spanned by $\mathbf{m}$ and $\mathbf{n}$. This plane is typically referred to as the *reference plane*. The exponential factor ensures decay with increasing depth from the surface $\mathbf{n}\cdot\mathbf{x}=0$, which is a necessary physical admissibility condition.

Substituting (2.18) into (2.8) and computing derivatives, we obtain

$$\frac{\partial \mathbf{u}}{\partial x_j} = -i\xi(m_j + pn_j)\,\mathbf{a}\,e^{-i\xi(\mathbf{m}\cdot\mathbf{x}+p\mathbf{n}\cdot\mathbf{x})}.$$

After simplification, this yields

$$\left(\sum_{j,l=1}^{3} C_{ijkl}(m_j+pn_j)(m_l+pn_l)\right)_{i\downarrow,\,k\to 1,2,3}\mathbf{a} = \mathbf{0}. \tag{2.19}$$

The notation $(\cdot)_{i\downarrow,\,k\to 1,2,3}$ indicates that the resulting quantity is arranged as 3×3 matrix with i denoting the row index (running downward) and k the column matrix (running horizontally).

To reformulate this compactly, we introduce three real 3×3 matrices:

$$\begin{aligned}
\mathbf{Q} &= \Big(\sum_{j,l=1}^{3} C_{ijkl}m_j m_l\Big)_{i\downarrow k\to 1,2,3},\\
\mathbf{R} &= \Big(\sum_{j,l=1}^{3} C_{ijkl}m_j n_l\Big)_{i\downarrow k\to 1,2,3},\\
\mathbf{T} &= \Big(\sum_{j,l=1}^{3} C_{ijkl}n_j n_l\Big)_{i\downarrow k\to 1,2,3}.
\end{aligned} \tag{2.20}$$

The major symmetry of the elasticity tensor ensures that

$$\mathbf{R}^T = \Big(\sum_{j,l=1}^{3} C_{ijkl}n_j m_l\Big)_{i\downarrow k\to 1,2,3}.$$

Hence, equation (2.19) can be rewritten as

$$\big[\,\mathbf{Q} + p(\mathbf{R} + \mathbf{R}^T) + p^2\mathbf{T}\,\big]\mathbf{a} = \mathbf{0}. \tag{2.21}$$

For the existence of a non-trivial displacement vector $\mathbf{a} \neq \mathbf{0}$, the scalar p must satisfy the associated characteristic equation

$$\det\big[\,\mathbf{Q} + p(\mathbf{R} + \mathbf{R}^T) + p^2\mathbf{T}\,\big] = 0. \tag{2.22}$$

This condition defines a sixth-degree (sextic) polynomial in p, known as the Stroh eigenvalue equation.

Lemma 1.

1. *The matrices* $\mathbf{Q}$ *and* $\mathbf{T}$ *are symmetric and positive definite.*
2. *The characteristic roots* $p_\alpha (1 \leq \alpha \leq 6)$*, i.e, the solutions to the sextic equation* (2.22)*, are not real and they occur in complex conjugate pairs.*

Proof. The symmetry of the matrices $\mathbf{Q}$ and $\mathbf{T}$ follows directly from their definitions in (2.21) together with the major symmetry property (2.5). To establish positive definiteness, consider any non-zero vector $\mathbf{v} = (v_1, v_2, v_3)^T \in \mathbb{R}^3$ and define

$$\varepsilon_{ij} = m_i v_j + v_i m_j.$$

Clearly, ε_{ij} is a non-trivial symmetric tensor. By applying the strong convexity condition (2.6) in combination with the minor symmetries (2.3), we obtain

$$0 < \sum_{j,l=1}^{3} C_{ijkl}\,\varepsilon_{ij}\varepsilon_{kl} = 4\sum_{i,k=1}^{3} v_i\Big(\sum_{j,l=1}^{3} C_{ijkl} m_j m_l\Big)v_k = 4(\mathbf{v}\cdot\mathbf{Q}\mathbf{v}),$$

which demonstrates that $\mathbf{Q}$ is positive definite. By following the same reasoning with $\mathbf{n}$ in place of $\mathbf{m}$, one concludes that $\mathbf{T}$ is also positive definite.

We now turn to part (2). The eigenvalues p_α $(1 \leq \alpha \leq 6)$ of (2.22) are non-real precisely when the matrix

$$\left(\sum_{j,l=1}^{3} C_{ijkl}(m_j + pn_j)(m_l + pn_l)\right)_{i\downarrow k \to 1,2,3} \quad \text{is invertible for all } p \in \mathbb{R}. \tag{2.23}$$

Define

$$\mathbf{w} = (w_1, w_2, w_3) = \frac{\mathbf{m} + p\mathbf{n}}{|\mathbf{m} + p\mathbf{n}|}, \quad p \in \mathbb{R}.$$

As p varies over the reals, $\mathbf{w}$ traces the unit circle. From part (1) of the lemma, it follows that the symmetric matrix

$$\left(\sum_{j,l=1}^{3} C_{ijkl} w_j w_l\right)_{i\downarrow k\rightarrow 1,2,3}$$

is positive definite. This property ensures the validity of (2.23). Finally, since the sextic equation (2.22) has real coefficients, its six roots occur as complex conjugate pairs. □

Next, we compute the traction on the surface defined by $\mathbf{n}\cdot\mathbf{x}=0$, which arises due to the displacement described in (2.18). Since the unit outward normal vector to this surface is $\mathbf{n}$, the traction $\mathbf{t}$ acting on $\mathbf{n}\cdot\mathbf{x}=0$ is expressed as:

$$\mathbf{t} = \left(\sum_{j=1}^{3} \sigma_{ij} n_j\right)_{i\downarrow 1,2,3}\Bigg|_{\mathbf{n}\cdot\mathbf{x}=0}. \tag{2.24}$$

Using the relationship between stress and displacement given in (2.4), this becomes:

$$\mathbf{t} = \left(\sum_{j,k,l=1}^{3} C_{ijkl}\frac{\partial u_k}{\partial x_l} n_j\right)_{i\downarrow 1,2,3}\Bigg|_{\mathbf{n}\cdot\mathbf{x}=0}. \tag{2.25}$$

By substituting (2.18) and (2.21), it follows that:

$$\begin{aligned}\mathbf{t} &= -\sqrt{-1}\xi\left(\sum_{j,l=1}^{3} C_{ijkl}(m_l + p n_l) n_j\right)_{i\downarrow 1,2,3} \mathbf{a} e^{-\sqrt{-1}\xi \mathbf{m}\cdot\mathbf{x}} \\ &= -\sqrt{-1}\xi[\mathbf{R}^T + p\mathbf{T}]\mathbf{a} e^{-\sqrt{-1}\xi\mathbf{m}\cdot\mathbf{x}}.\end{aligned} \tag{2.26}$$

We now define a vector $\mathbf{I}\in\mathbb{C}^3$ as:

$$\mathbf{I} = [\mathbf{R}^T + p\mathbf{T}]\mathbf{a}. \tag{2.27}$$

Thus, the traction on the surface $\mathbf{n}\cdot\mathbf{x}=0$ can be written as:

$$\mathbf{t} = -\sqrt{-1}\xi \mathbf{I} e^{-\sqrt{-1}\xi\mathbf{m}\cdot\mathbf{x}}. \tag{2.28}$$

Finally, we demonstrate that the relationships in (2.21) and (2.27) can be reformulated into a six-dimensional eigenvalue problem, commonly referred to as Stroh's eigenvalue problem. From part (1) of Lemma 1.1, we know that $\mathbf{T}^{-1}$ exists. Using

(2.27), we obtain:

$$p\mathbf{a} = -\mathbf{T}^{-1}\mathbf{R}^T\mathbf{a} + \mathbf{T}^{-1}\mathbf{I}. \tag{2.29}$$

Additionally, we have:

$$p\mathbf{I} = [p\mathbf{R}^T + p^2\mathbf{T}]\mathbf{a}.$$

Substituting (2.21) into this equation yields:

$$p\mathbf{I} = -[\mathbf{Q} + p\mathbf{R}]\mathbf{a}.$$

From (2.29), we further derive:

$$p\mathbf{I} = \mathbf{Q}\mathbf{a} - \mathbf{R}(-\mathbf{T}^{-1}\mathbf{R}^T\mathbf{a} + \mathbf{T}^{-1}\mathbf{I}) = [-\mathbf{Q} + \mathbf{R}\mathbf{T}^{-1}\mathbf{R}^T]\mathbf{a} - \mathbf{R}\mathbf{T}^{-1}\mathbf{I}. \tag{2.30}$$

Combining (2.29) and (2.30), we arrive at the following result:

Theorem 1. *Let*

$$\begin{bmatrix}\mathbf{a}\\ \mathbf{I}\end{bmatrix}$$

represent a column vector in $\mathbb{C}^6$*, where the first three components form a vector* $\mathbf{a} \in \mathbb{C}^3$ *satisfying* (2.21)*, and the last three components correspond to the vector* $\mathbf{I} \in \mathbb{C}^3$ *defined in* (2.27)*. Then, the following six-dimensional eigenrelation holds:*

$$\mathbf{N}\begin{bmatrix}\mathbf{a}\\ \mathbf{I}\end{bmatrix} = p\begin{bmatrix}\mathbf{a}\\ \mathbf{I}\end{bmatrix}, \tag{2.31}$$

where $\mathbf{N}$ *is the* 6×6 *real matrix given by:*

$$\mathbf{N} = \begin{bmatrix}-\mathbf{T}^{-1}\mathbf{R}^T & \mathbf{T}^{-1}\\ -\mathbf{Q} + \mathbf{R}\mathbf{T}^{-1}\mathbf{R}^T & -\mathbf{R}\mathbf{T}^{-1}\end{bmatrix}. \tag{2.32}$$

A solution

$$\begin{bmatrix}\mathbf{a}\\ \mathbf{I}\end{bmatrix}$$

to (2.31) *is referred to as a Stroh eigenvector, with* $\mathbf{a}$ *and* $\mathbf{I}$ *representing its displacement and traction parts, respectively. According to Lemma 1.1, the eigenvalues* p_α *(*$1 \le \alpha \le 6$*) of* $\mathbf{N}$ *are non-real and occur in complex conjugate pairs. This eigenvalue problem* (2.31) *is termed Stroh's eigenvalue problem.*

2.3 Rotational Invariance of the Stroh Eigenstructure

Let $\mathbf{a} \in \mathbb{C}^3$ denote the displacement part and $\mathbf{I} \in \mathbb{C}^3$ denote the traction part of a Stroh eigenvector. These quantities are defined by the relations (2.21) and (2.27), while the associated characteristic roots p are determined as solutions of (2.22). At first sight, the dependence of $\mathbf{a}$, $\mathbf{I}$, and p appears to be tied to the particular choice of two mutually orthogonal unit vectors $\mathbf{m}, \mathbf{n} \in \mathbb{R}^3$. However, we establish that $\mathbf{a}$ and $\mathbf{I}$ do not depend on $\mathbf{m}$ and $\mathbf{n}$ individually, but only through their vector product $\mathbf{m} \times \mathbf{n}$. In particular, the pair $(\mathbf{a}, \mathbf{I})$ is invariant under a simultaneous rotation of $\mathbf{m}$ and $\mathbf{n}$ about the axis defined by $\mathbf{m} \times \mathbf{n}$. Hence, $\mathbf{a}$ and $\mathbf{I}$ are invariant with respect to rotations within the reference plane. This invariance constitutes a fundamental structural property of the Stroh formalism. To formulate this rigorously, fix a pair of orthonormal vectors $\mathbf{e}_1, \mathbf{e}_2 \in \mathbb{R}^3$. Lets define $\mathbf{m}$ and $\mathbf{n}$ as the vectors obtained from $\mathbf{e}_1$ and $\mathbf{e}_2$, respectively, by rotation through an angle $\phi \in (0, 2\pi)$ about the axis $\mathbf{e}_1 \times \mathbf{e}_2$ so that

$$\mathbf{m} = \mathbf{m}(\phi) = \mathbf{e}_1 \cos\phi + \mathbf{e}_2 \sin\phi, \quad \mathbf{n} = \mathbf{n}(\phi) = -\mathbf{e}_1 \sin\phi + \mathbf{e}_2 \cos\phi \tag{2.33}$$

Then, the matrices $\mathbf{Q}$, $\mathbf{R}$, and $\mathbf{T}$ in (2.20), and consequently the scalar parameter p in (2.22), may be expressed as functions of ϕ:

$$\mathbf{Q} = \mathbf{Q}(\phi), \quad \mathbf{R} = \mathbf{R}(\phi), \quad \mathbf{T} = \mathbf{T}(\phi), \quad p = p(\phi). \tag{2.34}$$

In view of (2.32), we obtain

$$\mathbf{N} = \mathbf{N}(\phi) = \begin{bmatrix} -\mathbf{T}(\phi)^{-1}\mathbf{R}(\phi)^T & \mathbf{T}(\phi)^{-1} \\ -\mathbf{Q}(\phi) + \mathbf{R}(\phi)\mathbf{T}(\phi)^{-1}\mathbf{R}(\phi)^T & -\mathbf{R}(\phi)\mathbf{T}(\phi)^{-1} \end{bmatrix}. \tag{2.35}$$

Theorem 2. *Suppose that* $\mathbf{a}$ *and* $\mathbf{I}$ *satisfy* (2.21) *and* (2.27) *at* $\phi = 0$*, with* $p = p_1$ *being a solution of* (2.22) *at* $\phi = 0$*. Then, the same vectors* $\mathbf{a}$ *and* $\mathbf{I}$ *satisfy* (2.21) *and* (2.27) *for every* ϕ*. Moreover,* $p = p(\phi)$ *is governed by the Riccati equation*

$$\frac{d}{d\phi}p = -1 - p^2, \tag{2.36}$$

with the initial condition $p(0) = p_1$*.*

Proof. Let the derivative with respect to ϕ be denoted by $'$. From (2.33), we have

$$\mathbf{m}' = -\mathbf{e}_1 \sin\phi + \mathbf{e}_2 \cos\phi = \mathbf{n}, \quad \mathbf{n}' = -\mathbf{e}_1 \cos\phi - \mathbf{e}_2 \sin\phi = -\mathbf{m}.$$

Using (2.20), it follows that

$$\begin{aligned}
\mathbf{Q}' &= \left(\sum_{j,l=1}^{3} C_{ijkl} m_j m_l \right)_{i\downarrow k \to 1,2,3} \\
&= \left(\sum_{j,l=1}^{3} C_{ijkl} n_j m_l + \sum_{j,l=1}^{3} C_{ijkl} m_j n_l \right)_{i\downarrow k \to 1,2,3} = \mathbf{R} + \mathbf{R}^T, \\
\mathbf{R}' &= \left(\sum_{j,l=1}^{3} C_{ijkl} m_j n_l \right)_{i\downarrow k \to 1,2,3} \\
&= \left(\sum_{j,l=1}^{3} C_{ijkl} n_j n_l - \sum_{j,l=1}^{3} C_{ijkl} m_j m_l \right)_{i\downarrow k \to 1,2,3} = \mathbf{T} - \mathbf{Q}, \\
&= (\mathbf{R}^T)', \\
\mathbf{Q}' &= \left(\sum_{j,l=1}^{3} C_{ijkl} n_j n_l \right)_{i\downarrow k \to 1,2,3} \\
&= \left(\sum_{j,l=1}^{3} C_{ijkl} m_j n_l + \sum_{j,l=1}^{3} C_{ijkl} n_j m_l \right)_{i\downarrow k \to 1,2,3} = -(\mathbf{R} + \mathbf{R}^T).
\end{aligned} \tag{2.37}$$

Define

$$\mathbf{H}(\phi) = [\mathbf{Q} + p(\mathbf{R} + \mathbf{R}^T) + p^2 \mathbf{T}](\phi).$$

Differentiating and applying (2.37), we obtain

$$\begin{aligned}
\mathbf{H}'(\phi) &= [\mathbf{R} + \mathbf{R}^T + p'(\mathbf{R} + \mathbf{R}^T) + 2p(\mathbf{T} - \mathbf{Q}) + 2pp'\mathbf{T} - p^2(\mathbf{R} + \mathbf{R}^T)](\phi) \\
&= [-2p\mathbf{Q} + (1 + p' - p^2)(\mathbf{R} + \mathbf{R}^T) + 2p(1 + p')\mathbf{T}](\phi).
\end{aligned}$$

If $p(\phi)$ satisfies (2.36), the above reduces to

$$\mathbf{H}'(\phi) = -2p[\mathbf{Q} + p(\mathbf{R} + \mathbf{R}^T) + p^2 \mathbf{T}](\phi) = -2p(\phi)\mathbf{H}(\phi). \tag{2.38}$$

Now define

$$\mathbf{h}(\phi) = [\mathbf{Q} + p(\mathbf{R} + \mathbf{R}^T) + p^2 \mathbf{T}](\phi)\mathbf{a} = \mathbf{H}(\phi)\mathbf{a},$$

with $\mathbf{a}$ satisfying (2.21) at $\phi = 0$. Thus,

$$\mathbf{h}(0) = \mathbf{0}. \tag{2.39}$$

From (2.38), it follows that

$$\mathbf{h}'(\phi) = -2p(\phi)\mathbf{H}(\phi)\mathbf{a} = -2p(\phi)\mathbf{h}(\phi). \tag{2.40}$$

By uniqueness of solutions to the initial-value problem defined by (2.40) and (2.39), we deduce

$$\mathbf{h}(\phi) = [\mathbf{Q} + p(\mathbf{R} + \mathbf{R}^T) + p^2\mathbf{T}](\phi)\mathbf{a} = \mathbf{0}, \tag{2.41}$$

for all ϕ. Hence, $\mathbf{a}$ satisfies (2.21) for all ϕ. Next, let

$$\boldsymbol{\eta}(\phi) = [\mathbf{R}^T + p\mathbf{T}](\phi)\mathbf{a}. \tag{2.42}$$

By assumption,

$$\boldsymbol{\eta}(0) = \mathbf{I}.$$

Using (2.36) and (2.37), we find

$$\begin{aligned}\boldsymbol{\eta}'(\phi) &= [\mathbf{T} - \mathbf{Q} + p'\mathbf{T} - p(\mathbf{R} + \mathbf{R}^T)](\phi)\mathbf{a} \\ &= -[\mathbf{Q} + p(\mathbf{R} + \mathbf{R}^T) + p^2\mathbf{T}](\phi)\mathbf{a} = -\mathbf{h}(\phi).\end{aligned} \tag{2.43}$$

Since $\mathbf{h}(\phi) = \mathbf{0}$ from (2.41), we obtain

$$\boldsymbol{\eta}(\phi) = [\mathbf{R}^T + p\mathbf{T}](\phi)\mathbf{a} = \boldsymbol{\eta}(0) = \mathbf{I}, \tag{2.44}$$

for all ϕ. This proves that $\mathbf{I}$ satisfies (2.27) for all ϕ. □

Remark 1. *The Riccati equation (2.36) admits an explicit solution of the form*

$$p(\phi) = \tan(\phi_c - \phi), \tag{2.45}$$

where ϕ_c is a non-real constant determined by the initial condition

$$p(0) = p_1 = \tan(\phi_c),$$

or alternatively,

$$p(\phi) = \pm i.$$

In the proof of theorem 2, we assumed that $p(\phi)$ is a solution of the Riccati equation (2.36). In fact, it necessarily follows that $p(\phi)$ indeed satisfies (2.36). More precisely,

the relations (2.41) and (2.44) are equivalent to the eigenvalue problem (2.31), where the matrix $\mathbf{N} = \mathbf{N}(\phi)$ is defined in (2.35). Consequently, the vector

$$\begin{bmatrix} \mathbf{a} \\ \mathbf{I} \end{bmatrix} \in \mathbb{C}^6$$

is an eigenvector of $\mathbf{N}(\phi)$ associated with the eigenvalue $p(\phi)$. Hence, for each ϕ, the eigenvalue $p(\phi)$ is uniquely determined and satisfies the Riccati equation (2.36).

This observation allows us to reformulate theorem 2 in the framework of Stroh's eigenvalue problem (2.31), leading to the following result.

Theorem 3. *Let*

$$\begin{bmatrix} \mathbf{a} \\ \mathbf{I} \end{bmatrix}$$

be an eigenvector of $\mathbf{N}(0)$ *corresponding to the eigenvalue* $p = p_1$*. Then, for all* ϕ*, it holds that*

$$\mathbf{N}(\phi) \begin{bmatrix} \mathbf{a} \\ \mathbf{I} \end{bmatrix} = p(\phi) \begin{bmatrix} \mathbf{a} \\ \mathbf{I} \end{bmatrix}, \tag{2.46}$$

where the eigenvalue $p(\phi)$ *of* $\mathbf{N}(\phi)$ *satisfies the Riccati equation* (2.36) *with the initial condition* $p(0) = p_1$*.*

2.4 Forms of Basic Solutions in the Presence of Degeneracy in Stroh's Eigenvalue Problem

In general, the eigenvalue problem (2.31) admits six linearly independent eigenvectors, and the construction of solutions proceeds in a straightforward manner. However, when this is not the case, i.e., when the problem becomes *degenerate*, one must resort to the notions of generalized eigenvectors and Jordan chains. This complicates the structure of solutions: the displacement fields no longer take the simple exponential form (2.18), but instead involve additional terms that arise from the Jordan structure. The objective of this section is therefore to present the general forms of the basic solutions to (2.8), classified according to whether the eigenvalue problem (2.31) is simple, semisimple, or degenerate. The arguments provided here are primarily formal, with the emphasis on the systematic construction of admissible solution forms.

Preliminaries

It is well known that the six eigenvalues p_α $(1 \le \alpha \le 6)$ of (2.31) are non-real and occur in complex conjugate pairs. Without loss of generality, we adopt the convention

$$\operatorname{Im} p_\alpha > 0, \quad \alpha = 1, 2, 3,$$

where $\operatorname{Im} p$ denotes the imaginary part of p. Accordingly, we write

$$p_{\alpha+3} = \overline{p}_\alpha, \quad \alpha = 1, 2, 3, \tag{2.47}$$

where $\overline{p}_\alpha$ denotes the complex conjugate of p_α.

Let $\mathbf{m}, \mathbf{n}$ be orthogonal unit vectors in $\mathbb{R}^3$. Depending on the structure of the spectrum of $\mathbf{N}$, we distinguish between three situations: simple, semisimple, and degenerate eigenvalue problems. We now discuss each case in turn.

The Simple Case

We say that the eigenvalue problem (2.31) is *simple* if all six eigenvalues p_α are distinct. In this case, there exist six linearly independent eigenvectors

$$\begin{bmatrix} \mathbf{a}_\alpha \\ \mathbf{l}_\alpha \end{bmatrix}, \quad 1 \le \alpha \le 6,$$

which span $\mathbb{C}^6$. By virtue of (2.47), we may choose them to satisfy

$$\begin{bmatrix} \mathbf{a}_{\alpha+3} \\ \mathbf{l}_{\alpha+3} \end{bmatrix} = \overline{\begin{bmatrix} \mathbf{a}_\alpha \\ \mathbf{l}_\alpha \end{bmatrix}}, \quad \alpha = 1, 2, 3. \tag{2.48}$$

Under these assumptions, the general form of the basic solution to (2.8), which describes two-dimensional deformations in the half-space $\mathbf{n}\cdot\mathbf{x} \le 0$, satisfies the boundary condition

$$\mathbf{u} = \mathbf{a} e^{-\sqrt{-1}\xi \mathbf{m}\cdot\mathbf{x}} \quad \text{on } \mathbf{n}\cdot\mathbf{x} = 0$$

for some complex vector $\mathbf{a}$, and decays to zero as $\mathbf{n}\cdot\mathbf{x} \to -\infty$. This solution may be expressed as

$$\mathbf{u} = \sum_{\alpha=1}^{3} c_\alpha \mathbf{a}_\alpha e^{-\sqrt{-1}\xi(\mathbf{m}\cdot\mathbf{x} + p_\alpha \mathbf{n}\cdot\mathbf{x})}, \tag{2.49}$$

where $c_\alpha \in \mathbb{C}$ $(1 \le \alpha \le 3)$ are constants determined by the relation $\sum_{\alpha=1}^{3} c_\alpha \mathbf{a}_\alpha = \mathbf{a}$,

and $\xi > 0$. The associated traction vector on the surface $\mathbf{n} \cdot \mathbf{x} = 0$ is given by

$$\mathbf{t} = -\sqrt{-1}\xi \sum_{\alpha=1}^{3} c_\alpha \mathbf{I}_\alpha e^{-\sqrt{-1}\xi \mathbf{m}\cdot\mathbf{x}}. \tag{2.50}$$

The Semisimple Case

The eigenvalue problem (2.31) is called *semisimple* if there are repeated eigenvalues among $\{p_\alpha\}$, but the problem nonetheless admits six linearly independent eigenvectors. In this setting, the conventions (2.47) and (2.48) are again adopted. The form of the basic solution and the corresponding traction remain identical to those in (2.49)–(2.50). Thus, the essential difference from the simple case lies only in the multiplicity of eigenvalues, not in the structure of the solutions.

The Degenerate Case

We say that the eigenvalue problem (2.31) is *degenerate* (or *non-semisimple*) if multiple eigenvalues occur and the set of eigenvectors no longer spans the entire space $\mathbb{C}^6$. In such cases, one must augment the basis with generalized eigenvectors. The resulting solution forms are determined by the Jordan canonical structure of $\mathbf{N}$. Two canonical cases arise, which we denote as D1 and D2. These cases will now be examined in detail.

Case D1

There exists a nonsingular 6×6 matrix $\mathbf{P}$ such that $\mathbf{N}$ is similar to the Jordan normal form

$$\mathbf{P}^{-1}\mathbf{N}\mathbf{P} = \begin{bmatrix} p_1 & & & & & \\ & p_2 & 1 & & & \\ & & p_2 & & & \\ & & & \overline{p}_1 & & \\ & & & & \overline{p}_2 & 1 \\ & & & & & \overline{p}_2 \end{bmatrix},$$

where empty entries are zero. Let the α-th column of $\mathbf{P}$ be denoted by

$$\begin{bmatrix} \mathbf{a}_\alpha \\ \mathbf{I}_\alpha \end{bmatrix} \in \mathbb{C}^6, \quad 1 \le \alpha \le 6.$$

Then $\begin{bmatrix}\mathbf{a}_1\\ \mathbf{I}_1\end{bmatrix}$ and $\begin{bmatrix}\mathbf{a}_2\\ \mathbf{I}_2\end{bmatrix}$ are linearly independent eigenvectors associated with p_1 and p_2, respectively. The vector $\begin{bmatrix}\mathbf{a}_3\\ \mathbf{I}_3\end{bmatrix}$ is a generalized eigenvector corresponding to p_2, satisfying

$$\mathbf{N}\begin{bmatrix}\mathbf{a}_3\\ \mathbf{I}_3\end{bmatrix} - p_2\begin{bmatrix}\mathbf{a}_3\\ \mathbf{I}_3\end{bmatrix} = \begin{bmatrix}\mathbf{a}_2\\ \mathbf{I}_2\end{bmatrix}. \tag{2.51}$$

The convention (2.48) ensures that all six vectors $\begin{bmatrix}\mathbf{a}_\alpha\\ \mathbf{I}_\alpha\end{bmatrix}$ $(1 \le \alpha \le 6)$ are linearly independent and span $\mathbb{C}^6$.

Lemma 2. *The general form of the basic solution to (2.8) in Case D1, which describes a two-dimensional deformation in the half-space $\mathbf{n}\cdot\mathbf{x} \le 0$, satisfies the boundary condition $\mathbf{u} = \mathbf{a}e^{-\sqrt{-1}\xi\mathbf{m}\cdot\mathbf{x}}$ on $\mathbf{n}\cdot\mathbf{x} = 0$ for some $\mathbf{a} \in \mathbb{C}^3$, and decays as $\mathbf{n}\cdot\mathbf{x} \to -\infty$. It can be expressed as*

$$\begin{aligned}\mathbf{u} = &\sum_{\alpha=1}^{2} c_\alpha \mathbf{a}_\alpha e^{-\sqrt{-1}\xi(\mathbf{m}\cdot\mathbf{x}+p_\alpha\mathbf{n}\cdot\mathbf{x})}\\ &+ c_3\left(\mathbf{a}_3 - \sqrt{-1}\xi(\mathbf{n}\cdot\mathbf{x})\mathbf{a}_2\right)e^{-\sqrt{-1}\xi(\mathbf{m}\cdot\mathbf{x}+p_2\mathbf{n}\cdot\mathbf{x})},\end{aligned} \tag{2.52}$$

and the corresponding traction on $\mathbf{n}\cdot\mathbf{x} = 0$ is given by

$$\mathbf{t} = -\sqrt{-1}\xi\sum_{\alpha=1}^{3} c_\alpha \mathbf{I}_\alpha e^{-\sqrt{-1}\xi\mathbf{m}\cdot\mathbf{x}}, \tag{2.53}$$

where $c_\alpha \in \mathbb{C}$ $(1 \le \alpha \le 3)$ are constants determined from $\sum_{\alpha=1}^{3} c_\alpha\mathbf{a}_\alpha = \mathbf{a}$, and $\xi > 0$.

Proof. By considering the first three rows of the system (2.51), it follows from (2.32) that

$$-\mathbf{T}^{-1}\mathbf{R}^T\mathbf{a}_3 + \mathbf{T}^{-1}\mathbf{I}_3 - p_2\mathbf{a}_3 = \mathbf{a}_2.$$

This relation immediately yields

$$\mathbf{I}_3 = [\mathbf{R}^T + p_2\mathbf{T}]\mathbf{a}_3 + \mathbf{T}\mathbf{a}_2. \tag{2.54}$$

Next, examining the last three rows of the system, we obtain

$$[\mathbf{Q} + \mathbf{R}\mathbf{T}^{-1}\mathbf{R}^T]\mathbf{a}_3 - \mathbf{R}\mathbf{T}^{-1}\mathbf{I}_3 - p_2\mathbf{I}_3 = \mathbf{I}_2.$$

Substituting (2.54) into the above equation gives

$$[\mathbf{Q} + p_2(\mathbf{R} + \mathbf{R}^T) + p_2^2\mathbf{T}]\mathbf{a}_3 + [\mathbf{R} + p_2\mathbf{T}]\mathbf{a}_2 + \mathbf{I}_2 = \mathbf{0}. \tag{2.55}$$

Since the vector

$$\begin{bmatrix} \mathbf{a}_2 \\ \mathbf{I}_2 \end{bmatrix}$$

is an eigenvector of $\mathbf{N}$, it satisfies

$$\mathbf{I}_2 = [\mathbf{R}^T + p_2\mathbf{T}]\mathbf{a}_2.$$

Using this relation, the previous equation can be rewritten as

$$[\mathbf{Q} + p_2(\mathbf{R} + \mathbf{R}^T) + p_2^2\mathbf{T}]\mathbf{a}_3 + [\mathbf{R} + \mathbf{R}^T + 2p_2\mathbf{T}]\mathbf{a}_2 = \mathbf{0}. \tag{2.56}$$

Consequently, the system (2.51) is equivalent to the pair of equations (2.54) and (2.56). To verify that (2.51) satisfies the equilibrium equation (2.8), we observe that the first two terms

$$\sum_{\alpha=1}^{2} c_\alpha \mathbf{a}_\alpha e^{-\sqrt{-1}\xi(\mathbf{m}\cdot\mathbf{x}+p_\alpha\mathbf{n}\cdot\mathbf{x})}$$

on the right-hand side of (2.51) satisfy (2.8), since $\mathbf{a}_\alpha$ ($\alpha = 1, 2$) coincide with the displacement components of the Stroh eigenvectors introduced in Section 2.2. It therefore remains to verify that the final term

$$(\mathbf{a}_3 - \sqrt{-1}\xi(\mathbf{n}\cdot\mathbf{x})\mathbf{a}_2)e^{-\sqrt{-1}\xi(\mathbf{n}\cdot\mathbf{x})}$$

also satisfies (2.8).

To this end, we define

$$\tilde{\mathbf{u}} = (\tilde{u}_1, \tilde{u}_2, \tilde{u}_3) = \left(\mathbf{a}_3 - \sqrt{-1}\xi(\mathbf{n}\cdot\mathbf{x})\mathbf{a}_2\right) e^{-\sqrt{-1}\xi(\mathbf{m}\cdot\mathbf{x}+p_2\mathbf{n}\cdot\mathbf{x})}. \tag{2.57}$$

A direct computation yields

$$\sum_{j,k,l=1}^{3}\left(C_{ijkl}\frac{\partial^2\tilde{u}_k}{\partial x_j\partial x_l}\right)_{i\downarrow 1,2,3}=-\xi^2\Bigg(\sum_{j,l=1}^{3}C_{ijkl}m_jm_l+$$
$$+p_2\left(\sum_{j,l=1}^{3}C_{ijkl}m_jn_l+\sum_{j,l=1}^{3}C_{ijkl}n_jm_l\right)+p_2^2\sum_{j,l=1}^{3}C_{ijkl}n_jn_l\Bigg)_{i\downarrow k\rightarrow 1,2,3}$$
$$\times\left(\mathbf{a}_3-\sqrt{-1}\xi(\mathbf{n}\cdot\mathbf{x})\mathbf{a}_2\right)e^{-\sqrt{-1}\xi(\mathbf{m}\cdot\mathbf{x}+p_2\mathbf{n}\cdot\mathbf{x})}$$
$$-\xi^2\left(\sum_{j,l=1}^{3}C_{ijkl}\left(n_j(m_l+p_2n_l)+n_l(m_j+p_2n_j)\right)\right)_{i\downarrow k\rightarrow 1,2,3}\mathbf{a}_2e^{-\sqrt{-1}\xi(\mathbf{m}\cdot\mathbf{x}+p_2\mathbf{n}\cdot\mathbf{x})}.$$

Invoking (2.20) with $\mathbf{a}=\mathbf{a}_2$, the above expression reduces to

$$\xi^2\Big(-[\mathbf{Q}+p_2(\mathbf{R}+\mathbf{R}^T)+p_2^2\mathbf{T}]\mathbf{a}_3-[\mathbf{R}+\mathbf{R}^T+2p_2\mathbf{T}]\mathbf{a}_2\Big)e^{-\sqrt{-1}\xi(\mathbf{m}\cdot\mathbf{x}+p_2\mathbf{n}\cdot\mathbf{x})},$$

which vanishes identically by virtue of (2.56).

We now turn to the proof of (2.53). By (2.27) and (2.28), the first two terms

$$-\sqrt{-1}\xi\sum_{\alpha=1}^{2}c_\alpha\mathbf{I}_\alpha e^{-\sqrt{-1}\xi\mathbf{m}\cdot\mathbf{x}}$$

on the right-hand side of (2.53) represent the traction on the surface $\mathbf{n}\cdot\mathbf{x}=0$ generated by

$$\sum_{\alpha=1}^{2}c_\alpha\mathbf{a}_\alpha e^{-\sqrt{-1}\xi(\mathbf{m}\cdot\mathbf{x}+p_\alpha\mathbf{n}\cdot\mathbf{x})}$$

in (2.52). It therefore suffices to compute the traction corresponding to (2.57).

We find

$$\left(\sum_{j,k,l=1}^{3}C_{ijkl}\frac{\partial\tilde{u}_k}{\partial x_l}n_j\right)_{i\downarrow 1,2,3}$$
$$=-\sqrt{-1}\xi\left(\sum_{j,l=1}^{3}C_{ijkl}(m_l+p_2n_l)n_j\right)_{i\downarrow k\rightarrow 1,2,3}\left(\mathbf{a}_3-\sqrt{-1}\xi(\mathbf{n}\cdot\mathbf{x})\mathbf{a}_2\right)e^{-\sqrt{-1}\xi(\mathbf{m}\cdot\mathbf{x}+p_2\mathbf{n}\cdot\mathbf{x})}$$
$$-\sqrt{-1}\xi\left(\sum_{j,l=1}^{3}C_{ijkl}n_jn_l\right)_{i\downarrow k\rightarrow 1,2,3}\mathbf{a}_2e^{-\sqrt{-1}\xi(\mathbf{m}\cdot\mathbf{x}+p_2\mathbf{n}\cdot\mathbf{x})}.$$

Restricting this expression to the surface $\mathbf{n}\cdot\mathbf{x}=0$, we obtain

$$-\sqrt{-1}\xi\left([\mathbf{R}+p_2\mathbf{T}]\mathbf{a}_3+\mathbf{T}\mathbf{a}_2\right)e^{-\sqrt{-1}\xi\mathbf{m}\cdot\mathbf{x}},$$

which, by (2.54), is equal to

$$-\sqrt{-1}\xi\mathbf{I}_3e^{-\sqrt{-1}\xi\mathbf{m}\cdot\mathbf{x}}.$$

This completes the proof. □

Case D2

In this case, there exists a non-singular 6×6 matrix $\mathbf{P}$ such that $\mathbf{N}$ is similar to the Jordan normal form

$$\mathbf{P}^{-1}\mathbf{N}\mathbf{P}=\begin{bmatrix} p_1 & 1 & & & & \\ & p_1 & 1 & & & \\ & & p_1 & & & \\ & & & \overline{p}_1 & 1 & \\ & & & & \overline{p}_1 & 1 \\ & & & & & \overline{p}_1 \end{bmatrix}.$$

Let the α-th column of $\mathbf{P}$ be denoted by

$$\begin{bmatrix}\mathbf{a}_\alpha\\ \mathbf{I}_\alpha\end{bmatrix}\in\mathbb{C}^6,\quad 1\le\alpha\le6.$$

Then $\begin{bmatrix}\mathbf{a}_1\\ \mathbf{I}_1\end{bmatrix}$ is an eigenvector associated with the triple eigenvalue p_1, while $\begin{bmatrix}\mathbf{a}_2\\ \mathbf{I}_2\end{bmatrix}$ and $\begin{bmatrix}\mathbf{a}_3\\ \mathbf{I}_3\end{bmatrix}$ are generalized eigenvectors satisfying

$$\begin{aligned}\mathbf{N}\begin{bmatrix}\mathbf{a}_2\\ \mathbf{I}_2\end{bmatrix}-p_1\begin{bmatrix}\mathbf{a}_2\\ \mathbf{I}_2\end{bmatrix}&=\begin{bmatrix}\mathbf{a}_1\\ \mathbf{I}_1\end{bmatrix},\\ \mathbf{N}\begin{bmatrix}\mathbf{a}_3\\ \mathbf{I}_3\end{bmatrix}-p_1\begin{bmatrix}\mathbf{a}_3\\ \mathbf{I}_3\end{bmatrix}&=\begin{bmatrix}\mathbf{a}_2\\ \mathbf{I}_2\end{bmatrix}.\end{aligned}\tag{2.58}$$

As in the previous case, the convention (2.48) ensures that all six vectors span $\mathbb{C}^6$.

Lemma 3. *The general form of the basic solution to (2.8) in Case D2, describing two-dimensional deformation in the half-space $\mathbf{n}\cdot\mathbf{x}\le0$, satisfies the boundary condition*

$\mathbf{u} = \mathbf{a}e^{-\sqrt{-1}\xi\mathbf{m}\cdot\mathbf{x}}$ *on* $\mathbf{n}\cdot\mathbf{x} = 0$ *for some* $\mathbf{a} \in \mathbb{C}^3$*, and decays as* $\mathbf{n}\cdot\mathbf{x} \to -\infty$*. It can be expressed as*

$$\begin{aligned}\mathbf{u} = \Big(&c_1\mathbf{a}_1 + c_2\left(\mathbf{a}_2 - \sqrt{-1}\xi(\mathbf{n}\cdot\mathbf{x})\mathbf{a}_1\right)\\ &+ c_3\left(\mathbf{a}_3 - \sqrt{-1}\xi(\mathbf{n}\cdot\mathbf{x})\mathbf{a}_2 - \tfrac{1}{2}\xi^2(\mathbf{n}\cdot\mathbf{x})^2\mathbf{a}_1\right)\Big)e^{-\sqrt{-1}\xi(\mathbf{m}\cdot\mathbf{x}+p_1\mathbf{n}\cdot\mathbf{x})},\end{aligned} \tag{2.59}$$

and the corresponding surface traction on $\mathbf{n}\cdot\mathbf{x} = 0$ *is given by*

$$\mathbf{t} = -\sqrt{-1}\xi\sum_{\alpha=1}^{3} c_\alpha\mathbf{I}_\alpha e^{-\sqrt{-1}\xi\mathbf{m}\cdot\mathbf{x}}, \tag{2.60}$$

where $c_\alpha \in \mathbb{C}$ $(1 \le \alpha \le 3)$ *are constants determined from* $\sum_{\alpha=1}^{3} c_\alpha\mathbf{a}_\alpha = \mathbf{a}$*, and* $\xi > 0$*.*

Remark 2. *The expressions derived in Cases D1 and D2 demonstrate that, in the presence of degeneracy, the basic solutions to (2.8) contain polynomial terms in* $\mathbf{n}\cdot\mathbf{x}$*, in addition to the exponential factors present in the simple and semisimple cases. These polynomial terms arise directly from the structure of generalized eigenvectors and reflect the algebraic multiplicity of the eigenvalues. Consequently, the theory of Jordan chains plays a central role in the systematic construction of basic solutions under degeneracy.*

2.5 Degeneracy and Rotational Properties of the Stroh System

We now consider the degenerate form of the eigenvalue problem (2.31). When the Stroh matrix admits non-simple eigenvalues, the associated Jordan chains must be incorporated into the formulation. In this setting, the rotational invariance observed in the simple spectrum case no longer holds at the level of individual eigenvectors. Instead, the generalized eigenspaces transform in a structured manner under simultaneous rotations of $\mathbf{m}$ and $\mathbf{n}$ about $\mathbf{m}\times\mathbf{n}$.

Rotations are parameterized by ϕ according to (2.33), and the notation of (2.34)–(2.35) is retained. The analysis separates according to the algebraic type of degeneracy.

Case D1: A Double Eigenvalue with a Jordan Chain of Length Two

Theorem 4. *Let* $\begin{bmatrix}\mathbf{a}_\alpha\\ \mathbf{I}_\alpha\end{bmatrix}$ $(\alpha = 1, 2)$ *be linearly independent eigenvectors of* $\mathbf{N}(0)$ *associated with the eigenvalues* p_α $(\alpha = 1, 2)$*, and let* $\begin{bmatrix}\mathbf{a}_3\\ \mathbf{I}_3\end{bmatrix}$ *be a generalized eigenvector*

satisfying

$$\mathbf{N}(0)\begin{bmatrix}\mathbf{a}_3\\ \mathbf{I}_3\end{bmatrix} - p_2\begin{bmatrix}\mathbf{a}_3\\ \mathbf{I}_3\end{bmatrix} = \begin{bmatrix}\mathbf{a}_2\\ \mathbf{I}_2\end{bmatrix}. \tag{2.61}$$

Then the following hold:

1. *For all ϕ, the relations*

$$\mathbf{N}(\phi)\begin{bmatrix}\mathbf{a}_\alpha\\ \mathbf{I}_\alpha\end{bmatrix} - p_\alpha(\phi)\begin{bmatrix}\mathbf{a}_\alpha\\ \mathbf{I}_\alpha\end{bmatrix} = \mathbf{0}, \quad \alpha = 1, 2, \tag{2.62}$$

   are satisfied, where $p_\alpha(\phi)$ $(\alpha = 1, 2)$ are eigenvalues of $\mathbf{N}(\phi)$ determined by the Riccati equation (2.36) with initial conditions $p_\alpha(0) = p_\alpha$.

2. *By introducing*

$$\begin{bmatrix}\mathbf{a}_3(\phi)\\ \mathbf{I}_3(\phi)\end{bmatrix} = \exp\left(\int_0^\phi 2p_2(\psi)\, d\psi\right)\begin{bmatrix}\mathbf{a}_2\\ \mathbf{I}_2\end{bmatrix}, \tag{2.63}$$

   one obtains

$$\mathbf{N}(\phi)\begin{bmatrix}\mathbf{a}_3(\phi)\\ \mathbf{I}_3(\phi)\end{bmatrix} - p_2(\phi)\begin{bmatrix}\mathbf{a}_3(\phi)\\ \mathbf{I}_3(\phi)\end{bmatrix} = \begin{bmatrix}\mathbf{a}_2(\phi)\\ \mathbf{I}_2(\phi)\end{bmatrix}, \tag{2.64}$$

   for all ϕ.

Proof. Equation (2.62) follows directly from Theorem 3. To establish (2.64), we recall that the eigenrelation (2.52) is equivalent to (2.56) and (2.58). Hence, it suffices to verify the validity of the relations

$$[\mathbf{Q} + p_2(\mathbf{R} + \mathbf{R}^T) + p_2^2\mathbf{T}](\phi)\mathbf{a}_3(\phi) + [\mathbf{R} + p_2\mathbf{T}](\phi)\mathbf{a}_2 + \mathbf{I}_2 = \mathbf{0}, \tag{2.65}$$

and

$$\mathbf{I}_3(\phi) - [\mathbf{R}^T + p_2\mathbf{T}](\phi)\mathbf{a}_3(\phi) - \mathbf{T}(\phi)\mathbf{a}_2 = \mathbf{0}, \tag{2.66}$$

for all ϕ. Define the left-hand sides of (2.65) and (2.66) by $\mathbf{g}(\phi)$ and $\boldsymbol{\zeta}(\phi)$, respectively. From (2.61), one finds

$$\mathbf{g}(0) = \mathbf{0}, \quad \boldsymbol{\zeta}(0) = \mathbf{0}. \tag{2.67}$$

Differentiating (2.63) yields

$$\mathbf{a}_3'(\phi) = 2p_2(\phi)\mathbf{a}_3(\phi), \quad \mathbf{I}_3'(\phi) = 2p_2(\phi)\mathbf{I}_3(\phi). \tag{2.68}$$

Substitution into the governing relations (2.38), (2.41), and (2.43) shows that $\mathbf{g}'(\phi) = \mathbf{0}$, which, together with (2.67), implies $\mathbf{g}(\phi) = \mathbf{0}$ for all ϕ. An analogous calculation using (2.37), (2.43), and (2.68) establishes $\boldsymbol{\zeta}'(\phi) = 2p_2(\phi)\boldsymbol{\zeta}(\phi)$. Since $\boldsymbol{\zeta}(0) = \mathbf{0}$, it follows that $\boldsymbol{\zeta}(\phi) = \mathbf{0}$ for all ϕ, completing the proof. □

Case D2: A Triple Eigenvalue with a Jordan Chain of Length Three

Theorem 5. *Let* $\begin{bmatrix}\mathbf{a}_1\\ \mathbf{I}_1\end{bmatrix}$ *be an eigenvector of* $\mathbf{N}(0)$ *associated with the triple eigenvalue* p_1*, and let* $\begin{bmatrix}\mathbf{a}_\alpha\\ \mathbf{I}_\alpha\end{bmatrix}$ $(\alpha = 2, 3)$ *be generalized eigenvectors satisfying*

$$\mathbf{N}(0)\begin{bmatrix}\mathbf{a}_2\\ \mathbf{I}_2\end{bmatrix} - p_1\begin{bmatrix}\mathbf{a}_2\\ \mathbf{I}_2\end{bmatrix} = \begin{bmatrix}\mathbf{a}_1\\ \mathbf{I}_1\end{bmatrix}, \quad \mathbf{N}(0)\begin{bmatrix}\mathbf{a}_3\\ \mathbf{I}_3\end{bmatrix} - p_1\begin{bmatrix}\mathbf{a}_3\\ \mathbf{I}_3\end{bmatrix} = \begin{bmatrix}\mathbf{a}_2\\ \mathbf{I}_2\end{bmatrix}.$$

Then the following relations hold for all ϕ*:*

$$\begin{aligned}
\mathbf{N}(\phi)\begin{bmatrix}\mathbf{a}_1\\ \mathbf{I}_1\end{bmatrix} - p_1(\phi)\begin{bmatrix}\mathbf{a}_1\\ \mathbf{I}_1\end{bmatrix} &= \mathbf{0},\\
\mathbf{N}(\phi)\begin{bmatrix}\mathbf{a}_2(\phi)\\ \mathbf{I}_2(\phi)\end{bmatrix} - p_1(\phi)\begin{bmatrix}\mathbf{a}_2(\phi)\\ \mathbf{I}_2(\phi)\end{bmatrix} &= \begin{bmatrix}\mathbf{a}_1\\ \mathbf{I}_1\end{bmatrix},\\
\mathbf{N}(\phi)\begin{bmatrix}\mathbf{a}_3(\phi)\\ \mathbf{I}_3(\phi)\end{bmatrix} - p_1(\phi)\begin{bmatrix}\mathbf{a}_3(\phi)\\ \mathbf{I}_3(\phi)\end{bmatrix} &= \begin{bmatrix}\mathbf{a}_2(\phi)\\ \mathbf{I}_2(\phi)\end{bmatrix},
\end{aligned}$$

where $p_1(\phi)$ *is the eigenvalue of* $\mathbf{N}(\phi)$ *determined by the Riccati equation* (2.36) *with* $p_1(0) = p_1$*, and the generalized eigenvectors evolve according to*

$$\begin{aligned}
\begin{bmatrix}\mathbf{a}_2(\phi)\\ \mathbf{I}_2(\phi)\end{bmatrix} &= \exp\left(\int_0^\phi 2p_1(\psi)\,d\psi\right)\left(\begin{bmatrix}\mathbf{a}_2\\ \mathbf{I}_2\end{bmatrix} - \int_0^\phi \exp\left(-\int_0^\psi 2p_1(\theta)\,d\theta\right)d\psi\begin{bmatrix}\mathbf{a}_1\\ \mathbf{I}_1\end{bmatrix}\right),\\
\begin{bmatrix}\mathbf{a}_3(\phi)\\ \mathbf{I}_3(\phi)\end{bmatrix} &= \exp\left(\int_0^\phi 4p_1(\psi)\,d\psi\right)\begin{bmatrix}\mathbf{a}_3\\ \mathbf{I}_3\end{bmatrix}.
\end{aligned}$$

Corollary 1. *The classification of Stroh's eigenvalue problem* (2.31) *into simple, semisimple or degenerate cases is invariant under rotations of* $\mathbf{m}$ *and* $\mathbf{n}$ *about the vector product* $\mathbf{m} \times \mathbf{n}$*. Moreover, when the problem is degenerate, the structure of the Jordan*

chains associated with $\mathbf{N}$ *is also preserved under such rotations.*

Remark 3. *It follows that the character of Stroh's eigenvalue problem* (2.31) *whether simple, semisimple or degenerate is determined entirely by the elasticity tensor* $\mathbf{C} = (C_{ijkl})_{1\le i,j,k,l\le 3}$ *together with the unit normal* $\mathbf{m} \times \mathbf{n}$ *to the reference plane. In the degenerate case, these quantities also dictate the precise length of the Jordan chains. Thus, rotational dependence reveals a deeper structural rigidity of the eigenvalue problem: while the individual eigenvectors and generalized eigenvectors may transform non-trivially under rotations, the underlying algebraic structure encoded in the Jordan form remains invariant.*

2.6 Integral Representation of the Angular Average

An integral average over the rotation angle ϕ produces a spectrum-independent form of the Stroh eigenvalue problem. This formulation underlies the Barnett–Lothe integral approach and ensures the linear independence of the displacement components of eigenvectors and generalized eigenvectors, forming the basis for the subsequent definition of the surface impedance tensor.

We begin by letting $\mathbf{N}(\phi)$ be given by (2.35) and considering the angular average of its eigenvalue $p(\phi)$ over the full interval $[-\pi, \pi]$.

Lemma 4. *For each eigenvalue* $p(\phi)$ *of* $\mathbf{N}(\phi)$ *satisfying* $\operatorname{Im} p(0) > 0$*, we have*

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} p(\phi)\,d\phi = \sqrt{-1}. \tag{2.69}$$

Proof. By Remark 1, the eigenvalue satisfies either $p(\phi) = \tan(\phi_c - \phi)$ with $\operatorname{Im}(\tan\phi_c) > 0$, or $p(\phi) \equiv \sqrt{-1}$. In the latter case, the result (2.69) is immediate. In the former case, write $\phi_c = \phi_r + \sqrt{-1}\phi_i$ with $\phi_r, \phi_i \in \mathbb{R}$. Since $\operatorname{Im}(\tan\phi_c) > 0$, we must have $\phi_i > 0$. Then

$$\begin{aligned}\frac{1}{2\pi}\int_{-\pi}^{\pi} p(\phi)\,d\phi &= \frac{1}{2\pi}\big[\log\cos(\phi_c - \phi)\big]_{\phi=-\pi}^{\phi=\pi} \\ &= \frac{1}{2\pi}\Big[\log\Big(\cos(\phi_c-\phi_r)\cosh\phi_i + \sqrt{-1}\sin(\phi-\phi_r)\sinh\phi_i\Big)\Big]_{\phi=-\pi}^{\phi=\pi}.\end{aligned}$$

The argument of the complex-valued function

$$\cos(\phi-\phi_r)\cosh\phi_i + \sqrt{-1}\sin(\phi-\phi_r)\sinh\phi_i$$

increases by 2π as ϕ varies from $-\pi$ to π, because $\phi_i > 0$ and hence $\sinh \phi_i > 0$. This yields (2.69). □

Definition 1. *We define the* 6×6 *real matrix* $\mathbf{S}$ *to be the angular average of* $\mathbf{N}(\phi)$ *over* $[-\pi, \pi]$*:*

$$\mathbf{S} = \begin{bmatrix} \mathbf{S}_1 & \mathbf{S}_2 \\ \mathbf{S}_3 & \mathbf{S}_1^T \end{bmatrix} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathbf{N}(\phi)\, d\phi, \tag{2.70}$$

where the block matrices $\mathbf{S}_1, \mathbf{S}_2, \mathbf{S}_3$ *are* 3×3 *real matrices given by*

$$\mathbf{S}_1 = \frac{1}{2\pi} \int_{-\pi}^{\pi} -\mathbf{T}(\phi)\mathbf{R}(\phi)^T\, d\phi,$$

$$\mathbf{S}_2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathbf{T}(\phi)^{-1}\, d\phi, \tag{2.71}$$

$$\mathbf{S}_3 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(-\mathbf{Q}(\phi) + \mathbf{R}(\phi)\mathbf{T}(\phi)^{-1}\mathbf{R}(\phi)^T\right) d\phi. \tag{2.72}$$

Lemma 5. *The matrices* $\mathbf{S}_2$ *and* $\mathbf{S}_3$ *are symmetric. Moreover,* $\mathbf{S}_3$ *is positive definite.*

Proof. The integrands $\mathbf{T}(\phi)^{-1}$ and $-\mathbf{Q}(\phi) + \mathbf{R}(\phi)\mathbf{T}(\phi)^{-1}\mathbf{R}(\phi)^T$ are symmetric for all ϕ, which implies that their averages $\mathbf{S}_2$ and $\mathbf{S}_3$ are symmetric. By Lemma 1(1), the matrix $\mathbf{T}(\phi)$ is positive definite for each ϕ, hence so is its inverse $\mathbf{T}(\phi)^{-1}$. Therefore, $\mathbf{S}_2$, being the angular average of positive definite matrices, is also positive definite. □

We are now in a position to state the central result of this section.

Theorem 6. *Let* $\begin{bmatrix} \mathbf{a}_\alpha \\ \mathbf{I}_\alpha \end{bmatrix}$ *be an eigenvector or generalized eigenvector of* $\mathbf{N}(0)$ *corresponding to the eigenvalues* p_α $(\alpha = 1, 2, 3)$ *with* $\operatorname{Im} p_\alpha > 0$*. Then*

$$\mathbf{S} \begin{bmatrix} \mathbf{a}_\alpha \\ \mathbf{I}_\alpha \end{bmatrix} = \sqrt{-1} \begin{bmatrix} \mathbf{a}_\alpha \\ \mathbf{I}_\alpha \end{bmatrix}. \tag{2.73}$$

Proof. When $\begin{bmatrix} \mathbf{a}_\alpha \\ \mathbf{I}_\alpha \end{bmatrix}$ is an eigenvector of $\mathbf{N}(0)$, taking the angular average of both sides of (2.46) and applying (2.69) directly yields (2.73). This argument holds for eigenvectors corresponding to simple or semisimple eigenvalues, as well as for eigenvectors of degenerate eigenvalue problems.

Now suppose that the eigenvalue problem (2.31) is degenerate. We first consider Case D1. Let $\begin{bmatrix} \mathbf{a}_3 \\ \mathbf{I}_3 \end{bmatrix}$ be a generalized eigenvector of $\mathbf{N}(0)$, as in Theorem 4, corresponding

to the repeated eigenvalue p_2 with $\operatorname{Im} p_2 > 0$. From (2.63) and (2.64), we obtain

$$\mathbf{N}(\phi) \begin{bmatrix} \mathbf{a}_3 \\ \mathbf{I}_3 \end{bmatrix} - p_2(\phi) \begin{bmatrix} \mathbf{a}_3 \\ \mathbf{I}_3 \end{bmatrix} = \exp\left(\int_0^{\phi} -2p_2(\psi)\, d\psi \right) \begin{bmatrix} \mathbf{a}_2 \\ \mathbf{I}_2 \end{bmatrix}, \tag{2.74}$$

where $p_2(\phi)$ solves the Riccati equation (2.36) with initial condition $p_2(0) = p_2$.

By Remark 1, two cases arise: either $p_2(\phi) = \tan(\phi_c - \phi)$ with $p_2 = \tan\phi_c$, or $p_2(\phi) \equiv \sqrt{-1}$. In the latter case, the right-hand side of (2.74) reduces to

$$\exp(-2\sqrt{-1}\phi) \begin{bmatrix} \mathbf{a}_2 \\ \mathbf{I}_2 \end{bmatrix} = \left(\cos 2\phi - \sqrt{-1}\sin 2\phi\right) \begin{bmatrix} \mathbf{a}_2 \\ \mathbf{I}_2 \end{bmatrix},$$

whose angular average over $[-\pi, \pi]$ vanishes. In the first case, a similar calculation shows that the average is again zero, due to the periodicity of $\tan(\phi_c - \phi)$ and the condition $\operatorname{Im}\phi_c > 0$.

Thus, in both cases, the angular average of the right-hand side of (2.74) is zero, while the angular average of the left-hand side yields

$$\mathbf{S} \begin{bmatrix} \mathbf{a}_3 \\ \mathbf{I}_3 \end{bmatrix} - \sqrt{-1} \begin{bmatrix} \mathbf{a}_3 \\ \mathbf{I}_3 \end{bmatrix},$$

which proves (2.73) for Case D1. An analogous argument, based on Theorem 5, establishes the result for Case D2. □

We conclude that the angle-averaged Stroh eigenvalue problem takes the unified form (2.73), independent of whether the original problem (2.31) is simple, semisimple or degenerate. This unified structure underlies the Barnett–Lothe integral formalism and serves as the foundation for subsequent results, including the definition of the surface impedance tensor.

2.7 Surface Impedance Tensor

Consider an elastic half-space and the Stroh formalism applied to two-dimensional deformations. Let

$$\begin{bmatrix} \mathbf{a}_\alpha \\ \mathbf{I}_\alpha \end{bmatrix}, \quad \alpha = 1, 2, 3,$$

be linearly independent eigenvectors or generalized eigenvectors of the Stroh eigenvalue problem (2.31), corresponding to eigenvalues p_α with $\operatorname{Im} p_\alpha > 0$. The displacement components $\mathbf{a}_\alpha$ form a basis of $\mathbb{C}^3$, and their corresponding traction components $\mathbf{I}_\alpha$

define a linear mapping from boundary displacements to tractions.

We define the *surface impedance tensor* $\mathbf{Z}$ as the unique 3×3 matrix satisfying

$$\mathbf{I}_\alpha = \sqrt{-1}\,\mathbf{Z}\mathbf{a}_\alpha, \quad \alpha = 1, 2, 3, \tag{2.75}$$

or equivalently in matrix form,

$$\mathbf{Z} = -\sqrt{-1}\begin{bmatrix}\mathbf{I}_1 & \mathbf{I}_2 & \mathbf{I}_3\end{bmatrix}\begin{bmatrix}\mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3\end{bmatrix}^{-1}. \tag{2.76}$$

This definition provides a well-defined mapping from boundary displacements to the tractions required to sustain them, independent of the particular choice of eigenvectors.

Theorem 7. *The surface impedance tensor* $\mathbf{Z}$ *admits the representation*

$$\mathbf{Z} = \mathbf{S}_2^{-1} + \sqrt{-1}\,\mathbf{S}_2^{-1}\mathbf{S}_1, \tag{2.77}$$

where $\mathbf{S}_1$ *and* $\mathbf{S}_2$ *are the block matrices defined in* (2.72).

From (2.33), (2.34), (2.72), and the 2π-periodicity of $\cos\phi$ and $\sin\phi$, it follows that:

Corollary 2. *The matrices* $\mathbf{S}_1$, $\mathbf{S}_2$, *and hence the surface impedance tensor* $\mathbf{Z}$, *are invariant under simultaneous rotations of* $\mathbf{m}$ *and* $\mathbf{n}$ *about the axis* $\mathbf{m}\times\mathbf{n}$. *Consequently, these objects depend not on* $\mathbf{m}$ *and* $\mathbf{n}$ *individually, but only on their vector product* $\mathbf{m}\times\mathbf{n}$.

Theorem 8. *The surface impedance tensor* $\mathbf{Z}$ *is Hermitian (i.e.* $\mathbf{Z} = \overline{\mathbf{Z}}^T$*) and positive definite.*

Lemma 6. *Let* $\mathbf{S}$ *denote the angular average of* $\mathbf{N}(\phi)$ *as defined in* (2.70). *Then*

$$\mathbf{S}^2 = -\mathbf{I},$$

where $\mathbf{I}$ *is the* 6×6 *identity matrix.*

Proof. Let

$$\begin{bmatrix}\mathbf{a}_\alpha \\ \mathbf{I}_\alpha\end{bmatrix}, \quad \alpha = 1, 2, 3,$$

be linearly independent eigenvectors or generalized eigenvectors of $\mathbf{N}(0)$ corresponding to eigenvalues p_α with $\operatorname{Im} p_\alpha > 0$. Then, by theorem 6, we have

$$\mathbf{S}^2\begin{bmatrix}\mathbf{a}_\alpha \\ \mathbf{I}_\alpha\end{bmatrix} = -\begin{bmatrix}\mathbf{a}_\alpha \\ \mathbf{I}_\alpha\end{bmatrix}, \quad \alpha = 1, 2, 3. \tag{2.78}$$

Recalling (2.48), we take complex conjugates of both sides of (2.78), which shows that (2.78) holds for six linearly independent eigenvectors or generalized eigenvectors of $\mathbf{N}(0)$. This proves the lemma. □

Proof that $\mathbf{Z}$ *is Hermitian.* From the block representation (2.70), the $(1,2)$-block component of the identity (2.78) is

$$\mathbf{S}_1\mathbf{S}_2 + \mathbf{S}_2\mathbf{S}_1^T = \mathbf{O}.$$

Since $\mathbf{S}_2$ is symmetric and invertible, it follows that

$$\mathbf{S}_2^{-1}\mathbf{S}_1 = -(\mathbf{S}_2^{-1}\mathbf{S}_1)^T,$$

that is, $\mathbf{S}_2^{-1}\mathbf{S}_1$ is antisymmetric. Hence, by (2.77), $\mathbf{Z}$ is Hermitian. □

Remark 4. *Suppose that the elasticity tensor* $\mathbf{C} = (C_{ijkl})_{i,j,k,l=1,2,3}$ *satisfies the major symmetry condition* (2.5)*, but not necessarily the minor symmetries* (2.3)*. If, instead of the convexity condition* (2.6)*, we assume the strong ellipticity condition*

$$\textit{the matrix } \Big(\sum_{j,l=1}^{3} C_{ijkl}\,\xi_j\xi_l\Big) \textit{ is positive definite for all nonzero } \xi \in \mathbb{R}^3,$$

and if the "traction" on $\mathbf{n}\cdot\mathbf{x} = 0$ *is given by* (2.28)*, then all arguments presented in Sections 2.2–2.7 remain valid, with the exception of the positive definiteness of* $\mathbf{Z}$*.*

2.8 Hamiltonian formalism and Its connection with Stroh formalism

Hamiltonian mechanics, introduced by Sir William Rowan Hamilton in the nineteenth century Hamilton (1834), is a reformulation of classical mechanics that emphasizes energy-based descriptions of physical systems. Unlike Newtonian mechanics, which is primarily concerned with forces and accelerations, the Hamiltonian framework describes system dynamics in terms of generalized coordinates and their conjugate momenta. One of its principal strengths lies in the unified manner in which it treats a wide range of physical systems within a single theoretical structure. This formulation naturally incorporates constraints and symmetry properties, often leading to a more transparent and systematic derivation of the governing equations. Moreover, the Hamiltonian approach provides a powerful setting for the application of advanced

analytical techniques, including perturbation methods and stability analysis, making it particularly well suited for the study of complex and coupled dynamical systems.

2.8.1 Generalized coordinates

The dynamics of a mechanical system are described in terms of a set of *generalized coordinates*. For discrete systems, these are denoted by q_i, $i = 1, \ldots, n$, while for continuum systems they are represented by generalized field variables $q(\mathbf{x})$. The generalized coordinates uniquely specify the configuration of the system, and their time derivatives $\dot{q}_i$ or $\dot{q}(\mathbf{x})$ describe the rates of change of this configuration.

2.8.2 Lagrangian and equations of motion

Given the generalized coordinates and their time derivatives, the dynamics of the system are governed by the *Lagrangian*, defined as the difference between the kinetic energy T and the potential energy V of the system (Safko et al., 2002; Gurtin, 1982):

$$L(q_i, \dot{q}_i, t) = T(q_i, \dot{q}_i) - V(q_i). \tag{2.79}$$

The evolution equations follow from the Euler Lagrange equations,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0, \qquad i = 1, \ldots, n, \tag{2.80}$$

which constitute a system of second-order differential equations in time.

For continuum systems, the Lagrangian formulation is expressed in terms of a Lagrangian density $\mathcal{L}$,

$$\mathcal{L} = \mathcal{T} - \mathcal{W}, \tag{2.81}$$

where $\mathcal{T}$ denotes the kinetic energy density and $\mathcal{W}$ the strain energy density. The latter may include contributions from elastic, thermal, fluid, or chemical effects.

2.8.3 Generalized momenta and conjugate variables

Associated with each generalized coordinate q_i is a *generalized (canonical) momentum*, defined for discrete systems as

$$p_i = \frac{\partial L}{\partial \dot{q}_i}, \qquad i = 1, \ldots, n, \tag{2.82}$$

and, for continuum systems, as

$$p(\mathbf{x}) = \frac{\partial \mathcal{L}}{\partial \dot{q}(\mathbf{x})}. \tag{2.83}$$

The generalized momenta are energetically conjugate to the generalized coordinates and characterize the inertial response of the system. In general, they do not coincide with the physical linear momentum except in Cartesian coordinate systems.

In continuum mechanics, the conjugate variables are often stress-like or flux-like quantities, such as linear momentum density, fluid mass flux, or entropy-related variables, depending on the physical interpretation of the field q_i. Their introduction reflects the underlying energetic structure of the medium.

2.8.4 Hamilton's equation of motion

Hamiltonian is defined to be the Legendre transform of Lagrangian $L(q_i, \dot{q}_i, t)$ with respect to the q_i variables

$$H(q_i, p_i, t) = \sum_{i=1}^{n} p_i \dot{q}_i - L(q_i, \dot{q}_i, t), \tag{2.84}$$

where the generalized velocities $\dot{q}_i$ are understood to be expressed in terms of q_i and p_i.

For linear systems, the Hamiltonian (or Hamiltonian density in the continuum setting) typically represents the total energy of the system and is at most quadratic in the canonical variables.

Once the Hamiltonian has been introduced, the equations of motion can be written in canonical form as Hamilton's equations (Goldstein et al., 1950)

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i}, \tag{2.85}$$

which are equivalent to the Euler Lagrange equations but are first order.

If the Lagrangian does not depend explicitly on time, the Hamiltonian is conserved along the motion:

$$\frac{dH}{dt} = 0. \tag{2.86}$$

This conservation law expresses the invariance of the total energy of the system. Hamiltonian systems possess a geometric structure known as the symplectic form, which ensures the conservation of phase-space volume during evolution.

2.8.5 Hamiltonian interpretation of the Stroh formalism

The Stroh formalism, originally developed for anisotropic elasticity, admits a natural interpretation within the framework of Hamiltonian mechanics Fu (2007). In this interpretation, the Stroh equations arise as a canonical Hamiltonian system in which a spatial coordinate act as a time like variable. This viewpoint provides a unified

and physically transparent foundation for the formulation of elastic, poroelastic, and thermoporoelastic wave propagation problems. In these cases, additional field variables and conjugate fluxes can be systematically incorporated into the state vector without altering the canonical structure. Constraints such as incompressibility modify the form of the Hamiltonian matrix, but preserve the underlying Hamiltonian nature of the system.

Stroh-Hamiltonian formulation and impedance matrix

Nobili (2024) presented the Hamiltonian formulation of the Stroh formalism for anisotropic elasticity along with the derivation of the associated eigenvalue and energy structures, and demonstrated how the impedance matrix provides an efficient tool for analyzing wave propagation and localized boundary modes. To discuss this interpretation, we begin with a homogeneous anisotropic elastic medium for which the governing equations may be cast into the Stroh formalism by introducing a mixed state vector that combines kinematic and kinetic quantities. To this end, the stress potential is defined as

$$\hat{\boldsymbol{\phi}} = \int \hat{\mathbf{t}}_2 \, dx_1, \tag{2.87}$$

and the Stroh state vector is introduced as

$$\xi = \begin{bmatrix} \mathbf{u} \\ \hat{\boldsymbol{\phi}} \end{bmatrix}, \tag{2.88}$$

where $\mathbf{u}$ denotes the displacement vector. This choice allows the elastic field equations to be written as a first-order system.

With this definition, the Stroh formalism takes the compact form

$$\frac{\partial \xi}{\partial x_2} = \mathbf{N} \frac{\partial \xi}{\partial x_1}, \tag{2.89}$$

where $\mathbf{N}$ is the fundamental Stroh matrix containing the elastic properties of the medium. It admits the block structure

$$\mathbf{N} = \begin{bmatrix} N_1 & N_2 \\ N_3 & N_1^T \end{bmatrix}, \tag{2.90}$$

with the submatrices N_1, N_2, and N_3 defined in terms of the elastic stiffness tensors following the standard notation Ting (1996).

It is worth noting that the state vector ξ has mixed physical dimensions: the displacement vector has dimensions of length, whereas the stress potential has

dimensions of force per unit length. Consequently, N_1 is dimensionless, while N_2 and N_3 have dimensions of compliance and stress, respectively.

Introducing the constant matrix

$$\hat{I} = \begin{bmatrix} O & I \\ I & O \end{bmatrix}, \tag{2.91}$$

and using the symmetry properties of N_2 and N_3, the product $\hat{I}\mathbf{N}$ is readily shown to be symmetric,

$$\hat{I}\mathbf{N} = (\hat{I}\mathbf{N})^T. \tag{2.92}$$

Moreover, N_2 is positive definite and $-N_3$ is positive semidefinite, ensuring the physical admissibility and stability of the formulation.

To investigate wave propagation, traveling-wave solutions of the form

$$\xi = \Xi\, f(x_1 + p x_2) \tag{2.93}$$

are considered. Substitution into the Stroh system leads to the eigenvalue problem

$$\mathbf{N}\Xi = p\,\Xi, \tag{2.94}$$

where the eigenvalues p characterize wave slowness or attenuation, and the associated eigenvectors describe the corresponding displacement and traction fields.

The Hamiltonian structure of the formulation becomes apparent by defining the Hamiltonian density as the quadratic form

$$\hat{H} = \frac{1}{2}\,\xi \cdot \hat{I}\mathbf{N}\,\xi. \tag{2.95}$$

Since the Lagrangian density is independent of the coordinate x_2, an energy-type first integral follows, expressing conservation of energy flux across the medium.

For boundary value and surface-wave problems, it is convenient to introduce the edge impedance matrix $\mathbf{M}$, defined through the relation

$$\hat{\boldsymbol{\phi}} = i\,\mathbf{M}\mathbf{u}, \tag{2.96}$$

where i denotes the imaginary unit. Substitution into the Stroh system yields a matrix Riccati equation for $\mathbf{M}$,

$$N_3 + iN_1^T\mathbf{M} - \mathbf{M}N_1 + \mathbf{M}N_2\mathbf{M} = O. \tag{2.97}$$

This equation provides a direct and efficient means of determining the impedance matrix. The importance of this result lies in the analysis of localized wave modes. In particular, for traction-free boundary conditions at $x_2 = 0$, non-trivial solutions exist only if

$$\mathbf{Mu} = \mathbf{o}. \tag{2.98}$$

Thus, the dispersion relations for localized waves reduce to the evaluation of the impedance matrix, which may be computed efficiently using the integral representation originally introduced by Barnett and Lothe (1974).

2.8.6 Conclusion

This chapter has developed the core elements of the Stroh formalism within the framework of anisotropic linear elasticity. Beginning with the basic kinematic relations, constitutive equations, and equilibrium conditions, the governing equations were cast into the Stroh first-order system. This formulation provides a compact and effective representation of elastic fields and forms the basis for the analysis of wave propagation and boundary value problems in anisotropic media.

The mathematical structure of the Stroh system was examined in detail through its eigenvalue problem. The eigenvalues were classified into simple, semi-simple, and degenerate categories, and the corresponding eigenvectors and generalized eigenvectors were constructed using Jordan chains. Particular attention was paid to degenerate cases, where the dependence of generalized eigenvectors on the orientation of the reference plane becomes significant. The rotational invariance of the formalism was highlighted as a fundamental property that underpins its robustness.

To manage the complications associated with degeneracy, an angular averaging procedure was introduced. This approach allows the essential features of the Stroh solution to be retained while avoiding unnecessary directional dependence. Within this setting, the surface impedance tensor was defined and its main properties established. Its Hermitian and positive-definite character, together with invariance under rotations, makes it a powerful and reliable tool for the study of surface and interface phenomena in anisotropic elastic solids.

Collectively, these results provide a clear and rigorous foundation for the Stroh formalism. The classification of eigenvalues, the treatment of generalized eigenvectors, and the introduction of the surface impedance tensor form a consistent analytical framework that will be relied upon in the chapters that follow.

The latter part of the chapter placed this framework within the broader context of Hamiltonian mechanics. Starting from variational principles, the Lagrangian

formulation was introduced for both discrete and continuous systems, emphasizing the role of generalized coordinates and their conjugate variables. By means of the Legendre transformation, the governing equations were rewritten in Hamiltonian form, making their energetic structure and conservation properties explicit.

From this perspective, the Stroh equations emerge naturally as a spatial Hamiltonian system, in which displacement-type variables and traction or flux type quantities appear as canonical pairs, and a spatial coordinate plays the role of the evolution parameter. This interpretation offers a clear explanation for key features of the Stroh matrix, such as eigenvalue pairing and orthogonality relations, as direct consequences of the underlying symplectic structure.

Finally, the Hamiltonian viewpoint provides a flexible and systematic framework for extending the analysis to coupled continua. Its application to poroelastic and thermoporoelastic media demonstrates how additional physical fields and their associated fluxes can be incorporated without altering the essential structure of the formulation. This unified approach not only deepens physical understanding but also sets the stage for the developments presented in the subsequent chapters of this thesis.

Chapter 3

Fundamentals of Poroelasticity

3.1 Introduction

The mechanics of porous media, where mechanical deformation is intrinsically coupled with fluid flow, are central to geomechanics, biomechanics, hydrology, and many engineering applications. The theoretical framework was initially formulated by Terzaghi for one-dimensional consolidation problems Terzaghi (2013, 1943). It was subsequently generalized to three dimensions through the work of Biot (1941, 1956b). In Terzaghi's original formulation, soil deformation was attributed primarily to the rearrangement of soil particles, while the compressibility of both the pore fluid and the solid particles was considered negligible. For saturated soils, this assumption implies that any volumetric change can occur only through the relative movement of pore fluid with respect to the soil skeleton. This simplification has proven to be a reasonable representation of the mechanical behaviour of many soft soils, particularly clays and loose sands. Such materials can undergo significant deformations, often amounting to several percent, despite the high stiffness of their individual constituents, namely the solid particles and the pore fluid. Subsequent developments of the theory, notably those introduced by Biot, extended this framework by incorporating the compressibility of both the pore fluid and the solid matrix. This enhancement allows the theory to be applied to stiffer geomaterials, including sandstones and other porous rocks, which are of particular importance in the analysis and engineering of deep oil and gas reservoirs.

This chapter focuses on the framework of Biot's linear poroelasticity, which combines linear elasticity with Darcy's law of fluid flow. The discussion covers the kinematics of deformation, stress and strain definitions, constitutive relations, governing equations, boundary and initial conditions, classical and dynamic responses, and the classification of poroelastic behavior into reversible and irreversible forms. Furthermore, we provide an overview of extended poroelastic formulations that incorporate additional

multiphysics couplings such as viscoelasticity, thermal, and chemical effects.

3.1.1 Classical Poroelasticity

Poroelasticity is a continuum-mechanical theory describing the coupled interaction between deformation of a porous elastic solid skeleton and the flow of pore fluid. The classical formulation, developed by Biot (1941), combines momentum balance for the mixture, mass conservation of the pore fluid, and constitutive relations linking stress, strain, pore pressure, and fluid flux. When fluid flow is modeled using Darcy's law, the resulting governing equations describe the time-dependent coupling between mechanical deformation and pore pressure diffusion (Coussy, 2004). Depending on whether inertial effects are neglected or retained, the classical poroelastic framework gives rise to quasi-static and dynamic regimes.

3.1.2 Static (Quasi-Static) Poroelasticity

Static or quasi-static poroelasticity neglects inertial effects in both the solid and fluid phases. This approximation is valid for slow or long-term processes where mechanical equilibration occurs much faster than pore fluid movement. The resulting equations resemble coupled diffusion–elasticity systems where time dependence arises exclusively from fluid mass balance (Biot, 1941; Coussy, 2004).

Characteristics

- Momentum balance reduces to static equilibrium.
- Deformation is governed by the interaction between pore pressure and effective stress.
- Time-dependent behavior is dominated by fluid diffusion rather than inertial effects.

Relevance in Continuum Mechanics

The quasi-static theory generalizes Terzaghi's consolidation to three dimensions and is widely applied in geomechanics and soft-tissue mechanics (Coussy, 2004). It provides a tractable framework for analyzing settlement, swelling, compaction, and slow fluid–structure interactions.

3.1.3 Dynamic Poroelasticity

Dynamic poroelasticity incorporates inertial forces and kinetic energy contributions of the solid and fluid phases. This generalization, formulated by Biot in his seminal dynamic poroelasticity (Biot, 1956a,c), predicts the existence of multiple compressional wave modes and a shear wave. The interactions between the phases lead to dispersion, attenuation, and other wave phenomena characteristic of saturated media.

Characteristics

- Governing equations include solid and fluid accelerations.
- Wave propagation yields fast and slow compressional waves.
- Dissipation arises from viscous coupling and relative motion.

Relevance in Continuum Mechanics

Dynamic poroelasticity forms the basis for the continuum-mechanical description of seismic waves in saturated soils, acoustic waves in porous materials, and high-frequency excitation in biological tissues (Carcione, 2014; Coussy, 2004). It extends classical elastodynamics to multiphase mixtures.

3.2 Kinematics and Fundamental Variables

Poroelasticity explains the coupled mechanical and hydraulic behavior of a porous solid saturated with fluid, as formulated by (Biot, 1941, 1956b). By combining classical continuum mechanics with fluid flow in porous media, it provides a framework to analyze deformation–fluid interactions in soils, rocks, and biological tissues.

3.2.1 Stress Tensor

In continuum mechanics, the Cauchy stress tensor $\boldsymbol{\sigma}$ describes the internal forces on a material element which we already discussed in chapter 2. In porous materials, the total stress comes from both the solid structure and the pressure of the fluid filling the pores. The decomposition of this stress into solid and fluid contributions is a key concept in poroelasticity.

3.2.2 Strain Tensor

The deformation of the solid skeleton in a porous medium is characterized by the displacement vector $\mathbf{u}(\mathbf{x}, t)$. For small deformations, the strain is defined by the

infinitesimal strain tensor $\boldsymbol{\varepsilon}$, as introduced in chapter 2. Its trace, $\varepsilon_{kk} = \nabla \cdot \mathbf{u}$, describes the volumetric deformation, which plays an essential role in coupling solid deformation with changes in the content of pore fluid in poroelasticity.

3.2.3 The Principle of Effective Stress

A fundamental concept in poroelasticity is Biot's effective stress principle, which states that only part of the total stress contributes to the deformation of the solid skeleton. The effective stress tensor is defined as

$$\sigma'_{ij} = \sigma_{ij} + \alpha\, p\, \delta_{ij}, \tag{3.1}$$

where p is the pore fluid pressure, δ_{ij} is the Kronecker delta, and α is Biot's coefficient ($0 \leq \alpha \leq 1$). This generalizes Terzaghi's one-dimensional effective stress to three dimensions (Terzaghi, 2013).

For an isotropic material, the effective stress is related to the strain tensor via Hooke's law:

$$\sigma'_{ij} = 2G\, \varepsilon_{ij} + \lambda\, \varepsilon_{kk} \delta_{ij}, \tag{3.2}$$

where G and λ are the Lamé parameters of the solid skeleton. Substituting the definition of effective stress yields the total stress in terms of strain and pore pressure (Cheng, 2016a):

$$\sigma_{ij} = 2G\, \varepsilon_{ij} + \lambda\, \delta_{ij} \varepsilon_{kk} - \alpha\, p\, \delta_{ij}. \tag{3.3}$$

3.2.4 Biot's Coefficient

Biot's coefficient α is also commonly referred to as the Biot–Willis coefficient quantifies the contribution of pore pressure to the total stress in a fluid-saturated porous medium. It is defined in terms of compressibilities as

$$\alpha = 1 - \frac{C_s}{C_d}, \tag{3.4}$$

where $C_d = 1/K_d$ is the drained compressibility of the porous skeleton and $C_s = 1/K_s$ is the compressibility of the solid grains.

This definition of Biot's coefficient is widely used in rock mechanics and poromechanics (Biot and Willis, 1957; Geertsma, 1957a,b) and has also been applied to other porous materials, including biological tissues such as bone and skin (Coussy, 2004). For soft soils, the value of Biot's coefficient is typically close to unity ($\alpha = 1$)

reflecting the high compressibility of the porous skeleton. This definition is equivalent to the bulk-modulus form $\alpha = 1 - K_d/K_s$. where K_s is the bulk modulus of the solid grains and K_b is the bulk modulus of the drained porous skeleton.

3.2.5 Porosity

Porosity, denoted by ϕ, is defined as the fraction of void space within a porous medium. Mathematically, it is expressed as:

$$\phi = \frac{V_f}{V_t}, \tag{3.5}$$

where V_f is the volume of void space (pores) and V_t is the total volume of the medium. Typical porosity values range from 0.1 to 50 percent and influence how mineral properties, such as density and thermal conductivity, translate into bulk properties (Kuhlman and Matteo, 2017). Porosity, together with pore compressibility, largely determines a medium's storage capacity and controls the relationship between the bulk Darcy flux and the flow occurring within individual pores.

3.2.6 Permeability

Permeability, denoted by κ, is a measure of the ability of a porous medium to transmit fluids. It serves as the proportionality constant in Darcy's law, which describes how fluid flows through a porous material under an applied pressure gradient. In general, permeability is a second-order tensor, particularly in anisotropic media, capturing the directional dependence of flow.

In its simplest form for linear flow, Darcy's law relates the superficial fluid velocity ν to the pressure difference, fluid viscosity, and thickness of the medium:

$$\nu = \frac{\kappa}{\mu}\frac{\Delta P}{\Delta x}, \tag{3.6}$$

where ν is the superficial fluid flow velocity through the medium (i.e., the average velocity as if the fluid alone occupies the pores), κ is the permeability of the medium, μ is the dynamic viscosity of the fluid, ΔP is the applied pressure difference and Δx is the thickness of the porous medium.

Permeability is a key parameter for predicting fluid flow and solute transport in porous systems. It is typically estimated through flow experiments, in which the pore pressure of a fluid-filled rock is perturbed and the response is observed. In practice, measuring permeability is often more complex than measuring porosity, because it

depends not only on pore size but also on pore connectivity, tortuosity, and fluid–solid interactions (Kuhlman and Matteo, 2017).

3.2.7 Increment of Fluid Content

The constitutive relation for the increment of fluid content ζ, is related to volumetric strain and pore pressure by

$$\zeta = \alpha\, \varepsilon_{kk} + \frac{p}{M}. \tag{3.7}$$

Introducing Biot's modulus, M, which characterizes the compressibility of the solid–fluid system, we define it as

$$\frac{1}{M} = \frac{\alpha - \phi}{K_s} + \frac{\phi}{K_f}, \tag{3.8}$$

where ϕ is the porosity, K_s is the bulk modulus of solid grains, and K_f is the bulk modulus of the pore fluid. A high value of M corresponds to a relatively stiff system with limited compressibility (Wang, 2000).

3.2.8 Drained and Undrained Responses

Two limiting regimes of deformation govern the mechanical behavior and consolidation of porous media: drained and undrained responses (Merxhani, 2016; Biot, 1941; Coussy, 2004). These regimes are defined by the hydraulic boundary conditions imposed on the porous medium and by the characteristic time scale of loading.

Drained response. A drained response corresponds to deformation under constant pore fluid pressure. In this regime, fluid is allowed to flow freely across the boundaries of the control volume, so that no excess pore pressure develops during loading:

$$p = \text{const.}$$

Volumetric deformation is therefore accommodated primarily through fluid exchange with the surroundings, and the mechanical response is governed by the drained elastic properties of the solid skeleton.

Undrained response. An undrained response corresponds to deformation under constant fluid mass content. In this case, no fluid flux is permitted across the boundaries

of the control volume, and the increment of fluid content remains constrained:

$$\zeta = \text{const.}$$

As a consequence, mechanical loading induces changes in pore pressure, which contribute to the total stress through Biot's effective stress principle. The resulting mechanical response is stiffer than in the drained case and is characterized by the undrained elastic moduli of the porous medium.

3.3 Governing Equations of poroelasticity

The governing equations of linear poroelasticity are derived from the equilibrium equations for the porous skeleton, conservation of fluid mass, and constitutive relations coupling stress, strain, and pore pressure. The formulation presented here follows the classical theory of Biot (1941) and its modern developments in poromechanics (Coussy, 2004; Wang, 2000).

3.3.1 Constitutive relations

As in classical elasticity, the mechanical behaviour of a porous medium is described in terms of dynamic and kinematic quantities. The total stress tensor is denoted by σ_{ij}, while ε_{ij} represents the strain tensor of the porous solid skeleton. The sign convention adopted throughout this work follows that of elasticity, with tensile stresses taken as positive.

Extending the elastic framework to poroelasticity requires the introduction of an additional scalar dynamic variable, the pore fluid pressure p, together with its kinematic counterpart, the variation of fluid content ζ.

Assuming a reversible and linear (or incrementally linear) deformation process, the constitutive behaviour of a poroelastic material can be expressed in a form analogous to Hooke's law:

$$\sigma_{ij} = C_{ijkl}\,\varepsilon_{kl} - \alpha_{ij}\,p, \tag{3.9}$$

$$p = M\left(\zeta - \alpha_{ij}\,\varepsilon_{ij}\right), \tag{3.10}$$

where C_{ijkl} is the fourth-order elasticity tensor of the porous skeleton, α_{ij} is the Biot coupling tensor, and M is Biot's modulus. These quantities define the material parameters that govern the coupled mechanical and fluid response of the porous medium. The pore pressure p appears explicitly in the first relation to emphasize the role of the

coupling coefficients, while linearity is preserved through the linear dependence between p and ζ.

The symmetry of the stress and strain tensors imposes the following conditions on the constitutive coefficients:

$$C_{ijkl} = C_{jikl} = C_{ijlk} = C_{klij}, \tag{3.11}$$

$$\alpha_{ij} = \alpha_{ji}. \tag{3.12}$$

Consequently, a fully anisotropic poroelastic material has 21 independent elastic constants C_{ijkl}, six independent components of the Biot tensor α_{ij}, and one Biot modulus M, for a total of 28 independent constitutive parameters, compared with 21 for anisotropic elasticity (Cheng, 1997).

In many practical applications, the porous medium can be reasonably approximated as isotropic. This isotropic formulation is a special case of the general anisotropic relations. Here, the elastic behaviour of the solid skeleton is fully described by the Lamé parameters λ and G, while the Biot coupling tensor reduces to a scalar coefficient α. The constitutive relations simplify to

$$\sigma_{ij} = 2G\,\varepsilon_{ij} + \lambda\,\delta_{ij}\,\varepsilon_{kk} - \alpha\,\delta_{ij}\,p, \tag{3.13}$$

$$p = M\left(\zeta - \alpha\,\varepsilon_{kk}\right), \tag{3.14}$$

where ε_{kk} denotes the volumetric strain and δ_{ij} is the Kronecker delta. In this isotropic setting, the poroelastic response is governed by four independent material parameters: λ, G, α, and M (Cheng, 2016a).

3.3.2 Equilibrium equation

The mechanical equilibrium of a porous medium is described by the local stress balance, which states that the divergence of the total stress must balance the body forces acting on the material (Detournay and Cheng, 1993):

$$\sigma_{ij,j} + F_i = 0, \tag{3.15}$$

where $F_i = \rho g_i$ represents the body force per unit volume, ρ is the bulk density of the porous medium, and σ_{ij} are the components of the total stress tensor. The bulk density accounts for both the solid and fluid phases and is given by

$$\rho = (1-\phi)\rho_s + \phi\rho_f, \tag{3.16}$$

with ρ_s and ρ_f denoting the densities of the solid skeleton and the pore fluid, respectively, and ϕ being the porosity of the medium.

3.3.3 Darcy's law

Fluid transport in the pores is governed by Darcy's law, which reads:

$$q_i = -\frac{\kappa_{ij}}{\mu}\left(\frac{\partial p}{\partial x_j} - \rho_f g_j\right), \tag{3.17}$$

where q_i is the i-th component of the Darcy flux which is defined as a volumetric flow rate per unit area, κ_{ij} the intrinsic permeability tensor (reducing to $\kappa\,\delta_{ij}$ for isotropic media), μ the dynamic viscosity, p the pore pressure, ρ_f the fluid density, and g_j the j-th component of the body force vector per unit mass (e.g., gravity).

Darcy's law, together with the stress–strain and fluid content relations, provides the coupling between mechanical deformation and fluid flow in the porous medium.

3.3.4 Continuity Equation for the Fluid Phase

Considerations of mass conservation for a compressible fluid lead to the local form of the continuity equation:

$$\frac{\partial \zeta}{\partial t} + q_{i,i} = \gamma, \tag{3.18}$$

where γ is the source density, representing the rate of fluid volume injected per unit volume of the porous medium. It should be noted that this equation is in a linearized form, as variations in fluid density have been neglected.

3.4 Boundary and Initial Conditions

To ensure a poroelastic problem is well-posed, appropriate boundary and initial conditions must be specified. For the system of partial differential equations describing the coupled mechanical and fluid response, well-posed conditions guarantee the existence and uniqueness of the solution.

Mechanical Boundary Conditions

Boundary conditions for the solid skeleton can be of two types:

- **Dirichlet (displacement) conditions:** The displacement of the solid is

prescribed on a portion of the boundary:

$$\mathbf{u} = \bar{\mathbf{u}} \quad \text{on } \Gamma_u.$$

- **Neumann (traction) conditions:** The traction vector is specified on the boundary:

  $$\mathbf{t} = \boldsymbol{\sigma} \cdot \mathbf{n} = \bar{\mathbf{t}} \quad \text{on } \Gamma_t,$$

  or in component form:

  $$t_i = \sigma_{ij} n_j,$$

  where $\mathbf{n}$ is the outward unit normal vector to the boundary.

These conditions can be combined to form a mixed boundary value problem. Note that a finite domain with only Neumann conditions (tractions) is ill-posed: the solution is determined only up to arbitrary rigid body motions.

Hydraulic Boundary Conditions

For the fluid phase, boundary conditions are required for either pore pressure or flux:

- **Dirichlet (pressure) conditions:** The pore pressure is prescribed:

  $$p = \bar{p} \quad \text{on } \Gamma_p.$$

- **Neumann (flux) conditions:** The normal component of the fluid flux is prescribed:

  $$\mathbf{q} \cdot \mathbf{n} = \bar{q} \quad \text{on } \Gamma_q,$$

  or in component form:

  $$q_n = q_i n_i.$$

Initial Conditions

Initial conditions define the state of the medium at $t = 0$ and typically include:

- Initial displacement or stress field:

  $$\mathbf{u}(x, 0) = \mathbf{u}_0(x),$$

- Initial pore pressure or flux field:

$$p(x, 0) = p_0(x).$$

These initial fields must satisfy constraints such as equilibrium and compatibility. If the medium is initially in a steady, equilibrated state that satisfies the governing equations, the initial conditions can be considered as a reference, and only the perturbed response needs to be computed.

3.5 Reversible and Irreversible Responses

In continuum mechanics, the behavior of poroelastic media can be further distinguished according to whether the coupled solid-fluid response is reversible or irreversible. This classification is grounded in thermodynamic considerations, particularly the partitioning of mechanical work into stored energy versus dissipation. Reversible poroelasticity corresponds to conservative, fully recoverable processes, whereas irreversible poroelasticity includes dissipation mechanisms such as plasticity, viscoelasticity, damage, chemical reactions, or thermal conduction.

3.5.1 Reversible Poroelasticity

The phenomenon of reversible poroelasticity is characterized by materials that respond to deformation by a combination of both solid matrix and redistribution of pore fluids, both of which are completely reversible during the unloading. There is neither permanent deformation nor a dissipation of energy, and the constitutive relations are derivable out of one thermodynamic potential.

Characteristics

- The solid skeleton behaves in a purely elastic manner, and has neither plasticity nor irreversibility caused by damage.

- The evolution of the flow of fluids and the evolution of pore-pressure is reversible, without the incarceration of irreversible changes in porosity and permeability.

- No internal dissipation is generated, and the Clausius-Duhem inequality (Frémond, 2006) is identically satisfied.

- There exists a strain energy density function W such that

$$\sigma_{ij} = \frac{\partial W}{\partial \varepsilon_{ij}}, \qquad p = \frac{\partial W}{\partial \zeta}, \tag{3.19}$$

where ε_{ij} is the small-strain tensor and ζ is the variation in fluid content (Coussy, 2004; Coussy et al., 1998).

3.5.2 Irreversible Poroelasticity

Irreversible poroelasticity covers a wider range of models where dissipation is due to mechanical, thermal, chemical or rheological processes. In such cases, the unloading response fails to put the material in its original configuration, and the stress strain relation cannot be obtained by a single elastic potential. Dissipation is formally expressed through the reduced form of the Clausius-Duhem inequality Coussy (2004); Frémond (2006):

$$\mathcal{D} = \sigma_{ij}\dot{\varepsilon}_{ij} + p\dot{\zeta} - \dot{W} \geq 0, \tag{3.20}$$

where $\mathcal{D}$ denotes the rate of dissipation.

Irreversibility may arise from several sources, which can be grouped into distinct physical mechanisms.

Mechanical Irreversibility

Mechanical irreversibility occurs when the solid skeleton undergoes permanent deformation or structural deterioration. They are typically manifested in the form of plastic yielding, compaction, microcracking, continuum damage and the irreversible reorganization or collapse of the pore architecture. Constitutive frameworks that deal with this behaviour include plasticity theory, hardening variables, and damage variables; internal state variables that develop according to the suitable flow rules (Coussy, 2004; Detournay and Cheng, 1993).

Rheological Irreversibility

Time dependent material reactions, including viscoelastic relaxation, viscoplastic flow and creep, result in rheological irreversibility. Poroviscoelasticity is a sub-class of this behaviour that requires the introduction of extra internal variables or rate-dependent evolution equations in order to describe material memory and rate-dependent behaviour (Biot, 1956b; Coussy, 2004).

Hydraulic Irreversibility

Hydraulic irreversibility are the processes wherein the transport or saturation of fluids is path dependent. They include hysteresis of infiltration drainage, irreversible change in porosity or permeability, and the presence of more than one phase of fluid under confinement (Coussy, 2004; Bear, 2013).

Thermo-chemical Irreversibility

Thermal conduction and chemical reactions also introduce irreversibility. Examples include diffusion of heat to produce entropy, chemically driven swelling or dissolution and reactive transport that alters the composition of fluids or solids (Frémond, 2006; Coussy, 2004).

3.6 Extended and Coupled Forms of Poroelasticity

Classical poroelasticity provides a conceptual framework for modeling the coupled behavior of a porous solid and its interstitial fluid. However, real materials often exhibit additional physical processes that require extensions to the basic Biot formulation. These extended poroelastic models incorporate viscoelastic, thermal, chemical, or other multiphysics couplings, enabling the rigorous analysis of complex porous media. In this section, we discuss the principal extended forms and their governing equations, highlighting the physical mechanisms, constitutive relations, and typical applications.

3.6.1 Poroviscoelasticity

Poroviscoelasticity generalizes classical poroelasticity by introducing time-dependent viscoelastic behavior in the solid skeleton. Such models are essential for describing creep, stress relaxation, and other rate-dependent phenomena observed in polymers, biological tissues, and sedimentary rocks (Lakes, 2009).

The total stress in a poroviscoelastic medium can be expressed using a hereditary integral that accounts for the history-dependent response of the solid skeleton (Lakes, 2009; Cheng, 2016a),

$$\sigma_{ij}(t) = \int_0^t C_{ijkl}(t-\tau)\,\frac{d\varepsilon_{kl}(\tau)}{d\tau}\,d\tau - \alpha\, p\,\delta_{ij}, \tag{3.21}$$

where the convolution term represents the viscoelastic effective stress, $C_{ijkl}(t)$ is the time-dependent stiffness tensor, ε_{kl} is the strain tensor, α is Biot's coefficient, p is the pore fluid pressure, and δ_{ij} is the Kronecker delta.

The fluid mass balance retains its classical form,

$$\frac{\partial \zeta}{\partial t} + \nabla \cdot \mathbf{q} = 0, \tag{3.22}$$

where ζ denotes the increment of fluid content and $\mathbf{q}$ is the Darcy flux.

3.6.2 Thermoporoelasticity

Thermoporoelasticity describes the coupling between mechanical deformation, fluid flow, and temperature changes in a porous medium. Thermal effects can alter pore pressure, material strain, and fluid properties, making this framework essential in geothermal, energy, and thermo-hydro-mechanical applications (Coussy et al., 1998; Cheng, 2016b).

The constitutive and fluid mass balance equations are:

$$\sigma_{ij} = C_{ijkl}\,\varepsilon_{kl} - M\,\beta_{ij}\,\zeta - A_{ij}\,\theta, \tag{3.23}$$

$$p = M\left(-\beta_{ij}\varepsilon_{ij} + \zeta + 3\alpha_m\theta\right), \tag{3.24}$$

$$\frac{\partial \zeta}{\partial t} + q_{i,i} = 0, \tag{3.25}$$

together with the specific entropy:

$$s = A_{ij}\,\varepsilon_{ij} - 3M\,\alpha_m\,\zeta + \frac{C_\epsilon}{T_0}\,\theta. \tag{3.26}$$

Here, M is Biot's modulus, β_{ij} is the poroelastic coupling coefficient, A_{ij} is the thermal stress coefficient, $\theta = T - T_0$ is the temperature increment relative to the reference temperature T_0, and α_m is the volumetric thermal expansion coefficient.

3.6.3 Porochemoelasticity

Porochemoelasticity extends classical poroelasticity by incorporating chemical species and their coupling with the mechanical response of the porous solid. This framework enables the description of swelling, shrinkage, and material alteration driven by concentration gradients or chemical reactions, as commonly observed in biological tissues, gels, and reactive geomaterials (Cheng, 2016a,b).

The general constitutive relations for linear porochemoelasticity are written as (Ghassemi and Diek, 2003)

$$\sigma_{ij} = C_{ijkl}\,\varepsilon_{kl} - \alpha\,p\,\delta_{ij} + \sum_{\beta} \omega^{\beta}\,\mu^{\beta}\,\delta_{ij}, \tag{3.27}$$

where σ_{ij} is the total stress tensor (tension positive), ε_{ij} is the infinitesimal strain tensor, p denotes the pore pressure, and μ^β is the chemical potential of the βth fluid component. The coefficients ω^β characterize the chemo–mechanical coupling associated with chemical swelling or shrinkage.

The evolution of the pore volume fraction is given by

$$\zeta = \alpha\, \varepsilon_{kk} + Q\, p + \sum_\beta \beta^\beta\, \mu^\beta, \tag{3.28}$$

where ζ is the pore volume fraction, ε_{kk} denotes the volumetric strain, Q is the pore compressibility modulus, and β^β are porochemoelasticity coupling coefficients.

The fluid mass balance retains its classical form,

$$\frac{\partial \zeta}{\partial t} + \nabla \cdot \mathbf{q} = 0, \tag{3.29}$$

while the transport and evolution of chemical species are governed by appropriate species balance and reaction equations, which are not detailed here.

3.6.4 Other Generalized Forms

In addition to classical poroelasticity and chemomechanical generalizations, many generalized theories have been offered to include additional physical effects commonly found in porous media. These models are an extension of the canonical poroelastic formulation, whereby other couplings are added but the overall structure of the governing balance equations remains the same.

- **Porothermoviscoelasticity:** Combines poroelasticity with viscoelastic deformation of the solid skeleton and thermal effects. The model is especially relevant to situations with high temperatures (like geothermal reservoirs and nuclear waste repositories) where temporal-dependent mechanical behavior and thermally-controlled expansion are of the greatest concern (Coussy, 2004; Wang, 2000; Schanz, 2012).

- **Electro-poroelasticity:** Describes the interaction of the electric fields, mechanical deformation, and fluid flow in porous materials. These models are typically used in the analysis of electrokinetic processes, electro-osmosis, and geophysical exploration techniques (Pride, 1994; Coussy, 2004).

- **Poroelastoplasticity:** Generalizes the poroelastic skeleton to include irreversible deformation of the porous skeleton through plastic actions. This set of models finds extensive application in the fields of soil mechanics, reservoir compaction,

and fault mechanics, in which yielding and permanent deformation of the solid matrix are inevitable (Biot, 1941; Tang et al., 2015).

3.6.5 Conclusion

In this chapter, poroelasticity and its major extensions have been treated in a broad way with the underlying physical processes in the mutual interplay of deformation, fluid flow, and other field variables in porous media. Since the fundamental notions and principles of the poroelastic theory, the classical balance equations and constitutive equations were presented to provide the equations that constitute the basis of poromechanics.

Various models and types of poroelastic models were later presented with the importance of effective stress, fluid content, and material compressibility being highlighted in explaining the behavior of fluid-saturated porous solids. The flexibilities of the classical theory were then described to handle time dependence of behavior, thermal effects, chemical interactions and other multiphysical couplings to show the convenience and extensivity of poroelastic theory in geomechanics and biomechanics, and energy applications.

Generally, poroelasticity offers a coherent and rational structure to the modeling of coupled hydro-mechanical processes in porous media. The generalized theories mentioned in this chapter allow one to add more physical effects maintaining the essential framework of the governing equations to have a solid framework of theoretical development to come in the subsequent chapters.

Chapter 4

Reversible Poroelasticity under Incompressibility of Solid in a Poroelastic Material

4.1 Introduction

This chapter examines the reversible action of incompressible anisotropic poroelastic media in pursuance of the theoretical framework encompassed by Biot. We consider the behavior of solid deformation and fluid flow assuming that the system is perfectly drained. A Hamiltonian-based Stroh formalism is adopted to systematically describe the mechanics and conservation properties.

The findings discussed here have been published in the Journal "Mathematics" Arshad and Tibullo (2025).

4.2 Basic Assumptions and Governing Equations

Consider a two-dimensional linear anisotropic poroelastic material. We begin with the definition of seepage displacement, which represents the fluid motion relative to the solid given by

$$\mathbf{w} = f(\mathbf{U} - \mathbf{u}), \tag{4.1}$$

where $\mathbf{u}$ and $\mathbf{U}$ are the displacements of the solid and fluid phases, respectively, and f denotes the effective porosity of the medium, which is not uniform in general.

The volumetric strains of solid and fluid are assumed to be

$$e = \operatorname{div} \mathbf{u}, \quad \zeta = -\operatorname{div} \mathbf{w}.$$

For an incompressible solid skeleton, the additional constraint applies (Fu, 2007):

$$e = \operatorname{div} \mathbf{u} = 0. \tag{4.2}$$

Furthermore, the volumetric coupling between solid and fluid phases yields (Biot, 1962):

$$-\zeta = (\mathbf{U} - \mathbf{u}) \cdot \operatorname{grad} f + f\,\epsilon, \quad \epsilon = \operatorname{div} \mathbf{U}. \tag{4.3}$$

In the case of uniform porosity and incompressible solid, it becomes

$$\epsilon = -f^{-1}\zeta.$$

The total stress tensor $\mathbf{T}$ is expressed as

$$\mathbf{T} = \boldsymbol{\sigma} - \alpha p_f \mathbf{1} - p\mathbf{1}, \tag{4.4}$$

where $\boldsymbol{\sigma}$ is the stress in the solid skeleton, p_f is the fluid pressure, α is the effective stress coefficient, and p is a Lagrange multiplier enforcing solid incompressibility (Berger et al., 1999; Abousleiman and Ekbote, 2002).

The solid stress for an anisotropic material is

$$\boldsymbol{\sigma} = \tilde{\mathbb{C}}\mathbf{E} \tag{4.5}$$

where $\tilde{\mathbb{C}} = \tilde{c}_{ijkl}$ denote the rank-4 tensor of elastic moduli and $\mathbf{E} = \frac{1}{2}(\operatorname{grad} \mathbf{u} + \operatorname{grad} \mathbf{u}^T)$ such that $e = \operatorname{tr} \mathbf{E} = 0$.

In case of incompressible solid skeleton and compressible fluid flow, the fluid pressure is related to the fluid volumetric strain by

$$p_f = m\zeta, \tag{4.6}$$

where m is the compressibility modulus of the fluid. As already discuued in chapter 3 that for incompressible solids, $\alpha = r/m = 1$ (Schanz, 2009), with r denoting the volumetric coupling between solid and fluid.

Substituting the values of (4.5) and (4.6) into (4.4), the total stress simplifies to

$$\mathbf{T} = \tilde{\mathbb{C}}\mathbf{E} - r\zeta\mathbf{1} - p\mathbf{1}. \tag{4.7}$$

4.2.1 Fundamental force vectors

We present the notion of a fundamental force vector, which naturally occurs in the theoretical framework of incompressible anisotropic solids (Nobili, 2024; Fu, 2007). It

is defined as The traction vectors acting on planes normal to the coordinate directions are defined by

$$\mathbf{t}_\alpha = \mathbf{T}\mathbf{e}_\alpha, \qquad \alpha = 1, 2,$$

or, in component form,

$$(t_\alpha)_i = T_{ij}\delta_{j\alpha}.$$

The constitutive relation for the stress tensor is taken as

$$T_{ij} = c_{ijkl}\varepsilon_{kl} - r\,\zeta\,\delta_{ij} - p\,\delta_{ij},$$

where ε_{kl} denotes the infinitesimal strain tensor. Substituting into the definition of $\mathbf{t}_1$ gives

$$\begin{aligned}(t_1)_i &= T_{ij}\delta_{j1} \\ &= \big(c_{ijkl}\varepsilon_{kl} - r\,\zeta\,\delta_{ij} - p\,\delta_{ij}\big)\delta_{j1} \\ &= c_{i1kl}\varepsilon_{kl} - r\,\zeta\,\delta_{i1} - p\,\delta_{i1}.\end{aligned}$$

Expanding the strain components,

$$\begin{aligned}(t_1)_i &= c_{i1k1}\varepsilon_{k1} + c_{i1k2}\varepsilon_{k2} - r\,\zeta\,\delta_{i1} - p\,\delta_{i1} \\ &= c_{i1k1}\left(\frac{u_{k,1} + u_{1,k}}{2}\right) + c_{i1k2}\left(\frac{u_{k,2} + u_{2,k}}{2}\right) - r\,\zeta\,\delta_{i1} - p\,\delta_{i1}.\end{aligned}$$

Invoking the minor symmetry of the elasticity tensor, the above reduces to

$$(t_1)_i = c_{i1k1}u_{k,1} + c_{i1k2}u_{k,2} - r\,\zeta\,\delta_{i1} - p\,\delta_{i1}.$$

Similarly, for $\mathbf{t}_2$, we have

$$(t_2)_i = c_{i2k1}u_{k,1} + c_{i2k2}u_{k,2} - r\,\zeta\,\delta_{i2} - p\,\delta_{i2}.$$

Introducing the matrices

$$Q_{ik} = c_{i1k1}, \qquad R_{ik} = c_{i1k2}, \qquad T_{ik} = c_{i2k2},$$

the traction vectors may be written compactly as

$$\begin{aligned}\mathbf{t}_1 &= Q\,\mathbf{u}_{,1} + R\,\mathbf{u}_{,2} - r\,\zeta\,\mathbf{e}_1 - p\,\mathbf{e}_1, \\ \mathbf{t}_2 &= R^T\mathbf{u}_{,1} + T\,\mathbf{u}_{,2} - r\,\zeta\,\mathbf{e}_2 - p\,\mathbf{e}_2.\end{aligned} \tag{4.8}$$

where the Stroh-type matrices are specified by

$$Q_{ij} = c_{i1j1}, \qquad R_{ij} = c_{i1j2}, \qquad T_{ij} = c_{i2j2},$$

and a subscript after a comma denotes partial differentiation, e.g., $\mathbf{u}_{,1} = \partial\mathbf{u}/\partial x_1$.

Remark 5. *Both the matrices, Q and T, are symmetric as well as positive definite which is a condition that corresponds to the fact that the strain energy function must remain strictly positive at all admissible deformations. The scalar field p appears as a Lagrange multiplier enforcing the incompressibility constraint, thereby permitting the displacement gradients $\mathbf{u}_{,1}$ and $\mathbf{u}_{,2}$ to vary independently in the variational setting.*

The assumption of reversibility, as originally formulated by Biot and Willis (1957), is associated with the stored elastic potential W, which characterizes the recoverable part of the material response. Although the formal definition of W will be introduced in a later section, it is noted here that this property permits identification of the coupling coefficient appearing in the third terms of equation (4.8) with the corresponding expression in equation (4.6). In addition, the strong ellipticity principle stipulates that both Q and T are positive definite as well as the material parameter m takes strictly positive values, which in turn guarantees that the constitutive model is stable.

To simplify the analysis, here the cross-coupling contribution is estimated based on the assumption of isotropy. A more general and rigorous treatment that incorporates transverse anisotropy is provided in the seminal work of Biot (1955). Finally, we restrict the analysis to a two-dimensional setting by assuming invariance in the out-of-plane direction, i.e.,

$$\frac{\partial}{\partial x_3}() = 0,$$

which eliminates dependence on the x_3-coordinate while retaining the essential coupling mechanisms of the model.

4.2.2 Equilibrium Equations

The equilibrium equations governing the present problem can be expressed as

$$\mathbf{t}_{1,1} + \mathbf{t}_{2,2} = 0 \tag{4.9}$$

$$\operatorname{grad} p_f = 0 \implies \operatorname{grad}(m\xi) = 0 \tag{4.10}$$

The first of these corresponds to the mechanical equilibrium equation under the assumption of vanishing body forces. The second condition represents the hydraulic equilibrium, derived as the stationary form of Darcy's law.

4.2.3 Boundary conditions

On the boundary B, the system is subjected to the following conditions:

$$T\mathbf{n} = t_0, \quad p_f = p_{f_0} \quad \text{for} \quad x \in \partial B \tag{4.11}$$

Here, the traction vector is prescribed as t_0, while the fluid pressure is fixed to p_{f_0}. These boundary conditions allow the problem to be fully defined in both its mechanical and hydraulic aspects.

4.2.4 Lagrangian Formulation

We now provide a formal definition of the stored potential energy W, which underlies the Lagrangian description of the coupled solid–fluid system. This potential encapsulates both the elastic strain energy of the solid skeleton and the contribution associated with the compressibility of the pore fluid. It may be expressed as

$$\begin{aligned} W &= \frac{1}{2}\left(T \cdot \operatorname{grad}\mathbf{u} + p_f\,\xi\right) \\ &= \frac{1}{2}\left((\boldsymbol{\sigma} - \alpha p_f \mathbf{I}) \cdot \operatorname{grad}\mathbf{u} + p_f\,\xi\right) \\ &= \frac{1}{2}\left(\boldsymbol{\sigma}\cdot\mathbf{E} - \alpha p_f e + p_f\,\xi\right), \quad \text{since } \mathbf{1}\cdot\mathbf{E} = e, \\ &= \frac{1}{2}\left(\boldsymbol{\sigma}\cdot\mathbf{E} + m\zeta^2\right), \quad \text{assuming } e = 0. \end{aligned} \tag{4.12}$$

The stored energy functional is, therefore, divided into two different components: the elastic contribution that is attributed to deformation of the solid matrix, and the compressibility energy of the pore fluid. This formulation serves as the foundation for the subsequent Lagrangian representation of the governing equations.

To proceed further, we define the total energy functional of the body B which is then given by

$$\mathcal{L} = \int_B W\,dV - \int_{\partial B} (\mathbf{t}_0 \cdot \mathbf{u} - p_{f_0}\mathbf{n}\cdot\mathbf{w}) dS. \tag{4.13}$$

The first integral represents the internal energy stored in the body, while the boundary integral accounts for the external mechanical and hydraulic work.

From (4.13)

$$\mathbf{t}_0 \cdot \mathbf{u} - p_f \mathbf{n}\cdot\mathbf{w} = T\mathbf{n}\cdot\mathbf{u} - p_f(\mathbf{n}\cdot\mathbf{w}) = (\mathbf{T}\mathbf{u} - p_f\mathbf{w})\cdot\mathbf{n}$$

Applying the divergence theorem

$$\int_{\partial B}(\mathbf{t}_0 \cdot \mathbf{u} - p_f \mathbf{n} \cdot \mathbf{w})dS = \int_B \nabla \cdot (T - p_f \mathbf{w})dV$$

thus

$$L = \int_B \left(W - (\nabla \cdot (\mathbf{T} - p_f \mathbf{w}))\right) dV$$

expanding

$$\nabla \cdot (\mathbf{T}\mathbf{u}) = (\nabla \cdot \mathbf{T})\mathbf{u} + \mathbf{T} : \nabla \mathbf{u}$$
$$\nabla \cdot (p_f \mathbf{w}) = \nabla p_f \cdot \mathbf{w} + p_f \nabla \cdot \mathbf{w}$$

so

$$\begin{aligned} L &= \int_B \left(\frac{1}{2}\mathbf{T} \cdot \nabla \mathbf{u} + \frac{1}{2}p_f \xi - ((\nabla \cdot \mathbf{T})\mathbf{u} + \mathbf{T} : \nabla \mathbf{u} - \nabla p_f \cdot \mathbf{w} - p_f \nabla \cdot \mathbf{w})\right) dV \\ &= \int_B \left(\frac{1}{2}\mathbf{T} \cdot \nabla \mathbf{u} + \frac{1}{2}p_f \xi - \nabla \cdot \mathbf{T}\mathbf{u} - \mathbf{T} \cdot \nabla \mathbf{u} + \nabla p_f \cdot \mathbf{w} - p_f \xi\right) dV \\ &= -\int_B \left(\frac{1}{2}\mathbf{T} \cdot \nabla \mathbf{u} + \frac{1}{2}p_f \xi\right) dV \quad \because \nabla \cdot \mathbf{T} = 0, \quad and \quad \nabla p_f = 0 \end{aligned}$$

the functional may be reformulated as

$$\mathcal{L} = -\int_B L \, dV \tag{4.14}$$

where L denotes the Lagrangian density, defined as

$$L(\mathbf{u}_{,1}, \mathbf{u}_{,2}, \mathbf{w}_{,1}, \mathbf{w}_{,2}) = \frac{1}{2}\mathbf{T} \cdot \operatorname{grad} u + \frac{1}{2}p_f \xi - (r\xi + \tfrac{1}{2}p)\mathbf{u}_{i,i} \tag{4.15}$$

Remark 6. *The final term is introduced to offset the constraint of solid incompressibility while still accounting for fluid compressibility, which is a key feature of poroelastic media.*

Noting that

$$\mathbf{T} \cdot \operatorname{grad} u = \mathbf{t}_1 \cdot \mathbf{u}_1 + \mathbf{t}_1 \cdot \mathbf{u}_2 \tag{4.16}$$

the expression simplifies further by using the definition of fundamental force vectors to

$$\mathbf{T} \cdot \operatorname{grad} u = \mathbf{u}_{,1} \cdot Q\mathbf{u}_{,1} + 2\mathbf{u}_{,1} R\mathbf{u}_{,2} + \mathbf{u}_{,2} \cdot T\mathbf{u}_{,2} \tag{4.17}$$

Substituting this into the Lagrangian density yields

$$L = \frac{1}{2}\mathbf{u}_{,1} \cdot Q\mathbf{u}_{,1} + \mathbf{u}_{,1}R\mathbf{u}_{,2} + \frac{1}{2}\mathbf{u}_{,2} \cdot T\mathbf{u}_{,2} + \frac{1}{2}m\xi^2 - p\mathbf{u}_{i,i} - r\xi\mathbf{u}_{i,i} \tag{4.18}$$

Because the formulation admits the Stroh formalism, the density may be further written as

$$L = \frac{1}{2}\mathbf{u}_{,1} \cdot Q\mathbf{u}_{,1} + \mathbf{u}_{,1}R\mathbf{u}_{,2} + \frac{1}{2}\mathbf{u}_{,2} \cdot T\mathbf{u}_{,2} + \frac{1}{2}m(\mathbf{e}_1 \cdot \mathbf{w}_{,1} + \mathbf{e}_2 + \mathbf{w}_{,2})^2 - p\mathbf{u}_{i,i} - r\xi\mathbf{u}_{i,i} \tag{4.19}$$

Once the Lagrangian density is obtained, the Euler-Lagrange equations can be derived directly. They are expressed as

$$\frac{d}{dx_1}\frac{\partial L}{\partial u_{,1}} + \frac{d}{dx_2}\frac{\partial L}{\partial u_{,2}} = 0 \tag{4.20a}$$

$$\frac{d}{dx_1}\frac{\partial L}{\partial w_{,1}} + \frac{d}{dx_2}\frac{\partial L}{\partial w_{,2}} = 0 \tag{4.20b}$$

Upon simplification, this system reduces to

$$\frac{1}{2}(2Q\mathbf{u}_{,1} + R\mathbf{u}_{,2} - p\mathbf{e}_1 - r\xi\mathbf{e}_1)_{,1} + (\mathbf{u}_{,1} \cdot R + T\mathbf{u}_{,2} - p\mathbf{e}_2 - r\xi\mathbf{e}_2)_{,2} = 0 \tag{4.21a}$$

$$(m\xi)_{,1}\mathbf{e}_1 + (m\xi)_{,2}\mathbf{e}_2 = 0 \tag{4.21b}$$

These equations represent the coupled equilibrium conditions for the solid and fluid phases, expressed in variational form. They form the basis for further analysis using the Stroh formalism, which corresponds to equations of equilibrium (4.9) and (4.10) respectively.

4.3 Hamiltonian Formulation

When extending the Hamiltonian formalism to incompressible or nearly incompressible materials, an additional difficulty arises: the stiffness matrix corresponding to volumetric deformation becomes singular, since such materials cannot sustain hydrostatic stresses. To overcome this issue, we introduce regularization through modified material tensors, defined as

$$\hat{Q} = Q + \lambda\,\mathbf{e}_1 \otimes \mathbf{e}_1, \tag{4.22}$$

$$\hat{R} = R + \lambda\,\mathbf{e}_1 \otimes \mathbf{e}_2, \tag{4.23}$$

$$\hat{T} = T + \lambda\,\mathbf{e}_2 \otimes \mathbf{e}_2, \tag{4.24}$$

where $\lambda > 0$ is chosen sufficiently large to prevent loss of positive definiteness.

Remark: This modification guarantees that $\hat{T}$ remains positive definite, ensuring both numerical stability and the invertibility of the stiffness operators (Fu, 2007). In practical terms, the parameter λ acts as a volumetric penalization that regularizes the system while preserving the physical integrity of the incompressible formulation.

With these adjustments, the generalized force vectors given earlier in equation (4.8) can now be written as

$$\mathbf{t}_1 = (\hat{Q} - \lambda \mathbf{e}_1 \otimes \mathbf{e}_1)\mathbf{u}_{,1} + (\hat{R} - \lambda \mathbf{e}_1 \otimes \mathbf{e}_2)\dot{\mathbf{u}} - r\xi \mathbf{e}_1 - p\mathbf{e}_1 \tag{4.25a}$$

$$\mathbf{t}_2 = (\hat{R} - \lambda \mathbf{e}_1 \otimes \mathbf{e}_2)\mathbf{u}_{,1} + (\hat{T} - \lambda \mathbf{e}_2 \otimes \mathbf{e}_2)\dot{\mathbf{u}} - r\xi \mathbf{e}_2 - p\mathbf{e}_2 \tag{4.25b}$$

Solving for $\mathbf{t}_1$

$$\begin{aligned}
\mathbf{t}_1 &= \hat{Q}\mathbf{u}_{,1} - \lambda(\mathbf{e}_1 \otimes \mathbf{e}_1)\mathbf{u}_{,1} + (\hat{R} - \lambda \mathbf{e}_1 \otimes \mathbf{e}_2)\mathbf{u}_{,2} - r\xi \mathbf{e}_1 - p\mathbf{e}_1 \\
&= \hat{Q}\mathbf{u}_{,1} - \lambda(\mathbf{e}_1 \cdot \mathbf{u}_{,1})\mathbf{e}_1 + \hat{R}\dot{\mathbf{u}} - \lambda(\mathbf{e}_2 \cdot \mathbf{u}_{,2})\mathbf{e}_2 - r\xi \mathbf{e}_1 - p\mathbf{e}_1 \\
&= \hat{Q}\mathbf{u}_{,1} - \lambda \mathbf{u}_{1,1}\mathbf{e}_1 + \hat{R}\dot{\mathbf{u}} - \lambda \mathbf{u}_{2,2}\mathbf{e}_1 - r\xi \mathbf{e}_1 - p\mathbf{e}_1 \\
&= \hat{Q}\mathbf{u}_{,1} - \lambda(\mathbf{u}_{1,1} + \mathbf{u}_{2,2})\mathbf{e}_1 + \hat{R}\dot{\mathbf{u}} - r\xi \mathbf{e}_1 - p\mathbf{e}_1 \\
&= \hat{Q}\mathbf{u}_{,1} + \hat{R}\dot{\mathbf{u}} - r\xi \mathbf{e}_1 - p\mathbf{e}_1 \quad \because e = 0
\end{aligned}$$

Similarly, evaluating for $\mathbf{t}_2$

$$\begin{aligned}
\mathbf{t}_2 &= \hat{R}^T\mathbf{u}_{,1} - \lambda(\mathbf{e}_1 \otimes \mathbf{e}_2)\mathbf{u}_{,1} + (\hat{T} - \lambda \mathbf{e}_2 \otimes \mathbf{e}_2)\mathbf{u}_{,2} - r\xi \mathbf{e}_2 - p\mathbf{e}_2 \\
&= \hat{R}^T\mathbf{u}_{,1} - \lambda(\mathbf{e}_1 \cdot \mathbf{u}_{,1})\mathbf{e}_2 - \hat{T}\dot{\mathbf{u}} - \lambda(\mathbf{e}_2 \cdot \mathbf{u}_{,2})\mathbf{e}_2 - r\xi \mathbf{e}_2 - p\mathbf{e}_2 \\
&= \hat{R}^T\mathbf{u}_{,1} - \lambda \mathbf{u}_{1,1}\mathbf{e}_1 + \hat{T}\dot{\mathbf{u}} - \lambda \mathbf{u}_{2,2}\mathbf{e}_2 - r\xi \mathbf{e}_2 - p\mathbf{e}_2 \\
&= \hat{R}^T\mathbf{u}_{,1} - \lambda(\mathbf{u}_{1,1} + \mathbf{u}_{2,2})\mathbf{e}_2 + \hat{T}\dot{\mathbf{u}} - r\xi \mathbf{e}_2 - p\mathbf{e}_2 \\
&= \hat{R}^T\mathbf{u}_{,1} + \hat{T}\dot{\mathbf{u}} - r\xi \mathbf{e}_2 - p\mathbf{e}_2 \quad \because e = 0
\end{aligned}$$

gives

$$\mathbf{t}_2 = \hat{R}^T\mathbf{u}_{,1} + \hat{T}\,\dot{\mathbf{u}} - r\xi \mathbf{e}_2 - p\mathbf{e}_2,$$

so finally we get the fundamental force vectors in the form

$$\mathbf{t}_1 = \hat{Q}\,\mathbf{u}_{,1} + \hat{R}\,\dot{\mathbf{u}} - \frac{r}{m}p_f\mathbf{e}_1 - p\mathbf{e}_1, \tag{4.26a}$$

$$\mathbf{t}_2 = \hat{R}^T\mathbf{u}_{,1} + \hat{T}\,\dot{\mathbf{u}} - \frac{r}{m}p_f\mathbf{e}_2 - p\mathbf{e}_2, \tag{4.26b}$$

here, the additional terms proportional to p and p_f introduce the hydrostatic and

pore-fluid contributions, respectively.

4.3.1 Lagrangian Density in Terms of Transform Matrices

Solving the Lagrangian density with the modified matrices, we obtain

$$\begin{aligned}
L &= \frac{1}{2}\mathbf{u}_{,1}\cdot(\hat{Q}-\lambda\mathbf{e}_1\otimes\mathbf{e}_1)\mathbf{u}_{,1}+\mathbf{u}_{,1}\cdot(\hat{R}+\lambda\mathbf{e}_1\otimes\mathbf{e}_2)\dot{\mathbf{u}}\\
&\quad+\frac{1}{2}\dot{\mathbf{u}}\cdot(\hat{T}+\lambda\mathbf{e}_2\otimes\mathbf{e}_2)\dot{\mathbf{u}}+\frac{1}{2}m\xi^2-p\,\mathbf{u}_{i,i}-r\zeta\mathbf{u}_{i,i}\\
&=\frac{1}{2}\mathbf{u}_{,1}\cdot\hat{Q}\mathbf{u}_{,1}-\frac{\lambda}{2}(\mathbf{u}_{,1}\cdot\mathbf{e}_1)^2+\mathbf{u}_{,1}\cdot\hat{R}\dot{\mathbf{u}}+\lambda(\mathbf{u}_{,1}\cdot\mathbf{e}_1)(\dot{\mathbf{u}}\cdot\mathbf{e}_2)\\
&\quad+\frac{1}{2}\dot{\mathbf{u}}\cdot\hat{T}\dot{\mathbf{u}}+\frac{\lambda}{2}(\dot{\mathbf{u}}\cdot\mathbf{e}_2)^2+\frac{1}{2}m\xi^2-p\,\mathbf{u}_{i,i}-r\zeta\mathbf{u}_{i,i}\\
&=\frac{1}{2}\mathbf{u}_{,1}\cdot\hat{Q}\mathbf{u}_{,1}+\mathbf{u}_{,1}\cdot\hat{R}\dot{\mathbf{u}}+\frac{1}{2}\dot{\mathbf{u}}\cdot\hat{T}\dot{\mathbf{u}}+\frac{1}{2}m\xi^2-p\,\mathbf{u}_{i,i}\\
&\quad+\frac{\lambda}{2}\left[(\dot{\mathbf{u}}\cdot\mathbf{e}_2-\mathbf{u}_{,1}\cdot\mathbf{e}_1)^2\right]-r\zeta\mathbf{u}_{i,i}.
\end{aligned}$$

Thus the resulting Lagrangian density associated with this formulation is obtained as

$$\begin{aligned}
L &= \tfrac{1}{2}\,\mathbf{u}_{,1}\cdot\hat{Q}\,\mathbf{u}_{,1}+\mathbf{u}_{,1}\cdot\hat{R}\,\dot{\mathbf{u}}+\tfrac{1}{2}\,\dot{\mathbf{u}}\cdot\hat{T}\,\dot{\mathbf{u}}\\
&\quad+\tfrac{1}{2}m\big(\mathbf{e}_1\cdot\mathbf{w}_{,1}+\mathbf{e}_2\cdot\mathbf{w}_{,2}\big)^2-p\,\mathbf{u}_{i,i}-r\zeta\mathbf{u}_{i,i}.
\end{aligned}\tag{4.27}$$

Remark 7. *The expression above combines the elastic-inertial energy of the displacement field* $\mathbf{u}$ *(first three terms) with the contribution of the auxiliary field* $\mathbf{w}$ *(fourth term), while the final terms enforce incompressibility through the Lagrange multiplier* p *and account for pore-fluid effects through the parameter* $r\zeta$*. The introduction of* λ *thus provides a mathematically stable and physically consistent extension of the Hamiltonian formalism to incompressible materials.*

4.3.2 Conjugate Momenta

Subsequently, the conjugate momenta are obtained as

$$\mathbf{p}_1=\frac{\partial L}{\partial\dot{\mathbf{u}}}=\mathbf{u}_{,1}\hat{R}+\hat{T}\dot{\mathbf{u}}-p\mathbf{e}_2-r\zeta\mathbf{e}_2=\mathbf{t}_2 \tag{4.28}$$

$$\mathbf{p}_2=\frac{\partial L}{\partial\dot{\mathbf{w}}}=-p_f\mathbf{e}_2 \tag{4.29}$$

where

$$p_f = \zeta m$$

solving (4.26b) for $\dot{\mathbf{u}}$ leads to

$$\dot{\mathbf{u}} = \hat{T}^{-1}(\mathbf{t}_2 - \hat{R}^T\mathbf{u}_{,1} + r\frac{p_f}{m}\mathbf{e}_2 + p\mathbf{e}_2) \tag{4.30}$$

Via the scalar multiplication by $\mathbf{e}_2$ yields

$$\dot{\mathbf{u}} \cdot \mathbf{e}_2 = \hat{T}^{-1}(\mathbf{t}_2 - \hat{R}^T\mathbf{u}_{,1} + \frac{r}{m}p_f\mathbf{e}_2 + p\mathbf{e}_2) \cdot \mathbf{e}_2$$

Using $\mathbf{u}_{,1} \cdot \mathbf{e}_1 + \mathbf{u}_{,2} \cdot \mathbf{e}_2 = 0$ and eliminating p as in Fu (2007):

$$p = \zeta_1(\hat{T}^{-1}(\mathbf{t}_2 - \hat{R}^T\mathbf{u}_{,1}) + r\zeta - \mathbf{u}_{,1} \cdot \mathbf{e}_1) \tag{4.31}$$

where

$$\zeta_1 = \frac{1}{\mathbf{e}_2 \cdot \hat{T}^{-1}\mathbf{e}_2} \tag{4.32}$$

Reflecting on $\zeta = -\operatorname{div}\mathbf{w}$, we have

$$-\dot{\mathbf{w}} \cdot \mathbf{e}_2 = \mathbf{w}_{,\mathbf{1}} \cdot \mathbf{e}_1 + \frac{p_f}{m} \tag{4.33}$$

In this framework, we define the Hamiltonian density.

$$H = \mathbf{p}_1 \cdot \dot{\mathbf{u}} + \mathbf{p}_2 \cdot \dot{\mathbf{w}} - L \tag{4.34}$$

$$\begin{aligned} &= \mathbf{t}_2 \cdot \dot{\mathbf{u}} - \mathbf{p}_2 \cdot \left(\mathbf{w}_{,1} \cdot \mathbf{e}_1 + \frac{p_f}{m}\right) - \Big(\frac{1}{2}\mathbf{u}_{,1}\hat{Q}\mathbf{u}_{,1} + \mathbf{u}_{,1}\hat{R}\dot{\mathbf{u}} + \frac{1}{2}\dot{\mathbf{u}} \cdot \hat{T}\dot{\mathbf{u}} + \\ &+ \frac{1}{2}m(\mathbf{e}_1 \cdot \mathbf{w}_{,1} + \mathbf{e}_2 \cdot \mathbf{w}_{,2})^2 - p\mathbf{u}_{i,i} - r\zeta\mathbf{u}_{i,i}\Big) \end{aligned} \tag{4.35}$$

or

$$\begin{aligned} H &= \mathbf{t}_2\dot{\mathbf{u}} + p_f(\mathbf{w}_{,1} \cdot \mathbf{e}_1 + \frac{p_f}{m}) - (\tfrac{1}{2}\mathbf{u}_{,1}\hat{Q}\mathbf{u}_{,1} + \mathbf{u}_{,1}\hat{R}\dot{\mathbf{u}} + \tfrac{1}{2}\dot{\mathbf{u}}\hat{T}\dot{\mathbf{u}} \\ &+ \tfrac{1}{2}m\xi^2 - p\mathbf{u}_{i,i} + \tfrac{1}{2}r\xi(\mathbf{u}_{,1} \cdot \mathbf{e}_1 + \dot{\mathbf{u}} \cdot \mathbf{e}_2)) \\ &= \dot{\mathbf{u}}(\mathbf{t}_2 - \mathbf{u}_{,1}\hat{R} + p\mathbf{e}_2 - \frac{1}{2}r\xi e_2 - \frac{1}{2}\dot{\mathbf{u}}(\mathbf{t}_2 - \hat{R}^T\mathbf{u}_{,1} + r\xi\mathbf{e}_2 + p\mathbf{e}_2) - \\ &- p_f(\mathbf{w}_{,1} \cdot \mathbf{e}_1 + \frac{p_f}{m}) - \frac{1}{2}\mathbf{u}_{,1}\hat{Q}\mathbf{u}_{,1} - \frac{1}{2}m\xi^2 + p(\mathbf{u}_{,1} \cdot \mathbf{e}_1) - \frac{1}{2}r\xi\mathbf{u}_{,1}\mathbf{e}_1 \end{aligned}$$

by further simplifying, we obtain

$$H = \frac{1}{2}(\mathbf{t}_2 - \hat{R}\mathbf{u}_{,1} + p\mathbf{e}_2 + r\zeta\mathbf{e}_2)\hat{T}^{-1}(\mathbf{t}_2 - \hat{R}\mathbf{u}_{,1} + p\mathbf{e}_2 + r\zeta\mathbf{e}_2) - \frac{1}{2}\mathbf{u}_{,1}Q\mathbf{u}_{,1} - \\ - \frac{1}{2}m\zeta^2 - p\mathbf{u}_{,1}\cdot\mathbf{e}_1 + r\zeta\mathbf{u}_{,1}\mathbf{e}_1 - p_f\left(\mathbf{w}_{,1}\cdot\mathbf{e}_1 + \frac{p_f}{m}\right) \tag{4.36}$$

4.3.3 Canonical Equations

Canonical equations are generally grouped into two complementary sets, typically written in the compact form

$$\dot{\mathbf{q}} = \frac{\partial H}{\partial \mathbf{p}}, \qquad \dot{\mathbf{p}} = -\frac{\partial H}{\partial \mathbf{q}}, \tag{4.37}$$

where $\mathbf{q}$ and $\mathbf{p}$ denote the generalized coordinates and their conjugate momenta, respectively.

The first set describes the evolution of the kinematic variables. For the present system, this yields

$$\dot{\mathbf{u}} = \frac{\partial H}{\partial \mathbf{t}_2} = \hat{T}^{-1}\Big(\mathbf{t}_2 - \hat{R}\mathbf{u}_{,1} + p\mathbf{e}_2 + r\zeta\mathbf{e}_2\Big) \tag{4.38}$$

together with

$$\begin{aligned}\dot{\mathbf{w}}\cdot\mathbf{e}_2 &= -\frac{\partial H}{\partial p_f} \\ &= -\left(\tfrac{r}{m}\mathbf{e}_2\cdot\hat{T}^{-1}\left(\mathbf{t}_2 - \hat{R}^T\mathbf{u}_{,1} + \tfrac{r}{m}p_f\mathbf{e}_2 + p\mathbf{e}_2\right) - \tfrac{p_f}{m} + \tfrac{r}{m}\mathbf{u}_{,1}\cdot\mathbf{e}_1\right) \\ &= -\left(\mathbf{w}_{,1}\cdot\mathbf{e}_1 + \tfrac{p_f}{m}\right),\end{aligned} \tag{4.39}$$

which correspond directly to Eqs. (4.30) and (4.33).

The second set of canonical equations yields the equilibrium relations. In particular,

$$\dot{\mathbf{t}}_2 = \frac{\partial H}{\partial \mathbf{u}} = -\left[\hat{R}\hat{T}^{-1}\left(\mathbf{t}_2 - \hat{R}^T\mathbf{u}_{,1} + \tfrac{r}{m}p_f\mathbf{e}_2 + p\mathbf{e}_2\right) + \hat{Q}\mathbf{u}_{,1} - p\mathbf{e}_1 - \tfrac{r}{m}p_f\mathbf{e}_1\right]_{,1}, \tag{4.40}$$

which recovers Eq. (4.30). Here, the factor $\hat{T}^{-1}$ applied to the bracketed term corresponds to $\dot{\mathbf{u}}$, and, in combination with the first relation in (4.8), reduces to Eq. (4.9).

Finally, the canonical equation for the fluid variable reads

$$-\dot{p}_f\mathbf{e}_2 = -\frac{\partial H}{\partial \mathbf{w}} = (p_f\mathbf{e}_1)_{,1}, \tag{4.41}$$

which immediately recovers the corresponding balance relation in (4.10).

4.4 Conservation of the Hamiltonian Density

To assess whether the Hamiltonian density exhibits a conservative character, one must analyze its algebraic structure together with its dependence on the underlying field variables. The Hamiltonian density H can be written as

$$H = \frac{1}{2}\left(\mathbf{t}_2 - \hat{R}\mathbf{u}_{,1} + p\mathbf{e}_2 + r\zeta\mathbf{e}_2\right)\cdot\hat{T}^{-1}\left(\mathbf{t}_2 - \hat{R}\mathbf{u}_{,1} + p\mathbf{e}_2 + r\zeta\mathbf{e}_2\right) - \frac{1}{2}\mathbf{u}_{,1}\cdot Q\mathbf{u}_{,1}$$
$$- \frac{1}{2}m\zeta^2 p\,\mathbf{u}_{,1}\cdot\mathbf{e}_1 + r\zeta\,\mathbf{u}_{,1}\cdot\mathbf{e}_1 - p_f\left(\mathbf{w}_{,1}\cdot\mathbf{e}_1 + \frac{p_f}{m}\right), \tag{4.42}$$

where $\mathbf{t}_2$ represents the fundamental traction vector, $\hat{Q}, \hat{R}, \hat{T}$ denote generalized Stroh matrices adapted to the poroelastic setting, $\mathbf{u}_{,1}$ and $\mathbf{w}_{,1}$ are spatial derivatives of the displacement fields, $\zeta = -\mathrm{div}\,\mathbf{w}$ characterizes the volumetric fluid exchange, $p_f = m\zeta$ is the pore fluid pressure, p is a Lagrange multiplier enforcing incompressibility.

4.4.1 Independence of x_2

The expression of H contains no explicit dependence on the coordinate x_2. Its possible influence is only indirect, entering through the displacement fields and their derivatives. This ensures that translational invariance along x_2 holds, which is a prerequisite for conservation in the present framework.

4.4.2 Canonical Equations

A further requirement is that the canonical system generated by H reproduces the governing equilibrium relations. For the generalized coordinates, one obtains

$$\dot{\mathbf{u}} = \frac{\partial H}{\partial \mathbf{t}_2}, \qquad \dot{\mathbf{w}}\cdot\mathbf{e}_2 = -\frac{\partial H}{\partial p_f},$$

which coincide with equations (4.38) and (4.39).

For the conjugate momenta, the corresponding relations are

$$\dot{\mathbf{t}}_2 = -\frac{\partial H}{\partial \mathbf{u}}, \qquad -\dot{p}_f\mathbf{e}_2 = -\frac{\partial H}{\partial \mathbf{w}},$$

which reproduce equations (4.40) and (4.41). This agreement verifies that the Hamiltonian framework is fully consistent with the equilibrium description.

4.4.3 Physical Implications

The conservative nature of the Hamiltonian density has important physical consequences. Without dissipative mechanisms the dynamics of the system are reversible and the total energy is conserved. In a poroelastic medium, it implies that the pore fluid can be able to redistribute without permanent loss of energy. For wave propagation, the absence of dissipation ensures that waves maintain amplitude and frequency over time, allowing both forward and backward modes to persist indefinitely. At boundaries, conservation implies that energy is either reflected or transmitted in its entirety, giving rise to characteristic phenomena such as surface and edge modes. The mentioned properties highlight the importance of the Hamilton framework in marking regimes where the wave motions and fluid-solid interactions are non-dissipative, thus providing a framework of analyzing the phenomenon of stability and localization.

4.5 Conclusion

This chapter has developed a detailed model of the description of reversible poroelasticity in incompressible solids, based on the Biot's theory. Based on fundamental assumptions on solid incompressibility and fluid compressibility, total stress, constitutive relations, and equilibrium equations were systematically formulated. The fact that fundamental force vectors have been introduced, has enabled the representation of the governing equations succinctly and internally consistently, which in turn has made it possible to adhere to the strict conditions of the strong ellipticity, as well as the fact that positive-definite energetic storage must be provided.

A Lagrangian formulation was established, which includes explicitly both the elastic strain energy of the solid skeleton and the compressibility energy of the pore fluid. The resulting equations of Euler-Lagrange equations restore the coupled equilibrium equations which were the proper equations to the system, and thus the equations give a rigorous variational basis to the later analysis. Extending the framework to a Hamiltonian representation enabled the identification of canonical coordinates and conjugate momenta, yielding a fully consistent set of canonical equations that reproduce the mechanical and hydraulic balance laws.

The Hamiltonian density was shown to be conservative and to admit an integral quadratic representation, highlighting the underlying symplectic structure. This structure naturally accommodates the Stroh formalism, allowing the description of traveling wave solutions and energy propagation in poroelastic media. In general, the chapter demonstrates that the combination of the Lagrangian and Hamiltonian models provides a strong and integrated modeling paradigm of reversible and incompressible

poroelastic materials, and therefore, provides a platform on future studies of the stability, wave propagation, and local phenomena. In the next chapter, we will extend this approach to the Stroh–Hamiltonian formulation for thermoporoelasticity, which includes thermal processes in the coupled solid-fluid system.

Chapter 5

Analysis of Thermo-Coupled Poroelastic Anisotropic Materials

5.1 Introduction

The current chapter formulates the theoretical framework for reversible coupled anisotropic thermoporoelasticity, an approach where mechanical deformations, interaction between pores and fluids, and thermal interactions are connected. Starting with the generalized Duhamel-Neumann constitutive relations we obtain the equations governing anisotropy, poroelastic coupling and temperature dependence. The problem is developed into the variational basis by introducing the basic traction vectors in the problem and applying the principle of thermodynamic reversibility. The treatment then advances toward a Lagrangian and Hamiltonian formalism, allowing for a systematic reduction of the governing equations. Through Stroh's method, the problem is recast into a first-order differential system whose spectral properties establish the foundation for constructing fundamental solutions.

The findings discussed here have been presented at the International School of Multiscale Mathematical Models for Multi-Agent Systems (Multiscale Mathematical Models for Multi-Agents Systems).

5.2 Basic Assumptions and Governing Equations

The present analysis focuses on an orthogonal coordinate system $Ox_1x_2x_3$ defined in the Euclidean space $\mathscr{E}$. Let $\mathbf{n}$ denote the unit outward normal to a directed surface S, and introduce the orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, aligned with the coordinate axes. These vectors satisfy

$$\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}, \tag{5.1}$$

where δ_{ij} is the Kronecker delta. Summation over repeated indices will be assumed henceforth.

Consider a two-dimensional, homogeneous, anisotropic thermoporoelastic medium. The stress field in such a material is governed by the generalized Duhamel-Neumann constitutive relation (Coussy et al., 1998):

$$\mathbf{T} = \mathbb{C}\mathbf{E} - M\beta\xi\mathbf{I} - A\theta\mathbf{I}, \tag{5.2}$$

where $\tilde{\mathbb{C}} = \{\tilde{c}_{ijkl}\}$ is the fourth-order tensor of elastic moduli, $\mathbf{E}$ is the infinitesimal strain tensor obtained from the displacement vector field $\mathbf{u}$ as

$$\mathbf{E} = \tfrac{1}{2}(\operatorname{grad}\mathbf{u} + \operatorname{grad}\mathbf{u}^T), \quad e = \operatorname{tr}\mathbf{E}, \tag{5.3}$$

M is Biot's modulus, β is the poroelastic coupling coefficient, ξ represents the increment of fluid content, A is the isotropic thermal stress coefficient, and $\theta = T - T_0$ denotes the temperature increment with respect to the reference temperature T_0.

The pore pressure is given by

$$p_f = M(-\beta e + \xi + 3\alpha_m\theta), \tag{5.4}$$

where α_m is the volumetric thermal expansion coefficient. This formulation reflects the fundamental coupling between thermal, mechanical, and fluid effects in the medium.

The seepage displacement, characterizes the relative fluid-solid motion scaled by porosity, takes the form

$$\mathbf{w} = f(\mathbf{U} - \mathbf{u}), \tag{5.5}$$

where $\mathbf{U}$ is the fluid displacement vector, $\mathbf{u}$ is the solid displacement vector and f denotes the effective porosity. The associated kinematic relations are

$$e = \operatorname{div}\mathbf{u}, \quad \xi = -\operatorname{div}\mathbf{w}, \tag{5.6}$$

with e denoting the volumetric strain of the solid skeleton and ξ the fluid content variation.

For spatially varying porosity, the relation between the solid and fluid volumetric strains becomes

$$-\xi = (\mathbf{U} - \mathbf{u}) \cdot \operatorname{grad} f + f(\epsilon - e), \quad \text{with} \quad \epsilon = \operatorname{div}\mathbf{U}. \tag{5.7}$$

In the special case of constant porosity, this reduces to

$$\epsilon = e - f^{-1}\xi. \tag{5.8}$$

5.2.1 Fundamental Force Vectors

In the context of Stroh formulation, the fundamental traction vectors corresponding to planes normal to x_1 and x_2 are written as

$$\begin{aligned} \mathbf{t}_1 &= \mathbf{T}\mathbf{e}_1 = T_{ij}\delta_{j1}, \\ \mathbf{t}_2 &= \mathbf{T}\mathbf{e}_2 = T_{ij}\delta_{j2}. \end{aligned} \tag{5.9}$$

Solving for $\mathbf{t}_1$ gives

$$\begin{aligned} (t_1)_i &= (c_{ijkl}\varepsilon_{kl} - M\beta_{i1}\xi - A_{i1}\theta) \\ &= c_{i1kl}\varepsilon_{kl} - M\beta_{i1}\xi - A_{i1}\theta \\ &= c_{i1k1}\varepsilon_{k1} + c_{i1k2}\varepsilon_{k2} - M\beta_{i1}\xi - A_{i1}\theta \\ &= c_{i1k1}\left(\frac{u_{k,1} + u_{1,k}}{2}\right) + c_{i1k2}\left(\frac{u_{k,2} + u_{2,k}}{2}\right) - M\beta_{i1}\xi - A_{i1}\theta \\ &= c_{i1k1}u_{k,1} + c_{i1k2}u_{k,2} - M\beta_{i1}\xi - A_{i1}\theta, \end{aligned}$$

where we have used the minor symmetry of the elasticity tensor.

Similarly, for $\mathbf{t}_2$:

$$(t_2)_i = c_{i2k1}u_{k,1} + c_{i2k2}u_{k,2} - M\beta_{i2}\xi - A_{i2}\theta.$$

Thus, the traction vectors can be written compactly as

$$\begin{aligned} \mathbf{t}_1 &= \mathbf{T}\mathbf{e}_1 = Q\mathbf{u}_{,1} + R\mathbf{u}_{,2} - M\beta\xi\mathbf{e}_1 - A\theta\mathbf{e}_1, \\ \mathbf{t}_2 &= \mathbf{T}\mathbf{e}_2 = R^T\mathbf{u}_{,1} + T\mathbf{u}_{,2} - M\beta\xi\mathbf{e}_2 - A\theta\mathbf{e}_2, \end{aligned} \tag{5.10}$$

where $Q_{ik} = c_{i1k1}$, $R_{ik} = c_{i1k2}$, and $T_{ik} = c_{i2k2}$. Here, commas denote partial derivatives, e.g., $\mathbf{u}_{,1} = \partial\mathbf{u}/\partial x_1$.

Remark 8. *The stiffness matrices Q and T are symmetric and positive definite assuming strong ellipticity (Ting, 1996). The assumption of thermodynamic reversibility, proposed by Biot (Teagle-Hernandez et al., 2018), yields the existence of a stored elastic potential W. Strong ellipticity requires that Q and T remain positive definite and that $M > 0$.*

5.2.2 Equilibrium Equations

The equilibrium equations in the absence of body forces are expressed as

$$\mathbf{t}_{1,1} + \mathbf{t}_{2,2} = 0, \tag{5.11a}$$

$$\operatorname{grad} p_f = 0, \tag{5.11b}$$

the first equation above is the divergence of $\mathbf{T}$, representing the mechanical boundary condition, whereas the latter is the hydraulic boundary condition that is analogous to Darcy's law (Biot, 1962).

5.2.3 Boundary Conditions

We partition the boundary ∂B in to pairwise disjoint sets whose union is ∂B: $\partial B = S_u \cup S_t = S_p \cup S_w$, where displacement and traction boundary conditions are prescribed on S_u and S_t, respectively, while fluid pressure and fluid flux conditions are prescribed on S_p and S_w. We assume that $S_u \cap S_t = \varnothing$ and $S_p \cap S_w = \varnothing$. Specifically, we impose homogeneous boundary conditions

$$S_u : \mathbf{u} = \mathbf{0} \qquad \text{(Dirichlet)}, \tag{5.12}$$

$$S_t : \mathbf{t} = \boldsymbol{\sigma}\mathbf{n} = \mathbf{0} \qquad \text{(Neumann)}, \tag{5.13}$$

$$S_p : p_f = 0 \qquad \text{(Dirichlet)}, \tag{5.14}$$

$$S_w : \mathbf{n} \cdot \mathbf{w} = 0 \qquad \text{(Neumann)}, \tag{5.15}$$

5.2.4 Lagrangian Formulation

The stored elastic energy minus the external work is given by Eshelby's variational principle (Fu, 2007):

$$\mathcal{L} = \int_B W \, dV - \int_{\partial B} (\mathbf{t}_0 \cdot \mathbf{u} + p_{f_0} \, \mathbf{n} \cdot \mathbf{w}) \, dS, \tag{5.16}$$

with the stored energy density (Coussy et al., 1998):

$$\begin{aligned} W = {} & \tfrac{1}{2}\mathbf{u}_{,1} \cdot Q\mathbf{u}_{,1} + \mathbf{u}_{,1} \cdot R\mathbf{u}_{,2} + \tfrac{1}{2}\mathbf{u}_{,2} \cdot T\mathbf{u}_{,2} - A\theta e + 3\alpha_m M\xi\theta \\ & - M\beta e\xi + \tfrac{1}{2}M\xi^2 - \tfrac{1}{2}\frac{C_\epsilon}{T_0}\theta^2. \end{aligned} \tag{5.17}$$

Using the divergence theorem, the mechanical contribution satisfies

$$\int_B \boldsymbol{\sigma} : \nabla \mathbf{u} \, dV = \int_{\partial B} (\boldsymbol{\sigma}\mathbf{n}) \cdot \mathbf{u} \, dS - \int_B (\nabla \cdot \boldsymbol{\sigma}) \cdot \mathbf{u} \, dV.$$

Under the equilibrium condition $\nabla \cdot \boldsymbol{\sigma} = \mathbf{0}$ and the imposed homogeneous boundary conditions, the boundary term reduces to

$$\int_{\partial B} \mathbf{t} \cdot \mathbf{u}\, dS = \int_{S_u} \mathbf{t} \cdot \mathbf{0}\, dS + \int_{S_t} \mathbf{0} \cdot \mathbf{u}\, dS = 0,$$

so that

$$\int_B \boldsymbol{\sigma} : \nabla \mathbf{u}\, dV = 0.$$

Similarly, for the fluid contribution, the divergence theorem yields

$$\int_B (\nabla p_f \cdot \mathbf{w} - p_f \nabla \cdot \mathbf{w})\, dV = \int_{\partial B} p_f\, \mathbf{n} \cdot \mathbf{w}\, dS.$$

Using the homogeneous boundary conditions on S_p and S_w, the boundary integral vanishes. Recalling the definition of the variation of fluid content,

$$\xi = -\operatorname{div} \mathbf{w},$$

we obtain

$$\int_B (\nabla p_f \cdot \mathbf{w} + p_f \xi)\, dV = 0.$$

As a consequence, the total potential energy reduces to a purely volumetric functional,

$$\mathcal{L} = \int_B L\, dV, \tag{5.18}$$

where the Lagrangian density L depends on the displacement gradients, the fluid content variation ξ, and the temperature field θ, and is given by

$$\begin{aligned} L = {} & \tfrac{1}{2}\mathbf{u}_{,1} \cdot Q\mathbf{u}_{,1} + \mathbf{u}_{,1} \cdot R\mathbf{u}_{,2} + \tfrac{1}{2}\mathbf{u}_{,2} \cdot T\mathbf{u}_{,2} - A\theta e + 3\alpha_m M\xi\theta \\ & - M\beta e\xi + \tfrac{1}{2}M\xi^2 - \tfrac{1}{2}\frac{C_e}{T_0}\theta^2. \end{aligned} \tag{5.19}$$

The Euler-Lagrangian equations are stated as

$$\frac{d}{dx_1}\frac{\partial L}{\partial u_{,1}} + \frac{d}{dx_2}\frac{\partial L}{\partial u_{,2}} = 0 \tag{5.20a}$$

$$\frac{d}{dx_1}\frac{\partial L}{\partial w_{,1}} + \frac{d}{dx_2}\frac{\partial L}{\partial w_{,2}} = 0 \tag{5.20b}$$

Computing

$$\frac{\partial L}{\partial \mathbf{u}_{,1}} = Q\mathbf{u}_{,1} + R\mathbf{u}_{,2} - A\theta\mathbf{e}_1 - M\beta\xi\mathbf{e}_1$$
$$\frac{\partial L}{\partial \mathbf{u}_{,2}} = R^T\mathbf{u}_{,1} + T\mathbf{u}_{,2} - A\theta\mathbf{e}_2 - M\beta\xi\mathbf{e}_2$$

As a result, the final equations are written as

$$(Q\mathbf{u}_{,1} + R\mathbf{u}_{,2} - A\theta\mathbf{e}_1 - M\xi\beta\mathbf{e}_1)_{,1} + (R^T\mathbf{u}_{,1} + T\mathbf{u}_{,2} - A\theta\mathbf{e}_2 - M\xi\beta\mathbf{e}_2)_{,2} = 0 \quad (5.21a)$$
$$(3\alpha_m M\theta - M\beta e + M\xi)_{,1}\mathbf{e}_1 + (3\alpha_m M\theta - M\beta\mathbf{e} + M\xi)_{,2}\mathbf{e}_2 = 0 \quad (5.21b)$$

The first equation (5.21a) corresponds to the equilibrium equation (5.11a). Similarly, (5.21b) is consistent with (5.11b).

5.3 Hamiltonian Formulation

In order to establish a Hamiltonian description of the problem, we reinterpret the second spatial coordinate x_2 as a time-like variable, following the methodology presented in Fu (2007). Consequently, derivatives with respect to x_2 are denoted by an overdot for clarity. This reinterpretation enables us to treat the governing equations in a form analogous to dynamical systems, which in turn facilitates the application of Hamiltonian formalism.

To simplify subsequent derivations, we introduce the following modified second-order tensors:

$$\bar{Q} = Q - M\beta^2(\mathbf{e}_1 \otimes \mathbf{e}_1), \quad (5.22)$$
$$\bar{R} = R - M\beta^2(\mathbf{e}_1 \otimes \mathbf{e}_2), \quad (5.23)$$
$$\bar{T} = T - M\beta^2(\mathbf{e}_2 \otimes \mathbf{e}_2). \quad (5.24)$$

These modifications effectively account for the inertia-like contributions associated with the parameter $M\beta^2$. As a result, the original material tensors Q, R, and T are adjusted to $\bar{Q}$, $\bar{R}$, and $\bar{T}$, respectively, which allows the subsequent formulation to be expressed more compactly.

By substituting the redefined tensors into the constitutive relations, the fundamental force vectors originally given in equation (5.10) can be recast as

$$\mathbf{t}_1 = (\bar{Q} + M\beta^2\,\mathbf{e}_1 \otimes \mathbf{e}_1)\,\mathbf{u}_{,1} + (\bar{R} + M\beta^2\,\mathbf{e}_1 \otimes \mathbf{e}_2)\,\dot{\mathbf{u}} - M\beta\xi\,\mathbf{e}_1 - A\theta\,\mathbf{e}_1,$$
$$\mathbf{t}_2 = (\bar{R} + M\beta^2\,\mathbf{e}_1 \otimes \mathbf{e}_2)\,\mathbf{u}_{,1} + (\bar{T} + M\beta^2\,\mathbf{e}_2 \otimes \mathbf{e}_2)\,\dot{\mathbf{u}} - M\beta\xi\,\mathbf{e}_2 - A\theta\,\mathbf{e}_2.$$

Expanding $\mathbf{t}_1$ gives

$$\begin{aligned}\mathbf{t}_1 &= \bar{Q}\mathbf{u}_{,1} + M\beta^2(\mathbf{e}_1 \otimes \mathbf{e}_1)\mathbf{u}_{,1} + \bar{R}\dot{\mathbf{u}} + M\beta^2(\mathbf{e}_1 \otimes \mathbf{e}_2)\dot{\mathbf{u}} - (p_f + M\beta\xi - 3M\alpha_m\theta)\,\mathbf{e}_1 - A\theta\mathbf{e}_1 \\ &= \bar{Q}\mathbf{u}_{,1} + M\beta^2(\mathbf{e}_1 \cdot \mathbf{u}_{,1})\mathbf{e}_1 + \bar{R}\dot{\mathbf{u}} + M\beta^2(\mathbf{e}_2 \cdot \dot{\mathbf{u}})\mathbf{e}_1 - (p_f + M\beta\xi - 3M\alpha_m\theta)\,\mathbf{e}_1 - A\theta\mathbf{e}_1 \\ &= \bar{Q}\mathbf{u}_{,1} + \bar{R}\dot{\mathbf{u}} - p_f\beta\mathbf{e}_1 + 3M\alpha_m\theta\beta\mathbf{e}_1 - A\theta\mathbf{e}_1.\end{aligned}$$

Similarly, for $\mathbf{t}_2$:

$$\begin{aligned}\mathbf{t}_2 &= \bar{R}\mathbf{u}_{,1} + M\beta^2(\mathbf{e}_1 \otimes \mathbf{e}_2)\mathbf{u}_{,1} + \bar{T}\dot{\mathbf{u}} + M\beta^2(\mathbf{e}_2 \otimes \mathbf{e}_2)\dot{\mathbf{u}} - (p_f + M\beta\xi - 3M\alpha_m\theta)\,\mathbf{e}_2 - A\theta\mathbf{e}_2 \\ &= \bar{R}\mathbf{u}_{,1} + \bar{T}\dot{\mathbf{u}} - p_f\beta\mathbf{e}_2 + 3M\alpha_m\theta\beta\mathbf{e}_2 - A\theta\mathbf{e}_2.\end{aligned}$$

Remark 9. *Here, we used the property of dyadic products that* $(\mathbf{e}_i \otimes \mathbf{e}_j)\mathbf{v} = (\mathbf{e}_j \cdot \mathbf{v})\mathbf{e}_i$*, which allows the above simplification of the stress-resultant vectors.*

Hence, the fundamental force vectors reduce to

$$\mathbf{t}_1 = \bar{Q}\,\mathbf{u}_{,1} + \bar{R}\,\dot{\mathbf{u}} + 3M\alpha_m\theta\beta\,\mathbf{e}_1 - A\theta\mathbf{e}_1 - p_f\beta\mathbf{e}_1, \tag{5.25a}$$
$$\mathbf{t}_2 = \bar{R}^T\mathbf{u}_{,1} + \bar{T}\,\dot{\mathbf{u}} + 3M\alpha_m\theta\beta\,\mathbf{e}_2 - A\theta\mathbf{e}_2 - p_f\beta\mathbf{e}_2, \tag{5.25b}$$

where $\mathbf{t}_1$ and $\mathbf{t}_2$ denote the stress-resultant vectors along the two coordinate directions. The first two terms in each equation correspond to elastic and inertial contributions, while the remaining terms capture the effects of thermoelastic coupling and pore-fluid interactions.

5.3.1 Lagrangian Density in Terms of Transform Matrices

Having established these relations, the Lagrangian (5.19) can now be expressed explicitly as

$$L(\mathbf{u}_{,1}, \dot{\mathbf{u}}, \mathbf{w}_{,1}, \dot{\mathbf{w}}) = \tfrac{1}{2}\,\mathbf{u}_{,1}\cdot\bar{Q}\,\mathbf{u}_{,1} + \mathbf{u}_{,1}\cdot\bar{R}\,\dot{\mathbf{u}} + \tfrac{1}{2}\,\dot{\mathbf{u}}\cdot\bar{T}\,\dot{\mathbf{u}} + \tfrac{1}{2}M\beta^2\Big(\mathbf{u}_{,1}\cdot\mathbf{e}_1 + \dot{\mathbf{u}}\cdot\mathbf{e}_2\Big)^2 - A\theta e + 3\alpha_m M\xi\theta - M\beta\xi e + \tfrac{1}{2}M\xi^2 - \tfrac{1}{2}\frac{C_e}{T_0}\,\theta^2. \tag{5.26}$$

The above expression highlights the coupled nature of the problem: The quadratic terms in $\mathbf{u}_{,1}$ and $\dot{\mathbf{u}}$ describe the elastic and inertial effects associated with the structural displacements. The term involving $M\beta^2$ introduces additional quadratic coupling between generalized displacement gradients and velocities. The scalar terms involving θ and ξ capture the influence of temperature variations, thermal expansion, and pore-fluid interactions, weighted by the material constants A, α_m, M, β, and C_e/T_0.

5.3.2 Conjugate Momenta

The conjugate momenta are defined as

$$\mathbf{p_1} = \frac{\partial L}{\partial \dot{\mathbf{u}}}, \quad \mathbf{p}_2 = \frac{\partial L}{\partial \dot{\mathbf{w}}} \tag{5.27}$$

are immediately obtained for $\mathbf{p_1}$

$$\mathbf{p}_1 = \frac{\partial L}{\partial \dot{\mathbf{u}}} = \bar{R}^T\mathbf{u}_{,1} + \bar{T}\dot{\mathbf{u}} + M\beta^2 e\mathbf{e}_2 - M\beta\xi\mathbf{e}_2 - A\theta\mathbf{e}_2 \tag{5.28}$$

using the definition of p_f i.e $-p_f + 3\alpha_m M\theta = M\beta e - M\xi$, we get

$$\mathbf{p}_1 = \bar{R}^T\mathbf{u}_{,1} + \bar{T}\dot{\mathbf{u}} - p_f\beta\mathbf{e}_2 - \beta 3\alpha_m\theta\mathbf{e}_2 = \mathbf{t}_2 \tag{5.29}$$

for $\mathbf{p}_2$

$$\mathbf{p}_2 = \frac{\partial L}{\partial \dot{\mathbf{w}}} = -(3\alpha_m M\theta - M\beta e + M\xi)\mathbf{e}_2 = -p_f\mathbf{e}_2 \tag{5.30}$$

Solving (5.4) for ξ gives

$$\xi = \frac{p_f}{M} + \beta e - 3\alpha_m\theta \tag{5.31}$$

while solving (5.25b) for $\dot{\mathbf{u}}$ gives

$$\dot{\mathbf{u}} = \bar{T}^{-1}(\mathbf{t}_2 - \bar{R}^T\mathbf{u}_{,1} - 3\alpha_m M\theta\beta\mathbf{e}_2 + A\theta\mathbf{e}_2 + p_f\beta\mathbf{e}_2). \tag{5.32}$$

Scalar multiplication of (5.32) throughout by $\mathbf{e}_2$ gives

$$\dot{\mathbf{u}} \cdot \mathbf{e}_2 = \xi_1^{-1} \bar{T}^{-1} \bar{\mathbf{t}_2} \cdot \mathbf{e}_2 \tag{5.33}$$

where we have let the shorthand

$$\bar{\mathbf{t}}_2 = \mathbf{t}_2 - \bar{R^T} \mathbf{u}_{,1} + A\theta \mathbf{e}_2 - M\beta^2 \mathbf{e}_2 \cdot \mathbf{u}_{,1} \cdot \mathbf{e}_1 + M\beta\xi \mathbf{e}_2 \tag{5.34}$$

and as seen in (Fu, 2007), it is

$$\xi_1^{-1} = 1 + M\beta^2 \mathbf{e}_2 \cdot \bar{T}^{-1} \mathbf{e}_2 > 1 \tag{5.35}$$

whose last term is always positive by virtue of strong ellipticity.

In a similar fashion, in light of (5.6) and (5.32) we can define

$$-\dot{\mathbf{w}} \cdot \mathbf{e}_2 = \frac{p_f}{M} + \beta e - 3\alpha_m \theta + \mathbf{w}_{,1} \cdot \mathbf{e}_1 \tag{5.36}$$

We define Hamiltonian density as

$$H = \mathbf{t}_2 \cdot \dot{\mathbf{u}} + p_2 \cdot \dot{\mathbf{w}} - L. \tag{5.37}$$

Substituting $p_2 \approx p_f \mathbf{e}_2$ and using (5.36) for $-\dot{w} \cdot \mathbf{e}_2$, we can rewrite H as

$$\begin{aligned} H &= \mathbf{t}_2 \cdot \dot{\mathbf{u}} - p_f \mathbf{e}_2 \cdot \dot{\mathbf{w}} - L \\ &= \mathbf{t}_2 \cdot \dot{\mathbf{u}} + p_f \Big(\frac{p_f}{M} - 3\alpha_m \theta + w_{,1} \cdot \mathbf{e}_1 \Big) - L. \end{aligned}$$

Next, substituting the explicit expression for L from (5.19) and collecting terms, the Hamiltonian density can be written in blocks as

$$\begin{aligned} H = {} & \dot{\mathbf{u}}(\mathbf{t}_2 - \bar{R}^T \mathbf{u}_{,1} + p_f \beta \mathbf{e}_2 + A\theta \mathbf{e}_2) - \frac{1}{2} \dot{\mathbf{u}} \cdot \bar{T} \dot{\mathbf{u}} + \\ & + \frac{p_f^2}{M} + p_f \beta \mathbf{u}_{,1} \cdot \mathbf{e}_1 - 3\alpha_m p_f \theta + p_f \mathbf{w}_{,\mathbf{1}} \cdot \mathbf{e}_1 + A\theta \mathbf{u}_{,1} \cdot \mathbf{e}_1 - \\ & - 3\alpha_m \xi M \theta - \frac{1}{2} M \beta^2 \mathbf{e^2} + M\beta\xi e - \frac{1}{2} M \xi^2 + \frac{1}{2} \frac{C_\epsilon}{T_0} \theta^2 + \frac{1}{2} \mathbf{u}_{,1} \cdot \bar{Q} \mathbf{u}_{,1} \end{aligned} \tag{5.38}$$

Finally, using $\bar{T}^{-1}$ to write the kinetic term in standard Hamiltonian form, we have

$$\begin{aligned} H = &\frac{1}{2}(\mathbf{t}_2 - \bar{R}^T\mathbf{u}_{,1} + p_f\beta\mathbf{e}_2 + A\theta\mathbf{e}_2 - 3M\alpha_m\theta\beta\mathbf{e}_2)\cdot\bar{T}^{-1}(\mathbf{t}_2 - \bar{R}^T\mathbf{u}_{,1} + p_f\beta\mathbf{e}_2 \\ &+ A\theta\mathbf{e}_2 - 3M\alpha_m\theta\beta\mathbf{e}_2) + \frac{p_f^2}{M} + p_f\beta\mathbf{u}_{,1}\cdot\mathbf{e}_1 - 3\alpha_m p_f\theta + p_f\mathbf{w}_{,1}\cdot\mathbf{e}_1 \\ &+ A\theta\mathbf{u}_{,1}\mathbf{e}_1 - 3\alpha_m\xi M\theta - \frac{1}{2M}(-p_f + 3\alpha_m M\theta)^2 + \frac{1}{2}\frac{C_\epsilon}{T_0}\theta^2 + \frac{1}{2}\mathbf{u}_{,1}\cdot\bar{Q}\mathbf{u}_{,1} \end{aligned} \tag{5.39}$$

5.3.3 Canonical Equations

The canonical equations are defined as

$$\dot{\mathbf{q}} = \frac{\partial H}{\partial \mathbf{p}}, \quad \dot{\mathbf{p}} = -\frac{\partial H}{\partial \mathbf{q}}, \tag{5.40}$$

where the generalized coordinates and momenta are

$$\mathbf{q} = (\mathbf{u}, \mathbf{w}), \quad \mathbf{p} = (\mathbf{t}_2, p_f\mathbf{e}_2).$$

Using the Hamiltonian (5.39), the velocity $\dot{\mathbf{u}}$ is

$$\dot{\mathbf{u}} = \frac{\partial H}{\partial \mathbf{t}_2} = \bar{T}^{-1}\Big(\mathbf{t}_2 - \bar{R}^T\mathbf{u}_{,1} + p_f\beta\mathbf{e}_2 + A\theta\mathbf{e}_2 - 3\alpha_m M\theta\beta\mathbf{e}_2\Big). \tag{5.41}$$

The velocity component along $\mathbf{e}_2$ is

$$\begin{aligned} \dot{\mathbf{w}}\cdot\mathbf{e}_2 &= -\frac{\partial H}{\partial p_f} \\ &= \bar{T}^{-1}\Big(\mathbf{t}_2 - \bar{R}^T\mathbf{u}_{,1} + p_f\beta\mathbf{e}_2 + A\theta\mathbf{e}_2 - 3\alpha_m M\theta\beta\mathbf{e}_2)\cdot\mathbf{e}_2 \qquad (5.42) \\ &\quad + \frac{2p_f}{M} - 3\alpha_m\theta + \mathbf{w}_{,1}\cdot\mathbf{e}_1 - \xi - 3\alpha_m\theta \\ &= -\mathbf{w}_{,1}\cdot\mathbf{e}_1 - \frac{p_f}{m} - \beta e + 3\alpha_m\theta, \qquad (5.43) \end{aligned}$$

which confirms consistency with the relation $\xi = -\operatorname{div}\mathbf{w}$.

The evolution of the momentum $\mathbf{t}_2$ is

$$\begin{aligned} \dot{\mathbf{t}}_2 = &-\big(\bar{T}^{-1}\bar{R}^T(\mathbf{t}_2 - \bar{R}^T\mathbf{u}_{,1} + p_f\beta\mathbf{e}_2 + A\theta\mathbf{e}_2 - 3M\alpha_m\theta\beta\mathbf{e}_2) + \\ &A\theta\mathbf{e}_1 + \bar{Q}\mathbf{u}_{,1} + p_f\beta\mathbf{e}_1\big)_{,1} \end{aligned} \tag{5.44}$$

This corresponds to Equation (5.32), where multiplying the bracketed term by $\bar{T}^{-1}$ yields $\dot{\mathbf{u}}$.

Finally, the evolution of the pore pressure variable is

$$-\dot{p}_f \mathbf{e}_2 = -\frac{\partial H}{\partial \mathbf{w}_{,1}} = (p_f \mathbf{e}_1)_{,1}. \tag{5.45}$$

Remark 10. *The canonical equations are derived directly from the Hamiltonian* (5.39)*, and the verification of* $\dot{\mathbf{w}} \cdot \mathbf{e}_2$ *confirms consistency with the definition* $\xi = -\operatorname{div} \mathbf{w}$*. The resulting expressions provide a clear and complete representation of the system dynamics.*

5.4 Conservation of the Hamiltonian Density

To assess whether the Hamiltonian density exhibits a conservative character, one must analyze its algebraic structure together with its dependence on the underlying field variables. The Hamiltonian density H can be written as

$$\begin{aligned} H = \frac{1}{2}(\mathbf{t}_2 - \bar{R}^T \mathbf{u}_{,1} + p_f \beta \mathbf{e}_2 + A\theta \mathbf{e}_2 - 3M\alpha_m \theta \beta \mathbf{e}_2) \cdot \bar{T}^{-1}(\mathbf{t}_2 - \bar{R}^T \mathbf{u}_{,1} + p_f \beta \mathbf{e}_2 \\ + A\theta \mathbf{e}_2 - 3M\alpha_m \theta \beta \mathbf{e}_2) + \frac{p_f^2}{M} + p_f \beta \mathbf{u}_{,1} \cdot \mathbf{e}_1 - 3\alpha_m p_f \theta + p_f \mathbf{w}_{,1} \cdot \mathbf{e}_1 \\ + A\theta \mathbf{u}_{,1} \mathbf{e}_1 - 3\alpha_m \xi M\theta - \frac{1}{2M}(-p_f + 3\alpha_m M\theta)^2 + \frac{1}{2}\frac{C_\epsilon}{T_0}\theta^2 + \frac{1}{2}\mathbf{u}_{,1} \cdot \bar{Q}\mathbf{u}_{,1} \end{aligned} \tag{5.46}$$

where $\mathbf{t}_2$ represents the fundamental traction vector, $\hat{Q}, \hat{R}, \hat{T}$ denote generalized Stroh matrices adapted to the poroelastic setting, $\mathbf{u}_{,1}$ and $\mathbf{w}_{,1}$ are spatial derivatives of the displacement fields, $\zeta = -\operatorname{div} \mathbf{w}$ characterizes the volumetric fluid exchange, p_f is the pore fluid pressure, p is a Lagrange multiplier enforcing incompressibility.

5.4.1 Independence of x_2

The expression of H does not depend explicitly on the coordinate x_2. Any effect of x_2 can appear only indirectly through the displacement field , and their derivatives. As a result, the system remains invariant under translations along x_2, a condition required for conservation in this formulation.

5.4.2 Canonical Equations

Another condition to check is that the canonical system generated by H reproduces the governing equilibrium relations. For the generalized coordinates, one obtains

$$\dot{\mathbf{u}} = \frac{\partial H}{\partial \mathbf{t}_2}, \qquad \dot{\mathbf{w}} \cdot \mathbf{e}_2 = -\frac{\partial H}{\partial p_f},$$

which aligns with equations (5.41) and (5.43). Considering conjugate momenta, the corresponding relations are

$$\dot{\mathbf{t}}_2 = -\frac{\partial H}{\partial \mathbf{u}}, \qquad -\dot{p}_f \mathbf{e}_2 = -\frac{\partial H}{\partial \mathbf{w}},$$

which yield equations (5.44) and (5.45). This verifies that the Hamiltonian framework is fully consistent with the equilibrium description.

5.4.3 Physical Implication

The fact that the Hamiltonian density is conserved implies that the coupled thermoporoelastic system is developed in a fully reversible state in the absence of dissipative process. Energy is exchanged among the solid deformation, pore-fluid pressure and thermal fields without irreversible loss. Thus, the thermo-poroelastic wave modes are propagating without attenuation and do not lose either their amplitude or frequency, but, at boundaries, all the energy is reflected back or transmitted. This conservative framework provides a natural scheme of discovering non-dissipative regimes and studying the phenomenon of stability and localization in coupled thermoporoelastic media.

5.5 Conclusion

Throughout this chapter, we have developed a robust theoretical framework for reversible coupled anisotropic thermoporoelasticity, where the interactions between elastic deformation, pore-fluid flow, and thermal effects are analyzed within a unified framework. The model takes into account anisotropy in the elastic response, Biot-type poroelastic coupling, and temperature-dependent effects beginning with generalized Duhamel-Neumann constitutive relations. The variational structure, grounded in Eshelby's principle, results in a consistent energy formulation, ensuring thermodynamic reversibility and confirming well-posedness under the conditions of strong ellipticity.

The derivation of a Lagrangian density allowed us to cast the governing equations into a compact form, while reinterpretation of the spatial coordinate within the Hamiltonian formalism enabled a dynamical systems perspective. This reformulation not only explains the intrinsic symplectic form of the problem but it also makes it easier to study the stability of the problem and the admissibility of solutions. Moreover, the adoption of Stroh's reduction provided a systematic pathway to transform the field equations into a first-order differential system governed by a Hamiltonian matrix. The associated eigenvalue problem of this matrix discloses admissible decay modes and

fundamental solutions which form the basis of solving the problem of anisotropic porous thermoelastic media with respect to the boundary-value problems.

Overall, this chapter establishes the mathematical and physical foundations necessary for analyzing coupled thermoporoelastic processes in anisotropic continua. Linking the variational, Hamilton and Stroh methods provides the outcomes with a powerful analytical framework to understand wave propagation, stability and energy transfer in porous thermoelastic systems.

The following chapter will be an extension of this framework to the incompressible solid case of thermoporoelasticity. The extension allows the behavior of porous anisotropic materials to be captured where the solid skeleton is not subjected to volumetric deformation and as such, it adds more constraints to the coupled system. The incompressible formulation, which is based on the reversible one described in this chapter provides a more sophisticated account of incompressible elastic responses of materials, which is especially important in many geomechanical and biological applications.

Chapter 6

Incompressibility on coupled-thermoporoelastic anisotropic materials

6.1 Introduction

This chapter generalizes the theory of reversible coupled anisotropic thermoporoelasticity to incompressible solids, permitting to cover a broader scope of both materials and phenomena found in real practical and natural systems. Based on the generalized Duhamel-Neumann constitutive relations we obtain governing equations incorporating anisotropy, poroelastic coupling, dependence on temperature and the solid incompressibility constraint. The incompressibility condition requires the inclusion of a Lagrange multiplier which plays the role of the reactive pressure that pours the volumetric condition in the solid skeleton.

Proceeding from the definitions of fundamental traction vectors and the enforcement of thermodynamic reversibility, the chapter develops the variational formulation that naturally incorporates the incompressibility constraint. This framework facilitates the derivation of consistent boundary conditions and energy relations.This analysis is then developed to a Lagrangian and Hamiltonian formalism, modified to take the constraint structure of incompressible media into account, therefore, allowing the systematic reduction of the augmented equations of motion.

Lastly, the constrained system is transformed into a first-order differentiable expression by the Stroh formalism. This formulation provides a rigorous base to building fundamental solutions and the investigation of stability, wave propagation and localization effects in anisotropic incompressible thermoporoelastic solids.

6.2 Basic assumptions and Governing equations

We consider a three-dimensional Euclidean space E, equipped with an orthogonal coordinate system Ox_1, x_2, x_3. Let S be a directed surface embedded in this space, with $\mathbf{n}$ denoting the unit normal vector field that orients the surface. To describe physical quantities relative to this coordinate framework, we employ an orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, where each basis vector is aligned with the corresponding coordinate axis. The orthonormality condition is expressed as

$$\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij},$$

with δ_{ij} being the Kronecker delta. The second-order identity tensor in this space is denoted by I. In the chosen orthonormal basis, its components are given by

$$I = I_{ij} = \delta_{ij},$$

which again reflects the Kronecker delta structure. This definition plays a central role in tensorial formulations throughout the analysis Nobili (2024).
Consider a homogeneous linear two-dimensional thermoporoelastic material that characterizes anisotropic behavior with the seepage displacement, which describes the fluid's motion relative to the solid per unit volume of the poroelastic medium

$$\mathbf{w} = f(\mathbf{U} - \mathbf{u}) \tag{6.1}$$

where $\mathbf{U}$ is the fluid phase displacement, $\mathbf{u}$ is the solid phase displacement, f is the effective porosity that characterizes the interconnected pore space which is assumed uniform here.

The components of stress for the material under consideration are formulated by the Duhamel-Neumann constitutive law (Coussy et al., 1998).

$$T = \tilde{\mathbb{C}}\mathbf{E} - M\boldsymbol{\beta}\xi - \boldsymbol{A}\theta - p_s I \tag{6.2}$$

where $\tilde{\mathbb{C}}$ is the rank-4 tensors of elastic moduli whose components in the reference frame $\mathscr{E}$ are denoted by $\tilde{c}_{ijkl}$, p_s is a Lagrange multiplier introduced to permit independent variation of $\mathbf{u}_{,1}$ and $\mathbf{u}_{,2}$, owing to the fact that the solid is incompressible (Berger et al., 2017; Abousleiman and Ekbote, 2002). $\mathbf{E}$ is the linear strain tensor obtained from the displacement vector field $\mathbf{u}$ using the kinematic relations of continuum mechanics (Nowacki, 1975)

$$\mathbf{E} = \frac{1}{2}(\operatorname{grad}\mathbf{u} + \operatorname{grad}\mathbf{u}^T)$$

such that $\mathbf{e} = \operatorname{tr}\mathbf{E}$ with ξ is the change in fluid content defined by (Biot, 1941)

$$\xi = \frac{m}{p_{f_0}} \tag{6.3}$$

for which p_{f_0} is the initial value of the pore pressure p_f, while p_f is given as

$$p_f = M\xi + 3M\alpha_m\theta \tag{6.4}$$

and

$$\xi = -\operatorname{div}\mathbf{w}, \tag{6.5}$$

with ξ represent the variable of fluid content, which measures the change in fluid volume per unit volume of the reference configuration. Moreover, the condition of incompressibility for an elastic solid, we introduce the following kinematic relations.

$$e = \operatorname{div}\mathbf{u} = 0 \tag{6.6}$$

where e denote the volumetric strain of the solid phase. More precisely, the relationship between these variables is given by the following expression:

$$-\xi = (\mathbf{U} - \mathbf{u}) \cdot \operatorname{grad} f + f\epsilon, \tag{6.7}$$

with

$$\epsilon = \operatorname{div}\mathbf{U}$$

representing the fluid phase's volumetric strain. Under the assumption of uniform porosity, this expression simplifies to

$$\epsilon = -f^{-1}\xi. \tag{6.8}$$

6.2.1 Fundamental Force vectors

Proceeding further, the fundamental force (traction) vectors associated with planes normal to the coordinate directions x_1 and x_2 are defined as

$$\begin{aligned} \mathbf{t}_1 &= \mathbf{T}\mathbf{e}_1 = T_{ij}\delta_{j1}, \\ \mathbf{t}_2 &= \mathbf{T}\mathbf{e}_2 = T_{ij}\delta_{j2}. \end{aligned} \tag{6.9}$$

Solving for $\mathbf{t}_1$ yields

$$\begin{aligned}
(t_1)_i &= (c_{ijkl}\varepsilon_{kl} - M\beta_{ij}\xi - A_{ij}\theta - p_s)\,\delta_{j1} \\
&= c_{ijkl}\varepsilon_{kl}\delta_{j1} - M\beta_{ij}\delta_{j1}\xi - A_{ij}\delta_{j1}\theta - p_s\delta_{j1} \\
&= c_{i1kl}\varepsilon_{kl} - M\beta_{i1}\xi - A_{i1}\theta - p_s\delta_{i1} \\
&= c_{i1k1}\varepsilon_{k1} + c_{i1k2}\varepsilon_{k2} - M\beta_{i1}\xi - A_{i1}\theta - p_s\delta_{i1} \\
&= c_{i1k1}\left(\frac{u_{k,1}+u_{1,k}}{2}\right) + c_{i1k2}\left(\frac{u_{k,2}+u_{2,k}}{2}\right) - M\beta_{i1}\xi - A_{i1}\theta - p_s\delta_{i1} \\
&= c_{i1k1}u_{k,1} + c_{i1k2}u_{k,2} - M\beta_{i1}\xi - A_{i1}\theta - p_s\delta_{i1},
\end{aligned}$$

where the minor symmetry of the strain tensor has been used.

Similarly, for $\mathbf{t}_2$ one obtains

$$(t_2)_i = c_{i2k1}u_{k,1} + c_{i2k2}u_{k,2} - M\beta_{i2}\xi - A_{i2}\theta - p_s\delta_{i2}.$$

Thus, the fundamental force vectors take the compact form

$$\begin{aligned}
\mathbf{t}_1 &= \mathbf{T}\mathbf{e}_1 = Q\mathbf{u}_{,1} + R\mathbf{u}_{,2} - M\boldsymbol{\beta}\xi\mathbf{e}_1 - \boldsymbol{A}\theta\mathbf{e}_1 - p_s\mathbf{e}_1, \\
\mathbf{t}_2 &= \mathbf{T}\mathbf{e}_2 = R^T\mathbf{u}_{,1} + T\mathbf{u}_{,2} - M\boldsymbol{\beta}\xi\mathbf{e}_2 - \boldsymbol{A}\theta\mathbf{e}_2 - p_s\mathbf{e}_2,
\end{aligned} \tag{6.10}$$

Remark 11. *The submatrices Q and T must be positive definite under the assumption of strong ellipticity, ensuring the well-posedness of the resulting boundary-value problem.*

6.2.2 Equilibrium equations

The equilibrium equations for the current problem are

$$\mathbf{t}_{1,1} + \mathbf{t}_{2,2} = 0, \tag{6.11a}$$

$$\operatorname{grad} p_f = 0 \implies M\xi + 3M\alpha_m\theta = 0 \tag{6.11b}$$

6.2.3 Boundary Conditions

In this study, we restrict attention to the special case of an incompressible solid skeleton. The boundary ∂B is decomposed into pairwise disjoint subsets whose union equals ∂B, namely $\partial B = S_u \cup S_t = S_p \cup S_w$, where displacement and traction boundary conditions are prescribed on S_u and S_t, respectively, while fluid pressure and fluid flux conditions are imposed on S_p and S_w. The sets satisfy $S_u \cap S_t = \varnothing, \quad S_p \cap S_w = \varnothing$.

Specifically, we impose homogeneous boundary conditions

$$S_u : \mathbf{u} = \mathbf{0} \qquad \text{(Dirichlet)}, \tag{6.12}$$

$$S_t : \mathbf{t} = \boldsymbol{\sigma}\mathbf{n} = \mathbf{0} \qquad \text{(Neumann)}, \tag{6.13}$$

$$S_p : p_f = 0 \qquad \text{(Dirichlet)}, \tag{6.14}$$

$$S_w : \mathbf{n} \cdot \mathbf{w} = 0 \qquad \text{(Neumann)}, \tag{6.15}$$

These constraints are consistent with an incompressible solid skeleton, for which volume changes of the solid phase are prohibited, and the deformation–fluid interaction is governed purely by the mechanical loading and the prescribed pore pressure at the boundary.

6.2.4 Lagrangian formulation

Following Eshelby's formulation, the total energy of the system is given by the stored elastic potential energy diminished by the work performed by external forces acting on B:

$$L = \int_B W \, dV - \int_{\partial B} (\mathbf{t_0} \cdot \mathbf{u} + p_f \, \xi) \, dS, \tag{6.16}$$

with W expressed as (Coussy et al., 1998)

$$\begin{aligned} W = {} & \frac{1}{2}\mathbf{u}_{,1} \cdot Q\mathbf{u}_{,1} + \mathbf{u}_{,1} \cdot R\mathbf{u}_{,2} + \frac{1}{2}\mathbf{u}_{,2} \cdot T\mathbf{u}_{,2} - A\,\theta e + 3\alpha_m M\xi\theta - \xi M\beta e \\ & + \frac{1}{2}M\xi^2 - \frac{1}{2}\frac{C_\epsilon}{T_0}\theta^2 - p_s \, e, \end{aligned} \tag{6.17}$$

where $\theta = T - T_0$ is the temperature deviation from the ambient state and p_s acts as the Lagrange multiplier enforcing the incompressibility constraint.

The assumption of reversibility imposes strict conditions on the thermodynamic variables and their conjugate quantities. In the absence of dissipative mechanisms, the total energy remains constant throughout any mechanical deformation, an invariance naturally preserved in both Lagrangian and Hamiltonian formulations. For the present incompressible setting, a key implication is the spatial uniformity of the pore fluid pressure p_f across the entire medium. As demonstrated in Biot (1962), such uniformity characterizes the perfectly drained state, in which no relative motion occurs between the fluid and the incompressible solid skeleton. Under these boundary conditions, the coupling between solid deformation and pore pressure simplifies significantly: the mechanical response is driven solely by the imposed surface tractions, while the pore pressure remains constant and independent of the deformation field. This simplification

is particularly advantageous for variational formulations, as it reduces the complexity of the governing equations without sacrificing the essential features of solid–fluid interaction in incompressible porous media.

Under such conditions, the thermodynamic contribution to the stored energy is reduced to its essential form, with the specific heat at constant strain C_ϵ playing a central role through the term $-\frac{1}{2}\frac{C_\epsilon}{T_0}\theta^2$. Furthermore, by applying the divergence theorem, the total energy can be equivalently expressed as a single volume integral, facilitating subsequent variational analysis.

$$\mathcal{L} = \int_{\mathbf{B}} L dV \tag{6.18}$$

We define the Lagrangian density L as a functional that depends on the spatial gradients of the displacement field $\mathbf{u}$ and the divergence of an auxiliary field W, which is given by (Coussy et al., 1998)

$$\begin{aligned} L = &\frac{1}{2}\mathbf{u}_{,1}\cdot Q\mathbf{u}_{,1} + \mathbf{u}_{,1}\cdot R\mathbf{u}_{,2} + \frac{1}{2}\mathbf{u}_{,2}\cdot T\mathbf{u}_{,2} - A\theta\mathbf{e} + 3\alpha_m M\xi\theta - \xi M\boldsymbol{\beta}\mathbf{e} \\ &+ \frac{1}{2}M(\mathbf{w}_{,1}\cdot\mathbf{e}_1 + \mathbf{w}_{,2}\cdot\mathbf{e}_2)^2 - \frac{1}{2}\frac{C_e}{T_0}\theta^2 - p_s\mathbf{e} \end{aligned}$$

The Lagrangian equation of interest takes the forms

$$\frac{d}{dx_1}\frac{\partial L}{\partial u_{,1}} + \frac{d}{dx_2}\frac{\partial L}{\partial u_{,2}} = 0 \tag{6.19a}$$

$$\frac{d}{dx_1}\frac{\partial L}{\partial w_{,1}} + \frac{d}{dx_2}\frac{\partial L}{\partial w_{,2}} = 0 \tag{6.19b}$$

results in

$$\begin{aligned} &(Q\mathbf{u}_{,1} + R\mathbf{u}_{,2} - A\theta\mathbf{e}_1 - M\xi\boldsymbol{\beta}\mathbf{e}_1 - p_s\mathbf{e}_1)_{,1} \\ &\quad + (R^T\mathbf{u}_{,1} + T\mathbf{u}_{,2} - A\theta\mathbf{e}_2 - M\xi\boldsymbol{\beta}\mathbf{e}_2 - p_s\mathbf{e}_2)_{,2} = 0 \end{aligned} \tag{6.20}$$

which coincides with the equilibrium equation (6.11a). In the same way (6.19b) contributes

$$(3\alpha_m M\theta + M\xi)_{,1}\mathbf{e}_1 + (3\alpha_m M\theta + M\xi)_{,2}\mathbf{e_2} = 0 \tag{6.21}$$

which aligns with (6.11b).

6.3 Hamiltonian formalism

In line with the methodology applied earlier, we proceed here in a similar manner. So, for the Hamiltonian formalism, x_2 is viewed as a time-like coordinate (Fu, 2007). Consequently, derivatives with respect to x_2 are presented by a dot superscript. Let the transformed matrices be defined as

$$\bar{Q} = Q - \lambda(\mathbf{e}_1 \otimes \mathbf{e}_1), \quad \bar{R} = R - \lambda(\mathbf{e}_1 \otimes \mathbf{e}_2), \quad \bar{T} = T - \lambda(\mathbf{e}_2 \otimes \mathbf{e}_2) \tag{6.22}$$

where λ is a positive constant (specifically large enough) added here on account of incompressible materials, the inability of the material to sustain hydrostatic stresses results in the singularity of the stiffness matrix for volumetric deformation. To avoid this singularity, λ is added to the diagonal terms of the stiffness matrices. This ensures that $\bar{T}$ remains positive definite, which is significant for the stability of numerical computations and the invertibility of matrices (Fu, 2007).

By substituting the redefined tensors into the constitutive relations, the fundamental force vectors originally given in equation (6.10) can be expressed as

$$\begin{aligned}
\mathbf{t}_1 &= (\bar{Q} + \lambda \mathbf{e}_1 \otimes \mathbf{e}_1)\mathbf{u}_{,1} + (\bar{R} + \lambda \mathbf{e}_1 \otimes \mathbf{e}_2)\dot{\mathbf{u}} - M\beta\xi\mathbf{e}_1 - A\theta\mathbf{e}_1 - p_s\mathbf{e}_1, \\
\mathbf{t}_2 &= (\bar{R} + \lambda \mathbf{e}_1 \otimes \mathbf{e}_2)\mathbf{u}_{,1} + (\bar{T} + \lambda \mathbf{e}_2 \otimes \mathbf{e}_2)\dot{\mathbf{u}} - M\beta\xi\mathbf{e}_2 - A\theta\mathbf{e}_2 - p_s\mathbf{e}_2.
\end{aligned}$$

We first expand $\mathbf{t}_1$:

$$\begin{aligned}
\mathbf{t}_1 &= \bar{Q}\mathbf{u}_{,1} + \lambda(\mathbf{e}_1 \otimes \mathbf{e}_1)\mathbf{u}_{,1} + \bar{R}\dot{\mathbf{u}} + \lambda(\mathbf{e}_1 \otimes \mathbf{e}_2)\dot{\mathbf{u}} - (p_f + M\beta\xi - 3M\alpha_m\theta)\beta\mathbf{e}_1 - A\theta\mathbf{e}_1 - p_s\mathbf{e}_1 \\
&= \bar{Q}\mathbf{u}_{,1} + \lambda(\mathbf{e}_1 \cdot \mathbf{u}_{,1})\mathbf{e}_1 + \bar{R}\dot{\mathbf{u}} + \lambda(\mathbf{e}_2 \cdot \dot{\mathbf{u}})\mathbf{e}_1 - p_f\beta\mathbf{e}_1 - M\beta^2\xi\mathbf{e}_1 + 3M\alpha_m\theta\beta\mathbf{e}_1 - A\theta\mathbf{e}_1 - p_s\mathbf{e}_1 \\
&= \bar{Q}\mathbf{u}_{,1} + \bar{R}\dot{\mathbf{u}} - p_f\beta\mathbf{e}_1 + 3M\alpha_m\theta\beta\mathbf{e}_1 - A\theta\mathbf{e}_1 - p_s\mathbf{e}_1.
\end{aligned}$$

Similarly, for $\mathbf{t}_2$:

$$\begin{aligned}
\mathbf{t}_2 &= \bar{R}\mathbf{u}_{,1} + \lambda(\mathbf{e}_1 \otimes \mathbf{e}_2)\mathbf{u}_{,1} + \bar{T}\dot{\mathbf{u}} + \lambda(\mathbf{e}_2 \otimes \mathbf{e}_2)\dot{\mathbf{u}} - (p_f + M\beta\xi - 3M\alpha_m\theta)\beta\mathbf{e}_2 - A\theta\mathbf{e}_2 - p_s\mathbf{e}_2 \\
&= \bar{R}\mathbf{u}_{,1} + \lambda(\mathbf{e}_1 \cdot \mathbf{u}_{,1})\mathbf{e}_2 + \bar{T}\dot{\mathbf{u}} + \lambda(\mathbf{e}_2 \cdot \dot{\mathbf{u}})\mathbf{e}_2 - p_f\beta\mathbf{e}_2 - M\beta^2\xi\mathbf{e}_2 + 3M\alpha_m\theta\beta\mathbf{e}_2 - A\theta\mathbf{e}_2 - p_s\mathbf{e}_2 \\
&= \bar{R}\mathbf{u}_{,1} + \bar{T}\dot{\mathbf{u}} - p_f\beta\mathbf{e}_2 + 3M\alpha_m\theta\beta\mathbf{e}_2 - A\theta\mathbf{e}_2 - p_s\mathbf{e}_2.
\end{aligned}$$

As a result, the fundamental force vectors simplify to

$$\mathbf{t}_1 = \bar{Q}\mathbf{u}_{,1} + \bar{R}\dot{\mathbf{u}} + 3M\alpha_m\theta\boldsymbol{\beta}\mathbf{e}_1 - A\theta\mathbf{e}_1 - p_f\boldsymbol{\beta}\mathbf{e}_1 - p_s\mathbf{e}_1, \tag{6.23a}$$

$$\mathbf{t}_2 = \bar{R}^T\mathbf{u}_{,1} + \bar{T}\dot{\mathbf{u}} + 3M\alpha_m\theta\boldsymbol{\beta}\mathbf{e}_2 - A\theta\mathbf{e}_2 - p_f\boldsymbol{\beta}\mathbf{e}_2 - p_s\mathbf{e}_2. \tag{6.23b}$$

6.3.1 Lagrangian Density in Terms of Transform Matrices

Rewriting the Lagrangian density using the transform matrices, we have

$$L = \frac{1}{2}\mathbf{u}_{,1}\cdot(\bar{Q}+\lambda\mathbf{e}_1\otimes\mathbf{e}_1)\mathbf{u}_{,1} + \mathbf{u}_{,1}\cdot(\bar{R}+\lambda\mathbf{e}_1\otimes\mathbf{e}_2)\dot{\mathbf{u}} - \frac{1}{2}\dot{\mathbf{u}}\cdot(\bar{T}+\lambda\mathbf{e}_2\otimes\mathbf{e}_2)\dot{\mathbf{u}} - M\beta\xi e - A\theta e + 3\alpha_m M\xi\theta - p_s e - \frac{1}{2}\frac{C_\epsilon}{T_0}\theta^2 + \frac{1}{2}M\xi^2.$$

Expanding the terms associated with the transform matrices:

$$L = \frac{1}{2}\mathbf{u}_{,1}\cdot\bar{Q}\mathbf{u}_{,1} + \frac{\lambda}{2}(\mathbf{u}_{,1}\cdot\mathbf{e}_1)^2 + \mathbf{u}_{,1}\cdot\bar{R}\dot{\mathbf{u}} + \lambda(\mathbf{u}_{,1}\cdot\mathbf{e}_1)(\dot{\mathbf{u}}\cdot\mathbf{e}_2) - \frac{1}{2}\dot{\mathbf{u}}\cdot\bar{T}\dot{\mathbf{u}} - \frac{\lambda}{2}(\dot{\mathbf{u}}\cdot\mathbf{e}_2)^2 - M\beta\xi e - A\theta e + 3\alpha_m M\xi\theta - p_s e - \frac{1}{2}\frac{C_\epsilon}{T_0}\theta^2 + \frac{1}{2}M\xi^2.$$

The terms involving λ can be grouped to highlight the quadratic form:

$$L = \frac{1}{2}\mathbf{u}_{,1}\cdot\bar{Q}\mathbf{u}_{,1} + \mathbf{u}_{,1}\cdot\bar{R}\dot{\mathbf{u}} - \frac{1}{2}\dot{\mathbf{u}}\cdot\bar{T}\dot{\mathbf{u}} + \frac{\lambda}{2}\Big[(\mathbf{u}_{,1}\cdot\mathbf{e}_1)^2 - 2(\mathbf{u}_{,1}\cdot\mathbf{e}_1)(\dot{\mathbf{u}}\cdot\mathbf{e}_2) + (\dot{\mathbf{u}}\cdot\mathbf{e}_2)^2\Big] - M\beta\xi e - A\theta e + 3\alpha_m M\xi\theta - p_s e - \frac{1}{2}\frac{C_\epsilon}{T_0}\theta^2 + \frac{1}{2}M\xi^2.$$

Finally, writing the Lagrangian in compact form as a function of the generalized coordinates and velocities:

$$\begin{aligned} L(\mathbf{u}_{,1},\dot{\mathbf{u}},\mathbf{w}_{,1},\dot{\mathbf{w}}) = {} & \frac{1}{2}\mathbf{u}_{,1}\cdot\bar{Q}\mathbf{u}_{,1} + \mathbf{u}_{,1}\cdot\bar{R}\dot{\mathbf{u}} - \frac{1}{2}\dot{\mathbf{u}}\cdot\bar{T}\dot{\mathbf{u}} - A\theta\mathbf{e} \\ & + 3\alpha_m M\xi\theta - M\boldsymbol{\beta}\xi\mathbf{e} + \frac{1}{2}M(\mathbf{w}_{,1}\cdot\mathbf{e}_1 + \mathbf{w}_{,2}\cdot\mathbf{e}_2)^2 \\ & - \frac{1}{2}\frac{C_e}{T_0}\theta^2 - p_s\mathbf{e}. \end{aligned} \tag{6.24}$$

6.3.2 Conjugate Momenta

The conjugate momenta thus takes the form

$$\mathbf{p}_1 = \frac{\partial L}{\partial\dot{\mathbf{u}}}, \quad \mathbf{p}_2 = \frac{\partial L}{\partial\dot{\mathbf{w}}} \tag{6.25}$$

For $\mathbf{p_1}$

$$\mathbf{p}_1 = \frac{\partial L}{\partial\dot{\mathbf{u}}} = \bar{R}^T\mathbf{u}_{,1} + \bar{T}\mathbf{u}_{,2} - M\boldsymbol{\beta}\xi\mathbf{e}_2 - A\theta\mathbf{e}_2 - p_s\mathbf{e}_2 \tag{6.26}$$

by virtue of the definition of p_f i.e $-p_f + 3\alpha_m M\theta = -M\xi$, we get

$$\mathbf{p}_1 = \bar{R}^T \mathbf{u}_{,1} + \bar{T}\mathbf{u}_{,2} - 3\alpha_m M\boldsymbol{\beta}\theta\mathbf{e}_2 - p_f\boldsymbol{\beta}\mathbf{e}_2 - A\theta\mathbf{e}_2 - p_s\mathbf{e}_2 = \mathbf{t}_2 \tag{6.27}$$

For $\mathbf{p}_2$

$$\mathbf{p}_2 = \frac{\partial L}{\partial \dot{\mathbf{w}}} = -(3\alpha_m M\theta + M\xi)\mathbf{e}_2 = -p_f\mathbf{e}_2 \tag{6.28}$$

Computing (6.4) for ξ gives

$$\xi = \frac{p_f}{m} - 3\alpha_m\theta \tag{6.29}$$

while solving (6.23b) for $\dot{\mathbf{u}}$ gives

$$\dot{\mathbf{u}} = \bar{T}^{-1}(\mathbf{t}_2 - \bar{R}^T\mathbf{u}_{,1} + M\boldsymbol{\beta}\xi\mathbf{e}_2 + A\theta\mathbf{e}_2 + p_f\beta\mathbf{e}_2 + p_s\mathbf{e}_2). \tag{6.30}$$

perform scalar multiplication on (6.30) with $\mathbf{e}_2$ gives

$$\dot{\mathbf{u}}\cdot\mathbf{e}_2 = \left(\bar{T}^{-1}(\mathbf{t}_2 - \bar{R}^T\mathbf{u}_{,1} + M\boldsymbol{\beta}\xi\mathbf{e}_2 + A\theta\mathbf{e}_2 + p_f\beta\mathbf{e}_2 + p_s\mathbf{e}_2)\right)\cdot\mathbf{e}_2 \tag{6.31}$$

Since

$$\mathbf{u}_{,1}\cdot\mathbf{e}_1 + \dot{\mathbf{u}}\cdot\mathbf{e}_2 = 0$$

So

$$\begin{aligned}
-\mathbf{u}_{,1}\cdot\mathbf{e}_1 &= \left(\bar{T}^{-1}(\mathbf{t}_2 - \bar{R}^T\mathbf{u}_{,1} + M\beta\xi\mathbf{e}_2 + A\theta\mathbf{e}_2 + p_f\beta\mathbf{e}_2\right)\cdot\mathbf{e}_2 + \mathbf{e}_2\cdot\bar{T}^{-1}\mathbf{e}_2 p_s \\
-p_s\mathbf{e}_2\cdot\bar{T}^{-1}\mathbf{e}_2 &= \left(\bar{T}^{-1}(\mathbf{t}_2 - \bar{R}^{-1}\mathbf{u}_{,1} + M\beta\xi\mathbf{e}_2 + A\theta\mathbf{e}_2 + p_f\beta\mathbf{e}_2\right)\cdot\mathbf{e}_2 + \mathbf{u}_{,1}\cdot\mathbf{e}_1 \\
p_s &= \zeta_1\left(\bar{T}^{-1}(\bar{R}^T - \mathbf{t}_2 - M\beta\xi\mathbf{e}_2 - A\theta\mathbf{e}_2 - p_f\beta\mathbf{e}_2) - \mathbf{u}_{,1}\cdot\mathbf{e}_1\right)
\end{aligned} \tag{6.32}$$

with

$$\zeta_1 = \frac{1}{\mathbf{e}_2\cdot\bar{T}^{-1}\mathbf{e}_2} \tag{6.33}$$

The last term is always positive under the assumption of strong ellipticity.

By definition, the Hamiltonian density is

$$H = p_1\cdot\dot{\mathbf{u}} + p_2\cdot\dot{\mathbf{w}} - L, \tag{6.34}$$

where the generalized momenta satisfy

$$\begin{aligned} p_f &= M\xi + 3M\alpha_m\theta \\ p_f &= M(-w_{,1}\cdot\mathbf{e}_1 + w_{,2}\cdot\mathbf{e}_2) + 3M\alpha_m\theta, \quad \text{since } \xi = -\operatorname{div}\mathbf{w}. \end{aligned}$$

From (6.4) and (6.5), it follows that

$$-\dot{\mathbf{w}}\cdot\mathbf{e}_2 = \frac{p_f}{M} - 3\alpha_m\theta + w_{,1}\cdot\mathbf{e}_1. \tag{6.35}$$

Substituting the expressions for p_1, p_2, $\dot{\mathbf{u}}$, $\dot{\mathbf{w}}$ and L into (6.34), we get

$$\begin{aligned} H &= \mathbf{t}_2\cdot\dot{\mathbf{u}} - p_f\,\mathbf{e}_2\cdot\dot{\mathbf{w}} - L \\ &= \mathbf{t}_2\cdot\dot{\mathbf{u}} + p_f\Big(\frac{p_f}{M} - 3\alpha_m\theta + \mathbf{w}_{,1}\cdot\mathbf{e}_1\Big) - L \\ &= \mathbf{t}_2\cdot\dot{\mathbf{u}} + \frac{p_f^2}{M} - 3\alpha_m\theta\,p_f + p_f\,\mathbf{w}_{,1}\cdot\mathbf{e}_1 - L. \end{aligned}$$

Expanding L and collecting terms leads to

$$\begin{aligned} H = \dot{\mathbf{u}}\cdot\Big(\mathbf{t}_2 - \bar{R}^T\mathbf{u}_{,1} + M\beta\xi\mathbf{e}_2 + A\theta\mathbf{e}_2 + p_s\mathbf{e}_2\Big) - \frac{1}{2}\dot{\mathbf{u}}\cdot\bar{T}\dot{\mathbf{u}} + \frac{p_f^2}{M} - 3\alpha_m\theta\,p_f + p_f\,\mathbf{w}_{,1}\cdot\mathbf{e}_1 \\ - M\beta\xi\,\mathbf{u}_{,1}\cdot\mathbf{e}_1 - A\theta\,\mathbf{u}_{,1}\cdot\mathbf{e}_1 - 3\alpha_m M\xi\theta + p_s\,\mathbf{u}_{,1}\cdot\mathbf{e}_1 + \frac{1}{2}\frac{C_\epsilon}{T_0}\theta^2 - \frac{1}{2}M\xi^2 + \frac{1}{2}\mathbf{u}_{,1}\cdot\bar{Q}\mathbf{u}_{,1}. \end{aligned}$$

Finally, in compact form:

$$\begin{aligned} H = \Big(\mathbf{t}_2 - \bar{R}^T\mathbf{u}_{,1} + M\beta\xi\mathbf{e}_2 + A\theta\mathbf{e}_2 + p_s\mathbf{e}_2\Big)\cdot\bar{T}^{-1}\Big(\mathbf{t}_2 - \bar{R}^T\mathbf{u}_{,1} + M\beta\xi\mathbf{e}_2 + A\theta\mathbf{e}_2 + p_s\mathbf{e}_2\Big) \\ + \frac{p_f^2}{M} - 3\alpha_m\theta\,p_f + p_f\,\mathbf{w}_{,1}\cdot\mathbf{e}_1 - M\beta\xi\,\mathbf{u}_{,1}\cdot\mathbf{e}_1 - A\theta\,\mathbf{u}_{,1}\cdot\mathbf{e}_1 - 3\alpha_m M\xi\theta \\ + p_s\,\mathbf{u}_{,1}\cdot\mathbf{e}_1 + \frac{1}{2}\frac{C_\epsilon}{T_0}\theta^2 - \frac{1}{2}M\xi^2 + \frac{1}{2}\mathbf{u}_{,1}\cdot\bar{Q}\mathbf{u}_{,1}. \end{aligned} \tag{6.36}$$

6.3.3 Canonical Equations

The canonical equations are defined as

$$\dot{\mathbf{q}} = \frac{\partial H}{\partial p}, \quad \dot{\mathbf{p}} = -\frac{\partial H}{\partial q}. \tag{6.37}$$

From the Hamiltonian (6.36), the generalized velocity $\dot{\mathbf{u}}$ is

$$\dot{\mathbf{u}} = \frac{\partial H}{\partial \mathbf{t}_2} = \bar{T}^{-1}\Big(\mathbf{t}_2 - \bar{R}^T\mathbf{u}_{,1} + M\beta\xi\mathbf{e}_2 + A\theta\mathbf{e}_2 + p_s\mathbf{e}_2\Big), \tag{6.38}$$

which corresponds directly to $\dot{\mathbf{u}}$.

Correspondingly, the generalized velocity of the fluid component along $\mathbf{e}_2$ is

$$\begin{aligned}\dot{\mathbf{w}} \cdot \mathbf{e}_2 &= -\frac{\partial H}{\partial p_f} \\ &= -\frac{p_f}{M} - 3\alpha_m \theta + \mathbf{w}_{,1} \cdot \mathbf{e}_1 \\ &= -\mathbf{w}_{,1} \cdot \mathbf{e}_1 - \frac{p_f}{M} + 3\alpha_m \theta, \end{aligned} \tag{6.39}$$

which verifies consistency with $\xi = -\operatorname{div} \mathbf{w}$.

The evolution of the generalized force $\mathbf{t}_2$ is

$$\begin{aligned}\dot{\mathbf{t}}_2 &= \frac{\partial H}{\partial \mathbf{u}_{,1}} \\ &= \bar{R}^T \bar{T}^{-1} \Big(\mathbf{t}_2 - \bar{R}^T \mathbf{u}_{,1} + M\beta\xi \mathbf{e}_2 + A\theta \mathbf{e}_2 + p_s \mathbf{e}_2 \Big) + \bar{Q} \mathbf{u}_{,1} - M\beta\xi \mathbf{e}_1 - A\theta \mathbf{e}_1 + p_s \mathbf{e}_1, \end{aligned} \tag{6.40}$$

this agrees with Equation (6.30), where multiplying the bracketed term by $\bar{T}^{-1}$ results in $\dot{\mathbf{u}}$ and in consideration of first equation of (6.10), leads to (6.11a). In the same manner, the evolution of the fluid momentum is

$$-\dot{p}_f \, \mathbf{e}_2 = -\frac{\partial H}{\partial \mathbf{w}_{,1}} = (p_f \, \mathbf{e}_1)_{,1}. \tag{6.41}$$

that is exactly equal to (6.11b).

6.4 Conservation of Hamiltonian Density

The Hamiltonian density H can be written as

$$\begin{aligned} H = &\Big(\mathbf{t}_2 - \bar{R}^T \mathbf{u}_{,1} + M\beta\xi \mathbf{e}_2 + A\theta \mathbf{e}_2 + p_s \mathbf{e}_2 \Big) \cdot \bar{T}^{-1} \Big(\mathbf{t}_2 - \bar{R}^T \mathbf{u}_{,1} + M\beta\xi \mathbf{e}_2 + A\theta \mathbf{e}_2 + p_s \mathbf{e}_2 \Big) \\ &+ \frac{p_f^2}{M} - 3\alpha_m \theta \, p_f + p_f \, \mathbf{w}_{,1} \cdot \mathbf{e}_1 - M\beta\xi \, \mathbf{u}_{,1} \cdot \mathbf{e}_1 - A\theta \, \mathbf{u}_{,1} \cdot \mathbf{e}_1 - 3\alpha_m M \xi \theta \\ &+ p_s \, \mathbf{u}_{,1} \cdot \mathbf{e}_1 + \frac{1}{2} \frac{C_\epsilon}{T_0} \theta^2 - \frac{1}{2} M \xi^2 + \frac{1}{2} \mathbf{u}_{,1} \cdot \bar{Q} \mathbf{u}_{,1}. \end{aligned} \tag{6.42}$$

where $\mathbf{t}_2$ represents the fundamental traction vector, $\hat{Q}, \hat{R}, \hat{T}$ denote generalized Stroh matrices adapted to the poroelastic setting, $\mathbf{u}_{,1}$ and $\mathbf{w}_{,1}$ are spatial derivatives of the displacement fields, $\zeta = -\operatorname{div} \mathbf{w}$ characterizes the volumetric fluid exchange, p_f is the pore fluid pressure, p is a Lagrange multiplier enforcing incompressibility.

6.4.1 Independence of x_2

As described in last two chapters, It can be seen that H has no explicit dependence on the coordinate x_2 and it remains invariant under translation along x_2.

6.4.2 Canonical Equations

Similarly, as in the previous cases, the generalized coordinates are consistent with equations (6.38) and (6.39). Consequently, the relations obtained for the conjugate momenta agree with equations (6.40) and (6.41). This confirms that the Hamiltonian framework is fully consistent with the equilibrium description.

6.4.3 Physical Implications

For a coupled thermoporoelastic material with an incompressible solid skeleton, conservation of the Hamiltonian density reflects a constrained energetic structure in which volumetric deformation of the solid is suppressed. As a result, the pore pressure acts as a Lagrange multiplier, enforcing incompressibility, and the mechanical-thermal coupling is realized through isochoric deformation and temperature variations. Without the dissipative processes, redistribution of energy can only take place through fluid pressure and thermal field adjustments, which results in the formation of non-attenuating constrained wave modes and a clearly manifested elastic reflection and transmission at material interfaces. This theoretical model separates the role of incompressibility on the localization and stability effects inherent to thermoporoelastic materials.

6.5 Integral Representation and Symplectic Structure

The integral representation of the Hamiltonian density provides a global perspective on conservation laws and reveals its underlying symplectic structure, closely linked with the Stroh formalism. Central to this representation is the fundamental matrix $\hat{N}$ (Ting, 1996), which possesses a symmetric block form:

$$\hat{N} = \begin{bmatrix} N_3 & N_1^T \\ N_1 & N_2 \end{bmatrix},$$

with the 3×3 sub-blocks defined as

$$N_1 = -\bar{T}^{-1}\bar{R}^T, \qquad N_2 = \bar{T}^{-1}, \qquad N_3 = \bar{R}\bar{T}^{-1}\bar{R}^T - \bar{Q}. \tag{6.43}$$

Here, N_2 is positive definite, while $-N_3$ is positive semidefinite, ensuring the stability of the associated quadratic form.

The Hamiltonian density admits the compact quadratic representation

$$H = \tfrac{1}{2}\,\zeta \cdot \hat{I}\hat{N}\zeta,$$

where ζ is a vector of mixed physical dimensions: the displacement-like component has units of length, while the conjugate part has the dimension of force per unit length. Consequently, N_1 is dimensionless, N_2 represents compliance (inverse stress), and N_3 has the dimension of stress.

The quadratic structure is made explicit using the 6×6 constant matrix (Ting, 1996)

$$\hat{I} = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix},$$

which, together with the symmetry of N_2 and N_3, yields the symmetric fundamental matrix

$$\hat{I}\hat{N} = \begin{bmatrix} N_3 & N_1^T \\ N_1 & N_2 \end{bmatrix} = (\hat{I}\hat{N})^T.$$

6.5.1 Traveling Waves and Eigenvalue Problem

For traveling wave solutions of the form $\zeta = \Xi f(x_1 + px_2)$, the governing equations reduce to a right eigenvalue problem,

$$\hat{N}\,\Xi = p\,\Xi,$$

with eigenvector Ξ and eigenvalue p associated with propagation characteristics. Substitution into the integral representation of the energy yields

$$E = \int_\Sigma \tfrac{1}{2}\,\zeta \cdot \hat{I}\hat{N}\zeta \, dx_1 dx_3,$$

which highlights how wave energy is encoded in the quadratic form.

6.5.2 Conservation and Symplectic Structure

The crucial feature of the Hamiltonian is its independence of the "time-like" coordinate x_2. As a consequence, the total energy satisfies

$$\frac{dE}{dx_2} = 0,$$

expressing conservation along the evolution direction. This invariance arises directly from the canonical equations together with the symmetry of $\hat{N}$. Equivalently, the conservation law can be written as

$$\int_{\Sigma} \zeta \cdot \hat{I}\hat{N}\zeta \, dx_1 dx_3 = \text{constant},$$

where $\zeta = [\phi, u]^T$ collects the unknown field variables. This structure is intrinsically symplectic, ensuring that energy flux is preserved across the medium.

6.5.3 Connection to the Stroh Formalism

The Stroh formalism appears naturally within this Hamiltonian framework. The evolution equation takes the form

$$\frac{\partial}{\partial x_2}\zeta = \hat{N}\,\frac{\partial}{\partial x_1}\zeta,$$

with $\hat{N}$ identified as the fundamental block matrix of anisotropic elasticity. This provides a direct relationship between the Hamilton formulation, its symplectic structure and the classical Stroh expression, relating the conservation laws with the wave propagation in an anisotropic and poroelastic material.

6.6 Conclusion

This chapter develops the classical poroelastic model of Biot by introducing thermal effects to a coupled thermoporoelastic model that can be used in anisotropic materials with specific attention to materials with an incompressible solid skeleton. The model is built using the principle of reversibility such that it can be treated in a Hamilton based Stroh-like formalism, thus allowing systematic and thermodynamically consistent treatment of non-dissipative thermo-fluid structure interactions.

The incorporation of thermal effects into the classical poroelastic model modified to incompressible solid matrices allows finding the thermodynamically conjugate pairs of variables. The result is a coherent account of the coupled thermal, mechanical and fluid response in terms of energy, which is critical to the correct representation of the physics of the low compressibility porous media. The Hamiltonian inspired Stroh-like formulation proved to be a robust and elegant tool for examining anisotropic thermoporoelastic systems, particularly under conditions of perfect drainage and thermal equilibrium. Both the reversible dynamics and incompressibility of solids simultaneously provide the state variables with strict constraints on the allowable

admissible states of the system, which lower the effective degrees of freedom and unveil hidden conservation laws and symmetries. These features highlight the deep interplay between thermodynamic structure and kinematic restrictions.

The findings demonstrate the effectiveness of the Hamiltonian framework in modeling fully coupled thermo–fluid–mechanical behavior in porous materials with nearly incompressible solid phases. This establishes a rigorous theoretical basis for analytical studies and experimental interpretation, with relevance to materials such as saturated soils, hydrogels, and biological tissues. Although the present formulation is limited to reversible processes, it provides an essential reference point for understanding anisotropic porous materials under solid incompressibility. While this constraint simplifies certain aspects of the governing equations, it also introduces nontrivial coupling mechanisms that require careful treatment in more general settings.

Chapter 7

Summary and Future Work

This chapter provides a summary of the work conducted within the framework of this thesis and presents potential future research directions.

7.1 Summary

The motivation and scope of the thesis were presented in Chapter 1. The chapter indicated the necessity of an integrated and vibrant homogeneous framework to investigate anisotropic elastic and porous materials, especially where more than a single physical field of study like fluid flow and temperature are used. The thesis goals were then established, with special focus on the Hamiltonian interpretation of the Stroh formalism and its generalization to poroelastic and thermoporoelastic systems in the case of reversibility and incompressibility.

The chapter 2 offered the groundwork of the Stroh formalism and its relationship with the Hamilton mechanics. The chapter opened with the governing equations of linear anisotropic elasticity and gave the eigenvalue problem of Stroh and studied how eigenvalues are classified into simple, semi-simple and degenerate, as well as the structure of eigenvectors and Jordan chains. Rotational invariance and the properties of generalised eigenvectors in degenerate cases in particular received special consideration. Angular averaging and the surface impedance tensor were proposed as critical mathematical tools which define the main properties of Hermiticity, positive definiteness and rotational invariance. Basing on this, the chapter also determined the theoretical connection between the Stroh formalism and the Hamiltonian mechanics. Beginning with variational principles, a Lagrangian formulation (using generalised coordinates) of discrete and continuum systems was formulated and the Hamiltonian formulation derived by the Legendre transformation. This disclosed the canonical form, conservation laws and symplectic properties of the linear conservative systems. Among

the key lessons was that the Stroh formalism can be viewed as a spatial Hamiltonian system, to which canonical pairs are constituted out of displacements and tractions. This coherent perspective offers a strong structure of studying coupled continua and gives the conceptual basis of all the further extensions of the formalism.

In Chapter 3 the background on poroelasticity and extensions was given in detail. The classical poroelasticity theory was revised containing the basic balance equations, the constitutive equations, and the coupling between solid deformation and pore-fluid movement. Further extensions to thermoporoelasticity and other multiphysical models were then mentioned, showing how further field variables like temperature could be included into a consistent theoretical approach. This chapter laid down the physical background and modeling assumptions required by other chapters of the developments.

Chapters 4 to 6 mainly consist of the original contributions of this thesis. The chapters dwell on the formulation and the implementation of the Stroh-Hamiltonian formalism to multiphysics porous materials subjected to deterministic mechanical and thermal forces.

A Stroh-Hamiltonian framework of reversible anisotropic poroelasticity was derived in Chapter 4 with an incompressible solid skeleton. Beginning with the theory put forward by Biot, the governing equations were formulated through a variational approach resulting in a form of a Hamiltonian representation. Canonical variables and conjugate momenta were found and the resulting Hamiltonian density proved to be quadratic and conservative. The formulation did naturally fit the Stroh formalism allowing the study of wave propagation and energy transport in incompressible poroelastic media.

The framework was extended to reversible coupled-thermoporoelasticity in Chapter 5. Generalized Duhamel-Neumann constitutive relations were added to include thermal effects, and a coherent variational framework grounded on the Eshelby principle had been developed. This allows the uniform treatment of the elastic, fluid and thermal fields in the same framework and maintains the consistency of energy as well as the characteristic properties of reversibility and strong ellipticity.

In Chapter 6, the theory was further specialised to the coupled-thermoporoelastic materials of incompressible solid medium. The reduced system of admissible state variables was found by the joint assumptions of reversibility and solid incompressibility, and further symmetries and conservation laws were discovered. The resulting Stroh-like Hamilton formulation furnished a rigorous programme of understanding fully coupled thermo-fluid-mechanical behaviour in anisotropic porous media which include saturated soils, hydrogels, and biological tissues.

In general, this thesis proves that the Hamiltonian interpretation of the Stroh

formalism provides a potent and unifying perspective on the dynamics of elastic, poroelastic, and thermoporoelastic systems. The work provides physical interpretation and mathematical framework of the analysis of coupled multiphysics problems in anisotropic continua by bridging variational principles, Hamilton mechanics, and the Stroh approach.

7.2 Further Work

The current work is limited to reversible processes and idealised assumptions on materials. Some noteworthy extensions may be taken into consideration in the future research. The next logical thing would be the introduction of irreversible effects like heat conduction, fluid viscosity and dissipation in fluid flow. These mechanisms could only be included in the Hamilton framework by generalizing it to non-conservative processes, such as generalized formulations or metriplectic formulations.

The other direction is also the relaxations of the incompressibility of the solid skeleton. The consideration of finite solid compressibility would expand the applicability of the theory to a larger variety of geomaterials and biological tissues, and also provide some other coupling mechanisms, which would be subject to close consideration.

Future research can also address nonlinear kinematics, dynamic porosity and finite deformation effects, that are especially pertinent in finite strain applications with soft porous media. Regarding the application, the current theoretical concept can be applied to numerical approaches and experimental research to examine the effects of wave propagation, stability, and localization in the intricate porous media.

In general, the findings of this thesis give a strict theoretical basis on which more general and realistic coupled poroelastic and thermoporoelastic behaviour models can be constructed, and maintaining the energetic and structural insight of the Stroh Hamiltonian model.

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