

University of Salerno

Department of Mathematics

Doctoral dissertation

Characterizing cognitive role-playing in a rationality perspective to move students towards the scientific enculturation

Ph.D. in Mathematics
Cycle: XXXVIII

Supervisors:
Prof.ssa Cristina Coppola
Prof.ssa Giovannina Albano

Candidate:
Angela Donatiello

Coordinator:
Prof. Vincenzo Tibullo

2024/2025

a Laura e Pasquale
a Vincenzo
costruttori con me di sogni e possibilità

Acknowledgements

Quando ho intrapreso il percorso di dottorato, non avevo ancora ben chiaro cosa avrebbe davvero rappresentato per me, quali cambiamenti interiori avrebbe portato, come avrebbe contribuito a modificare il mio sguardo da insegnante attenta e osservatrice dei processi di classe a ricercatrice accorta e ponderata, che quei processi avrebbe cercato di interpretare.

Fortunatamente in ogni percorso di vita non si è mai soli e oggi desidero ringraziare tutti coloro che mi hanno accompagnato in questo viaggio, alcuni lasciando un segno importante e visibile anche se per un breve tempo, altri continuando ad accompagnarmi ancora oggi con la profondità di chi impara a starmi accanto conoscendo tutti i miei limiti e le mie fragilità.

Desidero innanzitutto ringraziare le mie Tutor, la prof.ssa Cristina Coppola e la prof.ssa Giovannina Albano che per me sono molto più che supervisors di dottorato, rappresentando piuttosto vere e proprie guide illuminate ed accoglienti, che mi hanno sostenuto nei momenti di sconforto, credendo sempre fortemente in me. Hanno accolto il mio entusiasmo, aiutandomi a canalizzarlo nella giusta direzione e lasciando che scoprissi giorno per giorno quanto la passione per la ricerca si stesse facendo strada in me. Il confronto costante con loro è sempre fonte di grande ispirazione, perché come sempre dico "solo se dialogo con qualcuno riesco realmente a pensare". Grazie Cristina per tutte le volte che le cose da fare sembravano tante e temevo potessero sopraffarmi, ma tu hai sempre saputo ascoltarmi e rincuorarmi con dolcezza, ricordandomi con il sorriso che "un passo alla volta" si fa tutto e anche di più. Grazie Giovanna per avermi guidato fin qui con fermezza, passione e cura, avendo sempre chiara la rotta e accogliendo le mie proposte con un deciso "mi hai convinto". Grazie per la fiducia che avete sempre posto e ancora ponete in me.

Ringrazio inoltre di cuore il prof. Paolo Boero per i suoi consigli preziosi, i ricchi momenti di confronto e le lunghe conversazioni di ricerca in cui non mi ha mai fatto sentire semplicemente una dottoranda in formazione, ma molto spesso una ricercatrice alla pari con cui confrontarsi. Di questo sono davvero immensamente grata, anche per il suo sguardo visionario e la passione con cui guida noi giovani generazioni di ricercatori.

Il mio ringraziamento va alla prof.ssa Laura Branchetti e al prof. Osama Swidan per aver dedicato il loro tempo prezioso alla lettura della mia tesi, offrendomi spunti di riflessioni e suggerimenti importanti che mi hanno consentito di strutturare al meglio la presentazione di questa dissertazione.

Un viaggio ha bisogno spesso di approdi sicuri in cui riparare e per questo sono profondamente grata al gruppo di ricerca MINDER Lab di cui faccio parte, i meravigliosi Didamat(ti), perché sono per me approdo e riparo, abbraccio, famiglia, crescita, casa. Annamaria, Gemma, Anna, Luca, Loredana, Rosalia e Angela, insieme con Giovanna e Cristina, sono felice e grata di sapervi parte imprescindibile del mio cammino. Luca, a te va un ringraziamento speciale, perché non era affatto scontato pensare di poter trovare nel mio collega di scrivania un compagno di viaggio con cui crescere, confrontarsi, ridere, piangere, capirsi al volo al primo sguardo, ritrovarsi ore a parlare di ricerca per poi dirsi semplicemente: "Che bello che ci sei!". Grazie per i momenti intensi di karaoke, le camminate per schiarirsi le idee, le colazioni prima di ogni missione, le misurazioni condivise di aiuole, rampe e fontane e la consapevolezza di un'amicizia che resta salda nel tempo.

Ringrazio poi la meravigliosa, ricca e appassionata comunità di dottorandi e giovani ricercatori dell'AIRDM che mi ha insegnato cosa significa essere davvero gruppo, uno spazio in cui non ti senti mai realmente solo, ma trovi accoglienza autentica e consigli che sanno di buono. Grazie Giuseppe, per la tua presenza discreta e silenziosa, ma costante; grazie Agnese, che al mio sentirmi come qualcuno che brancola nella nebbia mi hai ricordato che "nella ricerca il sentiero non è mai tracciato e che la nebbia si dipana perché quel sentiero lo stai tracciando tu"; grazie Alessandra, per il tuo illuminante ed accogliente sorriso con cui sai offrire il giusto equilibrio alle cose; grazie Bianca, per aver condiviso con me pezzi di ricerca e di vita anche nel mio piccolo borgo di Padula; grazie Gaia, per la tua lucidità e la forza accogliente che si sprigiona dal tuo solo esserci; grazie Carlotta, per la sincerità schietta e le risate autentiche; grazie Gabriella, Sara, Francesco, Ginevra, Giada, Lele, Margherita, Lorenzo, Daniele, Chiara, Helena, Silvia, Laura, per il confronto autentico, le risate buffe e le sane follie.

Un grande grazie va anche all'altra comunità che mi ha accolta e accompagnata in questi anni, gli amici dell'Aula 27 del Dipartimento di Matematica, un'aula che ogni anno cambia volto, ma che resta sempre un luogo sicuro in cui sentirsi a casa: Kinza, Paola, Giulia, Verdiana, Andrea, Matteo, Pia, Sabina, Alissa, Sebastiano, Roberto, Rosamaria, Salvatore, Federico, Michele, Carmine, Federica con Kelly, Gabriella, Maria Assunta, Enrico, Serena, Antonio, Vincenzo, Marialaura semplicemente grazie per gli allegri Secret Santa, i caffè condivisi, i cubi di Rubik e rompicapi matematici con mia figlia Laura, le risate irriverenti e gli abbracci sinceri.

Un grazie di cuore anche al prof. Gianni Vincenzi, per aver sempre creduto in me fin dai tempi dell'università e avermi sempre incoraggiato ad intraprendere il percorso di dottorato.

Un grazie di vero cuore lo rivolgo alle docenti Rosalba Garrisi e Vitangela Tortoriello e alle studentesse e agli studenti delle classi del Liceo Scientifico "Carlo Pisacane" di Padula e "Pomponio Leto" di Teggiano per aver accolto nelle loro aule Promoter, Pest, Blogger e Boss sentendoli propri e facendoli vivere in modo nuovo; grazie per la vostra gentile e aperta disponibilità.

Grazie a Marta, amica sincera e profonda, che anche a distanza di chilometri è in grado di sostenermi nei piccoli e grandi dilemmi quotidiani, offrendomi sguardi da differenti prospettive. Grazie per la tua leale lucidità e l'amicizia autentica.

Grazie al mio caro amico Armando, perché nei momenti di sintesi in cui provo a comunicare il cuore della mia ricerca, mi ricordi con le parole di Feynman che "una cosa la conosci veramente solo se sei in grado di spiegarla ad un bambino".

Grazie a tutti gli amici vicini e lontani che in questi anni mi hanno sempre sostenuto, incoraggiandomi a non mollare e ascoltando i miei racconti sulla ricerca con sguardo curioso e attento. Grazie Alessandra, Alessandro, Silvia, Anna Maria, Andrea, Carmela, Rino, per la vostra presenza viva anche nei silenzi.

Un grande grazie desidero rivolgerlo alle mie compagne di storie Francesca, Malvina, Gemma, Caterina, Raffaella, Laura, Alessandra e a tutte le visionarie sognatrici di Nati per Leggere Campania, per avermi insegnato che le storie narrate e illustrate costituiscono "sistemi cognitivi complessi" in grado di far nascere riflessioni e provocare emozioni.

Un grazie speciale va a mio fratello Piercarlo, riferimento stabile nei momenti di impasse, guida ferma e lucida. Grazie perchè sei sempre in grado di farmi riflettere, ponendomi le giuste domande. Non sono mai riuscita ad immaginare scelte importanti senza un confronto reale e approfondito con te. Grazie per esserci sempre e consentirmi di osservare le cose da prospettive nuove.

Grazie ai miei genitori, Filomena e Ulderico, le mie radici, i miei punti fermi, le colonne portanti e i fari a cui rivolgere sempre lo sguardo. Grazie per l'esempio di dedizione al lavoro, lealtà e senso pieno di responsabilità che mi avete donato. Ogni mio piccolo traguardo parla anche un po' di voi, perché in esso è riflesso il vostro profondo e costante insegnamento.

Grazie a mia nonna Serafina, che all'età di 101 anni, mi insegna che non si smette mai di imparare e che la curiosità per le piccole e grandi cose non ha età anagrafica, ma risponde solo all'esigenza di sentirsi profondamento vivo. Grazie per il tuo esempio e la tua dolcezza infinita.

Ma il ringraziamento più grande va alla mia stupenda famiglia, a mio marito Vincenzo e ai miei figli Laura e Pasquale, perché senza di loro nulla avrebbe acquisito il giusto senso. Grazie Vincenzo, perché senza di te non avrei mai preso quel treno per partecipare al concorso di dottorato, grazie per continuare a sostenere le mie scelte ed essere il motore che mi lancia anche quando io metto un timoroso freno. Grazie per essere il marito cooperativo che "non mi aiuta in casa o con i bambini", ma se ne fa carico in modo responsabile, perché come dici sempre tu "la casa e la famiglia sono sulle spalle di entrambi". Grazie per il tuo supporto continuo, per gli abbracci che calmano, per le mani che asciugano il mio volto. Grazie per accogliere le mie fragilità con il sorriso, grazie anche per gli scontri altamente stimolanti sul piano culturale in cui ci teniamo fortemente testa a vicenda e che mi aiutano a pensare. E infine grazie a voi, Laura e Pasquale, per aver gioito e sofferto con me in questo viaggio. Si dice che i bimbi amplifichino il sentire dei propri genitori, allora vi dico ancor più grazie perché avete fatto vostre le mie emozioni, senza che ciò fosse dovuto. Grazie per i vostri abbracci felici ad ogni notizia bella che riguardasse il mio dottorato; grazie per avermi ispirato nella stesura dei fumetti e aver iniziato a disegnarne di vostri, creando nuove storie; grazie per il vostro entusiasmo e la profonda sensibilità che vi caratterizza; grazie per la vostra rumorosa creatività senza la quale la vita sembrerebbe molto, ma molto più vuota.

Contents

Introduction **11**
Rationale . . . 11
Research aim and research hypothesis . . . 15
Overview . . . 17

1 Proving as a cultural process **21**
1.1 Mathematical proof as theoretical validation . . . 25
1.2 A look at the Geometric and Algebraic Proofs . . . 27

2 Proving as a rational process **31**
2.1 Our conceptualization of rationality . . . 35

3 Methodological background for design **37**
3.1 Storytelling as role-playing . . . 37
3.1.1 The 'Inside-Out' model as a methodological and didactical device 40
3.1.2 Our conceptualization of storytelling . . . 45

4 Methodological background for analysis **47**
4.1 Rational Mathematical Template . . . 47
4.2 Narrative research . . . 48
4.2.1 Categorical-Content Perspective . . . 49

5 Research Questions **53**

6 Extending the Conceptual Framework **57**
6.1 Coordinating locally as networking strategy . . . 58

7 A study to characterise cognitive role-playing **61**
7.1 Methods . . . 61
7.1.1 Participants . . . 61

7.1.2 Setting . . . 62
7.1.3 Task . . . 62
7.1.4 *A priori* analysis to characterise the cognitive roles in Habermas'perspective . . . 63
7.1.5 *A posteriori* qualitative analysis to test and refine the roles characterization . . . 65
7.2 Findings . . . 65
7.2.1 An *priori* characterisation of the cognitive roles . . . 65
7.2.2 A posteriori analysis: how the a priori characterisation shed light on chat protocols . . . 67
7.3 Discussion . . . 72

8 Euclid Experiment - Methods **77**
8.1 Design on Inside-Out towards Inside-Out 2 . . . 79
8.1.1 The context of Euclid Experiment . . . 82
8.1.2 Selection of the tasks and their a priori analysis . . . 85
8.1.3 Comics design and implementation into Moodle platform . . . 94
8.1.4 Methodological choices for didactical mediation . . . 98
8.1.5 A new Digital Toolkit for Proof: Dynamic Euclidean Bank of Theory . . . 100
8.1.6 An example of integrating comics with the mathematical problems 104
8.1.7 Summarizing the difference between Inside-Out and Inside-Out 2 models . . . 108
8.1.8 Phases of the classroom activity . . . 111
8.2 Collected data . . . 111
8.3 Methodological choices for data analysis . . . 113

9 Euclid Experiment - Preparatory activities **117**
9.1 Introduction . . . 117
9.2 Cloze activity: a new perspective . . . 118
9.3 Research question . . . 119
9.4 Methods . . . 119
9.5 The episode . . . 124
9.6 Data analysis and results . . . 126
9.7 Discussion . . . 130

10 Euclid Experiment - Chats' analysis **133**
10.1 Episode 1.1 . . . 133

10.1.1 Excerpts from Actors'chat . . . 134
10.1.2 Excerpts from Outlookers'chat . . . 139
10.2 Episode 1.2 . . . 143
10.2.1 Excerpts from Actors'chat . . . 144
10.2.2 Excerpts from Outlookers'chat . . . 145
10.3 Episode 1.3 . . . 146
10.3.1 Excerpts from Actors'chat . . . 146
10.3.2 Excerpts from Outlookers'chat . . . 151
10.4 Episode 1.4 . . . 157
10.4.1 Excerpts from Actors'chat . . . 158
10.4.2 Excerpts from Outlookers'chat . . . 166
10.5 Episode 1.5 . . . 173
10.5.1 Excerpts from Actors'chat . . . 174
10.6 Episode 2 . . . 177
10.6.1 Excerpts from Actors'chat . . . 177
10.6.2 Excerpts from Outlookers'chat . . . 190
10.7 Episode 3 . . . 194
10.7.1 Excerpts from Actors'chat . . . 194
10.8 Summarising the findings . . . 204

11 The cases of Giorgio and Estella **217**
11.1 Students profiles . . . 218
11.2 Methods . . . 218
11.3 Results . . . 220
11.4 Discussion . . . 236

12 The case of Blogger **237**
12.1 Students profiles . . . 238
12.2 Methods . . . 238
12.3 Results . . . 240
12.4 Discussion . . . 242

13 Discussion and conclusion **245**
13.1 Answer to Research Questions . . . 245
13.2 Conclusion . . . 253
13.3 Limitations . . . 254
13.4 The flavour of 256

Bibliography **259**

Appendix **269**
Comics 269

Introduction

Il modo naturale di concepire una dimostrazione è costruirla come un racconto [...]. Ogni singola dimostrazione poi è la storia di un'avventura, di un viaggio in un paese sconosciuto.
Gabriele Lolli, Matematica come narrazione

Rationale

The approach to proof most commonly used in schools focuses almost exclusively on the final product, on formal mathematical proof seen as a product (Boero et al., 1996a), to be learned through pre-packaged texts in textbooks, leaving out the whole process of creative thinking necessary for finding a good idea that leads to proof (Douek, 2007). Balacheff (1991) recognises that some students engage in proving because it is required by the teacher, not because they perceive proof as necessary in their own activity. Furthermore, the highly formalized nature of schooling leads students to attribute validity only to proofs written in a strictly formal manner, though often difficult to understand, rather than to more intuitive and comprehensible proofs that are less formalized (Wittmann, 2020). Furthermore, Douek (2007) claims that the purely formal proof model, not based on semantic content of propositions, but only on their "operational status" (Duval, 1991) is not in line with the modus operandi of a mathematician in the process of checking the validity of a statement or proof. Even mathematicians, before arriving at a formal proof, which is necessary for communicating results to the scientific community, go through a profound and complex process of arguing and proving, producing proofs that are "humanly understandable and humanly checkable" but often very far from a purely formal proof (Thurston, 1995). Only in rare cases where the proof requires a chain of algebraic transformations might the proving process coincide with the formal product (Douek, 2007). In most cases, however, the proving process is very different from the deductive process required in the formal writing of a proof. This is the case, for example, with Euclidean geometry theorems, whose proving process is almost never determined by a linear deductive

chain, but often uses analogies, inductive approaches and abductive processes (Antonini & Mariotti, 2008; Boero, 1999; Boero et al., 2007). The process of producing a proof is therefore dynamic and often requires communication and sharing with a small working community, before arriving much later at a mathematical proof that meets the requirements of correctness shared by the scientific community. Both the mathematical proof as a product and the proving process should not be understood as stable and unchangeable processes, but rather as culturally situated activities conditioned by the level of awareness of the person proving, whether a mathematician or a student (Douek, 2007; Morselli & Boero, 2009).

Hatano and Wertsch (2001) place great emphasis on the constitutive role of culture in the development of the human mind, as

> the special medium of human life consisting of a set of interrelated artifacts, shared to some extent among members of the community and often inherited over generations. These artifacts include physical tools, common sense knowledge and beliefs, social organizations, and conventional patterns of behavior associated with the physical, symbolic, and social tools. (Hatano & Wertsch, 2001, p.79)

This definition is thus situated within the socio-cultural perspective on learning (Cobb, 2007; Vygotsky, 1978). Within this framework, culture is not viewed as a static set of contents or knowledge, but as a dynamic system of practices, tools, and shared meanings that mediate human action and thought. In line with this perspective, Schoenfeld consider learning as a process through which individuals engage with patterns of behavior and, over time, develop forms of interaction and participation within a given social context (Schoenfeld, 2016). Hatano and Wertsch (2001) place strong emphasis on the idea of participation in practice, since through interaction with "socially rooted and historically developed activities" (Vygotsky, 1978, p.57) the process of internalization of cultural forms of behaviour takes shape, enabling the development of complex concepts and higher psychological functions. This occurs through a gradual, slow, and continuous developmental process that allows these functions to become internal.

In line with this perspective, we thus conceptualise learning as the progressive appropriation of such tools (whether they be physical, communicative or theoretical) and ways of acting, through participation in communities of practice and the internalisation of their forms of thinking and acting (Hatano & Wertsch, 2001). We refer to this progressive appropriation as a continuous spiral process through which learning proceeds expansively from one form to another, by overcoming contradictions and tensions that emerge from the confrontation between the perspective of the collective subject engaged in learning and the culturally and historically developed forms of knowledge through

which the transformation of knowledge is dialectically shaped (Engeström & Sannino, 2010).

In particular in the mathematics field, in line with Schoenfeld (2016) and Boero (1999), we describes learning in a problem-solving situation as a cultural and cognitive protracted apprenticeship into mathematics, through which one becomes a mathematician in a deep sense by adopting the values and worldview of a particular community. Therefore, the classroom should evolve into a community in which "mathematical sense-making" is a daily practice, enabling students to develop the mindset of the professional mathematician.

This perspective opens up an important epistemological dimension, as it frames learning in terms of "enculturation", that is the interpersonal and interactive process through which people become part of a mathematical culture by appropriating its values and practices (Bishop, 1991; Schoenfeld, 2016), that we conceptualise as "an intersubjectively shared lifeworld, thereby securing at the same time the horizon within which everyone can refer to one and the same objective world" (Habermas, 1998, p.315).

Morselli and Boero (2009), in line with this point of view, describe the approach to theorems as a process of *scientific enculturation*, consisting in the development of a type of behaviour, more specifically rational behaviour according to Habermas' perspective (1998), which involves the intentional search for proving strategies, the control of epistemic aspects and the communication of results through rules shared by the mathematical community.

Furthermore, from our point of view, the notion of *enculturation* allows us to see an ongoing process from an expansive perspective (Engeström & Sannino, 2010), as the term "enculturation" already conveys a sense of movement that leads to a dialectical view of learning as a continuously evolving process, taking place through forms of mediation and cultural multi-voices negotiation in which communication "oriented towards understanding" (Habermas, 1998, p.315) works within the collective Zone of Proximal Development (Engeström, 2014).

Thus, scientific enculturation into the Culture of Theorems should be understood as something that is gradually taking shape, in a sort of "joint enterprise" (Bishop, 1991, p.126) that involves all participants in the learning process, both students and teachers.

We consider that, for learning to be effective and productive, it is essential that learners actively participate in cultural development, continually encountering new problems within socially rich contexts of interaction (Hatano & Wertsch, 2001). This highlights the need to propose open and authentic problems that cannot be immediately solved through predefined schemes (Liljedahl et al., 2016; Schoenfeld, 1985), but which, from a Vygotskian perspective, engage students in a dialectical process

between constructive and creative moments that drive advancement and moments of adherence with Culture (Schneuwly et al., 2022).

Accessing to *Culture of Theorems* means developing "specific skills related to the production of conjectures and the proof of conjectures produced taking into account elements of theoretical knowledge"(Boero, 1999; Boero et al., 2007).

Therefore, we focus on rational proving processes and, in line with Albano and colleagues (2021a) and Morselli and Boero (2009), we consider such processes as a special case of problem-solving, intentionally oriented towards achieving a mathematical proof, as students are required to explore a situation, then produce a conjecture and develop a mathematical proof in order to validate the conjecture.

For Hatano and Wertsch (2001), students' participation in social and cultural practices takes place also, and above all, through language (Vygotsky, 1934/2008), of which narrative is one of the forms that allow human beings to make sense of the world they are about to come to know (Hatano & Wertsch, 2001).

In conceptualizing storytelling activity, Zazkis and Liljedahl (2019) place emphasis on the idea of interactive storytelling in educational contexts, particularly within mathematics classrooms, by engaging students in listening to and constructing stories in which open, highly engaging, and motivating mathematical problems are embedded. These problems address mathematical issues of increasing complexity while remaining coherently situated within a consistent narrative framework

According to this point of view, we conceptualise storytelling as

> not simple artistic telling, but an interactive telling. The teacher provides the frame, but the students become participants in the telling by deciding how to extend the story [...] making the students themselves the heroes or the actors [...] and introducing role playing (Zazkis & Liljedahl, 2019, pp.69-70)

In line with the view that participation in social practices is a collective activity in which expertise is distributed among participants, each contributing their own specific competences, we adopt the perspective of role-playing as "a distributed system of cognition" (Hatano & Wertsch, 2001, p.80), in which students work together as a team that shares its own ongoing expertise and role-based features toward a common goal. Balacheff (2008) acknowledges the essential relationship between proof and language, stating that "Language is at the core of proof and argumentation [...] especially in mathematics because of the non-physical nature of mathematical objects" (p. 502), and highlighting the "dialogic nature of proof production" (p. 507). The methodological and didactical device of Inside-Out Model (Albano et al., 2020c, 2021a) allows participants to become immersed in collaborative problem-solving activities through their identification

with the characters in a story that can be implemented in a digital environment. Participants' actions are guided by the goals and cognitive role of the character they play. As it operationalises Inside-Out model, our study assumes that roles played by students in a collaborative problem-solving activity fit various cognitive functions that come into play in the process and that are active in the mind of a mathematician engaged in a problem-solving activity. These cognitive functions can be embodied by the students as characters of a digital story-problem—Pest, Boss, Promoter, Blogger— giving rise to the Inside-Out model, with the aim of fostering the awareness in learners of the structure and organisation of mathematical thinking (Albano et al., 2021a). The underlying hypothesis is that, through social practice, students internalize these functions, which shift from being externally assigned to being self-initiated.

Many studies that have addressed the proving process from the perspective of rationality (Boero & Azrou, 2023; Boero & Turiano, 2022, 2023; Turiano & Boero, 2019; Urhan & Bülbül, 2023a, 2023b; Zhuang, 2023) essentially focus on individual proving activities, where, when collective activity is present, it mainly takes the form of teacher-orchestrated mathematical discussions aimed at conceptualising or systematising knowledge. Although there are studies analysing the proving process from the perspective of Habermas' rationality within collaborative learning settings (Urhan & Zengin, 2025), these involve undergraduate mathematics students engaged in proving activities with some moments of computer-supported teamwork, while the exploratory activity remains essentially individual.

In contrast, in the Inside-Out model, communication evolves in a dialectical interplay between self-communication and intra-group interaction towards external communication addressed to the mathematical community during the whole proving process, within a continuous process that places language at its core. The existing literature thus reveals a gap concerning the analysis in Habermas' rationality perspective of proving processes within cognitive role-playing contexts played in a digital storytelling situation.

Research aim and research hypothesis

Our research problem is hence to investigate whether and how digital storytelling activities based on cognitive role-playing may contribute to developing awareness of the rational thought processes of secondary school students involved in proving activities, in particular in the euclidean geometry area.

Following the evolution of the film Inside Out with Inside Out 2 [1], where the

[1] https://www.pixar.com/inside-out-2

exploration of Riley's inner world is significantly expanded, similarly we explores the mental functions illustrate nell'Inside-Out model, highlighting their gradual internalization by students through reflection on conjecturing, proving, and processes at meta-level. Our Inside-Out 2 does not introduce new cognitive functions; instead, it broadens the inquiry by reframing it through a lens of rationality (Habermas, 1998). In line with the extant literature (Boero, 2017; Boero & Azrou, 2023; Morselli & Boero, 2009; Morselli & Boero, 2010; Urhan & Bülbül, 2023a, 2023b), we aim to extend the use of Habermas' rationality construct to collaborative problem solving wherein students are able to create a space to develop a shared communicative environment based on mutual understanding. In this context, a central role is played by the Guru, who in the previous model interacted only with the Promoter, whereas in the new model acts as a rational mediator, actively participating in the Actors' chat. Its interventions are designed a priori, according to the Teacher Rational Questioning Framework (Zhuang & Conner, 2022), in order to foster the emergence of strategic behaviours, theoretical control over the statements produced, and the formulation of clear, shared, and rigorous communication.

Our research contributes thus to investigating, in a rationality perspective, the impact of digital storytelling on the scientific enculturation process of students within the Culture of Theorems, in particular when the students are involved in a cognitive role-playing (Albano et al., 2021a), during the collaborative proving process in Euclidean geometry by means of a highly immersive narrative approach in a digital environment.

We hypothesise that storytelling, alongside with the Inside-Out model and its roles that embody the cognitive functions at play in the mind of a mathematician during problem solving, have the potential to act as effective methodological tools for extricating the various rational components of the proving process, thus promoting the process of scientific enculturation of students into the Culture of Theorems, in particular, in the field of Euclidean geometry.

In light of a careful reflection on the differences between proofs in algebra and those in geometry—concerning the nature and characteristics of the mathematical objects involved, and consequently the nature of the reasoning underpinning proofs in the two domains—we choose to design the new experiment within the field of Euclidean geometry. This choice leverages the presence of a more explicit and structured theory, organized as a coherent and deductive corpus, in order to support students' scientific enculturation into the culture of theorems, in a context that presents greater structural difficulties for learners, through the use of cognitive role-playing. The Euclidean axiomatic system shifts attention from purely empirical arguments rooted in practical science toward the conceptualization of abstract entities, placing logical deduction at

the center and idealizing objects from the physical world through the use of arguments often closely connected to intuition (Harel & Sowder, 2007). Not everything in the Euclidean axiomatic system is fully formalized and, unlike modern axiomatics, Euclid gives greater importance to content than to form. Geometrical proofs and constructions are therefore not detached from the meaning of the mathematical objects and properties involved, but maintain a strong connection with their semantic content. This makes Euclid's *Elements* an example of a "well-organized body of knowledge" that crosses generations, with which learners could come into contact through participation in practice (Hatano & Wertsch, 2001, p.79). Such a system of knowledge includes both the theoretical domain (axioms, definitions, theorems) and the meta-theoretical one (e.g., the logical-formal structures of deduction and inference) (Mariotti, 2006) and can be regarded as a "reference corpus" (Douek, 2007, p.171), characterized by a logical hypothetico-deductive structure that differs significantly from formal logical derivation (Rav, 1999). Rather, it is closely related to the understanding and systematization of knowledge within a body of knowledge historically and culturally transmitted, accepted by the mathematical community, and communicable externally (Mariotti, 2006).

Overview

This thesis is structured as follows:

Chapter 1: The first chapter frames the discussion on the process of proving and on what is meant by mathematical proof from a cultural and epistemological perspective, highlighting the definition of proof to which we adhere.

Chapter 2: The second chapter presents the theoretical construct of rationality as developed by Jürgen Habermas, together with its reinterpretation within mathematics education. We also provide our own conceptualization of rationality, offering an extended definition grounded in Habermas's perspective and in its educational interpretations.

Chapters 3 and 4: In the third and fourth chapters we present the methodological background divided into two parts. In the third one, we present methodology for design, in particular we define and discuss the Inside-Out Model as a methodological and didactic device on which the entire study is designed. In the fourth chapter, we introduce methodology for data analysis at a theoretical level, in particular the Rational Mathematical Template as an analytical tool based on Habermas'rationality perspective and the Categorical–Content Perspective as developed by Lieblich and colleagues.

Chapter 5: In this chapter, starting from the research hypothesis presented in the Introduction, the research questions we aim to investigate in our study are

operationalized and contextualized.

Chapter 6: Recognizing that, in order to address the research questions, the sole framework of Habermas' rationality is not sufficient, and that the Inside-Out Model represents for us not only a methodological tool for design but also an integral part of the theoretical framework, in this chapter we extend the conceptual background. We do so by attempting to combine different theoretical constructs through a process of local coordinating as a networking strategy.

Chapter 7: This chapter presents a pilot study in which we characterize the cognitive roles of the Inside-Out Model from a rationality perspective through an a priori analysis, to better understand the emergence of rational behaviour in secondary school students during cognitive role-playing. The aforementioned characterization of cognitive roles is then tested through an a posteriori analysis, with the aim of further refining it and thereby designing a new, broader experiment grounded in this characterization from a rationality perspective. The chapter concludes with a discussion of the preliminary results that emerged from this pilot study.

Chapter 8: This chapter represents the methodological core of the thesis, as it presents the entire methodology related to the Euclid Experiment. In addition to outlining the context of the new study, we describe all the methodological choices for the design, ranging from decisions concerning the construction and implementation of comic-based storyboards to the development of a new digital toolkit for proof, as well as the choices related to didactic mediation by the teacher/researcher through the Teacher Rational Questioning Framework. The chapter also includes the whole a priori analysis of the mathematical tasks proposed to students. Furthermore, it presents the methodological choices for data analysis, consistently adopting the construct of rationality developed by Habermas as the interpretative lens.

Chapter 9: This chapter presents a study concerning a preparatory activity for the proving process, planned and implemented in the classroom outside the digital platform, with the aim of fostering an initial exploratory approach to proof focused on the perpendicular bisector of a segment. This chapter presents the results of an exploratory study related to the cognitive unity of theorems and the cloze tool.

Chapter 10: This chapter presents a fine-grained analysis of all the synchronous discursive interactions that took place in the digital chats of the Actors/Onlookers groups in one of the two classes involved in the activity, as well as the comments on the drawings posted on the shared board on Padlet, using a Categorical–Content analysis. The chapter includes the transcripts of the chat conversations, together with their corresponding coded categorical analyses, related to the seven storytelling episodes designed and implemented on the Moodle platform. It concludes with a summary of

the results that emerged from the analysis, including the analytical categories identified and a preliminary discussion of these findings.

Chapter 11: This chapter focuses on the evolution process followed by two students. In particular, we adopt a case study as a research approach in order to progressively narrow the focus of the study, like a funnel, concentrating on the process that has already been shown through the detailed analysis of the chats, moving from the participation level of Onlookers to that of Actors. The chapter concludes with a brief discussion of the results of this study.

Chapter 12: In this chapter, we conduct a qualitative case study in which we observe the evolution of the Blogger role within a group engaged in conjecturing and proving activities. We investigate how enacting the Blogger role, understood in a collective sense through the Actor/Onlooker dynamic, can support the emergence of rationality in 10th-grade students' collective arguments involved in geometric proving processes, and we discuss the results that emerged.

Chapter 13: Finally, in this last chapter, starting from the results that emerged from the various studies presented in this thesis, we provide a general discussion that synthesizes and interprets the different findings within a unified and coherent framework. This chapter also presents some perspectives for future research developments.

Chapter 1

Proving as a cultural process

Approaching studies on the proving process from an epistemological perspective, we immediately notice the emergence of multiple dichotomies, as well as many different perspectives, leading to diverse interpretations of these processes.

First, we wish to draw attention to the fundamental distinction that emerged from the beginning of the twentieth century between logical deduction and formal derivation. While the former is characteristic of a deductive axiomatic system in which propositions are grounded in the truth of previously accepted statements, as in the Euclidean axiomatic system, still maintaining a strong connection with semantic content (Harel & Sowder, 2007), formal deduction is instead a syntactic object derived from axioms through logical rules of inference (Rav, 1999), which completely detach the process from the meaning inherent in the mathematical object (Wittmann, 2020). Hanna (1990) highlights how the emphasis placed on the purely formal rigor of logical derivation has strongly influenced both university courses and approaches to proof in secondary education over the decades, although the acceptability of a proof by mathematicians depends more "for their semantic rather than their syntactic qualities" (Hanna, 1990, p.8), giving greater value to "humanly understandable and humanly checkable proofs" (Thurston, 1995, p.34) rather than to proofs structured according to purely formal standards.

While Duval (1991), in line with the idea of deductive reasoning based solely on logical inferences and therefore understood as formal derivation, focuses on the operational status of propositions stripped of their semantic content, Boero and colleagues (2010) consider this view "not adequate for the school approach to theorems and proof" (p.197), recalling Thurston (1995)'s position concerning the semantic value of content both in the construction and in the checking of proofs.

In order to categorize proof schemes and better understand the different approaches to the proving process adopted by students, Harel and Sowder (2007)

carry out an accurate epistemological analysis of the distinction between Euclidean axiomatics—understood as an abstraction of the physical world and therefore still connected to the semantic value of statements, both in construction and proof—and modern Hilbertian axiomatics, which is purely focused on formal derivation. By *proof scheme*, they mean "of what constitutes ascertaining and persuading for that persona or community" (Harel & Sowder, 2007, p.810), that is, what allows one to remove doubts about the truth of a statement in order to convince oneself, and what allows one to remove the interlocutors' doubts about the truth or falsity of a statement. They classify these schemes within a taxonomy distinguishing among *External Conviction proof schemes*, based on the authority of the teacher; *Empirical proof schemes*, based on quantitative measures, repeated examples, inductive approaches, or perceptions; and *Deductive proof schemes*, which include transformational schemes related to generality and logical inference, explicitly distinguishing between Greek axiomatics and modern axiomatics (Harel & Sowder, 2007).

The distinction between empirical and deductive approaches is also present in Balacheff (1987), who refers to them as *preuve pragmatique* and *preuve intellectuelle*. This distinction is mainly grounded in the role of action and language within the validation process (Balacheff, 2010; Balacheff, 1987). Pragmatic proofs are characterized by the use of real actions on concrete objects, which Balacheff (1987) considers typical of the early stages of knowledge genesis. They rely on practice rather than theoretical discourse, being anchored to facts and actions and based on what Vergnaud (2009) defines as *théorèmes-en-acte*, namely properties used by subjects in problem solving without necessarily being explicitly formulated or theoretically justified. Intellectual proofs, instead, mark a shift away from direct action toward a focus on the properties of objects and the logical relations among them (Cabassut et al., 2012). In this distinction between pragmatic and conceptual proof, Cabassut and colleagues (2012) identify an influence of Piagetian psychology, with the idea that action necessarily precedes thought. Accordingly, Balacheff (2010) recognizes the need for learners "to change their intellectual posture," moving from action to discourse. However, de Villiers (1990) points out that the discovery of new results does not always occur through empirical approaches alone, but may also arise deductively by generalizing already known properties. In some cases, investigating the validity of a conjecture instead requires the search for counterexamples and the use of what de Villiers (1990) calls quasi-empirical methods.

The view of proof to which students are still too often exposed, and which traditionally continues to persist in schools, is almost exclusively related to the idea of proof as *verification* of the truth of a statement, as if mathematical proof only had the

function of providing certainty (de Villiers, 1990). However, proof should not necessarily be understood as "a prerequisite for conviction, to the contrary, conviction is probably far more frequently a prerequisite for the finding of a proof" (de Villiers, 1990, p.18). The *need for certainty* arises from the desire to verify a statement, often through empirical methods closely connected with sense making (Zaslavsky et al., 2021), similarly to pre-Greek mathematics, in which conjectures were validated through empirical evidence (Harel & Sowder, 2007). Even when a formal proof is available, students often still need to engage in empirical attempts in order to become convinced of the truth of the statement (Zaslavsky et al., 2021).

Empirical proofs, however, while useful for confirmation and verification, do not always succeed in explaining why a conjecture is true (de Villiers, 1990). The consideration of such methods as proofs constitutes an important epistemological obstacle that needs to be addressed in order to generate in students an *intellectual need* to move toward non-empirical forms of validation (Stylianides & Stylianides, 2022). The explanatory function, related to "explaining why," highlights another fundamental human need, namely the *need for causality* (Zaslavsky et al., 2021), which drives learners to search for the reasons underlying the truth or falsity of a statement, also from a theoretical perspective. The issue of the explanatory power of proof is clearly emphasized by Hanna (1989), according to whom "a proof that explains must provide a rationale based upon the mathematical ideas involved: the mathematical properties that cause the asserted theorem or other mathematical statement to be true" (Hanna, 1989, p.47); that is, it must be grounded in the semantic content of the proposition. However, because of the empty formalism to which they are constantly exposed, students and preservice teachers often fail to recognize intuitive or visual proofs as mathematically valid, even though they perceive them as more explanatory (Wittmann, 2020). Indeed, students simultaneously hold two different conceptions of proof: a more formal and syntactic one, which they believe is necessary to obtain good grades from the teacher, and a more empirical and narrative one, which they would adopt as their personal approach (Healy & Hoyles, 2000).

Mariotti (2006) emphasizes that explanation should not be kept separate from, or disconnected from, conviction, since students engaged in proving activities should be encouraged to "explain their arguments to a peer or to the whole class, including the teacher, also to convince themselves of their truth" (p. 198).

The term "explanation" is also used by Balacheff, although from a somewhat different perspective from that of Hanna (1990). In the process of constructing a mathematical proof, Balacheff (2010) distinguishes three dimensions of proof: explanation, concerning oneself; proof, concerning a small community; and

mathematical proof, concerning the academic mathematical community. Explanation is viewed as an attempt to achieve certainty and clarification by making the truth of a proposition understandable to oneself through the use of everyday language. This inner dialogue is considered an integral part of the proving process. The transition from explanation to proof occurs when reasoning is validated and accepted by a specific community using its own language, situated between everyday language and the formal language of mathematics. For Balacheff (2010), proof is not fixed, since it may evolve over time with the ultimate aim of achieving understanding. Mathematical proof, on the other hand, is a highly structured form of communication, governed by specific rules and widely accepted within mathematical culture; it thus becomes an external form of communication addressed to the broader mathematical community. Language is therefore a crucial element for Balacheff (2008), who, referring in particular to the interpretation that Habermas (1998) gives of communication in the validation of statements, recognizes it as an essential point of convergence among the multiple conceptions of proof, considering it "at the core of proof and argumentation in general, [...] because of the non physical nature of the mathematical objects" (Balacheff, 2008, p.502).

In examining the factors that lead students and learners to recognise the role and importance of proving, Zaslavsky and colleagues (2021) identify also a *need for communication*, which involves re-organising ideas into a coherent and logical structure. This allows both the reorganization of discourse in a systematic way "to better express what they have in mind" (p.222), and the possibility "to organise and communicate a body of knowledge," such as Euclid's *Elements*.

The communicative function of proof is also addressed by de Villiers (1990), who emphasizes the social interaction process involving "the negotiation of not only the meanings of concepts, but implicitly also the criteria for an acceptable argument" (p.22).

Cognitive unity of Theorems

The relationship between the argumentation process and the proving process represents another dichotomy that needs to be considered.

For his part, Duval (1991) presents a cognitive perspective on the relationship between argumentation and formal proof, clarifying that deductive reasoning does not operate in the same way as argumentation, since argumentative reasoning is grounded in semantic content, whereas deductive reasoning is based on inferential roles concerning the epistemic value of proof.

However, several experimental studies showed that constructing a mathematical proof becomes more accessible to students when it takes place within the process

of formulating a conjecture and producing argumentation for the plausibility of the statement they previously proposed (Boero et al., 1996a, 2007). When students move on to developing a rigorous proof, they reorganise their entire cognitive process into a coherent chain of deductive reasoning. The delicate shift from generating an argumentation to producing a formal proof can reveal a continuity between these two phases, a phenomenon referred to as the *Cognitive Unity of Theorems* (Boero et al., 1996a).

However, this continuity does not always occur, especially when students must move from inductive or abductive reasoning to deductive inference, as particularly evident in the geometry field (Pedemonte, 2007), but also into the algebraic domain.

Indeed, Pedemonte (2007) claims that cognitive unity is not always sufficient to generate continuity between the arguing and proving processes, but a structural distance sometimes occurs when moving from an abductive line of reasoning to a deductive proof.

In the geometric domain, when moving from abductive argumentation to deductive writing, students may fail more easily, since they carry over the abductive reasoning developed during the argumentative phase, thus showing *referential* continuity within cognitive unity, despite the presence of a structural rupture (Pedemonte, 2002). In the algebraic domain, instead, due to the fact that the deductive structure is particularly strong, students may find it more natural to rely on the deductive framework typical of algebraic proofs rather than on the abductive steps that could emerge in argumentative reasoning (Pedemonte, 2008). In this case, the structural rupture of cognitive unity does not compromise formal deductive writing.

1.1 Mathematical proof as theoretical validation

From this perspective, proof construction can be conceptualise as a dynamic and communicative process that only later leads to the writing of the proof according to forms accepted by the mathematical community. Proof fulfills a central communicative function, into the need of a systematic reorganization of discourse aimed at making thought arguments more explicit and understandable both to oneself and to others (Zaslavsky et al., 2021).

We therefore consider a notion of mathematical proof as "having an irreducible semantic content, and distinguish it from derivation, which is a syntactic object of some formal system" (Rav, 1999, p.11).

In line with the perspective of Boero and colleagues (2010), we therefore conceive the proving process as "a cognitive, culturally situated activity" (p.193) involving the exploration and search for reasons supporting the validity of a statement, the

development of an argumentation that justifies the conjecture produced, and finally the production of a proof as a written text according to deductive rules and culturally imposed constraints of the mathematical community, while duly taking into account the delicate transition between argumentation and proof.

Proving can be seen as a socially situated practice in which meanings and acceptability criteria are negotiated within a community (de Villiers, 1990). Accordingly, we adopt a cultural perspective that acknowledges classroom context and allows, at least temporarily, proofs that are valid within the classroom community (Stylianides, 2007). Therefore, neither mathematical proof as a product nor the process through which it is constructed should be conceived as fixed or immutable; rather both are to be understood as culturally and socially situated activities (Boero et al., 2010), shaped by the level of awareness of the individual engaged in proving, whether a mathematician or a student. In this process the teacher plays a crucial role both as Agent of the Culture of Mathematics (Zaslavsky et al., 2021), leading students to explore mathematical knowledge, and as a cultural mediator to move students towards standards of theoretical validation, acting as guarantor for mathematical community and classroom community (Mariotti, 2006).

In line with Mariotti's theoretical construct of Theorem (2006), we understand a theorem as a triple consisting of a Statement, a Proof, and a Theory, where theory is conceived both as knowledge (axioms, definitions, and theorems) and as meta-knowledge of the proving process (types of proofs and rules of inference). The statement produced and the proof acquire meaning within a reference theory, responding to the need for epistemic control. Mathematical proofs derive their meaning and rigor from their reference theory rather than from logical structure alone.

Balacheff (2010), citing Habermas (1998), indeed recalls that "at stake is not truth but validity of a statement within a well defined theoretical context" (p.5 preprint), emphasizing the need for a transition from the idea of proof to the idea of *validation*.

The rigor of a proof, therefore, lies less in its logical structure than in the *theoretical validation* of the statement itself. This perspective offers a level of formalism that goes beyond mere logical deduction, yet finds its meaning in relation to the theory and thus maintains a connection with semantic content. The construction of a proof is therefore a dynamic process and often also requires a communicative effort of sharing within a restricted working community, only much later leading to a version of mathematical proof that meets the criteria of correctness shared by the scientific community. We adopt a cultural perspective that takes into account the context of the specific classroom, balancing between honesty towards the discipline and acknowledgement of mathematical meaning in the specific context.

This view is consistent with Stylianides' (2007) notion of proof situated within a specific classroom context and at a given time.

According to Stylianides (2007), by proof in the context of a classroom community at a given time we mean a mathematical argument that has the following three characteristics:

1. It uses statements accepted by the classroom community (*set of accepted statements*) that are true and available without further justification;

2. It employs forms of reasoning (*modes of argumentation*) that are valid and known to, or within the conceptual reach of, the classroom community; and

3. It is communicated with forms of expression (*modes of argument representation*) that are appropriate and known to, or within the conceptual reach of, the classroom community.

   (Stylianides, 2007, p.291).

For illustrative purposes, by *set of accepted statements* we mean definitions, axioms, and theorems; by *modes of argumentation* we mean the application of logical rules and inference, the use of definitions to deduce statements, the construction of counterexamples, and the development of reasoning based on accepted premises; finally, by *modes of argument representation* we mean linguistic, verbal, pictorial, symbolic, graphical, or tabular representations.

It is important to note that, in order to avoid the risk of viewing the classroom community as "constructing" its own mathematics, with its own rules and statements, from our perspective it is essential to situate the definition of proof proposed by Stylianides (2007), as already done by Reid and Vallejo Vargas (2019), within the construct of Theorem introduced by Mariotti (2006). This allows both the theoretical and meta-theoretical dimensions to be taken into account, framing proof within a culturally shared theoretical context and the proving process as "a cognitive, culturally situated activity" (Boero et al., 2010, p.193).

1.2 A look at the Geometric and Algebraic Proofs

In the previous paragraphs have outlined a general picture of the proving process, but now we want focus on the significant differences emerge between proofs in the algebraic domain and those in the geometric domain. These differences concern the nature of the objects involved, the structure of reasoning, and the relationship between argumentation and the writing of the mathematical proof.

The mathematical objects involved in algebraic proofs are numerical and algebraical expressions, letters, and numbers that allow for calculations and symbolic manipulations, governed by rules of syntactic transformation. Mariotti (2005) emphasizes the need to approach algebraic proof by focusing not so much on the operational aspects of transformations, which risk reducing algebra to mere computation (Balacheff, 2010), but rather on the equivalences that, capturing the fundamental properties of addition and multiplication, can be taken as an axiomatic system for an algebraic theory, viewing a sequence of algebraic transformations as a proving process that establishes the equivalence of two expressions.

In geometry, the main objects under consideration, the figures, cannot be understood either as purely abstract concepts or as mere images, since they possess a dual nature whose two aspects—conceptual and figural—are interconnected in "a permanent interplay between conceptual and imaginative dynamics" (Fischbein, 1993, p.144). While the conceptual aspect refers to theoretical properties, such as definitions or validated propositions, the figural aspect essentially concerns the shape of the figure and its spatial properties. This dual nature, in some cases—such as in proofs by contradiction or reductio ad absurdum—makes the approach to geometric proof particularly complex (see, for example, Antonini and Mariotti, 2008).

Although proof by contradiction in the algebra domain can create difficulties for students when confronting with the rules of the "absurd world" (Antonini & Mariotti, 2006), since the absurdity of the assumption conflicts with the commonly used theory, in cases where such a proof does not disrupt the underlying theory students tend to accept it more comfortably (e.g., the proof of the irrationality of $\sqrt{2}$), as, despite the false assumption, they can continue to apply the rules of algebra in the usual way (Antonini & Mariotti, 2008).

In the geometric domain, the perspective changes completely, as students must deal with the dual conceptual–figural nature of geometric figures (Fischbein, 1993), which may sometimes come into conflict—especially in indirect proofs—with the construction of an impossible figure. Indeed, indirect mathematical proof, by relying on logical rules of inference (referred to by Antonini and Mariotti (2008) as a meta-theorem), leads to the rejection of the impossible figure, thus generating a conflict (Antonini & Mariotti, 2008). This conflict can be overcome in the indirect argumentation produced by students, since in such cases the figure is not rejected but rather manipulated and modified in accordance with the underlying theory, resulting in a kind of compromise between the need for evidence and the formal rigor of a proof (Antonini, 2019).

Urhan and Bülbül (2023a) highlight a further difference between the algebraic

and geometric domains, again concerning the nature of the object, but specifically in relation to its dimensionality. They clarify that, whereas in arithmetic and algebra numbers and operations have a dimension that can essentially be described as discrete, in the context of Euclidean geometry one deals with continuous objects, such as points, lines, angles, planes, and surfaces.

Another aspect to consider concerns the structure of the proving process. In the geometric domain, proofs tend to be more discursive, whereas in algebra the structure of proof is typically characterized by syntactic transformations of expressions, whose steps are validated through a sequence of chained transformations (Mariotti, 2005).

Algebraic and geometric proofs also differ in how a breakdown in the cognitive unity of theorems may occur when moving from the argumentative phase to the writing of the mathematical proof. In geometry, there is often a difficulty in transitioning from the abductive argumentation, produced during exploration, to a proof structured in a deductive way, as students find it more challenging to reorganize abductive arguments into a formal deductive chain, thus revealing a structural gap between abductive argumentation and deductive proof (Pedemonte, 2007). In some cases of geometric proof, the abductive arguments used in the argumentative phase lead to an incorrect formulation of the deductive proof (Pedemonte, 2007, 2008). This gap is less problematic in the algebraic domain, as the already strongly deductive structure of algebraic proof—linked to symbolic manipulation—allows students to reorganize arguments (even if abductive or inductive in nature) into a deductive chain more effectively (Pedemonte, 2008).

Moreover, in the algebraic domain, students sometimes show generally higher levels of success than in the geometric domain, both in validating and in constructing proofs (Healy & Hoyles, 1999). However, in algebra students are not always aware of the theorems they use; the underlying theory thus often appears hidden and implicit, as algebra is conceived as a language with excessive focus on syntactic aspects, leading to a loss of the meaning of proof and reducing it to a mere grammatical structure of formulas connected by rules of calculation (Pedemonte, 2008).

Chapter 2

Proving as a rational process

From an epistemological perspective, it is important to take into account the relationship between proof and language, which stems from the inherently dialogical nature of the proving process (Balacheff, 2008). Indeed, language lies at the core of argumentation and proving processes (p.502), becoming constitutive of the rationality of proving, which for Balacheff (2008) is realised through making decisions, performing judgments, discussing, and arguing.

According to Habermas (1998), we recognise that rationality emerges in distinct forms: an opinion is rational if it can be justified in a given context, an action is rational if it is consciously oriented to an aim and achieved by intentionally chosen means, and communication is rational when it seeks mutual understanding, through the careful use of language to ensure an effective communication. In this context, three dimensions of rationality can be distinguished: epistemic rationality (ER), which refers to the acceptability and validity of a statement in a given context; teleological rationality (TR), which refers to the intentionality of the subject who acting and making decisions by intentionally chosen means in relation to the aim; communicative rationality (CR), which "is expressed in the unifying force of speech oriented towards understanding" (Habermas, 1998, p.315). Therefore, communicative rationality emerges as an act aimed at mutual understanding—a form of agreement based on understanding rather than mere consensus—that allows the other to comprehend and subsequently accept or reject a verbal act with full freedom, if necessary (Habermas, 1998).

Within epistemic rationality Habermas (1998) distinguishes the explicit knowing, namely knowledge for which one is able to explain the "why", providing justifications and reasons, from implicit or practical knowing, which Habermas (1998) associates with the procedural act of how to do something. Furthermore, epistemic rationality does not imply the truth of a judgement, but rather its "justified acceptability in a given context" (Habermas, 1998, p.312). Guala and Boero (2017) acknowledge that Habermas's

requirements for epistemic rationality fit properly the definition of "Theorem" proposed by Mariotti (2006), particularly with respect to theory, including within it also the rules of inference and the meta-theory.

However, such dimension of rationality cannot be understood independently of action and language; rather, it achieves its full realization only when it is consciously employed in goal-directed actions, that is, in relation to teleological rationality.

Boero and Planas (2014) strongly emphasize the intentionality of action and the reflective stance of the subject towards it. For Habermas (1998), awareness of action is not only manifested in conscious strategic choices, but also guides the subject to enact further actions that can deliberately enable the achievement of the intended goal. Intentionality is thus a key feature of Habermasian rationality, which allows this construct to be adapted to the analysis of human behaviour in social contexts during conjecturing and proving activities (Boero, 2006). Indeed, Boero (2006) identifies in the subject's intentionality in relation to goals a crucial aspect that should be further addressed in educational contexts, enabling learners to "experiencing possible resources [...] and progressively becoming aware of them" (p.191).

This perspective introduces a significant epistemological dimension by interpreting learning as a process of "enculturation" through which individuals enter a mathematical culture by appropriating its practices and values (Bishop, 1991; Schoenfeld, 2016). In this sense, mathematical culture can be understood as "an intersubjectively shared lifeworld, thereby securing at the same time the horizon within which everyone can refer to one and the same objective world" (Habermas, 1998, p.315). Morselli and Boero (2009), in line with this point of view, describe the approach to theorems as a process of *scientific enculturation*, consisting in the development of a type of behaviour, more specifically rational behaviour according to Habermas' perspective (1998), which involves the intentional search for proving strategies, the control of epistemic aspects and the communication of results through rules shared by the mathematical community. In the discursive practices of the proving process, Morselli and Boero (2010) identify aspects of communicative rationality in the conscious selection of communication means and in the adherence to the communicative norms of mathematical culture, particularly recognising the importance of communication also within the writer–interpreter dynamic (Boero & Morselli, 2010). The communicative act therefore relies on intersubjective recognition, expressed through the dialogical nature of discourse, while enabling the individuals involved to refer to a shared horizon of meaning (Habermas, 1998).

For Boero and Turiano (2022), however, the shared horizon should not be understood merely as a requirement for communication, but rather as a broad educational goal that is constructed through rational discursive practices and the

use of cultural mediators such as Rational Mathematical Templates (RMTs)(Boero, 2022; Boero & Turiano, 2020, 2022) which enable students to become familiar with rational practices within the proving process. RMTs are also employed as analytical tools to identify the presence or absence of rationality components in students' written productions (Boero & Azrou, 2023).

Drawing on Boero (2006) and Morselli and Boero (2009), many studies have operationalised Habermas' construct of rationality in order to analyse students' behaviour in problem-solving contexts, particularly to highlight the difficulties students encounter in proving activities by identifying absences, lacks, and weaknesses, as well as by defining significant observables within analytical categories. In particular, Boero and Morselli (2010) adapt Habermas' construct of rationality to algebraic language and modelling, introducing two additional criteria to characterise epistemic rationality: modelling requirements, related to the control of algebraic formalisation in coherence with the situation being modelled, and systemic requirements, relating to transformation rules governing the system of signs commonly referred to as algebraic language, together with the appropriate use of methods for solving equations and inequalities. On the other hand, Urhan and Bülbül (2023a) conduct a detailed a priori analysis of tasks proposed in Euclidean geometry, interpreting the expected steps in terms of rationality and thereby identifying criteria to categorise the presence or absence of rationality components among prospective teachers attending a university course. Furthermore, Urhan and Bülbül (2023b), through the analysis of university students' behaviour in differential equations problem-solving activities, identify criteria capable of showing how a lack in one or more components of rationality affects the other components. More specifically, with regard to teleological rationality, Urhan and Bülbül (2023a, 2023b) formulate criteria related to the enactment of the actor's free decision-making, the achievement of outcomes through intentionally chosen and applied means, and the purposive justification of the adopted solution strategy. Concerning epistemic rationality, their attention is directed towards the correctness of judgements and the justification of the different solving steps. Finally, in relation to communicative rationality, they identify criteria connected to the production of valid and comprehensible discourse, together with the appropriate use of mathematical language and symbols in the construction of formal proofs. The authors articulate these criteria across different mathematical domains (algebra, geometry, and differential equations), in relation to the specific epistemological features of the problems under consideration.

Also the break in cognitive unity of the theorems highlights the lack of rationality components (Boero, 2017) within the proving process, especially in the cases wherein

these components involve the teleological aspects of the reasoning, as abduction or induction strategies, emerged within the creative process of producing of conjecture (Morselli & Boero, 2009). Therefore, the cognitive unit provides an analytical lens that complements the components of rationality by revealing their limitations.

On the other hand, Boero and Azrou (2023) introduce an additional focus on meta-knowledge of proof as a concept according to Vergnaud (2009), as well as on meta-knowledge of the proving process, analysing the written productions of undergraduate mathematics students attending a mathematics education laboratory. In this way, they extend the Rational Mathematical Template of Proof to the meta-level, thereby providing an interpretative lens for both the knowledge level and the meta-knowledge level.

However, although for Habermas (1998) communicative rationality is realised within social interactions, Boero and Planas (2014) highlight a limitation of the construct of rationality with respect to the negotiation of valid statements and the social construction of strategies, thus pointing to the need to address these aspects through specific studies. As an example, Goizueta (2014) attempt to adapt Habermas' construct to the analysis of conversational interactions related to the emergence of conditions of epistemic validity.

Attention to rational processes in activities involving the negotiation of meanings is present in studies in which the Habermas' rationality construct is employed to analyse the crucial role of the teacher in fostering and supporting the emergence and development of argumentation and proof competencies, as well as theoretical validation within proving processes, especially in group dynamics (Branchetti & Morselli, 2019; Conner, 2017; Cusi & Morselli, 2016; Zhuang & Conner, 2022). Building on a study concerning the teacher's supportive actions in collective classroom argumentative practices (Conner, 2017) and on the proposal of Douek (2014) to introduce rational questioning into classroom argumentative practices, Zhuang and Conner (2022) develop a framework grounded in Habermas' construct of rationality, the Teacher Rational Questioning Framework (TRQF), aimed at supporting students' participation in collective argumentation. The framework encourages students to theoretically justify the arguments they advance, guides them in reflecting on the strategic choices they make, and supports the use of appropriate mathematical language and suitable representations. On the other hand, Zhuang (2023) also employ the TRQF to investigate graduate students' proof validation competencies within transition-to-proof courses designed to prepare them for more advanced mathematics courses. They emphasize the need to strengthen practices of requesting theoretical justification for each step involved in proof validation, in order to make students' awareness of each rationality component

involved in the process visible. Furthermore, Cusi and Morselli (2016) integrate Habermas' construct of rationality with the Model of Aware and Effective Attitudes and Behaviours theoretical construct in order to analyse the different roles assumed by the teacher during classroom discussions aimed at supporting the development of algebraic thinking. Finally, Branchetti and Morselli (2019) investigate, from the perspectives of rationality and identity, the dynamics of agreement and disagreement that may emerge within collective argumentation processes during complex problem-solving activities, also taking into account the role played by the teacher in either fostering or constraining genuine agreement among the students involved.

2.1 Our conceptualization of rationality

From a careful review of the literature, we observed that in mathematics education the framework proposed by Habermas is mainly used to describe rational behaviour in terms of three dimensions—rational acting, rational communication, and rational validation—while, in our view, there more attention should be given to the notion of self-reflection. In particular, limited emphasis is placed on the idea that, in communicating with oneself, the subject engages in a dialogical form of action (Habermas, 1998, p.308). The idea of thinking as communication with oneself, understood in a dialogical sense, is also present in Sfard (2001), who refers to a human activity of a social nature that originates as an internal and private form of interpersonal communication. However, the different aspect highlighted by Habermas concerns "the relation of the first person *to herself*, on the basis of the adoption of the perspective of a second-person perspective *on me*" (Habermas, 1998, p. 308). The dialogical dimension thus manifests itself in an openness to the perspective of the other, in a sort of inner dialogue that fosters meta-reflection. This enables the rational subject to "distance themselves from themselves" by opening up to the perspective of the other, freeing themselves from egocentric intuitions in favour of freedom of choice. This view, in which the individual distances themselves from their own perspective, is characteristic of the argumentative process that Habermas associates with discursive rationality. According to Habermas' thinking, which identifies argumentative praxis as "a reflexive form of communicative action" (Habermas, 1998, p.311), reflection can be considered from a cognitive point of view as a cognitive shift from egocentric thinking to accepting the point of view of others, which is pivotal in the argumentative process (Albano et al., 2024b).

This allows agents to refer to a common objective world within a shared horizon (Boero & Planas, 2014; Boero, 2022; Boero & Turiano, 2020, 2022). Self-reflection of

the knowing, acting and speaking agent can be seen as an inner dialogue with oneself, perceived as someone outside oneself, and thus it occurs within a communicative act. Self-reflection thus assumes a dialogical nature, in which one perceives oneself from a dual perspective, both as self and other, in a kind of inner discourse that takes shape within communicative action (Albano et al., 2024b). The subject does not remain confined to a monological perspective, but recognises the need for intersubjective validation (Habermas, 2014), in a form of decentralisation process. Within this reflexive self-relation (epistemic, practical, and moral), we can also recognise the reflective stance of the subject towards their own beliefs, actions, and forms of communication, taking into account the perspective of others and understanding themselves as interconnected with a collective dimension (Habermas, 1998, p.310). This perspective thus opens up the possibility of meta-level awareness. Boero et al. (2010) Boero and Turiano (2020) also interpret communicative rationality as an act of openness toward otherness, understood both as other members of the community and as Culture. Moreover, Branchetti and Morselli (2019) highlight the crucial role of the communicative and epistemic dimensions in enabling individuals not to remain anchored to their own beliefs, but to recognize the validity of a judgment in terms of its justified acceptability within a given context. Nevertheless, we argue that greater emphasis should be placed on the subject's self-reflection. This form of self-reflection, in which the subject perceives themselves as other, led us to reconsider a definition of rationality that encompasses both aspects discussed.

Drawing on Habermas' perspective, we conceptualise rationality at two interconnected levels (Albano et al., in press):

H1) as rational behaviour manifested in acting, communicating, and validating mathematical statements within the proving process, thereby supporting students' enculturation into the culture of theorems (Habermas, 1998; Morselli & Boero, 2009);

H2) an internal and conscious form of reflection that occurs in a dialogical manner, in which attention is directed primarily to others' viewpoints, not only to one's own, with the aim of achieving mutual understanding (Albano et al., 2024b; Habermas, 1998). This level is a meta-level operating across all three H1 dimensions.

Chapter 3

Methodological background for design

In this chapter, we present the methodological background on which our research methodology is grounded. In particular, we first outline the methodological and didactic device for the design, and subsequently the methodological framework for data analysis.

3.1 Storytelling as role-playing

Storytelling has always been a fundamental form of human communication, carrying deep-rooted meanings connected to the psychological and emotional sphere. It can take many forms, ranging from highly theatrical presentations to approaches that directly engage the audience (Zazkis & Liljedahl, 2024)(A. Donatiello, trad. p.62).

Egan (1989) attributes to stories the capacity to engage students through a narrative rhythm grounded in *binary opposites*, that is, forms of conflict that capture attention and thereby enable the communication of complex concepts by appealing to the emotional dimension. According to Egan, separating the cognitive from the affective domain "is a product of a restricted conception of learning" (Egan, 1989, p.29), since our experience of the world is inherently both cognitive and affective.

Zan (2011), taking into account the emotional dimension, places particular emphasis on the role of story-problems and on the difficulties students encounter when solving so-called words-problems, typically found in textbooks or formulated by teachers. According to Zan, one of the main sources of difficulty lies in the externally imposed nature of such problems, as students are often required to solve situations defined by others. This externalization frequently leads to goals or purposes that learners do not perceive as their own. As a result, a narrative rupture may occur,

which needs to be bridged through storytelling and by situating the problem within the student's real-world context (Zan, 2011, 2012). When the coherence among narrative elements breaks down, comprehension becomes ineffective. For Zan, the problem should therefore arise from the situation experienced by the story's protagonist, so as to appear convincing and meaningful to the student. This idea is formalized in the *C&Q* (Context–Question) theoretical construct (Zan, 2012). This construct highlights that, to make a story effective for understanding a mathematical problem, it must combine a coherent narrative structure with a clear and meaningful connection between context and question. The question, therefore, should not be artificially imposed, but embedded within a well-structured narrative situation that evolves over time, includes at least one active character, and presents details consistent with the story's context. Unfortunately, in traditional classroom explanations, teachers often rely again excessively on so-called "rules" or adopt an uncritical use of textbooks, resulting in the proposal of many repetitive exercises and very few opportunities for genuine problem solving.

Zazkis and Liljedahl (2019), positioning themselves within this perspective, discuss the need to offer students authentic problem-solving activities that do not boil down to simple exercises but instead represent real problems, namely problematic situations in which the solution is neither immediate nor reducible to the mere application of formulas or procedures. As already clarified (see Introduction), moreover, in conceptualizing storytelling as an activity, they add a further element to this view, namely the idea of interactive storytelling in educational contexts, particularly within mathematics classrooms. Indeed, Zazkis and Liljedahl (2019) propose a shift in perspective entirely, by engaging students in listening to and constructing stories in which open, highly engaging, and motivating mathematical problems are embedded. These problems address mathematical issues of increasing complexity while remaining coherently situated within a consistent narrative framework. The role of storytelling is therefore to convey complex concepts in a way that is engaging and captivating for students, so that they feel actively involved in the mathematical activity (Albano & Pierri, 2017). According to this point of view, we conceptualise storytelling as

> not simple artistic telling, but an interactive telling. The teacher provides the frame, but the students become participants in the telling by deciding how to extend the story [...] making the students themselves the heroes or the actors [...] and introducing role playing (Zazkis & Liljedahl, 2019, pp.69-70)

The origins of role-playing, already present in primitive ritualised forms (Kipper, 1996), can be traced back to psychodrama, a therapeutic technique devised by the psychiatrist Jacob L. Moreno (1889–1974), who in 1934 first coined the term

"role-playing" to refer to "free role play", namely an improvisational practice that belonged to what Moreno called the "Theatre of Spontaneity," for which we might recognise the *Commedia dell'Arte* as its precursor (Scheiffele, 2008).

Role-playing has later also been introduced in educational contexts, in particular within mathematics classroom, because to enhance the effectiveness of collaborative and cooperative learning, an accurate design of collaboration is required: in this setting, the assignment of roles to each group member became essential (Pesci, 2009). This holds even more strongly in computer-supported environments, where the need to pre-structure and regulate both social and cognitive processes becomes critical (King, 2007; Weinberger et al., 2009).

Like all simulation techniques, role-playing seeks to recreate in the classroom, thus in a protected setting, problems and events similar to those of real life. The difference from psychotherapy of Moreno is that, while the latter involves enacting personal aspects, role-playing stages cognitive or social roles more generally. Therefore, the actual focus of role-playing as a learning activity is the dramatization of role-based behaviors.

This idea of dramatization can partly be traced back to the work an actor does on a character as presented in the Stanislavskij method. Written between 1916 and 1938 and published posthumously in 1957, the work on the character consists of four phases: knowledge, re-enactment, embodiment, and communicative power. Knowledge refers to the preparatory stage, involving an initial approach to the role and the analysis of what Stanislavskij calls the *given circumstances*. Re-enactment, on the other hand, is the phase of the creative act in which the actor begins to achieve, even if still unconsciously, the character's aim. This leads to embodiment, which represents the stage of becoming aware, where character awareness begins to develop, even if still naive and not fully mature, and finally to communicating the character's script through words (Stanislavskij, 1988). Although in role-playing the script is not fully predetermined but outlined only as a *canovaccio*, the affinity with theatrical communication play is possible, and role-playing can therefore be understood as a "mode of communication in interpersonal interactions" (Kipper, 1996, p. 100) between two or more individuals assuming different roles, thus involving the adoption of behaviors within an imagined situation.

In line with the view that participation in social practices is a collective activity in which expertise is distributed among participants—each contributing their own specific features—we adopt also the perspective of role-playing as "a distributed system of cognition" (Hatano & Wertsch, 2001, p.80), in which students work together as a team that shares role-based features toward a common goal, that is as a collective activity in

which students are engaged as active participants in a model that we call Inside-Out [1] (Albano et al., 2021a).

3.1.1 The 'Inside-Out' model as a methodological and didactical device

The methodological and didactical device of the Inside-Out model (Albano et al., 2021a) allows participants to become immersed in collaborative problem-solving activities by identifying with characters in a story whose actions are guided by the goals and role of the character they play. This model assumes that the roles played by students in a collaborative problem-solving activity fit various cognitive functions that come into play in the process and that are active in the mind of a mathematician engaged in a problem-solving activity. In an interactive story-problem, these cognitive functions can be embodied by the students as characters—Pest, Boss, Promoter, Blogger— giving rise to the Inside-Out model, with the aim of fostering the awareness in learners of the structure and organisation of mathematical thinking (Albano et al., 2021a). The underlying hypothesis is that, through social practice, students internalize these functions, which shift from being externally assigned to self-initiated. Students are involved in cooperative co-construction of a substantially new object of knowledge, and the learning that emerges within the groups can be located in the group's zone of proximal development (Aceto et al., 2024).

The Inside-Out model is therefore based on cognitive role-playing and was developed with the aim of allowing students to experience the cognitive functions involved in the mind of a mathematician when solving a problem (Albano et al., 2021a). The cognitive roles of the Inside-Out model are defined as follows:

* Promoter, i.e. prospecting function, who, as new questions arise, takes in charge the exploration that is necessary to proceed with the problem-solving process;
* Boss, i.e. planning function, who takes care of maintaining control of the resolution process with respect to both the previous results and the targeted goal; in addition, Boss takes care of the involvement of the group and the collaborative setting that must arise between the characters;
* Blogger, i.e. publishing function, who takes care of communication, sharing with the others and with the expert both the resolution process and the results achieved, in compliance with the rules shared by the mathematical community;

[1] https://www.pixar.com/inside-out

* Pest, i.e. validating function, who is responsible for validating the solving process, with reference to the theoretical aspects and by checking the outcomes;

* Guru, i.e. knowledge and wisdom function, who represents the expert to be turned to in the event of an impasse; it symbolises wisdom and access to external resources that may be needed during problem solving.

The model requires the students to play the roles in groups of four members, with each group containing each of the characters from the Inside-Out model. To facilitate internalisation, students will have to experience all the different cognitive roles during the problem-solving, taking turns in the transition among different phases of the problem-solving activity on two different levels: Actors or Onlookers. For every phase of the problem-solving activity—that is for every episode along the story—one group of students are the Actors, with each student taking on the role of each of the characters in the story. The other groups are the Onlookers, that is, every student takes charge of observing a specific character in the story and reflects on how the observed character behaves with respect to both the mathematical problem and the role they are playing, reflecting in turn on their own possible way of living out the assigned cognitive role. The groups' membership and the roles played within them are subjected to rotation. For example, if in the first episode a group played as Actors, in the second episode they will play as Onlookers and a group that played as Onlookers will play as Actors. Also, if a student played as Boss in the first episode, in the second episode they will play in a role other than Boss.

A further reflection is needed on the relationship between the dual level of participation, Actors/Onlookers. Some neuroscience research (Umiltà et al., 2001, 2008) has shown that mirror neurons are activated when the observed action has a real goal. This allows for a resonant response in the observer, which in the theatrical domain may be due to the Stanislavskian notion of subtext—that is, all those intentions, choices, or strategies not explicitly written in the theatrical text but enacted by the actor to make the action true, real, and believable (Sofia, 2013). Thus, these intentions are projected onto spectators who absorb them and transform them in light of their own personal experiences and reflections (Cruciani, 2006, p. 98).

> The activity of the theatrical spectator constitutes a very particular form of experience. It is not the result of passive one-way perception; rather, it should be understood as a circular and creative process of reworking and building one's own dramaturgical "truths," triggered by the stimuli and cues deliberately crafted by the actor. The spectator is physically called into play and becomes involved in a shared space of action. The "spectator

> in play" is thus engaged in a dense network of stimuli and intersubjective resonances that characterize the phenomenological specificity not only of theatre, but of all forms of live performance [2] (Sofia, 2013, p.73)

Digital Interactive Storytelling in Mathematics

The interactive story-problem can be implemented in a digital environment, in the form of a succession of episodes which are part of a consistent narrative in which the students act as characters of the digital story moved by an aim and their actions to achieve the aim are guided by the role assigned. Moreover, in the Inside-Out model, the development of the narrative supports the development of the mathematical problem (Bruner, 2009). In particular, Albano and colleagues (2020c, 2020b, 2024c) embed the characters of the Inside-Out Model in a digital story-problem that enables interactive participation at multiple levels within a digital environment, specifically a Moodle platform, according to the design principles of the methodological-technological architecture of Digital Interactive Storytelling in Mathematics (DIST-M).

We would like to note that within the literature on storytelling, particular attention is given to Digital Storytelling, which combines the potential of narrative with the affordances offered by the digital environment. While Robin (2008) refers to Digital Storytelling as a computer-based tool that enables storytelling through the use of images, text, and audio, Moutsios-Rentzos and colleagues (2019) places greater emphasis on the cooperative dimension, which is enhanced through shared digital writing. In the systematic review conducted by Deslis and colleagues (2025), digital storytelling (DST) in mathematics education is not viewed as a mere auxiliary activity, but rather as a robust and multimodal pedagogical approach of which Deslis et al. propose a specific operational definition

> "The individual or collaborative process by which learners, educators or researchers create, interpret, or share narratives—combining textual, visual, and/or auditory digital media— to construct, communicate, contextualise, and reflect upon mathematical ideas, practices, or experiences, with the aim of supporting conceptual understanding and emotional engagement of teachers and/or students". (Deslis et al., 2025, pp. 1548/3)

In DIST-M, students are therefore guided through a comic-strip story-problem divided into episodes, by taking on the cognitive roles of Inside-Out model, operating within the mind of a mathematician (Boss, Promoter, Peste, Blogger) and interacting on a dedicated platform using accounts associated with each character. Indeed, to foster

[2] Translated by the author of this thesis from the original Italian version

immersivity in the digital story, in the Moodle learning environment profiles with names and icons for each character are implemented. In this way, users interact in the virtual environment always and only through the characters they are playing at that moment.

Within DIST-M (Albano et al., 2024c), comics do not only stimulate students' engagement through the visual senses, as is the case in Deslis and colleagues (2023), but rather, they constitute an interactive context, initially proposed by the teacher/researcher as a *canovaccio* in the tradition of the *Commedia dell'Arte* (following a careful a priori analysis of the selected mathematical problems), which is subsequently extended through digital chats where the comics-narrative continues by means of students. This development occurs through the free interpretation and improvisation of students within an intensive and mediated role-playing activity.

The activity is understood as socially and culturally mediated on various aspects. In the first place, the methodological-technological architecture of DIST-M is based on the assumption that learning is the result of a dialogical process of negotiation within a digital environment, in which argumentation emerges in its dialogical dimension, as a form of organisation and development of knowledge, which evolves within a process of intersubjective discursive negotiation in a digital environment (Albano et al., 2022b). Furthermore, the set of characters of the Inside-Out model also includes the figure of the Guru as a cultural mediator (Albano et al., 2020a, 2021b). The Guru's role is multifaceted: on the one hand, it provides access to external resources; on the other, it acts as a guide supporting the resolution of impasse situations. For this latter purpose, the Guru maintains a privileged relationship with the Promoter, who can consult it in case of difficulty.

In the initial phase, the work is designed to take place only among peers (Boss, Peste, Blogger, Promoter), without direct intervention from the expert. The Guru remains in the background, communicating privately with the Promoter. The latter turns to the Guru when the group has exhausted its own resources, while the Guru can intervene indirectly, through the Promoter, to support the problem-solving process or help overcome moments of impasse, addressing not only mathematical aspects but also organizational ones. The group work concludes when the group has developed a shared response to the problem, to be sent via email to the Guru (Albano et al., 2024c).

We would also like to note that the digital environment allows the multilevel participation of Actors/Onlookers, enabling students to interact exclusively in written form in digital chats with reference to their assigned roles and observing and reflecting both on the cognitive roles played and on the mathematical problem, as this fosters reflective thinking during and after the interactive activity itself, in line with the principles of writing-to-learn (Albano et al., 2024a).

The discussion about the resolution of the problem takes place through the integration into the digital environment of various technological tools that foster the sharing of ideas (for example whiteboard or shared document). In this way, the students can go back and forward in the story and throughout the whole process, "accessing the previous chats, or whiteboard, or any material produced, and work on a shared document to produce a collective mathematical narrative" (Albano et al., 2024a, p.7).

We would like to note that, the point of view of digital storytelling as a multimodal approach (Deslis et al., 2025), language plays a fundamental role. The digital platform "allows the use not only of a wide range of semiotic systems, but also of a great variety of linguistic registers" (Albano & Ferrari, 2013, p. 39), such as a more colloquial register that enables the negotiation of meanings, and a more literate one, which refers to more stable and conventional forms of communication (Ferrari, 2017). The possibility of using a more colloquial register in chats among actors and onlookers, and a more formal, literate register in external communication when sending the email to the Guru, thus allows students to interact across multiple communicative levels.

In summary of the design principle, Albano and colleagues (2024c) propose a framework that operates across four distinct levels:

1. **Macro-level - Mathematical meta-narrative**. Students are involved in solving a problem arising from the situation narrated in the story and therefore from the protagonist's specific needs, in line with Zan's *C&Q* construct (2012), through a sequence of comic-strip narrative episodes that mirror corresponding phases in the mathematical problem-solving process;

2. **Micro-level - Cognitive role-playing**. Students perform the cognitive roles mentioned above, personifying the cognitive functions that come into play in the mind of a solver (mathematician or student) during a problem-solving activity and that interact in a polyphony of voices (Bachtin, 1968; Bartolini Bussi, 1996), each of which enters the collective discourse by bringing in the thinking of each individual cognitive role;

3. **Meso-level - Actor/Onlookers**. In this case, reference is made to the dual level of students' participation, as they will sometimes act as actors and at other times as Onlookers, that is, as observers of the process, paying attention both to the mathematical problem and to the role played by the actors corresponding to the same role observed by the Onlookers.

4. **Meta-level - Reflection**. During all phases of the activity, and especially at the end of the learning path, students are asked to engage in retrospective reflection

through structured logbooks and questionnaires, as well as through the collective writing of a narrative that reconstructs the entire experience. This process takes place by re-reading the comics, chats, and logbooks as a shared classroom resource, in order to develop an awareness of the mathematical meaning of the experienced narrative (Lolli, 2018).

3.1.2 Our conceptualization of storytelling

In light of the presentation of the model, we now return to the definition of Zazkis and Liljedahl (2019), presented in Sec. 3.1, in order to better frame the perspective of storytelling involved in the Inside-Out model.

We conceptualise storytelling, and thus the digital interactive storytelling in mathematics, as:

* *not simple artistic telling, but an interactive telling*: This aspect is highlighted in the model through the direct interaction of students with each other and with the mathematical problem, within a narrative context in a digital environment. The interaction takes place on multiple levels—both in the interpretation of cognitive roles and in the rotation between Actors and Onlookers—and occurs primarily through the digital environment.

* *The teacher provides the frame, but the students become participants in the telling by deciding how to extend the story*: The comics story-problem is presented as a *canovaccio*, framed by the teacher and/or the researcher to provide the inputs necessary to initiate the storytelling; it is then further developed and extended by the students within the chats, as they interact using accounts associated with individual characters, thereby enhancing immersion.

* *making the students themselves the heroes or the actors*: Students are engaged at the dual level of Actors/Onlookers, thus becoming first-hand protagonists of the narrated story.

* *and introducing role playing*: Student participation is based on cognitive role-playing, in which each character embodies specific cognitive functions that come into play in the mind of a mathematician—or more generally, a problem solver—during problem-solving activity.

Chapter 4

Methodological background for analysis

4.1 Rational Mathematical Template

The Rational Mathematical Template (RMT) (Boero & Turiano, 2020) is defined as a pair consisting of a mathematical entity (for example, a theorem, a definition, or a counterexample) and a rational process, understood as a specific instance, in a rationality perspective (Habermas, 1998), formulated in relation to the mathematical entity under consideration. It initially emerges as a mediating tool within classroom relationships between students and teacher, or among students themselves, to develop awareness in defining process (Boero & Turiano, 2020), in the generation of counter-examples (Boero, 2022), and in the process of proving (Boero & Turiano, 2022, 2023). In particular, Boero and Turiano (2020) recognise in the RMT the advantage of going beyond concept formation understood from the perspective of Vergnaud (2009), additionally enabling students to experience and thus internalize also "how-to guidelines and move towards a goal to be achieved" (p. 144). RMTs should therefore be understood as "dynamic-evolving objects of teaching and learning" (Boero & Turiano, 2022, p.84), addressing the need for shared references capable of informing collective activities and for a mediating tool that supports the development of mastery of mathematical entities, for example proof (as a mathematical entity) and proving (as a rational process). The RMT is thus recognised as an interactional space in which not only cognitive aspects, but also affective ones, come into play, fostering autonomy and agency (Boero & Turiano, 2020).

The need to strike a balance between a careful design of instructional sequences and attention to the difficulties encountered by students calls into play the dual role of

the teacher-researcher as both a promoter of the development of mastery of the RMT and an interpreter of students' productions in terms of rationality, with the aim of recalibrating and adapting the intervention (Boero & Turiano, 2023).

Beyond its mediating role, the RMT is in fact also used as an analytical tool to interpret, from a rationality perspective, the evolution of students' behaviour in classroom discursive practices, in particular the progressive development of the three components of the rational process (Boero & Azrou, 2023; Boero & Turiano, 2022). Boero and Turiano (2023) consider the process of proving to be rational if the various phases of which it is composed, although not necessarily conceived sequentially (exploration for the elaboration of a conjecture, argumentation supporting the conjecture formulated, and the writing of the proof in relation to the rules shared by the mathematical community), are developed consciously in relation to the aim, while taking into account epistemic, teleological, and communicative aspects. In particular, Boero and Turiano (2022) consider, for epistemic rationality, the verification of the truth of statements and the validation of reasoning concerning the mathematical object; for teleological rationality, the relevance and effectiveness of the strategies adopted in relation to the goal; and for communicative rationality, the choice of communicative means oriented toward mutual understanding, both with oneself and with Others (the Culture), within a given cultural and social context. It is therefore possible to investigate in students' written productions the lack or weakness of rationality dimensions, which become manifest in erroneous or incomplete utterances, as well as in the absence of theoretical justifications or in their implicit reference to them (Boero & Turiano, 2022).

As already envisaged by Boero (2022), through the RMT of proof it is also possible to focus attention on the meta-level of the proving process. In particular, Boero and Azrou (2023) analyse aspects of the proving process related to meta-knowledge of proof as a product and meta-knowledge of the proving process, as well as the interplay between levels, for example concerning the logical aspects of the proving process (direct proof, proof by contradiction, proof by contraposition, proof by induction, ...), legitimacy within the reference theory, the correctness of symbolic or verbal representations of implications, the relationship between the exploratory strategy (for example, abductive reasoning) and the logical organization of the formal proof, and the choice and use of systems of representation of meta-knowledge.

4.2 Narrative research

Narrative research is an analytical method within qualitative research, developed to explore and understand the inner world of people through their narratives. According

to the definition provided by Lieblich and colleagues (1998), it refers to any study that collects, "uses and analyzes narrative materials" (p.2), such as researchers' field notes, conversations among study participants, diaries, or stories obtained through interviews.

Lieblich and colleagues (1998) distinguish various types of narrative analysis, corresponding to different ways of interpreting and analyzing a narrative, identifying two dimensions, *holistic–categorical* and *content–form*, which are considered as intersecting.

As concerns the first dimension, while the holistic approach considers the narrative as a whole, the categorical approach fragments it by extracting sentences or words and coding them into categories. The latter is particularly useful for investigating phenomena across groups of participants, whereas the holistic approach is often preferred in single-case analyses. The second dimension concerns whether the focus is placed on the meanings conveyed in the narrative or on the way the narrative is constructed, independently of its content.

This gives rise to four distinct approaches to narrative analysis:

- Holistic-Content Perspective;
- Holistic-Form Perspective;
- Categorical-Content Perspective;
- Categorical-Form Perspective.

While the Holistic–Content Perspective involves an overall reading of the data, emphasizing the content emerging from the narrative, the Holistic–Form Perspective also provides a broad and comprehensive view but focuses instead on the form of the narrative—its grammatical structures, expressions, and metaphors—independently of the content. By contrast, the Categorical–Content Perspective enables a more detailed and specific analysis, examining particular sentences or words and emphasizing the meaning of the narrative content. Finally, the Categorical–Form Perspective also entails a detailed analysis, but it focuses primarily on form and stylistic features rather than on content and its meaning.

In the next section, we will examine in detail the Categorical–Content Perspective, which will be the focus of the empirical studies described in the following chapters.

4.2.1 Categorical-Content Perspective

The categorical–content perspective draws on a classical method of qualitative analysis known as *content analysis*, through which the collected data are "processed analytically [...] by breaking the text into relatively small units of content."(Lieblich et al., 1998, p.112).

Below, we briefly describe the paradigmatic and illustrative phases through which Lieblich and colleagues (1998) synthesize the analytical process, presenting them as a model to draw inspiration from, but not to be understood either rigidly or as a fixed sequence of steps. Rather, this method needs to be interpreted as "a circular procedure that involves careful reading, suggesting categories, sorting the subtext into categories, generating ideas for additional categories or for refinement of the existing ones" (Lieblich et al., 1998, p.113). The phases of the method can be summarized as follows:

1. *Selection of the Subtext.* This initial phase aims to focus attention only on those portions of the text that are essential to the study and aligned with the research hypotheses and questions. This subtext thus becomes the new dataset for the content analysis.

2. *Definition of the Content Categories.* This phase leads to the definition of categories or themes by identifying recurring patterns, sentences, utterances, words, or expressions that can characterize a given theme. Categories may be small and detailed—reflecting a meticulous search for specific features in the data—or broader and more easily applicable, though with the risk of overlooking some important aspects within a large dataset. Moreover, categories can be theory-driven, that is, defined a priori on the basis of well-established theoretical constructs that guide the researcher in identifying relevant textual elements consistent with the adopted framework; alternatively, they may emerge directly from the selected data. A careful reading, as free as possible from theoretical preconceptions, can allow distinctive features to emerge from the data through a process of successive refinement. It is important to note, however, that even data-driven categories are never entirely free from interpretation, as researchers inevitably approach the data with their own perspectives, often informed by underlying theoretical frameworks.

3. *Sorting the Material into the Categories.* This phase represents one of the most delicate moments of the analysis, as, through a cyclical process, individual elements—sentences, words, or utterances—are assigned to the categories that are progressively being defined. Categories, in fact, are not fixed and stable from the outset; rather, they are continuously refined throughout the entire analytical process. For this reason, phases 2 and 3 can be considered interconnected and interchangeable. This stage can also be supported by the use of different colors to distinguish among various segments of the text.

4. *Drawing Conclusion From the Results.* At the end of the process of assigning data to categories, the results can be collected and synthesized in various ways. Data

may be organized in tabular form, indicating the frequency with which specific phrases occur, or the contents of each category may be presented narratively, describing their characteristics in order to provide an overview of the features observed in the groups analyzed. At this stage, by summarizing the results, it is also possible to examine or test the research hypotheses formulated at the beginning of the study.

Chapter 5

Research Questions

In the Introduction we highlighted our need to investigate whether storytelling, through roles embodying the cognitive functions involved in a mathematician's mind during a problem-solving activity, may contribute as an effective methodological tool for making explicit the different Habermas' rational components of the proving process, thereby fostering students' enculturation into the Culture of Theorems, particularly within the domain of Euclidean geometry. Aware that the process of scientific enculturation is a long-term one and that the movements towards it, which might be observed through our study, may not necessarily represent stable individual internalisation processes, in light of the theoretical background presented in the previous sections, our research aim can be redrafted in the following overarching research programme:

How may the integration of digital storytelling as a cognitive role-playing with Habermas' construct of rationality foster, or fail to foster, the process of scientific enculturation into the Culture of Theorems, understood as a rational process, by prompting proving processes?

This first question represents the overarching research programme of our study and has a theoretical relevance, as it investigates the possibility of integrating storytelling with Habermas' construct of rationality in order to support students in proving processes. Retaining scientific enculturation as the broader horizon, the term 'integration' takes on a broad meaning for us, encompassing both the design-oriented question and the analytical-empirical question. We interpret Habermas's construct of rationality both as a theoretical framework upon which to design the behaviours of characters in digital storytelling – and thus to ground the actions and aims of cognitive role-play – and as an analytical lens through which to interpret and analyse students' synchronous discursive interactions in digital chats, email and paper and

pencil worksheets. Therefore, in order to address this overarching and central research programme throughout the entire study, we further articulate it into more specific and operational sub-questions.

RQ1: *How to characterise the cognitive roles of Blogger, Pest, Promoter and Boss according to Habermas' rationalities and related studies and focused on the domain of mathematics, in order to foster the emergence of the components of rationality in the proving process?*

Our aim is to carry out an a priori analysis from a theoretical standpoint, in order to characterize cognitive roles of Inside-Out model within a rationality perspective.

RQ2: *How the use of the above characterisation can shed light on student behaviour emerging from their synchronous discursive interactions occurring in a chat in a digital problem solving situation?*

We are interested in investigating how the a priori characterization of cognitive roles can be used as an analytical lens to interpret students' behavior while engaged in proving processes during a cognitive role-playing activity, with particular attention to the emergence of components of rationality.

RQ3: *How the above role-playing setting supports the development of modes of theoretical validation?*

Our aim is to conduct a detailed and fine-grained analysis of the synchronous discursive interactions, as well as of the shared productions that emerged during the role-playing activity, in order to investigate the possible evolution of students' behaviors toward forms of theoretical validation. In this analysis, we are also interested in taking into account the role of the Guru as a mediator of rational processes.

RQ4: *What connections can be or not be identified between cognitive roles and the dimensions of rationality?*

Finally, we would like to clarify the emerging connections between the cognitive roles of the Inside-Out Model and the dimensions of rationality, in order to better map the dynamics of interrelation existing between them and emerging within proving processes.

Furthermore, ìn the chapter describing the preparatory activity of the Euclid Experiment (see chapter 9), an exploratory study related to the cognitive unity of theorems and the cloze tool is presented. In this case, the specific research questions,

which are not directly related to the main RQ but rather concern that particular activity, are not included in this section and are instead presented directly within the corresponding chapter.

Chapter 6

Extending the Conceptual Framework

To account for the complexity of cognitive role-playing in proving processes, we recognise that the rationality framework alone is not enough to capture all the aspects involved in the thinking of a mathematician, as well as of a student, engaged in conjecturing and proving activities. Although Habermas' rationality framework is essentially used in the literature to analyse exactly these processes, this is mainly done to analyse written protocols produced individually (Boero & Azrou, 2023; Boero & Turiano, 2020, 2022, 2023) or through the analysis of transcripts of classroom discussions (Urhan & Zengin, 2024). The use of Habermas' framework to analyse verbal discursive interactions has been considered in the works of Branchetti and Morselli (2019) and Goizueta and colleagues (2014), but always based on transcripts of video recordings of small-group work. In synchronous discursive interaction taking place in a digital chat during a cognitive role-playing activity, students are required to experience firsthand the personification of the processes occurring in the mind of a solver engaged in a proving activity. Moreover, cognitive role-playing involves participants at a dual level of participation, fostering a reflective process, through attentive and participatory observation, highlighted by several studies (Albano, 2023; Albano et al., 2022a, 2024a; Miranda et al., 2023). Furthermore, in his epistemological analysis of the nature of mathematical proof, Balacheff (2008) recalls, drawing on the studies of Mariotti and colleagues (1997), that when a student is engaged in constructing a mathematical proof understood as a process, many aspects emerge that are shared with the activity of a mathematician involved in a proving process

The Inside-Out Model, presented in Section 3.1.1, cannot therefore be reduced to a mere methodological and didactic tool; rather, it becomes an integral part of the

conceptual framework, together with some selected constructs from the field of proof and proving.

To address Research Question RQ1, it is thus necessary to attempt to combine different theoretical constructs into a coherent framework capable of capturing the nuances of the complex process underlying rational behaviour in the face of open conjecturing and proving tasks, up to the formal writing of a proof as a form of external communication within the mathematical community.

Therefore, our research starts from the Inside-Out model, wherein students collaboratively solve some problems, and each of them contributes by acting mainly from a specific point of view, given by assigned roles, as detailed in section 3.1.1. Looking at previous pilots of the model, it seemed to us that the way of acting fostered by the cognitive roles evoke Habermas' rationalities. As we are interested in using the model in activities concerning argumentation and proof, we do furthermore take into account more references, associated to the Theorem construct of Mariotti (2006), the needs for proof identified by Zaslavskij and colleagues (2021), the Balacheff's dimensions of proof (Balacheff, 2010), and finally, the cognitive unity of the theorem introduced by Boero et al. (1996b).

6.1 Coordinating locally as networking strategy

We argue that each aspect featured in the conceptual background aligns with Habermas' rationality. Specifically, the Inside-Out model (sec. 3.1.1) with the cognitive functions of a person handling a problem, appearing as roles in the model, can be related to Habermas' approach (Chapter 2), as the roles are intended as parts of one and the same individual, and then their interactions unfold a dialogic inner discourse which promotes self-reflection and self-awareness (Albano et al., in press). This perspective is in tune with the two levels of our conceptualisation of rationality (see H1 and H2 in sec. 2.1), in which are interrelated since self-reflection of the knowing, acting, and speaking agent (H1) can be understood as an internal dialogue in which the individual engages with oneself as if addressing another person (H2). According to this perspective, such a conceptualisation of rationality can be regarded as foundational to the structure of the Inside-Out Model since its roles can be interpreted as different parts of a single individual, who engages in an internal dialogue among these parts, thereby promoting processes of self-reflection (Albano et al., 2024b). In this sense, it unfolds within a communicative act. Self-reflection therefore takes on a dialogical character, as the subject adopts a dual perspective—both self and other—engaging in an internal discourse that develops within communicative action (Albano et al., 2024b).

Moreover, considering the construct of Theorem by Mariotti (2006) and Mariotti and colleagues (1997), in the relationship among statement, proof, and reference theory we recognize a strong connection between the idea of the theoretical validation of a statement within a given geometrical system and the epistemic dimension of Habermas' rationality. Within the discursive practice of the proving process, an epistemic dimension can therefore be identified, related to the "conscious validation of statements according to shared premises and legitimate ways of reasoning" (Boero et al., 2010, p.188).

Zaslavsky and colleagues (2021), in investigating the factors driving students—and, more generally, learners—to recognise the role and importance of the proving process, identifies three needs that are particularly relevant in the teaching and learning of proof: the need for certainty (related to the desire to verify and become convinced of the validity of a given statement, also through the use of empirical methods), the need for causality (related to the search for an explanation of why a given statement is true), and the need for communication (which emerges in the requirement to reorganise concepts in a logical and structured form). These three needs lead us to identify a connection with Habermas' three dimensions of rationality—teleological, epistemic, and communicative—although it should be noted that, while the three roots of rationality are closely interrelated in a dialectical way, the three needs described by Zaslavsky and colleagues (2021) appear to be distinguishable and not dialectically interconnected.

Furthermore, in line with Habermas' point of view, Balacheff (2010) acknowledges the epistemological value that proof plays in the construction of knowledge as communication oriented towards understanding. Language is no longer understood simply as a means for social interactions but, rather, as a tool enabling the transition from pragmatic truth to the theoretical validity of a mathematical proof.

Finally, Boero establishes a link between cognitive unity and rationality, examining the breakdown of the cognitive unity of theorems highlighted by Pedemonte (2007, 2008) in relation to the lack of rationality components, particularly the teleological aspects of reasoning, such as abduction or induction strategies, that play a role in the creative process of conjecture formation.

In this effort to connect different constructs in order to build a coherent framework, we choose to adopt one of the networking strategies highlighted by Prediger and colleagues (2008). We recall that the networking strategies of combining and coordinating "are mostly used for a networked understanding of an empirical phenomenon or a piece of data" (Prediger et al., 2008, p.172). In particular, coordinating theories involves drawing on analytical elements from different theoretical

frameworks in order to investigate a specific research problem within a given empirical context. In this perspective we consider the aforementioned conceptual backgrounds, coordinating them locally to develop an analytical tool as a lens to analyse students' behaviour in a collaborative problem-solving activity, as conjecturing and proving processes.

Chapter 7

A study to characterise cognitive role-playing

Within a larger study that investigates how cognitive role-playing in digital storytelling can foster students' proving processes, in this chapter we present a pilot study focused on proving as a rational process, contributing to the field of teaching and learning of proof. In particular, drawing on the conceptual background presented in the previous chapter, we choose to use the components of Habermas' construct of rationality to characterise the roles students assume while engaging in proving, considering both a knowledge and a meta-knowledge perspective. Moreover, we use this characterisation as a lens to analyse students' synchronous discursive interactions in a digital environment during a proving activity, with the aim of highlighting the emergence of rational behaviours activated through cognitive role-playing. This pilot study is also useful for testing the aforementioned characterization of cognitive roles, with the aim of refining it further and thus designing a new, broader experiment based on this characterization from a rationality perspective. The findings presented in this chapter are the result of the collaboration with Prof. Giovannina Albano and Prof. Cristina Coppola and presented in a paper entitled *"The emergence of rational behaviour in the proving process: towards Inside-Out 2"* submitted on Educational Studies in Mathematics (Albano et al., in press).

7.1 Methods

7.1.1 Participants

The experimentation we deal with in this chapter was conducted within the research project of national interest PRIN "Digital Interactive Storytelling in

Mathematics: a competence-based social approach"[1]. It was part of a pilot study and involved a group of 26 students attending the second year of an Italian Liceo Scientifico Scienze Applicate (i.e. 10th grade). The school is located in a small town near a main provincial centre. The school's socio demographic is mixed. Throughout the experiment, the researcher—one of the authors of this paper—acted as Guru. The mathematics teacher attended and observed the activities.

7.1.2 Setting

The experimentation took place in the period January 2020-February 2020 and was carried out in presence, during the school hours, in two laboratory classrooms of the school with PC workstation. For a detailed description of the design of the experiment, see Albano and colleagues (2020c). The students were immersed in the digital story "Here come the Aliens", implemented in a Moodle platform. The students and the researcher were characters of the story, and they could communicate exclusively in written form via Moodle chats, without speaking directly to each other. The story evolved depending on the characters' interactions. According to the design principles of the methodological-technological architecture of Inside-Out model, the story evolves in the form of a succession of episodes that are part of a linear narrative, wherein students act as characters of the digital story moved by a goal. and their actions to achieve the goal are guided by the assigned roles (see section 3.1.1). After an initial roles' assignment by the researcher, in each subsequent episode, students are required to take on a role not previously played. To foster immersivity in the digital story, Moodle profiles have been implemented according to names and icons of the characters, in a way that encourages students to identify with the characters. Within the story, the characters have been involved in a cooperative task in which they are expected to explore a situation, to develop a conjecture and to produce a proof of the produced statement. The digital environment allows participants to be engaged in solving a problem on the two levels -Actors and Outlookers- as described in Section 3.1.1 along the episodes in which the story develops.

7.1.3 Task

The task concerns the following problem: *Choose four consecutive natural numbers, multiply the two intermediate numbers, multiply the two extremes, and subtract the results. What do you get?* (Mellone & Tortora, 2015). The mathematical problem was chosen as it is open and suitable to support students' introduction to algebraic

[1]PRIN2015, Prot. 20155NPRA5 (February 2017 – February 2020)

modelling and development of argumentative and proving competencies. The goal for students is to produce a conjecture (i.e. the result is always 2) and to prove it, through the algebraic representation of the generic quatern as n n+1 n+2 n+3, and of the request as (n+1) (n+2) - n(n+3) and the treatment of the latter leading to obtaining the constant 2. According to a didactical choice, aimed at balancing and exploiting both logical thinking and narrative thinking in story-problem (Zan, 2011), students did not receive the problem in the form stated above, but they had to face the problem that came up in the story asking. In the narrative scenario, students, immersed in the context of the story, are presented with a sheet as the one in Figure 7.1. The students are then naturally prompted to explore those numbers, and from their exploration the mathematical problem they are asked to solve emerges naturally.

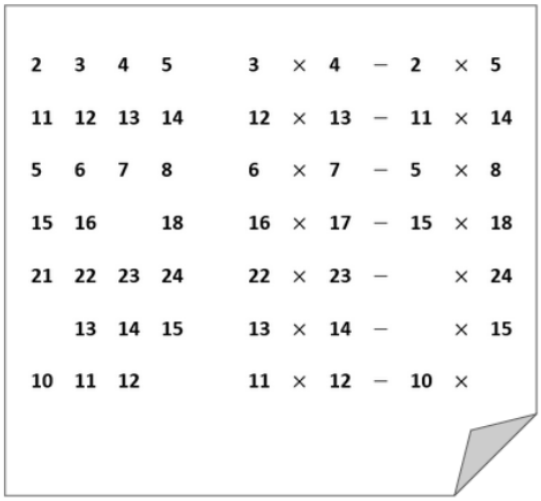


Figure 7.1: Strings of numbers received from aliens in digital story

7.1.4 *A priori* analysis to characterise the cognitive roles in Habermas'perspective

The a priori analysis to characterise the cognitive roles of the Inside-Out model—Promoter, Boss, Pest, Blogger—was made through local coordinating (Prediger et al., 2008) of the different theoretical approaches, as presented in section 6, in order to develop an analytical lens to analyse students' behaviour in a collaborative activity of problem-solving. The resulting indicators in Table 7.1 are elaborated from the works of Albano and colleagues (2021b), Urhan and Bülbül (2023a, 2023b), Balacheff (2010), Boero and Azrou (2023), Boero (2017), Mariotti (2006) and Pedemonte (2007, 2008).

Starting from this, we consider cognitive roles on different levels: knowledge level, metaknowledge level and interplay between the levels. The terms knowledge level and meta-knowledge level were introduced and conceptualised by Boero and Azrou (2023) to denote, respectively, the knowledge of mathematical concepts and patterns and the knowledge of the structure of mathematical theories and formal logic. The interplay between the levels concerns the relationship between the other two levels within the moment of transition from a level to another one. Referring to the knowledge level, the a priori analysis is mainly developed through the criteria identified by Urhan and Bülbül (2023a, 2023b) on students' difficulty in the proof and proving process and which are inherent to the different components of rationality. In particular, from the point of view of the teleological rationality, Urhan and Bülbül (2023a, 2023b) develop criteria based on implementation of the free choice of a deciding actor, results based on deliberately selected and implemented means and purposive justification for the solution method. From the point of view of epistemic rationality, they focused in turn on the accuracy of the judgements and the justification of the resolution steps. Finally, so far as to communicative rationality, they selected criteria about valid and comprehensible speech and the correct use of language and mathematical symbols to implement a formal mathematical proof. The authors detail these criteria in several areas of mathematics (algebraic, geometric, differential analysis), linked to the epistemological domain in which the problems are posed. In constructing the a priori analysis as an analytical tool, we decided to identify, among these criteria, some invariants that were independent of the specific mathematical domain involved. Regarding the level of meta-knowledge and the relationship between the two levels we refer mainly to the work of Boero and Azrou (2023) who employed the Rational Mathematical Template (RMT) as a theoretical construct to analyse students' behaviours in proving and proof processes. They claim that a rational proving process needs a systematic attention to epistemic, teleological and communicative aspects, considered in their complexity and interconnection. RMT consists of a couple (rational process; mathematical object) in which the mathematical object can be a mathematical proof, considering mathematical proof as a concept, according to Vergnaud's point of view (2009), while the rational process aims to produce an instance on the mathematical object.

For the meta-knowledge level, we also take into account of Balacheff's dimensions of proof (2010). This helped us to highlight the internal and external role of the communicative act in the proving process, producing an explanation of the validity of a statement from the subject's own perspective, achieving a proof accepted and valued by a closed community and writing a mathematical proof, in line with formal rules and cultural constraints shared into the mathematical Culture.

7.1.5 *A posteriori* qualitative analysis to test and refine the roles characterization

Using Table 7.1 that arose out of the characterisation of cognitive roles, we conduct a qualitative analysis of the discursive synchronous interactions of the students participating in the study, adopting a categorical-content analysis (Lieblich et al., 1998), in which the categories are theory-based. In accordance with the construct of rationality of Habermas (1998), the a priori characterization in Table 7.1 thus provides a guide for coding the interventions. In particular, after selecting excerpts from collected data, we identify those communicative exchanges amongst the students from which the reference to the dimensions of rationality emerged, according to the theory-informed operational indicators presented in Table 7.1, identifying in the students' words significant references to the dimensions of rationality. The analytical categories considered are Promoter (PRO), Pest (PE), Blogger (BLO), Boss (BO), across different levels: knowledge level (KL), meta-knowledge level (MK), and interplay between the two levels (IP). Subsequently, we assign the data to each analytical category distinguishing the different levels at which the analysis is conducted (knowledge level, meta-knowledge level, and the interplay between levels). Finally, we provide a discursive synthesis of the results derived from the analysis. Accordingly, the analysis focuses on sentences, verbs, verbal expressions, terms, and words that can be traced back to the dimensions of rationality.

7.2 Findings

7.2.1 An *priori* characterisation of the cognitive roles

As explained in the Section 7.1, we use the assumed different theoretical approaches of the conceptual background to a priori characterise the cognitive roles of the Inside-Out model, Promoter, Boss, Pest, Blogger. The results, expressed as operational indicators, are shown in Table 7.1. In the columns, we indicate the characteristic elements of the conceptual background in terms of operational indicators, corresponding in the rows of Table 7.1 to each role of the Inside-Out model. Moreover, within Table 7.1, we standardise the terms used according to the level to which each indicator refers (e.g., to refer to monitoring, we used "checking" at the knowledge level, "awareness of" at the meta-knowledge level, and "control" at the interplay level). Furthermore, we clarify each indicator by specifying the theoretical references, in order to better define its epistemological status.

Although in Table 7.1 each role is associated with some specific rationality components, these operational indicators, commonly linked to one role, may be

performed or experienced by other characters. This phenomenon reflects the dynamic nature of cognitive processes and suggests us to use this characterisation as an analytical lens, without risking circularity in the analysis.

Table 7.1: Roles' characterisation

	Operational indicators of analysis		
Roles	**Knowledge level** **KL**	**Meta - knowledge level** **MK**	**Intra-levels Interplay** **IP**
Promoter **PRO**	**Recourse** to memory in the act of idea generation (*cognitive roles*, Albano et al., 2021) **Generating** ideas through intuition and analogy (*teleological rationality*) **Performing or recalling** actions oriented towards the aim (*teleological rationality*, Habermas, 1998) **Choosing** and **using** the appropriate knowledge, methods and/or theorems to reach the aim. (*teleological rationality,* Urhan & Bülbül, 2023a, 2023b) **Establishing** correct relationships between the variables and the mathematical objects and concepts involved. (*teleological rationality*, Urhan & Bülbül, 2023a) **Choosing** suitable methods to prove the statement and expression given in the problem, **selecting** tools that serve the aim and **using** these tools in the process in a purposive manner. (*teleological rationality,* Urhan & Bülbül, 2023a)	**Awareness of the need for a** functional connection of proving strategies with their aim (proof), with reference to possible schemes of proof for the logical organisation of proof. (*teleological rationality,* Boero & Azrou, 2023) **Awareness of the need** for appropriate systems of representing meta-knowledge, e.g. implicative arrows. (*epistemic and communicative rationality,* Boero & Azrou, 2023) **Awareness of** consistency of different lines of reasoning (e.g. semantic and/or syntactic proofs) (*cognitive roles*, Albano et al., 2021) **Awareness in** choosing and using of mathematical means or tools appropriate to reach the aim (*teleological rationality,* Urhan & Bülbül, 2023b)	**Provides the contributions** of a constructive nature that allow the process to move forward (*cognitive roles,* Albano et al., 2021) **Control of** consistency in the cognitive unity construct and (in relation to the aim) the discontinuity that may emerge in this construct in the transition from proving to proof. (*teleological rationality,* Boero, 2017, Pedemonte, 2007, 2008)
Pest **PE**	**Validation of statements** from accepted premises and using correct reasoning rules (*epistemic rationality*, Habermas, 1998) **Checking** the requirements of truth of statements and validity of reasoning (*cognitive roles,* Albano et al., 2021). **Checking** the consistency of the patterns of use of the concepts, procedures and conceptual properties used and the correctness of the representations of the concepts (*epistemic rationality*, Urhan & Bülbül, 2023a) **Checking for** the effectiveness (or lack) of the strategic choices in relation to the aim (*epistemic and teleological rationality,* Urhan & Bülbül, 2023a) **Checks** the lack of operation mistakes while structuring the process. (*epistemic rationality*, Urhan & Bülbül, 2023a) **Checking** on constructions mathematically correct and valid representations. (*epistemic rationality*, Urhan & Bülbül, 2023b) **Justify** all the steps of reasoning, although sometimes with implicit references to the theory or using set of accepted statements (*epistemic rationality*, Urhan & Bülbül, 2023b) **Checking** the justify of all the steps of reasoning (*epistemic rationality*, Urhan & Bülbül, 2023b) **Checking** the soundness of all reasoning steps. (*epistemic rationality*, Urhan & Bülbül, 2023b)	**Meta-knowledge of proof** and reference of validity of proof to the epistemological assumptions of a theory (*epistemic rationality*, Boero & Azrou, 2023, Mariotti, 2006) **Awareness of** the logical aspects of the chosen demonstrative schemes and their consistency in relation to the reference theory (*epistemic rationality*, Boero & Azrou, 2023; Mariotti, 2006) **Awareness of** the accuracy of deductive chains and symbolic and verbal representations *(epistemic and communicative rationality*)	**Control of** the ambiguity of the statements made and the quality of the communication of logical steps (*communicative rationality*) . **Control of** epistemic lacks in the misinterpretation of an algebraic transformation or calculation or choices that do not correspond to the identified demonstrative scheme, i n the transition from knowledge level to meta-knowledge and vice versa (*epistemic rationality*)

Blogger **BLO**	**Writing correctly and giving clear explanations** regarding the steps taken during the process (also involving visual and dynamic aspects of representations) (*communicative rationality,* Urhan & Bülbül, 2023a) **Using** mathematical language and symbols correctly and in accordance with the aim, thereby producing formal proof (*cognitive roles*, Albano et al., 2021; *communicative rationality,* Urhan & Bülbül, 2023a) **Using the** symbolic language of mathematics properly (*communicative rationality,* Urhan & Bülbül, 2023b) **Facilitating** the reading and understanding of problem solving by verbal and/or textual explanations (*communicative rationality,* Urhan & Bülbül, 2023b) **Providing** the reasons for the correctness of the drawings and the transitions within and/or between the representations (*communicative rationality,*Urhan & Bülbül, 2023b)	**Internal role in itself:** Effectiveness of communication with herself in the proving process, producing an explanation or a justification of the validity or not validity of a statement from the subject's own perspective (*communicative rationality,* Balacheff, 2010; Boero & Azrou, 2023) **Internal role in the group as a mediator of communication:** Communicating with others in a valid and comprehensible way, and hence activating the writer-interpreter dynamics (*communicative rationality,* Urhan & Bülbül, 2023b) Effectiveness of communication with others to achieve the status of proof, shared and accepted by a small community (*communicative rationality,* Balacheff, 2010) **External role for the Culture:** Effectiveness of communication with the Culture, in the formal writing of the mathematical proof, taking cultural format constraints into account. (*cognitive roles*, Albano et al., 2021; *communicative rationality,* Balacheff, 2010, Boero & Azrou, 2023)	**Use of** symbolic and verbal representations concerning implications or deductive chains. (*communicative rationality,*Boero & Azrou, 2023)
Boss **BO**	Keeps the goal fixed, **stimulating** the activation of all cognitive functions (*cognitive roles*, Albano et al., 2021). **Operational guide** that leads the actions of group (*cognitive roles*, Albano et al., 2021)	**Monitors** and **directs** the process (*cognitive roles*, Albano et al., 2021) **Awareness of dialectical dimension** amongst three rationality roots (*communicative, epistemic and teleological rationalities*) **Systematic taking care** of epistemic, teleological and communicative requirements for the used mathematical knowledge throughout the proving process. (*communicative, epistemic and teleological rationalities*, Boero & Azrou, 2023)	**Control of** the need for a move to the next level of the proving process (*cognitive roles*, Albano et al., 2021) **Cares for** and **prompts** the moments **of transition** from process to product (*cognitive roles*, Albano et al., 2021)

7.2.2 A posteriori analysis: how the a priori characterisation shed light on chat protocols

In the following, we present a qualitative analysis of the discursive synchronous interactions of a group of students produced during the problem-solving process. We report some excerpts of these interactions [2] and of their analysis carried out through the lens of the characterisation presented in the previous Section (Table 1). We present the excerpts into four "Scenes", each of them taken from one of the episodes that constituted the whole activity.

[2]Excerpts featured in this paper are direct translations from the Italian original language

Scene 1 - from Episode 1

4	Boss:	guys, do you have any ideas?
7	Promoter:	reading the numbers horizontally have multiplied the middles with the extremes
8	Blogger:	yes, in the fourth line, since they are all successive numbers, we could think that the missing number is 17
10	Promoter:	yes, in the fifth is $21 * 24$
16	Promoter:	in the seventh [line] the missing number is 13
19	Boss:	let's try doing the calculations
21	Pest:	in the fifth line we have to do this calculation $22*23-21*24$?
23	Boss:	the result is always 2
27	Promoter:	then having 4 successive numbers and doing the subtraction between the multiplication of the extremes and the extremes the result is 2
28	Promoter:	of the extremes and the middles
29	Boss:	blogger what do you think: have we reasoned correctly?
33	Blogger:	Could $(n+1*n+2)-(n+3*n+4)$ also be OK?
34	Blogger:	Wait, I made a mistake in the writing
35	Blogger:	$(n*n+1)-(n+2*n+3)$
37	Blogger:	I made a mistake again
42	Promoter:	guys there is a tool that allows us to try this for all rows and for numbers that we can enter
43	Boss:	for example in the first line n=1 so $(3*4)-(2*5)$
55	Pest:	the result is always 2
60	Pest:	$(n2*n3)-(n1*n4)=2$ is this OK?
63	Boss:	I don't think so because it looks like you multiply n by 2,3 or 4
68	Blogger:	$(n+1)*(n+2)-(n)*(n+3)$ can you see if that's OK too?!
94	Guru:	Can you explain it better?
96	Boss:	we thought that by taking four consecutive numbers and multiplying the extremes and the averages and subtracting the multiplication of the extremes from the multiplication of the averages, the result will always be 2

Looking at Scene 1, Boss stimulates the activation of cognitive functions, prompting the development of the process [BO-KL] (4, 29), while it takes care of directing the process, keeping the goal fixed and taking care of the moments of transition from process to product [BO-MK] (19). Boss finally makes a check on the epistemic component, being in charge of a form of interpersonal communication that fosters understanding and allows the process to move forwards [PE-KL; BLO-MK]

(23, 43), in a coordinated care of the entire rationality. Promoter and Blogger provide instead contributions of a constructive nature that allow the process to advance [PRO-IP] (7, 8) and Promoter establishes relationships between numbers using conjectural solving strategies based on analogies [PRO-KL] (10,169). Furthermore, it develops an initial conjecture, building on the numerical examples and abstracting, in a dialectical dimension between the epistemic and teleological components of rationality [PE-KL; PRO-IP] (27). This scene also reveals a process of verifying the effectiveness of both intrapersonal and interpersonal communication, which we interpret as indicative of a meta-knowledge level within the Blogger's role [BLO-MK] (33–37). Specifically, Pest (60) employs a formal notation involving indices to clarify its understanding of the positioning of elements to be multiplied—an approach that diverges from the conventional linear notation of consecutive numbers. This moment proves crucial for intrapersonal communication, facilitating a process of self-clarification and communicative regulation [BLO-MK]. The epistemic dimension is further reinforced by Boss (63), who undertakes the validation of mathematically sound constructions and representations [PE-KL]. In the transition from knowledge to meta-knowledge, Boss also identifies epistemic gaps, as revealed through a misinterpretation of an algebraic transformation inconsistent with the established proving scheme [PE-IP]. This leads to a shift in Blogger's mode of communication within the group, prompting the use of formal mathematical language [BLO-KL] (68). Additionally, Pest evaluates the truth conditions of statements and the validity of the reasoning involved [PE-KL] (55), while Boss contributes to the communicative effectiveness toward others (96), by verbally explaining the group's process to the teacher [BLO-MK].

Scene 2 - from Episode 2

5	Blogger:	then guys we have to find a way to say in words that the result is always 2
9	Promoter:	4 consecutive numbers so x,x+1,x+2,x+3
10	Boss:	guys, do you have any ideas?
13	Promoter:	the 2 times 3 minus the 1 times 4 thus
16	Promoter:	$(x+1)*(x+2)-(x)*(x+3)$
17	Blogger:	=2
19	Boss:	so taking 4 consecutive numbers and subtracting the product of the averages the product of the extremes is always 2
21	Blogger:	let's do a test with large numbers
27	Promoter:	ok $(50+1)*(50+2)-(50)*(50+3)$
28	Promoter:	$51*52-50*53$
31	Promoter:	2652-2650
33	Promoter:	it comes out 2

36	Promoter:	But we have to find a way to prove it's always true
43	Blogger:	$(x+1)*(x+2)-(x)*(x+3)$ by multiplying the middles and extremes and then subtracting them the result is always 2

Promoter makes a reflection on the concept of the consecutive that enables it to activate the process in formal writing [BLO-KL] (9): it gives a boost from a semiotic point of view. Successively it acts by caring for consistency between different ways of reasoning, moving from a semantic approach (13) to a syntactic writing of the statement [PRO-MK] (16) (Weber & Alcock, 2004). The check on the validity of the reasoning followed, involving the epistemic dimension, initially takes place with an empirical approach. Promoter, however, ultimately perceives the necessity of a proof that is independent of the particular examples, activating a teleological dimension of rationality at the meta level [PRO-MK] (36), anxiously anticipating potential requests from the Guru.

Scene 3 - from Episode 3

8	Promoter:	we have to write the formula we found with the symbols
9	Blogger:	what do you suggest?
10	Pest:	shall we start by opening the app?
14	Promoter:	I'd say start with "for every x that is different from 0"
16	Promoter:	x must be a natural number, right?
19	Boss:	promoter do you have any ideas?
21	Promoter:	you can't enter the "not equal" symbol
22	Blogger:	in the app how do you spell x different from 0?
119	Pest:	Do we also add x not equal to 0?
120	Promoter:	I know, but we need the "not equal" sign
122	Blogger:	do I write x not equal to 0, in the email?
127	Pest:	you could maybe tell him that the not equal sign was missing in the app so we wrote it in words

In Scene 3, the students revisit the same task, this time attempting to formulate a written conjecture using the vocabulary provided by a digital app. This digital tool allows students to write their conjecture in symbols in order to formalise it. In fact, it has symbols and letters that facilitate the formalization of the conjecture in mathematical language. Although it is not strictly necessary to include as a condition that x is non-zero, the students decide to analyse the property only within natural numbers (without zero) and therefore decide to exclude zero from the conjecture of the theorem (14, 16). Students, however, find that the app does not contain any symbols

indicating "not equal" (21, 22, 119, 120) and they are thus led to try to formulate such a concept in words [BLO-KL] (122, 127). Here, a communicative need emerges, whereby the constraints of the digital tool prompt students to engage in a semiotic transformation, transitioning from formal to verbal language, i.e. "a transition between the representations", in line with Urhan and Bülbül' s indicators (2023b), present in the Table 1 [BLO-KL]. Furthermore, Pest moves to the meta level of the communicative rationality, as he suggests motivating the choice of writing in words for the lack observed in the digital tool [PE-MK] (127), with "awareness of the accuracy of symbolic and verbal representations" in line with the lens of the characterisation presented in Table 1.

Scene 4 - from Episode 4

51	Promoter:	Gianmaria (i.e. Guru) told me: 'who can tell me that the first member is always equal to 2, whatever x I take?
52	Pest:	let's take an example
54	Promoter:	$3 * 4 - 2 * 5$
57	Promoter:	Gianmaria said we have already tried examples, now we have to show that it works without substituting any value for x
60	Pest:	if you do the computation without substituting a value for x you get 2
61	Promoter:	so if we try to carry out the equation with one unknown without substituting any number for x, we get 2
62	Pest:	Exactly
67	Boss:	so the formula is true because even if you try infinite examples, the result will always be 2 and also because if you want to carry out the equation without substituting x, the result is always 2
85	Guru:	what have you done materially to arrive at saying that the result of $(x + 1) * (x + 2) - x * (x + 3)$ is always 2?
87	Promoter:	we previously multiplied $(x + 1) * (x + 2)$
89	Promoter:	and it got $x^2 + 3x + 2$
92	Promoter:	then we multiplied $x * (x + 3)$
94	Promoter:	$-x^2 - 3x$
97	Promoter:	because the minus in front changes the signs inside the parenthesis
103	Guru:	can we say which property allows us to say that $x * (x + 3) = x^2 + 3x$?
108	Promoter:	x the monomial is multiplied by both members of the polynomial
114	Pest:	distributive property

Pest (60) verifies the coherence of conceptual and procedural usage schemes and [PE-KL] (114) assesses the conceptual properties employed during the proving process. Additionally, Pest validates statements based on accepted premises (62). In parallel, Promoter focuses on evaluating the logical structure of the selected proving strategies and their internal consistency within the reference theoretical framework, with particular attention to semantic aspects [PE-KL] (87–97; 108). Moreover, Boss and Promoter employ verbal representations to express logical relationships, notably through the use of implication markers such as 'so' [PRO-IP] (61, 67) and causal connectors like 'because' (67, 97).

7.3 Discussion

Following the evolution of the film Inside Out with Inside Out 2 [3], where the exploration of Riley's inner world is significantly expanded, similarly in this chapter we explored various cognitive functions that are active in the mind of a mathematician engaged in a problem-solving activity and that can be embodied by the students as characters, Pest, Boss, Promoter, Blogger, highlighting their gradual internalization by students through reflection on conjecturing, proving, and processes at meta-level. Our Inside-Out 2 does not introduce new cognitive functions; instead, it broadens the inquiry by reframing it through a lens of rationality (Habermas, 1998).

In our previous studies (Albano et al., 2020c, 2021a, 2024b), we identified certain roles that represent cognitive functions that come into play in mathematical problem-solving. These roles were used in storytelling activities, becoming characters that were played by the students in order to foster the construction of a mathematician's attitude towards problematic situations. The RQs we explored in this chapter fit squarely within this strand. RQ1 concerns the characterising the cognitive roles of Blogger, Pest, Promoter and Boss according to Habermas rationalities. In this respect, we performed an a priori analysis by means of indicators based on the works of Albano and colleagues (2021a), Balacheff (2010), Boero (2017), Mariotti (2006), Pedemonte (2007, 2008), Urhan and Bülbül (2023a, 2023b), Boero and Azrou (2023), and we obtained a characterisation of cognitive roles given by Table 7.1, which we named Inside-Out 2 Model. Indeed, similarly to the film, the new model expands the world of a mathematician's thinking, going into depth of the cognitive roles in terms of Habermas' three dimensions of rationality, as detailed in the following . With regard to Promoter, we refer to both the teleological and the epistemic component of rationality (Habermas, 1998), taking into account the dialectic relationship between them. Furthermore, we

[3] https://www.pixar.com/inside-out-2

consider the meta-knowledge level of the proving process (Boero & Azrou, 2023), accepting as valid also temporary proving strategies, such as analogies and/or abduction, which cannot be validated in the production of a final product (proof), but which provide the incentive for reasoning during the creative act and during argumentation process. In this delicate transition from proving to proof, also by means of abductive reasoning, a break of cognitive unity of theorems may occur (Boero, 2017). Thus, the Promoter takes care of checking for consistency in the cognitive unity construct, being aware, in relation to the aim, of the discontinuity that may emerge in this construct. The role of Boss, instead, is evocative of the idea of a manager aware of the functions of other roles, i.e. it embodies a meta-organisational function of awareness of the entire rationality. Boss is concerned with the functionality and direction of the proving process but also takes care of the moments of transition from process to product. From the perspective of rationality, Boss appears as the figure who embodies all rationalities, controlling them. The rationalities involved in the epistemic and communicative dimensions are associated with the roles of Pest and Blogger, based on the theoretical frameworks previously outlined. So far as the Blogger role, we propose a distinction at the meta-knowledge level between internal communication—understood as an internal dialogue—and external communication—aimed at mutual understanding—in line with the perspectives of Habermas (1998) and Boero and Azrou (2023). Moreover, drawing on Balacheff's (2010) three dimensions of proof we highlighted that the construction of a proof ought to be seen as a communicative act with oneself, within a small community, with the mathematical community, particularly within the Blogger role. According to the evolving character of the proof, this process involves an intermediate stage within a restricted community that employs an evolving mathematical language. Concerning the Pest role, it ensures the accuracy and internal consistency of the use of concepts, procedures, and representations at the knowledge level. Conversely, at the meta-knowledge level, it assumes a regulatory function by monitoring the logical coherence of the demonstrative schemes employed and their consistency with the reference theoretical framework (Boero & Azrou, 2023).

RQ2 concerns the application of the a priori characterisation to analyse the students' behaviour emerging from their synchronous discursive interactions occurring during a problem-solving activity structured as Inside-Out model (Albano et al., 2021a). In the exploration and conjecture formulation phases, the rationality components acted mainly on the knowledge level, while there is a movement to the meta-knowledge level in the formalisation and initial proving phases, with the exception of the Boss who worked almost always on the meta level in taking care of the rationality requisites of the whole process. Therefore, data analysis supports the a priori analysis, as Boss stimulates the activation of cognitive functions, taking care of directing the process, keeping

the goal fixed and minding the moments of transition from process to product and Promoter gives inputs of a constructive nature that allow the process to advance, using conjectural strategies based on analogies. Furthermore, he acts in the dialectic between the epistemic and teleological dimensions of rationality and caring for consistency among different lines of reasoning. Both the generative act and the advancement of the process in Habermas can be found in the dialectical development of the three components of rationality, moving from control of what one has done to forward action. Promoter, who is concerned with the forward thrust, is in turn distinct from the Boss who has to control the process moving forwards; the two, ultimately, are engaged in a dialectical relationship. This also should make it possible to explain certain moments of overlapping roles of Promoter and Boss. The data analysis confirms the a priori framework and led to two refinements regarding the Blogger role. Indeed, our analysis suggests a refined view of the Blogger role, distinguishing between self-reflective dialogic communication and external dialogic communication, in line with Habermas (1998) and Boero and Azrou (2023). Consistent with Balacheff's views (2010), we argue that proof construction in this role occurs within a restricted community, wherein shared meanings and mathematical language evolve through interaction. The data reveal evidence of epistemic control carried out by the Pest role. Pest ensures that the mathematical expressions are properly structured algebraic manipulations are valid, and statements are logically consistent, aligning with the framework proposed by Urhan and Bülbül (2023a, 2023b). Furthermore, Pest engages with the epistemology of mathematics by critically evaluating both the clarity of the statements made and the quality of communication in conveying logical reasoning (Boero & Azrou, 2023). As such, Pest's rationality is defined by both epistemic and communicative dimensions.

This type of analysis was made possible by the inclusion of the Guru role—serving as a cultural mediator—introduced during the 2020 pilot study, as argued by Albano and her colleagues (2020a). The new role of the Guru into the Inside-Out 2 Model will be the focus of further exploration in our research. The constraint of providing students with a limited set of mathematical symbols within a digital App (Scene 3) gave rise to a communicative need. As a result, students were prompted to perform a semiotic transformation, shifting from formal language to verbal language—an example of representational transition consistent with the indicators identified by Urhan and Bülbül (2023b) and illustrated in Table 7.1. Finally, we wish to clarify a fundamental distinction between the typical cognitive functions of the human brain and the cognitive roles represented by the characters in the Inside-Out model. Although each role is associated with characteristic functions, in practice, the functions commonly linked to one role may be enacted or experienced by other characters. This phenomenon

reflects the dynamic nature of cognitive processes and suggests promising avenues for future research, including potential implications for neuroscience, particularly in light of emerging theories on brain plasticity. The analysis of the findings presented in this study brings to light that the Inside-Out 2 Model also incorporates a preliminary systemic model (Fig. 7.2), which we named Cognitive Role Interconnection Model (CRIM), for describing the interconnections amongst roles within the perspective of the rationality guiding our inquiry.

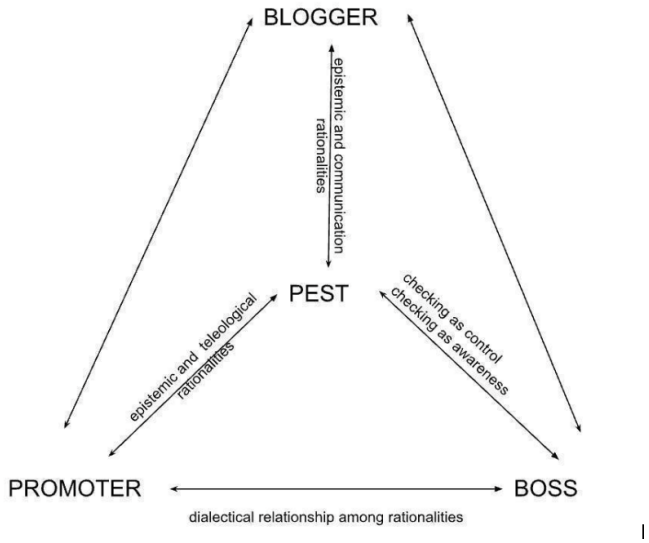


Figure 7.2: Cognitive Role Interconnection Model

This model could make the interconnections among the different roles evident to students, enabling them to recognise, within their own actions, the transitions from one role to another during the process and to develop an awareness that the boundaries among the different roles are neither fixed nor can they be, as they represent different yet cooperating facets of a single whole. Within the broader research, the Inside-Out 2 Model, with the incorporated CRIM model, will be the subject of further studies, presented in the next chapters, with the aim to refine the Inside-Out 2 Model presented in this chapter to explore further emerging aspects in greater depth.

Chapter 8

Euclid Experiment - Methods

The results of the pilot study encouraged us to design and implement a new experiment based on the methodological and didactic device of the Inside-Out model, in order to investigate and foster students' rational process of enculturation into the Culture of Theorems (Morselli & Boero, 2009).

In light of the reflection developed in Sec.1.2 on the differences between proofs in algebra and those in geometry—concerning the nature and characteristics of the mathematical objects involved, and consequently the nature of the reasoning underpinning proofs in the two domains—we choose to design the new experiment within the field of Euclidean geometry.

To foster the development of awareness of the importance of theoretical thinking in mathematics, Turiano and Boero (2019) identify in Euclidean geometry a genuine opportunity to support the process of students' enculturation into the Culture of Theorems, as this domain offers both students and teachers the opportunity to engage in the interplay between geometric constructions and proving processes within a well-organized and consistent "corpus" of knowledge (Mariotti, 2006). While construction activities allow students to handle geometric figures, including through paper and pencil explorations, to become familiar with the use of diagrams that are useful during the conjecturing phase (Boero & Turiano, 2022, 2023), geometric proving enables them to engage with the notion of *theoretically situated* truth (Mariotti, 2006), developing awareness also into meta-theoretical issues (Boero & Azrou, 2023).

Thus, we choose to design our teaching experiment within the domain of Euclidean geometry, as we believe that engaging with the cognitive and dialogical roles of the Inside-Out model in this domain can foster the process of *scientific enculturation* into the Culture of Theorems—understood as a rational process, in the perspective of Morselli and Boero (2009)—and contribute to the development of the attention toward *theoretical validation* "which tightly links proof, theorem and theory" (Balacheff, 2008, p.507).

This is because cognitive roles support interaction throughout the entire proving process, making explicit certain processes that might remain implicit in individual exploration and enabling more precise mediation by the teacher-researcher.

Our choice thus leverages the presence of a more explicit [1] and structured theory, organized as a coherent and deductive "corpus", in order to support students' scientific enculturation into the culture of theorems, in a context that presents greater structural difficulties for learners (see for example Pedemonte, 2007, 2008), through the use of cognitive role-playing.

The Euclid Experiment belongs to the field of qualitative scientific research studies, situating it within a given context, through also an increasingly in-depth and focused analysis, including progressive zooms on case studies as specific instances, both on students and on particular cognitive roles (Cohen et al., 2007). This study is supported by multiple collected data and by a conceptual background developed through a networking strategy (sec.6).

The design phase unfolded over several months [2], from September 2024 to December 2024, during which the mathematical problems to be proposed to the students were selected and subjected to a careful a priori analysis. Then, based on the new characterization of cognitive roles that emerged as a priori analysis and was tested in the pilot study (see sec.7), the characters and the comics storyline were developed, so that each character would reflect the cognitive role already characterized in terms of rationality.

Once the design phase was completed, an implementation phase of the experiment was initiated. During this stage, agreements were signed with the two host schools, and explicit approval was requested from and granted by the Ethics Committee of the Department of Information and Electrical Engineering and Applied Mathematics at the University of Salerno. For this reason, the study involves the anonymous collection of digital interaction data and the anonymized observation of discussions through observation grids by the researcher. The data were labeled using codes identifying fictitious digital accounts associated with cognitive roles rather than with individual students, thus avoiding the use of identifying or sensitive personal data. The implementation was completed with the activation on the Moodle platform of all the episodes of the digital storytelling.

[1] We are aware of the presence of many implicit elements also in Euclid's corpus; nevertheless, compared to the algebraic domain, we believe that for students the Euclidean structure appears more evident and explicit as a theoretical domain

[2] The design required long and careful reflection, including fruitful discussions with Prof. Paolo Boero, whom we thank for the rich and productive exchanges that allowed us to shed new light on the experiment under study.

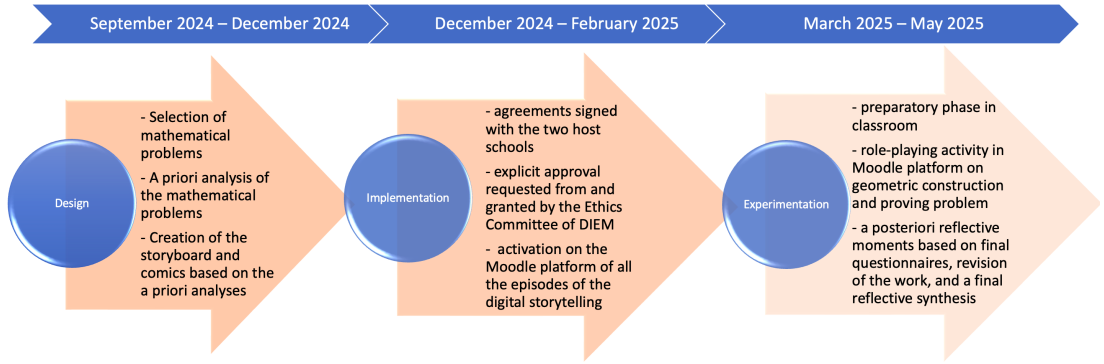


Figure 8.1: Chronology of the design and implementation of the study

From March 2025 to May 2025, the classroom experimentation was carried out. It began with a preparatory phase aimed at familiarizing students with processes of conjecturing and proving, followed by the actual role-playing activity conducted on a digital platform, with a posteriori reflective moments based on final questionnaires, revision of the work, and on-line final reflective synthesis. In this chapter, we present the design principles underlying the development and implementation of the Euclid Experiment, along with a detailed account of the methodological choices concerning the selection and formulation of tasks and the construction of the storytelling. The a priori analyses of the selected tasks will also be presented, in order to highlight issues of an epistemological nature, as well as the methodological choices related to didactical mediation.

Subsequently, we outline the methods for the analysis, including the methodological choices concerning both data collection and data analysis.

8.1 Design on Inside-Out towards Inside-Out 2

Drawing on Inside-Out model, we choose to implement a new experiment with 10th grade students of secondary school (see sec. 8.1.1) in which learners, by performing characters in a comics story-problem implemented on the Moodle platform, interact and collaborate in digital chats at the dual level of Actors/Onlookers (see sec. 3.1.1) to formulate conjectures, search for theoretical justification for the statement produced, and collectively formalise the proof in field of Euclidean Geometry.

The characters in the story therefore correspond to the cognitive roles of the Inside-Out model, but now are characterized from a perspective of rationality (Albano et al., in press). Indeed, the a priori characterization developed in the pilot study (see

sec.7.1.4 and sec.7.2) serves also as a basis for designing the characters of the Inside-Out model in the Euclid Experiment comics. In sec. 8.1.3 we describe the methodological choices that guided us in the construction and implementation of the comic strips in a digital environment, using table 7.1 as a design tool.

According to the principles of Inside-Out model, the narrative evolves in the form of a succession of episodes (developed as comics) in which the students act as characters in the story moved by an aim and whose actions are guided by the specific cognitive role assigned. The students will be divided into groups, of which one group we named Actor Group and the other we named Outlookers Groups. These groups discuss and argue in two different chats in the Moodle Platform. The Actors and Outlookers groups varies from episode to episode turning roles and partecipation levels.

Compared to the previous model, we choose to refer to the observing students as “Outlookers” rather than “Onlookers", although this form is less common and not fully correct from a linguistic and grammatical point of view. This choice is based on the root of the term “outlook” which means “vision”. The Outlookers, as in the first model, are students who observe in an active and reflective way; however, in this new version we aim to emphasize more strongly the idea that observation is carried out through a perspective-based form of observation. In fact, while the Actor Group is directly involved with solving the proposed task, the Outlookers, in addition to observing the cognitive roles corresponding to their own assigned role (the engagement with the problem, the actions, and the contributions) solve themselves the problem, making notes in their dedicated chat even if they agree with the solution choices and the content produced by the Actor Group ("I agree, but ..."; "I would have solved it this way ..."), and upload their own diagrams to the shared board, in order to further enhance participation and engagement of the Outlookers and thus support an observation that is even more focused on the ongoing proving process.

The four characters of Inside-Out Model in the story are joined by a fifth character, Euclid-Guru, played by the author of this thesis in the role of teacher-researcher as a cultural mediator, whose inputs and contributions were guided by the Teacher Rational Questioning Framework (Zhuang & Conner, 2022), in order to foster the emergence of rationality components in the students involved in the experiment. In sec.8.1.4 we describe in detail the methodological choices adopted for didactic mediation.

While in the pilot study the task was of algebraic nature, in the Euclid Experiment the tasks are of geometric nature. This entails a shift in perspective, as proof in algebraic and geometric contexts requires different forms of attention, as previously discussed in detail in sec. 1.2. In sec.8.1.2 we present the selected tasks and the epistemological and didactic reasons for our choices together with the a priori analysis of the mathematical

tasks proposed.

The coherent and deductive nature—although incomplete—of the "Euclidean corpus" indeed us an important historical-cultural framework in which the deductive organization of the entire knowledge system does not hinder the understanding of the meaning of the concepts involved (Mariotti, 2006). In sec.8.1.5 we describe the methodological choices that lead us in the development of a new dynamic tool that fosters a cultural approach to Euclidean theory and supports students' conjecturing and proving process.

Moreover, the choice of a geometric context leads us to reflect further on meta-mathematical narration. Indeed, we recall that in the pilot study the design was structured according to the principles of narrative coherence proposed by Zan (2011, 2012), with a problematic situation emerging naturally from the story, from a real, physical, and contingent need of the protagonist. This prompted us to consider the possibility of constructing a story that also arises directly from the context for tasks in the geometric domain (e.g., by imagining concrete objects or real-life situations that might evoke geometric figures, such as cart wheels to introduce circles). However, such an approach seemed to conflict with the idea of a geometric figure as conceptualized by Fischbein (1993). For Fischbein, in fact, a geometric figure arises from the "integration of conceptual and figural properties in a unitary mental structure, with the predominance of conceptual constraints over figural ones" (Fischbein, 1993, p.156). When handling geometric figures, students may naturally encounter conflicts between the dimensionless and ideal nature of the concept and the spatial properties of the figure. However, this should be addressed by designing activities that emphasize the predominance of conceptual properties over figural ones (p. 155). A cart wheel, for example, has non-negligible dimensions that would require a prior step of abstraction before engaging in geometric processes of conjecturing and proving. For this reason, in the Euclid Experiment we chose not to have tasks emerge directly from a physical, concrete, and contingent need of the protagonist. Instead, we decided that they would be proposed to the group by the Guru within a narrative framework that remains coherent and serves a clear narrative purpose, while ensuring that the geometric object under consideration remains faithful to its nature as a *figural concept*, according to Fischbein's perspective (1993). Although this choice may appear to diverge from the C&Q construct of Zan (2012) (see sec. 3.1), we choose not to force the emergence of the geometric problem from the narrative situation; instead, we construct a coherent narrative context that frames the problem situation, while maintaining student engagement, in line with definition of storytelling of Zazkis and Liljedahl (2019) that we choose to adopt (see sec.3.1.2).

In sec. 8.1.6 we present some examples that show the relationship between comics

and open mathematical problems proposed to students, developed in order to design the construction of the comics story-problem and the didactic mediation in a way that is coherent with the selected open problems.

Finally in sec.8.1.7, we describe our new model, named Inside-Out 2, highlighting the differences from the first model in relation to the choices that we made.

8.1.1 The context of Euclid Experiment

In this section, we present the environmental and school context in which the Euclid Experiment took place, along with a description of the classes involved and the experimental setting that was designed

The environment of the two secondary schools[3].

The two secondary schools that hosted the teaching experiment, Liceo Scientifico "Carlo Pisacane" in Padula and Polo Liceale "Pomponio Leto" in Teggiano, are both located in the Campania region in the south of Italy, specifically in the Province of Salerno. The territory of the Province of Salerno is highly extensive and diverse, ranging from densely populated areas to the north of the city of Salerno to areas in the southern part of the province where the population is distributed across small towns, often located about 10 km apart from one another.

The schools in question are both located within the territory of the Vallo di Diano, in the south-eastern area of the Province of Salerno. The Vallo di Diano is a plateau covering approximately 130 km^2 at an elevation of 450 m above sea level, enclosed by Apennine mountain ranges and crossed by the Tanagro River. During the Pleistocene, the valley was occupied by a lake, which later dried up due to the karst nature of the area and was subsequently reclaimed. Over time, this led to the development of urban settlements situated on the slopes of the mountains surrounding the valley or on small elevations, resulting in a distinctive territorial layout. Schools are typically located in the upper parts of these small municipalities, at around 660 m above sea level, i.e., in areas classified as mountainous. In particular, the two municipalities hosting the schools involved in the study, Padula and Teggiano, are situated at 699 m and 635 m above sea level, respectively.

The economy of the Vallo di Diano is predominantly based on the agricultural sector (supported by a long history of land reclamation). However, in the larger urban centers there are also more diversified economic activities, mainly linked to small and

[3]Informations contained in this paragraph are drawn from the Treccani Digital Encyclopedia https://www.treccani.it/enciclopedia/vallo-di-diano/ and from the *Piano dell'Offerta Formativa*(PTOF), Educational Offering Plan, of the two secondary schools involved

medium-sized enterprises (including crafts, mechanical engineering, food production, textiles, chemicals, and electronics) as well as to the social and healthcare sector. In addition, there is a growing tertiary/tourism sector, with a strong focus on commerce and services, particularly aimed at enhancing the natural, historical, and cultural heritage of the entire area.

The catchment area of the two schools involved is highly heterogeneous, encompassing the various municipalities of the Vallo di Diano, as well as some areas of the Cilento and nearby towns in the Basilicata region. Consequently, it reflects a wide range of family backgrounds and needs. The transport network generally meets the users' basic needs; however, it does not provide continuous connections among the inland areas or with the nearest major centers of interest. Furthermore, the distance from major infrastructures and the lack of numerous connections (both rail and air) sometimes constitute an objective limitation. Students reach the schools by buses operated by local companies, which run mainly during school hours and terms, while the absence of a widespread and comprehensive transport network makes public connections with the provincial capital difficult.

Both schools are currently engaged in a significant process of didactic and methodological innovation, aiming to provide meaningful opportunities for students' cultural development. The "Carlo Pisacane" Scientific High School in Padula is equipped with classrooms featuring modular desks designed for cooperative learning, a library, a science laboratory, a physics laboratory, a language laboratory, and two computer labs. In addition, the school has a set of mobile digital devices (tablets and laptops) that can be used directly in the classrooms. This allows the laboratory to be effectively brought into the classroom in a highly flexible way, enabling the integration of traditional teaching with laboratory-based activities.

The "Pomponio Leto" High School in Teggiano has joined the Indire Avanguardie Educative network, implementing a teaching approach that allows for the redesign and reconfiguration of learning spaces. This model enables students to move between different classrooms from one lesson to another, thereby making full use of the available spaces. This school is equipped, in addition to its various computer, language, and science laboratories, with a digital laboratory featuring modular desks and a supply of laptops and tablets available directly within the lab.

Both schools adopt shared departmental assessment grids and rubrics for evaluating and certifying student learning outcomes, as well as a common vertically articulated departmental curriculum planning framework.

Participants

The experiment involved 37 students of two 10th grade classes of two secondary schools in the South of Italy. In particular, 18 students of the "C. Pisacane" Scientific High School in Padula (Salerno, Italy) and 19 students of the "P. Leto" Scientific High School – Applied Sciences Track in Teggiano (Salerno, Italy). The activity was led by the author of this study, as a teacher-researcher, whilst the classroom-teacher supported taking the role of onlooker.

The students were already familiar with the fundamental objects of Euclidean geometry, the triangle congruence criteria, and the definition of the perpendicular bisector. Moreover, although presented and guided by the classroom teacher, the students had encountered some mathematical proofs in the context of Euclidean geometry, and had also attempted to use triangle congruence criteria to prove some statements proposed in the textbook as proof-based exercises. However, students did not usually carry out systematic exploratory activities relating to geometric proofs. The approach to Euclidean geometry was often limited to the study of geometric properties, definitions, postulates, and theorems, focusing mainly on the knowledge of statements or, at most, on the proof of a few propositions considered particularly significant by the classroom teachers. An exploratory approach to proof as an open problem—entailing the formulation and validation of conjectures—is not a practice with which students were familiar.

With regard to tools, the students are familiar with the compass, but they use it almost exclusively for technical drawing rather than for mathematical purposes.

Setting

The teaching experiment carried out between March and May 2025 in two 10th grade classes, for a total of 20 hours for each class, during which students engaged in problem-solving activities in Euclidean geometry, with a specific focus on Books I and III of Euclid's Elements, played as characters of Inside-Out model on Moodle platform.

At the "Carlo Pisacane" Scientific High School, we had access to tablets connected to the school's wireless network directly in the students' classroom. This provided a significant advantage in terms of space use, as it was not subject to availability constraints or scheduling with other teachers. It also allowed students to experience the activity as an integral part of their regular class program, without the potential distractions associated with moving to a laboratory classroom. At the "Pomponio Leto" Scientific High School, on the other hand, activities on the Moodle platform could only be carried out within the two dedicated computer/multimedia labs. This required students to move frequently and necessitated booking the spaces in advance,

coordinating sessions with the availability of the labs in relation to the schedules of other classes.

Before starting the actual experimentation, we presented the project to the students, agreeing—together with the classroom teachers—on a proposed working and assessment method, outlining the objectives of the pathway, the phases of the work, and the commitment by the classroom teacher not to assess the ongoing activities through traditional evaluation methods, but rather to adopt formative assessment approaches as specified within the didactic agreement itself [4].

8.1.2 Selection of the tasks and their a priori analysis

Euclid's *Elements* provide us with an example of a "well-organized body of knowledge" that crosses generations, with which learners could come into contact through participation in practice (Hatano & Wertsch, 2001, p.79). Such a system of knowledge includes both the theoretical domain (axioms, definitions, theorems) and the meta-theoretical one (such as the logical-formal structures of deduction and inference) (Mariotti, 2006), as well as all those empirical evidences that are not necessarily defined at a formal level or are sometimes left implicit, a feature present in Euclid'*Elements*. Indeed, "the hypothetical-deductive structure of the *Elements* cannot be fully assimilated to an axiomatic system in the modern sense. For instance, purely syntactic interpretations are certainly not admissible. Greek geometry deals with objects whose basic properties are strongly tied to an unavoidable intuitive component" (Acerbi, 2007, p.227). We can refer to such a system of knowledge as a "reference corpus" (Douek, 2007, p.171) with a logical hypothetico-deductive structure that differs significantly from the deductive structure of formal logical derivation (Rav, 1999). Rather, it is closely related to the understanding of the knowledge under study and its systematization within a body of knowledge accepted by the mathematical community and capable of being communicated externally as knowledge historical and cultural transmitted (Mariotti, 2006).

The need to reconcile the difficulties that students face in a given domain with the need to connect new knowledge to the prior knowledge that students already possess framed the selection of tasks (Schoenfeld, 1985, 2016). In Schoenfeld's point of view (1985) "it is a particular relationship between the individual and the task that makes the task a problem for that person" (p.74). The accessibility of the task should therefore not be understood in terms of simplifying the problem situation, but rather the latter should be conceptualized as "an intellectual impasse" (p.74). Therefore, didactical situations

[4] This proposal was inspired by a assessment method already adopted by Prof. P. Boero and Prof. F. Morselli at the University of Genoa

must be sufficiently rich to allow exploration, while remaining connected to internal resources of students, understood as what Schoenfeld (2016) defines as the *Knowledge Base*.

Our choice of tasks focused on construction problems and their related theoretical justification, as well as on the production and proving of conjectures concerning the Euclidean circle. In particular, the tasks address the exploration of circles passing through one point, two points, and three non-collinear points, including the determination of their centers, as well as the investigation of the relationship between central angles and inscribed angles.

The implementation of each selected open problem—whether related to the preparatory classroom activities or embedded in the platform in the form of comics—was consistently preceded by an a priori analysis at two levels:

- *a priori analysis of the mathematical problem*, adopting the student's perspective in order to identify possible difficulties and conceptual hurdles inherent in the problem, with the aim of considering a possible expansion or contraction of the instructional design;
- *a priori analysis of the possible solution strategies and related mediation* students may adopt when faced with the proposed problem situation, in order to more precisely calibrate the intervention of the teacher-researcher in the classroom.

Below we present a summary of the proposed tasks, together with the corresponding a priori analysis we conducted. All tasks are presented to students as open exploratory tasks, but are summarized here in their final version. All the propositions, definitions, common notions, and postulates cited are drawn from the version of the *Elements* of Acerbi (2007).

Preparatory tasks

To enable students to work confidently with the selected problems, we designed a set of preparatory activities aimed at familiarizing them with conjecturing and proving processes. From the preparatory tasks onward, students are informed of the existence of a Bank of Theory provided by Euclid himself, which will be discussed in more detail in Sec. 8.1.5. The tasks associated with these preparatory activities involve the exploration and proof of properties of the perpendicular bisector of a segment.

Task 1P: *All points on the perpendicular bisector of a segment AB are equidistant from the endpoints A and B of the segment, namely: IF C belongs to the perpendicular bisector of segment AB THEN AC is congruent to CB.*

After sharing with the class the definition of the perpendicular bisector of a segment and asking them to explore its graphical representation in search of relevant properties, students are then asked: "What happens if a point P lies on the perpendicular bisector of AB?" It is expected that students, after conjecturing that point P is equidistant from the endpoints A and B, denoting by M the midpoint of segment AB, will identify the congruence of segments MA and MB and the perpendicularity of PM to segment AB, based on the definition of perpendicular bisector. Furthermore, students are expected to recognize the possibility of applying a congruence criterion for right triangles (or triangles more generally) in order to validate the conjecture they have produced, namely that P is equidistant from A and B.

Task 2P: *Every point in the plane that is equidistant from points A and B lies on the perpendicular bisector of segment AB; namely: IF AC = CB, THEN C lies on the perpendicular bisector of AB.* Initially, Euclid's Proposition 5 from the *Elements*, concerning the congruence of the base angles in an isosceles triangle, is not made available to students. This choice is motivated both by the subtle construction underlying the proposition and by the intention of directing students' attention toward an indirect line of reasoning, encouraging them to explore different possible cases and engage with the formulation of a secondary statement S* and its direct proof (Antonini, 2008). The task is therefore introduced through an open question: 'Are we really sure that all points in the plane equidistant from A and B lie on the perpendicular bisector of AB? Could there exist a point C that is equidistant from A and B but does not lie on the perpendicular bisector?' Students are expected to explore all possible positions of point C under the condition that it does not lie on the perpendicular bisector of AB, as illustrated in Figure 8.2.

- If C were to lie to the left of the line perpendicular to segment AB passing through A (see C'in Fig. 8.2), then angle $\angle$ CAB would certainly be obtuse, as it is greater than a right angle (Def. 11, Book I of the *Elements*). Therefore, CA $<$ CB, since in any triangle the side opposite the greater angle is longer (Proposition 19, Book I of *Elements*).

- If C were to lie on the strip of the plane between the lines perpendicular to AB segment passing through A and B, draw the line CQ parallel to AB, meeting the perpendicular bisector at point Q. Then $\triangle$ AHC is right, angled at H, so $\angle$ACH is less than a right angle, whereas $\angle$ACQ is obtuse, since it is supplementary to an acute angle (Prop. 13 - Book I). Therefore, students should recognize in this case that the segment AC is less than the segment AQ, since in the triangle $\triangle$ACQ

the side AQ is opposite the larger angle (Prop. 19 - Book I). Considering then the triangle △CQB, ∠CQB is obtuse since it is greater than a right angle (Def. 11 - Book I), and thus again QB < CB. At this point, students should be led to state that, since AQ ≅ QB by definition of perpendicular bisector, it follows that AC < AQ ≅ QB < CB.

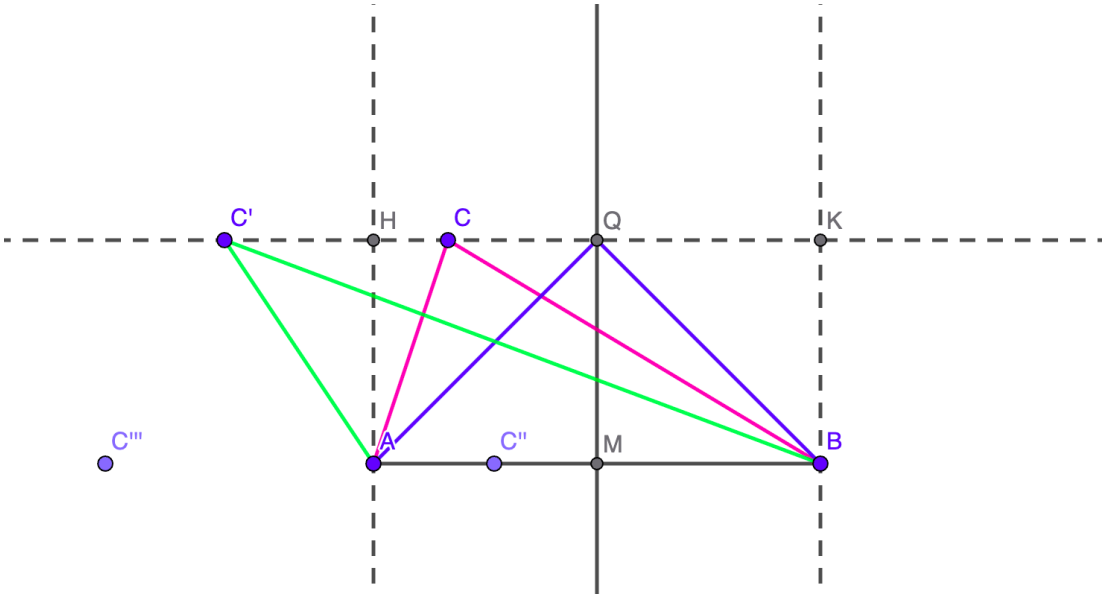


Figure 8.2: A possible strategy for the second preparatory task

Students may also be led to explore what happens if point C lies on segment AB or on the line containing segment AB. The latter case would be recognized rather straightforwardly by students (see C"'). On the other hand, having point C on segment AB, assumed to be equidistant from A and B but different from the midpoint of AB, would raise the subtle issue of the uniqueness of the midpoint. In Proposition 10 of Book I (To bisect a given finite straight line), Euclid constructs the midpoint of a segment but makes no explicit reference to its uniqueness, as if this aspect were left implicit. In this case, uniqueness can be proved by means of a *reductio ad absurdum*, leading to a contradiction with the hypothesis of equidistance, using Common Notion 5 (The whole is greater than the part), that is, by making explicit that AC<AM, AM=MB, MB<CB. Following this path, students would thus be led to construct a new statement S*, namely *IF C does not lie on the perpendicular bisector of AB, THEN AC and CB are not equal*, which is the statement of the contrapositive of the original claim. The proof presented as an a priori analysis is in fact the direct proof of the

contrapositive. The delicate issue of returning to the original statement S therefore remains open, through the passage to the meta-theorem (Antonini & Mariotti, 2008).

Task 3P: Initially an open task will be proposed: *Let us consider two points C and C' both equidistant from endpoints of the segment AB. Determine the line joining them.* Subsequently, students will be asked to produce a formal proof of the conjecture, ossia *IF two points C and C' are both equidistant from endpoints of the segment AB, THEN the line passing through those two points is the perpendicular bisector of the segment AB.* We expect that the conjecture can be formulated without major difficulty, as students have already worked on previous preparatory tasks involving the perpendicular bisector. Regarding the search for a theoretical justification and the subsequent formal deductive proof, this requires the use of Proposition 8 and 4 from Book I of Euclid's *Elements*, as well as the use of Common Notion 3.

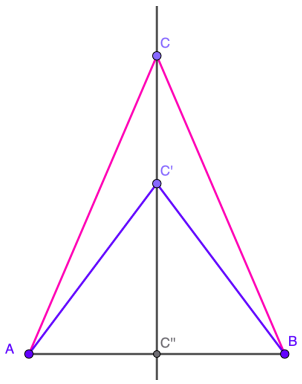


Figure 8.3: Figure of the third preparatory task

In particular, referring to Figure 8.3, the triangles △AC'C ≅ △BC'C by Proposition 8 of Book I of the *Elements*, since they have sides respectively AC ≅ CB and AC' ≅ BC' by hypothesis, and also share the side CC'. This implies that the angles ∠AC'C" ≅ ∠BC'C", since they are supplementary to congruent angles. Moreover, △AC'C" ≅ △BC'C" by Proposition 4 of Book I, since AC' ≅ BC' and C'C" is a common side. This leads to the validation of the conjecture.

During the argumentation phase, students might also proceed by reasoning backwards; abductive processes may emerge, and they should therefore be appropriately supported in such reasoning.

Tasks embedded in digital storytelling

The tasks within the storytelling are presented in an open and evolving format. Indeed, tasks follow the narrative and are interconnected with the comic strips themselves. To present their a priori analysis, below we now choose to describe tasks sometimes in a more concise form. In Sec. 8.1.6 we then present an example of how the task has been embedded within the storytelling.

Task 1.1:*How many circles pass through a point P?* The task of Episode 1.1 is an exploratory task, in which students are asked to formulate a conjecture, supporting it with valid arguments, about how many circles pass through a point in the plane. Students should be encouraged to explore graphically, using a compass on paper. It is expected that students will rather easily realize that there are infinitely many such circles. The theoretical validation of the conjecture essentially relies on the infinitude of the plane and on Postulate 3 of the *Elements*: For any point and radius, a circle can be drawn.

Task 1.2: *What is the difference between drawing a circle using a glass and drawing it with a compass?* The task of Episode 1.2 is designed with the aim of drawing students' attention to the centers of the different circles constructed, and thus to the importance of using a compass. Although the role of artefacts is not specifically addressed in our study, in the comic strips, students are in fact invited to explore the different semiotic potential of a compass and a glass(Mariotti, 2018), in order to foster awareness of the different mathematical knowledge that emerges through the use of different instruments. This task also aims to investigate the familiarity of students with the use of the compass in mathematics as a tool that enables attention to be drawn to the centre of the circle.

Task 1.3: *How many circles pass through two given points A and B? Where are their centers?* This is a conjecturing task that represents an epistemic obstacle for students. This leads us to envisage, within the mediation of the comic strips, several possible approaches. One of these is a more empirical approach, based on the use of the compass and on the search for circles passing through both points, thereby discovering the alignment of the centers. It is expected that students carry out trials with a compass, trying to position it so as to grasp the two given points, and thus draw a first circle satisfying the condition. The mediation in the comic strips also suggests looking for the center of the constructed circle and then trying to intuit where the centers of the other circles may lie. Another approach is instead more theoretical, starting from a

privileged circle having AB as a diameter and therefore its center at the midpoint of AB, and then attempting to move the center in order to obtain other circles still passing through points A and B. By the end of the episode, students should conjecture the existence of infinitely many circles passing through two points, with their centers lying on the perpendicular bisector of segment AB. Some students may also be guided from a theoretical perspective by the properties of the perpendicular bisector previously proven.

Task 1.4: *Is it possible to find circles passing through A and B whose center does not lie on the perpendicular bisector of AB?*. Once the conjecture of Episode 1.3 has been developed, students will be prompted with a task aimed at leading them to prove it. This task was initially conceived in the following formulation: *IF C is the center of a circle passing through A and B, THEN C lies on the perpendicular bisector of AB.* However, such a formulation would have presupposed the proof of Proposition 1 of Book III of the *Elements*. Indeed, stating that the center of a circle passing through A and B lies on the perpendicular bisector would require the prior proof of the existence of such a center. For this reason, it was decided to reformulate the statement in its negated form ("If C does not belong to the perpendicular bisector of AB, then C is not the center of any circle passing through A and B"). Students will therefore be encouraged to explore and attempt to construct a circle passing through the two given points A and B, while placing the compass at a point outside the perpendicular bisector of segment AB. This moment represents for students an initial encounter with an impossible figure. It is expected that students will find it difficult to construct such a circle and will therefore conjecture its non-existence. Some students may also recall the characteristic property of the perpendicular bisector, in particular the implication explored and collectively proven in task 2P. It is therefore expected that students will support the conjecture with a *constructive argumentation*, using as theoretical support the result proven in task 2P and referring to the definition of a circle (Def. 15 and Def. 16 - Book I of *Elements*)

Task 1.5:*All points on the perpendicular bisector of AB are centers of circles passing through A and B.* In Episode 1.5 students are asked to prove the converse of the previous proposition, namely that every point belonging to the perpendicular bisector of AB is the center of a circle passing through A and B, i.e. that if C is a point on the perpendicular bisector of AB, then C is the center of a circle passing through A and B. The proof is based on the proposition demonstrated in task 1p and on the definition of a circle (Def. 15 and Def. 16 - Book I of the *Elements*)

Task 2: *How many circles pass through three non-collinear points? What property does the point O that has been found have?* In this new task, students are led to construct the circle passing through three points, providing a theoretical justification for the construction performed. Initially, an open question is posed regarding what happens when considering not two, but three not aligned points A, B and C. The idea is to guide students to draw a single chord (for example, the one with endpoints A and B) and adjust the compass so as to find the unique circle that also passes through C. The intention is to have them work first with the compass, starting from the center of a circle passing through A and B and trying, by physically adjusting the opening of the compass, to "capture" point C as well. This attempt to grasp point C necessarily requires modifying the chosen center point and therefore takes place through a bodily-physical adjustment (where "physical" refers only to the use of the hands) involving the geometric figure. Such adjustments made with the hands and the compass may suggest the idea of a conceptual selection, i.e., they may support the intuition of also constructing the perpendicular bisector of chord BC. Starting from the idea of infinitely many circles passing through A and B and trying to find one that also passes through C by "adjusting the aim" and attempting to capture point C, students should construct the perpendicular bisector of segment AB and move along it in search of a point that can serve as the center of the circle passing through A, B, and C.

In particular, students should recognize the characteristic property of the perpendicular bisector of a segment, namely that:

- The center O lies on the perpendicular bisector of AB; therefore, by the previously proven Property 1 of the perpendicular bisector, we have that OA = OB
- Moreover, the center O of the circle also lies on the perpendicular bisector of BC; therefore, we also have that OB = OC
- Finally, by Common Notion 1 ('Things equal to the same thing are also equal to one another'), we have that OA = OB = OC, and therefore OA = OC

At this point, another rather delicate issue arises. Students might in fact think that the definition of a circle is sufficient to complete the proof, namely that Definitions 15 and 16 of the Elements are enough. In reality, this is not sufficient to conclude the proof, since these definitions do not guarantee the existence and, above all, the uniqueness of such a circle. What comes into play is in fact Postulate 3: *And that with any center and radius a circle can be drawn.* It is this postulate that allows us to construct the unique circle passing through A, B, and C, with its center located at the intersection of the perpendicular bisectors.

Task 3: *Relationship between the central angle and inscribed angles subtending the same arc.* In this Episode, a problematic situation is presented that requires students to formulate a conjecture regarding the relationship between central angles and inscribed angles and to construct its corresponding proof. We decided to focus the students' attention on a particular case, namely one in which one side of the angle coincides with the diameter. Students may carry out auxiliary constructions, or they may measure the angles in question using a protractor, or attempt to work on the given figure (see Fig. 8.4) by extracting relevant information from it. Having previously noticed that AB is a diameter and therefore passes through the center O, it is likely that students will easily recognize that OA, OB, and OC are all radii of the circle and thus congruent to one another. This may lead students to reason about the isosceles triangle $\triangle$AOC.

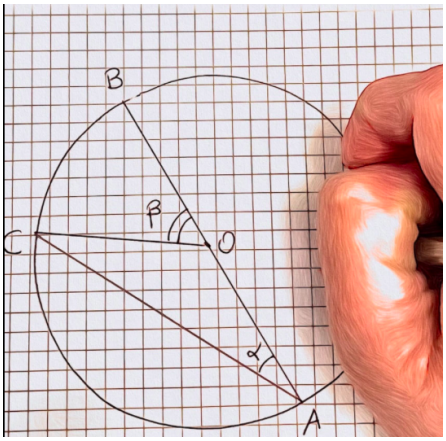


Figure 8.4: Figure of the third storytelling task

In addition to Proposition 5 of Book I, the proof involve the definition of isosceles triangle (Definition 20 of Book I); the theorem on the sum of the interior angles of a triangle (on which a subsequent discussion could be carried out, reflecting on a possible proof involving parallelism but not the exterior angle theorem); the Proposition 13 of Book I (If a straight line standing on a straight line makes angles, it will either make two right angles or angles equal to two right angles), which is necessary to establish that angle β is supplementary to angle AOC and finally,Common Notion 3 (And if equals are subtracted from equals, the remainders are equal), which is necessary to establish that $\beta = 2\alpha$, since $\beta + \gamma = 180°$ and $\gamma + 2\alpha = 180°$

8.1.3 Comics design and implementation into Moodle platform

The selected geometric tasks were presented in narrative form through a sequence of comic episodes entitled *Euclid and the escape to Beta! Geometry will save us!*, set in a dystopian world in which the inhabitants of the planet Beta decide to help the surviving humans out of gratitude for the classical Greek culture they had learned in previous centuries from the Earthlings themselves. This help is offered under the condition that humans demonstrate their ability to actively collaborate by solving ancient geometry problems and communicating responsibly with each other, as shown in Figure 8.5.

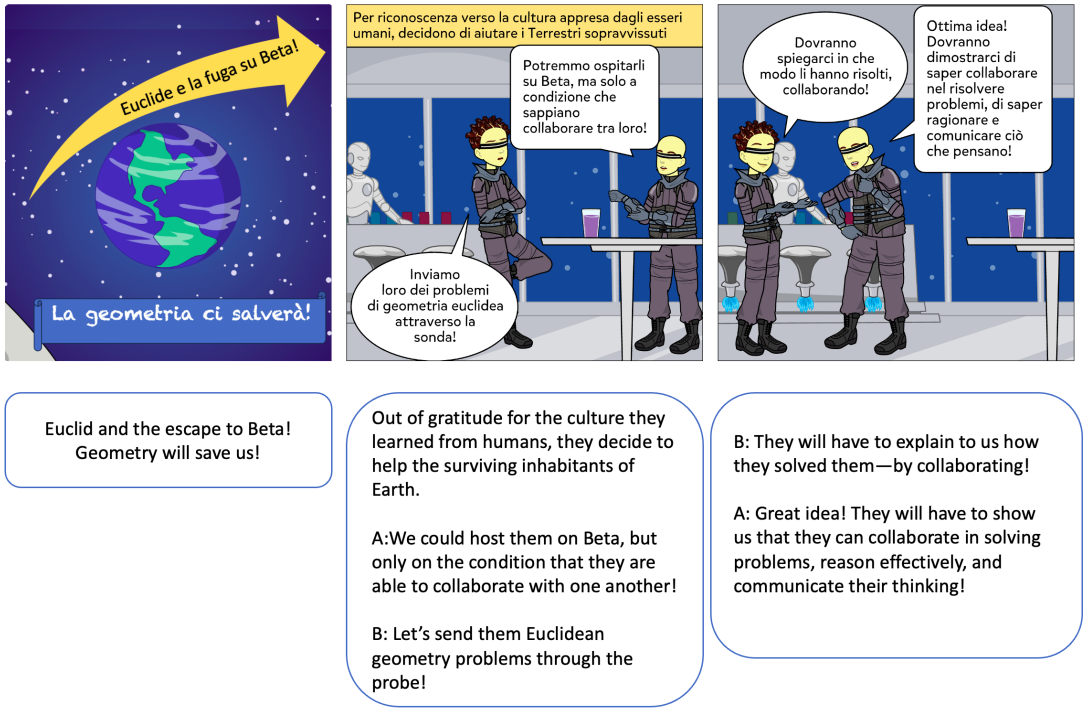


Figure 8.5: Some comics taken from the introductory episode of the comic-story

One of the main challenges was the enactment of roles within the comic strips, in light of the a priori characterization presented in Table 7.1 and the a priori analysis carried out on the selected tasks. For storytelling to be effective in supporting the emergence of rational behavior in students, it is necessary to strike a balance between a priori design and free interpretation of the assigned roles by students. It is important, in fact, that not everything is anticipated in the comic strips, so as to allow students to interpret the roles in a free and autonomous way; at the same time, however, the

comics must be structured so that the characters act according to the cognitive roles characterized a priori within a perspective of rationality. How far should the comics go? How much should be made explicit in them? How much should instead be left to emerge in the chats? This phase required careful and prolonged reflection.

The first choice we were called to make concern the characters. In this story, and with reference to the aforementioned cognitive roles, the selected names are as follows: Kyle is the Boss, Tommy is the Blogger, Ellie is the Promoter, and Shona is the Pest. In particular, for the Boss role, we chose a non-binary name, Kyle, and this aspect was reflected both in the character's clothing and in their expressions. The rationale behind this choice lies in the intention to avoid identifying the Boss role with a male character, as had occurred in previous implementations, and instead to allow the figure of a leader to be experienced comfortably by every student. Moreover, compared to the previous model, a female character was chosen for the Promoter role, in order to avoid reinforcing a traditional gender stereotype that associates the generation of solutions in a problem solving situation with male figures.

The narrative also includes a fifth character, Euclid (Guru), played by the teacher-researcher, who represents access to external resources and plays a significant role as a cultural mediator throughout the entire experience. For narrative coherence with the dystopian setting presented in the comic story, Euclid cannot, of course, appear as a human figure; instead, Euclid appears as a hologram created by the alien inhabitants of Planet Beta and sent to Earth via the @Euclid probe. The protagonists of the story therefore interact with an electronic probe, which presents itself to them in the form of a hologram (Fig. 8.6). The use of the hologram device also serves another purpose: by presenting itself to the group as a hologram of the great mathematician, rather than as a flesh-and-blood personification, Euclid can take on the role of a cultural mediator—as a voice of culture—without inhibiting free expression of students, thus helping to break possible distorted views of the expert associated with traditional didactic contract (Brousseau, 2002).

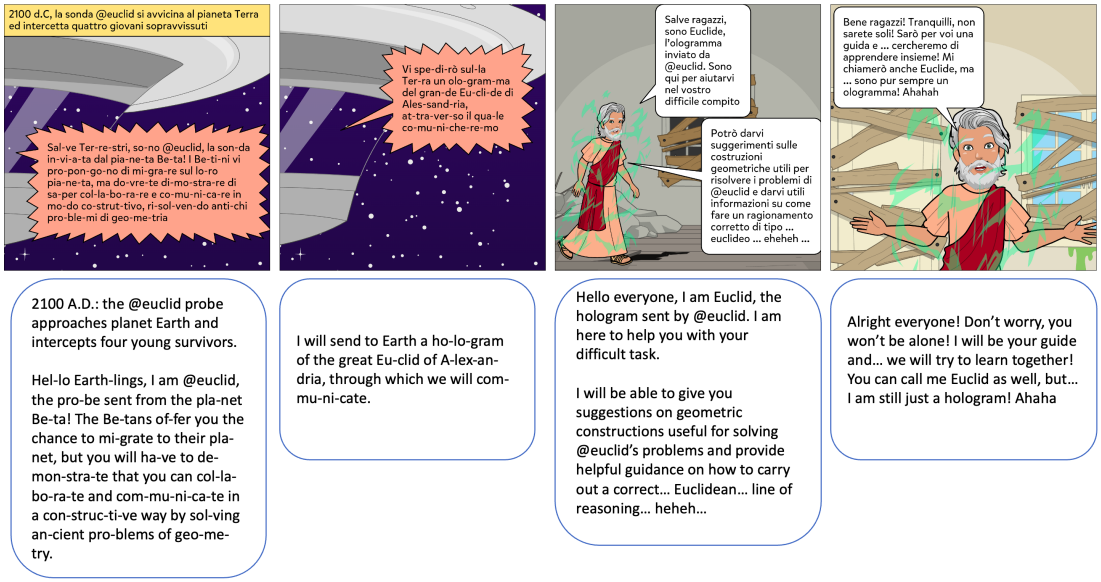


Figure 8.6: The hologram of Euclid sent on the Earth by @euclid probe

Descriptions of the four characters as presented in the comic strips are reported in the following figure 8.7.



Figure 8.7: The four characters of digital comics-story

In Fig. 8.8 we present two examples in which some of Tommy-Blogger's ways of communicating, as described in Table 7.1, are illustrated through comic strips. In

the first panel, referring to Episode 1.1, Tommy-Blogger recaps in order to clarify the situation to himself. In the last two panels, taken from Episode 1.3, instead, an instance of communication is shown that serves to make explicit, both to oneself and to others, the purpose of one's actions. Moreover, in the second panel, there is also a Kyle-Boss feature as an activator of cognitive roles.

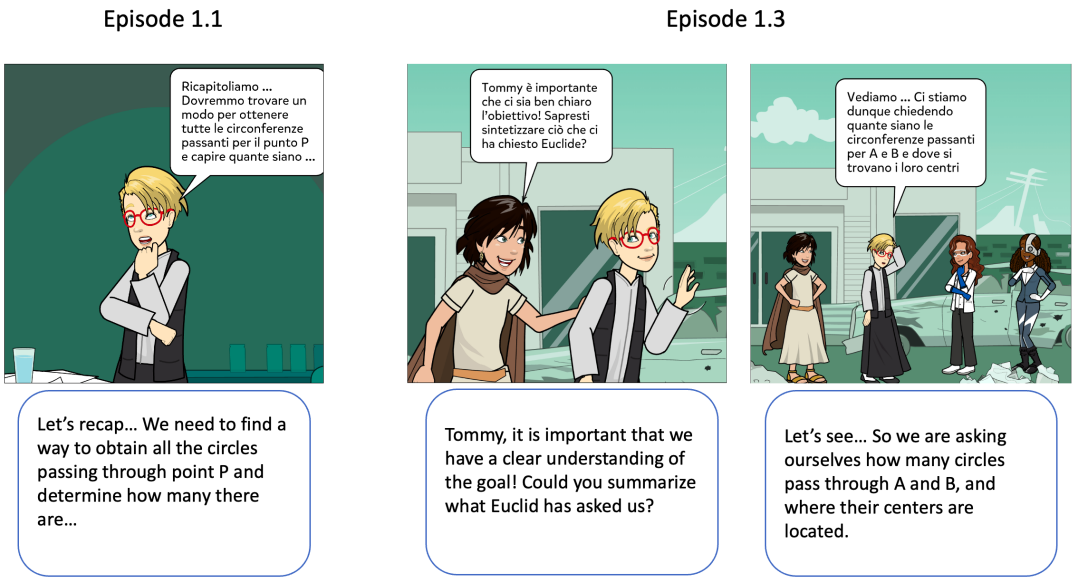


Figure 8.8: Some Blogger's ways of communication and a Boss' feature

In Figure 8.9, we present two situations drawn from the comics that illustrate how the a priori characterization in Table 7.1 was instantiated in the roles of Ellie-Promoter and Shona-Pest. In the first panel, taken from Episode 2, in response to Euclid's request to determine how many circles can be drawn through three non-collinear points, Ellie-Promoter proposes a strategy: to look, among the infinitely many circles passing through two points (a task addressed in previous episodes), for one that also passes through the third point. In the second and third panels of Figure 8.9, two examples of Shona-Pest's interventions are shown both taken from Episode 2. One of these is related to monitoring the effectiveness of strategic choices with respect to the goal, and the other concerning emerging awareness—also visible in the comics—of the need for theoretical justification of a geometrical property.



Figure 8.9: Promoter and Pest roles in actions

The entire implementation is carried out within a Moodle platform of the Department of Information and Electrical Engineering and Applied Mathematics (DIEM) at the University of Salerno. Each episode of the digital storytelling implemented as a Moodle Book resource, within which the comics—created using the software Pixton and organized into subchapters—are embedded, together with all the digital resources needed both to enable interaction (such as chats and digital whiteboards) and to provide access to external materials. During platform-based work, students are not required to communicate verbally with their peers, but to interact exclusively through digital chats, to promote deliberate and reflective communication. The access credentials to the platform were distributed confidentially and the passwords were changed in each subsequent session to minimize, as much as possible, the likelihood that students could track access patterns or recognise their peers in the roles they were playing.

8.1.4 Methodological choices for didactical mediation

To support the development of scientific concepts from a Vygotskian perspective—and thus their internalization (Vygotsky, 1978)—while carefully attending to the epistemological issues of the mathematical theories involved in their formation

(as, for instance, in the proving process within Euclidean geometry), particular attention should be paid to didactic mediation. The role of teacher is crucial for introducing students "to a theoretical perspective which is needed for a systematic view of mathematical theorems" (Mariotti et al., 1997). In this context, mathematical discussion constitutes a central construct for orchestrating a polyphony of voices around a mathematical object, where the teacher's voice assumes the role of Culture (Bartolini Bussi, 1996). Moreover, there is experimental evidence showing that, in open-problem tasks, students sometimes consider it enough to provide only a line of reasoning that justifies their conjecture, without reorganizing such reasoning into a formal deductive structure (Mariotti & Maracci, 1999). The role of mediation therefore appears to be crucial.

In relation to computer-based cooperative learning, King (1989) highlights that groups characterized by more frequent verbal interactions and a higher number of questions tend to achieve greater learning success.

In line with the idea that "awareness and intentional use of knowledge are promoted through the direct mediation of the teacher" (Boero et al., 2010, p.180), and drawing on Habermas's construct of rationality to foster the development of rational behavior in students, Douek (2014) identifies rational questioning as a didactic toolkit that "enriches the potential of mathematical discussion and allows improving its planning and management" (p.210). It thus represents an effective means for organizing mathematical discussion in order to promote the emergence of rational behavior in students. Building on the proposal of Douek (2014) to introduce rational questioning into classroom argumentative practices, Zhuang and Conner (2022) develop the Teacher Rational Questioning Framework (TRQF), turning it into an operational tool. A question is defined as a *rational question* if it "contains at least one component of rationality" (Zhuang & Conner, 2022, p.348). They thus construct a classification of teacher interventions in the classroom that may refer to one or more forms of rationality, in line with the dialectical interconnection that characterizes the three dimensions of Habermas' rationality (1998).

Cusi and Morselli (2016) also explored the role of the teacher as an activator of rational processes, but in this case with a primary focus on the algebraic domain.

To prompt the epistemic, teleological, and communicative components of rationality (Habermas, 1998), for the Euclid Experiment we conducted an a priori analysis of Euclid-Guru's interventions based on the Teacher Rational Questioning Framework (Zhuang & Conner, 2022).

Throughout the whole experiment, the framework thus provided the basis for the design and implementation of the interventions carried out by the teacher-researcher,

author of this thesis, both within synchronous digital interactions in Moodle chats—where he acted in the role of Euclid—and during phases of whole-class discussion (Donatiello et al., 2026). The questions posed were designed to prompt students' reflection on the choice of problem-solving strategies, the theoretical validation of the statements produced, and the need for shared and rigorous communication. Within this context, the teacher-researcher took on the role of a *rational mediator*, refraining from providing direct answers and instead fostering reflective processes, as well as argumentative peer interaction, through prompts such as: "Why do you agree/disagree with your classmate's statement?", "Why did this method work/not work?", "What term would you use to refer to . . . ?" (Donatiello et al., in press).

Finally, the opportunity afforded by the methodological and didactic device of the Inside-Out model and by the digital environment to unobtrusively observe everything that occurs in the chats—such as the strategies adopted by students, the actions performed and the communicative modes enacted during the process—combined with the possibility of choosing the most appropriate moment to intervene, allows for a carefully considered reflection on instructional interventions (Albano et al., 2021b, 2024c). This facilitates the application of the a priori analysis of the teacher-researcher's interventions, and thus the possibility of providing prompts that are accurate and appropriate to the specific instructional moment.

8.1.5 A new Digital Toolkit for Proof: Dynamic Euclidean Bank of Theory

One of the tools that Euclid-Guru can make available to students, upon explicit request, is the *Dynamic Euclidean Bank of Theory* (DEBT).

We designed this tool inspired by the Digital Toolkit for Proof (DTP) (Albano et al., 2021c), which is presented as a "Bank of Theory" integrated into the Moodle platform and which consists of a set of fixed tiles containing definitions and theorems useful for constructing the required proofs, as well as other movable digital tiles that can be arranged and aligned to form sentences in response to the proposed task, developed as an extension of the Interactive Semi-Open Question (Albano et al., 2017).

In line with the DTP, the DEBT also aims to stimulate "the need to make explicit references to its elements referring to inferring either by hypothesis or by construction" (Albano et al., 2021c, p.12), in order to foster a shift toward theoretical validation and support the development of the *literate register* in the use of mathematical language.

Our Dynamic Euclidean Bank of Theory (DEBT) is built on a column-structured Padlet, in which the Premises (Postulates and Common Notions) from Euclid's *Elements* are included, along with the 23 Definitions from Book I of the *Elements*, as well as a

selection of propositions identified through the a priori analysis of the mathematical tasks and considered necessary for the proposed proofs, as shown in Figure 8.10.

The approach adopted for the construction of the DEBT is grounded in a historical-cultural perspective, adhering as closely as possible to the Euclidean text, while occasionally introducing more modern variations to facilitate students' understanding.

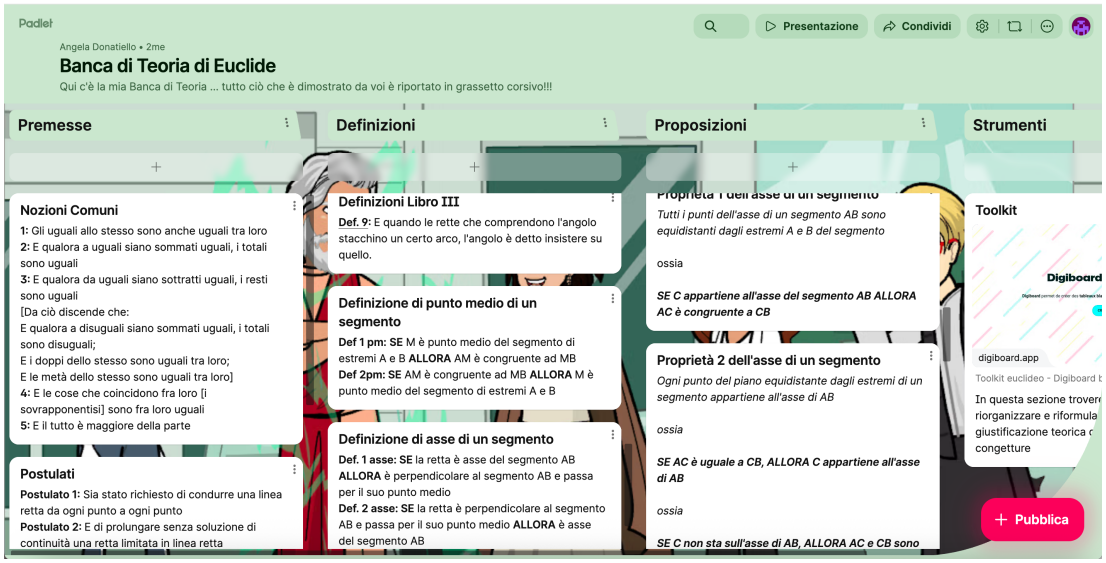


Figure 8.10: Dynamic Euclidean Bank of Theory

The formulation of the Postulates is drawn from translation from the Greek text by Acerbi (2007), as are the 23 Definitions of Book I. With regard to the Common Notions, Acerbi (2007) provides translations of nine in total; however, in the Introduction to the Greek Euclidean Corpus related to the Elements, he notes that Proclus himself did not acknowledge the authenticity of the ninth common notion (namely, that two lines do not enclose an area), as it pertains specifically to the geometric domain rather than having the general character of the others. Moreover, Acerbi (2007) points out that Proclus considered the fifth common notion to be derived from the second (as well as the sixth), and that the fourth common notion—attributed to Pappus of Alexandria—can also be obtained from the preceding ones. We therefore choose to include explicitly only five Common Notions in the Bank of Theory, while referring to the others as consequences of the preceding ones.

The 23 definitions from Book I of the *Elements* are provided in their original formulation, consistent with the version by Acerbi (2007). However, for the definitions

of the perpendicular bisector and the midpoint of a segment, a more contemporary formulation has been adopted, as it is more effective for understanding the causal structure. In this sense, we remain aligned with the structural approach of the DTP (Albano et al., 2021c), namely by using the conditional formulation "IF ... THEN" and, more specifically, expressing the first part of the definition through the implication "NAME $\implies$ PROPERTY" and the second part through "PROPERTY $\implies$ NAME". Each of these parts is identified by a code (e.g. Def.1 pm denotes the first part of the definition of midpoint).

By way of example, the definition of the perpendicular bisector of a segment as presented in the DEBT is provided below:

- **Def.1 pb**: IF a line is the perpendicular bisector of segment AB, THEN it is perpendicular to segment AB and passes through its midpoint.
- **Def.2 pb**: IF a line is perpendicular to segment AB and passes through its midpoint, THEN it is the perpendicular bisector of segment AB.

As for the propositions, we choose to present them mostly using a modern formulation. Only a few propositions are stated in the style of Book I, essentially those strictly necessary (or slightly more) for the proofs that students are required to tackle.

The Dynamic Euclidean Bank of Theory (DEBT), although inspired by the DTP, introduces a substantial modification with respect to the previous toolkit, as it is not a static object designed exclusively a priori by researchers, but rather a dynamic tool that evolves over time (Donatiello et al., 2026). The theorems proved by the students are incorporated into the DEBT and highlighted in bold italics to emphasize that they have been produced by the students themselves, so that in subsequent episodes students can find the DEBT enriched with their own results.

Example of a proposition embedded in the DEBT and whose proof was produced collaboratively between students and the teacher/researcher during one of the preparatory activities of the Euclid Experiment:

- Every point in the plane that is equidistant from the endpoints of a segment lies on the perpendicular bisector of AB

  namely

- IF AC is equal to CB, THEN C lies on the perpendicular bisector of AB

  namely

- IF C does not lie on the perpendicular bisector of AB, THEN AC and CB are not equal

We decide to present this proposition in multiple formulations. Indeed, in this case, the statement contains the word "every," which hides a conditional form of the type "if ... then" that can be difficult for students to handle. Therefore, we choose to write it in a conditional form and also in a formulation using the contrapositive as an equivalent statement.

In a research report for PME49, we initially referred to the DEBT as the Euclidean Theory Bank (ETB) (Albano et al., 2026b), but subsequently we decide to change its name to better highlight its dynamic nature.

The dynamic nature of the tool is due both to the integration within it of the propositions proven step by step by the students—which are added as new content and highlighted with a different formatting—and to the presence, within the Padlet, of a column connected to a digital whiteboard on which tiles are arranged (see Fig. 8.11). These tiles allow students to reorganize and reformulate, according to logical schemes, the theoretical justification of statements produced as conjectures. This design choice enables students to view the Postulates, Definitions, and Propositions of the Theory Bank on the same screen where the movable digital tiles are also displayed. Indeed, the whiteboard with tiles is integrated into the Padlet and opens in a smaller overlapping window, allowing students to switch between the Padlet and the whiteboard—and vice versa—with simple mouse movements.



Figure 8.11: Toolkit into DEBT

Unlike other tools designed to support students in proof processes in digital environments, where feedback is provided directly by the digital system (Miyazaki et al., 2017), in our DEBT collaboration and interactions among students is instead encouraged and feedback is eventually provided by the Euclid-Guru, acting as a cultural and rational mediator present in the chat.

8.1.6 An example of integrating comics with the mathematical problems

To make even more explicit the way in which the tasks were implemented within the comic-strip narrative, we choose to present below an example related to Episode 2.

Initially Euclid presents the group with a new challenge, asking them to consider not two points, but three non-collinear points, as shown in Figure 8.12. "Let us consider three non-collinear points. How many circles would you be able to draw through the three points?"

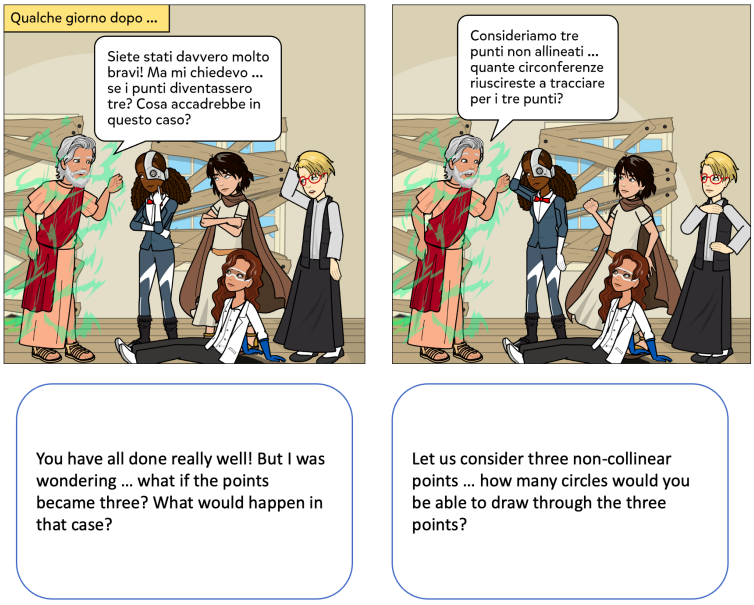


Figure 8.12: Comics from Episode 2

We choose to pre-assign on a sheet the three points A, B, and C in such a way that they do not form an isosceles triangle and that the center of the circle does not lie

on the sides of the triangle determined by the three points, since such configurations could be critical for students. Moreover, we choose to assign the labels A, B, C from left to right, with segment AB slightly longer than segment BC, in order to encourage the choice of AB as the segment for constructing the first perpendicular bisector. To avoid stepping outside the storytelling, and thus not to break the narrative, we also choose to include such instructions directly within the comic strip itself, by showing the particular configuration of the three points in a specific panel, displayed in Figure 8.13.

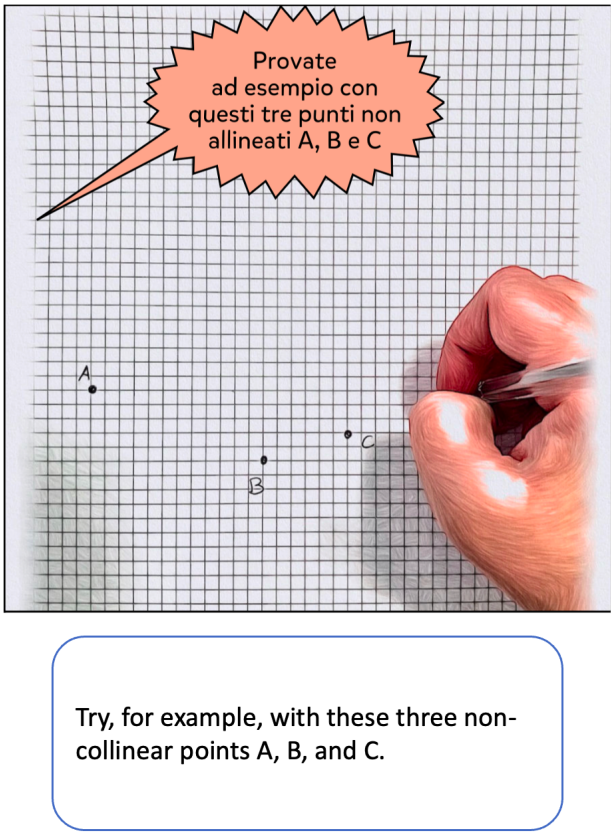


Figure 8.13: Comics from Episode 2

By suggesting through the comic strips that students draw upon their prior knowledge, namely by searching for the circle through A, B, and C among the infinitely many circles passing through points A and B, the idea is to guide students to construct only one chord (for example, the one with endpoints A and B) and then try, using the compass, to adjust their aim and grasp that unique circle which also passes through C (see Fig. 8.14).

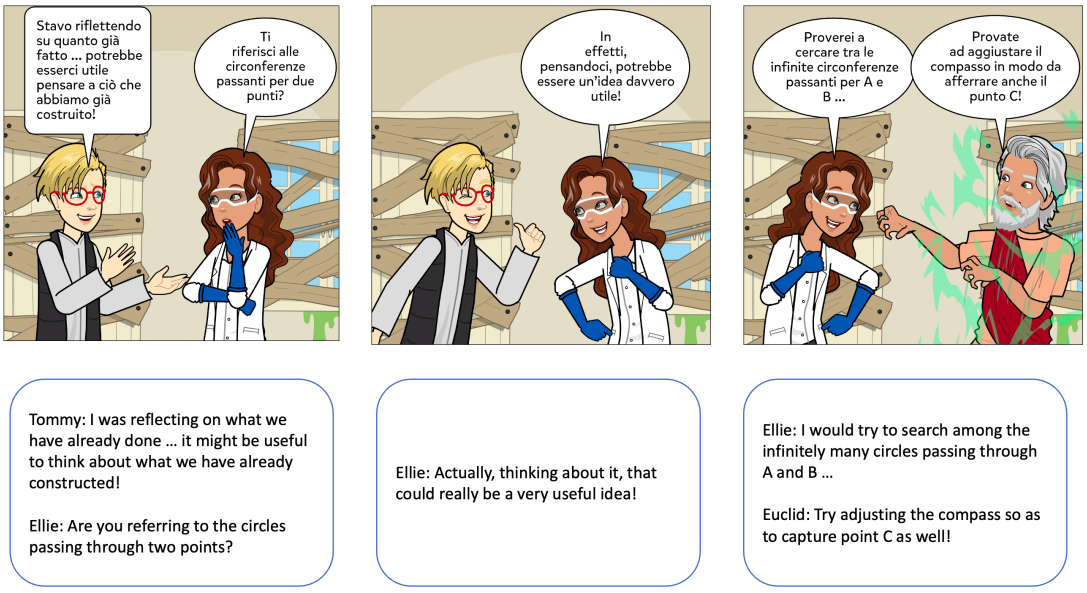


Figure 8.14: Comics from Episode 2

At this point, however, two possibilities may occur:

- Students may in fact find the point only empirically and draw a circle with center on the perpendicular bisector of AB that also passes through C, but without full awareness of why this happens, i.e. without realizing that such a center also lies on the perpendicular bisector of BC. In this case, it will be necessary to guide them to a meta-level of reasoning, while still allowing a high degree of autonomy in their actions. The initiative should be left to the students, although they may not succeed in completing the task independently. One possible question to ask in the chat in such a situation could be: "Good, I see that you are trying many different centers and have found some that work. Why do they work? Have you thought about it? Why does the circle with that particular center also pass through C?"

- Students may instead already think of constructing the perpendicular bisector of BC as well and identifying the center as the intersection of the two perpendicular bisectors. In this case, students should be asked to reflect on the intersection: "Do they intersect? Why do they intersect?" It is expected that students will state that this occurs because the points are not collinear; otherwise the perpendicular bisectors would be parallel and therefore could not intersect. The delicate issue of the impossibility of finding a common point of the two bisectors when the three points are collinear is based on Euclid's Fifth Postulate, which in its original

formulation states - Postulate 5: And that, if a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles - this issue could be discussed jointly at a later stage (Acerbi, 2007).

The problem now becomes how to lead students to move beyond this particular configuration and generalize to any configuration of three points; this may in fact represent the very trigger and motivation for the need for a theoretical validation. The choice of pre-assigning the three points and the initial segment also has the advantage of making it possible to later raise the question: 'Are we sure that by choosing the perpendicular bisectors of AC and BC we obtain the same center and the same circle that we found starting from AB and BC?' Such a question leads to reaffirming the uniqueness of the circle through three non-collinear points, which will naturally be discussed and mediated through the chat. For this reason, a specific comic-strip panel was included in which Shona-Pest raises exactly this doubt, as shown in Fig. 8.15.

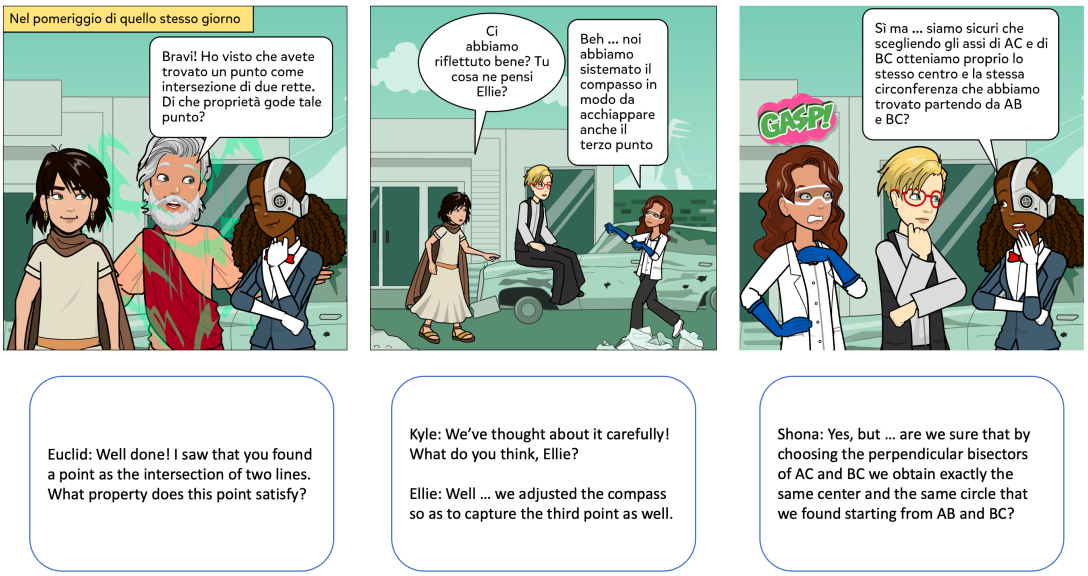


Figure 8.15: Comics from Episode 2

It is expected that students clearly explain their reasoning, thus moving in a very precise way from evidence to theoretical validation of their conjecture. To support this process, the comic strips have Shona-Pest explicitly highlight the importance

of theoretical validation and Kyle remind that the Theory Bank contains all the information they need (see Fig. 8.16).

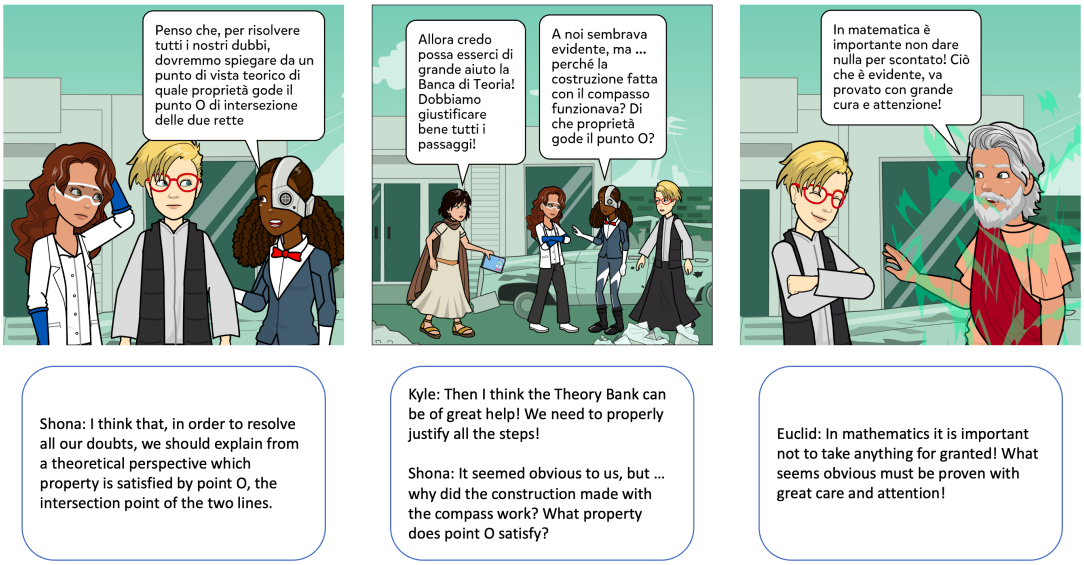


Figure 8.16: Comics from Episode 2

8.1.7 Summarizing the difference between Inside-Out and Inside-Out 2 models

Following the evolution of the film Inside Out with Inside Out 2, where the exploration of Riley's inner world is significantly expanded, similarly we would to offer a more in-depth exploration of the identified mental functions. The educational value of such an exploration lies in the contribution it offers to teachers and students in reflecting on certain aspects of the conjecturing and proving process, including at a meta-level, which should be gradually internalized in order to achieve success. Our Inside-Out 2 does not introduce new cognitive functions; instead, it broadens the inquiry by reframing it through a lens of rationality (Habermas, 1998).

Therefore, we design within the Inside-Out framework, differentiating ourselves from previous studies in some fundamental aspects. In particular:

1. To foster greater involvement of the Outlookers in the activity, they are invited to work actively on the mathematical problem, also sharing their own drawings on the shared Padlet board. Although their main role remains essentially meta-reflective, in which careful observation of the behaviours and strategic choices adopted by the

actor group is privileged, a greater attention to the mathematical problem will enhance their participation, contributing to the internalisation of the observed choices. To emphasise this aspect of reflective observation, we choose to use the term Outlookers to refer to the observers, instead of the more usual "Onlookers", since the term Outlookers contains the root outlook, i.e. vision. The observer is therefore seen as a fully active participant with an open gaze towards meta-reflection, supported by participatory and active observation.

2. The mathematical objects involved in algebraic proofs are numerical and algebraical expressions, letters, and numbers that allow for calculations and symbolic manipulations, governed by rules of syntactic transformation. In the geometric domain, the perspective changes completely, as students must deal with the dual conceptual–figural nature of geometric figures (Fischbein, 1993), which may sometimes come into conflict—especially in indirect proofs—with the construction of an impossible figure. The choice of a geometric context led us to further reflect on meta-mathematical narration. While previous studies grounded tasks in concrete narrative situations, this approach appeared problematic in geometry from Fischbein's perspective of geometric figures as *figural concepts*, where conceptual properties must prevail over figural ones. Real objects, such as cart wheels, would require an additional abstraction step before engaging in conjecturing and proving. For this reason, in the Euclid Experiment, tasks are not generated directly from contingent physical needs of the protagonist, but are introduced by the Guru within a coherent narrative framework. In this way, storytelling supports engagement while preserving the conceptual nature of geometric reasoning. Although this partially diverges from the C&Q construct of Zan (2012), it remains consistent with the storytelling perspective adopted from Zazkis and Liljedahl (2019).

3. Unlike previous teaching experiments, the tasks were selected in accordance with the school curriculum and were therefore situated within a specific moment of the school year, consistently with the disciplinary planning established by the classroom teacher. This choice was made in order to foster students' ownership of the activities and to avoid the perception of the tasks as being external to the curricular pathway.

4. The Guru, while remaining the group's reference point for access to external resources, will no longer maintain a privileged relationship solely with the Promoter. Instead, the Guru will be fully integrated into the actors' group chat, interacting with all members of the group. At times, in order to stimulate the

involvement of the Outlookers with the mathematical problem, the Guru will also be allowed to intervene in the Outlookers' group chat. This necessary modification had already emerged in the collaboration between researchers and teachers in a previous implementation of the model. Moreover, this change is expected to support the internalisation of the Guru's role within the group, as the continuous and direct observation of the mediation enacted by the Guru—through the Teacher Rational Questioning Framework (Zhuang & Conner, 2022)-will enable students to internalise a practice that is socially experienced and shared (Vygotsky, 1978), namely that of rational questioning. Furthermore, the Guru will be conceived as an Agent of the Culture of Mathematics (Zaslavsky et al., 2021), guiding students in the exploration of mathematical knowledge, as the Voice of Culture, within the polyphonic interplay of voices (Bachtin, 1968; Bartolini Bussi, 1996) in which the epistemological substrate finds expression through the figure of the Guru, and as a cultural mediator who supports their progression toward standards of theoretical validation, acting as a guarantor for both the mathematical community and the classroom community (Mariotti, 2006);

5. One of the tools that Euclid-Guru can make available to students, upon explicit request, is the Dynamic Euclidean Bank of Theory (DEBT). We designed this tool inspired by the Digital Toolkit for Proof (DTP) (Albano et al., 2021c), which is presented as a "Bank of Theory" integrated into the Moodle platform. The Dynamic Euclidean Bank of Theory (DEBT), although inspired by the DTP, introduces a substantial modification with respect to the previous toolkit, as it is not a static object designed exclusively a priori by researchers, but rather a dynamic tool that evolves over time. The theorems proved by the students are incorporated into the DEBT and highlighted in bold italics to emphasize that they have been produced by the students themselves, so that in subsequent episodes students can find the DEBT enriched with their own results.

6. For the Boss role, we chose a non-binary name, Kyle, and this aspect was reflected both in the character's clothing and in their expressions. The rationale behind this choice lies in the intention to avoid identifying the Boss role with a male character, as had occurred in previous implementations, and instead to allow the figure of a leader to be experienced comfortably by every student. Moreover, compared to the previous model, a female character was chosen for the Promoter role, in order to avoid reinforcing a traditional gender stereotype that associates the generation of solutions in a problem solving situation with male figures.

8.1.8 Phases of the classroom activity

The Euclid Experiment was structured in several phases, not necessarily always in this sequence:

1. An initial preparatory phase, in which students are involved in exploring, formulating conjectures and subsequently proving geometric properties relating to the perpendicular bisector of a line segment, discussing and working together with the teacher/researcher, in order to become familiar with the proving process within the Euclidean context and with cognitive roles, through a role-playing simulation.

2. Individual work consisting of a cloze activity-completing a theoretical justification of a statement-during which students work independently on worksheets that provide prompts and information for the complete or partial rewriting of a mathematical proof related to the Euclidean geometry topics addressed. This phase will be described in detail in chapter 9.

3. A collective problem-solving activity in the field of Euclidean geometry, through a highly immersive narrative approach in a digital environment, in which students enact the cognitive roles of the Inside-Out 2 model, characterised by a rational perspective, in the dual-level Actor/Outlooker, with rotation of roles and levels from one episode to the next;

4. Individual reflective activities on paper worksheets for the rewriting of mathematical proofs, shared collectively in the Moodle chats and then sent by email to Euclid-Guru as external communication towards Culture by Blogger Actor;

5. Collective discussion sessions with the teacher/researcher aimed at conceptualizing critical issues or stabilizing knowledge;

6. Individual on-line revision of the work and written on-line final reflection, as activities for formative assessment.

8.2 Collected data

For the Euclid Experiment, multiple types of data were collected, as approved by the Ethics Committee of the Department of Information and Electrical Engineering and Applied Mathematics at the University of Salerno. In particular, all synchronous discursive interactions taking place in the Actors' chats and in the Outlookers' chats

were collected. These chats, in their original version, are in Italian and were fully translated into English for transcription in this thesis work. In addition to the digital chats, the drawings shared on the Padlet integrated into the Moodle platform were also collected, together with the captions written as comments, which were translated into English. While the chats allow for the recording of all synchronous discursive interactions occurring among student groups, including the exact timing of these interactions, the Padlet drawings compensate for the lack of graphical visualization in the chat environment, promoting multimodal rather than purely verbal interaction.

We also collected and analysed the emails sent to Euclid-Guru at the end of each episode experienced on the platform, conceived as a moment for writing the mathematical proof and for its external communication, in accordance with the norms shared by the mathematical Culture.

Classroom discussions and all different activities carried out during the experiment were not audio-recorded or video-recorded for reasons related to privacy and the approval requirements of the Ethics Committee. However, the activities were always carefully observed both by the author of this thesis in the role of teacher/researcher and by the classroom teachers to keep track of any relevant emerging aspects during the activities. The verbal contributions, gestures, and behaviours that allowed identifying the emergence of different components of reasoning were systematically collected and recorded by the teacher-researcher and classroom teachers using a specially designed observation grid to support the subsequent data analysis (Donatiello et al., 2026), as shown in Figure 8.17, which presents the translation of original Italian version.

These observations were then enriched by field notes and memos that the author of this thesis recorded during and at the end of each activity to track and document the entire empirical investigation in the form of written reflections. Although memoing originated and has been developed most extensively within Grounded Theory, it has potential in all qualitative research, as it represents an effective tool to support the researcher's reflection both in the design phase and in the actual analysis phase (Birks et al., 2008). Despite its fairly established and widespread use in qualitative research across various fields, it still appears to be somewhat under-addressed as a data collection tool in mathematics education.

Finally, we collected the students paper worksheets, completed during the cloze activity in the preparatory phase and during the individual reflective activities, in which they engaged in the autonomous writing of the mathematical proof.

Systematic Observation Grid

Phase ..... Date ..... School ..... Student .....

For each indicator, specify how its presence is recognized by noting actions, questions, and statements, or any other observable student behavior (such as gestures, gaze, particularly meaningful hand movements, etc.) that has been noticed.

Significant questions (specify which ones)	
Shows curiosity – Listens actively (indicate which observable signs)	
Responds actively during discussion (specify how)	
Seeks solutions to non-routine problems	
Applies prior knowledge to new situations	
Shares her/him proposed solution with the class	
Explains own ideas to the class in a collaborative way	
Explains the reasoning behind own claims	
Adopts a collaborative approach towards different ideas	
Uses communication for a variety of purposes (to inform, but also to motivate and convince)	
Can articulate ideas and thoughts clearly	

Various observations on the work during the activity:

Figure 8.17: Systematic Observation Grid

8.3 Methodological choices for data analysis

Paper worksheets and email

To analyze the paper worksheets produced at the end of the preparatory session and the emails sent to Euclid-Guru at the end of each storytelling episode, we choose to use the Rational Mathematical Template of proof (RMT) (Boero & Azrou, 2023; Boero & Turiano, 2022, 2023) as an analytical tool to examine and interpret the presence or absence of components of rationality (epistemic, teleological, and communicative) in the cloze texts or in the email texts. This methodological choice was presented at a theoretical level in sec. 4.1.

We briefly recall that the epistemic component pertains to opinions grounded in a given theoretical framework, the teleological component relates to intentional choices of goal-oriented strategies, and the communicative component involves the use of language to foster understanding within a community.

Table 9.1 provides a paradigmatic example of the application of the RMT to the analysis of a written text, specifically a cloze text. Accordingly, we identify statements that are either supported or unsupported by theoretical justification (epistemic component), statements that indicate the presence or absence of strategic choices (teleological component, and statements that reflect either limited or rich exposition in line with norms shared within the mathematical community (communicative component)—as exemplified in the paradigmatic case shown in Table 9.1 (where students' writing appears in italics).

Synchronous digital discoursive interactions

For the a posteriori qualitative analysis of the synchronous discursive interactions occurring in the digital chats, we adopt a categorical-content analysis (Lieblich et al., 1998), in which the categories are partly theory-based and partly derived from suggestions emerging directly from the data during the analysis.

Following the methodology described in Sec. 4.2.1, the analysis will be structured according to the following steps:

1. *Selection of the Subtext.* We select excerpts from all the collected chat episodes, numbering the lines that are essential to the study. This subtext thus becomes now the new our dataset for the content analysis.

2. *Definition of the Content Categories.* We define the analytical categories, describing them through operational indicators, which are partly theory-informed and partly emergent from the data. Since the pilot study was focused on an algebraic domain, whereas the Euclid Experiment addressed geometric proofs, the a priori analysis, already developed and presented in Table 7.1 (Albano et al., 2024b, in press), requires further refinement that necessarily takes into account the substantial differences between the two domains (see sec. 1.2). The theory-based indicators are grounded in and elaborated from prior research in mathematics education, including Albano and colleagues (2021a), Urhan and Bülbül (2023a, 2023b), Balacheff (2010), Boero and Azrou (2023), Boero (2017), Mariotti and colleagues (1997), Mariotti (2006), and Pedemonte (2007, 2008). In accordance with the construct of rationality of Habermas (1998), the a priori characterization in Table 7.1 thus provides a guide for coding the interventions. The analytical categories considered are Promoter (PRO), Pest (PE), Blogger (BLO), Boss (BO), Euclid (EU), and Outlooker (OU), across different levels: knowledge level (KL), meta-knowledge level (MK), and interplay between the two levels (IP). Although it may initially appear that using roles both as didactical devices representing

cognitive functions and as analytical categories could entail a potential risk of circularity—whereby behaviors are interpreted through categories that the design itself has induced—it is important to bear in mind the dynamic and non-fixed nature of roles. Indeed, even if each role is associated with specific components of rationality and with specific operational indicators, the last ones are performed or experienced by all participants; therefore, the characterization of roles can be used as an analytical lens while avoiding circularity in the analysis.

3. *Sorting the Material into the Categories.* We assign the data to each analytical category through identifying codes that allow us to distinguish both the categories and the different levels at which the analysis is conducted (knowledge level, meta-knowledge level, and the interplay between levels). At the knowledge level, we consider all behaviors related to, for example, exploring the proposed mathematical problem, searching for strategies, producing conjectures, generating ideas, selecting and using appropriate tools for the task, checking the truth and validity of statements, verifying conceptual properties, justifying the steps taken, and using mathematical language correctly—in other words, all behaviors that can be described using operational verbs. At the meta-knowledge level, we instead include all behaviors that denote awareness of the choices made and the logical aspects of a mathematical proof, also in relation to the reference theory, as well as communication oriented toward understanding, directed both toward oneself and toward others. Instead, the interplay between levels refers to the relationship linking the other two levels within the moment of transition from a level to another one. We choose to support this stage by using six different colours to distinguish among various segments of the text, with which we highlight passages, sentences, verbs, and adjectives that appear relevant and that we associate with a given category. Accordingly, the analysis focuses on sentences, verbs, verbal expressions, terms, and words that can be traced back to the codified actions. Alongside the highlighted segments, we also choose to write the operational criteria that guide our identification of that particular category, each associated with a specific colour.

4. *Drawing Conclusion From the Results.* Finally, we provide a discursive synthesis of the results derived from the analysis. We therefore choose not to calculate frequencies or repetitions, but instead to reorganize what emerged from the data in a narrative form, so as to make the entire process that took place in the analyzed chat clearer to the reader.

Chapter 9

Euclid Experiment - Preparatory activities

9.1 Introduction

Before starting the role-playing activity on the Moodle platform, we plan and implement a series of preparatory sessions in both classes, in order to foster an initial exploratory approach to the proving process. The activities focused on conjecturing and proving properties of the perpendicular bisector of a segment, as well outlined in Section 8.1.2.

The preparatory phase serves multiple purposes for all participants involved in the experiment. It is designed to allow students to become familiar with proving processes, as well as to engage with the different cognitive roles they will need to perform during the role-playing activity (see section 8.1). It also aims to introduce students to the figure of the Euclid-Guru as a cultural mediator and as a means of accessing external resources provided by Euclid, such as the Dynamic Euclidean Bank of Theory (see section 8.1.5). In addition, it offers an opportunity to test the interactions between the class and the teacher-researcher acting in the role of Euclid-Guru [1], guided by the Teacher Rational Questioning Framework (Zhuang & Conner, 2022) (see section 8.1.4), before the role-playing in Moodle platform. Last but not least, the preparatory phase also aims to investigate the extent to which the cloze activity can function as a mediating tool to support the cognitive unity of theorems by making it explicit.

The preparatory phase consists of two meetings. The first meeting focused on the proof of a canonical statement concerning the perpendicular bisector of a segment, collectively constructed and discussed under the guidance of the teacher-researcher. In

[1] The role of Euclid-Guru was played by the author of this study

the second meeting, on which this chapter is focused, the students were engaged in proving a non-canonical statement concerning the same topic, taking advantage of the cloze activity. While providing a brief description of the activity carried out during the first preparatory session, as revealed in the memos and field notes collected, in this chapter we focus more closely on the activity implemented during the second meeting, in which students worked on proving a new statement concerning the perpendicular bisector of a segment, initially proposed as an open exploratory problem, with aim to explore and test the cloze activity conceived as a tool to make the cognitive unity of theorems explicit [2] (Albano et al., 2026a).

9.2 Cloze activity: a new perspective

Boero and Turiano (2022, 2023), with the aim of analysing the progressive development of 10th grade classes students' rational behaviour within proving process and construction of mathematical proof, planned and implemented proving activities structured in different steps: construction of problems and theoretical justification, proving of produced conjectures, collective discussions to share and refine the text, analysing and refining some written productions and finally cloze activity on proof text. The cloze activity in mathematical proof consists of completing a theoretical justification of a statement, "where all the steps were reported, but their justification were lacking" (Boero & Turiano, 2022, 2023; Turiano & Boero, 2019). Therefore, by *theoretical justification* we mean the explicit reasons (i.e. definitions, propositions, theorems) within the reference theory that make each step of the proving process admissible. Cloze is traditionally used as a moment of awareness and reflection aimed at stabilizing and transferring at the meta-level, and in an organized form, after the sharing of a proof negotiated by the whole class (Boero & Turiano, 2022, 2023; Turiano & Boero, 2019). Boero and Azrou (2023), in order to foster development students' awareness of logical aspects of a formal deductive proof text, suggest implementing cloze activities, as complete a proof text specifying justifications of epistemic nature, related to definitions, axioms and theorems. In this chapter we contribute to cognitive unity research, presenting a teaching episode, situated within a broader experimental study, in which the cloze activity is intentionally used immediately following the *structurant argumentative phase* (Pedemonte, 2007), in which students' develop a theoretical justification of conjecture, differently from what Boero and Turiano did. This is because

[2] The findings presented in this chapter will be published in a research report entitled *"Cloze activity as a mediator to make the cognitive unity of theorems transparent: a study with secondary school students"* and presented in june 2026 within ERME Topic Conference (ETC24) - *Argumentation and Proof in the Teaching and Learning of Mathematics and Interaction with Other Disciplines* in Milan

we aim to investigate to what extent cloze activity can be a mediating tool to support cognitive unity of theorems making it transparent, in order to foster the overcoming of the structural gap between argumentation phase and formal proof (Duval, 1991).

9.3 Research question

To this aim, we formulate the following research questions:
RQ1: To what extent the cloze activity can be as a mediator to support cognitive unity of theorems making it transparent?
RQ2: How Habermas' construct of rationality can be used to analyse the mediating role of cloze activity?

9.4 Methods

Context of research

The activity described in this chapter concerns a part of the preparatory phase of the broader teaching experiment, named Euclid Experiment (see Chapter 8), in 10th grade classes, from March to May 2025, for a total of 20 hours of intervention per class. The experiment involved 18 students of the Liceo Scientifico "C. Pisacane" in Padula (Salerno, Italy) and 19 students of the Liceo Scientifico Scienze Applicate "P. Leto" in Teggiano (Salerno, Italy). The activity was led by the teacher-researcher (the author of this study) whilst the classroom-teacher supported taking the role of outlooker. The broader experiment, as shown in the section 8.1, concerned problem-solving activities in Euclidean geometry based on the methodological and didactic device Inside-Out, with focus on Books I and III of Euclid's Elements. The students were expected to interact and collaborate on digital chats to formulate conjectures, search for theoretical justification for the statements produced, and collectively formalise the mathematical proof. They had limited prior experience with conjecture production and proving activities: their previous encounters with geometric proofs had been restricted to studying and reproducing complete proofs presented by the teacher. The preparatory phase consists in two meetings and one of its main aims is exactly to familiarise them with these processes.

The first preparatory meeting: a narrative description

During the first preparatory meeting, students worked on the characteristic property of the perpendicular bisector, collectively constructing the conjecture, and jointly engaging within the proving process of the two implications. In both classes, the

definition of the perpendicular bisector emerged quite easily, as it was already familiar to the students, but when we asked to explore the figure to identify properties of the perpendicular bisector, students initially struggled, as if they were not accustomed to exploratory tasks. It was therefore necessary to encourage them to take risks, to experiment, and to sketch freely—trying out ideas and making tentative constructions on paper. When students stated that every point on the perpendicular bisector is equidistant from the endpoints A and B of the segment, we asked to clarify what they meant by "equidistant", and they specified that the segments AP and PB were considered to be "equal". This enabled to initiate a brief discussion that introduced a strong historical-cultural perspective: although in geometry the term "equal" may refer to different notions (e.g., congruent, equivalent), we agreed to use the term "equal," noting that Euclid himself used this term in the Elements to denote congruent geometric objects. Once the exploratory phase was completed and the conjecture had been formulated students were encouraged to seek a justification for it by explaining why they considered the proposed statement true. In both classes students immediately referred to the first triangle congruence criterion, arguing sometimes also that "they are mirror images of each other" or that "if we overlap them, they coincide." To address the converse implication, we challenged the students by asking whether all points equidistant from the endpoints A and B of a segment must necessarily lie on the perpendicular bisector of AB: "Are we really sure that all points in the plane equidistant from A and B lie on the perpendicular bisector of AB? Could there exist a point in the plane equidistant from A and B that does not lie on the perpendicular bisector of AB?" The immediate and collective response from students in both classes was a clear "no". To justify this claim, one student suggested reasoning ad absurdum, assuming that the point does not lie on the perpendicular bisector; students had already encountered this type of reasoning in their prior learning. It also emerged that a point "equidistant" from A and B cannot lie outside the perpendicular bisector of AB, since "otherwise it would be closer to one or the other." This observation proved useful in directing attention to the difference between the two segments AC and CB. Students thus engaged in reasoning ad absurdum, using it to exclude all possible cases of points C that are equidistant from A and B but do not lie on the perpendicular bisector of AB. In both classes, the first situation proposed concerned a point C not lying on the perpendicular bisector but located on segment AB. This represented a crucial moment: in arguing that equidistance no longer holds in this case, students referred to the uniqueness of the midpoint of a segment, stating that "there is only one midpoint." The epistemological reflection on this issue is discussed in Section **??**. In the second class, students also considered what would happen if the point lay on the

line containing segment AB, but outside the segment itself; in this case as well, they used Common Notion 5, this time referring to it without the need for further external intervention. Students then adopted the strategy of exploring the case of a point located first on, and then beyond, the line perpendicular to the segment and passing through one of its endpoints. This allowed them to visualise triangles involving the segments AC and CB, with angles that were either right or greater than a right angle. Although the strategic dimension was clearly evident in students' actions, it required epistemic support. In particular, it was necessary to explicitly recall Proposition 19 from Book I of Euclid's Elements, as it was not readily accessible in students' memory. Recognising this moment of impasse, we invited students to consult the Dynamic Euclidean Bank of Theory (see Section 8.1.5), prompting them by saying, "Let's see what Euclid tells us—let's consult his theory bank". This prompted active engagement, as students quickly recognised that the proposition provided information about the side of a triangle opposite the larger angle. Referring to Euclidean theory as a "bank," personified in the figure of Euclid, led one student to use the symbol $\sim$*Euclid* each time the DEBT was consulted, as a kind of validation criterion for the statements produced. Beyond the specific strategy adopted, through this process of elimination students thus formulated a new statement, the secondary statement S*. However, they proceeded by constructing a direct proof of the contrapositive, without explicitly referring to the corresponding meta-theorem. Auxiliary constructions, such as drawing a line parallel to segment AB through point C, or a line perpendicular to AB through one of its endpoints, were proposed by the students themselves, sometimes spontaneously and at other times prompted by questioning strategies (e.g., "How can we be sure that this is an obtuse angle?" or "It would be useful to have a larger angle here as well, but how can we show it? How can we construct one?").

Design e Task of the second preparatory meeting

In the second meeting, on which this chapter is focused, the students were engaged in proving a non-canonical statement concerning the same topic, taking advantage of the cloze activity. During the experiment, students also had access, upon request, to the Euclidean Theory Bank (Albano et al., 2021c), which contains postulates, axioms, and selected definitions and theorems from Euclid's Elements, presented in conditional form, as presented in section 8.1.5.

The task proposed in the second meeting, presented below and discussed in this paper, was designed to allow for a potential structural gap between students' argumentation and the subsequent construction of a formal proof. Indeed, initially an open task is proposed: *Let us consider two points C and C' both equidistant from*

endpoints of the segment AB. Determine the line joining them. Subsequently, students are asked to produce a formal proof of the conjecture through a cloze activity.

ACTIVITY - Axis of a segment
Worksheet 1 - CLOZE test of knowledge and theoretical justification of a geometric proof

In the previous group activity, we discovered and conjectured that if two points are both equidistant from the endpoints A and B of a segment, then the line passing through those two points is the axis of segment AB.

1) We considered two points C and C' equidistant from A and B.
2) We considered the point C'' where the line passing through C and C' intersects the segment AB

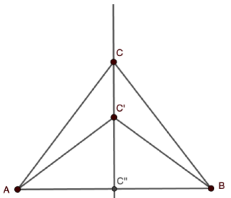
The following text is a proof of the fact that if two points are both equidistant from the endpoints A and B of a segment, then the line passing through those two points is the axis of the segment AB.

The following proof consists of two steps: first, we prove that the two triangles $A\widehat{C}C$ and $B\widehat{C}C$ are congruent and then we prove that the two triangles $A\widehat{C}C''$ and $B\widehat{C}C''$ are congruent and hence we prove that the line through C and C' is the axis of the segment AB.

Complete the text with the missing parts (you can also add symbols to the figure).

Some completions will require you to insert mathematical elements or references to segments or angles. Other completions will require you to insert theoretical justifications such as terms such as "by hypothesis" or references to specific theorems or properties.

Proof:



Let us consider two points C and C' both equidistant from the points A and B.

We want to prove that the line passing through C and C' is the axis of the segment AB, i.e. that AC'' and BC'' are congruent and the angles $A\widehat{C}'C'$ and $B\widehat{C}'C'$ are both right angles.

Let us consider the triangles $A\widehat{C}C$ and $B\widehat{C}C$ they:

- have _____ in common;
- AC' and BC' are congruent _____;
- _____ by hypothesis

Consequently, the triangles $A\widehat{C}C$ and $B\widehat{C}C$ are congruent by
_____

From this it follows that the angles $A\widehat{C}C$ and $B\widehat{C}C$ are congruent since
_____

Let us now consider the two triangles $A\widehat{C}'C'$ and $B\widehat{C}'C'$. They have:
- _____ by hypothesis
- _____ in common
- The angles $A\widehat{C}'C''$ and $B\widehat{C}C''$ congruent, since
_____

Therefore, the two triangles $A\widehat{C}'C'$ and $B\widehat{C}'C'$ are congruent by
_____

and thus the segments _____ and _____ are also congruent

as well as the angles _____ and_____ which are therefore _____ since adjacent and congruent.

Figure 9.1: Cloze activity, translated from the original Italian version

The design of the second meeting was articulated into the following phases:

1. Presentation and submission of the task to the students;
2. Engagement of students in producing collectively a conjecture under the guidance of the teacher-researcher by means of the Teacher Rational Questioning Framework (Zhuang & Conner, 2022) to induce students' contributions from a rationality perspective (see section 8.1.4); at this stage, the teacher-research can also support the students by means of a constructive argumentation (Pedemonte, 2007);
3. Collective construction of a theoretical justification in a classroom discussion;

4. An individual cloze activity (Figure 9.1) to produce the formal proof, in which some steps of the proof were provided and students were required to complete them by supplying the appropriate theoretical justifications; in other cases, the theoretical justification was given, and students were asked to articulate the corresponding mathematical elements.

We note that after step 3 it was deliberately chosen not to proceed with the construction of a collectively negotiated formal proof, but to have a paper-and-pencil cloze activity, with the specific aim of making the distinction between argumentation and proof more explicit, thereby reducing the risk that students might confuse these two processes (Boero et al., 2007).

The formal proof of the conjecture, structured as a deductive chain, requires the use of Propositions 8 and 4 respectively from Book I of Euclid's Elements, as well as the use of Common Notion 3 (see section 8.1.2).

Data collection

We collected various kinds of data: the written productions completed on the worksheet shown in Figure 1, produced during the phase 4 in which students reorganized their arguments into a formal proof; systematic observation grids, where the teacher-researcher and the classroom-teacher annotated all students' words, gestures, movements deemed relevant; teacher–researcher's memos, including a narrative of the events along with her own educational reflections they prompted. The memos were daily documented to maintain an accurate record of what took place throughout the whole process with the students.

Tool of analysis

We chose to use the Rational Mathematical Template (RMT) of proof (Boero & Azrou, 2023; Boero & Turiano, 2022, 2023) as analytical tool to analyse and interpret the attendance or the lack of rationality components (epistemic, teleological and communicative) into the cloze text. We briefly recall that the epistemic component concerns opinions grounded in a given theoretical context, the teleological component concerns the intentional choices of aim-oriented strategies and the communicative component concerns the use of language for understanding within a community.

Table 9.1: Paradigmatic example for the analysis

Transcripts from worksheet	**Analysis**
Consideriamo i triangoli $A\hat{C}C$ e $B\hat{C}C$ essi: - hanno C'C in comune; - AC' e BC' sono congruenti perché C' è equidistante - AC e BC sono congruenti per ipotesi Let us consider the triangles AC'C and BC'C they: - have *CC'* in common; - AC' and BC' are congruent *because C' is equidistant*; - *AC and BC are congruent* by hypothesis	In the theoretical justification, the student shows a *communicative lack*, as she/he does not explicitly justify the congruence of AC′ and BC′ by referring to the given hypothesis and writing "by hypothesis" in an explicit way. However, she/he implicitly brings back this information by writing that "C′ is equidistant".
Quindi i due triangoli $A\hat{C}'C$ e $B\hat{C}'C$ sono congruenti per il secondo criterio di congruenza Consequently, the triangles AC''C' and BC''C' are congruent by *The second criterion of congruence*	This claim reveals an *epistemic lack*: the student invokes an incorrect criterion for triangle congruence (the one involving two angles and a side) instead of the appropriate Proposition 4 from Book I of Euclid's *Elements*.
Da ciò segue che gli angoli $A\hat{C}C$ e $B\hat{C}C$ sono congruenti poiché sono "specchiati" From this it follows that the angles AC'C and BC'C are congruent since *They are mirrored*	This claim reflects the presence of a *teleological dimension*: the student employs an argument characteristic of the exploratory phase as a strategic means to achieve the intended goal. However, it also reveals an *epistemic lack*, as the argument lacks an explicit reference to the underlying theory.

Thus we look for statements that are either supported or not from theoretical justification (epistemic component), statements that reveal the presence or absence of strategic choices (teleological component), and statements that display either a lack or richness in exposition in accordance with norms shared within the mathematical community (communicative component)—as this following paradigmatic example presented in Table 9.1 (note that the students' writing has in italic)—and compare them with the argument statements recorded into systematic observation grid and reported into teacher-researcher's memos.

9.5 The episode

Students were asked to consider two points, C and C', both equidistant from points A and B, the endpoints of a segment. The teacher–researcher sketched the figure on the

whiteboard and then posed a challenge, asking students to identify the line that joins the two points C and C' . The conjecture that this line was the perpendicular bisector of the segment emerged immediately in both classes, as students quickly reasoned that, since both points are equidistant form A e B, they have to lay on the perpendicular bisector of segment AB and so the line must itself be the perpendicular bisector. It therefore became necessary to ask students how they could be certain of this claim, prompting them to articulate more explicitly what it meant to justify that the line through C and C' was indeed the perpendicular bisector of segment AB. A collective structurant argumentation was then initiated, with attention systematically drawn to the specific elements needed to make the conjecture explicit—namely, that AC" was congruent to C"B and that angles AC"C' and BC"C' were right angles. In both classes, it was also necessary, at the beginning of the process, to redirect attention to the fact that C" —the potential midpoint of AB—was initially only the intersection point between line CC' and segment AB, and that its status as midpoint required proof. This scaffolding enabled students to activate their reasoning processes and initiate the substantive argumentative work. At one point in the first class, a student, hereinafter named Victor, intuited that establishing the congruence of triangles AC"C' and BC"C' would be essential for proving the conjecture. Victor reached this insight through an abductive process during the argumentative phase, stating: "Suppose, for the sake of argument, that C" is already the midpoint, even though we haven't proven it; if we find that triangles AC"C' and BC"C' are congruent, then C" must be the midpoint." The student appears to ascribe to his reasoning a sense of surprise characteristic of a premise that seems intuitively evident yet still demands verification (Pierce, 1934). This move places the student in an abductive mode of reasoning that enables him to pursue the intended goal. This step provided the class with the necessary input to engage in a collective backward-oriented argumentative process, iteratively identifying what was needed to reach the conjecture. The ensuing discussion allowed other students to contribute to the reasoning process. Victor continuing with an abductive process focused the attention on the angles AC'C" and BC'C" and said: "If we have that the line C'C" is a bisector of the angle AC'B, then we can consider the two triangles AC'C" e BC'C" as congruent". A student, named Marcus, took up the suggestion to prove the congruence of triangles AC"C' and BC"C', focusing on the congruence of two sides and noting that AC' and BC' were congruent by hypothesis. Another student, named Manuel, observed that CC' was common to both triangles and suggested that it would be useful if line C'C" were the bisector of angle AC'B. This activated a further student, named Nicole, who then directed attention to the angle at C', recalling a triangle congruence criterion involving two sides and the included angle: "Since we have two congruent sides, we can find the angle between them".

At this point, while the class was focusing on the angles at C', another student, named Max, shifted attention beyond the two triangles under analysis, suggesting drawing a line parallel to AB to "see two triangles at the top". Although this parallel line was not required for the formal proof, it played a meaningful role in the students' reasoning by allowing them to consider the configuration from a different perspective. When the teacher–researcher asked whether drawing the parallel line was actually necessary, a student, named Daniel, replied: "we already have them," referring to triangles ACC' and BCC'. This comment reactivated Victor, who had previously engaged in abductive reasoning. He argued that the congruence of sides AC and BC, along with AC' and BC' and the common side CC', allowed the application of the third triangle congruence criterion (Proposition 8, Book I, of Euclid's Elements) to establish the congruence of triangles ACC' and BCC': "If we consider triangles ACC' and BCC', we have AC congruent to CB, CC' as a common side, and AC' congruent to C'B; therefore, by the third congruence criterion, if all sides are known, the two triangles are congruent". The teacher–researcher then asked, "What does this situation tell us about what we still need?" Manuel immediately noted that there is a straight angle at C': "angle AC'C is congruent to BC'C, and together with angles AC'C" and BC'C", they form two angles summing to 180°". The argumentative process in the second class unfolded somewhat differently, perhaps because the abductive inference that had given strong momentum to the reasoning in the first class did not emerge. Nevertheless, students engaged in a similar process, although slightly more guidance from the teacher-researcher was required—particularly in prompting precise naming of geometric elements and clarity in referencing specific angles. The backward construction carried out in the argumentative phase was not subsequently formalized into a collectively negotiated linear deductive structure. Instead, students were immediately provided with a paper-and-pencil cloze activity (Figure 9.1) in which some steps of the proof were pre-inserted and they were asked to complete them by supplying the appropriate theoretical justifications, while in other instances the justification was given and students were required to provide the corresponding mathematical elements.

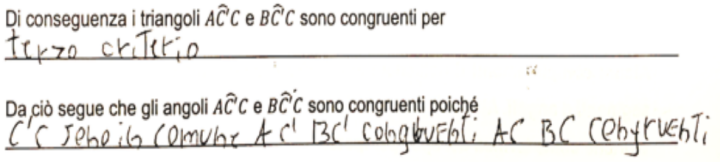
9.6 Data analysis and results

In this section we present the analysis of excerpts from paper-and-pencil worksheets in terms of rationality, comparing them with the argument used by students during the argumentative phase, in order to highlight how the cloze proposed in phase 4 can support the emergence of the cognitive unity of theorems, despite being in one of the cases of structural distance theorised by (Pedemonte, 2007). The excerpts are

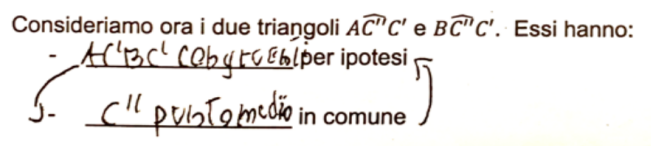
transcribed and analysed into the following tables: Table 9.2 relating to the student Victor; Table 9.3 relating to the student Marcus; Table 9.4 relating to the student Nicole and Table 9.5 relating to the student Manuel.

Table 9.2: Victor's worksheet and analysis

Transcripts from Victor's worksheet



Consequently, the triangles AC'C and BC'C are congruent by *Third criterion*
From this it follows that the angles AC'C and BC'C are congruent since *C' C in common AC' BC' congruent AC BC congruent*

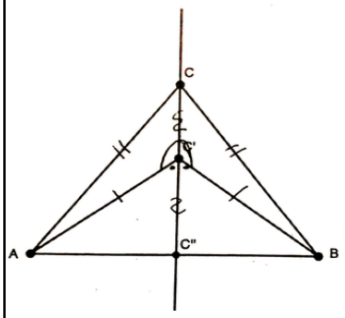


Let us now consider the two triangles AC''C' and BC''C' . They have:
- *AC' BC' congruent* by hypothesis
- *C'' midpoint* in common

Victor reorganises his reasoning in a non-linear manner that is inconsistent with the requirements of the cloze activity, demonstrating a communicative lack, as he leaves implicit the justification regarding the congruence of angles as corresponding elements in congruent triangles. Nevertheless, he shows a cognitive continuity with the arguments used in the argumentation phase described above, visible into claims "C'C in common AC' BC' congruent AC BC congruent" Victor correctly introduces some mathematical elements related to congruent sides, while simultaneously showing an epistemic lack by identifying C" as a midpoint. Subsequently, he uses arrows to reverse the theoretical justifications provided in the worksheet. This behaviour leads to an incorrect description of the mathematical elements involved in the formal proof, but reveals a cognitive continuity with the argumentative process undertaken previously. From a teleological dimension, indeed he remains immersed in the abductive process carried out in the preceding argumentative phase.

Table 9.3: Marcus's worksheet and analysis

Transcripts from Marcus' worksheet



Consideriamo i triangoli $A\hat{C}'C$ e $B\hat{C}'C$ essi:

- hanno C'C in comune;
- AC' e BC' sono congruenti per ipotesi;

IMG_4918.jpg AC e CB sono congruenti per ipotesi

Di conseguenza i triangoli $A\hat{C}'C$ e $B\hat{C}'C$ sono congruenti per il terzo criterio di congruenza

Da ciò segue che gli angoli $A\hat{C}'C$ e $B\hat{C}'C$ sono congruenti poiché i due triangoli sono congruenti con il lato dell'angolo AĈ'C e CĈ'B in comune

Let us consider the triangles $A\hat{C}'C$ and $B\hat{C}'C$ they:

- have CC' in common;
- AC' and BC' are congruent *by hypothesis*
- *AC and BC are congruent* by hypothesis

Consequently, the triangles $A\hat{C}'C$ and $B\hat{C}'C$ are congruent by *third criterion of congruence*

From this it follows that the angles $A\hat{C}'C$ and $B\hat{C}'C$ are congruent since
the two triangles are congruent with the side of the angles AC'C and CC'B in common

Let us now consider the two triangles $A\hat{C}''C'$ e $B\hat{C}''C'$. They have:

- $AC' \cong C'B$ by hypothesis
- $C'C''$ in common
- The angles $A\hat{C}'C''$ and $B\hat{C}'C''$ congruent, since *extensions of the angles AC'C and CC'B and fall on the same point*

Therefore, the two triangles $A\hat{C}''C'$ e $B\hat{C}''C'$ are congruent by *1° criterion of congruence*

Consideriamo ora i due triangoli $A\hat{C}''C'$ e $B\hat{C}''C'$. Essi hanno:

- AC' ≅ C'B per ipotesi
- C'C'' in comune
- Gli angoli $A\hat{C}'C''$ e $B\hat{C}'C''$ congruenti, poiché prolungamenti degli angoli AC'C e CC'B e cadono nel punto in comune

Quindi i due triangoli $A\hat{C}''C'$ e $B\hat{C}''C'$ sono congruenti per il 1° criterio di congruenza

Marcus displays a coherent organisation of the formal proof structure. In the first part of the proof, epistemic dimension attendances are evident, as he introduces the required mathematical elements together with the expected theoretical justifications. However, when explaining why the angles AC'C and BC'C are congruent, a communicative lack emerges: he does not explicitly state that the angles at stake correspond in the two congruent triangles. However, this information is evident from the

statement "with the side of the angles AC'C and BC'C in common", suggesting a mirror image—in the teleological dimension— in relation to the side in common as can also be seen from the marks drawn by Marcus on the figure. In the second part of the proof, the required mathematical elements are again present, as well as the congruence of the sides AC' and BC'. This is consistent with his reasoning in the previous phase, in which, to justify the congruence of triangles AC"C' and BC"C', he focused on the congruence of AC' and BC' based on the given hypothesis. However, a communicative lack emerges: to indicate that the angles AC'C" and BC'C" are supplementary to congruent angles, the student states that these angles are "extensions of the angles AC'C and CC'B and fall on the same point", implicitly referring to the shared vertex.

Table 9.4: Nicole's worksheet and analysis

Transcripts from Nicole' worksheet
Da ciò segue che gli angoli $A\hat{C}'C$ e $B\hat{C}'C$ sono congruenti poiché TUTTI I LATI SONO CONGRUENTI E QUINDI ANCHE GLI ANGOLI Consequently, the triangles AC'C and BC'C are congruent by *all the sides are congruent and then also the angles* - Gli angoli $A\hat{C}'C''$ e $B\hat{C}'C''$ congruenti, poiché HANNO DUE LATI CONGRUENTI The angles AC'C'' and BC'C'' are congruent, since *have two congruent sides*

In this excerpt we note the attendance of the epistemic dimension at the meta-level, as well as of the communicative dimension, because Nicole chooses appropriate systems of representing meta-knowledge by using a deductive chain highlighted by the terms "and then". Nicole's contribution during the argumentative phase—namely, shifting the focus to angles AC'C" and BC'C"—results, in the subsequent written phase of the formal proof, in a contribution of a teleological nature. The statement "they have two congruent sides" is explicitly oriented toward the goal of establishing the congruence of the angles. In this sense, Nicole reuses the same line of reasoning employed in the argumentative phase.

Table 9.5: Manuel's worksheet and analysis

Transcripts from Manuel' worksheet

Let us now consider the two triangles $A\widehat{C''}C'$ e $B\widehat{C''}C'$. They have:

- $AC' \cong BC'$ by hypothesis
- $C''C'$ in common
- The angles $\widehat{AC'}C''$ e $B\widehat{C'}C''$ congruent, since *They respectively form, together with angles AC'C and BC'C, straight angles of 180° and AC'C and BC'C are congruent, in short they are their supplementary [angles].*

Therefore, the two triangles $A\widehat{C''}C'$ e $B\widehat{C''}C'$ are congruent by *1° criterion of congruence*

and thus the segments AC'' e C''B are also congruent

as well as the angles AC''C' e BC''C' which are therefore *right angles* since adjacent and congruent.

Consideriamo ora i due triangoli $A\widehat{C''}C'$ e $B\widehat{C''}C'$. Essi hanno:

- $AC' \cong BC'$ per ipotesi
- $C''C'$ in comune
- Gli angoli $\widehat{AC'}C''$ e $B\widehat{C'}C''$ congruenti, poiché formano rispettivamente con $A\hat{C}C'$ e $B\hat{C}'C$ angoli di 180° ed $A\hat{C}'C$ e $B\hat{C}'C$ sono congruenti, in sostanza sono i supplementari

Quindi i due triangoli $A\widehat{C''}C'$ e $B\widehat{C''}C'$ sono congruenti per il 1° criterio di congruenza

e dunque sono congruenti anche i segmenti AC'' e $C''B$

e gli angoli $\hat{AC''}C'$ e $\hat{BC''}C'$ che pertanto risultano retti poiché adiacenti e congruenti

This excerpt shows an attendance of epistemic nature. Even if Manuel does not recall explicitly the Common Notion 3 of Euclid's Element, he puts forward his theoretical justification taking into account the mathematical elements involved in his reasoning. The logical implication still remains implicit, but the main sense of claim is clarified, at communicative level, by the expression "in short they are their supplementary". Manuel reuse the same argument claimed into argumentative phase, showing a cognitive continuity with the previous phase.

9.7 Discussion

With respect RQ2, we analysed and interpreted selected excerpts from students' paper-and-pencil worksheets through the lens of Habermas' rationality, specifically using the Rational Mathematical Template of proof (Boero & Azrou, 2023; Boero & Turiano, 2022, 2023), searching and interpreting the attendances or the lacks of rationality components (epistemic, teleological and communicative) into the cloze text. We then compared these excerpts with the arguments produced by students during the

argumentative phase. The analysis of four students' worksheets and the comparison with their argumentative statements suggest to us that a cognitive continuity between the two phases can be possible, fostering by cloze activity in formal proof writing. In Victor's case, the analysis revealed epistemic and communicative lacks in the written formal proof, as the student remained grounded, at teleological level, in the abductive reasoning which he himself developed in the preceding argumentative phase. Instead, in Marcus', Nicole's and Manuel's cases, the analysis highlights a coherent organisation of the formal proof structure, with the emergence of both epistemic and teleological attendances. Nevertheless also their written texts are consistent with their previous argumentative reasoning. Indeed the cloze task supplies the formal transformation from abductive reasoning proposed into collective argumentation by Victor, to deductive reasoning required, providing the students with some necessary logical inferences in advance. This allows us to observe how they reorganised their arguments, bringing to light elements of continuity between the argumentative phase and the formal proof and to highlight how the cloze proposed in the proof phase could foster the emergence of cognitive unity of theorems, even in a situation that corresponds to one of the structural distance cases theorised by (Pedemonte, 2007).

With respect RQ1, the findings suggest that the cloze, in the effort to systematise and reorganise the whole process, functions as a tool that supports the delicate transition from argumentation to proof, acting as a mediator that makes the cognitive unity of the theorems transparent and bridges the structural gap (Duval, 1991). In relation to traditional use of the cloze as a moment of awareness and reflection to stabilizing and transferring at the meta-level a shared proof in an organized form, this study shows a new perspective for cloze activity. In the analysed cases always emerged a recalling, into cloze text, of the statements claimed in the argumentative phase, almost as if each student were anchored to their own thinking process. Such use of the cloze should, however, remain limited, as there is a risk that students may come to view it as a necessary tool for accessing the formal proof and consequently struggle to detach themselves from it. Its use should therefore never undermine the development of students' autonomy in the formal construction of proofs. Our contribution, situated within the line of research on the construct of cognitive unity of theorems, may support a revisiting and further elaboration of this construct. It highlights aspects whose awareness could assist students in gaining insight into their own reasoning processes and the proof schemes they employ, even when a structural gap exists between the method adopted during the argumentative phase and the deductive structure of the proof as a formal product.

Chapter 10

Euclid Experiment - Chats' analysis

This chapter presents the qualitative analysis of the synchronous discursive interactions that took place in the Actors and Outlookers chats, collected during the experimentation. This interactions are sometimes supported by drawings and pictures that students shared on a Padlet and explanatory comments written as captions to their pictures. In the following sections we analysis both the interactions on chats and the comment on padlet. In this chapter, we present the analysis of only one of the two classes involved in the experiment, as the students in this class completed all the designed episodes.

As described in detail in Chapter 8 on methodological choices (8.3), the analysis is articulated as a Categorical-Content Analysis (Lieblich et al., 1998) in which some operational indicators that characterise categories are theory-informed and some others emerged by data. In the next sections we present the line by line process of analysis (*micro-analysis*) in which words, sentences and utterances are highlighted, separated into categories and described by operational indicators, at the aim to making transparent the whole analysis process.

This analysis is sometimes preceded and supported by a global description, aimed at reconstructing the overview of the teaching episode under study. This description emerges by the memos collected during the experimentation by the author, in the role of teacher-researcher, as well as by the systematic observations carried out by both the author and the classroom teachers through dedicated observation grids.

10.1 Episode 1.1

In this episode, students are required to solve the first task, previously introduced in Section 8.1.2, by assuming the roles of their assigned characters and interacting via chat, each other and with Euclid portrayed by the teacher-researcher.

In these initial excerpts, the absence of a fully developed proof understood as theoretical validation is still evident, as students do not explicitly refer to theorems and axioms included in Euclid's Theory Bank. Nevertheless, the arguments produced by the students—also prompted by the interventions of the teacher-researcher in the role of Guru as a rational mediator—can be considered acceptable, since their epistemic value is justified and supported by the semantic value of the arguments themselves. The students only implicitly refer to the underlying Euclidean theory (e.g., "since the points of the plane are infinite"), without yet making such references explicit; for instance, Euclid's Postulate 3, stating that "a circle can be drawn with any center and radius," is not explicitly invoked.

10.1.1 Excerpts from Actors'chat

Scene 1- Actors

1	[10:02]	Kyle-Boss:	Right then, everyone, let's get to work! Jot down any ideas you have
2	[10:02]	Tommy-Blogger:	Let's start thinking
5	[10:03]	Kyle-Boss:	So let us first consider the Euclid's request
6	[10:04]	Ellie2-Promoter:	Let's start by considering a point p
8	[10:04]	Tommy-Blogger:	We need to find how many circles pass through p
9	[10:05]	Tommy-Blogger:	Let's start by considering a point p and drawing some circles passing through it
11	[10:05]	Ellie2-Promoter:	We can draw many of them
12	[10:05]	Kyle-Boss:	If you have any ideas from a graphical point of view send us your drawings that might be useful
13	[10:06]	Tommy-Blogger:	But passing through p?
15	[10:08]	Euclid-Guru:	How can we obtain all the circumferences passing through point P?
17	[10:10]	Ellie2-Promoter:	P must be a point on the circle, which is common to all circles passing through p
21	[10:15]	Shona-Pest:	But couldn't the circles passing through a point P be infinite?
25	[10:16]	Euclid-Guru:	Shona, what do you mean by infinite? How do you find them?
27	[10:17]	Tommy-Blogger:	I also believe they are infinite, but proving it with a [drawing] is almost impossible
30	[10:18]	Kyle-Boss:	How would you do it?
31	[10:18]	Kyle-Boss:	Come on, guys, we will find a way

In the utterance "Let's get to work" we observe Kyle-Boss acting as an activator of

the different roles [1] (BO-KL), whereas the statement "let us first consider the request" [5] shows Kyle-Boss, in this case, acting as a group coordinator in order to direct the process (BO-MK).

In statement [6], we identify a teleological dimension, as Ellie2, in the role of Promoter, adopts a strategic, goal-oriented approach (PRO-KL) and she is followed by Tommy-Blogger, who in [9] shifts towards the teleological dimension, particularly in his request to draw some circles passing through point P (PRO-KL). In [11], Ellie2-Promoter communicates a personal discovery to the group, namely the possibility of drawing many circles passing through P (BLO-MK).

In [12], Kyle-Boss directs the process ("If you have any ideas...") while also providing operational guide to the group ("send your drawings") (BO-MK). Tommy-Blogger then expresses a need for clarification [13] as a communicative requirement addressed to himself, operating at that moment at a meta-knowledge level of the communicative dimension (BLO-MK).

Euclid-Guru's intervention in [15] prompts the teleological dimension, which emerges in particular through the question "How can we obtain...?" (EU-TD). This allows Ellie2-Promoter in [17] to shift towards the communicative aspect, especially in self-communication (BLO-MK), as evidenced by the verbal expression "P must be a point," which is necessary to foster idea generation.

In [21], Shona-Pest puts forward a conjectural proposal, offering a constructive contribution that enables the process to move forward (PRO-IP). This statement leads Shona-Pest to shift towards the teleological dimension. Building on Pest's input, Euclid-Guru in [25] provides a stimulus that activates both the epistemic dimension ("what do you mean by infinite?") (EU-ED) and the teleological dimension ("how do you find them?") (EU-TD).

Scene 1 concludes with Kyle-Boss who acts with a questioning activity, stimulating the teleological dimension [30] (EU-TD; OU-IP) as we note in the utterance "How would you do it?", while also attending to affective aspects [31] by encouraging the other group members ("come on, guys") and positioning himself as part of the ongoing process ("we will find a way") (BO-IP).

Scene 2 - Actors

1 [8:50] Kyle-Boss: Right, everyone! Let's pick up where we left off.
2 [8:51] Tommy-Blogger: We were working out how many circles pass through a point P
3 [8:51] Tommy-Blogger: Any ideas?

5 [8:52] Kyle-Boss: We had already supposed that there were an infinite number and that it was impossible to prove this graphically
6 [8:52] Euclid-Guru: Why do you say infinite?
7 [8:52] Euclid-Guru: Can you justify why?
8 [8:54] Ellie2-Promoter: Because we can draw infinitely many circles passing through P, all with different centers.
9 [8:55] Kyle-Boss: What do you notice from the drawings?
10 [8:55] Euclid-Guru: Interesting, Ellie! Do you agree with Ellie?
14 [8:56] Shona-Pest: From the drawings, we can see that all the circles we draw passing through a point P can also have different diameters
16 [8:56] Euclid-Guru: How could you justify what ellie2 said?
17 [8:56] Tommy-Blogger: We need to find a way to prove this not in a graphic way
19 [8:57] Kyle-Boss: We know that there are infinite points in the plane
20 [8:58] Kyle-Boss: Come on, guys! Don't worry, there has to be a way.
22 [8:59] Ellie2-Promoter: We can say that since there are infinite points on a plane, there will be infinite centers of circles passing through p.
23 [9:00] Kyle-Boss: Right, what do you think?
24 [9:00] Tommy-Blogger: I agree with ellie
28 [9:02] Shona-Pest: But how do we prove it?
31 [9:03] Euclid-Guru: You had already found a good theoretical justification
32 [9:03] Euclid-Guru: Agree on what to write
37 [9:05] Tommy-Blogger: Let's see how to write it
39 [9:05] Kyle-Boss: And if we take the distance from the center to point P as a radius, we will obtain a circle, one for each center taken: infinite points on the plane as centers and therefore there would be infinite circles
42 [9:06] Tommy-Blogger: Let's try to write it better, but we're on the right track
43 [9:07] Kyle-Boss: Well, Tommy, this is definitely your job, you're really good with words
44 [9:07] Tommy-Blogger: It's a team effort
50 [9:09] Ellie2-Promoter: We could say that: infinite circles pass through a point p because, since there are infinite points on the plane, it is possible to draw infinite circles, all with different centers and different radii

In [1] Kyle-Boss stimulate the activation of all cognitive roles and direct the process, as the sentence "Let's pick up where we left off" shows (BO-KL).

In [2], Tommy-Blogger addresses an internal communication within the group's restricted community, with the aim of recalling what the group had elaborated up to that moment and subsequently assumes a role of mediator of communication that is usually played by the teacher-researcher in the figure of Euclid (BLO-MK). Indeed, in [3], through the question "Any ideas?", Tommy-Blogger stimulates the teleological dimension aimed at activating the strategic function (EU-TD; OU-IP). This, in turn, leads Kyle-Boss to provide a recap of the information derived from the group [5], acting on the communicative dimension, as evidenced by the expression "We had already supposed ..." (BLO-MK).

In [6] and [7], Euclid-Guru instead stimulates the epistemic component through rational questioning aimed at stimulating the search for a theoretical justification of the proposed hypothesis (EU-ED). In [8], Ellie2-Promoter, in an effort to respond to Euclid-Guru's request, therefore provides a constructive contribution that allows the process to move forward. This is reflected in the sentence "all [circles] with different centers," which reveals an initial attempt toward the formulation of a justification (PRO-IP).

In Kyle-Boss's request "What do you notice from the drawings?" [9], an action of activation and direction of attention is present, : he indeed tries to focus the group members' attention on the drawings uploaded on the Padlet (BO-IP). Euclid-Guru, taking up Kyle-Boss's proposal, thus supports the emergence of the epistemic component through the question "Do you agree. . . ?" [10] (EU-ED) and the question "How could you justify. . . ?" [16] (EU-ED).

In [14] Shona-Pest establishes correct relationship between the mathematical objects and concepts involved, by stating that "all the circles [. . .] can also have different diameters." In doing so, she performs a shift at the knowledge level of the Promoter role (PRO-KL).

In the Tommy-Blogger's sentence [17] we recognise the presence of all three dimensions of rationality. In the expression "We need to find a way", a teleological dimension emerges, highlighted by the terms "need" and "find" as indicating the need to explore a strategy (PRO-KL). At the same time, the expression "to prove this" brings out the epistemic dimension, oriented towards the need to search for a proof (that still takes the form of a theoretical justification) (PE-IP). Finally, in the expression "not in a graphic way", we identify a communicative dimension at the meta-knowledge level, related a type of representation to be used in order to communicate the reasoning to the mathematical community (BLO-MK).

In [19], two rationality dimensions are present, both teleological and epistemic. Indeed, from a teleological point of view, Kyle-Boss draws on memory in the act of

generating ideas (a practice typical of the cognitive role of the Promoter) (PRO-KL). This emerges from the terms "We know that". However, this recourse to memory relies on mathematical knowledge that is consistent with the reference theory, allowing an epistemic dimension to emerge (PE-KL). Although there is no explicit reference to the theory, at this stage there are implicit references to mathematical knowledge that provide coherence to the reasoning, as in "there are infinite points in the plane". In [20], Kyle-Boss instead attends to emotional aspects that may arise during moments of impasse (BO-IP).

In [22], Ellie2-Promoter makes a significant contribution to the advancement of the process by providing support of a conceptual nature and drawing on mathematical knowledge (PRO-IP), while performing a logical inference. Indeed, the statement "We can say that *since there are* infinite points on a plane, *there will be* infinite centers of circles passing through p" highlights, although still implicitly, a conditional structure with a clear distinction between premise and consequence (BLO-IP).

After the group's evident agreement with the statement put forward by Ellie2, Shona-Pest in [28] expresses the need to validate this statement in a way that can be recognized as a proof, both by the small classroom group and by the wider mathematical community. This is evident from the words "But *how* do we prove it?" (PE-KL).

Euclid in [31] and [32] thus intervenes in support of the group, endorsing the soundness of the justification that has been developed and encouraging agreement on a shared written formulation. In this case, Euclid's intervention does not take the form of questioning; nevertheless, it fosters a communicative dimension oriented toward mutual understanding, as it acts on the search for shared agreement within the small group (EU-CD).

In [39], we recognize the presence of a first theoretical reference, although still in an implicit form. In the expressions "if we take the distance from the center to point P as a radius" and "obtain a circle, one for each center taken", there is a reference to Euclid's Third Postulate (*that a circle can be drawn with any center and radius*), although this reference is not yet accompanied by full theoretical awareness (PE-KL). In [39] Kyle-Boss also uses, even if implicitly, a conditional structure in which the premise and the consequence are evident, as emerges from the expressions"if ... we will obtain ... and therefore" (BLO-IP).

Subsequently, we recognize a communicative dimension in the following interventions by Tommy-Blogger in [42] and [44]. In the sentence "Let's try to write it better", we identify a request for clear communication that aims to align as closely as possible with the standards required by the mathematical community, while in the expression "It's a team effort" we can observe the need for negotiation involving the

entire group (BLO-MK).

The episode concludes with an intervention by Ellie2-Promoter, who in [50] attempts to formulate a theoretical justification that synthesizes all the reasoning developed by the group. In the expression "We could say that", we can observe a communicative dimension at the level of Blogger's meta-knowledge (BLO-MK), oriented toward achieving effective communication with others and toward the search for a theoretical justification that is accepted and validated by the small working-group community. In particular, in the excerpt "since there are infinite points on the plane, it is possible to draw infinite circles", we note a first logical inference (BLO-IP), while in the excerpt "all with different centers and different radii" we observe the implicit emergence of theoretical elements (PE-KL).

10.1.2 Excerpts from Outlookers'chat

Scene 1 - Outlookers

In this subsection we present some excerpts by outlookers'chat, in which emerge the interventions of the two students, named Estella and Giorgio, in the roles Ellie-Promoter-GO1 and Shona2-Pest-GO1 respectively. In questi estratti del gruppo Osservatori si nota talvolta un linguaggio più libero e colloquiale, talvolta con espressioni gergali.

1	[10:02]	Ellie2-Promoter-GO1:	Eh, but in my opinion they are traced.
2	[10:04]	Ellie2-Promoter-GO1:	Ellie does not speak in the other group
3	[10:05]	Ellie-Promoter-GO1:	Guys, let's take any point and start drawing circles passing through P, even of different sizes
4	[10:05]	Ellie2-Promoter-GO1:	Let's do it too
5	[10:07]	Tommy-Blogger-GO2:	How are you getting on?
6	[10:07]	Ellie2-Promoter-GO1:	In my opinion, they are infinite.
7	[10:07]	Ellie2-Promoter-GO1:	I drew and made lots of them
8	[10:07]	Kyle-Boss-GO1:	If you have any drawings, upload them to the board
9	[10:08]	Ellie-Promoter-GO1:	I agree, but I think we need to verify in some way. We can't reach any conclusions without first being sure.
10	[10:08]	Shona2-Pest-GO1:	I also think they are infinite, the circles can be of any size and in any position, as long as they come into contact with point P.
11	[10:09]	Ellie2-Promoter-GO1:	But sorry, how do you prove it?
12	[10:09]	Kyle-Boss-GO2:	Guys, they are infinite

14	[10:10]	Ellie2-Promoter-GO1:	But how do you actually prove it, like using formulas and that kind of stuff?
15	[10:11]	Tommy-Blogger-GO1:	Drawing
16	[10:11]	Tommy-Blogger-GO2:	If by drawing them infinitely many come out, that's already a proof for me
17	[10:12]	Shona2-Pest-GO1:	You will never be able to fit an infinite number of them on a sheet of paper, but the more you draw, the more you realise that more and more can fit.
18	[10:13]	Shona2-Pest-GO1:	There is no exact limit, as a circumference can be of multiple sizes
19	[10:13]	Ellie-Promoter-GO1:	Guys, I've put something on the board, but I can keep drawing them indefinitely
20	[10:15]	Ellie2-Promoter-GO1:	The Ellie over there [in the other group] don't talk much
21	[10:16]	Kyle-Boss-GO1:	Kyle, on the other hand, is the one who writes the most and tries to encourage the others, just like the character said
22	[10:17]	Ellie2-Promoter-GO1:	You draw them and, okay, you see that there are many.
23	[10:17]	Ellie2-Promoter-GO1:	But how do you prove it theoretically?
24	[10:18]	Ellie2-Promoter-GO1:	Isn't this a kind of postulate?
25	[10:18]	Ellie-Promoter-GO1:	Guys, can we consult the theory bank to see if there's anything?
26	[10:18]	Shona2-Pest-GO1:	Through a point P, infinitely many circles pass, because in my opinion any point in the plane can be taken as the center of a circle,
27	[10:18]	Shona2-Pest-GO1:	and as we know, the plane is infinite

In [2] it emerges that Ellie2-Promoter-GO1 is observing the corresponding role within the actors' group, taking note of the behavior displayed (OU-KL).

In the utterance of Ellie-Promoter-GO1 [3], a teleological dimension emerges, highlighted in particular by the expressions "let's take any point" and "start drawing", in which goal-oriented actions can be identified (PRO-KL).

In [4], Ellie2-Promoter-GO1 stimulates the group with respect to the proposed mathematical content (BO-KL), while in [6] and [7] she brings out a teleological dimension through the activation of an inductive strategy (PRO-KL).

In [8], Kyle-Boss-GO1 directs the process (BO-MK), whereas in [9] Ellie-Promoter-GO1 conveys the need for certainty, as emerged from the phrases "we need to verify in some way" and "without first being sure" (PRO-KL).

Shona2-Pest-GO1 in [10] shows agreement (OU-MK) with the conjecture advanced by Ellie2-Promoter, as can be seen in the statement "I also think they are infinite" providing, in support of her claim, empirical methods of an inductive nature (PRO-KL), as evidenced in the utterance "can be of any size and in any position", with a movement towards generalization, shown by term "any".

In [11], we identify an epistemic dimension, expressed as a request to verify the truth conditions of the statement made (PE-KL), which Ellie2-Promoter-GO1 further highlights in [14], as shown in the questions "how do you prove it?" and "how do you actually prove it?" However, there is a clear misconception regarding the epistemological nature of proof, as suggested by the phrase "like using formulas and that kind of stuff," which thus indicates an epistemic shortcoming at the level of meta-knowledge about proof (lack PE-MK).

In [15] and [16], the answers of Tommy-Blogger-GO1 and Tommy-Blogger-GO2 highlight a teleological dimension in the choice of empirical methods (PRO-KL), as indicated by the verb "Drawing." However, once again, they reveal an epistemic lack at the level of meta-knowledge of proof (LACK PE-MK), as can be observed in the expression "that's already a proof for me."

In [17], Shona2-Pest-GO1 points out the limitations of the empirical method adopted, performing a check on the coherence of the selected procedures and on the correctness of the representations (PE-KL), as evidenced by the statement "You will never be able to fit an infinite number of them on a sheet of paper". At the same time, however, she also relies on an inductive reasoning process, as suggested by the phrase "the more you draw, the more you realise that more and more can fit" (PRO-KL).

In [19], Ellie-Promoter-GO1 informs the group that she has shared a drawing on the board, which is reproduced here in Figure 10.1, highlighting a teleological dimension as goal-oriented action (PRO-KL), emerging from the statement "I can keep drawing them indefinitely". In particular, in the drawing on Padlet, Ellie-Promoter-GO1 includes a comment as a caption, which reads: *"Even though there are only 4, if I keep drawing I can do it indefinitely"*. This contribution highlights a *constructive argumentation* that supports the formulation of the conjecture and describes the possibility of a goal-oriented action that can continue indefinitely (PRO-KL). In her comment, a conditional structure is also evident, highlighted by "if I keep drawing I can do it" (BLO-IP).

In [20], Ellie2-Promoter-GO1's attention is directed toward the role played by her same type of role in the actors' group, making an observation about behavior (OU-KL), as evidenced by the statement "The Ellie don't talk much", while in [21], Kyle-Boss-GO1 reflects on the way his counterpart is interpreting the role and adhering to the character's traits (OU-MK), as shown in the expressions "[Kyle] tries to encourage the others" and

"just like the character said."

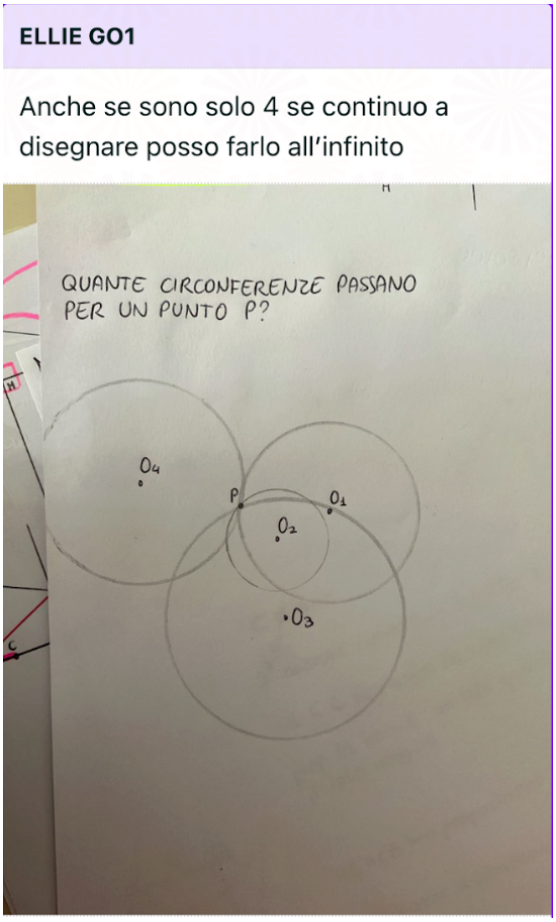


Figure 10.1: Ellie-go1's picture on the board

In [22] and [23], although Ellie2-Promoter-GO1 shows agreement with the conjecture, as indicated by the statement "okay, you see that there are many" (OU-MK), she highlights the limitation of the empirical method, expressing a need for causality, as evidenced by the question "But how do you prove it theoretically?" (PE-IP).

In [24], she then draws on epistemic memory (PRO-KL), as evidenced by the expression "Isn't this a kind of postulate?"

In [25], Ellie-Promoter-GO1 brings out the need to access the Dynamic Euclidean Bank of Theory, requesting clarification regarding the possibility for the Outlookers to use a tool available to the Actors, highlighting the need of selecting tools that serve the aim (PRO-KL).

In [26], an epistemic component emerges as validation of a statement, as Shona2-Pest-GO1 provides a justification for the conjecture put forward, as indicated by the term "because" (PE-KL). However, this justification relies on teleological arguments in which goal-oriented actions appear (PRO-KL), as can be seen in the phrase "any point in the plane *can be taken* as the center of a circle".

Finally, in [27], Shona2-Pest-GO1 refers to statements accepted by the classroom community and therefore not questioned by anyone, as evidenced by the utterance "and as we know" (PE-KL).

Scene 2 - Outlookers

1	[8:50]	Tommy-Blogger-GO1:	Infinitely many circles pass through a point P because infinitely many centers can be drawn
2	[8:52]	Shona2-Pest-GO1:	The plane is infinite, and any point in the plane can be the center of a circle
3	[8:53]	Ellie-Promoter-GO1:	Euclid is right, it needs to be proved
10	[9:03]	Shona2-Pest-GO1:	The problem is precisely to prove that the circles passing through a point P are infinite

In Scene 2, some themes already observed in the previous scene reappear. In particular, in [1], both an epistemic and a teleological component emerge, as Tommy-Blogger-GO1 provides a justification for the conjecture, visible in "because" (PE-KL), but does so using empirical methods as goal-oriented actions, as evidenced by the phrase "infinitely many centers can be drawn" (PRO-KL).

In [2], Shona2-Pest-GO1 states results that are accepted and shared by the community, though implicitly, as in "The plane is infinite" (PE-KL), whereas in [3] and [10], Ellie-Promoter-GO1 and Shona2-Pest-GO1 show an epistemic need for validity of the statement produced (PE-IP), as indicated by "it needs to be proved" and "The problem is precisely to prove that".

10.2 Episode 1.2

In this Episode, the discursive interactions that occurred in the Actors and Outlookers chats and developed around Task 1.2 are analyzed. This task was proposed to the students in order to foster a focus on the representation of the center and, consequently, on the possibility of using a compass for the construction of circles in the subsequent Episodes.

10.2.1 Excerpts from Actors'chat

1	[09:17]	Kyle-Boss:	Right, let's get started
2	[09:17]	Ellie2-Promoter:	Here we are with a new problem
3	[09:18]	Tommy-Blogger:	Yes, a new problem that asks us to find the difference between a circle constructed with a compass and one constructed with a glass
4	[09:18]	Kyle-Boss:	Certainly, a compass allows us to create circles of different sizes compared to a glass, unless we use glasses of different sizes
5	[09:19]	Ellie2-Promoter:	First of all, we can say that in a circle drawn with a glass, we do not know the center, while in a circle drawn with a compass, we do
6	[09:19]	Kyle-Boss:	What do you think about what I have said?
7	[09:19]	Tommy-Blogger:	However, with a compass, you have to point.
8	[09:19]	Tommy-Blogger:	You have to point at the center.
9	[09:19]	Kyle-Boss:	Right
10	[09:20]	Tommy-Blogger:	With a glass, we have to draw a circumference around a hypothetical center
11	[09:20]	Kyle-Boss:	I think Ellie2's answer is the right one
12	[09:21]	Ellie2-Promoter:	So, with the glass, we draw circles that are all the same and "without" a centre, while with the compass, we know the center and can vary the radius
13	[09:21]	Kyle-Boss:	I agree
14	[09:21]	Kyle-Boss:	What do the others think?
16	[09:22]	Tommy-Blogger:	I agree
17	[09:22]	Ellie-Promoter:	Guys, we're very good at this
19	[09:23]	Euclid-Guru:	Ellie, I think so too! You're very good!

In [1], Kyle-Boss activates the roles of the different characters involved (BO-KL), while in [3] Tommy-Blogger operates at the communicative level by recalling for the group the task proposed in the comics (BLO-KL).

In [5], we identify a Promoter action that provides a constructive contribution to the advancement of the process (PRO-IP). In particular, in the expressions "drawn with a glass" and "drawn with a compass" we identify goal-oriented actions (teleological dimension) (PRO-KL). This subsequently enable Ellie2 to shift toward the epistemic component, which emerges in the excerpt "we do not know the center". This statement highlights the control over a conceptual property that is characteristic of the Pest role (PE-KL).

In [8] and [10], particularly in the excerpts "with a compass, you have to point [...] at the center" and "with a glass, we have to draw [...] around", we recognize both

a teleological component (in the intentionally chosen action) (PRO-KL) and an internal communicative component, understood as a form of self-clarification (BLO-MK). The use of the term "hypothetical" instead highlights an awareness of an absence, namely of a mathematical object that exists but is not visible, such as the center of a circle (PE-MK).

This also emerges in [12], where the term "without," presented in the written text in quotation marks, refers to the awareness of an object that one knows exists but cannot see (PE-MK).

Ellie2-Promoter provides another constructive contribution, as evidenced by the excerpts "that are all the same" and "can vary the radius," highlighting further elements of the two different types of representation (PRO-IP).

The episode concludes with Euclid-Guru supporting the group from an emotional perspective, as shown in [19] (BO-IP).

10.2.2 Excerpts from Outlookers'chat

From this episode onward, we choose no longer to indicate the specific role played by the Outlookers, but only the name and the group (GO1 or GO2) to which they belong, in order to avoid overburdening the text.

3	[09:18] Ellie2-GO1:	In my opinion, the difference is that with a compass we have the centre, whereas with a glass we do not
7	[09:19] Shona2-GO1:	Well, with a glass we can draw a circle of only one size and without a center, whereas with a compass you have a reference point and can produce circles of different sizes
8	[09:19] Ellie-GO1:	In addition, with a compass we can obtain different radii, whereas with a glass the radius is always the same
9	[09:20] Tommy-GO2:	Okay, Ellie2 made a correct statement
12	[09:21] Ellie2-GO1:	Ellie2 in the other group developed the same line of reasoning as mine
20	[09:23] Ellie-GO1:	Not only Ellie2, the group over there is working well; I think they may have already found the solution.

In [3], an epistemic dimension emerges concerning the control of the conceptual property of the center of the circle (PE-KL), taken up by Shona2-GO1 in [7], anticipating an argument that almost simultaneously will appear in the actors' group: the awareness that circle's representation with a glass deprives us of knowledge of the center, almost as if there is an absence (PE-MK), as emerging by the term "without".

In [8], Ellie-GO1 makes a constructive contribution that allow the process to move forward (PRO-IP), as can be seen from the sentence "with a compass we can obtain different radii", anticipating a topic that will later be introduced in the actors group.

Attention therefore shifts to observing the roles played in the other group, as in [9], [12] and [20] (OU-KL).

10.3 Episode 1.3

In this Episode, a conjecturing task is proposed to the students. In particular, Euclid asks them to determine how many circles pass through two given points, A and B, and where their centers are located. During this episode, the students arrive at the formulation of a conjecture, also supported by a justification that appears acceptable, but without yet making explicit reference to the theory The argumentation collectively produced by the students therefore constitutes a *constructive argumentation* (Pedemonte, 2008), aimed at formulating a conjecture.

10.3.1 Excerpts from Actors'chat

Scene 1 - Actors

During the episode, two issues related to epistemology emerge. In particular in this scene emerge one concerning the mathematical content, as some students express doubts about the alignment of two or more points, thereby revealing an epistemic gap in their understanding of the mathematical properties involved.

1	[8:58]	Kyle-Boss:	Alright everyone, let's begin!
2	[8:58]	Tommy-Blogger:	Good morning, let's summarise what Euclid has said
4	[9:00]	Tommy-Blogger:	We have to figure out how many circles pass through two points
5	[9:00]	Ellie2-Promoter:	And where their centers are
6	[9:01]	Ellie2-Promoter:	Shall we try to do some practical proof?
8	[9:06]	Tommy-Blogger:	Guys, we can draw more than one of them graphically
9	[9:06]	Ellie2-Promoter:	Kyle, we were wrong;there isn't just one circle that passes through two points
10	[9:07]	Kyle-Boss:	Are you saying that infinitely many pass through them?
11	[9:07]	Ellie-Promoter:	Through two non-collinear points, there passes only one
12	[9:08]	Kyle-Boss:	I would say that for two collinear points there is only one circle, but for two non-collinear points there are

		infinitely many
14 [9:10]	Euclid-Guru:	Kyle, could you explain to me more clearly when two points are collinear and when they are not?
18 [9:12]	Euclid-Guru:	Can you give me an example of two points that are not collinear?
21 [9:16]	Ellie2-Promoter:	For two collinear points, there is only one circle that passes through them, because there is only one possible center, the two points being the diameter
22 [9:17]	Euclid-Guru:	Are you sure that two points that are not collinear exist?
23 [9:17]	Euclid-Guru:	Draw them
25 [9:18]	Ellie2-Promoter:	If you try to draw them, they don't exist, guys
29 [9:24]	Tommy-Blogger:	Since two points are always collinear infinitely many circles pass through them
30 [9:25]	Shona-Pest:	How can you prove it?

In [1], Kyle-Boss acts on the activation of the different roles, giving input to the group work (BO-KL). In [2] and [4], Tommy-Blogger refers back to the proposed task to help the group understand the problem, as can be seen from the verbal expression "let's summarise" (BLO-KL).

In [6], Ellie2-Promoter prompts strategic choices in order to encourage exploration of the problem. These choices take the form of empirical methods, as emerges from the phrase "some practical proof" (PRO-KL).

In Tommy-Blogger's intervention in [8], a dialectic of the three dimensions of rationality emerges (BO-MK), as Tommy-Blogger communicates to the group the result of the exploration carried out graphically, describing the actions performed (BLO-MK), this communication highlights the choice of the exploratory strategy, as shown by the term "Graphically" (PROM-KL), and the simultaneous control action on the emerging conjecture, as can be noted from the expression "we can draw more than one" (PE-KL).

In [9], Ellie2-Promoter performs a check on reasoning and the correctness of representations (PE-KL), as can be seen from the utterances 'we were wrong' and "there isn't just one ... that pass". This intervention suggests to Kyle-Boss a possible conjecture [10] as a constructive contribution that allows the process to move forward (PRO-IP)

In Ellie-Promoter's intervention [11], there is a noticeable lack of both communicative and epistemic types, highlighting an incorrect use of language (LACK BLO-KL) and an inconsistency regarding the mathematical concepts involved (LACK PE-KL), as evidenced by the phrase "Through two non-collinear points".

In [12], Kyle-Boss, following the error made by Ellie, still highlights a

communicative lack (LACK BLO-KL), which prompts Euclid-Guru to encourage reflection. In [14], in fact, Euclid asks "could you explain to me more clearly", where "clearly" stimulates the communicative component (EU-CD), while the term "explain" refers to an epistemic need (EU-ED). Moreover, in [18], Euclid-Guru also addresses the teleological component by asking "Can you give me an example...?"

Finally, in [21], Ellie2-Promoter uses the term "collinear" with a lack only of the communicative type (LACK BLO-KL), since it implicitly assumes that the alignment of the two points A and B is relative to a third point, namely the center of the circle having segment AB as its diameter. This is evident from the verbal expression "the two points being the diameter".

In [22] and [23], Euclid-Guru prompts the epistemic component by asking "Are you sure that two points ... exist?" (EU-ED), and the teleological component, as we can recognise in "Draw them" (EU-TD). Ellie2-Promoter thus performs a check on the possibility of representation, invoking the goal-oriented action that enables such verification (PE-KL; PRO-KL), as illustrated in the excerpt: "If you try to draw them, they don't exist".

In [29], Tommy-Blogger highlights a structure in which the premise and consequence are clearly evident, thus evoking a deductive-type structure (BLO-IP), as we observe in the excerpt "Since ... many circles pass," with the "then" left entirely implicit. This prompts Shona-Pest in [30] to question the validity of Tommy-Blogger's statement, showing a need for causality (PE-IP), but also engaging in a form of questioning that seeks the epistemic component (EU-ED, OU-IP).

Scene 2 - Actors

During this scene, technical problems prevent students from uploading their images to the shared digital boards. This limitation was thus transformed into an opportunity, because I asked the students to comment on the drawings they were unable to upload to the platform, by describing in detail, in the chat, everything they had drawn on paper. In other words, I asked them to make their drawings visible through verbal description, also as an opportunity to refine their mathematical language. While sharing photos certainly accelerates communication, it relies on immediate visualization, which can sometimes lessen the responsibility to provide a clear verbal account, leading students to offer hasty explanations. The technical issue that occurred on this occasion enabled students to make a greater effort to describe their drawings clearly.

1	[9:31]	Euclid-Guru:	Describe in detail the drawings you are making
3	[9:37]	Ellie2-Promoter:	I tried making some drawings, and this leads me to think that there are infinitely many, since there are really so many. We could explain this by saying that it is possible because we can consider A and B as the diameter or as chords of the circle, more or less distant from each other, varying the radius of the circle and then its center.

In [1], Euclid-Guru prompts a detailed description of the drawings made on paper, thereby stimulating the communicative component (EU-CD) as a reflective action.

In [3], Ellie2-Promoter describes the drawing produced, highlighting in the description the three components of rationality. In particular, the phrase "I tried making some drawings" emphasizes the strategy followed as a goal-oriented action (PRO-KL). In the expression "since there are really so many", we can note a teleological component, as a need for certainty using empirical methods (PRO-KL), which leads to the formulation of the conjecture ("and this leads me to think that"). In the expression "We could explain this by saying", a dialectic emerges between the epistemic dimension (PRO-KL) and the communicative dimension, understood as an explanation to get clarity for oneself (BLO-MK). Ellie2-Promoter then makes an attempt to validate the produced conjecture, with the emergence of an epistemic dimension related to the reference theory, although still in implicit form, as seen in the utterance "because we can consider A and B as the diameter or as chords of the circle" (PE-KL). Finally, in the expression "varying the radius of the circle and then its center" a dynamic visualization of the drawing on paper emerges (BLO-KL), with the use of verbal representations ("then") concerning implications (BLO-IP)

Scene 3- Actors

The other epistemological aspect emerged in this episode, instead, is more closely related to the process. In this scene students brings to light the need to understand the meaning of the term to conjecture, thus pointing to an epistemic difficulty related to method rather than to content.

1	[9:42]	Euclid-Guru:	Try to conjecture: where could their centers be?
2	[9:48]	Shona-Pest:	What does it mean to conjecture?
4	[9:50]	Euclid-Guru:	Conjecturing means imagining a statement like a theorem ... making a claim that you believe could be true, which you will then need to prove

5	[9:50]	Kyle-Boss:	It means making a hypothesis
6	[9:51]	Euclid-Guru:	That's correct, Kyle, but the statement you think might be true still needs to be proven somehow
8	[9:53]	Ellie2-Promoter:	We may have some idea about the centers, they must be on the line perpendicular to the line connecting the two points
9	[9:53]	Euclid-Guru:	Ah, I see... have you drawn it?
10	[9:54]	Tommy-Blogger:	Yes, starting from the midpoint of the segment that joins a and b.
11	[9:57]	Ellie2-Promoter:	So, since this line is made up of infinitely many points, we can have infinitely many centers and therefore infinitely many circles passing through a and b.
12	[10:01]	Euclid-Guru:	What line are you thinking of?
13	[10:02]	Ellie2-Promoter:	The line perpendicular to the segment joining a and b and passing through their midpoint
19	[10:10]	Tommy-Blogger:	Through two points pass infinitely many circles, since it is possible to have infinitely many centers, which lie on the line perpendicular to the chord joining the two points and passing through its midpoint
20	[10:11]	Euclid-Guru:	Which line is this?
21	[10:11]	Tommy-Blogger:	The perpendicular bisector of segment

In [1], Euclid-Guru activates the teleological component, as emerges from the phrase "where could their centers be?" (EU-TD), introducing for the first time the verb "to conjecture". This prompts Shona-Pest [2] to request clarification about the meaning of the term "to conjecture", acting on the communicative dimension in the form of questioning, as shown in the question "What does it mean to conjecture?" (EU-CD).

In [4], Euclid-Guru provides an explanation of the meaning of the term "to conjecture" (BLO-KL), also shifting the focus to the epistemic aspects of meta-knowledge of proof, in relation to the need to validate a statement (PE-MK).

In [5], Kyle-Boss looks for a synonym of "to conjecture", using the expression "It means making a hypothesis", adopting a meaning of the term "hypothesis" that is closer to its use in the experimental sciences rather than to the meaning it assumes in mathematics. This leads Euclid-Guru [6] to emphasize the idea of a statement that is assumed to be true but necessarily requires proof, as emerges in the phrase "the statement you think might be true still needs to be proven" (PE-MK).

Ellie2-Promoter in [8], in the act of generating ideas ("We may have some idea

about the centers"), provides a constructive contribution that enables the process to move forward (PRO-IP), establishing correct relationships among the mathematical objects involved, as emerge in the expression "[the centers] must be on the line perpendicular" (PRO-KL).

In [10], Tommy-Blogger provides an additional clarification about the line represented, highlighting its visual and dynamic aspect, as emerges from the utterance "starting from the midpoint of the segment" (BLO-KL).

In [11], Ellie2-Promoter performs a shift toward the communicative dimension, both as a need to clarify the reasoning developed up to that point (as indicated by the opening 'So') (BLO-MK), and by using a conditional structure highlighted by the terms "Since ... we can have ... and therefore ..." (BLO-IP).

In [12] and [20], Euclid-Guru further stimulates the communicative component by asking for more detail about the line the students are referring to (EU-CD), as can be seen in the phrases "What line are you thinking of?" and "Which line is this?"

Finally, in [19], Tommy-Blogger reorganizes the reasoning into a discourse that enables communication external to the Culture (BLO-MK), still in the form of an acceptable justification supported by a control of the conceptual properties emerging from the figure, as can be seen in "since it is possible to have infinitely many centers, which lie on the line" (PE-KL), but without yet making explicit reference to the theory.

10.3.2 Excerpts from Outlookers'chat

Scene 1- Outlookers

1	[8:59]	Shona2-GO1:	In my opinion, the situation remains unchanged, always infinite
2	[8:59]	Ellie-GO1:	Guys, let's start drawing, but let's do it with compasses so we can find the centre too
3	[9:01]	Ellie-GO1:	Guys, there are two circumferences that I can draw
5	[9:06]	Ellie-GO1:	In the meantime, let's continue. I don't think the drawing alone is enough. Tommy, can you come up with something?
6	[9:07]	Ellie-GO1:	Over on the other side, they just said that more than one can be drawn, but they didn't say the exact number
7	[9:09]	Shona2-GO1:	They are infinite because the segment of non-aligned points AB can be the chord of any circle of any size
8	[9:10]	Shona-GO2:	I agree with Shona2
13	[9:14]	Ellie-GO1:	Guys, let's upload the drawings with the names
15	[9:18]	Ellie-GO1:	Guys, Shona2 is right, they are infinite

17 [9:20]	Shona2-GO1:	Exactly. Because the segment formed by the two points AB can be the chord of any circle of any size with any centre
19 [9:22]	Ellie2-GO1:	One straight line always passes through two points
20 [9:24]	Shona2-GO1:	Yes, but the segment formed by AB can be a chord of the circle
21 [9:29]	Shona2-GO1:	The aligned points form a segment; this segment will always lie inside the circle, and if A and B are tangent to the circle, since they must belong to the circle, these points will form chords, each circle larger than the other
22 [9:32]	Ellie2-GO1:	I agree with Shona2
23 [9:33]	Ellie-GO1:	I also agree with Shona2

In [1] Shona2-GO1 formulates a conjecture but does not provide the reasons behind her idea. Therefore, a contribution based on intuition ("In my opinion") can be recognized, though still without justification (PRO-KL). In [2] a teleological dimension emerges as goal-oriented action, as we can observe from the Ellie-GO1's utterance "but let's do it with compasses" (PRO-KL), while she leads the actions of group ("let's start drawing") (BO-KL). At the same time, we also recognize an awareness of collective knowledge born from observing processes in the previous episode (OU-MK), which helps in choosing the actions to be taken (PRO-KL), as shown in "so we can find the centre too" . Ellie-GO1 in [5] thus assumes the role of group activator (BO-KL) in stating "let's continue," a role of checking regarding chosen representation as shown in the sentence "I don't think the drawing alone is enough" (PE-KL), and a role of activating the communication function in asking Tommy-Blogger, "Can you come up with something?" (BO-KL). Subsequently, in [6] Ellie-GO1 makes a critical observation about the behavior of the group of actors (OU-MK), as we can see in the excerpts "they just said that" and "but they didn't say," while in [13] Ellie-GO1 again takes on the role of Boss by guiding the group's actions, as can be inferred from the sentence "Guys, let's upload the drawings with the names" (BO-KL). In excerpt [7], Shona2-GO1 then provides a justification for the fact that circles are infinite, which leads Ellie-GO1 in [15] to reconsider her previous assumption, as shown in the sentence "Shona2 is right, they are infinite," demonstrating epistemic control acting on error regulation (PE-KL). In [17], Shona2-GO1 offers an even more accurate justification of the conjecture made, providing a constructive contribution that allows the process to move forward (PRO-IP), as observed in the excerpt "can be the chord of any circle of any size with any centre," highlighted by repetition of the term "any" for three times, which guides generalization. Finally, in [19] Ellie2-GO1 shows control over the epistemic error that emerged within

the group of actors regarding the alignment of two points, as highlighted by the term "always" (PE-KL). This control occurs through an implicit reference to Euclid's First Postulate.

In [20] and [21], Shona2-GO1 puts forward arguments in support of her idea in an effort to communicate insight to the group and lead the others to accept own position (BLO-MK), namely that segment AB can be a chord of infinitely many circles passing through A and B, as can be inferred from the excerpt "the segment formed by AB can be a chord of the circle". In making this effort, a communicative lack emerges in the use of the term "tangent" to mean "belonging to" (LACK-BLO-KL).

In [22] and [23], Ellie2-GO1 and Ellie-GO1 show agreement with Shona2-GO1's statement ("I agree" and "I also agree") (OU-MK).

Scene 2- Outlookers

1	[9:38] Ellie-GO1:	Guys, for the drawing, I started by drawing the two points. The first circle is the one with segment AB as its diagonal, therefore the points are the ends of the diagonal and the center of the circle is the midpoint. Then, as Shona2 said, the circles are infinite because segment AB can be their chord, so I was able to draw many of them and I can still continue
2	[9:39] Ellie2-GO2:	If we draw a circle and take two points on it, we cannot draw any others that pass through those two points, so it is only one.
3	[9:41] Ellie-GO1:	How come I was able to draw so many? They are infinite, there is not just one

In [1], following the suggestion given by Euclid in the actors' group, Ellie-GO1 describes in detail to the group the followed process to produce the drawing on paper, as emerges from the sentence "Guys, for the drawing, I started by drawing the two points" (BLO-MK). In providing this description, a communicative lack can be identified in the inappropriate use of the term "diagonal" with the meaning of "diameter" as can be inferred from the utterance "the points are the ends of the diagonal" (LACK-BLO-KL). In the excerpt "the center of the circle is the midpoint" Ellie-GO1 identifies a relationship between the geometric object "circle" and the midpoint of the segment, which thus becomes its center (PRO-KL). Finally, in the sentence "I was able to draw many of them" a goal-oriented action emerges (PRO-KL), while in the excerpt "and I can still continue" an attempt at generalization can be discerned (PRO-IP).

Finally, in [3], Ellie-GO1 shifts into the role of Euclid-Guru, prompting the teleological dimension through questioning and reproducing the observed behaviors of

the Guru. This emerges from the sentence, "How come I was able to draw so many?" (EU-TD; OU-IP)

Scene 3- Outlookers

1	[9:43] Ellie-GO1:	Guys, two points are always aligned. You can talk about aligned points if there are three or four, in any case more than two, but two points are always aligned
2	[9:44] Shona2-GO1:	Ellie1 is right
3	[9:45] Shona-GO1:	But if the points are always aligned, the circumference passing through them is only one
4	[9:46] Ellie-GO2:	Instead, in my drawing, I only made three circumferences. One has the segment as its diameter and the other two have the chord as their diameter
5	[9:47] Ellie-GO1:	Guys, but I managed to draw more than four, and now you're telling me it's one. Try drawing it. The segment AB can be the chord of the circle
6	[9:47] Shona2-GO2:	If they are aligned because there are two, not three, four, five... two form a segment, so this segment will be the diameter of one circle and the chord of infinite other circles
7	[9:49] Ellie-GO2:	But it cannot be the chord of infinite circumferences
8	[9:50] Shona2-GO1:	Why not? A circumference can be of infinite sizes
9	[9:51] Ellie-GO1:	It is possible, it is the chord of circumferences that are above and below (or on the sides if you have drawn them one above the other)
10	[9:57] Ellie-GO1:	Guys, Elle2 actor made a correct observation: the centers of the circles are aligned I've just noticed it in my drawing as well
11	[10:02] Ellie-GO1:	Guys, I think we've solved the problem. If Shona wants, she can summarize everything in a statement, adding the fact that the centers are aligned
12	[10:04] Shona2-GO1:	The circles passing through two points, which are necessarily aligned, are infinitely many because these points form a segment; this segment, with its endpoints lying on the circle, will be the diameter of one circle and the chord of infinitely many other increasingly larger circles
13	[10:05] Ellie2-GO2:	And the center?
14	[10:05] Shona2-GO1:	With a compass, so we know the center of each circumference
15	[10:06] Ellie-GO1:	Ah, true, the aligned centers

16 [10:06] Kyle-GO2: The centers form a perpendicular line
17 [10:11] Shona2-GO1: They will form a line perpendicular to the midpoint
18 [10:11] Ellie2-GO1: So the perpendicular bisector
19 [10:12] Ellie-GO1: There are no lines perpendicular to points
20 [10:14] Ellie-GO1: Guys, a point is dimensionless it's not like a segment, for which I can say it is perpendicular to another line. It doesn't form a 90-degree angle when a line passes through it
21 [10:16] Ellie-GO1: The line where the centers are aligned is perpendicular to segment AB, not to the midpoint

n [1], Ellie-GO1, with reference to the issue that had previously emerged in the actors' group, monitors the ambiguity of statements made by others, as shown by the utterance, "You can talk about aligned points if there are three or four" (PE-IP). Moreover, in the expression "in any case more than two" she performs a check on the epistemic lack that had emerged in the previous discussions (PE-IP). In [2], Shona2-GO1 shows agreement with Ellie-GO1's statement (OU-MK)

In [5], Ellie-GO1 responds to the objections raised by other outlookers ([3] and [4], in which epistemic and communicative lacks emerge). Her response highlights a teleological component as a goal-oriented action, as shown in "but I managed to draw more than four" (PRO-KL). She subsequently stimulates this teleological component in the other group members, acting as Euclid-Guru, as we can see in the sentence "Try drawing it" (EU-TD; OU-IP).

In [6], Shona2-GO1 shows an epistemic dimension of control over the conceptual property related to the alignment of points, as shown in "If they are aligned [it is] because there are two, not three, four, five... two form a segment" (PE-KL).

In [8], Shona2-GO1, with the question "Why not?", prompts reflection on the statement that was made, thereby stimulating the epistemic component and reproducing the observed behaviors of the Guru (EU-ED; OU-IP). Instead, in the sentence "a circumference can be of infinite sizes" a new attempt at generalization emerges (PRO-IP), particularly highlighted by the words "infinite sizes".

In [9], Ellie-GO1 looks for arguments to make her thinking understandable to the others, also using a form of communication that relies on visual aspects, as shown in the excerpt "that are above and below (or on the sides if you have drawn them one above the other)" (BLO-MK). In this excerpt, Ellie-GO1 specifically describes the drawing she produced on paper and later shared on Padlet, as we can see in Figure 10.2, where the circles are visible above or below segment AB, or on the sides if the points are represented one below the other.

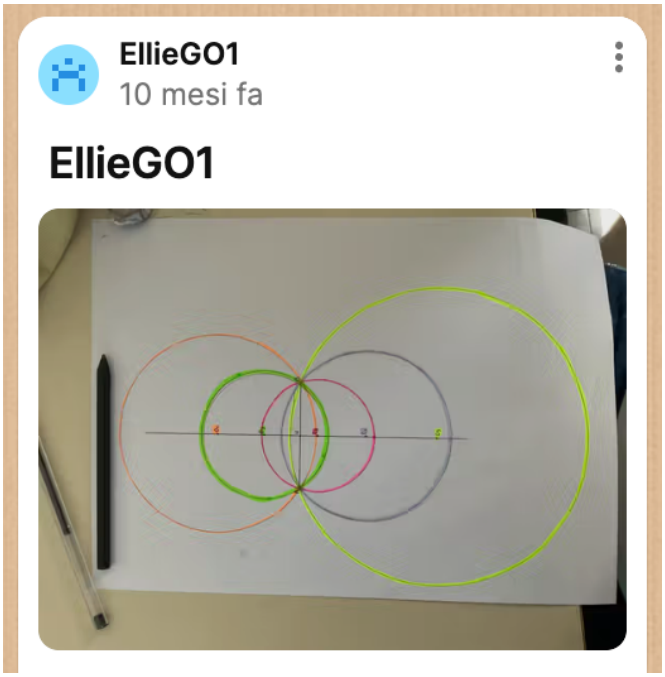


Figure 10.2: Ellie-GO1's drawing of circles passing through two points

Subsequently, Ellie-GO1 shifts her attention to what is happening in the actors' group [10]. In particular, she engages in a reflection on the mathematical process observed, which enables to make the same observation about own representation (OU-MK), as shown by expressions such as "a correct observation" or "I've just noticed it in my drawing".

Furthermore, Ellie-GO1 in [11] acts according to the characteristics of the leader Boss, activating roles and providing a guide on the actions to be taken, as evidenced by "If Shona wants, she can summarise everything in a statement, adding the fact that the centres are aligned" (BO-KL).

Prompted by Ellie-GO1's requests, in [12] Shona2-GO1 summarizes the conjecture produced in a communication addressed to the group (BLO-MK), supporting it with constructive arguments as a justification of the conjecture itself (PRO-IP), as shown in the sentence: "The circles passing through two points [...] are infinitely many *because* these points form a segment [...] will be the diameter of one circle and the chord of infinitely many other increasingly larger circles." Moreover, in this intervention Shona2-GO1 reiterates the concept that two points are always aligned, working on the epistemic component (PE-KL), as emerges from the excerpt "two points, which are necessarily aligned".

In response to Ellie2-GO2's question about the centers of the circles, Shona2-GO1 answers "With a compass," which allows her to perform a check on the geometric knowledge characterizing the centers of the infinitely many circles (PE-KL), as we can see from the sentence "so we know the center of each circumference" in [14] and from the statement "They will form a line perpendicular to the midpoint" in [17]. In the latter sentence, although the intended meaning is clear, a communicative shortcoming appears in the expression "perpendicular to the midpoint" (LACK-BLO-KL). This is immediately pointed out by Ellie-GO1 in [19] with the verbal expression "There are no lines perpendicular to points" thereby highlighting an epistemic control over geometric properties and on the validation or invalidation of statements(PE-KL).

In support of her statement, Ellie-GO1 concludes the scene with two interventions in which the communicative dimension toward the group emerges (BLO-MK), with the aim of mediating knowledge by making it understandable to others and seeking a shared meaning [20] and [21]. In particular, in the sentence "Guys, a point is dimensionless" we can observe an implicit reference to theory (PE-KL), while in the expressions "it's not like a segment" and "It doesn't form a 90-degree angle" we recognize the act of establishing relationships between mathematical objects (PRO-KL), always with the aim of making the communication of her thinking clearer and activating the writer-interpreter dynamics(BLO-MK). In this final intervention by Ellie-GO1, a dialectic among the three dimensions of rationality thus emerges.

10.4 Episode 1.4

The comic strips in 1.4 pose a provocation to the students. Euclid asks, in fact, "Could there exist a circle passing through A and B with its center not on the perpendicular bisector?" Both the actors and the observers immediately picked up compasses and sheets of paper and tried to draw one.

It was a very interesting moment, because I could see the students standing still in front of their sheets: they wanted to try, yet they were thoughtful; their faces looked puzzled. They were unable to draw a circle passing through A and B with its center not on the axis. They set to work, but the sense of unease caused by the impossibility of drawing it was tangible.

Furthermore, it is notable that in this episode, the group of Actors asked Euclid for access to the Dynamic Euclidean Bank of Theory for the first time and using the Digital Toolkit integrated into Bank.

Some excerpts from the two chats are presented in the following subsections.

10.4.1 Excerpts from Actors'chat

Scene 1- Actors

1	[8:50]	Tommy-Blogger:	So guys, Euclid asked us to draw a circle passing through A and B but with its center not on the perpendicular bisector
2	[8:51]	Ellie2-Promoter:	Shall we start with some graphical attempts?
3	[8:51]	Tommy-Blogger:	We can give it a try
4	[8:52]	Tommy-Blogger:	Euclid also told us that we might find something interesting in the theory bank
6	[8:59]	Tommy-Blogger:	I'm trying to draw it
7	[9:00]	Kyle-Boss:	Me too, but I can't manage to draw it so that it doesn't have its center on the perpendicular bisector.
10	[9:02]	Euclid-Guru:	Yeah, I know, you're struggling! I can see that! Why do you think you can't draw it?
11	[9:03]	Euclid-Guru:	Try to think about why you're finding it difficult
12	[9:04]	Kyle-Boss:	Maybe we're imagining the drawing incorrectly
13	[9:04]	Euclid-Guru:	What were we asked to do? Let's try to repeat it. Tommy, could you clarify for everyone what is required?
14	[9:05]	Shona-Pest:	Because if we move the center of the circle, it won't pass through both points A and B — it will only pass through one of them
16	[9:05]	Euclid-Guru:	What does it mean for a circle to pass through A and B?
17	[9:06]	Euclid-Guru:	Or rather, what do you hypothesize? What conjecture would you make in response to the request?
18	[9:07]	Euclid-Guru:	If you are asked to find a circle passing through A and B with its center not on the perpendicular bisector, what would you feel confident asserting?
23	[9:12]	Kyle-Boss:	So, to recap, for us it's impossible to draw a circle through A and B that doesn't have its center on the perpendicular bisector
26	[9:14]	Tommy-Blogger:	But there could be [1] something theoretical if we can't do it graphically
29	[9:15]	Euclid-Guru:	Great, let's work with the theory. Kyle says: it's impossible to draw a circle through A and B that doesn't have its center on the perpendicular bisector of AB... try to rephrase the sentence

[1] In the source Italian text, the verb occurs within a real conditional construction featuring the future indicative in the apodosis. Accordingly, it has been translated here using the modal form "could be".

30 [9:16]	Tommy-Blogger:	There does not exist any circle passing through points A and B that does not have its center on the perpendicular bisector
32 [9:20]	Euclid-Guru:	What does it mean to say that C is not on the perpendicular bisector?
33 [9:21]	Euclid-Guru:	Make it explicit
34 [9:21]	Tommy-Blogger:	That the center does not belong to the axis
35 [9:21]	Ellie2-Promoter:	Because a circle, in order to pass through two points, must have its center equidistant from the two points, and we know that the set of points equidistant from the endpoints of a segment is the perpendicular bisector
36 [9:22]	Euclid-Guru:	But if I go into the details and write that C does not belong to it, what does it mean and how can I express this sentence better?
37 [9:22]	Ellie2-Promoter:	That C is not equidistant from the endpoints
38 [9:23]	Tommy-Blogger:	If C is not on the perpendicular bisector, then it is not the center of the circle
39 [9:23]	Euclid-Guru:	Why do you say that the center must be equidistant from A and B?
40 [9:23]	Tommy-Blogger:	Because a circle is the set of all points that are at a distance r from the center O
41 [9:24]	Ellie2-Promoter:	Because if it weren't, the circle would not pass through both points
44 [9:25]	Kyle-Boss:	Let's not get discouraged. We are definitely on the right track!
62 [9:53]	Euclid-Guru:	Tommy, can you try to provide a complete summary of everything? That is, write what you have conjectured, how this idea arose, and why you think it is true
63 [9:55]	Tommy-Blogger:	So we started by trying to draw these circles, but we realized we couldn't, so we began to think about whether theory could help us. We concluded that if C does not lie on the perpendicular bisector, then it cannot be the center of the circle. We also recalled the definition of a circle: it is the set of all points that are at a distance r from the center O

In [1], Tommy-Blogger repeats for the group the task proposed by Euclid (BLO-KL), while in [2], Ellie2-Promoter suggests starting the work using exploratory empirical methods, as reflected in the expression "graphical attempts" (PRO-KL)

In [4], Tommy-Blogger communicates to the group a suggestion given by Euclid in the comics, acting as an intermediary between the group and Culture as a mediator of communication (BLO-MK).

In [6], Tommy-Blogger shifts to the strategic dimension, communicating to the group the choices of actions undertaken (PRO-KL), in relation to which Kyle-Boss [7] provides feedback related to the failure of the graphical representation attempt, thus carrying out a control over the choices made, as reflected in the excerpt "but I can't manage to draw it so that it doesn't have its center on" (PE-KL)."

In [10], there is an intervention by Euclid-Guru, who initially acts on the emotional co-participation of the teacher-researcher with the difficulties encountered by the students in searching for such a circle, as can be seen in the sentence "Yeah, I know, you're struggling! I can see that!" (BO-IP). This intervention then turns into questioning that stimulates the teleological dimension, as we can observe in the question "Why do you think you can't draw it?" (EU-TD). The prompt in [11] finally becomes an intervention that guides reflection on the causes of the difficulties encountered, thereby stimulating the epistemic component (EU-ED), as we can see from the sentence "Try to think about why you're finding it difficult".

This leads Kyle-Boss to check the effectiveness (or lack thereof) of the chosen strategy in relation to the aim (PE-KL), as reflected in the utterance "Maybe we're imagining the drawing incorrectly" [12], and leads Shona-Pest [14] to carry out another form of checking, this time focusing on the effects of the representation procedure used in relation to the aim (PE-KL), as we can observe in the expressions "because" and "if we move ... it won't pass".

Several mediating interventions by Euclid-Guru therefore become necessary. In particular, in [13] Euclid-Guru prompts the communicative dimension (EU-CD), as we can see from the sentences "Let's try to repeat it" and "Tommy, could you clarify for everyone what is required?"; whereas in [16] he stimulates the emergence of the epistemic dimension (EU-ED) ("What does it mean ...?") in order to enable students to reflect on the meaning of a point belonging to a circle. Finally, in [17] and [18], Euclid-Guru prompts the formulation of a conjecture, acting on the teleological component of rationality (EU-TD), as reflected in the sentences "What conjecture would you make in response to the request?" and "what would you feel confident asserting?".

In Kyle-Boss's intervention [23], a dialectic of the three rationalities can be identified. In particular, in the excerpt "So, to recap", we recognize a communicative dimension as a need for clarification with oneself (BLO-MK), whereas in the excerpt "for us it's impossible to draw a circle through A and B", we can observe a teleological dimension as goal-oriented action (PRO-KL). Finally, in "that doesn't have its center

on the perpendicular bisector", an epistemic dimension emerges as the monitoring of a conceptual property (PE-KL).

Tommy-Blogger's contribution [26] represents a decisive moment in which the need for theoretical validation emerges for the first time. In the expression "But there could be something theoretical if we can't do it graphically", we can first observe the use of a real conditional construction ("There could be ... if we can't"), which in the original Italian text appears through a future tense expressing hypothetical reality. There is therefore not yet full awareness of the underlying theory; nevertheless, a form of process control emerges in the form of a need for causality (PE-IP). The use of the future tense as a real conditional suggests that this contribution lies in the interplay between the two levels.

In [29], Euclid-Guru takes up the opportunity provided by these interventions to prompt a reformulation of the statement made by Kyle-Boss in [23], thus acting on the communicative dimension, as evidenced by the sentence "try to rephrase the sentence" (EU-CD).

This leads Tommy-Blogger to move from the formulation "it's impossible to draw a circle", which is still empirical in nature, towards a formulation expressed in proper mathematical language [30], namely "There does not exist any circle" (BLO-KL).

In order to stimulate an epistemic reflection within the group and thereby foster a more conscious shift toward the underlying theory, Euclid-Guru in [32] and in [33] asks the students to make explicit the meaning of a point not belonging to the perpendicular bisector of segment AB, thus working on both the epistemic dimension (EU-ED) and the communicative dimension (EU-CD), as evidenced by the sentences "What does it mean to say ...?" and "Make it explicit".

Ellie2-Promoter responds to Euclid by formulating a justification [35] whose epistemic value is supported by a conditional structure, even if implicit ("in order to ... must have ...") (BLO-IP), and by reference to statements accepted by the classroom community (PE-KL), as evidenced by "we know that".

Euclid-Guru therefore continues to prompt the communicative component, asking, "If I write that C does not belong to it, how can I express this sentence better?" [36] (EU-CD), which allows Tommy-Blogger to use the conditional structure in the "If ... then" form (BLO-IP) for the first time, as observed in the sentence "If C is not on the perpendicular bisector, then it is not the center of the circle" [38].

In [39], Euclid-Guru decides to prompt the epistemic component, asking Ellie2-Promoter to provide a justification for the necessity of the center being equidistant from the two points A and B (EU-ED). While Ellie2-Promoter responds to this prompt in [41] by still operating on the teleological component of rationality—namely, by

invoking the possible failure of a goal-oriented action (PRO-KL) ("the circle would not pass through both points")—Tommy-Blogger instead responds in [40] by validating the statement through the use of a definition (PE-KL), even if in a still partially implicit form, as shown in "Because a circle is the set of all points that are at a distance r from the center O".

In [44], Kyle-Boss encourages the other members of the group on an emotional level (BO-IP).

Finally, in [62], Euclid prompts meta-communication understood as communication about the process (EU-CD), as can be observed in the excerpt "can you try to provide a complete summary of everything?" and in the use of the terms "what", "how", and "why".

This leads Tommy-Blogger to reconstruct the entire process that unfolded within the group through an intervention in which the three dimensions of rationality emerge. In particular, in the initial term "So" we recognize the need for communication aimed at self-clarification (BLO-MK). In the utterances "we started by trying to draw" and "we realized we couldn't", we can observe the teleological dimension related to the strategic choices adopted and the corresponding failures encountered (PRO-KL). Meanwhile, in the conditional structure "if ... then", a communicative dimension emerges, related to the use of verbal representations to express implications (BLO-IP). Finally, in the sentence "we began to think about whether theory could help us", we can identify an epistemic component at the level of meta-knowledge, highlighting the emergence of awareness concerning the underlying theory (PE-MK), which is even more evident in the sentence "We also recalled the definition of a circle", a concrete reference to theoretical elements (PE-KL).

Scene 2 - Actors

This scene took place one week after the previous one; therefore, I first suggested rereading the comic strips and then resuming the discussion in the chat. During this scene, the students also use the Toolkit for Proof available within the Euclidean Dynamic Bank of Theory

2	[8:54]	Tommy-Blogger:	So, we were saying that if a point C of a circle does not belong to the perpendicular bisector, then it is not the center of the circle
3	[8:55]	Tommy-Blogger:	Because we had been asked to try to draw a circle passing through A and B whose center was not on the perpendicular bisector
4	[8:55]	Ellie2-Promoter:	And we had said that it was graphically impossible

5	[8:56] Kyle-Boss:	And we had to prove that it was impossible, right?
6	[8:57] Tommy-Blogger:	We were discussing what to write in the toolkit
7	[8:58] Ellie2-Promoter:	Shall we try, Tommy?
11	[9:04] Tommy-Blogger:	We had said that if a circle does not have its center C on the perpendicular bisector of AB, then AC is different from BC
12	[9:05] Tommy-Blogger:	And then it is not a circle, because we know that a circle is the set of all points that are at a distance r from the center
13	[9:05] Euclid-Guru:	Why can you make that step? Why can you assert what you have written?
14	[9:06] Kyle-Boss:	Thanks to the theory bank
15	[9:06] Kyle-Boss:	And the proof that we provided ourselves
16	[9:06] Euclid-Guru:	What exactly is in the Theory Bank?
17	[9:07] Kyle-Boss:	Proofs about the perpendicular bisector of a segment
18	[9:07] Euclid-Guru:	Which one exactly? Consult the Theory Bank
19	[9:07] Tommy-Blogger:	Properties 1 and 2 of the perpendicular bisector of the segment
20	[9:08] Euclid-Guru:	1 or 2?
21	[9:08] Tommy-Blogger:	2
22	[9:10] Euclid-Guru:	Try to write it in the toolkit
24	[9:12] Euclid-Guru:	Remember that the toolkit is just a tool; Tommy could also try to rewrite here in the chat what you are trying to compose in the toolkit
25	[9:14] Euclid-Guru:	I read def. 2... what is the difference between the definition and the proposition?
26	[9:14] Euclid-Guru:	Please consult the theory bank carefully
27	[9:15] Tommy-Blogger:	If C does not belong to the perpendicular bisector, then it is not the center of the circle; therefore, if C does not belong to the perpendicular bisector of AB, then AC is different from BC
28	[9:15] Euclid-Guru:	for...
29	[9:15] Tommy-Blogger:	By the second property of the perpendicular bisector of a segment
30	[9:16] Euclid-Guru:	Good, and therefore? Remember the objective
35	[9:22] Tommy-Blogger:	It is not possible to draw a circle passing through A and B whose center is not on the perpendicular bisector of the segment, by the second property of the perpendicular bisector of a segment
36	[9:22] Euclid-Guru:	Write everything in the email, all the steps. Like a real proof
38	[9:23] Kyle-Boss:	Okay, let's start writing the first part

39 [9:23] Ellie2-Promoter: Add the second property

In [2] and [3], Tommy-Blogger, in his need to summarize what has been said so far ("So"), communicates with the group (BLO-MK) using the conditional structure "If ... then" (BLO-IP), while simultaneously highlighting an epistemic lack in the incorrectness of the relationship between the mathematical objects involved (LACK-PE-KL), as shown by "a point C of a circle".

In [4] we see the recognition of the failure of a goal-oriented action (PE-KL), as Ellie2-Promoter states that "it was graphically impossible", while Kyle-Boss in [5] highlights the need for a move to the next level of the proving process (BO-IP), as we can see from the excerpt "we had to prove that it was impossible".

Ellie2-Promoter has a shift to the role of Boss, as in [7] she activates the communication function, as we can see from the utterance "Shall we try, Tommy?"

Tommy-Blogger in [6] acts on the communicative dimension in an attempt to negotiate with the group (BLO-MK), as shown by the sentence "We were discussing what to write ..." and later in [11] he continues working on internal group communication (BLO-MK), using the conditional structure "if ... then" (BLO-IP) while simultaneously providing a constructive contribution to the proving process (PRO-IP), by reformulating the concept of equidistance given in the previous scene by Ellie2-Promoter, as seen in "then AC is different from BC". Tommy-Blogger's intervention continues in [12], where we recognize an implication expressed through the verbal representation "and then" (BLO-IP) and an epistemic contribution emerging from the phrase "because we know that" concerning the definition of a circle, although still in implicit form (PE-KL).

Euclid-Guru therefore prompts the epistemic component with a questioning move aimed at shifting attention to the underlying theory, as shown by the questions "Why can you make that step?" and "Why can you assert what you have written?" (EU-ED)

Kyle-Boss therefore shows an awareness of the choice and use of mathematical means appropriate to the goal (PRO-MK), referring in particular to the theorems they themselves proved during the preparatory activities, as evidenced by the phrases "the proof that we provided ourselves" and "Proofs about the perpendicular bisector of a segment" ([15] and [17]).

Euclid-Guru, in an attempt to elicit greater precision in communication, intervenes with requests such as "What exactly is in the Theory Bank?" [16] or "Which one exactly?" [18], thereby prompting the communicative dimension (EU-CD).

In [22] and in [24], Euclid-Guru instead suggests using the movable tiles of the toolkit available within the Dynamic Euclidean Bank of Theory to write the proof. In particular, in [25], by observing the tiles used in the toolkit (Fig. 10.3) and noticing

an inconsistency highlighted by "I read def. 2"[2], Euclid-Guru prompts meta-epistemic awareness by asking "what is the difference between the definition and the proposition?" (EU-ED).



Figure 10.3: Tiles used into the Toolkit - Scene 2 - Actors

In [27], Tommy-Blogger rewrites in the chat what was completed using the toolkit tiles, adding an element that is absent from the toolkit, namely the term "therefore". In this, we recognize a transition element from conjecture to proof, marking a move to a subsequent step in the proving process (BO-IP).

In [29], Tommy-Blogger, prompted by Euclid, carries out a check of the conceptual properties, precisely indicating the property of the perpendicular bisector of a segment to which they are referring and which is included in the Dynamic Euclidean Bank of Theory[3] (PE-KL).

Finally, Euclid-Guru in [36] prompts the group toward a meta-reflection on the proof by stimulating both the epistemic and communicative components (EU-ED; EU-CD), in particular by asking them to write "all the steps." Euclid uses the term

[2]Translating the figure: If not C on perpendicular bisector then not a center circle; if C is not on the perpendicular bisector of AB then AC is not equal to CB for def. 2 of perpendicular bisector

[3]Below is a translation of the way this property is stated in the Dynamic Euclidean Bank of Theory: Every point in the plane that is equidistant from the endpoints of a segment lies on the perpendicular bisector of AB; that is, IF AC is equal to CB, THEN C lies on the perpendicular bisector of AB; that is, IF C does not lie on the perpendicular bisector of AB, THEN AC and CB are not equal.

"real" to call students back to the idea of a rigorous proof, that is, one that complies with the standards of the mathematical community.

10.4.2 Excerpts from Outlookers'chat

The difficulty experienced by the students was evident and tangible also in the Outlookers group, especially during the first few minutes.

In the following excerpts, the integrated analysis of the Outlookers group interactions is presented along with the analysis of the visual representations they upload to the shared Padlet.

Scene 1 - Outlookers

1	[8:51] Ellie-GO1:	Alright, let's start by drawing the two points again
2	[8:51] Shona2-GO1:	No, a circle that is not on the perpendicular bisector doesn't work for me
3	[8:53] Ellie-GO1:	It's similar to the problem we addressed on the first day; we have to prove it with a statement. A drawing alone is not enough to show it.
5	[8:55] Ellie2-GO1:	Where is the Theory Bank?
6	[8:55] Shona2-GO1:	I was about to ask that
7	[8:55] Tommy-GO2:	We need to ask a clear and precise question about what we want to obtain
8	[8:58] Ellie-GO1:	We can try asking if there is a proposition or a notion that deals with the centers of circles perpendicular to a segment
9	[8:58] Shona2-GO1:	So, a circle with a center not on the perpendicular bisector technically will not pass through points A and B, because if the center moves, the circle moves as well
10	[8:59] Ellie-GO1:	And this can be shown with a drawing
11	[9:02] Ellie-GO1:	Euclid, is there a proposition or a notion that we can draw on to solve the problem?
12	[9:08] Shona2-GO1:	We have the center of a circle, O, with points tangent to it, A and B, and the distance between OB and OA is the same since they form radii and all radii are equal. Therefore, for the center of the circle to include both points, it must be perfectly central, on the perpendicular bisector; otherwise, it can only be considered a circle in empty space.
19	[9:13] Ellie2-GO2:	I took the points, drew the segment and the perpendicular bisector, then I took a point that is not on the perpendicular bisector line and drew a circle.

	This circle passes through only one of the two points.
22 [9:14] Ellie2-GO1:	So, the center is always on the perpendicular bisector?
23 [9:14] Shona2-GO1:	Yes
24 [9:14] Ellie2-GO2:	So, the center cannot not be on the perpendicular bisector
25 [9:15] Shona2-GO1:	Right
26 [9:15] Ellie2-GO1:	So, the answer is that the center must be necessarily on the perpendicular bisector
29 [9:19] Ellie2-GO1:	I sent the drawing
30 [9:19] Ellie2-GO1:	In the end, it proves what we said.
31 [9:20] Ellie-GO1:	A drawing alone is not enough; a statement must be formulated
32 [9:20] Shona2-GO1:	A circle with center O will have the following distances, OA and OB, if on the perpendicular bisector, since on the perpendicular bisector a circle can be drawn with points A and B tangent; outside the perpendicular bisector, the circle will be any circle in the plane, not touching points A or B
33 [9:21] Ellie-GO1:	But we can do it ourselves as well; it's useful for understanding
34 [9:21] Tommy-GO2:	"If on the perpendicular bisector, since on the perpendicular bisector" what would it mean exactly?
35 [9:22] Shona2-GO1:	I'll write it better
36 [9:23] Ellie2-GO2:	As can be seen from the various uploaded drawings,it is not possible to draw a circle with a center that is not on the perpendicular bisector passing through A and B, because the circle that would be obtained will always pass through only one of the two points.
37 [9:25] Shona2-GO1:	We have a point C; we draw it off the perpendicular bisector, that is, at any point in the plane that is not on the perpendicular bisector, and since the distance of the radii from center C is the same in every direction, it can touch only one of the points but not both at the same time, because the center does not coincide with the perpendicular bisector.
39 [9:26] Ellie-GO1:	So, Tommy, I draw the two points on the plane and connect them with a segment, find the midpoint, and draw the perpendicular bisector of the segment. I start drawing a point (which will be the center of our circle) and draw the circle passing through one of the points. Since the center of the circle you chose was outside the

	perpendicular bisector, then the circle will pass through only one of the two points of the segment, just like any other circle whose center is outside the bisector.
43 [9:46] Shona2-GO1:	So, we can state that a circle called "s" with center C not on the perpendicular bisector will not have points A or B tangent to it, since a circle is a set of points equidistant from the center O; therefore, if the center is drawn off the perpendicular bisector, it will be just any center, and "s" will be just any circle in the plane.
48 [9:56] Ellie-GO1:	Shona2 just needs a little adjustment: a circle with center C outside the perpendicular bisector will not have points A and B on it, because a circle, being a set of points at a distance r from O, if O (in our case C) does not lie on the perpendicular bisector, the circle with center C will pass through only one or none of the points, and the segment AB will not be the chord or the diameter of the circle

In [1] we recognize a goal-oriented action, in which Ellie-GO1 states "Let's start by drawing the two points again" (PRO-KL), while in [2] Shona2-GO1 immediately encounters difficulty in being unable to carry out such a representation. In the expression "a circle doesn't work for me" we can in fact recognize a teleological conflict, related to checking the effectiveness (or lack thereof) of an action in relation to the aim (PE-KL).

In [3], Ellie-GO1 almost immediately recognizes an analogy with the second proof collectively addressed during Preparatory Phase 1, in which the students proved that if a point is equidistant from the endpoints, then it lies on the perpendicular bisector of a segment, and thus demonstrated that if a point does not lie on the perpendicular bisector, then the equidistance between the point and the endpoints no longer holds. In the excerpt "It's similar to the problem" we can indeed observe the recognition of an analogy (PRO-KL), while a communicative lack emerges in the use of the term "statement" to refer instead to a written argument, a verbal justification (LACK-BLO-KL).

In the subsequent interventions, Ellie-GO1 shows a need for theoretical support that is functional to her goal [8], as emerges from "We can try asking if there is a proposition or a notion that deals with the centers ..." (PE-IP). This can also be observed in the response she gives in [10] to Shona2-GO1 and in the, even if vague, request she makes in [11]. Indeed, Shona2-GO1 in [9] engages in a communication with herself in order to clarify her thinking (BLO-MK), searching for explanations that may

be convincing but that still remain linked to a strategy of empirical and dynamical visualisation (PRO-KL), as we can see from the utterance "because if the center moves, the circle moves as well". Ellie-GO1's response, on the other hand, highlights how her control has now shifted from empirical evidence ("And this can be shown with a drawing") to the search for theoretical validation, as we can see in the sentence "Euclid, is there a proposition or a notion that we can draw on to solve the problem?" (PE-IP).

In [12], Shona2-GO1 communicates with the group in order to convince the others of her reasoning. Although the communication appears weak, especially from a formal point of view, and presents several communicative lacks (LACK-BLO-KL) (such as the use of the term "tangent" to mean "belonging to" or the final part of the intervention, which seems unclear, "otherwise, it can only be considered a circle in empty space"), an epistemic dimension nevertheless emerges (PE-KL), with the implicit reference to the definition of a circle ("all radii are equal") and with the checking of the conceptual property related to equidistance ("the center of the circle to include both points, it must be perfectly central, on the perpendicular bisector"). We also recognise a communicative component in the "Therefore" which connects the two sentences as a verbal representation of implication (BLO-IP).

In [19], Ellie2-GO2 therefore provides an important contribution both in the observers' chat and on the board (Fig. 10.4), clarifying and explaining how she carried out the construction and which difficulties she encountered.

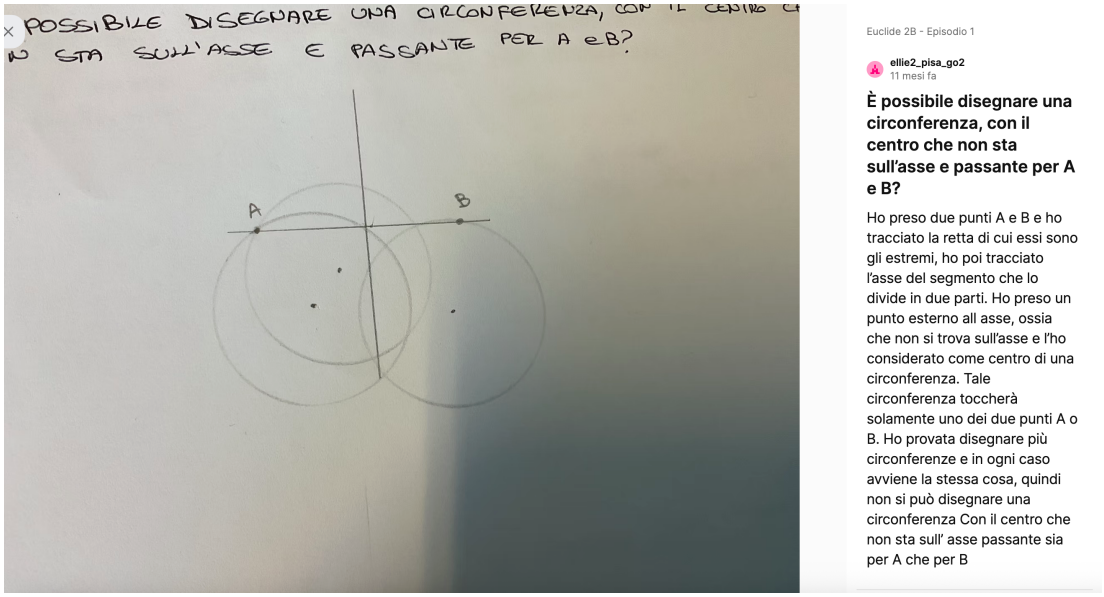


Figure 10.4: Episode 1.4 - Ellie2-go2's picture on the board

The translation of the comment that Ellie2-GO2 added in the Figure 10.4 is also provided below.

> I took two points, A and B, and drew the line segment of which they are the endpoints. I then drew the perpendicular bisector of the segment, which divides it into two equal parts. I took a point outside the perpendicular bisector, that is, a point not lying on it, and considered it as the center of a circle. This circle will pass through only one of the two points, A or B. I tried drawing several circles, and the same thing happens in every case; therefore, it is not possible to draw a circle with its center not on the perpendicular bisector that passes through both A and B.

In this comment, we recognise a detailed description of the actions carried out and oriented toward the goal (PRO-KL), highlighted by the expressions "I took," "I drew," "I then drew," and "I took a point outside". We also recognize a communicative component aimed at clarifying a concept, reformulating the sentence in more explicit terms in order to communicate with others in search of mutual understanding, as we can see in the excerpt "I took a point outside the perpendicular bisector, that is, a point not lying on it" (BLO-MK). Finally, an epistemic component emerges in the checking of the correctness of the representation in relation to the goal (PE-KL), as shown by the expressions "This circle will pass through only one of the two points" and "it is not possible to draw".

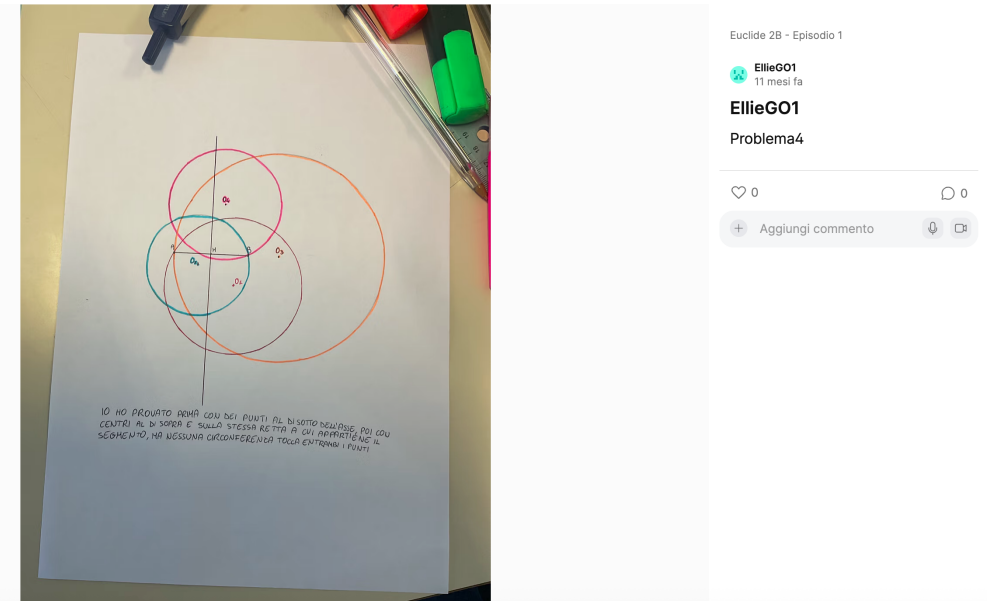


Figure 10.5: Episode 1.4 - Ellie-go1's picture on the board

Furthermore, in Figure 10.5 we have also shared the drawing produced on Padlet by Ellie-GO1, whose comment is transcribed here in translation from the original Italian.

> I first tried with points below the bisector, then with centers above it and on the line to which the segment belongs, but no circle passes through both points.

In this approach, we recognize a teleological dimension (PRO-KL), in the strategic choice of taking points "below" or "above" the segment or "on the line to which the segment belongs," explicitly recalling what occurred in the second proof of the preparatory activity 1, concerning the perpendicular bisector of the segment, namely when the students proved that if a point does not lie on the perpendicular bisector of AB, then there is no equidistance from points A and B.

We recall that in the preparatory phase the students had limited themselves to constructing the statement S* and thus carrying out a direct proof of the contrapositive, but without making a real transition to the original statement, or at least without understanding its equivalence with the main statement.

In this phase, however, an important event occurs in the Outlookers group, as some Outlookers move to the meta-statement. In particular, in [22] Ellie2-GO1 asks, "So, the center is always on the perpendicular bisector?" and subsequently in [24] Ellie2-GO2 states, using a double negation, "So, the center cannot not be on the perpendicular bisector", while Ellie2-GO1 in [26] concludes by stating, "So, the answer is that the center must necessarily be on the perpendicular bisector."

The expressions "is always," "cannot not be," and "must be" suggest an initial awareness towards the meta-knowledge about the proof, with a reference to the meta-statement and to a logical inference of modus tollens, even if still implicit (PE-MK).

In [31], Ellie-GO1 once again acts on the epistemic component by checking for the effectiveness (or lack thereof) of the strategic choices in relation to the aim and by bringing out a need for theoretical validation, as we can see from "A drawing alone is not enough" (PE-KL; PE-IP). In [33], she takes responsibility for the group's work (BO-KL), encouraging the others to act ("But we can do it ourselves as well"), also with a view to achieving a deeper understanding from an epistemic perspective and reproducing the observed behaviors of the Guru (EU-ED; OU-IP), as emerges from the phrase "it's useful for understanding".

In [36], Ellie2-GO2, referring to the drawings on Padlet in support of her reasoning ("As can be seen from the various uploaded drawings"), in particular those in Figure

10.4 and Figure 10.5, summarizes what had been said up to that point, communicating both with herself and with the group (BLO-MK).

Shona2-GO1, on the other hand, works on the communicative aspect, checking and attempting to improve the quality of the communication (PE-IP), as can be seen in [35] "I'll write it better". Indeed, in [37] Shona2 retraces the thread of her argument. In particular, we recognize a teleological dimension as goal-oriented action, highlighted by the expressions "we draw it off the perpendicular bisector" and "it can touch only one of the points" (PRO-KL); a communicative dimension in the clarification aimed at mutual understanding, as we can see in "that is, at any point in the plane that is not on the perpendicular bisector" (BLO-MK); and finally an epistemic dimension with implicit reference to theory, as emerges from "since the distance of the radii from center C is the same in every direction" (PE-KL).

In [39], Ellie-GO1 engages in a communication with Tommy, in which teleological elements emerge, highlighted by the goal-oriented actions "I draw," "connect them," "find," and "draw" (PRO-KL), as well as elements of conditional structure, such as "since ... then" as a verbal representation of implication (BLO-IP).

Finally, again in [48], Ellie-GO1 takes responsibility for the communicative aspect, prompting Shona2-GO1 toward deeper clarify of expression and reproducing the observed behaviors of the Guru (EU-CD; OU-IP) in relation to what was stated in [43], as emerges from the phrase "Shona2 just needs a little adjustment".

Scene 2 - Outlookers

In this scene, the Outlookers no longer work on the mathematical problem, but focus essentially on observing what happens in the Actors group. Some excerpts mainly concerning Ellie-GO1 are reported below.

1	[8:51] Ellie-GO1:	So, let's try to recap what we said last time
3	[8:56] Ellie-GO1:	We tried to prove that the centers of the circles passing through points A and B lie on the perpendicular bisector of segment AB, so we started by drawing circles with centers outside the perpendicular bisector.
5	[8:59] Ellie-GO1:	For those who made drawings, upload them, or let's try to comment on how we did ours.
6	[9:00] Ellie-GO1:	In the meantime, let's look at what the actors are doing and try to comment.
8	[9:03] Ellie2-GO1:	I don't know; in my opinion, the one from last time was correct.
9	[9:05] Ellie-GO1:	Evidently it wasn't complete; if he said to look at the drawings made last time, then something is missing.
16	[9:22] Ellie-GO1:	Guys, they've solved it, they're writing the email

	to Euclid. Tommy did a good job, but the others didn't participate actively, especially the Ellies, who basically didn't say anything.
18 [9:26] Ellie-GO1:	Maybe because they worked more actively last time in solving the problem, and today's task is Tommy's, but they could have said a bit more.

In [1], Ellie-GO1 activates roles by acting as Boss, as emerges from the phrase "let's try to recap" (BO-KL), while in [3] we identify a contribution of meta-knowledge about the proof, as an initial awareness emerges of the attempt to prove a statement starting from ad absurdum hypothesis (PE-MK), as we can see from the excerpts "We tried to prove that" and "so we started by drawing circles with centers outside the perpendicular bisector".

Ellie-GO1 then acts as Boss, operating on meta-functional aspects, directing the process and prompting the other members to upload and comment on the drawings on Padlet [5], as emerges from "For those who made drawings, upload them, or let's try to comment" (BO-MK), and further directing the group's reflection and attention to the actions carried out by the actors [6], as we can see in "let's look at what the actors are doing" (BO-IP).

Moreover, in [9] we recognize a reflective action highlighted by a conditional structure on meta-thinking, as we can see in the utterance "if he said to look at ... then something is missing" (OU-MK).

In [16] Ellie-GO1 makes an observation about the role and behavior adopted by the actors as a noting, as emerges from the phrase "Tommy did a good job, but the others didn't participate actively" (OU-KL), while in [18], Ellie-GO1 makes a reflective and critical observation of the different roles and the behavior adopted, as emerges from the excerpts "Maybe because they worked more actively last time", "today's task is Tommy's" and "they could have said a bit more" (OU-MK).

10.5 Episode 1.5

In this episode, the students were asked to prove that every point lying on the perpendicular bisector of segment AB is the center of a circle passing through A and B. Unlike the other episodes, this one did not conclude with sending an email to Euclid, but with an individual writing of the proof on a paper worksheet, after having nevertheless collaborated in the chat and argued collectively.

10.5.1 Excerpts from Actors'chat

1	[9:31]	Kyle-Boss:	Alright, let's get started
2	[9:32]	Kyle-Boss:	Euclid asks us to prove that every point on the perpendicular bisector of AB is the center of a circle passing through A and B
3	[9:32]	Tommy-Blogger:	He also told us that we can consult the Theory Bank
5	[9:33]	Ellie2-Promoter:	Graphically, we had already done some proofs in Episode 1.3
9	[9:35]	Kyle-Boss:	So, let's try to work from there
10	[9:37]	Euclid-Guru:	What do you think it means that EVERY POINT on the perpendicular bisector IS ...?
11	[9:38]	Tommy-Blogger:	That C belongs to the perpendicular bisector; C would be the center
13	[9:41]	Euclid-Guru:	Can you try to rewrite it here, Tommy, in full... what you just said? ... and maybe fix the grammar a bit as well?
14	[9:42]	Kyle-Boss:	If C lies on the perpendicular bisector, then it is the center of the circle
16	[9:43]	Euclid-Guru:	Try to reason about why this statement is true
17	[9:43]	Tommy-Blogger:	But wasn't that basically what we were asked in the other episode?
18	[9:45]	Euclid-Guru:	Tommy, do you think it's exactly the same thing?
20	[9:47]	Euclid-Guru:	Let me try to recall it: earlier we proved that if C is not on the perpendicular bisector of AB, then it is not the center of a circle passing through A and B
21	[9:47]	Euclid-Guru:	We now want to prove that if C lies on the perpendicular bisector of AB, then it is the center (of the circle through A and B)
22	[9:47]	Tommy-Blogger:	So, if C lies on the perpendicular bisector, it is the center
23	[9:48]	Euclid-Guru:	Exactly, as Kyle had also written... so it seems to me that it's different from before, don't you think?
24	[9:48]	Kyle-Boss:	Yes, because now C lies on the perpendicular bisector
25	[9:49]	Kyle-Boss:	If it does not lie on it, it is impossible; whereas if it lies on it, there will be infinitely many circles
26	[9:49]	Euclid-Guru:	In episode 3, you had indeed deduced that there were infinitely many and that C had to lie on the perpendicular bisector ... but now we are asking the opposite ... if C lies on the perpendicular bisector of AB, are we sure that it must necessarily

		be the center of a circle through A and B?
27 [9:50]	Euclid-Guru:	That is, if C is any point on the perpendicular bisector ...
28 [9:50]	Euclid-Guru:	Let yourselves be helped by the properties you have already proved
29 [9:51]	Ellie2-Promoter:	Yes, it's enough that the radius of the circle is large enough to reach the points A, C, and B
30 [9:52]	Kyle-Boss:	Property 1 of the perpendicular bisector of a segment states that if C lies on the perpendicular bisector of segment AB, then AC and CB are congruent
32 [9:53]	Tommy-Blogger:	So the circle can be drawn because they are at the same distance from that point
33 [9:54]	Kyle-Boss:	So C is the center of the circle passing through A and B

In [1] Kyle-Boss activates the roles, while in [2] shifts to the communicative dimension, recapping the task to make it understandable to the group (BLO-KL), as we note in the excerpt "Euclid asks us to prove that", just as Tommy-Blogger does in [3]

In [5], we can observe a dialectic between the dimensions of rationality, as Ellie2-Promoter addresses the group to recall what had been done up to that point (BLO-KL) ("we had already done"), while also referring to empirical methods (PRO-KL), as we note in the excerpt "Graphically ... some proofs".

Kyle-Boss then works on directing the process, as we note in [9], particularly in the sentence "let's try to work from there" (BO-MK).

In [10], Euclid-Guru makes a questioning intervention that operates on the dialectic of two dimensions of rationality, both the epistemic and the communicative, with the aim of stimulating interpretative processes on the communicative side and thus leading the students to rewrite the statement in a conditional form (EU-ED; EU-CD), as we can recognize from the excerpt "What [...] it means that EVERY POINT [...] IS ...?". Tommy-Blogger's first response is ambiguous and imprecise; therefore, in [13] Euclid-Guru asks for a rewriting of what has just been said, once again acting on the communicative aspect, as shown in the sentences "Can you try to rewrite it here?" and "maybe fix the grammar a bit as well?" (EU-CD). This mediation by Euclid leads Tommy-Blogger to reformulate his previous statement according to a conditional structure; in particular, we note the use of "If ... then" (BLO-IP).

At that point, Tommy-Blogger becomes confused and mixes up the current statement with the one from episodes 1.3 and 1.4, that is, with its inverse and the contrapositive of the inverse [17], highlighting a deficiency of an interpretative nature

that is both epistemic, regarding the meta-theory of proof, and communicative regarding the logical structure of implications (LACK-PE-MK; LACK-BLO-IP)."

Euclid-Guru therefore shifts the attention first to the epistemic dimension, asking in [16] "Try to reason about why this statement is true" (EU-ED), and then to the communicative dimension in [18] and [23], prompting reflection on the differences related to the logical structure of the two statements (EU-CD), as we note from "it seems to me that it's different from before, don't you think?". Moreover, Euclid also activates the teleological dimension, particularly in [28], acting as a strategic guide, as we note from the sentence "Let yourselves be helped by the properties you have already proved" (EU-TD)."

Kyle-Boss therefore performs a shift to the epistemic dimension in [24], checking the validity of the reasoning (PE-KL), as evidenced by the expression "because now C ...".

Ellie2-Promoter therefore engages in a teleological reflection (PRO-KL), in which the expression "it's enough that the radius ... is" in [29] clearly aims to answer the question posed again by Euclid in [26], namely, "are we sure that it must necessarily be the center of a circle through A and B?"

In [30], however, Kyle-Boss highlights a dialectic among the three dimensions of rationality. In particular, the teleological dimension is evident from the choice of the geometric property useful for the purpose ("Property 1 of the perpendicular bisector of a segment") (PRO-KL); the communicative dimension emerges both from the need to convey its conceptual content ("states that") (BLO-KL) and from the presence of the conditional structure ("if ... then") (BLO-IP); finally, the epistemic dimension is highlighted by the checking of the conceptual property itself ("AC and CB are congruent") (PE-KL).

Similarly, Tommy-Blogger in [32] also highlights a dialectic among the three dimensions of rationality, as the "So" allows an implicative step (BLO-IP) that, however, leads to an empirical conclusion, as we note in "the circle can be drawn" (PRO-KL), subjected to epistemic checking with implicit reference to the constancy of the radius (PE-KL), as we see in the excerpt "because they are at the same distance from that point". This reference to the radius, although implicit, allows Kyle-Boss to conclude with a final implication ("So") (BLO-IP) that enables validation of the initial statement starting from accepted premises, as we note in "C is the center of the circle passing through A and B" (PE-KL)."

10.6 Episode 2

In this episode, Euclid asks the students to determine how many circles pass through three non-collinear points and, finally, to construct the unique circle that satisfies the required condition, providing a theoretical justification for the construction procedure followed. Compared to the previous episodes, in this episode, the roles and the actors–outlookers levels are rotated.

In this episode, strong mediation by Euclid was at times necessary in order to guide the students toward a conscious use of the Dynamic Euclidean Bank of Theory.

10.6.1 Excerpts from Actors'chat

Scene 1 - Actors

In this first scene, the students worked in a very empirical way, constructing the only circle that satisfies the required conditions, but without a deep awareness of the underlying theoretical reasons.

1	[8:58]	Kyle-Boss:	Come on, guys, let's try to solve the problem.
2	[9:01]	Tommy-Blogger:	So, Euclid asked us how many circles pass through three non-collinear points.
4	[9:05]	Shona-Pest:	I've drawn one.
6	[9:05]	Tommy-Blogger:	Me too, and it doesn't seem possible to draw any others.
9	[9:08]	Shona-Pest:	Look at the board, I've drawn one that passes through all three.
11	[9:09]	Ellie2-Promoter:	I don't think there's any way to draw another one.
12	[9:09]	Shona-Pest:	I think so too. But why only one?
13	[9:09]	Ellie-Promoter:	And it seems to me that the only possibility is that there is just one.
14	[9:10]	Ellie2-Promoter:	Exactly, but we have to prove it.
15	[9:11]	Euclid-Guru:	Good, I see that you are trying and have found one that works, but... why does it work? Have you asked yourselves?
16	[9:14]	Euclid-Guru:	It seemed to me that initially Ellie and Tommy were discussing something...
17	[9:15]	Euclid-Guru:	Yes... Tommy was reflecting on what had already been constructed in the past
18	[9:16]	Shona-Pest:	We know that infinitely many circles pass through two points
20	[9:17]	Ellie2-Promoter:	But why does only one circle pass through three

		points?
21 [9:17]	Shona-Pest:	Good question
22 [9:19]	Shona-Pest:	But how can what we discovered last time help us?
23 [9:21]	Euclid-Guru:	It seemed to me that Ellie had told me that she wanted to start from that... or am I wrong?
24 [9:22]	Shona-Pest:	One of those infinitely many circles also passes through C

Kyle-Boss in [1] encourages the group ("Come on, guys"), activating roles related to the assigned task ("let's try to solve the problem") (BO-KL), while Tommy recaps the task to make it understandable to the rest of the participants [2] (BLO-KL).

In [4] and [9], we identify a goal-oriented action, as emerges from the expression "I've drawn one" (PRO-KL), while Tommy-Blogger in [6] carries out a check on the possible constructions (PE-KL), with the emergence of the difficulty of representing different circles, as we note in the sentence "doesn't seem possible to draw any others". We also observe this in [11], where Ellie2-Promoter states "I don't think there's any way to draw another one", and in [13], where Ellie-Promoter says "And it seems to me that the only possibility is that there is just one", carrying out a check on the possible representations (PE-KL).

In [12], Shona-Pest then highlights the need to understand the reasons why the possible circle is only one, as emerges from the words "But why only one?" (PE-IP), and again in [14] Ellie2-Promoter expresses a need for causality in the sentence "but we have to prove it" (PE-IP).

An intervention by Euclid-Guru therefore becomes necessary, as he prompts the teleological component by asking "why does it work?" (EU-TD). Moreover, in [16] and [17], Euclid redirects the group's attention to the comics, as emerges from the utterances "initially Ellie and Tommy were discussing" and "Tommy was reflecting on what had already been constructed in the past". This allows Shona-Pest to draw on epistemic memory in the act of generating ideas (PRO-KL), as we can note in [18], particularly in the phrase "We know that".

In [20], we once again recognize a need for causality (PE-IP), as Ellie2-Promoter shows a need to understand the underlying reasons for the conjecture that has emerged, as we can see from the excerpt "But why does only one circle...?".

In her act of problematizing, Shona-Pest thus takes on the role of Euclid-Guru [22], as she poses a question to the group, prompting the teleological component (EU-TD; OU-IP), as emerges from the statement "But how can what we discovered last time help us?" and in [24], once again prompted to reflect by rereading the comics [23], she finally provides a constructive contribution (PRO-IP), as we can see from the

excerpt "One of those ... also passes through ...".

Scene 2 - Actors

In this scene, the students encounter the same difficulty previously faced when trying to represent more than one circle passing through three given non-collinear points. Moreover, at this stage, the first attempts at construction appear on Padlet, with the search for the perpendicular bisectors of the segments obtained by joining the three points. There is also encouragement from Euclid to describe these constructions in detail in words, in order to foster the emergence of further epistemological obstacles or the glimmer of visual intuitions conducive to the development of awareness.

1	[8:56]	Shona-Pest:	Yesterday we had left off by saying that one of the infinitely many circles passing through A and B also passes through C
2	[8:57]	Ellie2-Promoter:	And by trying, we managed to find only one
4	[8:58]	Euclid-Guru:	How did you find it? Could you explain to me how you constructed it?
8	[9:05]	Euclid-Guru:	If I'm not mistaken, Ellie had an idea earlier; she told me that she intended to look among the infinitely many circles passing through A and B for one that would also pass through C.
9	[9:07]	Ellie2-Promoter:	We could try that, but with the compass, larger or smaller circles do not match with all three points.
10	[9:05]	Euclid-Guru:	How can we construct it?
11	[9:08]	Shona2-Pest:	Starting from the points, I made the midpoint between A and B and then between B and C, and I connected them to form another point above, which will be the center of the circle.
12	[9:08]	Euclid-Guru:	Explain yourself better
13	[9:09]	Euclid-Guru:	What do you mean by you "connected them to form another point above"?
14	[9:09]	Shona2-Pest:	I extended the lines of the midpoints and created a point where they joined, as can be seen from the drawing.
16	[9:10]	Tommy-Blogger:	To find the circle, I took the three non-collinear points and built the midpoint of AB and the midpoint of BC, extended the segments, and found the point where they intersect, which is the center of the circle.
18	[9:12]	Euclid-Guru:	Tommy, can you describe in words the construction

		you made, which is visible in the drawings?
20 [9:21]	Tommy-Blogger:	I took the three non-collinear points A, B, and C. Then I connected A to B and B to C, found the midpoint M of AB and the midpoint M' of BC, and extended the segments until they met at a point O, which is the center of the circle. I placed the compass at O and drew the unique circle that passes through the three non-collinear points.
22 [9:22]	Euclid-Guru:	How did you extend them? What do you mean? What did you construct?
23 [9:22]	Tommy-Blogger	Those are construction arcs for the midpoint
25 [9:23]	Euclid-Guru:	Those segments... those "extensions", as you call them... do they remind you of anything?
26 [9:23]	Shona-Pest:	That the center of the circle is on the perpendicular bisector
28 [9:25]	Ellie2-Promoter:	Now we need to understand what to do with this information
29 [9:25]	Ellie-Promoter:	So the only circle that passes through all three points is the one whose center is at the intersection of the perpendicular bisectors
30 [9:26]	Ellie-Promoter:	And thus, since there is only one intersection point, there will be only one circle
31 [9:26]	Euclid-Guru:	You see, you have reached an important point... let's see how this interesting story continues

Shona-Pest performs a shift toward Blogger [1], as she recaps for the group the state of the work up to that moment (BLO-KL). In particular, this emerges from the expression "Yesterday we had left off by saying that". In [2], we also identify the effort to search for new circles (PRO-KL), highlighted by "And by trying", as well as an action of checking the possible representations (PE-KL), since Ellie2-Promoter states "we managed to find only one".

Euclid-Guru therefore prompts both the communicative component (EU-CD) ("Could you explain to me") and the teleological component (EU-TD), highlighted by two excerpts, namely "How did you find it?" and "how you constructed it" [4]. Furthermore, in [8] Euclid-Guru refers to what was happening in the comics, using them as a mediator to stimulate reflection within the group, as we notice in "If I'm not mistaken, Ellie had an idea earlier; she told me that ...".

This stimulates in Ellie2-Promoter a checking of the representations in [9] (PE-KL), but guided by strategic intentions (PRO-KL). We thus recognize a dialectic between two dimensions of rationality, both epistemic and teleological, as emerges from

the expression "larger or smaller circles do not match with all three points".

To Euclid-Guru's question "How can we build it?" (EU-TD), Shona2-Pest responds by describing the actions followed to build the drawing uploaded to Padlet and visible in Figure 10.6. In particular, in [11] we recognize goal-oriented actions (PRO-KL), highlighted by the expressions "I made the midpoint" and "I connected them," which lead to a constructive contribution of a teleological nature, in which the dynamic character of the construction process followed becomes apparent ("I connected them to form another point above") (PRO-IP).

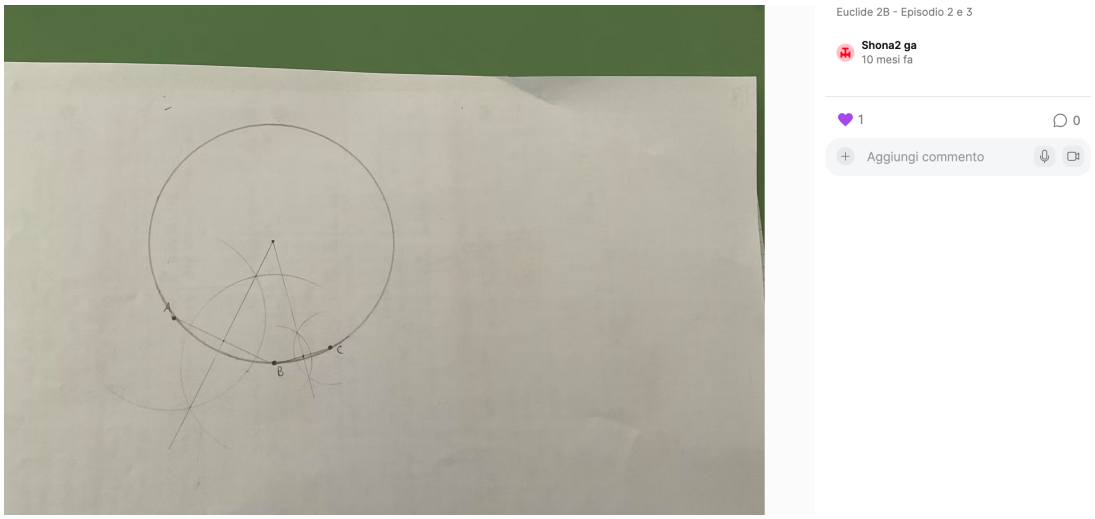


Figure 10.6: Picture on Padlet of Shona2-Pest - Episode 2

Euclid-Guru therefore asks for further clarification, prompting both the communicative dimension (EU-CD), as emerges from the utterance "Explain yourself better" [12], and the epistemic dimension, with regard to the search for the meaning of an expression left implicit, as in "What do you mean by" (EU-ED) [13]. Moreover, in [18] Euclid-Guru prompts the communicative component by also asking Tommy to provide a detailed account of the construction shared on Padlet, as emerges from the excerpt "describe in words the construction you made" (EU-CD).

In [14] we identify a dialectic among the three dimensions of rationality. In particular, although marked by communicative lack (LACK-BLO-KL) ("the lines of the midpoints"), Shona2-Pest makes more explicit the construction shown in Figure 10.6 to "create" point O, highlighted by goal-oriented actions ("I extended" and "I created a point") (PRO-KL), while at the same time highlighting a checking of the representation produced (PE-KL), as we observe in "where they joined".

In [16], Tommy-Blogger also communicates to the group the entire process followed to construct his representation, visible in Figure 10.7, highlighting a teleological dimension, as emerges from the phrase "To find the circle", by describing the actions carried out and goal-oriented ("I took", "built", "extended", "found") (PRO-KL). Finally, in the last expression ("which is the center of the circle"), we recognize a dialectic between an epistemic dimension, understood as checking the geometric property of the constructed object (PE-KL), and a constructive contribution that allows the process to move forward (PRO-IP).

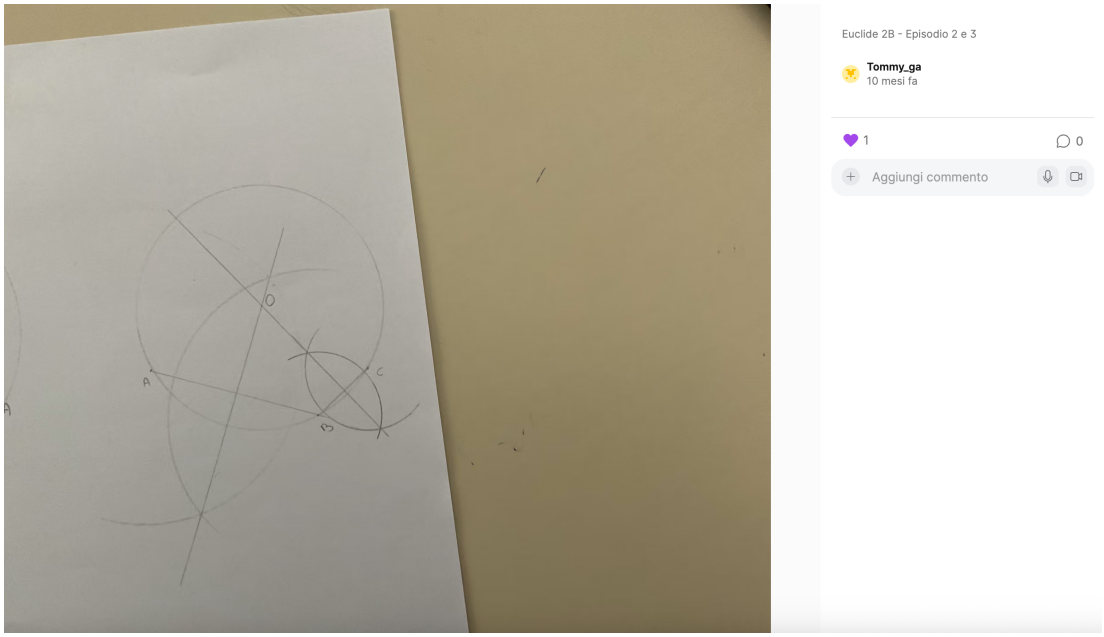


Figure 10.7: Picture on Padlet of Tommy-Blogger - Episode 2

In responding to Euclid's requests, in [20] Tommy-Blogger produces an even more precise communication than the previous one, oriented toward understanding (BLO-MK), yet with a strong teleological value. Indeed, we recognize numerous goal-oriented actions, as highlighted by the terms "I took," "I connected," "found," "extended," "I placed," and "drew" (PRO-KL). The teleological dimension is further emphasized by the expression "until they met at a point O," while we observe an epistemic dimension in the identification of the geometric property of the center, even if with an implicit reference to theory (PE-KL).

Euclid-Guru's attention thus shifts to the mathematical meaning of the lines drawn in the geometric constructions. To this end, in [22] Euclid encourages both

the teleological dimension (EU-TD), by asking "How did you extend them?", and the communicative dimension (EU-CD), in order to lead the students to name the objects involved, as we recognize in the question "What did you construct?".

Indeed, the descriptions provided by Shona2-Pest and Tommy-Blogger, especially expressions such as "I extended the lines of the midpoints and created a point where they joined", suggest that there is still no full awareness of the mathematical meaning of the lines drawn, and that the straightedge-and-compass constructions visible in Figure 10.6 and in Figure 10.7 were not carried out in order to construct the perpendicular bisectors of the segments, but only their midpoints, as emerges from Tommy-Blogger's expression in [23].

Therefore, a further intervention by Euclid-Guru becomes necessary, which in [25] stimulates the epistemic component of rationality through questioning oriented towards mathematical meaning to be retrieved from epistemic memory, as highlighted in the excerpt "those "extensions" as you call them ... do they remind you of anything?" (EU-ED).

This stimulates, first in Shona-Pest [26] and then in Ellie-Promoter [29], an awareness of the geometric and conceptual property of the point O that was found. In particular, we recognize a checking of the conceptual property of the center of the circle (PE-KL), as emerges from the expression "That the center of the circle is on the perpendicular bisector." Moreover, in the sentence "So the only circle that passes through all three points is the one whose center is at the intersection of the perpendicular bisectors," we observe a clear communication of the conjecture produced (BLO-MK) and we again identify a dialectic among the three dimensions of rationality. Indeed, we also recognise a teleological contribution indicating the aim (PRO-KL), as emerges from the excerpt "that passes through all three points", and an epistemic contribution as control of the conceptual property ("center is at the intersection of the perpendicular bisectors") (PE-KL). Furthermore, in the expression "the only circle ... is the one whose center," we also identify a constructive contribution that allows the process to move forward (PRO-IP).

Finally, in Ellie-Promoter's final statement [30], a deductive structure emerges, highlighted by the expression "And thus" at the beginning of the sentence. Subsequently, we also recognize a conditional structure, evidenced precisely by the terms "since ... there will be" (BLO-IP). In the excerpts "there is only one intersection point ... only one circle," we also identify a constructive contribution that advances the process (PRO-IP).

Scene 3 - Actors

At the end of the previous scene, Euclid-Guru invites the group to return to the comics,

as a new narrative sequence is unlocked. Having understood that exactly one circle passes through three non-collinear points, and having also identified where the center of such a circle is located, the comics now propose a sub-task: students are asked to determine which property the point O found possesses. In this phase, the students begin working towards a mathematical proof; they also ask to consult the Dynamic Euclidean Bank of Theory and construct the written proof as a product, again with the rational mediation of Euclid-Guru.

3 [9:33]	Euclid-Guru:	Why does the construction work? What property does point O have?
4 [9:33]	Tommy-Blogger:	Point O is equidistant from all the points on the circle
6 [9:35]	Ellie2-Promoter:	So OA, OB, and OC will be three radii
7 [9:37]	Ellie2-Promoter:	"Because A, B, and C are three points on the circle, and each point is equidistant from the center O
8 [9:37]	Euclid-Guru:	Why are they three radii? Why do you say that O is equidistant? How can you justify it from a theoretical point of view?
10 [9:41]	Tommy-Blogger:	Because it is the only point where the perpendicular bisectors intersect.
11 [9:41]	Euclid-Guru:	And how does this tell you about the equidistance?
12 [9:47]	Shona2-Pest:	Is there something in the bank of theory that explains the concept of equidistance between points?
13 [9:47]	Euclid-Guru:	I'm making my book available to you; consult it carefully!
14 [9:50]	Shona2-Pest:	Tommy and I think that the most useful is the 15
15 [9:51]	Ellie2-Promoter:	Def. 15 does indeed seem useful
16 [9:51]	Euclid-Guru:	Does it tell you about equidistance? What property does point O have?
17 [9:52]	Kyle-Boss:	I think it's Property 1 of the perpendicular bisector of a segment
18 [9:53]	Euclid-Guru:	Kyle, can you tell us what it states?
19 [9:54]	Kyle-Boss:	If C lies on the perpendicular bisector of segment AB, then AC is congruent to CB
20 [9:55]	Euclid-Guru:	Good, so what can you say about point O?
21 [9:55]	Shona-Pest:	So in this case, O is congruent for both
22 [9:56]	Euclid-Guru:	Let's say good... to whom does O belong?
25 [9:56]	Ellie-Promoter:	To the perpendicular bisectors
26 [9:56]	Euclid-Guru:	Which ones? The perpendicular bisectors of what? How could we say it better?
27 [9:57]	Ellie-Promoter:	O belongs to the perpendicular bisector of AB and to the perpendicular bisector of BC
29 [9:58]	Shona-Pest:	Because they intersect at that very point

31 [9:58] Tommy-Blogger:	If O lies on the perpendicular bisectors of AB and BC, then it is equidistant from all points on the circle
32 [10:00] Euclid-Guru:	I'll pick up what Tommy is saying... "If O lies on the perpendicular bisectors of AB and BC then"... let's continue from here, step by step
33 [10:03] Tommy-Blogger:	If O lies on the perpendicular bisectors of AB and BC, then OA, OB, and OC are congruent
34 [10:04] Euclid-Guru:	Okay, but I'd like more detail ... for lying on the perpendicular bisector of AB, which segments are congruent?
35 [10:05] Tommy-Blogger:	If O lies on the perpendicular bisector of AB, then OA and OB are congruent
37 [10:06] Tommy-Blogger:	If O lies on the perpendicular bisector of BC, then OB and OC are congruent
38 [10:07] Ellie2-Promoter:	The points OB and OC are congruent; therefore, if OA is congruent to OB, we have OA = OB = OC
39 [10:07] Euclid-Guru:	So, why do you say that all three are congruent? After all, here you have shown them to be congruent two by two... is there something in my book that can help you?
40 [10:08] Ellie-Promoter:	Common Notion 1
42 [10:09] Euclid-Guru:	Excellent!!! Very well done!!! You're really doing a great job!!! Ahhh, but... in your opinion, why did the perpendicular bisectors intersect? I mean... could they have failed to intersect?
43 [10:10] Ellie-Promoter:	No, because the points are not collinear
45 [10:12] Euclid-Guru:	Hmm, I still have another doubt... you constructed the circle, you explained why they intersect, you proved that OA, OB, and OC are all congruent, but... why are we sure that such a circle exists and is unique?
46 [10:12] Ellie-Promoter:	Because the perpendicular bisectors intersect at only one point
47 [10:13] Euclid-Guru:	Yes, but ... it's more subtle ... who guarantees that there isn't another one and, above all, that it exists with that center and those three radii ... after all, I only have three radii ... why should I be sure that it holds for all the points on the drawn circle?

Euclid-Guru opens with two requests that support both the teleological dimension, as emerges from "Why does the construction work?" (EU-TD), and the epistemic

component, through the questioning "What property does point O have?" (EU-EP).

In Tommy-Blogger's response [4], we thus recognise a dialectic among the three dimensions of rationality, particularly encapsulated in the term "equidistant." Indeed, this term assumes a communicative value, as the use of appropriate mathematical language consistent with the purpose (BLO-KL); an epistemic value, in relation to the conceptual property to which it refers (PE-KL); and a teleological aspect, as a constructive contribution that allows the process to move forward (PRO-IP).

Indeed, this term leads Ellie2-Promoter to make a deductive step [6] and to state: "So OA, OB, and OC will be three radii", establishing correct relationships between the mathematical objects and the concepts involved (PRO-KL). In [7], she then justifies this statement by monitoring the geometric property of equidistance (PE-KL) and with an implicit reference to the theory (PE-KL), as emerges from the sentences "Because A, B, and C are three points on the circle" and "each point is equidistant from the centre O".

In [10], Tommy-Blogger provides an initial idea of an answer, establishing correct relationships between mathematical objects (PRO-KL), as we note from the excerpt "is the only point where the perpendicular bisectors intersect", although some theoretical aspects are still left implicit.

After successive prompts ([8] and [11]) from Euclid-Guru regarding the search for a theoretical validation (EU-ED), Shona2-Pest in [12] highlights the need for causality, asking for access to the Dynamic Euclidean Bank of Theory. In particular, in [14], in the excerpt "Think that the most useful is the 15" we note a teleological dimension in the selection of useful knowledge for the aim (PRO-KL) and an epistemic dimension as a justification of reasoning with reference to theory (PE-KL), although with a communicative shortcoming related to the lack of specific references regarding the nature of the identified theoretical element (Definition, Theorem, etc.).

This is clearly checked by Ellie2-Promoter in [15], as we can see from the phrase "Def. 15 does ... seem useful" (PE-KL), in which Ellie2 checks the justify of the steps of reasoning ("seem useful") and also shows agreement with what was stated, as indicated by "indeed" (OU-MK).

In response to Euclid-Guru's request regarding the property possessed by point O [16], Kyle-Boss identifies Property 1 of the perpendicular bisector.[4] In the sentence "I think it's Property 1 of the perpendicular bisector," we recognise a reference to the

[4]This property was proven by the students themselves in preparatory activity 1 and concerns one of the two implications of the characterization of the perpendicular bisector, namely: "IF C belongs to the perpendicular bisector of segment AB, THEN AC is congruent to CB." This property was inserted, in conditional form, into the Dynamic Euclidean Bank of Theory after being explored, conjectured, and proven by the class group.

epistemic dimension in the precise recourse to theory as a justification for the steps taken (PE-KL), as well as a teleological reference in the selection of knowledge useful for the purpose (PRO-KL).

Euclid-Guru therefore asks for clarification of what has been said, acting on the communicative component (EU-CD), as we can see from the question "what it states?". This prompts Kyle-Boss to state the property, as presented in the Bank of Theory [19] (BLO-KL), and leads Shona-Promoter to introduce a new element [21], although with a communicative lack, as emerges from the excerpt "O is congruent for both" (LACK-BLO-KL).

From this point on, Euclid-Guru's interventions to encourage students to provide more detail, trying to guide them toward unpacking the information in order to make all steps explicit, as we can observe in [22], [26], [32], and [34], and in particular in expressions such as "Let's say good," "How could we say it better?" and "step by step" (EU-CD).

In [27], we recognize in Ellie-Promoter the use of correct verbal expressions (BLO-KL), particularly in the phrase "belongs to ... and to", while in Tommy-Blogger we observe an inference ("then it is equidistant from all points on the circle") deduced from previous statements and presented through the use of a conditional structure,"If ... then" (BLO-IP). This statement is later rewritten by referring to well-defined mathematical objects and their properties, as shown in "then OA, OB, and OC are congruent" (PE-KL). At Euclid's request, this implication is further detailed into two different implications, as we observe in [35] and [37], where we recognize an appropriate selection of knowledge (PRO-KL), communication through the use of the conditional structure (BLO-IP) and a reference—even if implicit—to the underlying theory (PE-KL).

Tommy-Blogger's previous contributions lead Ellie2-Promoter to formulate a deductive chain in [38] (as we can see from "therefore") (BLO-IP), allowing her to offer a contribution on the congruence of the three segments AO, OB, and OC, although without a real explicit statement of the theoretical reasons supporting the implication (PE-KL). Ellie2, however, shows a communicative shortcoming by referring to OB and OC segments using the term "points" (LACK-BLO-KL).

In [39], Euclid-Guru attempts to push further towards awareness, asking students to make the arguments they have put forward more explicit, as we can see from the question "why do you say that all three are congruent?" and from the prompt "After all, here you have shown them to be congruent two by two". All this is supported by the encouragement to consult once again the Dynamic Euclidean Bank of Theory, as we recognize from the question "is there something in my book that can help you?" (EU-ED).

In line 40, we identify a teleological dimension, observable in the appropriate and intentionally chosen theoretical reference (PRO-KL).

In [42], Euclid-Guru encourages the students to reflect on the reasons that allow the point O to be found, as can be seen in the questions "why did the perpendicular bisectors intersect?" (EU-ED) and "could they have failed to intersect?" (EU-TD).This leads, in [43], Ellie-Promoter to respond "No, because the points are not collinear", in which we recognize a justification of a step by checking the validity of the reasoning (PE-KL).

Finally, Euclid-Guru stimulates the group on a final aspect related to the existence and uniqueness of the circle found, requesting a theoretical justification that involves the epistemic component (EU-ED), as we can see from the question "why are we sure that such a circle exists and is unique?" [45] and from the excerpt "who guaranties that there is no another one" [47].

Scene 4 - Actors

In this scene, the group draws together the results of the work carried out and shares the writing of the proof to be sent to Euclid by email.

2	[11:43] Ellie2-Promoter:	We were reflecting on why the OA, OB, and OC rays are congruent, and thanks to the theory bank, we might have figured it out
3	[11:47] Ellie-Promoter:	We had said that, according to the first common notion, the three sides were equal; in fact, OA = OB = OC
5	[11:51] Ellie2-Promoter:	We were now reflecting on this, but we need to answer why, for three non-collinear points, there is only one circle
7	[11:55] Ellie2-Promoter:	Maybe the third one could help us?
8	[11:55] Tommy-Blogger:	Postulate 3?
9	[11:56] Shona2-Pest:	Postulate 3: And that with every center and radius a circle is drawn
10	[11:57] Ellie2-Promoter:	If for every point we draw a circle, only one circle out of infinite circles will pass through the three points
11	[11:57] Tommy-Blogger:	Because from the center O only one circle starts that passes through all three non-collinear points
12	[11:58] Shona2-Pest:	We proved it using the third postulate
13	[11:58] Euclid-Guru:	By now you have clarified everything that was

	unclear ... having done the construction with the perpendicular bisectors, proved the pairwise congruence of the segments OA, OB, and OC, as well as the congruence of all three, and understood why the circle is unique
14 [11:59] Euclid-Guru:	You also explained why the perpendicular bisectors intersect ... really, everything is clarified
15 [11:59] Euclid-Guru:	Now, Tommy, could you write a summary that also includes all the theoretical references?
19 [12:15] Tommy-Blogger:	Through three non-collinear points A, B, and C there passes only one circle, because the perpendicular bisectors of segments AB and BC intersect at a single point O, which is the center of the circle. The segments OA, OB, and OC are pairwise congruent because if O lies on the perpendicular bisector of AB, then OA and OB are congruent. If O lies on the perpindicular bisector of BC, then OB and OC are congruent. According to Postulate 3, for every center and every radius, there exists only one circle

In Ellie2-Promoter's intervention in [2], we recognize the external communication of a reflection on the reasons that support the theoretical validation of a statement (BLO-MK; PE-MK), as we can see from the sentence "We were reflecting on why the OA, OB, and OC rays are congruent", as well as awareness in choosing and using the appropriate mathematical means or tools to achieve the aim (PRO-MK), as shown in the excerpt "thanks to the theory bank, we might have figured it out".

In [3], we observe a shift towards Euclid, as Ellie-Promoter refers back to statements made earlier ("We had said that") (EU-CD; OU-IP), also stating claims from Dynamic Euclidean Bank of Theory, as we can see in the utterance "according to the first common notion, the three sides were equal; in fact, OA = OB = OC" (BLO-KL).

In [5], Ellie2-Promoter shows a need for causality (PE-IP), as we can see in the excerpt "but we need to answer why" and in [7] she selects appropriate means to address the goal (PRO-KL), as we recognize in the question "Maybe the third one could help us?", referring to the third postulate.

Shona2-Pest therefore states Postulate 3 by reading directly from the Dynamic Euclidean Bank of Theory, as we observe in [9] (BLO-KL). Instead, in [10] Ellie2-Promoter shows communication through the use of verbal representations

concerning implications (BLO-IP), where the comma plays the role of the term "then". This implication is supported by visual evidence expressed through goal-oriented actions, as we can see in the terms "we draw" and "will pass through" (PRO-KL).

In [11] Tommy-Blogger provides a justification that is based on empirical evidence rather than on theoretical validity (PRO-KL), as emerges from the sentence "Because from the center O only one circle starts", where goal-oriented actions are highlighted through the terms "starts" and "passes".

By contrast, in [12] Shona2-Pest shows an awareness in relation to the theory, stating "We proved it using the third postulate" (PE-MK).

Finally, in [19] Tommy-Blogger proposes a summary that will then be sent to Euclid by email. In the opening sentence, a teleological dimension emerges through the awareness that it is possible to identify a unique circle whose center coincides with the single intersection point of the two specified perpendicular bisectors (PRO-KL). From an epistemic perspective, the argument includes theoretical justifications supported by visual evidence, for example the claim that "the perpendicular bisectors intersect at only one point because the points are not collinear", even though there is no explicit reference to Postulate 5 of Euclid's Elements (PE-KL). Nevertheless, the text reveals a stronger attention to the meta-theoretical aspects involved in constructing the proof. Several propositions are indeed expressed in conditional form, even if implicitly invoked to justify certain steps (e.g.,"if O lies on the perpendicular bisector of AB, then OA and OB are congruent") (BLO-IP; PE-KL), with the exception of Common Notion 1 and Postulate 3 of Euclid's Elements, which are explicitly mentioned (PE-KL). Overall, the communication reflects a greater effort to align with the standards of the mathematical community, through a structuring of the proof increasingly oriented toward the theoretical validation of the statement (BLO-MK; PE-MK).

10.6.2 Excerpts from Outlookers'chat

In the excerpts from the observers' chat in this second episode, the students sometimes focus on the proposed proof problem; however, in most cases their discussions mainly concern the behaviors observed in the characters associated with their own role and active in the actors' group.

Scene 1 - Outlookers

1	[9:02]	Shona2-GO1:	How are you getting on?
2	[9:04]	Shona-GO2:	A circle passes through only three non-collinear points
3	[9:05]	Shona2-GO1:	How do you prove it?
4	[9:07]	Shona-GO2:	It is a theorem of Euclidean geometry

5	[9:12]	Tommy-GO1:	Through three non-collinear points, exactly one circle can be drawn
6	[9:13]	Shona-GO2:	Shona, the actor, intervenes but does not represent the character's role
7	[9:14]	Tommy-GO2:	Tommy, on the other hand, does not intervene much
8	[9:15]	Shona2-GO2:	Shona is asking questions as she should, but not continuously
9	[9:15]	Kyle-GO2:	Kyle, on the other hand, has not shown up yet. It seems that Shona has taken his place
10	[9:18]	Shona-Go2:	A circle can pass through three points only if they are not collinear
11	[9:18]	Shona2-GO2:	But isn't it a theorem of geometry?
12	[9:20]	Tommy-GO2:	We have to prove it
13	[9:21]	Shona2-Go2:	But how?
14	[9:21]	Shona-Go2:	By drawing

In [3], Shona2-GO1 problematizes the statement by requesting a proof, thus showing an epistemic need for causality (PE-IP). In response, Shona-GO2 [4] refers to theory (PE-KL), but leaves it completely unexplained and implicit (LACK-PE-KL), as we observe in the statement "It is a theorem of Euclidean geometry."

In [5], Tommy-GO1 states theorems from the Dynamic Theory Bank of Euclid (BLO-KL), referring to the goal-oriented action "can be drawn" (PRO-KL).

From this point on, several interventions become more oriented toward observing the roles interpreted by the actors. In particular, in [6] Shona-GO2 makes an observation about the role and behavior adopted by actors, as a noticing, as we can see from the term "intervenes" (OU-KL), also accompanied by a critical reflection on the role regarding the behavior adopted, as an acknowledgment, as we note in "but does not represent the character's role" (OU-MK).

In [7], Tommy-GO2 instead performs a check on the passive behavior of the observed character (OU-KL), as we recognise in the phrase "not intervene much," as does Shona2-GO2 in [8]. Instead, in [9] Kyle-GO2 makes a critical comparison between the characters interpreted by the actors (OU-MK), which we recognise in the excerpt "It seems that Shona has taken his place." At this moment, the observers' attention is once again directed toward the mathematical problem. Shona-GO2 in [10] establishes relationships among the mathematical objects involved ("can pass ... only if") (PRO-KL), while Shona2-GO2 in [11] checks the justification of individual steps (PE-KL), as we can see in the sentence "But isn't it a theorem of geometry?". This leads Tommy-GO2 to express a need for causality (PE-IP), as we note in the utterance "We have to prove it," although Shona-GO2 in [14] then suggests doing so by still

operating with empirical methods (PRO-KL).

Scene 2 - Outlookers

In this scene, the observers are almost exclusively oriented toward observing the behaviors of the actors' group, reflecting on the roles they observe.

1	[9:00]	Ellie2-GO2:	What did you say yesterday?
2	[9:02]	Shona2-GO2:	We observed our characters
4	[9:02]	Tommy-GO2:	Let's try to solve the problem ourselves as well
5	[9:10]	Ellie2-GO2:	I did the same thing as Tommy, the actor
7	[9:11]	Shona2-GO2:	I also agree with what the actors are doing
8	[9:11]	Tommy-GO2:	I've uploaded the drawing on the board if you want to take a look
9	[9:12]	Shona-GO2:	Shona is faithfully representing her character [5]
10	[9:13]	Tommy-GO2:	Tommy is also putting in effort
11	[9:13]	Ellie2-GO2:	Ellie doesn't speak much
12	[9:13]	Shona2-GO2:	Tommy, I agree with your drawing
13	[9:14]	Shona2-GO2:	Shona is helping a lot in solving the problem
14	[9:18]	Euclid-Guru:	I really like the constructions you made and added to the board... but why do they work?
16	[9:23]	Tommy-GO2:	Because if we join the perpendicular bisectors of the segments connecting A and B, and B and C, we obtain the center of the circle passing through the three points, and so we also understand that, since there is a single center, there will be only one circle passing through three non-collinear points.
17	[9:24]	Shona2-GO2:	I agree with the actors' solution
18	[9:24]	Shona-GO2:	Are you sure?
19	[9:25]	Tommy-GO2:	Me too, Tommy is doing his job really well!
20	[9:25]	Ellie2-GO2:	Yes, also Shona, even if this was more of Ellie's task

In [2], Shona2-GO2, in responding to Ellie2-GO2's question, shows awareness of observing her own assigned role (PU-MK), as we can see in "We observed our characters." In [5], Ellie2-GO2 compares her actions with those of one of the observed actors, although not corresponding to her own character, as we can recognize in the sentence "I did the same thing as Tommy, the actor" (OU-KL). Meanwhile, in [7], Shona2-GO2 highlights a reflection on the process, expressing agreement with the choices made by

[5] In the original Italian version, the term "reflecting" is used, which here has been translated as "faithfully representing"

the actors and thus operating at a meta-level (OU-MK), as we can see in the expression "I also agree with what the actors are doing."

In [8], Tommy-GO2 communicates to the group that he has uploaded a drawing to the shared Padlet, thus describing the actions performed (BLO-MK). Meanwhile, Shona-GO2 in [9], Tommy-GO2 in [10] and [19], Ellie2-GO2 in [11], Shona2-GO2 in [13], and Kyle-GO2 in [15] make observations about the observed role, as a noticing (OU-KL). Shona2-GO2 in [12] and [17] also reflects on the observed process, both within the observers' group and in the actors' group, expressing agreement with the choices made (OU-MK), as we can see in "I agree with your drawing" and "I agree with the solution of the actors." Ellie2-GO2, in [20], instead makes a critical reflection at a meta-level on the role interpreted by one of the actors, recognizing that certain characteristics are more typical of one character than another (OU-MK), which we identify in the excerpt "Even if this was more of Ellie's task."

A brief intervention by Euclid-Guru in [14] draws attention to the constructions shared on Padlet and to why these constructions work, as we can recognise in the question "but why do they work?" (EU-TD). This prompts Tommy-GO2 to provide an articulated response [16] that highlights a dialectic among the different dimensions of rationality. In particular, in "Because" we recognise a check of the conceptual property with an effort to justify the reasoning (PE-KL). In "if we join ... we obtain" we observe a conditional structure as a form of implication (BLO-IP), although not grounded in theory but rather in goal-oriented strategic actions that contribute to moving the process forward (PRO-IP). Finally, we can note another structure that recalls a deductive chain (BLO-IP), as shown in the excerpt "and so we also understand that," which leads to a new implication expressed through verbal representations in "since there is a single center, there will be only one circle"

Shona-GO2 finally takes on the role of Euclid-Guru by asking "Are you sure?" (EU-ED; OU-IP).

Scene 3 - Outlookers

1	[9:31]	Tommy-GO2:	Outlookers, what property do you think point O has?
2	[9:31]	Tommy-GO2:	I would start by saying that it is equidistant from all the points on the circle
3	[9:32]	Tommy-GO2:	Do you agree?
4	[9:32]	Ellie2-GO2:	Yes!
5	[9:37]	Shona-GO2:	I agree
6	[9:41]	Tommy-GO2:	I think Tommy is doing a really great job; he's putting together and organizing all the information

In this brief scene, the focus is entirely on Tommy-GO2, who initially in [1] takes on Euclid's role asking the group the same question that Euclid posed to the actors: "what property do you think point O has?" (EU-ED; OU-IP). Immediately afterward, in [2], he engages in self-communication, seeking a convincing answer (BLO-MK), as seen in the excerpt "I would start by saying that." This inner dialogue then unfolds into a request for approval from the group, as we note in "Do you agree?" in [3], once again adopting the communicative dimension as a form of questioning, and finally observing the character Tommy-Blogger, the actor, at a meta-level, as shown in the sentence "he's putting together and organizing all the information" (OU-MK).

10.7 Episode 3

In this episode, Euclid-Guru proposes a new task to the group of actors, as indicated in section 8.1.2. Specifically, he asks them to identify the relationship between the central angle and the inscribed angle subtended by the same arc in the particular case where one side coincides with the diameter. Students are first asked to explore a figure to generate a conjecture about this relationship and then to produce a proof that validates the conjecture within the Euclidean theoretical system. In this episode, roles and levels of interpretation (actors-outlookers) are rotated once again; nevertheless, due to some missing students, it was necessary for certain students playing again at actors level, which they had already experienced in previous episodes.

10.7.1 Excerpts from Actors'chat

Scene 1 - Actors

In this scene, the group of actors explores the figure proposed by Euclid in the comics and begins to formulate conjectures about the possible relationship between angle α and angle β, also sharing some drawings on Padlet. The exploration takes place entirely on a theoretical level, without any physical manipulation of the figure, even though more than one student has reproduced it on paper. During the exploratory phase, students do not use a protractor to measure angles because they do not have one; at one point, someone even asks aloud for one, but no one in the class has it available. No paper folding is carried out, so the exploration remains purely theoretical. This, of course, does not support the immediate production of a conjecture, but it does help to focus attention on the geometric properties of the elements present in the figure. In this case, the formulation of the conjecture does not occur immediately as a statement of fact;

rather, it is supported and accompanied by a constructive argumentation (Pedemonte, 2008), during which the arguments provided by the students already show references to the theoretical context and are often supported by theorems or definitions.

2	[9:16]	Kyle-Boss:	Let's reflect on the question posed
3	[9:16]	Ellie-Promoter:	Let's start by making some drawings inspired by the photo
4	[9:21]	Kyle-Boss:	We know that, according to definition number 17, AB is a diameter
5	[9:21]	Kyle-Boss:	And it seems to me that AC is a chord.
7	[9:21]	Ellie-Promoter:	We can also see visually that a kind of triangle is formed, am I wrong?
8	[9:22]	Shona-Pest:	We need to understand what it has to do with the angles
9	[9:24]	Shona-Pest:	A chord is a segment joining two points on the circumference that does not pass through the center, so we know that AC is a chord
10	[9:24]	Ellie-Promoter:	It is an isosceles triangle: two sides are equal and one is different
11	[9:25]	Shona2-Pest:	Triangle AOC is isosceles since it has two equal oblique sides and a different base
12	[9:26]	Shona-Pest:	Why are the two oblique sides equal?
13	[9:27]	Ellie-Promoter:	Because it is isosceles
14	[9:27]	Shona-Pest:	We need to prove it
15	[9:27]	Shona2-Pest:	It is isosceles because the slanted sides are radii and therefore equidistant from O
16	[9:28]	Ellie2-Promoter:	Because if we extend the radius CO, it becomes a diameter and consequently will be equal to AO
17	[9:30]	Kyle-Boss:	We know that AB is a diameter, AC is a chord, and CO and AO are radii, which forms an isosceles triangle, but now, what is the relationship between angles alpha and beta?
18	[9:32]	Shona2-Pest:	Since the triangle is isosceles, we can also say that the angle at C is equal to the one at A
20	[9:35]	Euclid-Guru:	Good morning, I see that there are some interesting drawings on the board.
21	[9:36]	Kyle-Boss:	The straight angle at O is 180 degrees
22	[9:36]	Kyle-Boss:	And beta is an exterior angle of the isosceles triangle
23	[9:37]	Shona-Pest:	Which angle are you considering?
24	[9:37]	Kyle-Boss:	At the center, O, 180 degrees
25	[9:38]	Kyle-Boss:	Beta is part of this angle
26	[9:39]	Euclid-Guru:	Let's try to read what's written on Tommy's

drawing... I see a possible clue

27 [9:40] Tommy-Blogger: Let's consider the 180-degree angle at O, which is made up of beta plus the angle of triangle ACO

28 [9:41] Euclid-Guru: Tommy, could you also recap here what you put on the board? That way, you involve all of us

29 [9:44] Tommy-Blogger: So, triangle ACO is isosceles because both point A and point C are equally distant from O. The angle at C is congruent to alpha since the triangle is isosceles, and the sum of the triangle's interior angles is 2(alpha) plus the angle at O, which is equal to 180

31 [9:45] Euclid-Guru: Interesting ... how can this lead you to a relationship with angle beta?

35 [9:47] Shona2-Pest: β is 180 − the angle at O

36 [9:48] Tommy-Blogger: Let's try to separate the two angles and place them next to each other

37 [9:49] Ellie-Promoter: The entire angle at O should be 180

38 [9:53] Euclid-Guru: Tommy says: "So triangle ACO is isosceles because both point A and point C are equally distant from O. The angle at C is congruent to alpha since the triangle is isosceles, and the sum of the interior angles of the triangle is 2(alpha) plus the angle at O". Meanwhile, Shona2 says: "β is 180 minus the angle at O"... I have the feeling that you are veeeeery close

39 [9:55] Euclid-Guru: Anyway, how do you think the two angles relate to each other? Have you tried exploring or manipulating the figure?

40 [9:56] Ellie-Promoter: By eye, I'd guess that beta might be twice alpha, or in any case larger

41 [9:56] Shona2-Pest: In my opinion, we can say that 2(alpha) is equal to β, because if 180 − O equals β and 2(alpha) + O equals 180, then 2(alpha) = β

42 [9:57] Shona2-Pest: Therefore, Alpha will be half of B

43 [9:58] Tommy-Blogger: Shona 2 is right: Beta is equal to 2 Alpha

Kyle-Boss in [2] activates the roles, focusing attention on the goal (BO-KL), as emerges from the excerpt "Let's reflect on the question". At the same time, Ellie-Promoter in [3] suggests goal-oriented actions, highlighting empirical strategies, as we can see in the phrase "making some drawings" (PRO-KL).

In [4], Kyle-Boss recaps what had already been highlighted in the comics, referring to Definition 17 of the Dynamic Euclidean Bank of Theory concerning the diameter of a circle (BLO-KL), as we note from the excerpt "We know that ... AB is a diameter".

Furthermore, in the utterance "according to definition number 17" we recognise an adherence to the reference theory (PE-MK). In [5] Kyle-Boss makes a remark about another element present in the figure, showing a visual-type strategy, as we can see in "it seems to me that" (PRO-KL).

Ellie-Promoter in [7] also uses empirical visual methods to explore the figure, as we can notice from the expression "We can also see visually that" (PRO-KL), prompted by Shona-Pest, who checks the relevance of the argument and carries out a checking for the effectiveness (or lack) of the strategic choices in relation to the aim (PE-KL). This emerges from the excerpt "We need to understand what it has to do with".

In [9], Shona-Pest states the definition of a chord (BLO-KL), consequently checking knowledge and conceptual properties (PE-KL), as emerges from "so we know that AC is a chord". This allows both Ellie-Promoter and Shona2-Pest to focus their attention on triangle AOC. Both notice that it is isosceles, checking the conceptual property (PE-KL), as we can see in the phrases "two sides are equal and one is different" and "since it has two equal oblique sides and a different base".

In [12], we recognize a contribution from Shona-Pest that highlights a need for causality (PE-IP), posing a question that prompts the epistemic component within the group (EU-ED; OU-IP), as we can see in the question "Why are the two oblique sides equal?". Shona-Pest also reminds in [14] that the claim that AOC is an isosceles triangle requires a proof, once again highlighting a need for causality (PE-IP), as shown in "We ned to prove it". This prompt is taken up by Shona2-Pest who, in [15], provides a contribution in which the dialectic between the three dimensions of rationality emerges. In particular, Shona2-Pest provides a justification of the previously stated claim (PE-KL), as we can see in "It is isosceles because", offering a constructive contribution that allows the process to move forward (PRO-IP) ("the slanted sides are radii"), which leads to a deduction expressed through verbal representations ("and therefore") (BLO-IP). Finally, this justification is supported by a reference, albeit implicit, to theory, as shown in the excerpt "equidistant from O" (PE-KL).

Ellie2-Promoter in [16] also seeks a justification for the equidistance (PE-KL), but does so through empirical methods (PRO-KL), as we can observe in the expression "if we extend the radius CO, it becomes a diameter". In the expression "and consequently will be equal to AO", we instead recognise a verbal representation indicating a deductive structure ("and consequently") (BLO-IP) and an implicit reference to the definition of a circle ("will be equal to AO") (PE-KL).

In [17], Kyle-Boss recaps what has emerged up to that point, in order to facilitate understanding, as we can see in "We know that" (BLO-KL), and finally focuses attention on the goal, reiterating the objective with a questioning (EU-TD; OU-IP), as shown in

the expression "but now, what is the relationship between angles alpha and beta?".

In [18], Shona2-Pest, shifting the focus from the sides to the angles of triangle AOC, highlights a deductive structure, as we can see in "Since ... we can also say that" (BLO-IP), making use of Proposition 5 of Book I of the Dynamic Euclidean Bank of Theory, although still in an implicit form. This can be recognized in the sentence "Since the triangle is isosceles ... the angle at C is equal to the one at A" (PE-KL).

In [21], Kyle-Boss engages in self-directed communication to clarify his thinking, recalling statements accepted by the classroom community, in a form of thinking aloud, as we can see in "The straight angle at O is 180 degrees" (BLO-MK). In [22], [24], and [25], he provides constructive contributions to the advancement of the process (PRO-IP), as we can observe in the excerpts "And beta is an exterior angle of the isosceles triangle" and "Beta is part of this angle".

In [20] and [26], Euclid-Guru thus shifts the group's attention to the drawings uploaded on Padlet, prompting Tommy-Blogger in [28] to also recap in the chat what was written on the drawing (EU-CD). Up to that point, Tommy-Blogger had in fact been silent in the chat, intervening only in [27] with an exploratory contribution and establishing correct relationships among the mathematical objects involved (PRO-KL), as we can see in "Let's consider the 180-degree angle at O, which is made up of beta plus the angle of triangle ACO". Tommy-Blogger at the same time (from 9:22 to 9:29) [6] had produced and shared an annotated drawing in Padlet, as shown in Figure 10.8, already starting to develop a proof that draws on some theoretical aspects. Below is the translation of the comment from the original Italian language:

> AB is the diameter, but triangle AOC is isosceles because points C and A are equally distant from the center O. The angle at C is congruent to angle α, therefore the sum of the interior angles is $2\alpha + O = 180°$.

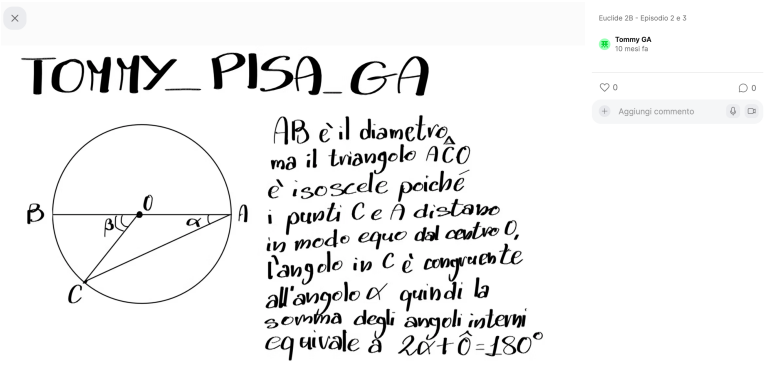


Figure 10.8: Episode 3 - Tommy-Blogger's figure on Padlet

[6] Recording of timings noted in the observation grid

In this comment, as well as in [29], we observe a deductive chain (BLO-IP). In particular, Tommy-Blogger provides a justification for the fact that triangle AOC is isosceles, referring to the definition, although not explicitly (PE-KL), as we can see in "triangle AOC is isosceles because points C and A are equally distant from the center O". Moreover, in [29], Tommy-Blogger further clarifies the reasoning that leads to considering the two angles at C and α as congruent, implicitly referring to the proposition 5 of Book I of the Dynamic Euclidean Bank of Theory about of isosceles triangles (PE-KL), as we can see in the expression "since the triangle is isosceles". Finally, Tommy-Blogger arrives at a relationship between angle α and angle at O, again implicitly referring to the theorem on the sum of the interior angles of a triangle, as we can see from "which is equal to 180" (PE-KL).

In [31] and [39], Euclid-Guru stimulates the group by prompting the teleological dimension, as we can see in the utterances "how can this lead you to a relationship with angle beta?" and "how do you think the two angles relate to each other?" (EU-TD).

In [35], we recognize a constructive contribution from Shona2-Pest, visible in the phrase "β is 180 the angle at O" (PRO-IP), while in [36], Tommy-Blogger attempts to carry out a goal-oriented manipulative action (PRO-KL), as we can see from the verbs used in the phrase "Let's try to separate ... and place them next to each other".

In [37], Ellie-Promoter checks a conceptual property, as shown in the phrase "the entire angle at O should be 180" (PE-KL). Instead, in [40], Ellie-Promoter provides a contribution that relies more on intuition and visualization, as shown in the expression "By eye, I'd guess that".

In Shona2-Pest's intervention in [41], we recognise a logical inference, specifically a modus ponens, highlighted by the terms "can say that ... because if ... then ..." (BLO-IP), and thus awareness of the logical aspects of the chosen demonstrative schemes also emerges (PE-MK), as we can see from the expression "In my opinion, we can say that". Moreover, in [42], a deductive chain arises, since a new implication follows from previous implications, as shown by the terms "Therefore" and "will be".

Finally, in [43], Tommy-Blogger shows agreement with Shona2-Pest's opinion, highlighting active observer behavior as a reflection on the process followed (OU-MK).

Scene 2 - Actors

In this scene, in response to the request to access the Dynamic Euclidean Bank of Theory, Euclid informs the group that the DEBT now contains updates, that is, newly added propositions, such as Proposition 5 from Book I of Euclid's Elements, which until now had not been made available within the Bank of Theory.

1	[9:59]	Shona2-Pest:	Can we access the theory bank?
2	[10:00]	Euclid-Guru:	Ah, sure... there are also some updates!
3	[10:04]	Tommy-Blogger:	Euclid, could Proposition 13 be useful to us?
4	[10:05]	Euclid-Guru:	Would you like to look it up in my book? I'll share it with you now... what do you think it might help you with?
5	[10:07]	Tommy-Blogger:	To prove that angle beta is equal to the sum of angles alpha and at C, since it is supplementary to the angle at O of triangle AOC
6	[10:09]	Euclid-Guru:	I have made my Bank available to you
7	[10:12]	Euclid-Guru:	I believe that your reasoning so far has been very good; now try to reformulate it with all the necessary theoretical references... also write it here step by step
8	[10:15]	Tommy-Blogger:	It could be like this: angle beta corresponds to twice angle alpha, and according to Proposition 13, which states that "if a line lying on another line forms angles, it will make either two right angles or angles equal to two right angles (supplementary)", therefore angles alpha and beta are supplementary, and beta is twice alpha

In [3], Tommy-Blogger chooses and uses appropriate knowledge as a theorem to reach the aim (PRO-KL), as we can see in the phrase "could Proposition 13 be useful ...?", immediately prompted by Euclid, who asks "what do you think it might help you with?", thereby stimulating the teleological dimension (EU-TD). In [5], Tommy-Blogger thus shows an awareness in choosing and using mathematical means appropriate to reach the aim, as we can recognize in the excerpt "To prove that ..." (PRO-MK).

Euclid's prompt to systematize with the proper theoretical references [7] leads Tommy-Blogger to perform an initial reorganization of thought [8], as emerges from "It could be like this:". In particular, in [8] we recognize a communicative dimension in stating a theorem from the Dynamic Theory Bank of Euclid (BLO-KL) and an epistemic dimension in referring in a awareness manner to a theory (PE-MK) ("according to Proposition 13, which states that "if a line lying on another line forms angles, it will make either two right angles or angles equal to two right angles (supplementary)""), with the additional clarification of the term "supplementary" in parentheses, as a translation of the archaic Euclidean formulation, almost as a form of communication with others in a valid and comprehensible way, and hence activating the writer-interpreter dynamics (BLO-MK). However, a communicative gap is also

recognizable on two levels: on one hand, the confusion between angle α and 2α, and on the other, the completely implicit steps that leave the subsequent implications entirely unexpressed (LACK-BLO-KL).

Scene 3 - Actors

In this final scene, students arrive at the writing of the proof to be sent to Euclid via email.

1	[10:41] Ellie-Promoter:	I added a detail to the drawing; I'm not sure if it could be a good line of reasoning
2	[10:41] Euclid-Guru:	Good, let's go through all the reasoning that Tommy and Shona2 had done... let's put the pieces together from the beginning, and for each step, precisely indicate the theoretical references that justify that step, and so on...
3	[10:44] Shona2-Pest:	We stated that the angle at alpha is equal to the angle at C according to Proposition 5, and that these two angles plus O must sum to 180. We also know that the sum of O and B must be 180 according to Proposition 13. From this, we can deduce that B is equal to 2 alpha because both B + O and 2 alpha + O sum to 180
4	[10:45] Euclid-Guru:	Ok ... that convinces me ... we're still missing the very first step and the theoretical justification for the last one
5	[10:57] Tommy-Blogger:	In the circle, segment AB is the diameter because, according to Definition 17, "the diameter of a circle is a straight line passing through the center and bounded on both sides by the circumference of the circle, which also divides the circle in half", and segment AC is a chord. Triangle AOC is isosceles because sides AO and CO are congruent, and therefore, according to Proposition 5, which states that "the angles at the base of an isosceles triangle are equal", the sum of these two angles (alpha and the angle at C) plus the angle at O of triangle AOC equals the sum of the interior angles of the triangle, which is 180. According to Proposition 13, the sum of angle O and beta must also be 180 since they are supplementary. Therefore, we can deduce that angle beta equals 2(alpha) because beta + angle O = 180 and 2(alpha) + angle O = 180, so alpha is half of

	beta
8 [11:01] Euclid-Guru:	Then I think the only thing missing is to explain why you make the last statement ... what lets you conclude that if 2 alpha + O = 180 and beta + O = 180, then 2 alpha = beta?
9 [11:05] Shona2-Pest:	2 alpha + O equals 180 because the sum of a triangle's interior angles is two right angles, beta + O equals 180 according to Proposition 13, and finally, 2 alpha equals B according to the first common notion
10 [11:06] Kyle-Boss	I'd say we're very close, if we haven't quite got there yet
11 [11:07] Euclid-Guru:	You're talking about the first notion ... hmm ... what does it state?
12 [11:08] Ellie-Promoter:	Things equal to the same thing are also equal to each other
13 [11:09] Euclid-Guru:	I'm not sure if the first notion really fits, it seems a bit different... I don't know
14 [11:11] Tommy-Blogger:	2(alpha) = B because if 180 − O equals β and 2 alpha + O equals 180, then 2(alpha) = β . Is this what you mean, Euclid? The one Shona wrote earlier
15 [11:12] Euclid-Guru:	So? What does this tell you from a theoretical point of view?
16 [11:13] Tommy-Blogger:	The third notion?
17 [11:13] Euclid-Guru:	Why? What does it state?
18 [11:14] Tommy-Blogger:	If equals are subtracted from equals, the remainders are equal.
20 [11:23] Tommy-Blogger:	In the circle, segment AB is the diameter because, according to Definition 17, "the diameter of a circle is a straight line passing through the center and bounded on both sides by the circumference of the circle, which also divides the circle in half", and segment AC is a chord. Triangle AOC is isosceles because sides AO and CO are congruent, and therefore, according to Proposition 5, which states that "the angles at the base of an isosceles triangle are equal", the sum of these two angles (alpha and angle at C) plus the angle at O of triangle AOC equals the sum of the interior angles of the triangle, which is 180. According to Proposition 13, which states that "if a line lying on another line forms angles, it will form either two right angles or angles equal to two right angles (supplementary)", the sum

> of angle O and beta must also be 180 since they are supplementary. From this, we can deduce that angle beta equals 2(alpha) because beta + angle O = 180,
>
> and 2(alpha) + angle O = 180, and according to Definition 3, which states that "if equals are subtracted from equals, the remainders are equal", we can also deduce that alpha is half of beta

In [3], we recognize a contribution from Shona2-Pest that highlights several dimensions of rationality. In particular, we can observe a need for communication as a requirement to reorganise reasoning into a logical structure (BLO-MK), which is especially evident in the expressions "We stated that ... and that ... We also know that ... From this we can deduce that". There is also a awareness of the need for theoretical validation of the implications made (PE-MK), as seen in the phrases "according to Proposition 5" and "according to Proposition 13". Furthermore, Shona2-Pest justifies the last step, as we can see from "because" (PE-KL).

Similarly, in [5], Tommy-Blogger reorganises the reasoning to communicate externally, by email, with Euclid-Guru (BLO-MK). We again notice a deductive structure with explicit and well-informed deductive chains, as emerges from "and therefore" and "Therefore, we can deduce that" (BLO-IP; BLO-MK). Furthermore, in this case, we can also observe numerous theoretical references supporting the reasoning, with an awareness of the need for explicit theoretical validation, as seen in the excerpts "because, according to Definition 17", "according to Proposition 5" and "According to Proposition 13" (PE-MK). These theoretical references are also stated one by one, taken literally from the Dynamic Euclidean Bank of Theory (BLO-KL). Finally, we observe a communicative and epistemic gap in the passage referring to the congruence of sides AO and CO, as the definition of a circle is left entirely implicit (LACK-PE-KL).

In [8], Euclid-Guru prompts the epistemic dimension, asking to make explicit the theoretical reasons for "if 2 alpha + O = 180 and beta + O = 180, then 2 alpha = beta?" (EU-ED). This stimulates Shona2-Pest in [9] to justify all steps of reasoning consciously and explicitly, stating the theorems used from the Dynamic Theory Bank of Euclid, as we can see in "because the sum of a triangle's interior angles is two right angles" and "according to Proposition 13" (PE-MK; BLO-KL). Finally, in [9], we recognize an epistemic lack ("according to the Common Notion 1"), cited in place of the Common Notion 3.

Euclid-Guru therefore seeks to stimulate reflection, prompting the communicative dimension in [11] and [17], as we can see in "what does it state?" (EU-CD) and the

epistemic dimensione in [15] ("What does this tell you from a theoretical point of view?") (EU-ED). This allows Tommy-Blogger to select Common Notion 3 from the DEBT ([16]) (PRO-KL) and correctly state it ([18])(BLO-KL), as recognized in the intervention "If equals are subtracted from equals, the remainders are equal".

Finally, in [20], Tommy-Blogger reorganizes the communication by sharing it with the group, in order to prepare the text to be sent by email to Euclid-Guru. The arguments already presented in [5] are reiterated. The reasoning is articulated through explicit and clear deductive chains (BLO-IP), as signaled by expressions such as "and therefore" and "Therefore, we can deduce that" that shows awareness of the logical aspects of the chosen demonstrative schemes and their consistency in relation to the reference theory (BLO-MK). Moreover, the argument is supported by multiple theoretical references, showing an awareness of the need for explicit theoretical validation, as evidenced by statements such as "because, according to Definition 17", "according to Proposition 5", and "according to Proposition 13" (PE-MK). These references are presented with accuracy by drawing on the Dynamic Euclidean Bank of Theory (BLO-KL). However, again two lacks emerge: an epistemic lack in the step concerning the congruence of segments AO and CO, where the definition of a circle is left entirely implicit (LACK-PE-KL) and a communicative lack in the end of contribution, in which Tommy-Blogger confuses the Common Notion 3 with the Definition 3 (LACK-BLO-KL).

10.8 Summarising the findings

In this section, we aim to provide a summary of the results obtained from the fine-grained analysis of the chats, based on the criteria outlined and clarified in the section 8.3. To this end, the analytical categories are presented below in tabular form, as shown in Table 10.1, Table 10.2 and Table 10.3, and detailed through operational indicators, which are partly theory-informed—further refining the a priori analysis already developed—and partly emergent from the data. The theory-based indicators are reported in the table along with their corresponding theoretical references, while the indicators emerging only from the data appear usually without theoretical references.

Table 10.1 presents the categories Promoter, Pest, Blogger, and Boss, which are thus detailed and further refined with respect to the prior a priori analysis, and enriched with evidence emerging directly from the data. Even some theory-based indicators, in fact, emerged from the fine-grained analysis of the data, thus requiring a further process of theoretical refinement. In addition to these, two further categories have emerged from the data, which we have named Outlookers (OU) and Euclid (EU), presented

respectively in Table 10.2 and Table 10.3.

Table 10.1: Categories of analysis and related operational indicators

	Operational indicators of analysis		
Categories of analysis	**Knowledge level KL**	**Meta - knowledge level MK**	**Intra-levels Interplay IP**
Promoter PRO	**Recourse** to memory in the act of idea generation (Albano et al., 2021) **Generating** ideas through intuition and analogy **Performing or recalling** actions oriented towards the aim (Habermas, 1998) **Choosing** and **using** the appropriate knowledge, methods and/or theorems to reach the aim. (Urhan & Bülbül, 2023a; 2023b) **Establishing** correct relationships between the mathematical objects and concepts involved. (Urhan & Bülbül, 2023a) **Choosing** suitable methods to prove the statement (Urhan & Bülbül, 2023a) **Selecting** tools that serve the aim and **using** these tools in the process in a purposive manner. (Urhan & Bülbül, 2023a) **Need for** certainty (Zaslavsky, 2021), **using** empirical methods	**Awareness of the need for a** functional connection of proving strategies with their aim (proof), with reference to possible schemes of proof for the logical organisation of proof. (Boero & Azrou, 2023) **Awareness of the need** for appropriate systems of representing meta-knowledge, e.g. implicative arrows. (Boero & Azrou, 2023) **Awareness of** consistency of different lines of reasoning (e.g. semantic and/or syntactic proofs) (Albano et al., 2021) **Awareness in** choosing and using of mathematical means or tools appropriate to reach the aim (Urhan & Bülbül, 2023b)	**Provides the contributions** of a constructive nature that allow the process to move forward (Albano et al., 2021) and that guide generalisation. **Control of** consistency in the cognitive unity construct and (in relation to the aim) the discontinuity that may emerge in this construct in the transition from proving to proof. (Boero, 2017, Pedemonte, 2007; 2008)
Pest PE	**Validation of statements** from accepted premises and using correct reasoning rules (Habermas, 1998) **Checking** the requirements of truth of statements and validity of reasoning (Albano et al., 2021). **Checking** the consistency of the patterns of use of the concepts, procedures and conceptual properties used and the correctness of the representations of the concepts (Urhan & Bülbül, 2023a) **Checking for** the effectiveness (or lack) of the strategic choices in relation to the aim (Urhan & Bülbül, 2023a) **Checks** the lack of operation mistakes while structuring the process. (Urhan & Bülbül, 2023a) **Checking** on constructions mathematically correct and valid representations. (Urhan & Bülbül, 2023b) **Justify** all the steps of reasoning, although sometimes with implicit references to the theory or using set of accepted statements (Urhan & Bülbül, 2023b) **Checking** the justify of all the steps of reasoning (Urhan & Bülbül, 2023b) **Checking** the soundness of all reasoning steps. (Urhan & Bülbül, 2023b)	**Meta-knowledge of proof** and reference of validity of proof to the epistemological assumptions of a theory (Boero & Azrou, 2023, Mariotti, 2006) **Awareness of** the logical aspects of the chosen demonstrative schemes and their consistency in relation to the reference theory (Boero & Azrou, 2023; Mariotti, 2006) **Awareness of** the accuracy of deductive chains and symbolic and verbal representations **Awareness of** the existence of a mathematical object through the perception of its absence	**Control of** the ambiguity of the statements made and the quality of the communication of logical steps. **Control of** epistemic lacks **Need for** causality (Zaslavsky, 2021)

Blogger **BLO**	**Writing correctly and giving clear explanations** regarding the steps taken during the process, (Urhan & Bülbül, 2023a) also involving visual and dynamic aspects of representations. **Using** mathematical language and symbols correctly and in accordance with the aim, thereby producing formal proof (Albano et al., 2021; Urhan & Bülbül, 2023a) **Stating** theorems or definitions from the Dynamic Theory Bank of Euclid **Recalling** for the group the task proposed **Using the** symbolic language of mathematics properly (Urhan & Bülbül, 2023b) **Facilitating** the reading and understanding of problem solving by verbal and/or textual explanations (Urhan & Bülbül, 2023b) **Providing** the reasons for the correctness of the drawings and the transitions within and/or between the representations (Urhan & Bülbül, 2023b)	**Internal role in itself:** Effectiveness of communication with herself in the proving process, producing an explanation or a justification of the validity or not validity of a statement from the subject's own perspective (Balacheff, 2010; Boero & Azrou, 2023) **Internal role in the group as a mediator of communication:** Communicate the process to others, by describing the actions performed Communicating with others in a valid and comprehensible way, and hence activating the writer-interpreter dynamics (Urhan & Bülbül, 2023b) Communicating with others searching mutual understanding and negotiation (Habermas, 1998) Effectiveness of communication with others to achieve the status of proof, shared and accepted by a small community (Balacheff, 2010) **External role for the Culture:** Effectiveness of communication with the Culture, in the formal writing of the mathematical proof, taking cultural format constraints into account. (Albano et al., 2021; Balacheff, 2010; Boero & Azrou, 2023) **Need for** communication as a reorganisation of information into a logical structure (Zaslavsky, 2021).	**Use of** symbolic and verbal representations concerning implications or deductive chains. (Boero & Azrou, 2023)
Boss **BO**	Keeps the goal fixed, **stimulating** the activation of all cognitive functions (Albano et al., 2021). **Operational guide** that leads the actions of group	**Monitors** and **directs** the process (Albano et al., 2021) **Awareness of dialectical dimension** amongst three rationality roots **Systematic taking care** of epistemic, teleological and communicative requirements for the used mathematical knowledge throughout the proving process. (Boero & Azrou, 2023)	**Control of** the need for a move to the next level of the proving process (Albano et al., 2021) **Activation and direction of** attention **Cares for** and **prompts** the moments **of transition** from process to product (Albano et al., 2021) **Takes care** of emotional aspects, supporting the group in moments of impasse

Table 10.2: Outlookers'category and related operational indicators

	Operational indicators of analysis		
Category of analysis	**Knowledge level KL**	**Meta - knowledge level MK**	**Intra-levels Interplay IP**
Outlooker OU	**Observation** about the role and behavior adopted by the actors as a noticing **Compares** their own actions with those observed in the group **Recalls** actions or goals learned through observing the processes	**Reflective and critical observation** of the different roles and the behavior adopted, as an acknowledgment (I would have done … in his/her place; intervenes but does not represent the character's role) **Reflection** on the mathematical process observed, showing agreement or disagreement **Awareness of** collective knowledge born from observing processes, which helps in choosing the actions to be taken **Reflective action** highlighted by a conditional structure on meta-thinking	**Reproducing** the observed behaviors of the Guru (internalization of the Guru)

Table 10.3: Euclid's category and related operational indicators

	Operational indicators of analysis		
Category of analysis	**Rational mediator**		
	Teleological dimension TD	**Epistemic dimension ED**	**Communicative dimension CD**
Euclid EU	Why does this method work? How can this lead you to …? How do you imagine … might be? Have you tried manipulating the figure? What do you think … might be useful for? Focus the group's attention on the objective Can you give me an example …? Why do you think you can't draw it? What is the difference between the method of … and that of …? Why does it work? How did you find it? How can we build it? How can this lead you to a relationship with angle beta? How do you think the two angles relate to each other?"	Request for theoretical justification How does this inform you about …? What properties does … have? Why do you agree with …? What does this tell you from a theoretical point of view? How could you justify …? Calls for greater detail in the proof Why not? Reflect on why you find it difficult What does it mean for a circle to pass through A and B? Encourages reflection on the metacognitive aspects of the proof What do you mean by …? Do they remind you of anything?	Do you agree? What does … say? How can we reformulate this better? How can we express it better? Fosters a communicative dimension oriented toward mutual understanding: "Agree on what to write" Focuses attention on shared representations Could you explain more clearly …? Explain yourself better

Requires reformulation with attention to theoretical references – EC

Focuses attention on statements made by other students – TC

We observed that all these behavior's indicators are not exclusive to a single role but are enacted, to varying degrees, by all participants, capturing evolving patterns rather than fixed behavioral assignments. We now intend to present a summary of the main findings that emerged from the analysis conducted, providing, as an illustrative example, some paradigmatic instances drawn from the Actors/Outlookers chats, in order to carry out a careful reflection on the processes described.

Finding 1: Internalization of the Guru's questioning interventions

The Euclid category is not intended primarily to analyse Euclid-Guru's actions, since this role is enacted by the author of the study in the dual position of teacher-researcher, and its interventions were designed a priori within the Theacher Rational Questioning Framework. However, this category is particularly relevant for us, as its operational indicators allow us to capture processes of internalization of the Guru's questioning practices within the students' activity, among actors and, even more especially, within the Outlookers group. The role played by Euclid has changed compared to the first model of Inside-Out. In the earlier version, the Guru represented access to resources and facilitated such access, but maintained a privileged dialogue only with the Promoter. In the second model, developed and implemented for the experimental study presented in Section 8, Euclid-Guru is instead embedded within the actors' chat as a *Rational Mediator*, interacting with the entire group. His constant presence within the actors' group not only enables more effective mediation with the students directly engaged in solving the problem, but also—and more importantly—allows the Actors and the Outlookers to witness his actions and internalize his interventions. Although each Outlooker was expected to focus primarily on their assigned role, we nonetheless observe that they display behaviors attributable to the observation of the actors and, in particular, of Euclid-Guru. In this sense, the Outlookers engaged with the entire process, developing a form of attentive and participatory observation.

Indeed, we observed that some students, by actively engaging with both the actors' contributions and Euclid-Guru's interventions, displayed forms of problematization. Specifically, they autonomously posed questions aimed at activating the three components of rationality, which we interpret as self-generated questioning and as moments of internalization of the Guru's role. Some students act in particular by stimulating the teleological dimension aimed at activating the strategic function (see Kyle-Boss in [30], Ep. 1 – Scene 1 – Actors, see Tommy-Blogger in [3], Ep. 1 – Scene 2 – Actors, see Ellie-GO1 in [3], Ep. 1.3 – Scene 2 – Outlookers). Other students, instead,

engage in questioning that prompts the epistemic component, highlighting a need for causality (see Shona-Pest in [12], Ep. 3 – Scene 1 – Actors; see Shona2-GO1 in [8], Ep. 1.3 – Scene 3 – Outlookers; see Shona-GO2 in [], Ep.2 - Scena 2 - Outlookers).

Nevertheless, Euclid's interventions do not always take the form of questioning. For example, in [31] and [32] (Ep. 1 – Scene 2), Euclid intervenes in support of the actors' group by validating the soundness of the justification developed and by encouraging the construction of a shared written formulation, without resorting to questioning. Anyway, this interventions promot a communicative dimension oriented toward mutual understanding, as it supports the negotiation of shared agreement within the small group. For this reason, We interpret the internalization of the Guru's interventions not only in terms of internalized questioning, but also in terms of actions and behaviors that reflect a reflective guiding stance (see Ellie-GO1 in [5], Ep. 1.3 – Scene 3 – Outlookers), an encouraging action in the pursuit of a deeper epistemic understanding (see Ellie-GO1 in [33], Ep. 1.4 – Scene 1 – Outlookers), and a taking of responsibility for the communicative dimension, prompting greater clarity of expression (see Ellie-GO1 in [ ], Ep. 1.4 – Scene 1 – Outlookers).

It is also for this reason that, in moments of internalization of Euclid's questioning/reflecting interventions, we have associated a double code, that identifies both Euclid's contribution in relation to the specific component of rationality, as well as the behaviors observed from the Guru, and which we interpret as a bridge towards the meta-knowledge level.

However, it is important to emphasise that for us the use of similar forms of questioning by some students, particularly Outlookers, does not necessarily indicate that the questioning has been fully and stably internalised, as internalisation has a specific meaning within the sociocultural perspective adopted in this thesis. Such forms of questioning might rather indicate the appropriation or re-enactment of behaviours similar to that of Euclid-Guru within the context of a particular activity. We consider it useful to make this clarification, because the process of internalisation is one that takes a long time to become fully stable.

Finding 2: Dialectical dimension between roots of rationality as movement towards theoretical validation and meta-level

The shift toward the meta-level is more evident in those interventions where the three dimensions of rationality do not remain clearly distinct and separate but instead emerge in a dialectical form (see Tommy-Blogger in [17], Ep.1 - Scene 2 - Actors). Indeed, in this case the expression "We need to find a way" shows a teleological dimension, highlighted by the terms "need" and "find" as indicating the need to explore a

strategy at knowledge level. At the same time, the expression "to prove this" brings out the epistemic dimension, oriented towards the need to search for a proof, although still in the form of a justification, that we recognise into interplay-level. Finally, in the expression "not in a graphic way", we identify a communicative dimension at the meta-knowledge level, related a type of representation to be used in order to communicate the reasoning to the mathematical community. This paradigmatic Tommy-Blogger's contribution shows the presence of all three dimensions of rationality closely interlinked with a shift towards the meta-level. When this occurs, we recognise a forward movement in the knowledge process (see Shona2-Pest in [15], Ep. 3 - Scena 1 - Actors), highlighted for example both by reflection on the reasons supporting the theoretical validation of a statement and by an increasing awareness in choosing and using mathematical means or tools appropriate to achieve the goal (Ellie2-Promoter in [2], Ep. 2 - Scene 4 - Actors). The shift toward the meta-level is also recognisable in interventions that guide toward generalization, highlighted by terms such as "any" or "more" mentioned three times consecutively (see Shona2-GO1 in [10] and [17], Ep. 1.1 – Scene 1 – Outlookers; see Shona2-GO1 in [17], Ep. 1.3 – Scene 1 – Outlookers). We recognise a dialectical interplay between three dimensions of rationality, which leads to a shift towards the theoretical perspective, also in the use of mathematical terms (as "equidistant") that assume communicative value, in accordance with the purpose for which they are used, epistemic value, in relation to the conceptual property to which it refers and teleological value, as a constructive contribution that allows the process to move forward (see [4], Ep. 2 - Scene 3 - Actors).

In some cases, the dialectic between the components of rationality, especially the epistemic and communicative dimensions, drives the process toward the search for theoretical validation, even when references to the underlying theory remain only partially explicit (see Shona2-Pest in [18], Ep. 3 – Scene 1 – Actors).

In other cases, we can observe a need for communication understood as a requirement to reorganize reasoning, with an awareness of the need for theoretical validation of the implications made in reference to the Dynamic Euclidean Bank of Theory, highlighted by expressions such as "according to Proposition 5" and "according to Proposition 13" (see Shona2-Pest in [3], Ep. 3 – Scene 3 – Actors; see Tommy-Blogger in [8], Ep. 3 – Scene 2 – Actors). Sometimes, the dialectical dimension among the components of rationality also highlights a awareness of the logical aspects underlying the chosen demonstrative schemes (see Shona2-Pest in [41], Ep. 3 – Scene 1 – Actors; see Tommy-Blogger in [5], Ep. 3 – Scene 3 – Actors) or makes it possible to identify a form of control over the conceptual properties involved, accompanied by an effort to justify one's reasoning through conditional structures, expressed as implications or chains of

deduction in verbal form, grounded in goal-oriented strategic actions as contributions that support the forward progression of the proving process (see Tommy-GO2 in [16], Ep. 2 – Scene 2 – Outlookers).

We also observed that these moments act as a driving force for the evolution of the proving process, also through a growing need for theoretical support that serves problem-solving goals, with a shift from reliance on empirical evidence towards seeking theoretical validation (see Ellie-GO1 in [8] - [9] - [10], Ep. 1.4 - Scene 1 - Outlookers). Therefore, we recognise a dialectic dimension between three roots of rationality also within summarizing the conjecture produced in a communication addressed to the group, supporting it with constructive arguments as a justification of the conjecture itself (see Shona2-GO1 in [12], Ep. 1.3 - Scena 3 - Outlookers).

We recognise the shift toward theoretical validation also in moments when students move from statements grounded in empirical evidence to more abstract forms of communication, for example from "It is impossible to draw it" to "There are not exist circle...". In this transition, we observe an increase in the epistemic value of their statements: they no longer refer to an operational failure, but instead formulate a claim grounded in theoretical properties (see Tommy-Blogger in [30], Ep. 1.4 – Scene 1 – Actors).

We identify the emergence of a need for theoretical validation also in situations in which students move away from empirical evidence in order to seek theoretical support to justify it, as for example in the real conditional construction "But there could be something theoretical if we can't do it graphically" in which the use of the future tense as a real conditional indicates an initial shift toward the meta-level, although not yet accompanied by full awareness (see Tommy-Blogger in [26], Ep. 1.4 – Scene 1 – Actors).

It is important to note that, while the actors' group still displays difficulties related to meta-theoretical aspects, confusing a statement with its inverse and with the contrapositive of the inverse, and showing interpretative shortcomings both of an epistemic nature (concerning the meta-theory of proof) and a communicative nature (concerning the logical structure of implications) (see Tommy-Blogger [17], Ep. 1.5 – Actors), the Outlookers' group, on the other hand, exhibits a significant shift toward the meta-knowledge level. This is evidenced by the utterances "So, the center is always on the perpendicular bisector" and "So, the center cannot not be on the perpendicular bisector" and "So, the answer is that the center must be necessarily on the perpendicular bisector" which reveal an emerging awareness of the meta-knowledge dimension of proof, in relation to the meta-statement and thus to the meta-theory (see Ellie2-GO1 in [22], [24], and [26], Ep. 1.4 – Scene 1 – Outlookers). With regard to reductio ad absurdum, in the actors' group only an attempt is

visible to construct the secondary statement related to a conjecture developed earlier, together with the direct proof of the contrapositive (see Tommy-Blogger in [36], Ep. 1.4 – Scene 1 – Actors), but the transition to the meta-statement never occurs.

Findings 3: Storytelling as a rational mediator of reflection

The storytelling implemented in this work operates across multiple levels. In addition to the narrative developed through the comics on the digital platform, it continues interactively in the chats, where students bring the characters, representing cognitive roles, to life through their interactions, thus moving the problematic situation beyond a static narrative. The narrative and immersive experience is primarily realized through the interplay of the different cognitive roles, as well as through the interaction between the two distinct profiles of Actors and Outlookers. For the Outlookers, Euclid's mediation sometimes occurs indirectly, through observing what takes place within the actors' group, comparing their own actions with those of the observed actors (see Ellie2-GO2 in [5], Ep. 2 – Scene 1 – Outlookers), engaging in reflections on the process unfolding in the other chat and showing agreement with the choices made by the actors (see Shona2-GO2 in [7], Ep. 2 – Scene 1 – Outlookers), or producing conjectures autonomously, guided solely by attentive observation of what occurs within the actors' group (see Shona2-GO1 in [12], Ep. 1.3 – Scene 3 – Outlookers).

The development of collective awareness and reflective activity emerging from the observation of the process is also fostered by the prompts provided by Euclid when requesting detailed descriptions of the drawings. Starting from Episode 1.3, Euclid begins to ask the actors' group to describe in detail the drawings uploaded on Padlet or produced on paper. This prompt is also taken up by some Outlookers, who provide constructive contributions in describing the representations produced. In these contributions, we can observe communicative efforts involving attempts to justify the constructions, the search for relationships among the geometric objects involved, and initial signs of deductive structuring (see Ellie-GO1 in Fig. 10.2 and in [1], Ep. 1.3 – Scene 2 – Outlookers). We also note contributions of a teleological nature, highlighting the strategic choices made and explicitly referring to geometric proofs previously shared in earlier phases (see Ellie-GO1 in Fig. 10.5 and in [39], Ep. 1.4 – Scene 1 – Outlookers).

In the active observation generated by the Actors-Outlookers dynamic, Outlookers internalise not only Euclid's role but also other roles, such as that of the Boss. For instance, we can identify interventions operating on meta-functional aspects, directing the process, and prompting other members to upload and comment on the drawings on Padlet (see Ellie-GO1 in [11], Ep. 1.3 – Scene 2 – Outlookers; see Ellie-GO1 in [5], Ep. 1.4 – Scene 2 – Outlookers).

The mediating role of storytelling is also enacted through Euclid's prompts to return to the comics during the narrative (see Euclid in [31], Ep. 2 – Scene 2 and see Scene 3). In some cases, once a given objective has been achieved, a new narrative sequence is unlocked, and Euclid-Guru invites students to leave the chat and return to the comics to read the newly available strips. This dynamic interplay of storytelling, characterized by a back-and-forth movement between the comics and the chat, between the a priori designed narrative and the one co-constructed synchronously, is made possible by a strong a priori analysis of both the mathematical problem and Euclid's interventions, thereby fostering the students' enculturation process. The back-and-forth between comics and chat, with theoretical references already embedded within the comics themselves (e.g., Def. 17 - Ep. 3), promotes the view of storytelling as a *mediator of reflection*, thus fostering the internalization of a process that has occurred socially and has been observed as an active spectator.

Findings 4: Connections between cognitive roles and dimensions of rationality

The analysis of the data also reveals some connections between cognitive roles and components of rationality. Although the roles were characterized a priori through by integrating different theoretical frameworks, many of which are already related to rationality, each role is not associated with only one dimension of rationality. Rather, they often display features that lie across two or more components of rationality, as already highlighted in (Albano et al., 2024b), in the case of the Pest role. The fine-grained analysis of the data from the Euclid Experiment has revealed some interesting connections that further refine the relationships between roles and rationality. In same case we recognise a dialectic between the epistemic and communicative dimensions, understood as an explanation aimed at achieving clarity for oneself (see, for example, [3], Ep. 1.3 – Scene 2 – Actors), while in other case we interpret the arguments presented by students as communication that reflects a greater effort to align with the standards of the mathematical community, through a structuring of the proof increasingly oriented toward the theoretical validation of the statement (see [19], Ep. 2 scene 4 - Actors; see [5] and [20], Ep. 3 - Scene 3 - Actors). We refer to this as *epistemic communication*, and we identify it as a feature of the Blogger role. Moreover, in the act of generating new ideas (a practice typical of the cognitive role of the Promoter), we recognise a new dialectical interplay between the teleological and epistemic rationality dimensions, emerging through recourse to memory. This recourse to memory, which we named *epistemic memory*, relies on mathematical knowledge that is consistent with the reference theory, allowing to retrieve theoretical results that enable

the process to move forward (see [19], Ep. 1.1 - Scene 2 - Actors). These moments of alternation between the teleological and epistemic dimensions, between adherence to culture and constructive moments of forward drive, in a Vygotskijan perspective, recall the idea of creative imagination as a dialectical movement of distancing and approaching to culture, as a free "envol" and a necessary adherence to Culture (Schneuwly et al., 2022).

In the following two-dimensional Table 10.4 we summarise the main relationships between cognitive roles and rationality dimensions emerged by data analysis.

Table 10.4: Relationships between roles and rationalities

	Relationship between roles and rationalities		
	Teleological dimension	**Epistemic dimension**	**Communicative dimension**
Promoter	**Performing or recalling** actions oriented towards the aim	**Epistemic memory:** Recourse to the epistemic memory in the act of idea generation	
Pest	**Checking** the effectiveness (or lack) of the strategic choices in relation to the aim	**Validation of statements** from accepted premises and using correct reasoning rules, in relation to reference theory	**Control of** the ambiguity of the statements made and the quality of the communication of logical steps. **Awareness of** the accuracy of deductive chains and symbolic and verbal representations
Blogger	Communicate the process to others, **by describing the actions performed** **Facilitating** the reading and understanding of problem solving by verbal and/or textual explanations	**Epistemic communication: "**Explain by saying", explanation aimed at achieving clarity for oneself **Communication as a reorganisation** of information into a logical structure	Communicating with others **searching mutual understanding**
Boss	**Activation and direction of** attention **Operational guide** that leads the actions of group		Communication that promotes the **activation of roles**

We can observe that, while in the roles of Pest and Blogger we identify a strong interconnection among the three dimensions of rationality, in the cases of Boss and Promoter we do not detect specific connections with all of them. In particular, the Promoter role shows natural connections with the teleological dimension, as it operates through goal-oriented actions, and also relates to the epistemic dimension through recourse to epistemic memory. However, we do not identify connections with the communicative dimension. Similarly, the Boss role is connected to the teleological dimension in directing and focusing the group's attention, acting as an operational guide that leads the group's actions. In addition, we recognize a connection with the communicative dimension in communication aimed at activating cognitive roles. However, we do not observe specific connections with the epistemic dimension. Indeed, even if the Boss engages in forms of control and monitoring, these are always related to the ongoing process and not specifically to epistemic issues, whose control is instead entrusted to the Pest role.

Chapter 11

The cases of Giorgio and Estella

After conducting a detailed and fine-grained analysis of all the synchronous discursive interactions that took place in the Actors' and Outlookers' chats of one of the two classes involved in the experiment, in this chapter we now focus our attention in particular on the evolution process followed by two students, hereinafter referred to as Giorgio and Estella. These students participated as Outlookers from Episode 1.1 to Episode 1.5, interpreting Giorgio the character Shona2-Pest-GO1 and Estella the character Ellie-Promoter-GO1. Furthermore, in Episode 2, Giorgio took on the role of Ellie2-Promoter as an Actor, while Estella did not participate due to absence. Finally, in Episode 3, both Giorgio and Estella participated as Actors, interpreting the characters Kyle-Boss and Tommy-Blogger, respectively.

The decision to focus on Giorgio and Estella as specific instances within a qualitative case study (Cohen et al., 2007) stems from the fine-grained analysis of the data and the results that emerged from it, which led us to zoom in on the developmental trajectory characterizing these two students.

Traditionally, case studies are regarded as an investigative approach useful for exploring features that can later be examined through more in-depth research (Bell & Waters, 2018). However, Yin (2009), by moving away from that misconception, attributes to case studies a specific explanatory value, arguing that they can support theoretical generalization through the in-depth analysis of a given phenomenon in a given context. Case studies, rather than aiming to capture the frequency of events, prioritise quality and intensity, focusing on a small number of significant and relevant instances, thereby enabling a deeper understanding of the underlying dynamics of situations and participants (Cohen et al., 2007). Therefore, we chose to use the case study as a research approach to progressively narrow the focus of the study, like a funnel, concentrating on the process that Estella, in particular, had already shown through a detailed analysis of the chats, moving from the role of Promoter as an Outlooker to that

of Blogger as an Actor. In this process, a strong focus on attentive observation in the Outlooker role emerged, along with a tendency toward forms of communication closely aligned with the idea of theoretical validation in the Actor role. The data analysis also highlighted interesting contributions from Giorgio, although with a lower level of engagement and participation compared to the other students. The process under investigation is inherently collective and, as evidenced by the fine-grained analysis, reveals a progressive shift of the group toward scientific enculturation in the Culture of Theorems, and forms approaching theoretical validation. Within this perspective, we believe that the case studies on Estella and Giorgio can be representative of a shared trajectory experienced by the different students in the class involved.

To address the research question RQ3, our aim is therefore to examine the process in a deeper detail and to show how, in a given context, the attentive and participatory observation of the Outlookers can contribute to the transition to the meta-level and to the development of awareness in the search for forms of theoretical validation and in the writing of mathematical proofs, taking cultural format constraints into account.

11.1 Students profiles

The classroom teacher describes Estella as a rigorous student, with strong determination and a marked tendency toward perfectionism, although, in the teacher's view, she sometimes shows some fragility. In any case, Estella fits the profile of a student who is committed, studies diligently, and strives to complete assigned tasks with care. The teacher, on the other hand, describes Giorgio as an intuitive student who achieves good results in written assessments, but who, during regular classroom lessons, tends not to speak up and remains silent due to his shyness.

11.2 Methods

By extracting from the fine-grained analysis of all the chats the already analysed and coded contributions of Estella and Giorgio from rationality perspective, and taking into account the memos collected by the author of this study in the role of teacher-researcher, supported by the classroom teacher, we conduct a qualitative case study (Cohen et al., 2007), choosing in particular to analyse these two students as specific instances of the process taking place in the class under investigation. We chose to collect in tables the representative excerpts of Estella's and Giorgio's interventions in the chats divided for each specific episode and to comment on them from time to time, also accompanying them, if necessary, with the notes collected in the memos. We then

report these contributions in a summary timetable that allows for the visualization of the diachronic evolution in relation to the three levels considered in the analysis: knowledge (KL), meta-knowledge (MK), and interplay between levels (IP). In the literature, there are many examples of timelines used to analyze the behavior of students engaged in problem-solving activities, as well as the development of the competences related to such activities (Schoenfeld, 1985), or to analyze in detail learning episodes characterized by the presence of multiple semiotic registers (Bagossi et al., 2022). In our case, drawing inspiration from these tools, we decide to construct a timeline that allow us to heuristically visualize the diachronic evolution of rational behavior, in terms of strategies adopted, epistemic control, and communication action, across the three levels knowledge (KL), meta-knowledge (MK), and interplay between levels (IP) considered in the fine-grained analysis of the chats. Unlike the aforementioned tools, our timeline does not segment time into minutes or seconds; rather, it is organized according to the different comic-based Episodes experienced by the students. Although this choice may seem to overlook temporal granularity, it enables us to obtain a broader view of the entire process as it unfolds over longer and more extended periods.

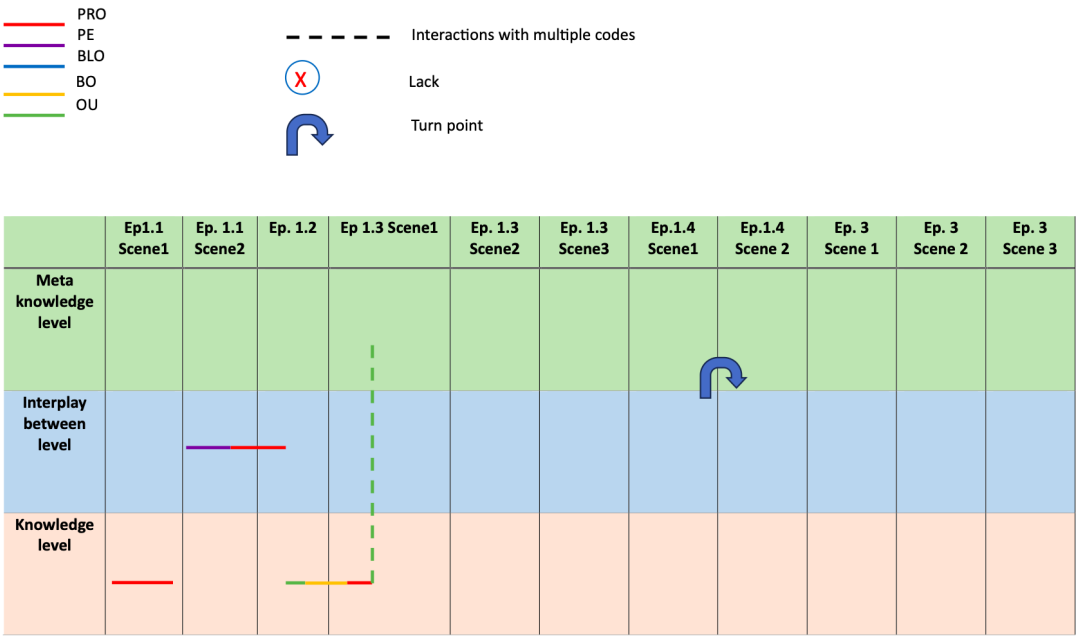


Figure 11.1: Illustrative example of our timeline

As shown in Figure 11.1, the contributions will be displayed in the summary timetable in relation to the categorical code assigned to them in the fine-grained analysis of the

chats and visualised by continuous horizontal lines in different colors depending on the category they refer to. Contributions associated with multiple codes will be indicated by a vertical dashed line, representing the dual interpretation that can be made of the data. Contributions analyzed as "lack" will be marked with a crossed circle, indicating the absence of a given feature in that specific context, while turning points will be highlighted by a curved arrow. Therefore, the diachronic analysis is structured by following the progression of the episodes and the students' contributions within each individual episode and subsequently by the visual representation in a summary tabular form, all always complemented by a narrative commentary in order to provide a clearer account of the overall process.

11.3 Results

Below, in Table 11.1, we report excerpts from Giorgio's contributions in Episode 1.1, coded and analysed through categories aforementioned.

Table 11.1: Excerpts from Giorgio's discursive interactions - Ep. 1.1

Episode 1.1 – Scene 1 – Outlookers

Speakers	Utterances	Codes	Operational indicators
Shona2-GO1	"I also think they are infinite"	OU-MK	shows agreement (OU-MK) with the conjecture advanced
Shona2-GO1	"the circles can be of any size and in any position"	PRO-IP	Movement toward generalization
Shona2-GO1	"You will never be able to fit an infinite number of them on a sheet of paper"	PE-KL	points out the limitations of the empirical method adopted, performing a check on the coherence of the selected procedures and on the correctness of the representations
Shona2-GO1	"the more you draw, the more you realise that more and more can fit"	PRO-KL PRO-IP	inductive reasoning process that guide generalization
Shona2-GO1	"because in my opinion"	PE-KL	epistemic component emerges as validation of a statement
Shona2-GO1	"any point in the plane can be taken as the center of a circle"	PRO-KL	justification relies on teleological arguments in which goal-oriented actions appear
Shona2-GO1	and as we know, the plane is infinite	PE-KL	Referring to statements accepted by the classroom community

Episode 1.1 – Scene 2 – Outlookers

Speakers	Utterances	Codes	Operational indicators
Shona2-GO1	"The plane is infinite"	PE-KL	states results that are accepted and shared by the community, though implicitly
Shona2-GO1	"The problem is precisely to prove that"	PE-KL	epistemic need for validity of the statement produced

In Table 11.1, we observe that Giorgio makes several moves toward attempts at generalization, although these are mainly supported by empirical evidence. His arguments are primarily teleological in nature, but we also record instances of checking within the epistemic dimension, even if sometimes with still implicit references to theory. In this first episode, his observational activity is reflected in expressing agreement with the conjecture produced by the actors' group.

Table 11.2: Excerpts from Giorgio's discursive interactions - Ep. 1.2

Episode 1.2 – Outlookers

Speakers	Utterances	Codes	Operational indicators
Shona2-GO1	"with a glass we can draw a circle of only one size and without a center"	PE-MK	the awareness that circle's representation with a glass deprives us of knowledge of the center, almost as if there is an absence

In Table 11.2, Giorgio shows a shift toward the meta-level of the epistemic dimension, as he reveals awareness of the existence of the center of a circle by highlighting its absence.

In the subsequent Table 11.3, Giorgio highlights further teleological contributions oriented toward generalization, as indicated by the term "any" repeated three times, thus contributing constructively to the process. Communicative efforts are also evident, aimed at making his ideas understandable to the group, despite some limitations due to the inappropriate use of certain terms.

We also record instances of epistemic checking, both with regard to the alignment of points and to the geometric properties of the required circles. The constructive contributions supporting the advancement of the process are also used as arguments to clarify his viewpoint to the rest of the group, in an effort to be understood. In this, we recognize a shift toward the meta-level of communication.

Giorgio also displays instances of attentive and reflective observation, in which he shows agreement with others' contributions and internalizes the Guru's questioning function.

Table 11.3: Excerpts from Giorgio's discursive interactions - Ep. 1.3

Episode 1.3 – Scene 1 – Outlookers

Speakers	Utterances	Codes	Operational indicators
Shona2-GO1	"In my opinion"	PRO-KL	Formulation of a conjecture based on intuition (without justification)
Shona2-GO1	"can be the chord of any circle of any size with any centers"	PRO-IP	Generalization and constructive advancement of the process
Shona2-GO1	"the segment formed by AB can be a chord of the circle"	BLO-MK	Effort to communicate the insight to the group and lead the others to accept own position
Shona2-GO1	"tangent"	LACK-BLO-KL	Communicative lack with inappropriate use of a term

Episode 1.3 – Scene 3 – Outlookers

Speakers	Utterances	Codes	Operational indicators
Shona2-GO1	"Ellie1 is right"	OU-MK	shows agreement with statement made
Shona2-GO1	"If they are aligned [it is] because there are two, not three, four, five… two form a segment"	PE-KL	Checking of the conceptual property related to the alignment of points
Shona2-GO1	"Why not?"	EU-ED OU-IP	stimulating the epistemic component by questioning and reproducing the observed behaviors of the Guru (internalization of the Guru) (internalization of the Guru)
Shona2-GO1	"a circumference can be of infinite sizes"	PRO-IP	Generalization of the process
Shona2-GO1	"The circles passing through two points [...] are infinitely many because these points form a segment [...] will be the diameter of one circle and the chord of infinitely many other increasingly larger circles."	BLO-MK PRO-IP	communication addressed to the group, supporting it with constructive arguments as a justification of the conjecture itself
Shona2-GO1	"two points, which are necessarily aligned"	PE-KL	Epistemic dimension on validation of statement
Shona2-GO1	"So we know the center of each circumference" and "They will form a line perpendicular to the midpoint"	PE-KL	check on the geometric knowledge
Shona2-GO1	"perpendicular to the midpoint"	LACK-BLO-KL	communicative shortcoming

In the following Table 11.4, we observe that during Episode 1.4 Giorgio operates across the three dimensions of rationality, teleological, epistemic, and communicative, although with some communicative gaps, mainly related to the inappropriate use of certain terms.

From a teleological perspective, we record strategic attempts that are often interrelated with the epistemic and communicative dimensions. In particular, we note efforts of communication with himself in order to clarify his own thinking, searching for

explanations that may be convincing but still remain linked to a strategy of empirical and dynamical visualization, that is, reflecting a need for certainty through empirical methods.

From an epistemic perspective, checking remains essentially at the meta-knowledge level, with references, albeit implicit, to the theory. At the communicative level, we observe a shift toward the meta-level, both in the use of terminology that suggests initial attempts at logical implication, and in communication with himself and with the group oriented toward achieving understanding.

Table 11.4: Excerpts from Giorgio's discursive interactions - Ep. 1.4

Episode 1.4 – Scene 1 – Outlookers

Speakers	Utterances	Codes	Operational indicators
Shona2-GO1	"a circle doesn't work for me"	PE-KL	a teleological conflict, related to checking the effectiveness (or lack thereof) of an action in relation to the aim
Shona2-GO1	"because if the center moves, the circle moves as well."	BLO-MK PRO-KL	communication with herself in order to clarify her thinking, searching for explanations that may be convincing but that still remain linked to a strategy of empirical and dynamical visualization – need for certainty using empirical methods
Shona2-GO1	"tangent" to mean "belonging to" "otherwise, it can only be considered a circle in empty space"	LACK-BLO-KL	communication appears weak, especially from a formal point of view, and presents several communicative lacks
Shona2-GO1	"all radii are equal" "the center of the circle to include both points, it must be perfectly central, on the perpendicular bisector"	PE-KL	implicit reference to the definition of a circle and with the checking of the conceptual property related to equidistance
Shona2-GO1	"Therefore"	BLO-IP	implications expressed through verbal representations
Shona2-GO1	"I'll write it better"	PE-IP	checking and attempting to improve the quality of the communication
Shona2-GO1	"we draw it off the perpendicular bisector" "it can touch only one of the points"	PRO-KL	a teleological dimension as goal-oriented action
Shona2-GO1	"that is, at any point in the plane that is not on the perpendicular bisector"	BLO-MK	a communicative dimension in the clarification aimed at mutual understanding
Shona2-GO1	"since the distance of the radii from center C is the same in every direction"	PE-KL	an epistemic dimension with implicit reference to theory

Table 11.5: Excerpts from Giorgio's discursive interactions - Ep.2

Episode 2 – Scene 1 – Actors

Speakers	Utterances	Codes	Operational indicators
Ellie2-Promoter	"I don't think there's any way to draw another one"	PE-KL	check on the possible constructions, with the emergence of the difficulty of representing different circles
Ellie2-Promoter	"but we have to prove it"	PE-IP	a need for causality
Ellie2-Promoter	"But why does only one circle pass through three points?"	PE-IP EU-ED OU-IP	a need for causality, as a need to understand the underlying reasons for the conjecture that has emerged

Episode 2 – Scene 2 – Actors

Speakers	Utterances	Codes	Operational indicators
Ellie2-Promoter	"And by trying"	PRO-KL	effort to search for new circles
Ellie2-Promoter	"we managed to find only one"	PE-KL	action of checking the possible representations
Ellie2-Promoter	"larger or smaller circles do not match with all three points"	PE-KL PRO-KL	checking of the representations, guided by strategic intentions - dialectic between two dimensions of rationality, both epistemic and teleological

Episode 2 – Scene 3 – Actors

Speakers	Utterances	Codes	Operational indicators
Ellie2-Promoter	"So OA, OB, and OC will be three radii"	PRO-KL	deductive step, establishing correct relationships between the mathematical objects and the concepts involved
Ellie2-Promoter	"Because A, B, and C are three points on the circle and each point is equidistant from the centre O"	PE-KL	justifies the statement by monitoring the geometric property of equidistance with an implicit reference to the theory
Ellie2-Promoter	"Def. 15 does ... seem useful"	PE-KL	Checking the justify of all the steps of reasoning
Ellie2-Promoter	"indeed"	OU-MK	shows agreement with what was stated
Ellie2-Promoter	"The points OB and OC are congruent;"	LACK-BLO-KL	Communicative lack about a term
Ellie2-Promoter	T"herefore, if OA is congruent to OB, we have OA = OB = OC"	PE-KL	Theoretical contribution on the congruence of the three segments AO, OB, and OC, although without a real explicit statement of the theoretical reasons supporting the implication
Ellie2-Promoter	"therefore"	BLO-IP	formulate a deductive chain

Episode 2 – Scene 4 – Actors

Speakers	Utterances	Codes	Operational indicators
Ellie2-Promoter	"We were reflecting on why the OA, OB, and OC rays are congruent"	BLO-MK PE-MK	external communication of a reflection on the reasons that support the theoretical validation of a statement
Ellie2-Promoter	"thanks to the theory bank, we might have figured it out"	PRO-MK	awareness in choosing and using the appropriate mathematical means or tools to achieve the aim
Ellie2-Promoter	"but we need to answer why"	PE-IP	need for causality
Ellie2-Promoter	"Maybe the third one could help us?"	PRO-KL	selecting appropriate means to address the goal
Ellie2-Promoter	"If for every point we draw a circle, only one circle out of infinite circles will pass through the three points"	BLO-IP	communication through the use of verbal representations concerning implications, where the comma plays the role of the term "then"
Ellie2-Promoter	"we draw" and "will pass through"	PRO-KL	visual evidence expressed through goal-oriented actions

In Episode 2, visible in the Table 11.5, Giorgio is engaged at the interpretative level of Actor, and it emerges that he works primarily on epistemic control, both at the knowledge level and within the interplay between levels. We observe, in particular, his control over the impossibility of drawing more than one circle passing through three points, as well as the need to seek theoretical validation to support the conjecture. This need also emerges through acts of questioning, reflecting the internalization of the Guru's problematizing actions within the epistemic dimension.

Giorgio's contributions also reveal a dialectical dimension among the roots of rationality, especially in the checking of possible representations guided by strategic choices. This control activity continues in subsequent scenes, where Giorgio justifies statements by checking geometric properties and providing theoretical contributions, albeit with implicit references to theory. We recognize the formulation of implications and deductive chains at the communicative level, although some gaps are evident in the inappropriate use of certain terms.

Several dialectical moments between different dimensions of rationality can be identified, with a shift toward the meta-level in the final scene of Episode 2, both in epistemic and teleological aspects. Indeed, a growing awareness emerges in the selection and use of appropriate means to achieve the aim, as well as an increasing need to draw on theory for the validation of the statements made.

Observational activity also continues during this episode, particularly through reflective contributions in which Giorgio expresses agreement with what others have stated.

Table 11.6: Excerpts from Giorgio's discursive interactions - Ep. 3

Episode 3 – Scene 1 – Actors

Speakers	Utterances	Codes	Operational indicators
Kyle-Boss	"Let's reflect on the question"	BO-KL	activates the roles, focusing attention on the goal
Kyle-Boss	"We know that ... AB is a diameter"	BLO-KL	recaps what had already been highlighted in the comics, referring to Definition 17 of the Dynamic Euclidean Bank of Theory concerning the diameter of a circle
Kyle-Boss	"according to definition number 17"	PE-MK	adherence to the reference theory
Kyle-Boss	"And it seems to me that AC is a chord"	PRO-KL	remark about another element present in the figure, showing a visual-type strategy
Kyle-Boss	"We know that AB is a diameter, AC is a chord, and CO and AO are radii, which forms an isosceles triangle"	BLO-KL	recaps what has emerged up to that point, in order to facilitate understanding
Kyle-Boss	"but now, what is the relationship between angles alpha and beta?"	EU-TD OU-IP	focuses attention on the goal, reiterating the objective with a questioning
Kyle-Boss	"The straight angle at O is 180 degrees"	BLO-MK	engages in self-directed communication to clarify his thinking, recalling statements accepted by the classroom community, in a form of thinking aloud
Kyle-Boss	"And beta is an exterior angle of the isosceles triangle" and "beta is part of this angle"	PRO-IP	constructive contributions to the advancement of the process

During the final Episode, whose excerpts can be read in Table 11.6, Giorgio works across all three dimensions of rationality, remaining mainly at the knowledge level, although we record some shifts toward the meta-knowledge level in both the epistemic and communicative domains. We also observe contributions of a constructive nature, as well as interventions in which Giorgio internalizes the role of the Guru, particularly within the teleological dimension.

Below, in Figure 11.2, we present the timeline of Giorgio's interventions over the long period of the classroom experimentation.

We can observe that Giorgio consistently works by engaging all three dimensions of rationality across the three levels: knowledge level, metaknowledge level, and interplay between levels. Control over the epistemic dimension initially remains at the knowledge level (during the Outlookers phase), and later shows some evidence also in the interplay between levels and at the meta-level (during the phase in which he played as an Actor), especially with regard to the emergence of the need to seek theoretical validation for the statements produced. With respect to the teleological dimension, we particularly

highlight those interventions in which we observe a movement toward generalization, which consistently appears through the repetition of terms such as "any" or "more," and which is emphasized by the constructive contributions in red within the interplay between levels. Communication is almost always aimed at clarifying one's own opinion, both with oneself and with others, and is oriented toward mutual understanding, with evidence of the use of verbal representations to indicate implications or deductive chains, although occasional communicative shortcomings are observed in the incorrect use of verbal expressions. We do not identify clear turning points in Giorgio's trajectory, as his behaviors consistently oscillate among the three levels. However, multiple points emerge in which a dialectic between the different dimensions of rationality becomes evident, highlighted by vertical dashed lines, which make explicit the complexity that can arise in the analysis of rational behavior. We also note instances of careful and reflective observation (at the meta-level), interspersed with moments of internalization of the Guru's behaviors (indicated by green segments in the interplay between levels).

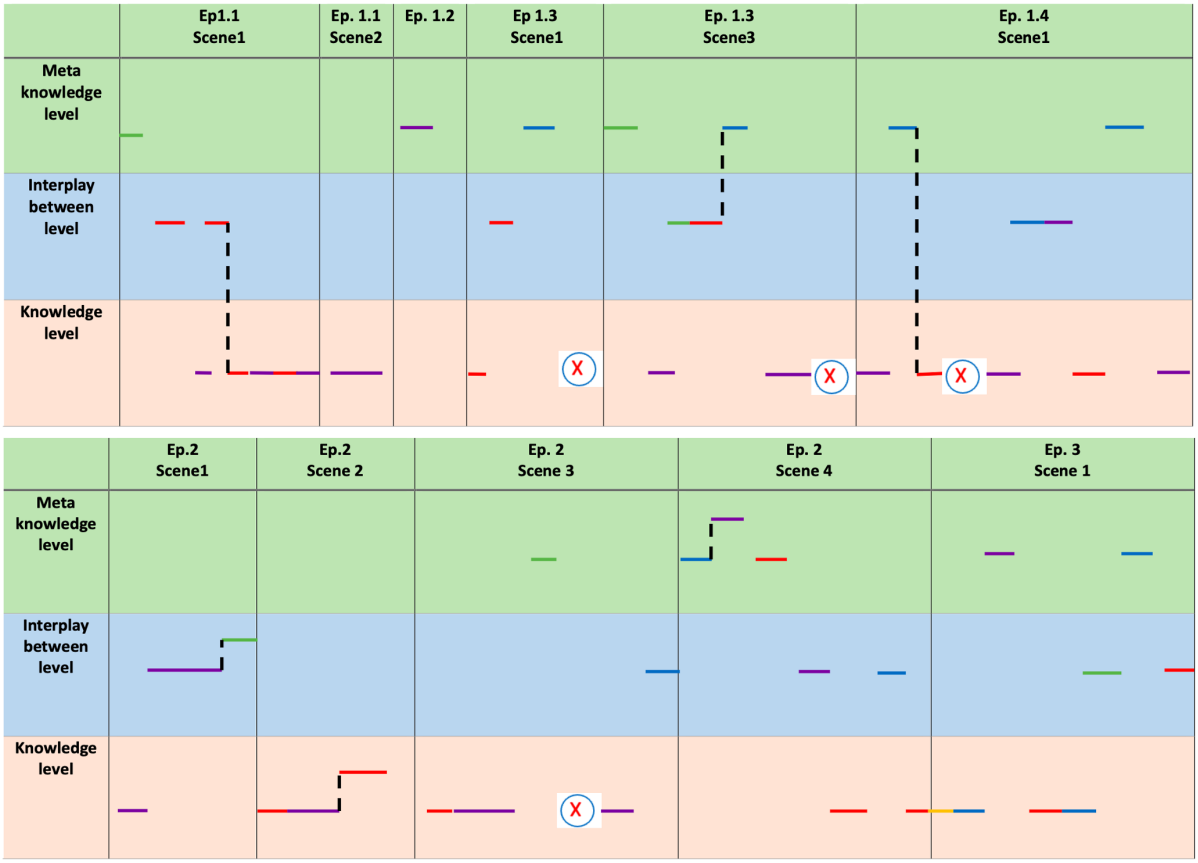


Figure 11.2: Timeline of Giorgio's interactions

In Table 11.7 below, we report excerpts from Estella's contributions in Episode 1.1, where she operates at the interpretative level of the Outlookers.

Table 11.7: Excerpts from Estella's discursive interactions - Ep. 1.1

Episode 1.1 – Scene 1 – Outlookers

Speakers	Utterances	Codes	Operational indicators
Ellie-GO1	"let's take any point" and "start drawing"	PRO-KL	teleological dimension - goal-oriented actions
Ellie-GO1	"we need to verify in some way" and "without first being sure"	PRO-KL	need for certainty
Ellie-GO1	"I can keep drawing them indefinitely"	PRO-KL	goal-oriented action
Ellie-GO1	"Even though there are only 4, if I keep drawing I can do it indefinitely"	PRO-KL	Describing the possibility of a goal-oriented action that can continue indefinitely
Ellie-GO1	"can we consult the theory bank to see if there's anything?"	PRO-KL	need of selecting tools that serve the aim

Episode 1.1 – Scene 2 – Outlookers

Speakers	Utterances	Codes	Operational indicators
Ellie-GO1	"it needs to be proved"	PE-IP	epistemic need for validity of the statement produced

In this episode, Estella displays an exploratory behavior consistent with her assigned role, acting on the teleological dimension. From the very first episode, a tendency emerges to describe the actions performed with accuracy, providing justifications in order to clarify, both for herself and for others, the reasons supporting the conjecture produced. Indeed, from the outset, Estella shows a need to validate the conjecture, even requesting access to the Dynamic Euclidean Bank of Theory. Estella does not limit herself to observing what takes place within the actors' group, but chooses to explore firsthand, also producing drawings on the shared Padlet, as shown in Fig. 10.1 and highlighting attempts at generalization justified at a graphical level.

In the subsequent Table 11.8, we report the few excerpts of Estella's contributions during Episode 1.2.

Table 11.8: Excerpts from Estella's discursive interactions - Ep. 1.2

Episode 1.2 – Outlookers

Speakers	Utterances	Codes	Operational indicators
Ellie-GO1	""with a compass we can obtain different radii"	PRO-IP	constructive contribution that allow the process to move forward
Ellie-GO1	"Not only Ellie2, the group over there is working well; I think they may have already found the solution"	OU-KL	observing the roles played in the other group

In this episode, Estella works on two levels: on one hand, she continues to work on the mathematical problem, providing a constructive contribution to the process, while on the other, she makes observations regarding the behavior of the Actors.

During Episode 1.3, Estella continues to work on two levels. As shown in Table 11.9, regarding the mathematical problem, Estella proceeds with the exploration and also conducts checks on the effectiveness of the choices made in order to validate the conjecture. Estella also displays behaviors in line with the Boss role, acting as an operational guide in directing the group's actions and prompting communication. On the Outlookers level, Estella shows agreement with what happens in the actors' group, bringing the collective knowledge born from the observation of the process into the Outlookers chat, and also making critical observations on the behavior observed in the actors' chat. These back-and-forth movements between working on the mathematical problem and actively observing what is happening in the actors' chat allow Estella to begin shifting to the meta-level. This occurs both in the communication with the group, where she always aims to clarify her actions and provide detailed descriptions of the drawings made (see Fig. 10.2), and in the internalization of Euclid-Guru's behaviors as she strives to stimulate reflection on her attempts to generalize the process. Furthermore, an effort to reach a generalization can also be observed, although it is still supported by empirical evidence.

Table 11.9: Excerpts from Estella's discursive interactions - Ep. 1.3 - Scene 1 and 2

Episode 1.3 – Scene 1 – Outlookers

Speakers	Utterances	Codes	Operational indicators
Ellie-GO1	"let's start drawing"	BO-KL	Operational guide that leads the actions of group
Ellie-GO1	"but let's do it with compasses"	PRO-KL	Goal-oriented action (teleological dimension)
Ellie-GO1	"so we can find the center too"	OU-MK PRO-KL	Awareness of collective knowledge born from observing processes and strategic choice of actions
Ellie-GO1	"let's continue"	BO-KL	Group activation and coordination
Ellie-GO1	"I don't think the drawing alone is enough"	PE-KL	checking regarding chosen representation
Ellie-GO1	"Can you come up with something?"	BO-KL	Activation of communication and incentive of others' contributions
Ellie-GO1	"they just said that"; "but they didn't say"	OU-MK	Critical observation of the actors group's behavior
Ellie-GO1	"Guys, let's upload the drawings with the names"	BO-KL	Operational guide that leads the actions of group
Ellie-GO1	"Shona2 is right, they are infinite"	PE-KL	Epistemic checking and revision of previous assumption (error regulation)
Ellie-GO1	"I agree";"I also agree"	OU-MK	Reflection on the mathematical process observed, showing agreement or disagreement

Episode 1.3 – Scene 2 – Outlookers

Speakers	Utterances	Codes	Operational indicators
Ellie-GO1	"Guys, for the drawing, I started by drawing the two points"	BLO-MK	describes in detail to the group the followed process to produce the drawing on paper
Ellie-GO1	"[…] segment AB as its diagonal, therefore the points are the ends of the diagonal"	LACK-BLO-KL	inappropriate use of a term with other meaning
Ellie-GO1	"the center of the circle is the midpoint"	PRO-KL	identifing a relationship between geometric objects
Ellie-GO1	"I was able to draw many of them"	PRO-KL	a goal-oriented action
Ellie-GO1	"and I can still continue"	PRO-IP	attempt towards a generalization
Ellie-GO1	"How come I was able to draw so many?"	EU-TD OU-IP	prompting the teleological dimension through questioning (internalization of the Guru)

In Episode 1.3, Scene 3 (see Table 11.10), Estella moves away from purely exploratory aspects, focusing more on epistemic checking and controlling the ambiguities of statements made by both the Actors and Outlookers groups. Reflective interventions at the meta level oriented toward observing the ongoing process are recorded, as well as contributions highlighting the internalization of the Euclid-Guru role, although not always through questioning activities. Estella also works by acting as a guide, prompting group members regarding actions to perform and thus assuming leadership within the

group. Finally, Estella addresses communicative aspects at the meta level to support the writer-interpreter dynamic.

Table 11.10: Excerpts from Estella's discursive interactions - Ep. 1.3 - Scene 3

Episode 1.3 – Scena 3 – Outlookers

Speakers	Utterances	Codes	Operational indicators
Ellie-GO1	"You can talk about aligned points if there are three or four"	PE-IP	Control of the ambiguity of the statements made
Ellie-GO1	"in any case more than two"	PE-IP	Control of epistemic lacks
Ellie-GO1	"but I managed to draw more than four"	PRO-KL	teleological component as a goal-oriented action
Ellie-GO1	"Try drawing it"	EU-TD OU-IP	stimulates the teleological component, acting as Euclid-Guru (internalization of the Guru)
Ellie-GO1	"a correct observation" or "I've just noticed it in my drawing"	OU-MK	Reflection on the mathematical process observed, which enables to make the same observation about own representation
Ellie-GO1	"If Shona wants, she can summarise everything in a statement, adding the fact that the centres are aligned"	BO-KL	activating roles and providing a guide on the actions to be taken
Ellie-GO1	"There are no lines perpendicular to points"	PE-KL	epistemic checking over geometric properties and on the validation or invalidation of statements
Ellie-GO1	"Guys, a point is dimensionless"	PE-KL	implicit reference to theory
Ellie-GO1	"it's not like a segment" and "It doesn't form a 90-degree angle"	PRO-KL BLO-MK	establishing relationships between mathematical objects with the aim of making the communication of thinking clearer and activating the writer-interpreter dynamics

Episode 1.4 represents an important turning point for Estella, as a stronger transition toward the meta-level becomes evident. In Table 11.11, we record several interventions that highlight the need to receive theoretical support functional to the intended goal, also due to the growing need to move from empirical evidence—recognized as "not enough" in relation to the goal—toward a theoretically grounded validation of the conjectures developed.

We also identify contributions that reveal the emergence of an initial awareness of meta-knowledge aspects of proof, for example those related to approaches to ad absurdum proof. At this stage, Estella also appears to take on responsibility for the group's work, with several interventions showing the internalization of the Euclid-Guru role, particularly in the awareness that certain actions may foster a deeper epistemic understanding for all group members, as well as in the request for greater clarity on the communicative level. Estella also displays behaviors of attentive observation and reflection with respect to the actions of the actors' group.

Table 11.11: Excerpts from Estella's discursive interactions - Ep. 1.4

Episode 1.4 – Scene 1 – Outlookers

Speakers	Utterances	Codes	Operational indicators
Ellie-GO1	"Let's start by drawing the two points again"	PRO-KL	a goal-oriented action
Ellie-GO1	"It's similar to the problem"	PRO-KL	recognition of an analogy
Ellie-GO1	"statement"	LACK-BLO-KL	use of the term "statement" to refer instead to a written argument, a verbal justification
Ellie-GO1	"We can try asking if there is a proposition or a notion that deals with the centers ..."	PE-IP	need for theoretical support that is functional to her goal
Ellie-GO1	"And this can be shown with a drawing" "Euclid, is there a proposition or a notion that we can draw on to solve the problem?"	PE-IP	control has now shifted from empirical evidence to the search for theoretical validation
Ellie-GO1	"A drawing alone is not enough"	PE-KL PE-IP	Checking for the effectiveness (or lack) of the strategic choices in relation to the aim and by bringing out a need for theoretical validation
Ellie-GO1	"But we can do it ourselves as well"	BO-KL	he takes responsibility for the group's work, encouraging the others to act
Ellie-GO1	"it's useful for understanding"	EU-ED OU-IP	encouraging with a view to achieving a deeper understanding from an epistemic perspective (internalization of the Guru)
Ellie-GO1	"I draw," "connect them," "find," and "draw"	PRO-KL	goal-oriented actions
Ellie-GO1	"since … then"	BLO-IP	verbal representation of implication
Ellie-GO1	"Shona2 just needs a little adjustment"	EU-CD OU-IP	takes responsibility for the communicative aspect, prompting toward deeper clarify of expression Reproducing the observed behaviors of the Guru (internalization of the Guru)

Episode 1.4 – Scene 2 – Outlookers

Speakers	Utterances	Codes	Operational indicators
Ellie-GO1	"let's try to recap"	BO-KL	stimulating the activation of cognitive functions
Ellie-GO1	"We tried to prove that" and "so we started by drawing circles with centers outside the perpendicular bisector"	PE-MK	contribution of meta-knowledge about the proof, as an initial awareness emerges of the attempt to prove a statement starting from ad absurdum hypothesis
Ellie-GO1	"For those who made drawings, upload them, or let's try to comment"	BO-MK	operating on meta-functional aspects, directing the process
Ellie-GO1	"let's look at what the actors are doing"	BO-IP	directing the group's reflection and attention
Ellie-GO1	"if he said to look at … then something is missing"	OU-MK	reflective action highlighted by a conditional structure on meta-thinking
Ellie-GO1	"Tommy did a good job, but the others didn't participate actively"	OU-KL	observation about the role and behavior adopted by the actors as a noting
Ellie-GO1	"Maybe because they worked more actively last time", "today's task is Tommy's" and "they could have said a bit more"	OU-MK	Reflective and critical observation of the different roles and the behavior adopted

Table 11.12: Excerpts from Estella's discursive interactions - Ep. 3 - Scenes 1, 2

Episode 3 – Scene 1 – Actors

Speakers	Utterances	Codes	Operational indicators
Tommy-Blogger	"Let's consider the 180-degree angle at O, which is made up of beta plus the angle of triangle ACO"	PRO-KL	exploratory contribution, establishing correct relationships among the mathematical objects involved
Tommy-Blogger	"AB is the diameter, but triangle AOC is isosceles because points C and A are equally distant from the center O. The angle at C is congruent to angle α, therefore the sum of the interior angles is 2α + O = 180°"	BLO-IP	a deductive chain (comment on the drawing)
Tommy-Blogger	"triangle AOC is isosceles because points C and A are equally distant from the center O"	PE-KL	justification for the fact that triangle AOC is isosceles, referring to the definition, although not explicitly
Tommy-Blogger	"The angle at C is congruent to alpha since the triangle is isosceles"	PE-KL	clarifies the reasoning that leads to considering two angles as congruent, implicitly referring to the proposition 5 of Book I of the Dynamic Euclidean Bank of Theory
Tommy-Blogger	"which is equal to 180"	PE-KL	relationship between angles, implicitly referring to the theorem on the sum of the interior angles of a triangle
Tommy-Blogger	"Let's try to separate the two angles and place them next to each other"	PRO-KL	carry out a goal-oriented manipulative action
Tommy-Blogger	"Shona 2 is right: Beta is equal to 2 Alpha"	OU-MK	shows agreement with Shona2-Pest's opinion, highlighting active observer behavior as a reflection on the process followed

Episode 3 – Scene 2 – Actors

Speakers	Utterances	Codes	Operational indicators
Tommy-Blogger	"could Proposition 13 be useful ...?"	PRO-KL	chooses and uses appropriate knowledge as a theorem to reach the aim
Tommy-Blogger	"To prove that …"	PRO-MK	shows an awareness in choosing and using mathematical means appropriate to reach the aim
Tommy-Blogger	"according to Proposition 13"	PE-MK	epistemic dimension in referring in a awareness manner to a theory
Tommy-Blogger	"… which states that 'if a line lying on another line forms angles, it will make either two right angles or angles equal to two right angles (supplementary)'"	BLO-KL; PE-MK	communicative dimension in stating a theorem from the Dynamic Theory Bank of Euclid
Tommy-Blogger	"supplementary"	BLO-MK	translation of the archaic Euclidean formulation, almost as a form of communication with others in a valid and comprehensible way, and hence activating the writer-interpreter dynamics
Tommy-Blogger	"angles alpha and beta are supplementary" "and beta is twice alpha"	LACK-BLO-KL	a communicative gap on two levels: the confusion between angles and leaving the implications entirely implicit

Episode 3 – Scene 3 – Actors

Speakers	Utterances	Codes	Operational indicators
Tommy-Blogger	"and therefore" “Therefore” “So”	BLO-IP	a deductive structure with explicit and well-informed deductive chains
Tommy-Blogger	"Therefore, we can deduce that"	BLO-MK	awareness of the logical aspects of the chosen demonstrative schemes and their consistency in relation to the reference theory
Tommy-Blogger	"because, according to Definition 17", "according to Proposition 5" and "According to Proposition 13"	PE-MK	numerous theoretical references supporting the reasoning, with an awareness of the need for explicit theoretical validation
Tommy-Blogger	"the diameter of a circle is a straight line passing through the center and bounded on both sides by the circumference of the circle, which also divides the circle in half” and “the angles at the base of an isosceles triangle are equal"	PE-KL	theoretical references stated one by one, taken literally from the Dynamic Euclidean Bank of Theory
Tommy-Blogger	“… because sides AO and CO are congruent”	LACK-PE-KL	epistemic gap due to the definition of circle left entirely implicit
Tommy-Blogger	The third notion?	PRO-KL	select Common Notion 3 from the Dynamic Euclidean Bank of Theory
Tommy-Blogger	"If equals are subtracted from equals, the remainders are equal"	BLO-KL	correctly state Common Notion 3 from the Dynamic Euclidean Bank of Theory

In Episode 3, Estella shifts from the participatory level of Outlooker to that of Actor, specifically taking on the role of Tommy-Blogger. At first, she appears silent for a while, not immediately intervening in the chat. This silence is due to her being focused and engaged in producing a commented graphical representation, which she later uploads to Padlet. In her drawing (Fig. 10.8), an initial intuition already emerges, guided by the search for properties that are valid at a theoretical level and that will also prove useful in the proving phase. No paper folding is carried out; therefore, the exploration remains entirely on a theoretical level, without physical manipulation of the figure. While this does not immediately support the production of a conjecture, it does help focus attention on the geometric properties of the elements involved in the figure. As shown in Table 11.12, Estella operates across all three dimensions of rationality, teleological, epistemic, and communicative, and does so across the three different levels (knowledge, meta-knowledge and interplay between levels) with a significant shift toward the meta-level. We recognize the presence of both theoretical and meta-theoretical arguments supporting the construction of the conjecture, with

references to Euclidean theory, initially implicit and then increasingly explicit. We still record some instances of reflective and active observation, although Estella's attention, especially in Scene 3, is primarily directed toward refining communication in the collaborative writing of the proof to be submitted to Euclid, thus oriented outward, despite some remaining communicative or epistemic gaps. Below, in Figure 11.3, the timeline of Estella's contributions across the different episodes is presented.

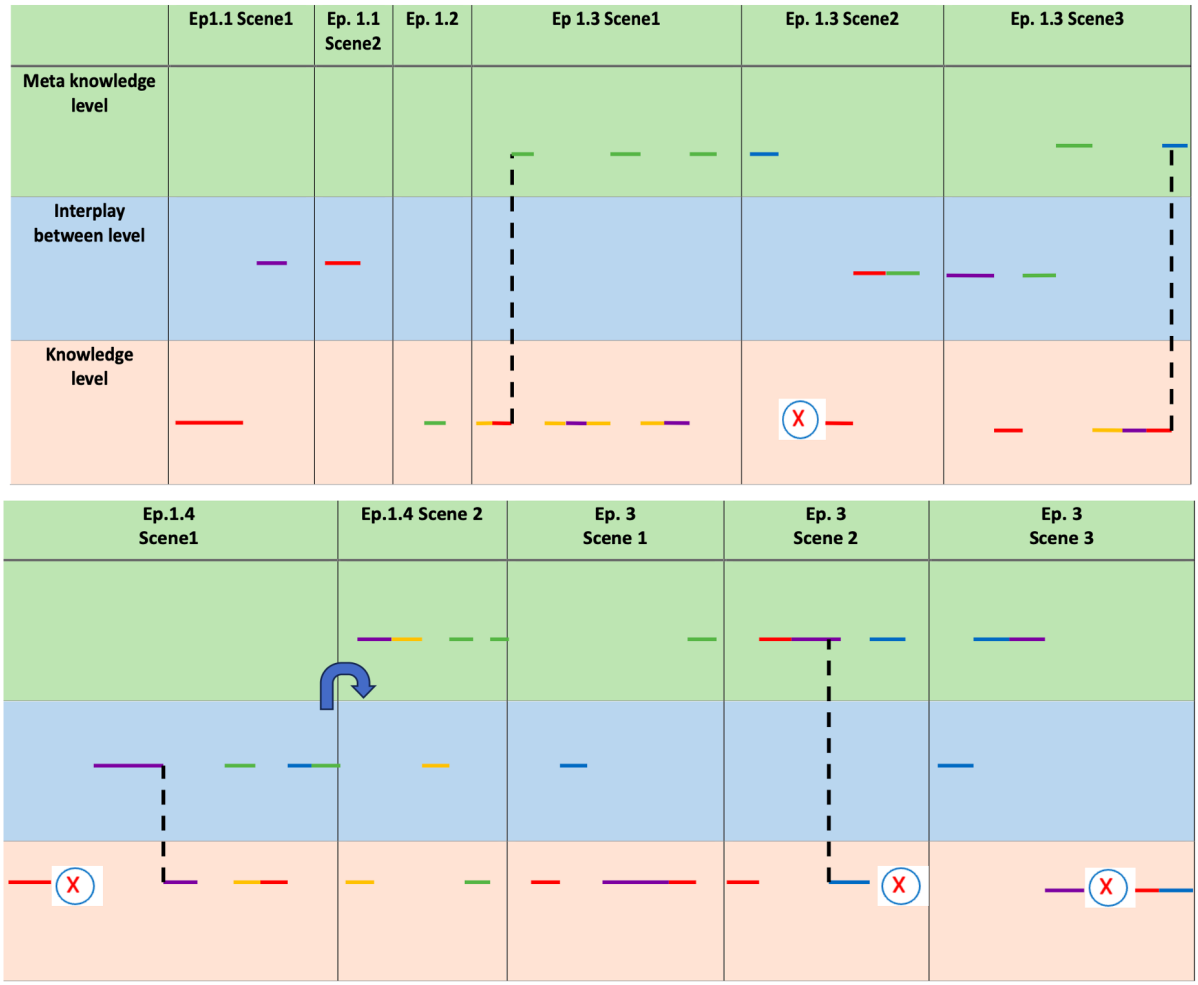


Figure 11.3: Timeline of Estella's interactions

From the timeline, we observe that in the initial Episodes, in which Estella operates at the interpretative level of Outlooker, her activity oscillates between engagement with the mathematical problem and attentive observation of the process. Her contributions at the meta-level are mainly focused on critical and reflective

observation of what occurs within the actors' group, while the dimensions of rationality emerge primarily at the knowledge level. We also record instances of observation at the interplay between the two levels, interpreted as internalization of the Guru's actions.

In the transition from Scene 1 of Episode 1.4 to Scene 2 of Episode 1.4, we identify a turning point, after which Estella's interactions at the meta-level begin to involve the dimensions of rationality as well. Reflective observation continues to be present even in Episode 3, although Estella now takes on the role of Actor. However, in the final episode, we more clearly identify contributions at the meta-level oriented toward communication with others, aimed at supporting the collaborative writing of the email to be sent to Euclid, with also contributions of an epistemic nature related to theoretical aspects.

11.4 Discussion

The diachronic analysis, carried out by focusing attention on two students in the class group through a qualitative case study, highlighted a shift toward the meta-level in the transition from the interpretative role of outlooker to that of actor. In particular, for the student Estella, we identified a well-defined and evident turning point, in which attentive and reflective observation enabled a shift of rational behavior toward the meta-level (Boero & Azrou, 2023). Although such a turning point is not clearly visible in Giorgio, the analysis nevertheless revealed, in his case as well, a development of awareness in the search for forms of theoretical validation (Mariotti, 2006), emerging mainly as a perceived necessity (Zaslavsky et al., 2021). For Estella, the analysis also highlighted a growing care in the writing of mathematical proofs, conceived as communication directed outward, toward the mathematical culture (Albano et al., 2021a; Zaslavsky et al., 2021), in compliance with the cultural and communicative constraints of the community of mathematicians (Balacheff, 2010; Boero & Azrou, 2023), and with careful attention to theory (Mariotti, 2006). The analysis also highlighted the dialectical complexity that characterizes the components of rationality which, according to the perspective of Habermas (1998), are closely interconnected. In moments of dialectic, a transition toward an increasing level of awareness often emerges, representing the forward movement of the process. We frequently observe, particularly in Giorgio and also in Estella, moments of alternation between the teleological and epistemic dimensions, as well as between adherence to culture and constructive impulses that drive the process forward. From a Vygotskijan perspective, these moments evoke the notion of creative imagination as a dialectical movement of distancing from and approaching culture, understood as a tension between a free "envol" and the necessity of adhering to Culture (Schneuwly et al., 2022).

Chapter 12

The case of Blogger

All our study is framed within the "approach to theorems" as a process of scientific "enculturation" (Morselli & Boero, 2009), consisting in the development of a form of rational behaviour, according to Habermas' perspective (1998). This process of scientific "enculturation" involves the intentional search for proving strategies, the control of epistemic aspects, and the communication of results through rules shared by the mathematical community. In mathematical proof, communication develops from self-communication to intra-group and then external communication addressed to the mathematical community, requiring explicit didactical attention (Ferrari, 2017). In line with this view, the Inside-Out Model includes the role of Blogger, responsible for communicating problem-solving processes and results according to mathematical norms (Albano et al., 2021a). For the Blogger role we distinguish two kinds of communication: inner communication, seen as an inner dialogue, and external communication, aimed at understanding, according to Habermas' perspective (1998). Moreover, the construction of a proof takes place as a social act, particularly for the Blogger role, thus requiring an intermediate phase within a restricted community whose shared language is still in the process of development (Albano et al., 2024b). In the previous chapter we analysed the development of Estella's behavior when she moving from the role of Promoter as an Outlooker to that of Blogger as an Actor, showing a tendency toward forms of communication closely aligned with the idea of theoretical validation. This aspect suggests us to focus on the progressive development of the Blogger's role, in particular by analysing how the communication of the students who played this role evolved when they sent the final mail to Euclid-Guru. For this reason, in this chapter [1] we conduct a qualitative case study (Cohen et al., 2007) in which we observe the evolution

[1]The findings presented in this chapter will be published in a research report entitled "Towards theoretical validation within cognitive role-playing: a high school case study" and that will be presented in PME49 Conference in Helsinki

of the Blogger's role within a group engaged in conjecturing and proving activities, investigating how playing the role of Blogger, understood in a collective sense through the Actor/Outlooker dynamic, can support the emergence of rationality in 10th-grade students' collective arguments involved in geometrical proving processes (Albano et al., 2026b). We chose to focus on Blogger as a specific instance, as this character acts as an intermediary between the group and mathematical culture, taking responsibility for reorganising and reformulating the group's work for external communication, while all actions remain collectively shared and negotiated within the group. To this aim, we would like to try to contribute to answer to the research question RQ3 presented into Chapter 5. Specifically, we analyse how collectively shared arguments evolve along the didactical path, looking at the evolution of Blogger work/interventions.

12.1 Students profiles

Across the episodes, the role of Blogger was enacted by three different students. The role of Blogger was played in the initial episodes (1.1, 1.2, 1.3, 1.4, and 1.5) by a student hereinafter named Eva, in episode 2 by a student hereinafter named Tina, and in episode 3 by the student already named Estella in the previous chapter. The three students were all characterised by a high level of accuracy and attentiveness and by active participation in the proposed activities. According to the teacher, all the three students show a school-related profile marked by commitment, rigorous study habits, and careful completion of assigned tasks. Consequently, they can be considered as sharing a largely comparable academic profile. Nevertheless some differences emerged: Eva was distinguished by strong participatory intuition and rigour, Tina by greater shyness, counterbalanced by a high level of preparation and Estella by strong determination and a tendency towards perfectionism, despite being perceived by the teacher as slightly less strong than the other two.

12.2 Methods

All the activities, described in Section 8.1 and analysed in Chapter 10, are developed through collaborative interactions in digital chats in which the students, playing the characters immersing in Moodle-platform, interact and collaborate in geometric problem-solving activities, communicating among themselves. The didactical path consists of various open-ended tasks presented in narrative form through a sequence of comic-strip episodes set in a dystopian world and requiring students to engage in conjecturing processes and, subsequently, in the construction of a proof aimed at the

theoretical validation of the formulated statements (see Section 8.1). The five tasks discussed in this chapter are shown in Fig. 12.1 and correspond to distinct episodes (named Ep. 1.1. etc.) of the story, but between one episode and the next the students are asked to work on the task under the guidance of the Guru, steering by Teacher Rational Questioning Framework (Zhuang & Conner, 2022). It is worthy to note that the requests evolve along the progression of the students' work, giving rise to further, and sometimes more specific, tasks.



Task Ep.1.1 Task Ep.1.3 Task Ep.1.4 Task Ep.2 Task Ep.3

1.1: How many circles pass through a point P?
1.3: Let's see... we are therefore asking ourselves how many circles pass through A and B and where their centres are located.
1.4: Is it possible to draw a circle passing through A and B, with a centre that is not on the perpendicular bisector of AB?
2: Let us consider three non-collinear points... how many circles could you draw for the three points?
3: Look closely at the figure. What relationship do you think exists between angles alpha and beta?

Figure 12.1: Tasks and their translations

The broader research project involves the analysis of multiple data sources, including synchronous discursive interactions in group chats, systematic observations by the teacher–researcher, and/or paper-based protocols. The present chapter focuses only on the emails sent to Euclid-Guru at the end of each episode or sub-episode of the comic-strip narrative. These emails represent the final outcome of a process of collective negotiation and rewriting within the group, carried out by the Blogger for external communication with Euclid, representing the Agent of Mathematical Culture. To observe the development of modes of theoretical validation (Mariotti, 2006) looking at the evolution of Blogger final interventions, we chose to analyse the emails sent by Blogger to Euclid, using the analytical tool of Rational Mathematical Template of proof (Boero & Azrou, 2023). Thus, within collectively shared written texts, we look for the three components of Habermas' rationality (teleological, epistemic and communicative) search statements describing actions highlighting intentional proving strategies search, opinions revealing epistemic control and communication of results according to norms shared within the mathematical community (Morselli & Boero, 2009). From Habermas'

point of view (1998) an action is rational when it is aim-oriented and is achieved by means that are intentionally chosen and implemented, an opinion is rational when it is possible to motivate it in a given context, a communication is rational when the use of language is oriented towards understanding.

12.3 Results

In the following, we report the transcripts and the analysis of the emails in response to Tasks of Episodes 1.1 and 1.3.

> 1.1 - Eva's email to Euclid: Infinite circles pass through a point p because, since there are infinite points on the plane, it is possible to draw infinite circles passing through [P], all with different radii and centres. Did we manage it?!
> 1.3 - Eva's email to Euclid: Infinite circles pass through two points because it is possible to have infinite centres lying on the line perpendicular to the chord connecting the two points and passing through its midpoint, i.e. the perpendicular bisector.

In the statement "it is possible to draw infinite circles passing through [P]"(1.1. email) or "it is possible to have infinite centers lying on the line perpendicular to the chord" (1.3 email) we acknowledge a teleological dimension, as an action aim-oriented. The epistemic dimension emerges from the recognition of the infinitely many points in the plane (1.1) or on the perpendicular bisector (1.3) as potential centers of circles through P or through A and B. The communication is yet simple, the arguments are mathematically correct, but no explicit theoretical references are present. In both emails indeed we note the implicit use of the Postulate 3 of Euclid's Elements, but at this stage, the students had not yet requested access to the Euclidian Theory Bank. Moreover, the students' texts still reveal a search for confirmation from Euclid, who is recognised as the expert.

In solving Tasks of Episode 1.4, students requested access to the Theory Bank. In the following the email text produced:

> 1.4 - Eva's email to Euclid: It is not possible to draw a circle passing through A and B that does not have its center on the perpendicular bisector of the segment due to property 2 of the perpendicular bisector of the segment, which tells us that if c is not on the perp. bisec. of AB, then AC and CB

> are unequal, and this cannot be possible because a circle is a set of points that are equidistant from the center.

In the claim "It is not possible to draw a circle passing through A and B that does not have its center on the perpendicular bisector" we observe the strategic effort made by the group in searching for possible representations that could satisfy the request. The reference to property 2 of the perpendicular bisector of a segment, which they had previously demonstrated and that is been included in the Theory Bank, shows an epistemic control expressed through a negative conditional form that shifts the communication from an implicit level to a language closer to the norms shared by the mathematical community. Finally, there is another implicit reference to the definition of a circumference as a set of points equidistant from a center.

We now report the transcript of the email texts produced in response to Tasks of Episode 2.

> 2 - Tina's email to Euclid: Through three non-collinear points A,B and C exactly one circle can be drawn, because the perpendicular bisectors of segments AB and BC intersect at a single point O, which is the center of the circle. The perpendicular bisectors intersect at only one point because the points are not collinear. The segments OA, OB, and OC are in pair congruent: if O lies on the perpendicular bisector of AB, then OA and OB are congruent; if O lies on the perpendicular bisector of BC, then OB and OC are congruent. According to the common notion 1, the three segments are equal, that is, OA=OB=OC. According to Postulate 3, for every center and every distance, there exists exactly one circle.

In the first sentence of the email, we recognise a teleological dimension in the awareness of being able to trace a unique circle that has as its center exactly the unique meeting point of the two perpendicular bisectors indicated. From an epistemic point of view, there are theoretical justifications guided by visual evidence ('the perpendicular bisectors intersect at only one point because the points are not collinear'), even if in this case there is no explicit reference to Postulate 5 of Euclid's Elements. However, we observe greater attention towards the theoretical aspects involved in the construction of the proof. Tina's text shows indeed several propositions presented in conditional form, even if invoked only implicitly to justify some steps (e.g., "if O lies on the perpendicular bisector of AB, then OA and OB are congruent") with the exception of Common Notion 1 and Postulate 3 of Euclid's Elements. The communication shows a greater effort to align with the standards of the mathematical community, through a

structuring of the proof that is increasingly oriented towards the theoretical validation of the statement.

We present now the transcript of the email texts produced in response to Tasks of Episode 3.

> 3 - Estella's email to Euclid: In the circle, segment AB is the diameter, since, according to Definition 17, "a diameter of a circle is a straight line drawn through the center and terminated in both directions by the circumference of the circle, and it also bisects the circle," whereas segment AC is a chord. Triangle $\overset{\triangle}{AOC}$ is isosceles because sides AO and CO are congruent; therefore, according to Proposition 5, which claims that "the angles at the base of an isosceles triangle are equal to one another," the sum of these two angles (angle α and the angle at C) together with the angle at O of triangle $\overset{\triangle}{AOC}$ is equal to the sum of the interior angles of a triangle, that is, 180°. According to Proposition 13, which states that "if a straight line standing on another straight line makes angles, it will make either two right angles or angles equal to two right angles," the sum of angle β and the angle at O is also equal to 180°, since they are supplementary. From this, it follows that angle β is equal to 2α, because β + angle at O = 180° and 2β + angle at O =180°. Finally, according to Common Notion 3, which states that "if equals are subtracted from equals, the remainders are equal," it can be deduced that angle α is half of angle β.

Estella's email shows a further change in the care taken in communication and in the conscious use of theoretical references involved in the proof, more adherent to mathematical norms. Propositions, common notions and definitions are no longer simply used, but are explicitly referred to from the Euclidean Theory Bank and quoted textually to support the entire argument. The final paragraph ("From this, it follows that") shows that Estella is exploiting the previous remarks to make new inferences aimed at proving the conjecture, towards a theoretical validation.

12.4 Discussion

The case study documented an evolution in the Blogger's arguments, moving from naïve justification to modes of theoretical validation within a geometrical system aligned with the standards of the mathematical community (Mariotti, 2006). We note

that the student playing Blogger in Episode 3 is very rigorous and precise; she is keen to do well, and this has certainly influenced her performance in the role. In addition, here we claims that her approach was also shaped by having previously experienced the roles of Promoter, Pest, and Blogger in sequence, the first two as an outlooker and the third as an actor. We suppose that the choice of role assignments undoubtedly influences the inquiry process, and the results are certainly affected by such choices. Nevertheless, at the level of the class as a whole, an increasingly greater precision was observed in the writing of the emails to Euclid. This led us to reflect on the different emails that were gradually sent to Euclid, which reveal a growing awareness both with regard to epistemic aspects of theoretical control and to the communicative component, which became increasingly careful and clear, with far fewer implicit elements than in the initial submissions. The analysis of the emails demonstrates a growing expository clarity in communication with Euclid, in line with the standards required by the mathematical community, together with an increasingly frequent reference to theoretical aspects to support the validity of the claims made. Based on the further analysis of all the chats, looking at the Actor/Outlooker levels and at the rotation of the students playing the role of Blogger, we can hypothesize that this evolution is a process involving the entire class group. Finally, we underline that the complete analysis of all the chats shows the key role of the guidance and the mediation of Euclid, as rational mediator, to allow the students to be successful in their moving towards forms of theoretical validation.

Chapter 13

Discussion and conclusion

This thesis explored the process of scientific enculturation within the Culture of Theorems, interpreted as a rational process (Morselli & Boero, 2009; Morselli & Boero, 2010) involving 10th-grade secondary school students, by analysing the emergence of rationality components within activities of digital storytelling as cognitive role-playing (Albano et al., 2024b, 2026b, in press), particularly in the domain of Euclidean geometry. Adopting a socio-cultural perspective in which the notion of "enculturation" is understood as the appropriation of the values of mathematical culture through social practice (Bishop, 1991; Hatano & Wertsch, 2001), we additionally interpret it within an expansive perspective (Engeström & Sannino, 2010), as a continuously evolving dialectical movement developing through forms of mediation and through the negotiation of multiple voices and meanings. Within this process, communication "oriented towards understanding" (Habermas, 1998, p.315) plays a central role inside the collective Zone of Proximal Development (Engeström, 2014).

In this chapter, we summarise and discuss the findings of our studies, providing a detailed response to the four research questions presented in Chapter 5. Subsequently, we discuss the limitations of our study and outline possible directions for future research.

13.1 Answer to Research Questions

Answer to Research Question 1

How to characterise the cognitive roles of Blogger, Pest, Promoter and Boss according to Habermas' rationalities and related studies and focused on the domain of mathematics, in order to foster the emergence of the components of rationality in the proving process?

RQ1 was initially addressed in the pilot study described in Chapter 7 and focuses

on the characterization of the cognitive roles of Blogger, Pest, Promoter, and Boss in relation to the proving process in the algebraic domain, through the lens of Habermas' rationalities. To address this question, we carried out an a priori analysis based on indicators derived from the works of Albano and colleagues (2021a), Balacheff (2010), Boero (2017), Mariotti (2006), Pedemonte (2007, 2008), Urhan and Bülbül (2023a, 2023b), and Boero and Azrou (2023). This analysis resulted in the characterization of the cognitive roles reported in Table 7.1, which we named the Inside-Out 2 Model. Analogously to the film, the model extends the representation of mathematician's thinking by deepening the interpretation of cognitive roles according to Habermas' three dimensions of rationality. In the Euclid Experiment, this characterization was adapted to the domain of Euclidean geometry and further refined from a theoretical perspective and was ultimately integrated with the operational criteria emerging from the analysis of the chats, which led to the categorization presented in Table 10.1.

Within this framework, the Boss role can be interpreted as a meta-organisational instance that oversees the functioning of the entire process. Boss is responsible for stimulating the activation of cognitive functions, maintaining attention on the goal, maintaining the direction and functionality of the proving process and managing the transitions from process to product and finally, taking care of emotional aspects, supporting the group in the impasse moments. In this sense, the role embodies an awareness of the rationality's dimensions involved in mathematical activity and coordinates the functions performed by the other roles.

The Promoter role, instead, mainly concerns the teleological and epistemic components of rationality (Habermas, 1998), considered in their dialectical interplay. Indeed, the Promoter role contributes constructive inputs that enable the advancement of the process, often through temporary conjectural strategies, such as analogies and abductive reasoning, accepted as productive elements of mathematical activity, even though they may not appear in the final proof product. These strategies nevertheless support reasoning and argumentation during the creative process, but in the transition from proving to proof, abductive reasoning may also generate breaks in the cognitive unity of theorems (Boero, 2017). In particular, in chapter 9, we analysis and discuss as the cognitive unity of theorem can be made transparent even in the transition from abduction conjecturing to written mathematical proof, through the use of cloze activity (Albano et al., 2026a).

The Blogger and Pest roles are mainly associated with the communicative and epistemic dimensions of rationality, coherently with the theoretical perspectives previously outlined. Blogger uses mathematical language and symbols correctly and in accordance with the aim, thereby producing formal proof, facilitates the reading

and understanding of problem solving by verbal and/or textual explanations and provides the reasons for the correctness of the drawings and the transitions within and/or between the representations. Drawing on Balacheff's (2010) dimensions of proof and in line with the evolving nature of proof, we interpret proof construction as a communicative act directed toward oneself, toward a small community, and toward the broader mathematical community. The process thus involves an intermediate stage situated within restricted communities where mathematical language progressively evolves through interaction. Therefore, regarding the Blogger role, we distinguish, at the meta-knowledge level, between an internal communication towards itself, understood as inner dialogue, an internal communication towards small group aimed at mutual understanding, understood the blogger role as a *mediator of communication*, and external communication towards the Culture consistently with the perspectives of Habermas (1998) and Boero and Azrou (2023).

Concerning the Pest role, at the knowledge level it ensures the correctness and internal consistency of concepts, procedures, and representations coherently with the framework proposed by Urhan and Bülbül (2023a, 2023b). At the meta-knowledge level, however, Pest assumes a awareness function by monitoring the consistency of the demonstrative schemes with the reference theoretical framework (Boero & Azrou, 2023; Mariotti, 2006). Moreover, Pest engages with the epistemology of mathematics by critically evaluating both the clarity of statements and the effectiveness of communication in conveying logical reasoning (Boero & Azrou, 2023).

Answer to Research Question 2

How the use of the above characterisation can shed light on student behaviour emerging from their synchronous discursive interactions occurring in a chat in a digital problem solving situation?

In response to RQ2, we initially tested the a priori characterization in a pilot study (see Chapter 7) through the analysis of students' behaviours emerging from synchronous discursive interactions during a problem-solving activity structured according to the Inside-Out model (Albano et al., 2021a). Subsequently, this characterization was further refined as an analytical tool, serving as the theoretical basis for the subsequent categorical analysis, as shown in Chapter 10.

The data analysis substantially supports the a priori framework. During the exploration and conjecture formulation phases, the rationality components mainly operated at the knowledge level, whereas in the formalisation and initial proving phases a shift toward the meta-knowledge level emerged. The advancement of the process and the generative act described in Habermas emerge precisely through the dialectical

interaction among the three dimensions of rationality, moving from retrospective control toward forward-oriented action. The data analysis also led to a refinement of the Blogger role compared with the original Inside-Out model, further supporting the a priori framework. More specifically, our findings suggest distinguishing between self-reflective dialogic communication and external dialogic communication, consistently with Habermas (1998) and Boero and Azrou (2023). In line with Balacheff's views (2010), proof construction within this role appears to take place inside a restricted community in which meanings and mathematical language progressively evolve through interaction. The analysis additionally provides evidence of epistemic control performed by the Pest role. Accordingly, Pest's rationality appears to involve both epistemic and communicative dimensions.

Answer to Research Question 3

How the above role-playing setting supports the development of modes of theoretical validation?

The cognitive role-playing in storytelling implemented in this study operates on multiple interconnected levels. Beyond the narrative initially developed through the comics embedded in the digital platform, role-playing continues dynamically within the chats, where students animate the characters associated with the cognitive roles through their interactions (Albano et al., 2024b). In this way, the problematic situation evolves beyond a fixed narrative structure and becomes progressively co-constructed through participation. The immersive and narrative dimension is mainly realized through the interaction among the different cognitive roles and through the interplay between the two profiles of Actors and Outlookers (Albano et al., 2024c).

Particular attention should be devoted to the role of Euclid-Guru, whose interventions were designed a priori through the Teacher Rational Questioning Framework (Zhuang & Conner, 2022). The role attributed to Euclid has undergone a significant transformation with respect to the first version of the Inside-Out model. In the original framework, the Guru primarily functioned as a mediator of access to resources, facilitating such access while maintaining a privileged interaction mainly with the Promoter. In the revised version, embedded within Inside-Out 2 model and implemented within the experimental study presented in Section 8, Euclid-Guru is instead situated directly within the actors' chat as a *Rational Mediator*, interacting with the whole group (Donatiello et al., 2026). His continuous presence within the actors' environment not only allows for a more effective mediation with the students directly involved in the problem-solving activity, but also enables both Actors and Outlookers to observe, appropriate, and progressively internalize his interventions.

The qualitative analysis showed that for the Outlookers, Euclid's mediation often occurs indirectly through observation both of the actors' contributions and of Euclid-Guru's interventions, displayed forms of problematization. By comparing their own actions with those of the observed actors, reflecting on the ongoing process in the parallel chat, expressing agreement or disagreement with the actors' choices, or autonomously formulating conjectures on the basis of attentive observation alone, the Outlookers engage actively with the proving process despite not being directly involved in it. More specifically, they autonomously generated questions aimed at activating the three dimensions of rationality. We interpret these behaviours both as forms of *self-generated questioning* and as evidence of an internalization process related to the Guru role. In this sense, the Outlookers did not act as passive observers, but rather engaged in a form of *attentive and participatory observation* involving the entire proving process. Consequently, in a vygotskjian perspective (Vygotsky, 1978), we interpret the internalization of the Guru's interventions not only in terms of internalized questioning, but also as the development of actions and behaviours reflecting a reflective guiding stance, an encouragement toward deeper epistemic understanding, and a growing responsibility for the communicative dimension oriented toward mutual understanding (Habermas, 1998), supporting the negotiation of shared meanings and agreements within the small group. Through the active observation fostered by the Actors–Outlookers dynamic, the Outlookers appear to internalise not only Euclid's role, but also other cognitive roles, such as that of the Boss. The Outlookers' group displays a significant shift also toward the meta-knowledge level, revealing an emerging awareness of the meta-knowledge dimension of proof, especially in relation to the meta-statement and, consequently, to the meta-theory (Antonini & Mariotti, 2008).

The diachronic analysis revealed a progressive shift toward the meta-level during the transition from the interpretative role of Outlooker to that of Actor, identifying some turning points, where attentive and reflective observation appeared to foster a movement of rational behaviour toward the meta-knowledge level (Boero & Azrou, 2023), as well as the development of an increasing awareness oriented toward the search for forms of theoretical validation (Mariotti, 2006), mainly experienced as a perceived necessity (Zaslavsky et al., 2021). Furthermore, the analysis showed a progressively stronger attention to the writing of mathematical proofs, conceived as a form of communication addressed to the mathematical culture (Albano et al., 2021a; Zaslavsky et al., 2021). This evolution appeared to be characterised by growing sensitivity to the cultural and communicative constraints shared within the mathematical community (Balacheff, 2010; Boero & Azrou, 2023), together with an increasing attention to the theoretical dimension of mathematical activity (Mariotti, 2006).

The shift toward the meta-level becomes particularly visible in those interventions where the three dimensions of rationality no longer appear as isolated components, but instead emerge through a dialectical interplay, in line with Boero and Azrou (2023). Such moments often correspond to a forward development in the knowledge process, visible both in the reflection on the reasons supporting theoretical validation and in the increasing awareness involved in selecting mathematical tools and representations appropriate to the pursued goal, in line with Habermas' rationality construct (1998). The transition toward the meta-level also emerges in interventions oriented toward generalization. In these cases, we identify a dialectical interplay among the three dimensions of rationality that progressively directs the activity toward a theoretical perspective, with a gradual shift from reliance on empirical evidence toward the search for theoretical validation, in line with Theorem's construct of Mariotti (2006).

The analysis further revealed the dialectical complexity underlying the different components of rationality which, according to Habermas (1998), are deeply interconnected. In particular, dialectical moments often appear to correspond to transitions toward higher levels of awareness, thereby supporting the forward development of the process.

We identified oscillations between the teleological and epistemic dimensions of rationality, as well as between adherence to established mathematical culture and constructive impulses oriented toward advancing the process. From a Vygotskijan perspective, such moments may be interpreted in terms of creative imagination, conceived as a dialectical movement of distancing from and approaching culture. More specifically, they reflect a tension between a free "envol" and the need to conform to Culture (Schneuwly et al., 2022).

The analysis revealed, at the level of the whole class group, a progressively increasing precision in the writing of the emails addressed to Euclid. Reflecting on the sequence of emails exchanged throughout the activity, we observed a growing awareness not only of epistemic aspects related to theoretical control, but also of the communicative dimension. Indeed, the students' communication became progressively clearer and more careful, with substantially fewer implicit elements than those present in the initial submissions. More specifically, the analysis of the emails documents a gradual development of expository clarity in communication with Euclid, increasingly aligned with the standards required by the mathematical community. Simultaneously, references to theoretical elements in support of the validity of the claims advanced became more frequent and more explicit, in line with Mariotti (2006) point of view. Furthermore, the broader analysis of all the chats—taking into account both the Actor/Outlooker dynamics and the rotation of students assigned to the Blogger role—suggests that

this evolution should not be interpreted merely as an individual development, but rather as a process involving the entire class group. Finally, the complete analysis of the chats highlights the crucial function of Euclid's guidance and mediation. In particular, Euclid's role as a rational mediator appears fundamental in supporting students' progression toward increasingly explicit forms of theoretical validation.

The mediating function of storytelling is also enacted through Euclid's invitations to return periodically to the comics. This continuous movement between comics and chat—between the a priori designed narrative and the synchronously co-constructed one—constitutes a central feature of the storytelling dynamic. Such interplay is made possible by a strong a priori analysis of both the mathematical problem and Euclid's interventions, thereby supporting students' enculturation processes. Furthermore, the alternation between comics and chat, together with the presence of theoretical references already embedded within the comics themselves, promotes storytelling as a *mediator of reflection*, fostering the internalization of a socially experienced and actively observed proving process (Vygotsky, 1978). Storytelling thus becomes a mediator of reflection, both through the comics designed a priori as a narrative framework and through the chat environment, where storytelling is interactively extended.

Answer to Research Question 4

What connections can be or not be identified between cognitive roles and the dimensions of rationality?

The analysis of the findings of pilot study brings to light that the Inside-Out 2 model also incorporates a preliminary systemic model, which we named Cognitive Role Interconnection Model (CRIM), for describing the interconnections amongst roles within the perspective of the rationality guiding our inquiry (Albano et al., in press). This model could make the interconnections among the different roles evident to students, enabling them to recognise, within their own actions, the transitions from one role to another during the process and to develop an awareness that the boundaries among the different roles are neither fixed nor can they be, as they represent different yet cooperating facets of a single whole.

From the results emerging from the Euclid Experiment, we refined this model by constructing a two-dimensional model that allows us to interpret the connections between cognitive roles and rationalities, as well as the gaps (see Table 10.4). The analysis of the data also brought to light several relationships between cognitive roles and components of rationality. Although the roles had already been characterised a priori through the integration of different theoretical frameworks (Albano et al., in press)—many of which are themselves connected to rationality—our findings suggest

that no role can be associated exclusively with a single rationality dimension. Rather, as already noted in (Albano et al., 2024b) with respect to the Pest role, cognitive roles frequently display features that simultaneously involve two or more dimensions of rationality.

The fine-grained analysis conducted within the Euclid Experiment further refined these relationships, revealing additional and more nuanced connections between cognitive roles and rationality. In some cases, we identify a dialectical interplay between the epistemic and communicative dimensions, particularly when explanations are produced in order to achieve clarity for oneself (Mariotti, 2006). In other cases, students' arguments appear as forms of communication increasingly oriented toward alignment with the norms of the mathematical community, through a structuring of proof progressively directed toward theoretical validation of the statement. We interpret this form of interaction between rationality dimensions as *epistemic communication*.

Moreover, within the generation of new ideas—a process typically associated with the Promoter role—we identify another dialectical interplay, this time between the teleological and epistemic dimensions of rationality. This interaction emerges through recourse to memory, which we define as *epistemic memory*. Such recourse relies on mathematical knowledge coherent with the underlying theory and enables students to retrieve theoretical results capable of advancing the process. These moments of alternation between teleological and epistemic dimensions, between adherence to mathematical culture and constructive forward-oriented impulses, recall, from a Vygotskijan perspective, the notion of creative imagination as a dialectical movement of distancing from and approaching culture, conceived simultaneously as a free "envol" and as a necessary adherence to Culture (Schneuwly et al., 2022).

The main relationships emerging from the data analysis between cognitive roles and dimensions of rationality are thus summarized in the two-dimensional Table 10.4, extending the previous CRIM model. The analysis carried out on the Euclid Experiment shows that the Pest and Blogger roles exhibit strong interconnections among all three dimensions of rationality, whereas the Boss and Promoter roles display more selective relationships.

In particular, the Promoter role is naturally connected to the teleological dimension, insofar as it operates through goal-oriented actions directed toward advancing the process. At the same time, it is linked to the epistemic dimension through recourse to *epistemic memory*. However, no specific connection with the communicative dimension emerged from the analysis.

Similarly, the Boss role appears strongly related to the teleological dimension through its function of directing the group's activity and maintaining attention on

the overall goal, thereby acting as an operational guide for the proving process. In addition, we identify a connection with the communicative dimension in those forms of communication aimed at activating and coordinating the cognitive roles within the group. By contrast, no specific connection with the epistemic dimension emerged. Indeed, although the Boss performs functions of control and monitoring, these concern the management of the ongoing process rather than epistemic issues themselves, whose regulation is instead mainly associated with the Pest role.

13.2 Conclusion

Some of the discussed findings go beyond the original a priori characterization, because students sometimes perform functions that are not expected from their assigned role, and the analysis shows that the same dimensions of rationality can appear across different cognitive roles. This is relevant for us, because it provide evidence of how the empirical study could contribute to the improvement and enhancement of the original model.

In responding to our overarching research programme, we now synthesise the overall process discussed in the previous sections through a diagram (see Fig. 13.1) that represents the interrelation between storytelling as cognitive role-playing and Habermas' dimensions of rationality. Within this diagram, two interconnected levels can be identified: the level of cognitive role-playing and the level of rationality. The considerations presented below emerge from the discussion developed in response to Research Question 3 (see Answer to RQ3).

Compared to the previous model, the Actors now receive a a *direct mediation* from the Guru, who no longer maintains a privileged relationship only with the Promoter, but is embedded within the actors' chat throughout the entire proving process, interacting continuously with the actors' group. At the same time, through *attentive and participatory observation*, the Outlookers receive an *indirect mediation* of the Guru's interventions, progressively internalising forms of rational questioning. Moreover, *reflective actions* such as writing to learn (Morgan, 2002), rereading the comics, and providing detailed descriptions of the drawings uploaded on Padlet, further stimulate attentive observation. This form of observation is not exclusive to the Outlookers, but also characterises those Actors who adopt a critical and dialectical stance throughout the process.

Furthermore, the level of rationality is deeply intertwined with that of role-playing through the a priori characterization of the cognitive roles developed from a rationality perspective (see Answer to RQ1). This characterization enables groups of students to

act coherently with the cognitive roles of the Inside-Out 2 model, which are themselves connected to specific dimensions of rationality (see Answer to RQ4).

Furthermore, the presence of a complex dialectical interplay among the different dimensions of rationality makes it possible both to identify moments of transition toward higher levels of awareness and to detect shifts toward the meta-level, particularly in the case of Outlookers, but also among Actors who had previously experienced participation as Outlookers. This progressively directs the activity toward a theoretical perspective, with a gradual shift from empirical evidence toward the search for *theoretical validation*, in line with Mariotti (2006).

We additionally interpret the oscillations between the epistemic and teleological dimensions of rationality as moments of "free envol" (Schneuwly et al., 2022), enabling the process to move forward while simultaneously requiring adherence to mathematical Culture. In this way, the activity supports a gradual and progressive process of scientific enculturation understood as a rational process, although such a process needs to be considered as "a long term *enculturation* process" (Morselli & Boero, 2010, p.219, italics added).

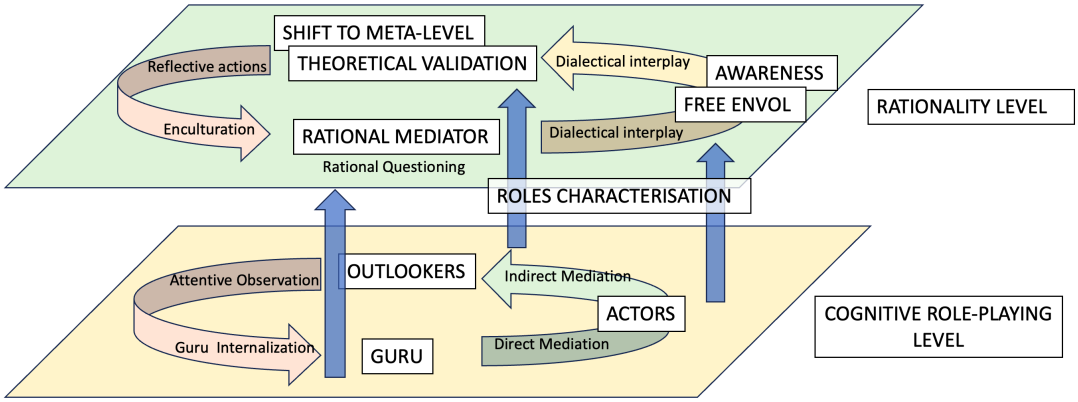


Figure 13.1: Storytelling as rational mediator of reflection

13.3 Limitations

The activity of digital storytelling as cognitive role-playing enables a highly immersive engagement of students through the extension of the narrative beyond the comics and into the Actors and Outlookers chats. In this way, it also promotes greater participation from more reserved students (as already observed in sec. 11.1) or as

emerged in one of the two classes involved in the study from retrospective students' reflections, an extract of which is given below:

> "At the beginning, I felt disoriented: this new way of working, based on comparison, discussion, and reflection, frightened me. I was afraid that, because of my introverted personality, I would not be able to achieve my goal. Indeed, from the very beginning, we were asked to participate very actively, and I feel satisfied that I managed to intervene several times. For me, this alone was already a great success.[...] In particular, the aspect I found most useful was the idea of having us interpret characters; in this way, I felt freer to express my opinions and reflections in the group chat under the name of the character I was playing, without being afraid of others' judgement or of saying something wrong"

Nevertheless, during the experimentation phases we observed difficulties in chat interaction for one student who, according to the classroom teacher, appeared particularly fragile from a cognitive perspective. This raises the question of to what extent the Inside-Out 2 Model actually supports the inclusion of students with objective or certified difficulties, or whether, conversely, it may contribute to rendering such students less visible to the attentive eye of the teacher. One of the critical issues identified concerns the specific form of mediation enacted in the chats by the teacher-researcher. Although the mediation was supported by the Teacher Rational Questioning Framework (Zhuang & Conner, 2022), the chat-based interaction highlighted a limitation related to the necessity of mediating exclusively through written language. In this form of communication, many visual and gestural aspects—which are fundamental for students' immediate reception of feedback—are inevitably lost. This issue may be particularly problematic for students with certified learning difficulties, who may feel disoriented or lost when confronted with a mode of interaction that is not well suited to their needs. Consequently, an interaction based exclusively on chat communication risks obscuring these dimensions and may therefore fail to fully support inclusion. Another limitation that deserves attention, and which also constitutes a stimulus for future reflections, concerns emotional aspects. Habermas' construct of rationality works particularly effective for interpreting both the construction of proof during the proving process and proof as a final product, thereby offering a snapshot of a broader enculturation process. Already Morselli and Boero (2009) recognised the need to conceptualise this construct within a broader cultural perspective, an attempt that this thesis pursued through the integration of rationality with culturally situated cognitive role-playing, although the activity described is limited to three months of work within the school curriculum and thus needs to be embedded within

a longer-term process. Nevertheless, Habermas' framework does not directly address the more specifically emotional dimensions that emerged from the data analysis. In particular, the operational indicator "Takes care of emotional aspects, supporting the group in moments of impasse", reported in Table 10.1 with reference to the Boss role, emerged directly from the data analysis, as highlighted by several students themselves. Furthermore, this aspect also appeared in the retrospective reflections requested to students at the end of the activities. A further issue to be explored therefore concerns how emotional dimensions emerging within a proving process—understood as a rational process—can be theoretically and analytically taken into account.

Among the study's limitations, we may also identify the fact that it essentially investigates the collective process of conjecturing, arguing, and proving, as well as the collaborative writing of proofs. Even the character of the Blogger is conceptualised as a collective character, as it incorporates the experiences previously developed by each student while acting as an outlooker within that specific role. Therefore, the dimension of learning as an interpersonal practice is adequately addressed in this research, whereas the analysis of the longer-term process of internalization appears to remain in the background.

To address this issue, students were asked to alternate collective activities embedded in role-playing settings with individual activities involving the writing or rewriting of proofs on personal worksheets.

Whereas the paper worksheets produced during the preparatory phase through the cloze activity were carefully analysed through the lens of the Rational Mathematical Template (see chapter 9), the paper worksheets produced during the individual proof-writing activities alternated with the role-playing activities will be the object of subsequent analyses. The analysis of these worksheets was not the direct and focused object of the present dissertation, but it certainly requires further investigation in future work, in order to better understand the process of internalization from a Vygotskian perspective.

These considerations open up several possible directions for future developments and research.

13.4 The flavour of ...

With regard to the reflection on the emotional dimension, the following excerpt from the responses to the final retrospective questionnaire is reported, as it highlights the emotional aspects associated with the character Kyle-Boss:

"Kyle somewhat represented the leader of the group, and I behaved

> accordingly; I entered into the character very naturally, also because she/he and I have many things in common. Like Kyle, I am always present in difficult moments, and I enjoy helping and encouraging people whenever they need support. During this episode, Kyle took control of the situation and tried to 'manage' everything in order to guide the group toward the correct resolution of the problem. In moments of discouragement, Kyle gave the others the strength to continue and not to lose hope, reminding them that every problem has a solution."

The character Kyle-Boss also gave rise to classroom discussions concerning the character's gender identity. During the preparatory phase, in which students were introduced to the different roles, they engaged in a lively discussion about Kyle-Boss's gender. The awareness of our design choice—namely, that Kyle-Boss had been conceived and developed as a non-binary character—surprised and engaged them deeply. Indeed, in the same excerpt reported above, we can observe that the student respects Kyle's non-binary identity by using the dual pronoun she/he.

In the retrospective reflections, the emotional dimension also involved feelings of fear and disorientation when confronted with a new approach to geometry, as well as the challenging nature of the activity, which fostered motivation. An excerpt from the students' retrospective reflections is reported below:

> "At the beginning, I felt disoriented by this new way of teaching Euclidean geometry, and this novelty terrified me because I was afraid I would not be able to accomplish anything [...] but I was willing to challenge myself".

The emphasis on students' possibility of perceiving themselves as capable of solving problems also emerged as a further aspect in the retrospective reflections and deserves to be addressed in future work. An excerpt is reported below:

> "I thought that the four surviving friends would have needed to collaborate and strengthen even further both their friendship and their knowledge in order to solve the problems [...] I would never have imagined that, thanks to these characters, we would have been able to solve problems *on our own*".

The emotional aspects represent a novel element with respect to the previous model and therefore require further in-depth investigation, which will be addressed in future research, involving affect.

Another area requiring further development concerns the role of the OutLooker. Compared with the first model, we chose to make this role more participatory and active by also encouraging the group of OutLookers to engage in solving the mathematical

problem and by allowing them to upload their drawings to Padlet, with the aim of fostering greater involvement. Nevertheless, one student, in the retrospective reflections, highlights the need to address an even more interactive design for the OutLookers who, not being directly involved in sending the final email to Euclid, must choose to solve the problem for their own sake, thus engaging in a process of full devolution going outside the standard didactical contract (Brousseau, 2002).

An excerpt from the students' retrospective reflections is reported below as an illustrative example, as it highlights precisely this aspect:

> "I truly enjoyed the role of actor, a little less the role of outlooker; I do not think the outlooker's role was useless or anything similar—on the contrary, I think it was very constructive to observe, not simply to look, because there is a great difference between the two verbs, "from the outside"—but I am not sure whether it was actually useful for the others. As a student, I admit that at times I did not fully engage in solving the tasks, using as an excuse the fact that I did not have to send the email. Therefore, if my opinion may be helpful, I would suggest, for the future, finding something that could motivate the outlookers so that they feel involved in the problems in the same way as the actors did".

In designing the interventions of the teacher-researcher, we employed the Teacher Rational Questioning Framework (Zhuang & Conner, 2022); however, while analysing the data, we realised that some interventions enacted by Euclid-Guru were also highly consistent with the Model of Aware and Effective Attitudes and Behaviours theoretical construct (Cusi & Morselli, 2016), such as the conception of the teacher as a Reflective and Strategic Guide or as an Activator of Anticipatory Thinking. In future work, it would be interesting to consider a combined use of the two frameworks.

Bibliography

Acerbi, F. (2007). *Euclide, tutte le opere*. Bompiani.

Aceto, M., De Santis, M. L., Donatiello, A., Lemmi, G., Manzoni, S., Montervino, C. R., Picariello, L., Scaramuzzino, V., & Vadalà, G. (2024). Due esperienze didattiche di early-algebra con il DIST-M: Dalla costruzione del problema all'implementazione in aula. *Didattica della matematica. Dalla ricerca alle pratiche d'aula*, *16*, 94–115.

Albano, G. (2023). Towards the construction of a mathematical identity through a reflective role-play in problem solving. *Quadrante*, *32*(2), 175–194.

Albano, G., Antonini, S., & Miranda, A. (2022a). Cognitive roles in cooperative problem solving at university level. In J. Hodgen, E. Geraniou, G. Bolondi, & F. Ferretti (Eds.), *Proceedings of the Twelfth Congress of the European Society for Research in Mathematics Education (CERME12)*, (16), (pp. 2331–2338). Free University of Bozen-Bolzano, Italy and ERME.

Albano, G., Antonini, S., & Miranda, A. (2024a). Digital experiences of mathematical cognitive functions in learning the basic concepts of general topology. *International Journal of Research in Undergraduate Mathematics Education*, *10*(3), 823–849.

Albano, G., Coppola, C., & Dello Iacono, U. (2021a). What does 'Inside Out' mean in problem solving? *For the Learning of Mathematics*, *41*(2), 32–36.

Albano, G., Coppola, C., Dello Iacono, U., & Pierri, A. (2020a). Online discursive interactions concerning mathematical issues within Digital Interactive Storytelling. In J. Ingram, K. Erath, F. Rønning, & A.K. Schüler-Meyer (Eds.), *Proceedings of the Seventh ERME Topic Conference on Language in the Mathematics Classroom*, (pp. 123.130). ERME / HAL Archive.

Albano, G., Coppola, C., & Donatiello, A. (2024b). Cognitive roles' characterisation of the Inside-Out Model in Habermas' rationality perspective: the cases of Pest and Blogger. In M. Ní Ríordáin, & K. Erath (Eds.), *Proceedings of the Sixteenth ERME Topic Conference on Language and Social Interaction in Mathematics Classrooms*, (pp.4–11). ERME / HAL Archive.

Albano, G., Coppola, C., & Donatiello, A. (2026a). Cloze activity as a mediator to make the cognitive unity of theorems transparent: A study with secondary school students. In A. Moutsios-Rentzos, O. Buchbinder, L. Branchetti, & V. Durand-Guerrier (Eds.), *MATHPIE2026 Argumentation and Proof in the Teaching and Learning*

of Mathematics and Interactions with Other Disciplines, ERME Topic Conference (ETC24), (pp.95–104). ERME.

Albano, G., Coppola, C., & Donatiello, A. (2026b). Towards theoretical validation within cognitive role-playing: A high-school case study. In H. Krzywacki, M. Hähkiöniemi, E. Haataja, M. Hannula P. Hästö (Eds.). *Proceedings of the 49th Conference of the International Group for the Psychology of Mathematics Education: Research Reports*, (pp.355–362).PME.

Albano, G., Coppola, C., & Donatiello, A. (in press). The emergence of rational behaviour in the proving process: Towards Inside-Out 2. *Educational Studies in Mathematics*.

Albano, G., Coppola, C., & Fiorentino, G. (2021b). The role of teacher in DIST-M activities. *International Journal for Technology in Mathematics Education*, *28*(3), 163–170.

Albano, G., Coppola, C., Iacono, U. D., Fiorentino, G., Pierri, A., & Polo, M. (2020b). Technology to enable new paradigms of teaching/learning in mathematics: The digital interactive storytelling case. *Journal of E-learning and Knowledge Society*, *16*(1), 65–71.

Albano, G., Coppola, C., & Polo, M. (2024c). Digital Interactive Storytelling in Matematica: un dispositivo metodologico per l'apprendimento sociale orientato alle competenze. *Didattica della matematica. Dalla ricerca alle pratiche d'aula*, *16*, 9–35.

Albano, G., Dello Iacono, U., & Fiorentino, G. (2020c). A Technological Storytelling Approach to Nurture Mathematical Argumentation. In H. Chad Lane, S. Zvacek, & J. Uhomoibhi (Eds.) *Proceedings of the 12th International Conference on Computer Supported Education (CSEDU 2020), Vol. 1* (pp. 420–427). SCITEPRESS.

Albano, G., Dello Iacono, U., & Mariotti, M. A. (2017). A computer-based collaboration script to mediate verbal argumentation in mathematics. In T. Dooley & G. Gueudet (Eds.), *Proceedings of the Tenth Congress of the European Society for Research in Mathematics Education (CERME 10)*, (pp.2507–2514). DCU Institute of Education and ERME.

Albano, G., Dello Iacono, U., & Mariotti, M. A. (2021c). An e-learning innovative approach for mathematical argumentative thinking. *International Journal for Technology in Mathematics Education*, *28*(1), 3–14.

Albano, G., & Ferrari, P. L. (2013). Linguistic competence and mathematics learning: The tools of e-learning. *Journal of e-Learning and Knowledge Society*, *9*(2), 27–41.

Albano, G., Mollo, M., Polo, M., & Marsico, G. (2022b). Dialogical interactions mediated by technology in mathematics education. *Dialogic Pedagogy: An International Online Journal, 10*, DT22–DT40.

Albano, G., & Pierri, A. (2017). Digital storytelling in mathematics: A competence-based methodology. *Journal of Ambient Intelligence and Humanized Computing*, *8*(2), 301–312.

Antonini, S. (2019). Intuitive acceptance of proof by contradiction. *ZDM*, *51*(5), 793–806.

Antonini, S., & Mariotti, M. A. (2006). Reasoning in an absurd world: Difficulties with proof by contradiction. In J. Novotná, H. Moraová, M. Krátká, & N. Stehlíková (Eds.). *Proceedings 30th Conference of the International Group for the Psychology of Mathematics Education*, *2*, (pp.65–72). PME.

BIBLIOGRAPHY

Antonini, S., & Mariotti, M. A. (2008). Indirect proof: What is specific to this way of proving? *ZDM*, *40*(3), 401–412.

Bachtin, M. (1968). *Dostoevskij. Poetica e stilistica.* Einaudi.

Bagossi, S., Swidan, O., & Arzarello, F. (2022). Timeline tool for analyzing the relationship between students-teachers-artifacts interactions and meaning-making. *Journal on Mathematics Education*, *13*(2), 357–382.

Balacheff, N. (2010). Bridging knowing and proving in mathematics an essay from a didactical perspective. In G. Hanna , H. N. Jahnke & H. Pulte (Eds.)*Explanation and Proof in Mathematics* (pp. 115–135). Springer.

Balacheff, N. (1987). Processus de preuve et situations de validation. *Educational Studies in Mathematics*, *18*(2), 147–176.

Balacheff, N. (1991). The benefits and limits of social interaction: The case of mathematical proof. In A.J. Bishop, S. Mellin-Olsen, & J. Van Dormolen (Eds). *Mathematical knowledge: Its growth through teaching* (pp. 173–192). Springer.

Balacheff, N. (2008). The role of the researcher's epistemology in mathematics education: An essay on the case of proof. *ZDM*, *40*(3), 501–512.

Bartolini Bussi, M. G. (1996). Mathematical discussion and perspective drawing in primary school: To Giovanni Prodi on occasion of his 70th birthday. *Educational Studies in Mathematics*, *31*(1), 11–41.

Bell, J., & Waters, S. (2018). *Doing your research project: A guide for first-time researchers.* McGraw-hill education (UK).

Birks, M., Chapman, Y., & Francis, K. (2008). Memoing in qualitative research: Probing data and processes. *Journal of research in nursing*, *13*(1), 68–75.

Bishop, A. (1991). *Mathematical enculturation: A cultural perspective on mathematics education* (Vol. 6). Springer Science & Business Media.

Boero, P., & Planas, N. (2014). Habermas' construct of rational behavior in mathematics education: New advances and research questions. In P. Liljedahl, C. Nicol, S. Oesterle, & D. Allan, (Eds.). *Proceedings of the Joint Meeting of PME38*, *1*, (pp.205–235). PME.

Boero, P. (1999). Argumentation and mathematical proof: A complex, productive, unavoidable relationship in mathematics and mathematics education. *Lettre de la Preuve. International Newsletter on the Teaching and Learning of Mathematical Proof*, *7*(8).

Boero, P. (2006). Habermas' theory of rationality as a comprehensive frame for conjecturing and proving in school. In J. Novotná, H. Moraová, M. Krátká, & N. Stehlíková (Eds.). *Proceedings of the 30th Conference of the International Group for the Psychology of Mathematics Education*, *2*, (pp.185–192). PME.

Boero, P. (2017). Cognitive unity of theorems, theories and related rationalities. T. In Dooley, & G. Gueudet (Eds.). *Proceedings of the Tenth Congress of the European Society for Research in Mathematics Education (CERME10)*, (pp.99–106). DCU Institute of Education and ERME.

Boero, P. (2022). Developing students' rationality by constructing and exercising rational mathematical templates: The case of counter-examples. J. Hodgen, E. Geraniou, G. Bolondi, & F. Ferretti (Eds.). *Proceedings of the Twelfth Congress of the European*

Society for Research in Mathematics Education (CERME12), (3), (pp.101-108). Free University of Bozen–Bolzano and ERME.

Boero, P., & Azrou, N. (2023). Analysis of proving in the Habermas' rationality perspective: The meta-knowledge–knowledge interplay. In P. Drijvers, C. Csapodi, H. Palmér, K. Gosztonyi, & E. Kónya (Eds.). *Proceedings of the Thirteenth Congress of the European Society for Research in Mathematics Education (CERME13)*, (4), (pp.72–79). Alfréd Rényi Institute of Mathematics and ERME.

Boero, P., Douek, N., Morselli, F., & Pedemonte, B. (2010). Argumentation and proof: A contribution to theoretical perspectives and their classroom implementation. In M. M. F. Pinto T. F. Kawasaki (Eds.)*, Proceedings of the 34th Conference of the International Group for the Psychology of Mathematics Education, vol. 1*, (pp.179–208). PME.

Boero, P., Garuti, R., & Lemut, E. (2007). Approaching theorems in grade VIII: Some mental processes underlying producing and proving conjectures, and conditions suitable to enhance them. In P. Boero (Ed.) *Theorems in school: From history epistemology and cognition to classroom practice* (pp.249–264). BRILL.

Boero, P., Garuti, R., Lemut, E., & Mariotti, M. A. (1996a). Challenging the traditional school approach to theorems: A hypothesis about the cognitive unity of theorems. In L. Puig, & A. Gutierrez (Eds.). *Proceedings of the 20th Conference of the International Group for the Psychology of Mathematics Education*, *2*, (pp.113–120). University of Valencia and PME.

Boero, P., Garuti, R., & Mariotti, M. A. (1996b). Some dynamic mental processes underlying producing and proving conjectures. In L. Puig, & A. Gutierrez (Eds.). *Proceedings of the 20th Conference of the International Group for the Psychology of Mathematics Education*, *2*, (pp. 121–128). University of Valencia and PME.

Boero, P., & Morselli, F. (2010). The use of algebraic language in mathematical modelling and proving in the perspective of Habermas'theory of rationality. In V. Durand-Guerrier, S. Soury-Lavergne, & F. Arzarello (Eds.). *Proceedings of the Sixth Congress of the European Society for Research in Mathematics Education (CERME 6)*, (pp.964–973). Institut National de Recherche Pédagogique and ERME.

Boero, P., & Turiano, F. (2020). Rational communication in the classroom: Students' participation and Rational Mathematical Templates. In J. Ingram, K. Erath, F. Rønning, & A.K. Schüler-Meyer (Eds.)*, Proceedings of the Seventh ERME Topic Conference on Language in the Mathematics Classroom*, (pp.139–146). ERME.

Boero, P., & Turiano, F. (2022). A case study on students' approach to euclidean proof in the rationality perspective. In C. Fernández, S. Llinares, A. Gutiérrez, & N. Planas (Eds.)*Proceedings of the 45th Conference of the International Group for the Psychology of Mathematics Education*, (pp.83–90). PME.

Boero, P., & Turiano, F. (2023). Proof and students' proving in euclidean geometry: Linguistic aspects and method issues in the rationality perspective. In A. Schüler-Meyer, J. Ingram, K. Erath (Eds.) *Proceedings of the Twelfth ERME Topic Conference on Language in the Mathematics Classroom*, (pp.22–29). ERME.

BIBLIOGRAPHY

Branchetti, L., & Morselli, F. (2019). The interplay of rationality and identity in a mathematical group work. In M. Hannula, G. Leder, F. Morselli, M. Vollstedt, Q. Zhang (Eds.). *Affect and Mathematics Education: Fresh Perspectives on Motivation, Engagement, and Identity. ICME-13 Monographs* (pp. 323–344). Springer.

Brousseau, G. (2002). The didactical contract: The teacher, the student and the milieu. *Theory of didactical situations in mathematics: Didactique des mathématiques, 1970–1990*, 226–249.

Bruner, J. (2009). *Actual minds, possible worlds* (Vol. 1). Harvard University Press.

Cabassut, R., Conner, A., İşçimen, F. A., Furinghetti, F., Jahnke, H. N., & Morselli, F. (2012). Conceptions of proof–in research and teaching. In G. Hanna, & M. de Villiers (Eds.). *Proof and Proving in Mathematics Education: The 19th ICMI study* (pp. 169–190). Springer.

Cobb, P. (2007). Putting philosophy to work. *Second Handbook of Research on Mathematics Teaching and Learning: A project of the National Council of Teachers of Mathematics*, *1*(1), 3–38.

Cohen, L., Manion, L., & Morrison, K. (2007). *Research methods in education.* (6th ed.). Routledge.

Conner, A. (2017). An application of habermas' rationality to the teacher's actions: Analysis of argumentation in two classrooms. In T. Dooley, & G. Gueudet (Eds.). *Proceedings of the Tenth Congress of the European Society for Research in Mathematics Education (CERME 10)* (pp. 123–130). ERME-DCU.

Cruciani, F. (2006). *Registi pedagoghi e comunità teatrali nel novecento (e scritti inediti).* Editoria & spettacolo,

Cusi, A., & Morselli, F. (2016). The teacher's role in promoting students' rationality in the use of algebra as a thinking tool. In C. Csíkos, A. Rausch, & J. Szitányi (Eds.). *Proceedings of the 40th Conference of the International Group for the Psychology of Mathematics Education,* Vol. 2 (pp. 187–194). PME.

Deslis, D., Moutsios-Rentzos, A., Kaskaouti, P., & Giakoumi, M. (2025). Digital storytelling in teaching and learning mathematics: A prisma systematic literature review. *Education Sciences*, *15*(11), 1548.

Deslis, D., Stylianides, A. J., & Jamnik, M. (2023). Primary school teachers' mathematical knowledge and views about lakatos-style proving activity: A latent profile analysis. In P. Drijvers, C. Csapodi, H. Palmér, K. Gosztonyi, & E. Kónya, (Eds.). *Thirteenth Congress of the European Society for Research in Mathematics Education (CERME13)* (pp. 104–111). Alfréd Rényi Institute of Mathematics; ERME.

de Villiers, M. (1990). The role and function of proof on mathematics. *Pythagoras*, *24*, 17–24.

Donatiello, A., Garrisi, R., & Tortoriello, V. (2026). Euclide e la fuga sul pianeta beta: Role-playing per la dimostrazione geometrica in ottica STEAM. In A.Amoroso, M. Rinaudo & M. Serio (Eds.)*, Le tecnologie emergenti al servizio dell'insegnamento della matematica e della fisica nella società di oggi Atti del XII Convegno Nazionale di Didattica della Fisica e della Matematica DI.FI.MA. 2025,* (pp.304-312). https://www.collane.unito.it/oa/items/show/283.

Donatiello, A., Garrisi, R., & Tortoriello, V. (in press). Il ruolo della mediazione razionale nel digital interactive storytelling in mathematics per favorire l'accesso alla cultura dei teoremi in geometria euclidea. In *Quaderni di Ricerca in Didattica. Atti delle Giornate di Studio dell'Insegnante di Matematica*.

Douek, N. (2014). Pragmatic potential and critical issues. In P. Liljedahl, C. Nicol, S. Oesterle, & D. Allan (Eds.). *Proceedings of the 38th Conference of the International Group for the Psychology of Mathematics Education, Vol. 1* (pp. 209–213). PME & UBC.

Douek, N. (2007). Some remarks about argumentation and proof. In Paolo Boero (Ed.).*Theorems in school: From history, epistemology and cognition to classroom practice* (pp. 163–181). Sense Publishers.

Duval, R. (1991). Structure du raisonnement déductif et apprentissage de la démonstration. *Educational studies in mathematics*, *22*(3), 233–261.

Egan, K. (1989). *Teaching as story telling: An alternative approach to teaching and curriculum in the elementary school*. University of Chicago Press.

Engeström, Y. (2014). *Learning by expanding: An activity-theoretical approach to developmental research*. Cambridge University Press.

Engeström, Y., & Sannino, A. (2010). Studies of expansive learning: Foundations, findings and future challenges. *Educational Research Review*, *5*(1), 1–24.

Ferrari, P. L. (2017). Language in argumentation and solution of problems with graphs. In T. Dooley, & G. Gueudet (Eds.). *Proceedings of the Tenth Congress of the European Society for Research in Mathematics Education (CERME 10)* (pp. 1276–1283). ERME-DCU.

Fischbein, E. (1993). The theory of figural concepts. *Educational Studies in Mathematics*, *24*(2), 139–162.

Goizueta, M. (2014). The emergence of validity conditions in the secondary mathematics classroom: Linking social and epistemic perspectives. In P. Liljedahl, C. Nicol, S. Oesterle, & D. Allan (Eds.). *Proceedings of the 38th Conference of the International Group for the Psychology of Mathematics Education, Vol. 1* (pp. 214–218). PME & UBC.

Goizueta, M., Mariotti, M. A., & Planas, N. (2014). Validating in the mathematics classroom. In S. Oesterle, C. Nicol, P. Liljedahl, & D. Allan (Eds.), *Proceedings of the Joint Meeting of PME 38 and PME-NA 36, Vol. 3* (pp. 169–173). PME.

Guala, E., & Boero, P. (2017). Cultural analysis of mathematical content in teacher education: The case of elementary arithmetic theorems. *Educational Studies in Mathematics*, *96*(2), 207–227.

Habermas, J. (1998). *On the pragmatics of communication*. MIT Press.

Habermas, J. (2014). *On the pragmatics of social interaction*. John Wiley & Sons.

Hanna, G. (1989). Proofs that prove and proofs that explain. In In G. Vergnaud, J. Rogalski, & M. Artigue (Eds.) *Proceedings of the 13th Conference of the International Group for the Psychology of Mathematics Education, Vol. 2* (pp. 45–51). PME.

Hanna, G. (1990). Some pedagogical aspects of proof. *Interchange*, *21*(1), 6–13.

Harel, G., & Sowder, L. (2007). Toward comprehensive perspectives on the learning and teaching of proof. In F. Lester, (Ed.). *Second Handbook of Research on Mathematics Teaching and Learning, Vol. 2* (pp. 805–842).

Hatano, G., & Wertsch, J. V. (2001). Sociocultural approaches to cognitive development. *Human Development*, *44*(2/3), 77–83.

Healy, L., & Hoyles, C. (1999). Students' performance in proving: Competence or curriculum. In I. Schwank, (Ed.). *European Research in Mathematics Education I.* (pp. 153–167). Forschungsinstitut für Mathematikdidaktik.

Healy, L., & Hoyles, C. (2000). A study of proof conceptions in algebra. *Journal for Research in Mathematics education*, *31*(4), 396–428.

King, A. (1989). Verbal interaction and problem-solving within computer-assisted cooperative learning groups. *Journal of Educational Computing Research*, *5*(1), 1–15.

King, A. (2007). Scripting collaborative learning processes: A cognitive perspective. In F. Fischer, I. Kollar, H. Mandl, & J.M. Haake (Eds.) *Scripting Computer-Supported Collaborative Learning: Cognitive, Computational and Educational perspectives, Vol. 6* (pp. 13–37). Springer.

Kipper, D. A. (1996). The emergence of role playing as a form of psychotherapy. *Journal of Psychodrama, Sociometry, and Group Psychotherapy*, *49*(3), 99–119.

Lieblich, A., Tuval-Mashiach, R., & Zilber, T. (1998). *Narrative research: Reading, analysis, and interpretation* (Vol. 47). Sage.

Liljedahl, P., Santos-Trigo, M., Malaspina, U., & Bruder, R. (2016). *Problem solving in mathematics education*. Springer Nature.

Lolli, G. (2018). *Matematica come narrazione*. Il mulino.

Mariotti, M. A. (2005). Proof and proving in algebra. *Lettre de la Preuve* http://www.lettredelapreuve.org/OldPreuve/Newsletter/06Hiver/Mariotti-GARME.pdf.

Mariotti, M. A. (2006). Proof and proving in mathematics education. In A. Gutiérrez & P. Boero (Eds.). *Handbook of Research on the Psychology of Mathematics Education: Past, present and future* (pp. 173–204). Brill.

Mariotti, M. A. (2018). From using artefacts to mathematical meanings: The teacher's role in the semiotic mediation process. In *Atas do XXIX Seminário de Investigação em Educação Matemática (XXIX SIEM)* (pp. 19–33). APM.

Mariotti, M. A., Bartolini Bussi, M., Boero, P., Ferri, F. e., & Garuti, R. (1997). Approaching geometry theorems in contexts: From history and epistemology to cognition. In E. Pehkonen (Ed.)*Proceedings of the 21th Conference of the International Group for the Psychology of Mathematics Education, Vol. 1* (pp. 180–195). PME.

Mariotti, M. A., & Maracci, M. (1999). Conjecturing and proving in problem-solving situations. In O. Zaslavsky (Ed.) *Proceedings of the 23rd Conference of the International Group for the Psychology of Mathematics Education (PME 23), Vol. 3* (pp. 265–272). PME.

Mellone, M., & Tortora, R. (2015). Ambiguity as a cognitive and didactic resource. In K. Krainer, & N. Vondrová (Eds.)*Proceedings of Ninth Congress of the European Society for Research in Mathematics Education (CERME 9)* (pp. 1434–1439). ERME.

Miranda, A., Picariello, L., & Coppola, C. (2023). Towards a designated undergraduate students' mathematics identity through a thinking group approach. In P. Drijvers, C. Csapodi, H. Palmér, K. Gosztonyi, & E. Kónya (Eds.). *Thirteenth Congress of the European Society for Research in Mathematics Education (CERME13)* (pp. 2491–2498). Alfréd Rényi Institute of Mathematics; ERME.

Miyazaki, M., Fujita, T., Jones, K., & Iwanaga, Y. (2017). Designing a web-based learning support system for flow-chart proving in school geometry. *Digital Experiences in Mathematics Education*, *3*(3), 233–256.

Morgan, C. (2002). *Writing mathematically: The discourse of'investigation'*. Routledge.

Morselli, F., & Boero, P. (2009). Habermas' construct of rational behaviour as a comprehensive frame for research on the teaching and learning of proof. In F.L. Lin, F.J. Hsieh, G. Hanna, & M. de Villiers (Eds.). *Proceedings of the ICMI Study 19 Conference: Proof and Proving in Mathematics Education, Vol 2* (pp. 100–105). National Taiwan Normal University, PME.

Morselli, F., & Boero, P. (2010). Proving as a rational behaviour: Habermas'construct of rationality as a comprhensive frame for research on the teaching and learning of proof. In V. Durand-Guerrier, S. Soury-Lavergne, & F. Arzarello (Eds.) *Proceedings of the Sixth Congress of the European Society for Research in Mathematics Education (CERME 6)* (pp. 211–220). ERME.

Moutsios-Rentzos, A., Kalavasis, F., & Meimaris, M. (2019). My relationship with mathematics: Multimodal realisations and realities. In A. Moutsios-Rentzos, A. Giannakoulopoulos, & M. Meimaris (Eds.) *Current trends in Digital Storytelling. Proceedings of International Digital Storytelling Conference 2018* (pp. 186–194). Club UNESCO Zakynthos.

Pedemonte, B. (2002). *Étude didactique et cognitive des rapports entre argumentation et démonstration dans l'apprentissage des mathématiques* [Doctoral dissertation, Université Joseph-Fourier-Grenoble I].

Pedemonte, B. (2007). How can the relationship between argumentation and proof be analysed? *Educational Studies in Mathematics*, *66*(1), 23–41.

Pedemonte, B. (2008). Argumentation and algebraic proof. *ZDM*, *40*(3), 385–400.

Pesci, A. (2009). Cooperative learning and peer tutoring to promote students' mathematics education. In L. Paditz, & A. Rogerson, (Eds.). *Proceedings of the 10th International Conference: Models in Developing Mathematics Education* (pp. 486–490). Dresden University of Applied Sciences.

Pierce, C. (1934). *Collected papers of Charles Sanders Pierce* (Vol. 5). Harvard University Press.

Prediger, S., Bikner-Ahsbahs, A., & Arzarello, F. (2008). Networking strategies and methods for connecting theoretical approaches: First steps towards a conceptual framework. *ZDM*, *40*(2), 165–178.

Rav, Y. (1999). Why do we prove theorems? *Philosophia Mathematica*, *7*(1), 5–41.

Reid, D. A., & Vallejo Vargas, E. A. (2019). Evidence and argument in a proof based teaching theory. *ZDM*, *51*(5), 807–823.

Robin, B. R. (2008). Digital storytelling: A powerful technology tool for the 21st century classroom. *Theory into practice*, *47*(3), 220–228.

Scheiffele, E. (2008). *The theatre of truth: Psychodrama, spontaneity and improvisation: The theatrical theories and influences of Jacob Moreno.* [Doctoral Dissertation, University of California, Berkeley, 1995].

Schneuwly, B., Leopoldoff, M., Silva, I., & Nunes Henrique Silva, D. (2022). *L.S.Vygotskij, Imagination: Textes choisis, avec des commentaires et des essais sur l'imagination dans l'oeuvre de Vygotskij* (B. Schneuwly, M. Leopoldoff, I. Silva, & D. Nunes Henrique Silva, Eds.). Peter Lang.

Schoenfeld, A. H. (1985). *Mathematical problem solving.* Academic Press, Inc.

Schoenfeld, A. H. (2016). Learning to think mathematically: Problem solving, metacognition, and sense making in mathematics (reprint). *Journal of Education*, *196*(2), 1–38.

Sfard, A. (2001). There is more to discourse than meets the ears: Looking at thinking as communicating to learn more about mathematical learning. *Educational Studies in Mathematics*, *46*(1), 13–57.

Sofia, G. (2013). Lo spettatore in gioco sottotesto, sottopartitura e meccanismo specchio. In M. Bortignon, K. Darici, & S. Imperiale (Eds.). *Il lettore in gioco. Finestre sul mondo della lettura, Vol 1* (pp. 69–74). Università Ca' Foscari.

Stanislavskij, K. S. (1988). *Il lavoro dell'attore sul personaggio* (F. Malcovati & A. Morpurgo, Eds.). Laterza Rome.

Stylianides, A. J., & Stylianides, G. J. (2022). Introducing students and prospective teachers to the notion of proof in mathematics. *The Journal of Mathematical Behavior*, *66*, 100957.

Stylianides, A. L. (2007). Proof and proving in school mathematics. *Journal for Research in Mathematics Education*, *38*(3), 289–321.

Thurston, W. P. (1995). On proof and progress in mathematics. *For the Learning of Mathematics*, *15*(1), 29–37.

Turiano, F., & Boero, P. (2019). Beyond direct proof in the approach to the Culture of Theorems: a case study on 10-th grade students' difficulties and potential. In U. T. Jankvist, M. van den Heuvel-Panhuizen, & M. Veldhuis (Eds.). *Proceedings of the Eleventh Congress of the European Society for Research in Mathematics Education (CERME 11)* (pp. 338–345). Centre pour la Communication Scientifique Directe-HAL-Inria.

Umiltà, M. A., Escola, L., Intskirveli, I., Grammont, F., Rochat, M., Caruana, F., Jezzini, A., Gallese, V., & Rizzolatti, G. (2008). When pliers become fingers in the monkey motor system. *Proceedings of the National Academy of Sciences*, *105*(6), 2209–2213.

Umiltà, M. A., Kohler, E., Gallese, V., Fogassi, L., Fadiga, L., Keysers, C., & Rizzolatti, G. (2001). I know what you are doing: A neurophysiological study. *Neuron*, *31*(1), 155–165.

Urhan, S., & Bülbül, A. (2023a). Analysis of mathematical proving in geometry based on habermas' construct of rationality. *Mathematics Education Research Journal*, *35*(4), 929–959.

Urhan, S., & Bülbül, A. (2023b). Habermas' construct of rationality in the analysis of the mathematical problem-solving process. *Educational Studies in Mathematics*, *112*(1), 175–197.

Urhan, S., & Zengin, Y. (2024). Investigating university students' argumentations and proofs using dynamic mathematics software in collaborative learning, debate, and

self-reflection stages. *International Journal of Research in Undergraduate Mathematics Education*, *10*(2), 380–407.

Urhan, S., & Zengin, Y. (2025). Students' proof construction through computer-supported collaborative learning: The perspectives of habermas' theory of rationality and duval's theory of registers of semiotic representation. *Science & Education*, *34*(5), 3689–3725.

Vergnaud, G. (2009). The theory of conceptual fields. *Human development*, *52*(2), 83–94.

Vygotsky, L. S. (1978). *Mind in society: The development of higher psychological processes* (M. Cole, V. John-Steiner, S. Scribner & E. Souberman, Eds.) (Vol. 86). Harvard University Press.

Vygotsky, L. S. (1934/2008). *Pensiero e Linguaggio.* (L. Mecacci Eds./Trad.) Editori Laterza (Original work published in 1934).

Weber, K., & Alcock, L. (2004). Semantic and syntactic proof productions. *Educational Studies in Mathematics*, *56*(2), 209–234.

Weinberger, A., Kollar, I., Dimitriadis, Y., Mäkitalo-Siegl, K., & Fischer, F. (2009). Computer-supported collaboration scripts: Perspectives from educational psychology and computer science. In N. Balacheff, S. Ludvigsen, T. de Jong, A. Lazonder, & S. Barnes (Eds.)*Technology-Enhanced Learning: Principles and products* (pp. 155–173). Springer.

Wittmann, E. C. (2020). When is a proof a proof? In E. C. Wittmann (Ed.) *Connecting Mathematics and Mathematics Education: Collected Papers on Mathematics Education as a Design Science* (pp. 61–76). Springer.

Yin, R. K. (2009). *Case Study Research: Design and Methods* (Vol. 5). SAGE Publications.

Zan, R. (2011). The crucial role of narrative thought in understanding story problems. In K. Kislenko (Ed.)*Current state of research on mathematical beliefs XVI. Proceedings of the MAVI-16 Conference* (pp. 331–348). Tallinn University.

Zan, R. (2012). La dimensione narrativa di un problema: il modello C&D per l'analisi e la (ri) formulazione del testo. *L'insegnamento della matematica e delle scienze integrate*, *35*(2), 107–126.

Zaslavsky, O., Nickerson, S. D., Stylianides, A. J., Kidron, I., & Winicki-Landman, G. (2021). The need for proof and proving: Mathematical and pedagogical perspectives. In G. Hanna and M. de Villiers (Eds.)*, Proof and Proving in Mathematics Education, New ICMI Study Series, Vol. 15* (pp. 215–230). Springer.

Zazkis, R., & Liljedahl, P. (2019). *Teaching mathematics as storytelling*. Brill.

Zazkis, R., & Liljedahl, P. (2024)(A. Donatiello, trad.). Insegnare matematica come narrazione. *Didattica della matematica. Dalla ricerca alle pratiche d'aula*, *16*, 61–92.

Zhuang, Y. (2023). An Application of Habermas Construct of Rationality to Support Students' Proof Validation Skills. In T. Lamberg, & D. Moss (2023). *Proceedings of the forty-fifth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education (Vol. 1)*. University of Nevada.

Zhuang, Y., & Conner, A. (2022). Teachers' use of rational questioning strategies to promote student participation in collective argumentation. *Educational Studies in Mathematics*, *111*(2), 345–365.

Appendix

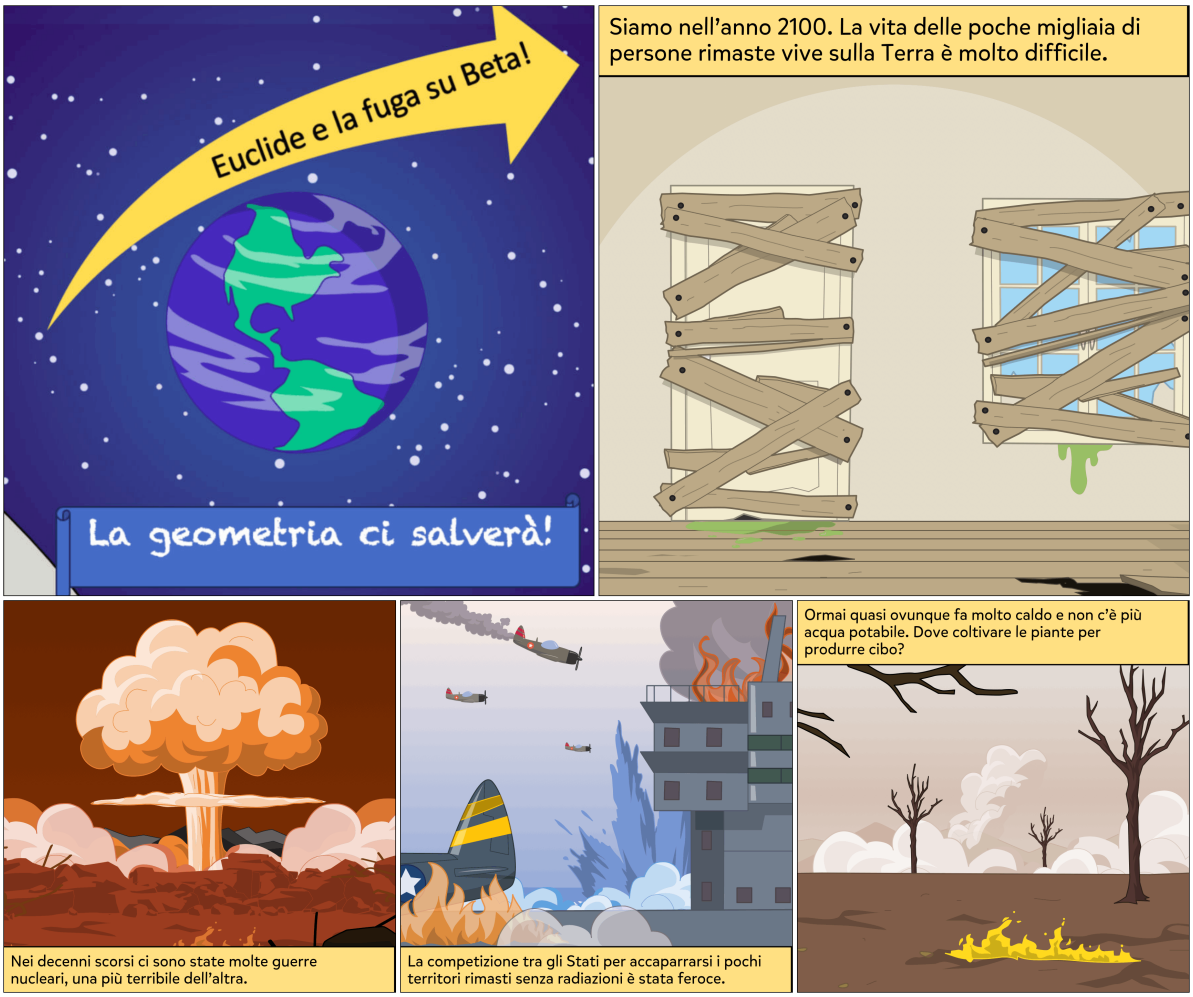


Figure 13.2: Comics Prelude 1



Figure 13.3: Comics Prelude 2

Appendix



Figure 13.4: Comics Prelude 3



Figure 13.5: Comics Episode 1.1



Figure 13.6: Comics Episode 1.2

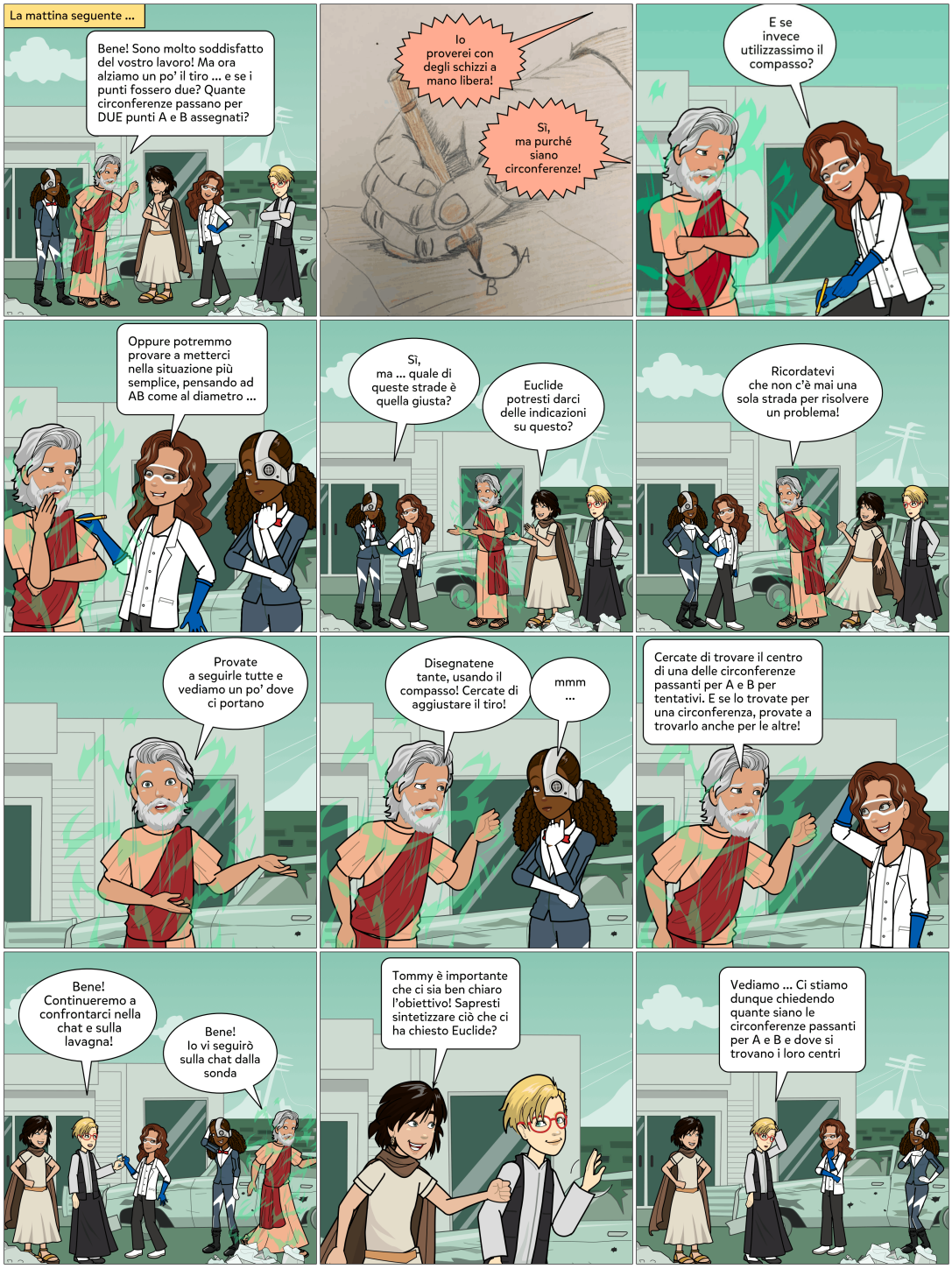


Figure 13.7: Comics Episode 1.3



Figure 13.8: Comics Episode 1.4

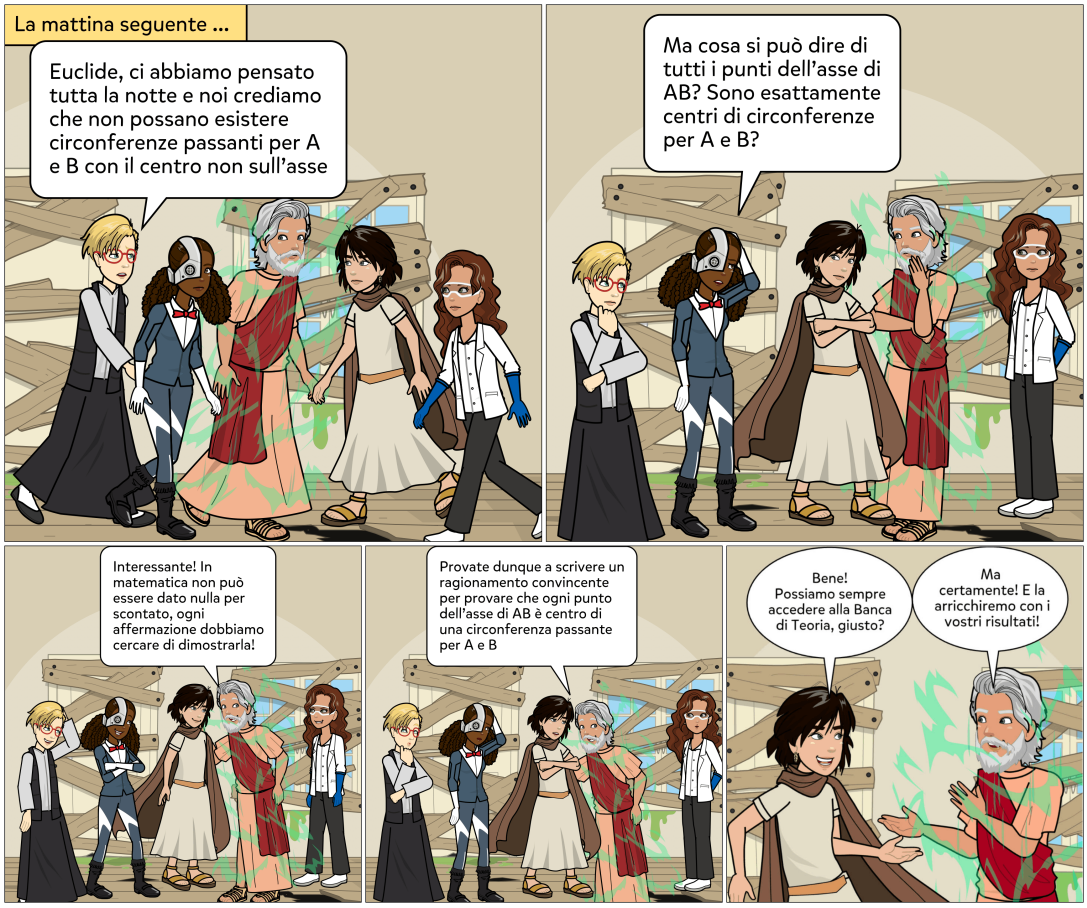


Figure 13.9: Comics Episode 1.5

Appendix

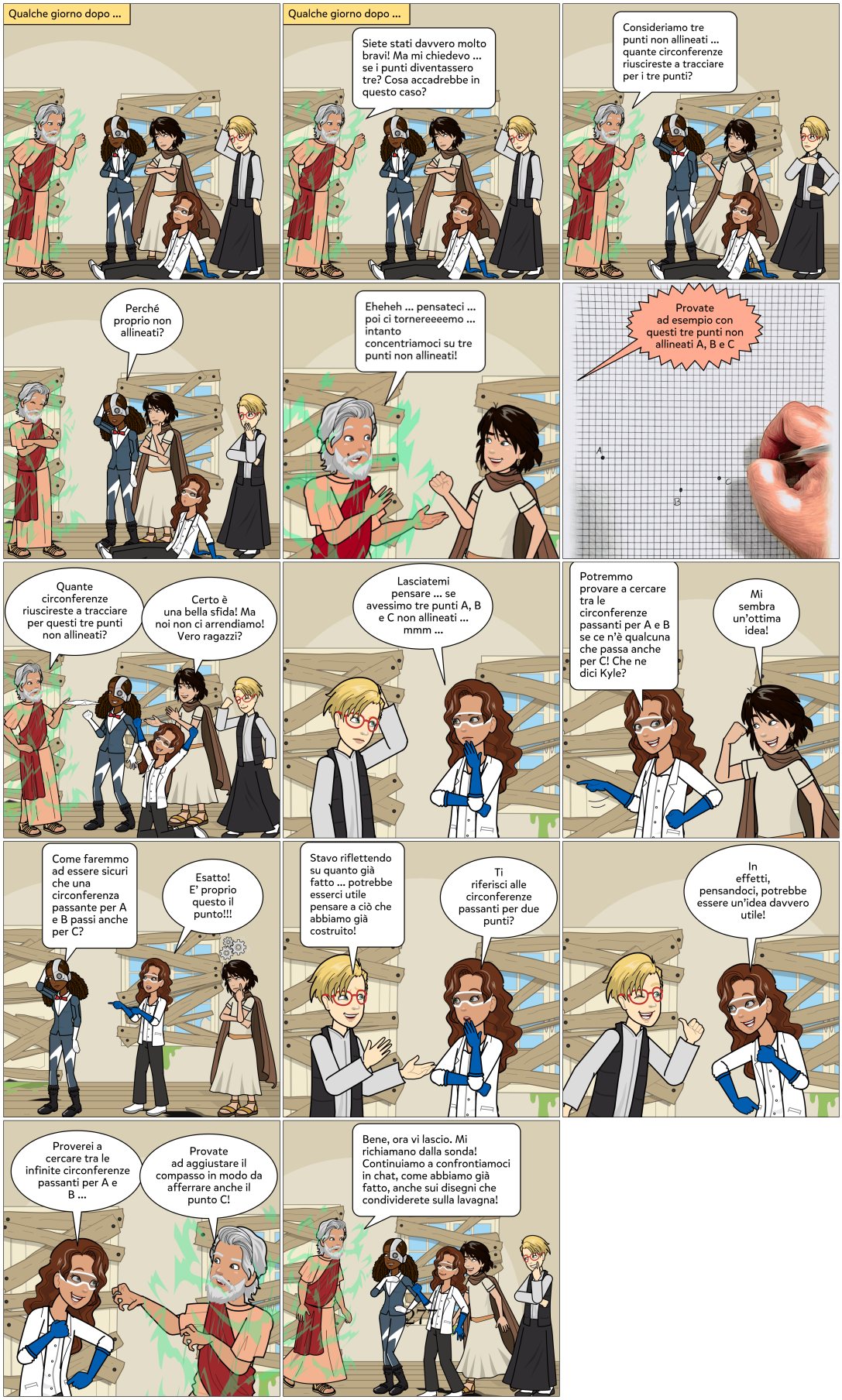


Figure 13.10: Comics Episode 2



Figure 13.11: Comics Episode 2

Appendix

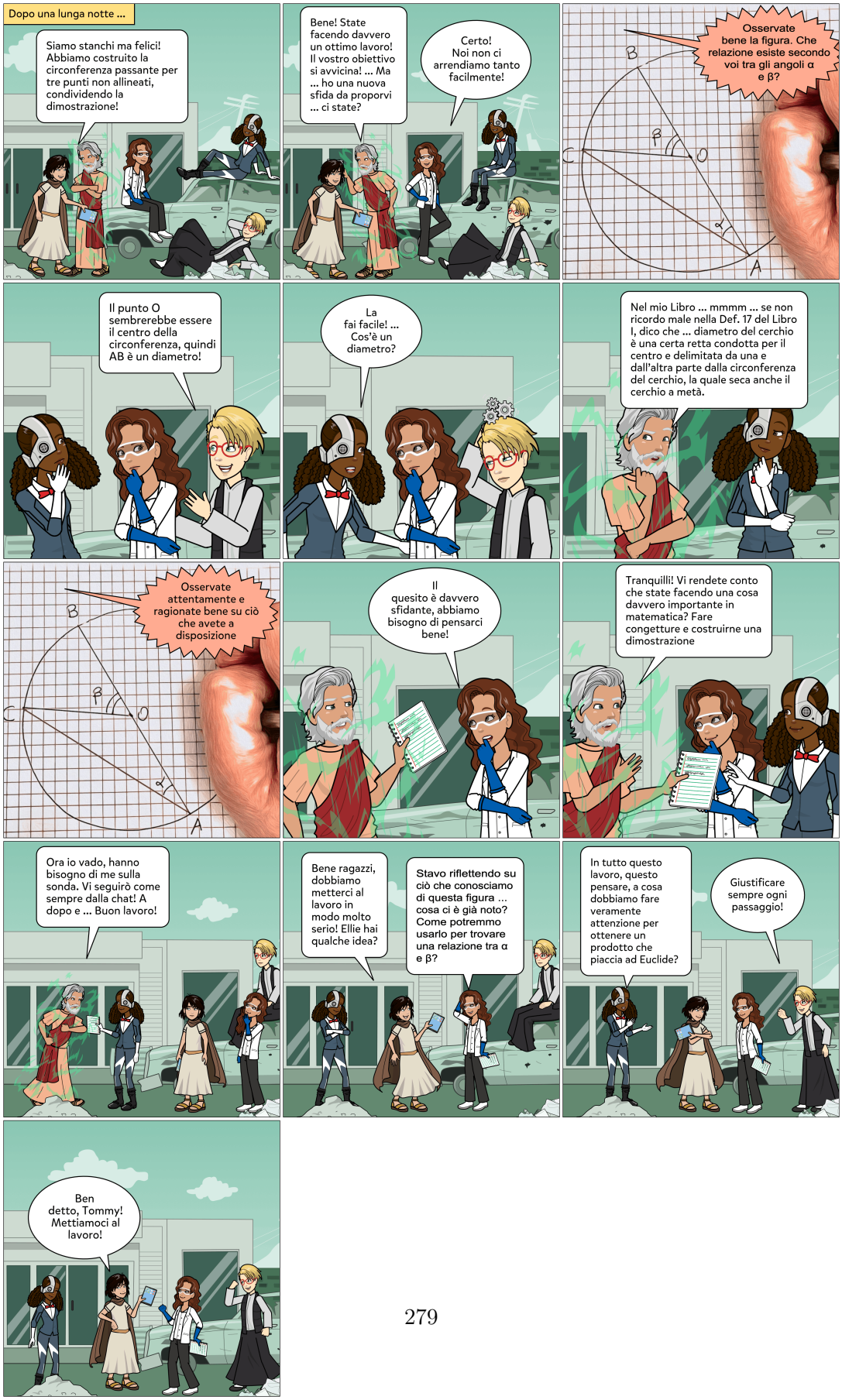


Figure 13.12: Comics Episode 3



Figure 13.13: Comics Episode 3