On a class of birth-death processes with time-varying intensity functions

Virginia Giorno[a], Amelia G. Nobile[a]

[a] *Dipartimento di Informatica, Università degli Studi di Salerno, Via Giovanni Paolo II n. 132, 84084 Fisciano (SA), Italy*
{giorno, nobile}@unisa.it

Abstract

In this paper, we investigate on a class of time-inhomogeneous birth-death chains obtained by applying the composition method to two time-inhomogeneous double-ended chains. Then, we consider the corresponding restricted birth-death process, with zero reflecting boundary. Finally, starting from the restricted process, we construct a time-inhomogeneous BD chain symmetric with respect to zero-state. We obtain closed form expressions for the transition probabilities and for the conditional moments; furthermore, the first-passage-time problem is also taken in consideration. Finally, various numerical computations are performed for periodic intensity functions.

Key words: Inhomogeneous birth-death chain, Transient distributions, First-passage time densities, Periodic intensity functions
2010 MSC: 60J80, 60J28, 65C50

1. Introduction and background

Continuous-time birth-death (BD) chains have been extensively used in various fields, such as physics, chemistry, engineering, biology, genetics, ecology, population growth dynamics and queueing systems.

In some instances, unrestricted continuous-time BD chains are taken in account (cf. Conolly [1], Di Crescenzo and Martinucci [2], Hongler and Parthasarathy [3], Pruitt [4], Tarabia and El-Baz [5]). In particular, double-ended BD chains can be employed to model several real systems, as the taxi-passenger queue (cf., for instance, Dobbie [6], Sharma and Nair [7], Tarabia [8], Conolly et al. [9]). Moreover, BD chains on the nonnegative integers are considered to model the states of molecules in chemical and physical models (cf., Buonocore et al. [10], Conolly et al. [11]), to describe population growth

https://doi.org/10.1016/j.amc.2020.125255

(see, Crawford and Suchard [12], Ricciardi [13]), to analyze the evolution of queueing systems (cf., Giorno et al. [14, 15], Lenin et al. [16], Medhi [17], Sharma [18]).

BD chains are also used to model systems subject to various types of disasters (see, Dharmaraja et al. [20], Di Crescenzo et al. [21], Economou and Fakinos [22, 23], Giorno and Nobile [24], Giorno et al. [25], Kapodistria et al. [26]). Moreover, various diffusion processes have been derived as continuous approximations of BD chains (cf., Abundo [19], Dharmaraja et al. [20], Di Crescenzo et al. [21, 27, 28]).

The determination of first-passage time (FPT) probability density function (pdf) and of its moments is of interest in applications. Unfortunately, closed form solutions for the transition probabilities and for the FPT densities are available only in special cases (cf. Di Crescenzo et al. [29], Di Crescenzo and Martinucci [30], Giorno and Nobile [31], Lenin and Parthasarathy [32], Poskroblo and Girejko [33]).

Time-inhomogeneous unrestricted and restricted BD Markov processes are frequently used to model a large number of real queueing systems and populations growth dynamics (cf., for instance, Giorno et al. [25, 34], Giveen [35], Di Crescenzo and Nobile [36], Zheng [37], Zeifman et al. [38]). Most of research on time-inhomogeneous queues and populations are devoted to the computation of the distribution of the state probabilities and of the limiting distribution, to the develop of various methods of approximation of their transient behavior and to the determination of first-passage time densities via analytical and numerical methods (cf. Crawford et al. [39]). Moreover, some studies on the ergodicity of time-inhomogeneous BD chains are considered in Scott and Isaacson [40] and in Zeifman and Isaacson [41].

In this paper, we construct a special class of time-inhomogeneous BD chains by using the composition method applied to two time-inhomogeneous double-ended chains. Then, we consider the corresponding restricted time-inhomogeneous BD process, with zero reflecting boundary. Finally, starting from the restricted process, we construct a time-inhomogeneous BD chain symmetric with respect to zero. For the considered processes, we obtain closed form expressions for the transition probabilities and for the conditional moments; furthermore, the first-passage-time problem is taken in consideration. The proposed approach allows to obtain stochastic models suitable to analyze unrestricted chemical queues, in which the motion of a molecule is modeled by means of a BD chain with time-varying intensity functions. Furthermore, the considered restricted BD processes are appropriate to describe unbounded populations subject to rapid growth, such as bacterial populations or cancer masses. Extensive numerical computations with MATHEMATICA are performed, by devoting special attention to the case of time-varying intensity functions. These cases are of interest in various applied fields, such as in queueing systems and population growth. For instance, in queueing models the presence of periodicity in the BD intensity functions expresses the existence of rush hours occurring on a daily basis (see, Di Crescenzo et al. [27, 36], Dong and Whitt [42], Giorno et al. [25, 34], Satin et al. [43], Whitt [44]), whereas in population dynamics it expresses the existence of predictable fluctuation in the growth due to seasonal or other regular environmental cycles (see, Thieme [45] and references therein).

1.1. Plan of the paper

In Section 2, we consider two time-inhomogeneous double-ended systems $\{X_1(t), t \geq t_0\}$ and $\{X_2(t), t \geq t_0\}$, whose state-space is $\mathbb{Z} = \{\ldots, -1, 0, 1, \ldots\}$, conditioned to start from j at time t_0, with time-dependent BD intensities $\lambda(t)$ and $\mu(t)$ for $X_1(t)$ and with BD intensities $\mu(t)$ and $\lambda(t)$ for $X_2(t)$. Making use of the composition method applied to $X_1(t)$ and $X_2(t)$, we construct a continuous-time BD chain $\{\widetilde{X}(t), t \geq t_0\}$, whose probabilities $\xi_{j,n}(t|t_0)$, $j, n \in \mathbb{Z}$, are a linear combination of the transition probabilities of $X_1(t)$ and $X_2(t)$, with weights depending on the initial condition (j, t_0) and on a regulation real parameter $\vartheta \in [0, 1]$.

In Section 3, we consider a special time-inhomogeneous BD chain $X(t)$ with birth and death intensity functions $\lambda_n(t)$ and $\mu_n(t)$, whose transition probabilities $f_{j,n}(t|t_0)$ identify with $\xi_{j,n}(t|t_0)$ for $j, n \in \mathbb{Z}$ and $t \geq t_0$. We distinguish two different cases: *(a)* $(j, t_0) = (0, 0)$ and *(b)* $\lambda(t)$ and $\mu(t)$ are proportional. Closed form expressions for the transition probabilities and for the FPT densities are obtained. Recently, a similar approach has been applied in Giorno and Nobile [46] to construct a class of time-inhomogeneous diffusion processes starting from some conditioned Gauss-Markov processes.

In Section 4, assuming that $\lambda(t)$ and $\mu(t)$ are proportional, we take into account the time-inhomogeneous restricted BD process $\{R(t), t \geq t_0\}$, with state-space $\mathbb{N}_0 = \{0, 1, \ldots\}$, 0 being a regular reflecting state. For $R(t)$, we determine the transition probabilities $p_{j,n}(t|t_0)$ for $j, n \in \mathbb{N}_0$ and $t \geq t_0$.

In Section 5, starting from $R(t)$ and assuming that $\lambda(t)$ and $\mu(t)$ are proportional, we consider the time-inhomogeneous BD chain $\{Y(t), t \geq t_0\}$ symmetric with respect to zero, having state-space $\mathbb{Z}$. We determine the transition probabilities $h_{j,n}(t|t_0)$ for $j, n \in \mathbb{Z}$ and $t \geq t_0$.

In the case of time-homogeneous BD processes, unrestricted or restricted, the methods based on the Laplace transforms (LT) and on their inversions are often used to determine the transition probabilities and the FPT densities. In the sequel of the paper, we will denote by $\widehat{\chi}(s)=\mathcal{L}_s(\chi(t))$ $(s\geq 0)$ the LT of the function $\chi(t)$.

2. Time-inhomogeneous double-ended chains: composition method

Let $\{X_i(t), t\geq t_0\}$ $(i=1,2)$ be a time-inhomogeneous double-ended BD chain having state-space $\mathbb{Z}$, conditioned to start from $j\in\mathbb{Z}$ at time t_0. We assume that births and deaths occur with intensity functions $\lambda_{i,n}(t)$ and $\mu_{i,n}(t)$ at time t, respectively. For the process $X_1(t)$ we set: $\lambda_{1,n}(t)=\lambda(t)$, $\mu_{1,n}(t)=\mu(t)$, $n\in\mathbb{Z}$, whereas for $X_2(t)$ we have: $\lambda_{2,n}(t)=\mu(t)$, $\mu_{2,n}(t)=\lambda(t)$, $n\in\mathbb{Z}$, with $\lambda(t)$ and $\mu(t)$ positive, bounded and continuous functions for $t\geq t_0$, such that $\int_{t_0}^{+\infty}\lambda(t)\,dt=+\infty$ and $\int_{t_0}^{+\infty}\mu(t)\,dt=+\infty$. Double-ended BD chains arise in many applications, such as taxi-passenger systems, financial markets, buyers and sellers in a financial market, assembly systems, perishable inventory systems and organ transplant systems (cf., for instance, Dobbie [6], Conolly et al. [9], Di Crescenzo et al. [27], Giveen [35]).

Let

$$f_{j,n}^{(i)}(t|t_0)=P\{X_i(t)=n|X_i(t_0)=j\},\qquad j,n\in\mathbb{Z}, t\geq t_0, i=1,2$$

be the transition probabilities of $X_i(t)$. For $i=1,2$, they satisfy the Kolmogorov forward equations and the related initial condition:

$$\begin{aligned}\frac{df_{j,n}^{(i)}(t|t_0)}{dt}&=\lambda_{i,n-1}(t)\,f_{j,n-1}^{(i)}(t|t_0)-\big[\lambda_{i,n}(t)+\mu_{i,n}(t)\big]\,f_{j,n}^{(i)}(t|t_0)\\&\quad+\mu_{i,n+1}(t)\,f_{j,n+1}^{(i)}(t|t_0),\quad j,n\in\mathbb{Z},\\ \lim_{t\downarrow t_0}f_{j,n}^{(i)}(t|t_0)&=\delta_{j,n},\end{aligned}\tag{1}$$

where $\delta_{j,n}$ is the Kronecker delta function. We denote by

$$\Lambda(t|t_0)=\int_{t_0}^{t}\lambda(\tau)\,d\tau,\qquad M(t|t_0)=\int_{t_0}^{t}\mu(\tau)\,d\tau,\quad t\geq t_0,\tag{2}$$

the cumulative BD intensity functions. For $t\geq t_0$, $j,n\in\mathbb{Z}$ and $i=1,2$ we have (cf. Di Crescenzo et al. [27], Giveen [35]):

$$f_{j,n}^{(i)}(t|t_0)=e^{-[\Lambda(t|t_0)+M(t|t_0)]}\left[\frac{\Lambda(t|t_0)}{M(t|t_0)}\right]^{(-1)^{3-i}(n-j)/2}I_{n-j}\Big[2\sqrt{\Lambda(t|t_0)\,M(t|t_0)}\Big],\tag{3}$$

where

$$I_\nu(z) = \sum_{m=0}^{+\infty} \frac{(z/2)^{\nu+2m}}{m!\,\Gamma(\nu+m+1)}, \qquad \nu \in \mathbb{R} \tag{4}$$

is the modified Bessel function of first kind. From (3), the following relation holds:

$$f_{j,n}^{(2)}(t|t_0) = \Big[\frac{M(t|t_0)}{\Lambda(t|t_0)}\Big]^{n-j} f_{j,n}^{(1)}(t|t_0), \qquad j,n \in \mathbb{Z}. \tag{5}$$

Furthermore, since (see, Abramowitz and Stegun [47], p. 376, n. 9.6.33)

$$\exp\Big\{\frac{s}{2}\Big(x+\frac{1}{x}\Big)\Big\} = \sum_{n=-\infty}^{+\infty} x^n\, I_n(s), \qquad x \neq 0, \tag{6}$$

by virtue of (3), we obtain the conditional mean and variance of $X_i(t)$:

$$\begin{aligned} &\mathrm{E}[X_i(t)|X_i(t_0)=j] = j + (-1)^{3-i}\big[\Lambda(t|t_0) - M(t|t_0)\big], \\ &\mathrm{Var}[X_i(t)|X_i(t_0)=j] = \Lambda(t|t_0) + M(t|t_0), \end{aligned} \tag{7}$$

for $t \geq t_0$ and $j \in \mathbb{Z}$. For $i=1,2$, let

$$\mathcal{T}_{j,n}^{(i)}(t_0) = \inf\{t \geq t_0 : X_i(t) = n\}, \qquad X_i(t_0) = j,\ j \neq n$$

be the FPT of $X_i(t)$ through $n \in \mathbb{Z}$ when starting from $j \in \mathbb{Z}$ and denote by $g_{j,n}^{(i)}(t|t_0) = dP\{\mathcal{T}_{j,n}^{(i)}(t_0) \leq t\}/dt$ its pdf. For $i=1,2$ the following integral equation holds:

$$f_{j,n}^{(i)}(t|t_0) = \int_{t_0}^{t} g_{j,k}^{(i)}(u|t_0)\, f_{k,n}^{(i)}(t|u)\, du \quad (j<k\leq n) \text{ or } (n \leq k < j). \tag{8}$$

2.1. Proportional BD intensity functions

We assume that $\lambda(t) = \lambda\,\varphi(t)$ and $\mu(t) = \mu\,\varphi(t)$, with $\lambda > 0$, $\mu > 0$ and $\varphi(t)$ positive, bounded and continuous function of $t \geq t_0$, such that $\int_{t_0}^{\infty} \varphi(t)\,dt = +\infty$. Denoting by $\varrho = \lambda/\mu$ and

$$\Phi(t|t_0) = \int_{t_0}^{t} \varphi(\tau)\, d\tau, \qquad t \geq t_0, \tag{9}$$

the transition probabilities (3) and the FPT pdf of $X_i(t)$ can be expressed as (cf. Di Crescenzo et al. [27]):

$$\begin{aligned} &f_{j,n}^{(i)}(t|t_0) = e^{-(\lambda+\mu)\Phi(t|t_0)}\, \varrho^{(-1)^{3-i}(n-j)/2}\, I_{n-j}\big[2\sqrt{\lambda\,\mu}\,\Phi(t|t_0)\big], \quad j,n \in \mathbb{Z}, \\ &g_{j,n}^{(i)}(t|t_0) = \frac{|n-j|\;\varphi(t)}{\Phi(t|t_0)}\, f_{j,n}^{(i)}(t|t_0), \qquad j,n \in \mathbb{Z},\ j \neq n. \end{aligned} \tag{10}$$

Moreover, by virtue of (5), from (10), one has:

$$g_{j,n}^{(2)}(t|t_0) = \frac{|n-j|\ \varphi(t)}{\Phi(t|t_0)}\ f_{j,n}^{(2)}(t|t_0) = \varrho^{-(n-j)}\ g_{j,n}^{(1)}(t|t_0), \quad j,n \in \mathbb{Z}, j \neq n. \tag{11}$$

2.2. Composition method

Let $0 \leq \vartheta \leq 1$ be a real number. For any fixed $t \geq t_0$, we assume that $U_{j,t_0}(t)$ is a random variable uniform in $(0,1)$, independent of $X_1(t)$ and $X_2(t)$. We consider the stochastic process $\{\widetilde{X}(t), t \geq t_0\}$ defined as

$$\widetilde{X}(t) = \begin{cases} X_1(t), & 0 \leq U_{j,t_0}(t) < Q_{j,t_0}(\vartheta) \\ X_2(t), & Q_{j,t_0}(\vartheta) \leq U_{j,t_0}(t) < 1, \end{cases} \tag{12}$$

where

$$Q_{j,t_0}(\vartheta) = \frac{\vartheta}{\vartheta + (1-\vartheta)\left[\frac{M(t_0|0)}{\Lambda(t_0|0)}\right]^j}, \qquad j \in \mathbb{Z}, t_0 \geq 0. \tag{13}$$

We note that ϑ is a regulation parameter that controls the relative weighting of the two components $X_1(t)$ and $X_2(t)$ in the definition of $\widetilde{X}(t)$. Then, denoting by

$$\xi_{j,n}(t|t_0) = P\{\widetilde{X}(t) = n | \widetilde{X}(t_0) = j\}, \qquad j,n \in \mathbb{Z}, t \geq t_0,$$

we have:

$$\begin{aligned} \xi_{j,n}(t|t_0) &= Q_{j,t_0}(\vartheta)\ f_{j,n}^{(1)}(t|t_0) + \left[1 - Q_{j,t_0}(\vartheta)\right]\ f_{j,n}^{(2)}(t|t_0) \\ &= \left\{Q_{j,t_0}(\vartheta)\ + \left[1 - Q_{j,t_0}(\vartheta)\right] \left[\frac{M(t|t_0)}{\Lambda(t|t_0)}\right]^{n-j}\right\} f_{j,n}^{(1)}(t|t_0), \end{aligned} \tag{14}$$

where relation (5) has been used. Note that the probability $\xi_{j,n}(t|t_0)$ is a mixture of the probabilities $f_{j,n}^{(1)}(t|t_0)$ and $f_{j,n}^{(2)}(t|t_0)$, given in (3). Then, the regulation parameter ϑ allows the flexibility in the shape of the probability $\xi_{j,n}(t|t_0)$. By virtue of (7), for $j \in \mathbb{Z}$ and $t \geq t_0$ from (14) one has:

$$\begin{aligned} &\mathrm{E}[\widetilde{X}(t)|\widetilde{X}(t_0) = j] = j + [\Lambda(t|t_0) - M(t|t_0)]\left[2\,Q_{j,t_0}(\vartheta) - 1\right], \\ &\mathrm{Var}[\widetilde{X}(t)|\widetilde{X}(t_0) = j] = \Lambda(t|t_0) + M(t|t_0) \\ &\qquad\qquad + 4\,Q_{j,t_0}(\vartheta)\left[1 - Q_{j,t_0}(\vartheta)\right][\Lambda(t|t_0) - M(t|t_0)]^2. \end{aligned} \tag{15}$$

Furthermore, we note that the infinitesimal rates of $\widetilde{X}(t)$ are:

$$\begin{aligned}
&\lim_{h\downarrow 0}\frac{\xi_{j,j+1}(t_0+h|t_0)}{h}=\lambda(t_0)\,\frac{\vartheta+(1-\vartheta)\frac{\mu(t_0)}{\lambda(t_0)}\left[\frac{M(t_0|0)}{\Lambda(t_0|0)}\right]^j}{\vartheta+(1-\vartheta)\left[\frac{M(t_0|0)}{\Lambda(t_0|0)}\right]^j}, \qquad j\in\mathbb{Z},\\
&\lim_{h\downarrow 0}\frac{\xi_{j,j-1}(t_0+h|t_0)}{h}=\mu(t_0)\,\frac{\vartheta+(1-\vartheta)\frac{\lambda(t_0)}{\mu(t_0)}\left[\frac{M(t_0|0)}{\Lambda(t_0|0)}\right]^j}{\vartheta+(1-\vartheta)\left[\frac{M(t_0|0)}{\Lambda(t_0|0)}\right]^j}, \qquad j\in\mathbb{Z}.
\end{aligned} \tag{16}$$

3. A special class of time-inhomogeneous BD chains

The object of this section is to determine the conditions so that (14) can be interpreted as the transition probability of a time-inhomogeneous BD chain $\{X(t), t\geq t_0\}$, with state-space $\mathbb{Z}$, conditioned to start from j at time t_0, characterized by BD intensity functions

$$\begin{aligned}
\lambda_n(t)&=\lambda(t)\,\frac{\vartheta+(1-\vartheta)\frac{\mu(t)}{\lambda(t)}\left[\frac{M(t|0)}{\Lambda(t|0)}\right]^n}{\vartheta+(1-\vartheta)\left[\frac{M(t|0)}{\Lambda(t|0)}\right]^n}, \qquad n\in\mathbb{Z},\\
\mu_n(t)&=\mu(t)\,\frac{\vartheta+(1-\vartheta)\frac{\lambda(t)}{\mu(t)}\left[\frac{M(t|0)}{\Lambda(t|0)}\right]^n}{\vartheta+(1-\vartheta)\left[\frac{M(t|0)}{\Lambda(t|0)}\right]^n}, \qquad n\in\mathbb{Z},
\end{aligned} \tag{17}$$

respectively, where $0\leq\vartheta\leq 1$. We assume that $\lambda(t),\mu(t)$ are positive bounded and continuous functions for $t\geq t_0$, such that $\int_{t_0}^{+\infty}\lambda(t)\,dt=+\infty$ and $\int_{t_0}^{+\infty}\mu(t)\,dt=+\infty$. By virtue of (16) and (17), for fixed (j,t_0) we note that

$$\lambda_j(t_0)=\lim_{h\downarrow 0}\frac{\xi_{j,j+1}(t_0+h|t_0)}{h}, \qquad \mu_j(t_0)=\lim_{h\downarrow 0}\frac{\xi_{j,j-1}(t_0+h|t_0)}{h}.$$

For $t\geq t_0$, from (17) we have: when $\vartheta=0$, then $\lambda_n(t)=\mu(t)$ and $\mu_n(t)=\lambda(t)$; when $\vartheta=1$, then $\lambda_n(t)=\lambda(t)$ and $\mu_n(t)=\mu(t)$; when $0<\vartheta<1$, then $\min\{\lambda(t),\mu(t)\}\leq\lambda_n(t)\leq\max\{\lambda(t),\mu(t)\}$ and $\min\{\lambda(t),\mu(t)\}\leq\mu_n(t)\leq\max\{\lambda(t),\mu(t)\}$. These considerations show that the BD intensity functions (17) are bounded by the intensity functions of $X_1(t)$ and $X_2(t)$. Furthermore, from (17) it follows:

$$\lambda_n(t)+\mu_n(t)=\lambda(t)+\mu(t), \qquad n\in\mathbb{Z}, t\geq t_0. \tag{18}$$

The transition probabilities of $X(t)$

$$f_{j,n}(t|t_0) = P\{X(t) = n|X(t_0) = j\}, \qquad j, n \in \mathbb{Z}, t \geq t_0$$

are solution of the Kolmogorov equations and the related initial condition

$$\begin{aligned} \frac{df_{j,n}(t|t_0)}{dt} &= \lambda_{n-1}(t)\, f_{j,n-1}(t|t_0) - \big[\lambda_n(t) + \mu_n(t)\big]\, f_{j,n}(t|t_0) \\ &\qquad +\mu_{n+1}(t)\, f_{j,n+1}(t|t_0), \quad j, n \in \mathbb{Z}, \\ \lim_{t\downarrow t_0} f_{j,n}(t|t_0) &= \delta_{j,n}, \end{aligned} \tag{19}$$

with $\lambda_n(t)$ and $\mu_n(t)$ defined in (17). We now consider two different cases: *(a)* $(j, t_0) = (0, 0)$, *(b)* $\lambda(t)$ and $\mu(t)$ are proportional.

Case (a)

Let $(j, t_0) = (0, 0)$. We consider the time-inhomogeneous BD chain $\{X(t), t \geq 0\}$, with BD intensity functions (17). We note that when $(j, t_0) = (0, 0)$, from (13) one has $Q_{j,t_0}(\vartheta) = Q_{0,0}(\vartheta) = \vartheta$, so that the transition probabilities (14) of the process $\widetilde{X}(t)$ are:

$$\begin{aligned} \xi_{0,n}(t|0) &= \vartheta\, f^{(1)}_{0,n}(t|0) + (1-\vartheta)\, f^{(2)}_{0,n}(t|0) \\ &= \Big\{\vartheta + (1-\vartheta)\Big[\frac{M(t|0)}{\Lambda(t|0)}\Big]^n\Big\}\, f^{(1)}_{0,n}(t|0), \qquad n \in \mathbb{Z}. \end{aligned} \tag{20}$$

We now prove that $f_{0,n}(t|0)$ coincide with (20) when $(j, t_0) = (0, 0)$.

Proposition 1. *Let $(j, t_0) = (0, 0)$. For the BD chain $X(t)$, with BD intensity functions (17), one has:*

$$\begin{aligned} f_{0,n}(t|0) = \xi_{0,n}(t|0) &= \Big\{\vartheta + (1-\vartheta)\Big[\frac{M(t|0)}{\Lambda(t|0)}\Big]^n\Big\}\, e^{-[\Lambda(t|0)+M(t|0)]} \\ &\quad\times \Big[\frac{\Lambda(t|0)}{M(t|0)}\Big]^{n/2} I_n\Big[2\sqrt{\Lambda(t|0)\, M(t|0)}\Big], \qquad n \in \mathbb{Z}, t \geq 0, \end{aligned} \tag{21}$$

with $\xi_{0,n}(t|0)$ given in (20).

Proof. We verify that $\xi_{0,n}(t|0)$, given in (20), satisfies (19) for $(j, t_0) = (0, 0)$, with $\lambda_n(t)$ and $\mu_n(t)$ defined in (17). First, by virtue of the initial conditions in (1), from (20) we have that $\lim_{t\downarrow 0} \xi_{0,n}(t|0) = \delta_{0,n}$. Moreover, by setting

$$\begin{aligned} \Psi_{0,n}(t|0) = \frac{d\xi_{0,n}(t|0)}{dt} &+ \big[\lambda_n(t) + \mu_n(t)\big]\, \xi_{0,n}(t|0) - \lambda_{n-1}(t)\, \xi_{0,n-1}(t|0) \\ &-\mu_{n+1}(t)\, \xi_{0,n+1}(t|0), \qquad n \in \mathbb{Z}, t \geq 0, \end{aligned} \tag{22}$$

by virtue of (1) and (5), from (20) one has:

$$\begin{aligned}\frac{d\xi_{0,n}(t|0)}{dt} &= -\big[\lambda(t)+\mu(t)\big]\,\xi_{0,n}(t|0) \\ &\quad +\lambda(t)\Big\{\vartheta+(1-\vartheta)\,\frac{\mu(t)}{\lambda(t)}\left[\frac{M(t|0)}{\Lambda(t|0)}\right]^{n-1}\Big\}\,f^{(1)}_{0,n-1}(t|0) \\ &\quad +\mu(t)\Big\{\vartheta+(1-\vartheta)\,\frac{\lambda(t)}{\mu(t)}\left[\frac{M(t|0)}{\Lambda(t|0)}\right]^{n+1}\Big\}\,f^{(1)}_{0,n+1}(t|0), \qquad n\in\mathbb{Z}. \end{aligned} \tag{23}$$

Hence, making use of (17), (18), (20) and (23) in (22), one obtains $\Psi_{0,n}(t|0)=0$ for all $t\geq 0$ and $n\in\mathbb{Z}$. For the uniqueness of the solution of (19), one has $f_{0,n}(t|0)=\xi_{0,n}(t|0)$ for all $t\geq 0$ and $n\in\mathbb{Z}$. $\square$

Under the assumptions of Proposition 1, by virtue of (15), for $t\geq 0$ the conditional mean and variance of $X(t)$ are:

$$\begin{aligned}&\mathrm{E}[X(t)|X(0)=0]=(2\vartheta-1)\,\big[\Lambda(t|0)-M(t|0)\big],\\ &\mathrm{Var}[X(t)|X(0)=0]=\Lambda(t|0)+M(t|0)+4\vartheta\,(1-\vartheta)\big[\Lambda(t|0)-M(t|0)\big]^2.\end{aligned} \tag{24}$$

Remark 1. *Under the assumptions of Proposition 1, by replacing ϑ with $1-\vartheta$ in (17) and (21), one has:*

$$\begin{aligned}&\lambda_n(t)\Big|_{\vartheta}=\mu_{-n}(t)\Big|_{1-\vartheta}, \qquad \mu_n(t)\Big|_{\vartheta}=\lambda_{-n}(t)\Big|_{1-\vartheta}, \qquad n\in\mathbb{Z}, t\geq 0,\\ &f_{0,n}(t|0)\Big|_{\vartheta}=f_{0,-n}(t|0)\Big|_{1-\vartheta}, \qquad n\in\mathbb{Z}, t\geq 0.\end{aligned}$$

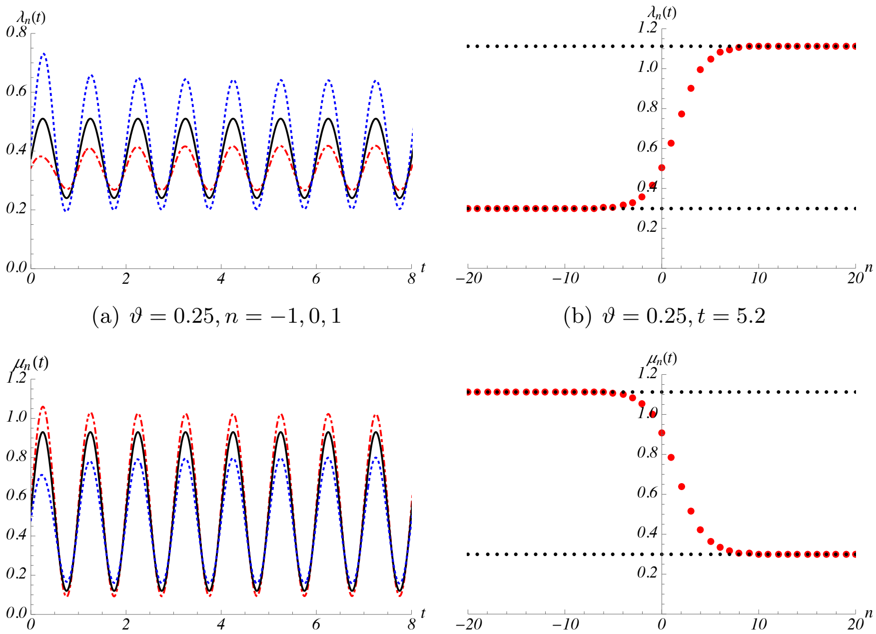
Moreover, by replacing ϑ with $1-\vartheta$ in (24), the conditioned means have opposite sign, whereas the conditional variances remain unchanged.

Example 1. We consider the BD chain $\{X(t), t\geq t_0\}$, with $t_0=0$, whose intensity functions $\lambda_n(t)$ and $\mu_n(t)$ are specified in (17), with

$$\lambda(t)=\lambda\left[1+a\,\sin\Big(\frac{2\pi\,t}{Q}\Big)\right], \qquad \mu(t)=\mu, \qquad t\geq 0, \tag{25}$$

where $\lambda>0$ is the average of the periodic function $\lambda(t)$ of period Q, a is the relative amplitude with $0\leq a<1$ and $\mu>0$. The associated cumulative BD intensity functions are:

$$\Lambda(t|0)=\lambda\Big[t+\frac{a\,Q}{\pi}\sin^2\Big(\frac{\pi t}{Q}\Big)\Big], \qquad M(t|0)=\mu\,t, \qquad t\geq 0.$$



(a) $\vartheta = 0.25, n = -1, 0, 1$

(b) $\vartheta = 0.25, t = 5.2$

(c) $\vartheta = 0.25, n = -1, 0, 1$

(d) $\vartheta = 0.25, t = 5.2$

Figure 1: For $\lambda = 0.6$, $\mu = 0.3$, $Q = 1$ and $a = 0.9$, the BD intensity functions (17), with $\lambda(t)$ and $\mu(t)$ given in (25), are plotted as function of t in (a) and (c) for $n = -1$ (red, dash-dotted curve), $n = 0$ (black, solid curve) and $n = 1$ (blue, dotted curve), whereas they are drawn as function of n for $t = 5.2$ in (b) and (d).

In Figure 1, for $\vartheta = 0.25$ the BD intensities $\lambda_n(t)$ and $\mu_n(t)$, specified in (17), with $\lambda(t)$ and $\mu(t)$ given in (25), are shown as function of t on the left by choosing $n = -1, 0, 1$ and as function of n on the right for $t = 5.2$; the dotted lines in (b) and (d) show the lower and upper bounds of the BD intensity functions (17) at $t = 5.2$. For fixed t, we note the S shape (sigmoidal behavior) of $\lambda_n(t)$ and the inverse sigmoidal behavior of $\mu_n(t)$. In Figure 2, for $\vartheta = 0.25$ the probabilities $f_{0,n}(t|0)$, given in (21), are represented as function of t (on the left) for $n = -1, 0, 1$ and as function of n (on the right) for $t = 5.2$. Finally, in Figure 3, the conditional mean $E[X(t)|X(0) = 0]$ and the conditional variance $\mathrm{Var}[X(t)|X(0) = 0]$, given in (24), are shown as function of t for some choices of ϑ. We note that the conditional mean oscillates around a line that decreases for $0 \leq \vartheta < 0.5$ and increases for $0.5 < \vartheta \leq 1$. $\diamond$

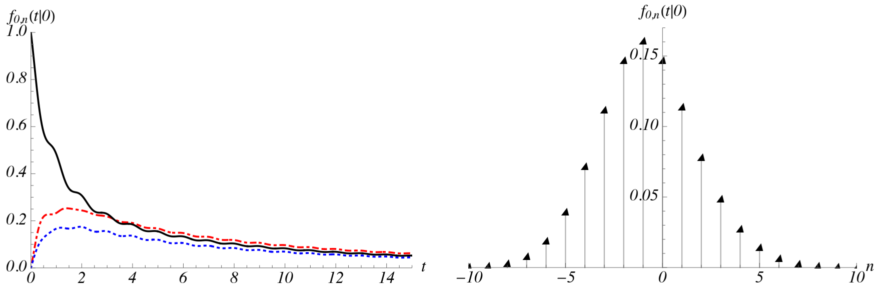


Figure 2: For $\lambda = 0.6$, $\mu = 0.3$, $Q = 1$, $a = 0.9$ and $\vartheta = 0.25$ in (25), the transition probabilities (21) are plotted as function of t for $n = -1$ (red, dash-dotted curve), $n = 0$ (black, solid curve) and $n = 1$ (blue, dotted curve) on the left and as function of n for $t = 5.2$ on the right.

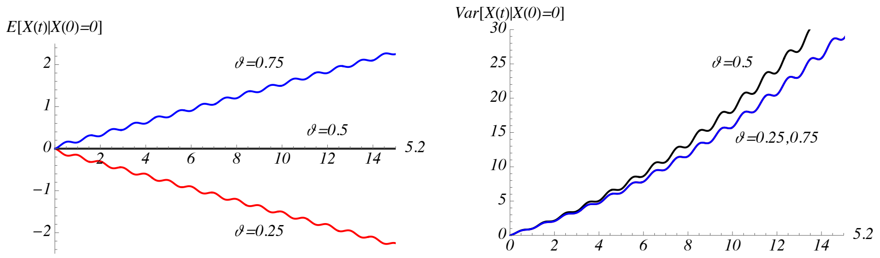


Figure 3: The conditional mean and the conditional variance (24) are plotted as function of t, with $\lambda = 0.6$, $\mu = 0.3$, $Q = 1$, $a = 0.9$ in (25) for some choices of ϑ.

Case (b)

We assume that $\lambda(t) = \lambda\,\varphi(t)$ and $\mu(t) = \mu\,\varphi(t)$. In this case, from (13) one has

$$Q_{j,t_0}(\vartheta) = \frac{\vartheta}{\vartheta + (1-\vartheta)\varrho^{-j}}, \qquad j \in \mathbb{Z}, t_0 \geq 0,$$

with $0 \leq \vartheta \leq 1$ and $\varrho = \lambda/\mu$. Furthermore, the probabilities (14) of the discrete process $\widetilde{X}(t)$ are:

$$\begin{aligned} \xi_{j,n}(t|t_0) &= Q_{j,t_0}(\vartheta)\; f^{(1)}_{j,n}(t|t_0) + \big[1 - Q_{j,t_0}(\vartheta)\big]\; f^{(2)}_{j,n}(t|t_0) \\ &= \frac{\vartheta + (1-\vartheta)\varrho^{-n}}{\vartheta + (1-\vartheta)\varrho^{-j}} f^{(1)}_{j,n}(t|t_0) \qquad j,n \in \mathbb{Z}, t \geq t_0, \end{aligned} \tag{26}$$

where the use of (5) has been made.

Recalling (17), we now take into account the time-inhomogeneous BD chain $\{X(t), t \geq t_0\}$, with BD intensity functions:

$$\begin{aligned} \lambda_n(t) &= \lambda\, \varphi(t)\, \frac{\vartheta + (1-\vartheta)\varrho^{-n-1}}{\vartheta + (1-\vartheta)\varrho^{-n}}, \qquad n \in \mathbb{Z}, \\ \mu_n(t) &= \mu\, \varphi(t)\, \frac{\vartheta + (1-\vartheta)\varrho^{-n+1}}{\vartheta + (1-\vartheta)\varrho^{-n}}, \qquad n \in \mathbb{Z}. \end{aligned} \tag{27}$$

In the following, we prove that the transition probabilities of $X(t)$, with BD intensity functions (27), coincide with (26).

Proposition 2. *For $X(t)$, with BD intensity functions (27), one has:*

$$\begin{aligned} f_{j,n}(t|t_0) = \xi_{j,n}(t|t_0) &= \frac{\vartheta + (1-\vartheta)\varrho^{-n}}{\vartheta + (1-\vartheta)\varrho^{-j}}\, e^{-(\lambda+\mu)\Phi(t|t_0)}\, \varrho^{(n-j)/2} \\ &\quad \times I_{n-j}\big[2\sqrt{\lambda\,\mu}\,\Phi(t|t_0)\big], \qquad j, n \in \mathbb{Z}, t \geq t_0, \end{aligned} \tag{28}$$

with $\Phi(t|t_0)$ given in (9). Hence, $X(t) \stackrel{d}{=} \widetilde{X}(t)$ for all $t \geq t_0$.

Proof. We prove that $\xi_{j,n}(t|t_0)$, given in (26), satisfies (19), with $\lambda_n(t)$ and $\mu_n(t)$ defined in (27). Recalling the initial conditions in (1), from (26) it follows $\lim_{t\downarrow t_0} \xi_{j,n}(t|t_0) = \delta_{j,n}$. Moreover, we consider the function:

$$\begin{aligned} \Psi_{j,n}(t|t_0) = \frac{d\xi_{j,n}(t|t_0)}{dt} &+ \big[\lambda_n(t) + \mu_n(t)\big]\, \xi_{j,n}(t|t_0) - \lambda_{n-1}(t)\, \xi_{j,n-1}(t|t_0) \\ &- \mu_{n+1}(t)\, \xi_{j,n+1}(t|t_0), \qquad j, n \in \mathbb{Z}, t \geq t_0. \end{aligned} \tag{29}$$

By virtue of (1), (5) and (27), from (26) it follows:

$$\begin{aligned} \frac{d\xi_{j,n}(t|t_0)}{dt} &= -\varphi(t)\,(\lambda+\mu)\, \xi_{j,n}(t|t_0) + \lambda\, \varphi(t)\, \frac{\vartheta + (1-\vartheta)\varrho^{-n}}{\vartheta + (1-\vartheta)\varrho^{-j}}\, f^{(1)}_{j,n-1}(t|t_0) \\ &\quad + \mu\, \varphi(t)\, \frac{\vartheta + (1-\vartheta)\varrho^{-n}}{\vartheta + (1-\vartheta)\varrho^{-j}}\, f^{(1)}_{j,n+1}(t|t_0), \qquad j, n \in \mathbb{Z}, t \geq t_0. \end{aligned} \tag{30}$$

Hence, making use of (26), (27) and (30) in (29), one obtains $\Psi_{j,n}(t|t_0) = 0$ for all $t \geq t_0$ and $j, n \in \mathbb{Z}$, implying that $\xi_{j,n}(t|t_0)$ is the solution of (19). For the uniqueness of the solution of (19), one has $f_{j,n}(t|t_0) = \xi_{j,n}(t|t_0)$ for $t \geq t_0$ and $j, n \in \mathbb{Z}$. Then, $X(t) \stackrel{d}{=} \widetilde{X}(t)$ for all $t \geq t_0$. $\square$

When ν is fixed and $|z|$ is large, the following asymptotic approximation holds (cf. Abramowitz [47]):

$$I_\nu(z) \sim \frac{e^z}{\sqrt{2\pi\, z}} \left[1 + \sum_{k=1}^{+\infty} \frac{(-1)^k}{(2z)^k}\, \frac{\Gamma(\nu+k+1/2)}{k!\,\Gamma(\nu-k+1/2)}\right], \tag{31}$$

so that, making use of (31) in (28), we note that a steady-state distribution for $X(t)$ does not exist.

By virtue of (15), for $t \geq t_0$ the conditional mean and variance of $X(t)$, with intensity functions (27), are:

$$\begin{aligned} \mathrm{E}[X(t)|X(t_0)=j] &= j + (\lambda-\mu)\,\Phi(t|t_0)\,\frac{\vartheta-(1-\vartheta)\,\varrho^{-j}}{\vartheta+(1-\vartheta)\,\varrho^{-j}}, \\ \mathrm{Var}[X(t)|X(t_0)=j] &= (\lambda+\mu)\,\Phi(t|t_0) + (\lambda-\mu)^2\,\Phi^2(t|t_0)\,\frac{4\,\vartheta\,(1-\vartheta)\,\varrho^{-j}}{[\vartheta+(1-\vartheta)\,\varrho^{-j}]^2}. \end{aligned} \tag{32}$$

Remark 2. *Under the assumptions of Proposition 2, for $\vartheta = 1/2$ from (27) one obtains*

$$\lambda_n(t) = \lambda\,\varphi(t)\,\frac{1+\varrho^{-n-1}}{1+\varrho^{-n}}, \quad \mu_n(t) = \mu\,\varphi(t)\,\frac{1+\varrho^{-n+1}}{1+\varrho^{-n}}, \qquad n \in \mathbb{Z},$$

so that $\lambda_n(t) = \mu_{-n}(t)$ for $n \in \mathbb{Z}$. Therefore, $X(t)$ is a symmetric BD chain with respect to zero and from (28) one has:

$$f_{j,n}(t|t_0) = \frac{\varrho^{n/2}+\varrho^{-n/2}}{\varrho^{j/2}+\varrho^{-j/2}}\,e^{-(\lambda+\mu)\Phi(t|t_0)}\,I_{n-j}\big[2\sqrt{\lambda\,\mu}\,\Phi(t|t_0)\big], \quad j,n \in \mathbb{Z}. \tag{33}$$

Remark 3. *Under the assumptions of Proposition 2, by replacing ϑ with $1-\vartheta$ and λ with μ, the expression of $f_{j,n}(t|t_0)$ does not change, i.e.*

$$f_{j,n}(t|t_0)\Big|_{\vartheta,\lambda,\mu} = f_{j,n}(t|t_0)\Big|_{1-\vartheta,\mu,\lambda}, \qquad j,n \in \mathbb{Z}, t \geq t_0.$$

For $t \geq t_0$ and $j,n \in \mathbb{Z}$, we denote by

$$\Omega_{j,n}(t|t_0) = -\tfrac{d}{dt}P\{X(t)<n|X(t_0)=j\} = \lambda_{n-1}(t)f_{j,n-1}(t|t_0) - \mu_n(t)f_{j,n}(t|t_0) \tag{34}$$

the probability current of $X(t)$ in n at time t, given $X(t_0)=j$. By virtue of (27) and (28), for $j,n \in \mathbb{Z}$ and $t \geq t_0$ from (34) one obtains:

$$\begin{aligned} \Omega_{j,n}(t|t_0) &= \varphi(t)\,e^{-(\lambda+\mu)\Phi(t|t_0)}\,\varrho^{(n-j)/2}\bigg\{\sqrt{\lambda\,\mu}\,\frac{\vartheta+(1-\vartheta)\varrho^{-n}}{\vartheta+(1-\vartheta)\varrho^{-j}} \\ &\times I_{n-1-j}\big[2\sqrt{\lambda\,\mu}\,\Phi(t|t_0)\big] - \mu\,\frac{\vartheta+(1-\vartheta)\varrho^{-n+1}}{\vartheta+(1-\vartheta)\varrho^{-j}}\,I_{n-j}\big[2\sqrt{\lambda\,\mu}\,\Phi(t|t_0)\big]\bigg\}. \end{aligned} \tag{35}$$

Let

$$\mathcal{T}_{j,n}(t_0) = \inf\{t \geq t_0 : X(t) = n\}, \qquad X(t_0) = j,\ j \neq n$$

be the FPT of $X(t)$ through $n \in \mathbb{Z}$, when starting from $j \in \mathbb{Z}$ and we denote by $g_{j,n}(t|t_0) = dP\{\mathcal{T}_{j,n}(t_0) \le t\}/dt$ its pdf. For $j, n \in \mathbb{Z}$ one has:

$$f_{j,n}(t|t_0) = \int_{t_0}^{t} g_{j,k}(u|t_0)\, f_{k,n}(t|u)\, du \quad (j < k \le n) \text{ or } (n \le k < j). \tag{36}$$

Proposition 3. *Under the assumptions of Proposition 2, it results:*

$$g_{j,n}(t|t_0) = \frac{|n-j|\;\varphi(t)}{\Phi(t|t_0)}\; f_{j,n}(t|t_0), \qquad j, n \in \mathbb{Z},\; j \ne n, \tag{37}$$

with $f_{j,n}(t|t_0)$ given in (28). Moreover, the FPT ultimate probability is:

$$P_{j,n} = \int_{t_0}^{+\infty} g_{j,n}(t|t_0)\, dt = \begin{cases} \dfrac{\vartheta + (1-\vartheta)\varrho^{-n}}{\vartheta + (1-\vartheta)\varrho^{-j}}, & (\varrho - 1)\,(n-j) \ge 0, \\ \varrho^{n-j}\, \dfrac{\vartheta + (1-\vartheta)\varrho^{-n}}{\vartheta + (1-\vartheta)\varrho^{-j}}, & (\varrho - 1)\,(n-j) < 0. \end{cases} \tag{38}$$

Proof. For $X(t)$, with intensity functions (27), making use of (28) in (36), one has:

$$f^{(1)}_{j,n}(t|t_0) = \int_{t_0}^{t} \Big[\frac{\vartheta + (1-\vartheta)\varrho^{-j}}{\vartheta + (1-\vartheta)\varrho^{-k}}\; g_{j,k}(u|t_0)\Big]\; f^{(1)}_{k,n}(t|u)\, du$$
$$(j < k \le n) \text{ or } (n \le k < j), \tag{39}$$

where (10) has been used. By comparing (8) for $i = 1$ with (39), one obtains

$$g_{j,n}(t|t_0) = \frac{\vartheta + (1-\vartheta)\varrho^{-n}}{\vartheta + (1-\vartheta)\varrho^{-j}}\; g^{(1)}_{j,n}(t|t_0), \qquad j, n \in \mathbb{Z},\; j \ne n,$$

from which, recalling (10) and (28), relation (37) follows. Then, by integrating (37), we obtain (38). $\square$

From (38), we note that the first passage is never a sure event for $0 < \vartheta < 1$. Instead, for $\vartheta = 0$ one has $P_{j,n} = 1$ when $(\varrho - 1)\,(n-j) \le 0$, whereas for $\vartheta = 1$ it follows $P_{j,n} = 1$ for $(\varrho - 1)\,(n-j) \ge 0$. Furthermore, when $\vartheta = 1/2$, from Remark 2 one has $f_{j,n}(t|t_0) = f_{-j,-n}(t|t_0)$ for $j, n \in \mathbb{Z}$; hence, from Proposition 3 it follows $g_{j,n}(t|t_0) = g_{-j,-n}(t|t_0)$ for $j, n \in \mathbb{Z}$ with $j \ne n$.

Example 2. We consider the BD chain $X(t)$, where $\lambda_n(t)$ and $\mu_n(t)$ are specified in (27), with $\lambda(t) = \lambda\,\varphi(t)$ and $\mu(t) = \mu\,\varphi(t)$, where $\lambda > 0$, $\mu > 0$ and

$$\varphi(t) = 1 + a\,\sin\Big(\frac{2\pi\,t}{Q}\Big), \qquad 0 \le a < 1, t \ge t_0. \tag{40}$$

Hence,

$$\Phi(t|t_0) = t - t_0 + \frac{a\,Q}{\pi}\Big[\sin^2\Big(\frac{\pi t}{Q}\Big) - \sin^2\Big(\frac{\pi t_0}{Q}\Big)\Big], \qquad t \ge t_0.$$

In Figure 4, for $\vartheta = 0.25$ the periodic BD intensity functions $\lambda_n(t)$ and $\mu_n(t)$, specified in (27) with $\varphi(t)$ given in (40), are represented as function of t on the left by choosing $n = -1, 0, 1$ and as function of n on the right for $t = 5.2$; the dotted lines in (b) and (d) show the lower and upper bounds of BD intensity functions (27). In Figure 5, for $\vartheta = 0.25$ (top) and for $\vartheta = 0.75$ (bottom), the probabilities $f_{j,n}(t|0)$, given in (28), are shown as function of t for $j = 1$ and $n = -1, 0, 1$ on the left and as function of n for $j = 1$ and $t = 5.2$ on the right. Moreover, in Figure 6, the conditional mean $E[X(t)|X(t_0) = j]$ and the conditional variance $\mathrm{Var}[X(t)|X(t_0) = j]$ given in (32), are plotted as function of t for $j = 1$ and $\vartheta = 0.25, 0.5, 0.75$. As in Example 1 the conditional means oscillate around a line. Finally, in Figure 7, the FPT density $g_{j,n}(t|t_0)$, given in (37) is represented as function of t for $\vartheta = 0.25$ on the left and $\vartheta = 0.75$ on the right, $j = 1$ and $n = -1, 0$. From (38), for $\vartheta = 0.25$ one has $P_{1,0} = 0.8$ and $P_{1,-1} = 0.7$; instead, for $\vartheta = 0.75$ it results $P_{1,0} = 0.571429$ and $P_{1,-1} = 0.357143$. $\diamond$

In Appendix A, some results on the LT for the time-homogeneous BD chain $X(t)$ are given. They will play an important role in determining the transition probabilities of BD processes described in the Sections 4 and 5.

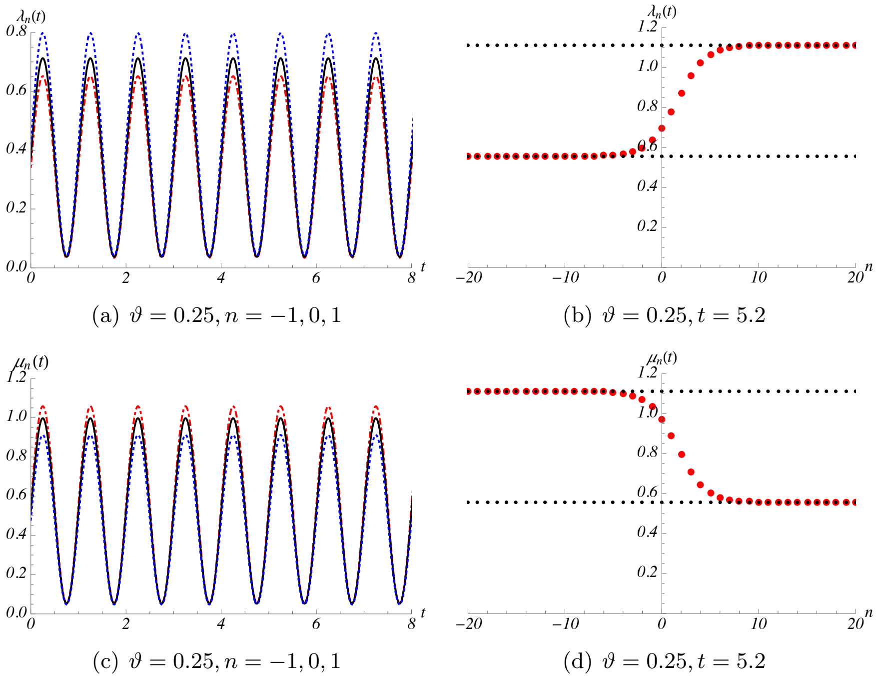
4. Special class of restricted time-inhomogeneous BD process

Let $\{R(t), t \ge t_0\}$ be a time-inhomogeneous BD process, whose state-space is $\mathbb{N}_0 = \{0, 1, \ldots\}$, with 0 reflecting state and

$$\begin{aligned} \lambda_n(t) &= \lambda\,\varphi(t)\,\frac{\vartheta + (1-\vartheta)\varrho^{-n-1}}{\vartheta + (1-\vartheta)\varrho^{-n}}, \qquad n \in \mathbb{N}_0,\\ \mu_n(t) &= \mu\,\varphi(t)\,\frac{\vartheta + (1-\vartheta)\varrho^{-n+1}}{\vartheta + (1-\vartheta)\varrho^{-n}}, \qquad n \in \mathbb{N}, \end{aligned} \tag{41}$$

where $0 \le \vartheta \le 1$, $\lambda > 0$, $\mu > 0$ and $\varrho = \lambda/\mu$. The transition probabilities of $R(t)$

$$p_{j,n}(t|t_0) = P\{R(t) = n|R(t_0) = j\}, \qquad j, n \in \mathbb{N}_0, t \ge t_0$$



(a) $\vartheta = 0.25, n = -1, 0, 1$ (b) $\vartheta = 0.25, t = 5.2$

(c) $\vartheta = 0.25, n = -1, 0, 1$ (d) $\vartheta = 0.25, t = 5.2$

Figure 4: For $\lambda = 0.6$, $\mu = 0.3$, $Q = 1$, $a = 0.9$, the BD intensity functions (27), with $\varphi(t)$ given in (40), are shown as function of t for $n = -1$ (red, dash-dotted curve), $n = 0$ (black, solid curve) and $n = 1$ (blue, dotted curve) in (a) and (c) and as function of n for $t = 5.2$ in (b) and (d).

are solution of the Kolmogorov equations and the related initial condition:

$$\begin{aligned} \frac{dp_{j,0}(t|t_0)}{dt} &= -\lambda_0(t)\, p_{j,0}(t|t_0) + \mu_1(t)\, p_{j,1}(t|t_0), \quad j \in \mathbb{N}_0, \\ \frac{dp_{j,n}(t|t_0)}{dt} &= \lambda_{n-1}(t)\, p_{j,n-1}(t|t_0) - \big[\lambda_n(t) + \mu_n(t)\big]\, p_{j,n}(t|t_0) \\ &\qquad + \mu_{n+1}(t)\, p_{j,n+1}(t|t_0), \quad j \in \mathbb{N}_0, n \in \mathbb{N}, \\ \lim_{t \downarrow t_0} p_{j,n}(t|t_0) &= \delta_{j,n}, \end{aligned} \tag{42}$$

with $\lambda_n(t)$ and $\mu_n(t)$ given in (41). Moreover, the probabilities $p_{j,n}(t|t_0)$ can be written in terms of $f_{j,n}(t|t_0)$ and of the probability current $\Omega_{j,n}(t|t_0)$ of the unrestricted BD chain $X(t)$ (see, Giorno and Nobile [31]):

$$p_{j,n}(t|t_0) = f_{j,n}(t|t_0) - \int_{t_0}^{t} \Omega_{j,0}(u|t_0)\, p_{0,n}(t|u)\, du, \qquad j, n \in \mathbb{N}_0, t \geq t_0. \tag{43}$$

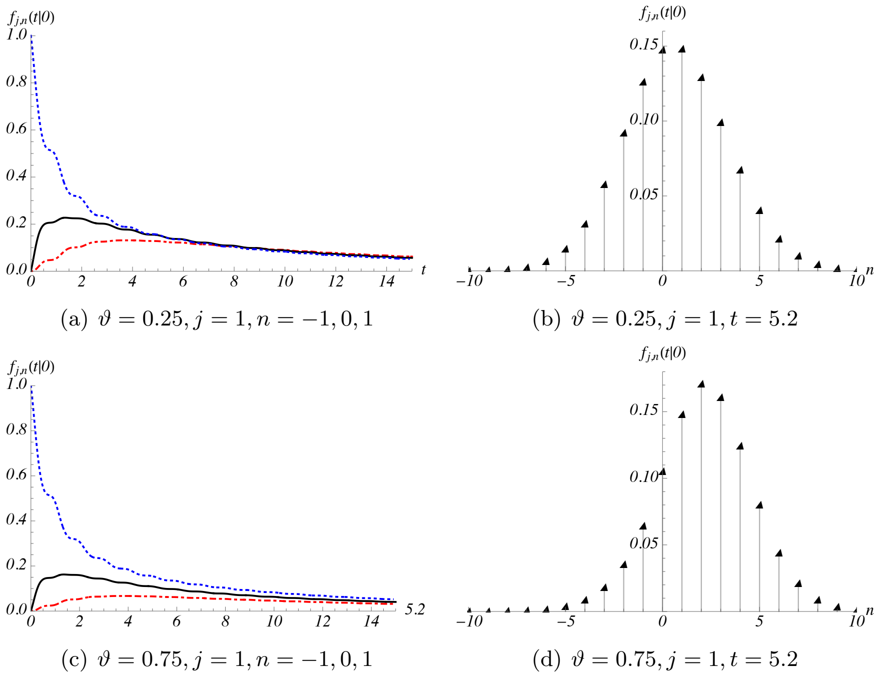


Figure 5: For $\lambda = 0.6$, $\mu = 0.3$, $Q = 1$, $a = 0.9$ in (40), the transition probabilities (28) are plotted as function of t with $n = -1$ (red, dash-dotted curve), $n = 0$ (black, solid curve) and $n = 1$ (blue, dotted curve) on the left and as function of n for $t = 5.2$ on the right.

Usually, the knowledge of $f_{j,n}(t|t_0)$ is not sufficient to determine the transition probabilities $p_{j,n}(t|t_0)$. Such probabilities are actually known only in few special cases, such as for the $M(t)/M(t)/1$ queueing system with proportional BD intensity functions and for the non-homogeneous Prendiville process (see, for instance, Giorno et al. [25], Zheng [37]).

Proposition 4. *For the restricted BD process $R(t)$, with BD intensity functions (41), one has:*

$$p_{j,n}(t|t_0) = \frac{\vartheta + (1-\vartheta)\,\varrho^{-n}}{\vartheta + (1-\vartheta)\,\varrho^{-j}}\, e^{-(\lambda+\mu)\Phi(t|t_0)} \Big\{ \varrho^{(n-j)/2}\, I_{n-j}\big[2\sqrt{\lambda\mu}\,\Phi(t|t_0)\big] + \frac{\beta}{\lambda}\,\varrho^{(n-j+1)/2}\, I_{n+j+1}\big[2\sqrt{\lambda\mu}\,\Phi(t|t_0)\big] + \varrho^{-j}\Big(\frac{\lambda}{\beta}\Big)^{n+j}\Big(1 - \frac{\lambda\mu}{\beta^2}\Big) \times \sum_{k=n+j+2}^{+\infty} \Big(\frac{\beta}{\lambda}\Big)^{k} \varrho^{k/2}\, I_k\big[2\sqrt{\lambda\mu}\,\Phi(t|t_0)\big] \Big\}, \qquad j, n \in \mathbb{N}_0, \tag{44}$$

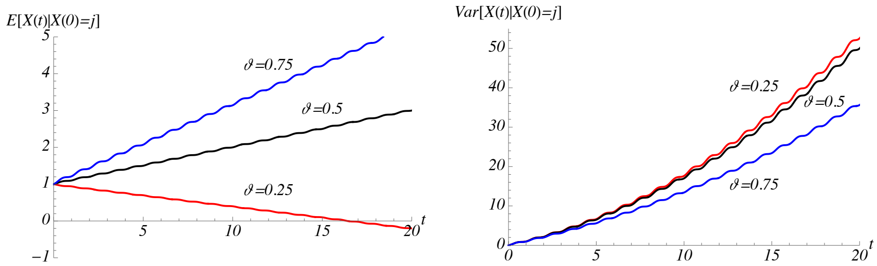
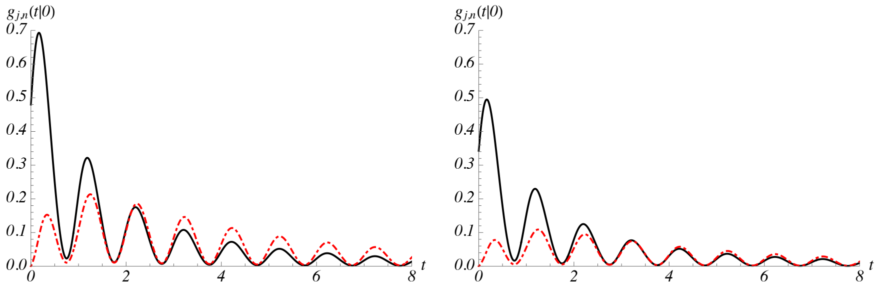


Figure 6: The conditional mean and the conditional variance (32) are drawn as function of t for $j = 1$, with $\lambda = 0.6$, $\mu = 0.3$, $Q = 1$ and $a = 0.9$ in (40).



(a) $\vartheta = 0.25$ (b) $\vartheta = 0.75$

Figure 7: The FPT densities (37) are shown as function of t for $j = 1$, with $\lambda = 0.6$, $\mu = 0.3$, $Q = 1$ and $a = 0.9$ in (40) with $n = 0$ (black, solid curve) and $n = -1$ (red, dashed curve).

where $\beta = \mu\,\vartheta + \lambda\,(1 - \vartheta)$.

Proof. It is obtained by starting from (42) and (43) and is given in Appendix B. □

We note that for $\vartheta = 1$ ($\vartheta = 0$), Eqs. (44) give the transition probabilities of the queueing system $M(t)/M(t)/1$ with arrival intensity function $\lambda\,\varphi(t)$ ($\mu\,\varphi(t)$) and service intensity function $\mu\,\varphi(t)$ ($\lambda\,\varphi(t)$) (see, for instance, Medhi [17] for $\varphi(t) = 1$ and Giorno et al. [25] for proportional intensity functions).

To study the limit behavior of the transition probabilities of $R(t)$, we use the definition (4) in (44) by obtaining:

$$
\begin{aligned}
p_{j,n}(t|t_0) &= \frac{\vartheta+(1-\vartheta)\,\varrho^{-n}}{\vartheta+(1-\vartheta)\,\varrho^{-j}}\, e^{-(\lambda+\mu)\Phi(t|t_0)} \Big\{ \varrho^{(n-j)/2}\, I_{n-j}\big[2\sqrt{\lambda\mu}\,\Phi(t|t_0)\big] \\
&+\frac{\beta}{\lambda}\,\varrho^{(n-j+1)/2}\, I_{n+j+1}\big[2\sqrt{\lambda\mu}\,\Phi(t|t_0)\big] + \varrho^{-j}\Big(\frac{\lambda}{\beta}\Big)^{n+j}\Big(1-\frac{\lambda\mu}{\beta^2}\Big) \\
&\times\Big[\exp\Big\{\Big(\beta+\frac{\lambda\mu}{\beta}\Big)\Phi(t|t_0)\Big\} - \sum_{m=0}^{+\infty}\frac{1}{m!}\Big[\frac{\lambda\mu\Phi(t|t_0)}{\beta}\Big]^m \sum_{r=0}^{m+n+j+1}\frac{[\beta\Phi(t|t_0)]^r}{r!}\Big]\Big\}
\end{aligned}
\tag{45}
$$

for $j,n \in \mathbb{N}_0$. Recalling (31), we note that the first two terms of (45) clearly go to 0 as t approaches $+\infty$ because $e^{-(\sqrt{\lambda}-\sqrt{\mu})\Phi(t|t_0)\}}/\sqrt{\Phi(t|t_0)}$ goes to 0 irrespective of the values of λ,μ and of the function $\varphi(t)$. Furthermore, the terms of the series in (45) are of the form $e^{-(\lambda+\mu)\Phi(t|t_0)}[\Phi(t|t_0)]^{m+r}$, implying that this series goes to zero as $t\to+\infty$. Moreover, when $0\leq\vartheta\leq 1$, one has:

$$
\lambda+\mu-\beta-\frac{\lambda\mu}{\beta} = \frac{\vartheta(1-\vartheta)(\lambda-\mu)^2}{\beta} \geq 0.
$$

Hence, for $0<\vartheta<1$ the process $R(t)$ does not admit a limit behavior. Instead, for $\vartheta=0$, $R(t)$ has a steady-state behavior when $\varrho>1$ and one has $q_n=\lim_{t\to+\infty}p_{j,n}(t|t_0)=(1-\varrho^{-1})\,\varrho^{-n}$ $(n\in\mathbb{N}_0)$. Finally, for $\vartheta=1$ the process $R(t)$ exhibits a steady-state behavior when $\varrho<1$ and one obtains $q_n=\lim_{t\to+\infty}p_{j,n}(t|t_0)=(1-\varrho)\,\varrho^{n}$ $(n\in\mathbb{N}_0)$. We note that the asymptotic results for $\vartheta=0$ and for $\vartheta=1$ are consistent with those obtained in Zeifman et al. [38] by using the truncation method.

Under the assumptions of Proposition 4, making use of (44) we obtain the *k-th* order conditional moment of $R(t)$:

$$
\begin{aligned}
\mathrm{E}[R^k(t)|R(t_0)=j] &= \frac{e^{-(\lambda+\mu)\Phi(t|t_0)}}{\vartheta+(1-\vartheta)\,\varrho^{-j}}\Big\{\sum_{n=1}^{+\infty} n^k\,[\vartheta+(1-\vartheta)\,\varrho^{-n}]\varrho^{(n-j)/2} \\
&\times\Big[I_{n-j}\big[2\sqrt{\lambda\mu}\,\Phi(t|t_0)\big]+\frac{\beta}{\sqrt{\lambda\mu}}\,I_{n+j+1}\big[2\sqrt{\lambda\mu}\,\Phi(t|t_0)\big]\Big] \\
&-\varrho^{-j/2}\Big(1-\frac{\beta^2}{\lambda\mu}\Big)\sum_{r=1}^{+\infty}\Big(\frac{\beta}{\sqrt{\lambda\mu}}\Big)^r I_{r+j+2}\big[2\sqrt{\lambda\mu}\,\Phi(t|t_0)\big] \\
&\times\sum_{n=1}^{r} n^k\Big(\frac{\lambda}{\beta}\Big)^n[\vartheta+(1-\vartheta)\,\varrho^{-n}]\Big\}, \qquad k=1,2,\ldots
\end{aligned}
\tag{46}
$$

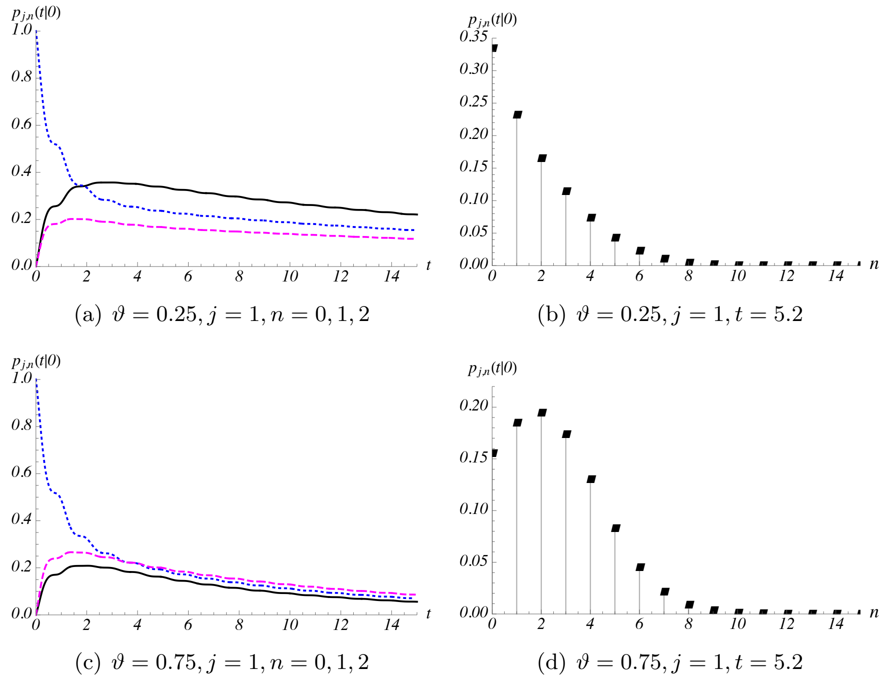
Remark 4. *Under the assumptions of Proposition 4, by replacing ϑ with $1-\vartheta$ and λ with μ the expression of $p_{j,n}(t|t_0)$ does not change, i.e.*

$$p_{j,n}(t|t_0)\Big|_{\vartheta,\lambda,\mu} = p_{j,n}(t|t_0)\Big|_{1-\vartheta,\mu,\lambda}, \qquad j,n \in \mathbb{N}_0, t \geq t_0.$$

Finally, for $R(t)$ with BD intensity functions (41), the downward FPT problem for $R(t)$ from j to n, with $j > n \geq 0$, is not affected by the reflecting boundary in the zero-state, so that the FPT density $g_{j,n}(t|t_0)$ is identified with (37).

Example 3. We consider the BD process $R(t)$ having $\lambda_n(t)$ and $\mu_n(t)$ specified in (41), where $\lambda(t) = \lambda\,\varphi(t)$ and $\mu(t) = \mu\,\varphi(t)$, with $\lambda > 0$, $\mu > 0$, and $\varphi(t)$ is given in (40) with $0 \leq a < 1$. In particular, for $\vartheta = 0$ ($\vartheta = 1$) the process $R(t)$ identifies with the $M(t)/M(t)/1$ queue with proportional periodic intensity functions that admits an asymptotic behavior as $\varrho > 1$ ($\varrho < 1$). Following Zeifman et al. [38], we set $d_k = \delta^{k-1}$ ($k \geq 1$), with $\delta > 1$, $\alpha_k(t) = \lambda_k(t) + \mu_{k+1}(t) - \delta\,\lambda_{k+1}(t) - (1/\delta)\,\mu_k(t)$ ($k \geq 0$) and $\alpha(t) = \inf_{k\geq 0} \alpha_k(t)$. Then, for $\vartheta = 0$ one has $\alpha(t) = (\delta-1)\varphi(t)(\lambda - \mu\,\delta)/\delta$ is a positive periodic function if and only if $\varrho > 1$ and $\delta \in (1, \rho)$. Instead, for $\vartheta = 1$ one has $\alpha(t) = (\delta - 1)\varphi(t)(\mu - \lambda\,\delta)/\delta$ is a periodic positive function if and only if $\varrho < 1$ and $\delta \in (1, \rho^{-1})$. Hence, when $\vartheta = 0$ and $\varrho > 1$, or $\vartheta = 1$ and $\varrho < 1$, the process $Z(t)$ is weakly ergodic and in Zeifman et al. [38] various bounds of truncation for the transition probabilities and for the conditional mean are obtained.

In Figure 8, for $\vartheta = 0.25$ (top) and $\vartheta = 0.75$ (bottom), the probabilities $p_{j,n}(t|0)$, given in (44), are shown as function of t for $j = 1$ and $n = 0, 1, 2$ on the left and as function of n for $j = 1$ and $t = 5.2$ on the right. We note as the regulation parameter ϑ influences the shape of the transition distribution. Moreover, by setting $\lambda = 0.6$ and $\mu = 0.3$, the process $R(t)$ admits a steady-state distribution if and only if $\vartheta = 0$, i.e $q_n = (1/2)^{n+1}$, $n \in \mathbb{N}$; instead, for $0 < \vartheta \leq 1$, the process $R(t)$ does not admit a limit behavior. In Figure 9, the conditional mean $E[R(t)|R(t_0) = j]$ and the conditional variance $\mathrm{Var}[R(t)|R(t_0) = j]$ obtained from (46), are represented as function of t for $j = 1$ and $\vartheta = 0.25, 0.5, 0.75$. The behavior of the conditional mean, showed in Figure 9, suggests that the process $R(t)$ can be used to model the number of cells in a population starting with a single cell, when the population is subject to a rapid growth. $\diamond$



(a) $\vartheta = 0.25, j = 1, n = 0, 1, 2$ (b) $\vartheta = 0.25, j = 1, t = 5.2$

(c) $\vartheta = 0.75, j = 1, n = 0, 1, 2$ (d) $\vartheta = 0.75, j = 1, t = 5.2$

Figure 8: For $\lambda = 0.6$, $\mu = 0.3$, $Q = 1$, $a = 0.9$ in (40), the transition probabilities (44) are plotted as function of t with $n = 0$ (black, solid curve), $n = 1$ (blue, dotted curve) and $n = 2$ (magenta, dashed curve) on the left and as function of n for $t = 5.2$ on the right.

5. Special class of symmetric BD chain with respect to zero-state

Let $\{Y(t), t \geq t_0\}$ be a BD chain with state-space $\mathbb{Z}$, characterized by BD intensity functions

$$\begin{aligned} \lambda_n(t) &= \mu_{-n}(t) = \lambda\,\varphi(t)\,\frac{\vartheta + (1-\vartheta)\varrho^{-n-1}}{\vartheta + (1-\vartheta)\varrho^{-n}}, \qquad n \in \mathbb{N}_0, \\ \mu_n(t) &= \lambda_{-n}(t) = \mu\,\varphi(t)\frac{\vartheta + (1-\vartheta)\varrho^{-n+1}}{\vartheta + (1-\vartheta)\varrho^{-n}}, \qquad n \in \mathbb{N}, \end{aligned} \tag{47}$$

with $0 \leq \vartheta \leq 1$, $\lambda > 0$, $\mu > 0$ and $\varrho = \lambda/\mu$. Since $\lambda_n(t) = \mu_{-n}(t)$ for all $n \in \mathbb{Z}$, the process $Y(t)$ is symmetric with respect to zero. The transition probabilities of the symmetric BD chain $Y(t)$

$$h_{j,n}(t|t_0) = P\{Y(t) = n | Y(t_0) = j\}, \qquad j, n \in \mathbb{Z}, t \geq t_0$$

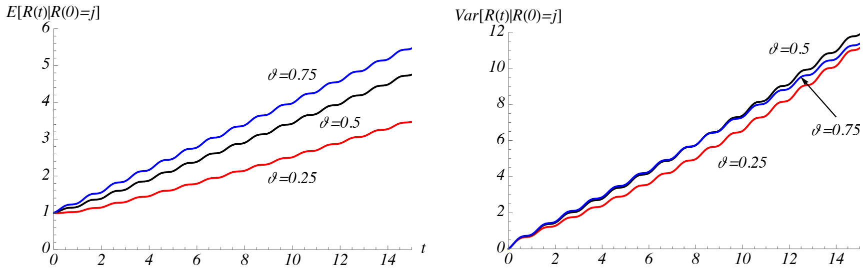


Figure 9: The conditional mean and the conditional variance (46) are plotted as function of t for $j = 1$, with $\lambda = 0.6$, $\mu = 0.3$, $Q = 1$ and $a = 0.9$ in (40).

satisfy the following relations:

$$\begin{aligned} & h_{j,n}(t|t_0) = h_{-j,-n}(t|t_0), \quad j, n \in \mathbb{Z}, \\ & h_{0,0}(t|t_0) + 2 \sum_{n=1}^{+\infty} h_{0,n}(t|t_0) = 1. \end{aligned} \tag{48}$$

Furthermore, for $t \geq t_0$ and $j, n \in \mathbb{Z}$, we consider the probability current of $Y(t)$:

$$\Theta_{j,n}(t|t_0) = -\tfrac{d}{dt} P\{Y(t) < n | Y(t_0) = j\} = \lambda_{n-1}(t)\, h_{j,n-1}(t|t_0) - \mu_n(t)\, h_{j,n}(t|t_0). \tag{49}$$

By virtue of the symmetry properties (48), from (49) one obtains:

$$\Theta_{0,0}(t|t_0) = \frac{1}{2} \frac{d}{dt} h_{0,0}(t|t_0), \qquad \Theta_{j,n}(t|t_0) = -\Theta_{-j,-n+1}(t|t_0), \quad j, n \in \mathbb{Z}. \tag{50}$$

Moreover, for $t \geq t_0$ one has:

$$\begin{aligned} & h_{j,n}(t|t_0) = h_{-j,-n}(t|t_0) = p_{j,n}(t|t_0) + \int_{t_0}^{t} \Theta_{j,0}(u|t_0)\, p_{0,n}(t|u)\, du, \;\; j, n \in \mathbb{N}_0, \\ & h_{j,n}(t|t_0) = \int_{t_0}^{t} g_{j,0}(u|t_0)\, h_{0,n}(t|u)\, du \qquad (j < 0 \leq n) \text{ or } (n \leq 0 < j), \end{aligned} \tag{51}$$

where $p_{j,n}(t|t_0)$ is the transition probability of the restricted BD process $R(t)$, $\Theta_{j,0}(t|t_0)$ and $g_{j,0}(t|t_0)$ are the probability current and the FPT density of $Y(t)$ to zero-state at time t starting from $Y(t_0) = j$, respectively. For $Y(t)$, one has $g_{j,0}(t|t_0) = g_{-j,0}(t|t_0)$, with $j \in \mathbb{N}$. Moreover, recalling (49) and making use of (51), for $t \geq t_0$ one obtains:

$$
\begin{aligned}
\Theta_{j,0}(t|t_0) &= \int_{t_0}^{t} g_{j,0}(u|t_0)\; \Theta_{0,0}(t|u)\, du, \qquad j > 0,\\
\Theta_{j,1}(t|t_0) &= \int_{t_0}^{t} g_{j,0}(u|t_0)\; \Theta_{0,1}(t|u)\, du, \qquad j < 0.
\end{aligned}
\tag{52}
$$

Proposition 5. *For the symmetric chain $Y(t)$ with BD intensity functions (47), when $j = 0$ one has:*

$$
\begin{aligned}
h_{0,n}(t|t_0) &= h_{0,-n}(t|t_0) = \Big[\vartheta + (1-\vartheta)\varrho^{-n}\Big]\; e^{-(\lambda+\mu)\,\Phi(t|t_0)}\,\varrho^{n/2}\\
&\times\Bigg\{ I_n\big[2\sqrt{\lambda\mu}\;\Phi(t|t_0)\big] + \frac{(\lambda-\mu)\,(2\vartheta-1)}{\mu(z_1-z_2)} \sum_{k=n+1}^{+\infty} (-1)^{k-n}\varrho^{(n-k)/2}\\
&\times (z_1^{k-n} - z_2^{k-n})\, I_k\big[2\sqrt{\lambda\mu}\;\Phi(t|t_0)\big]\Bigg\}, \qquad n \in \mathbb{N}_0,
\end{aligned}
\tag{53}
$$

where

$$
z_1, z_2 = \frac{-(\lambda-\mu)(1-2\vartheta) \pm \sqrt{(\lambda-\mu)^2(1-2\vartheta)^2 + 4\lambda\mu}}{2\mu}, \qquad z_1 > z_2. \tag{54}
$$

Proof. The proof follows by starting from (49) and from (51) and it is given in Appendix C. □

Proposition 6. *For $Y(t)$ with BD intensity functions (47), when $j \neq 0$ one has:*

$$
h_{j,n}(t|t_0) = h_{-j,-n}(t|t_0) = \frac{\vartheta + (1-\vartheta)\varrho^{n}}{\vartheta + (1-\vartheta)\varrho^{-j}}\; \frac{\varrho^{-j}\, h_{0,j-n}(t|t_0)}{\vartheta + (1-\vartheta)\varrho^{n-j}}, \qquad n \leq 0 < j, \tag{55}
$$

$$
\begin{aligned}
h_{j,n}(t|t_0) &= h_{-j,-n}(t|t_0) = p_{j,n}(t|t_0) - \frac{\vartheta + (1-\vartheta)\varrho^{-n}}{\vartheta + (1-\vartheta)\varrho^{-j}}\;\varrho^{-j}\\
&\times\Bigg\{ \frac{\alpha}{\lambda}\; \frac{h_{0,n+j+1}(t|t_0)}{\vartheta + (1-\vartheta)\varrho^{-n-j-1}} + \frac{\vartheta\,(1-\vartheta)\,(\lambda-\mu)^2}{\lambda\,\beta}\\
&\times \sum_{k=1}^{+\infty}\Big(\frac{\beta}{\lambda}\Big)^{k}\; \frac{h_{0,k+n+j+1}(t|t_0)}{\vartheta + (1-\vartheta)\varrho^{-k-n-j-1}}\Bigg\}, \qquad j, n \in \mathbb{N},
\end{aligned}
\tag{56}
$$

where $\alpha = \lambda\,\vartheta + \mu\,(1-\vartheta)$ and $\beta = \mu\,\vartheta + \lambda\,(1-\vartheta)$, with $p_{j,n}(t|t_0)$ and $h_{0,n}(t|t_0)$ given in (44) and (53), respectively.

Proof. The proof is shown in Appendix D by using (51) and (52). □

The transition probabilities (53), (55) and (56) of the process $Y(t)$, obtained in Propositions 5 and 6, can be expressed in an alternative form, as shown in the following proposition.

Proposition 7. *For $Y(t)$ with BD intensity functions (47), one has:*

$$h_{j,n}(t|t_0) = \frac{\vartheta + (1-\vartheta)\varrho^{-|n|}}{\vartheta + (1-\vartheta)\varrho^{-|j|}}\, e^{-(\lambda+\mu)\,\Phi(t|t_0)} \Big\{ \varrho^{(|n|-|j|)/2} I_{n-j}\big[2\sqrt{\lambda\mu}\,\Phi(t|t_0)\big] + \frac{(\lambda-\mu)\,(2\vartheta-1)}{\mu\,(z_1-z_2)} \sum_{k=|n|+|j|+1}^{+\infty} (-1)^{k-|n|-|j|} \varrho^{|n|-k/2} \times (z_1^{k-|n|-|j|} - z_2^{k-|n|-|j|})\, I_k\big[2\sqrt{\lambda\mu}\,\Phi(t|t_0)\big] \Big\}, \qquad j,n\in\mathbb{Z}, \tag{57}$$

where z_1 and z_2 are given in (54).

Proof. In Appendix E, starting from the results (53), (55) and (56), we derive the expression (57) for the transition probabilities $h_{j,n}(t|t_0)$ of the symmetric BD chain $Y(t)$. □

To analyze the limit behavior of the transition probabilities of $Y(t)$, we use (4) in (57):

$$h_{j,n}(t|t_0) = \frac{\vartheta + (1-\vartheta)\varrho^{-|n|}}{\vartheta + (1-\vartheta)\varrho^{-|j|}}\, e^{-(\lambda+\mu)\,\Phi(t|t_0)} \Big\{ \varrho^{(|n|-|j|)/2} I_{n-j}\big[2\sqrt{\lambda\mu}\,\Phi(t|t_0)\big] + \frac{(\lambda-\mu)\,(2\vartheta-1)}{\mu\,(z_1-z_2)}\, \varrho^{|n|} \sum_{m=0}^{+\infty} \frac{[\lambda\Phi(t|t_0)]^m}{m!} (-1)^{-m-|n|-|j|} \times \Big[z_1^{-m-|n|-|j|} \Big(e^{-\mu\, z_1\,\Phi(t|t_0)} - \sum_{r=0}^{m+|n|+|j|} \frac{-[\mu\, z_1 \Phi(t|t_0)]^r}{r!} \Big) - z_2^{-m-|n|-|j|} \Big(e^{-\mu\, z_2\,\Phi(t|t_0)} - \sum_{r=0}^{m+|n|+|j|} \frac{-[\mu\, z_2 \Phi(t|t_0)]^r}{r!} \Big) \Big] \Big\} \tag{58}$$

for $j,n\in\mathbb{Z}$. When $0\le\vartheta\le 1$, by virtue of (54) and (C.8), one has:

$$\lambda+\mu+\mu\, z_1 + \frac{\lambda}{z_1} > 0, \qquad \lambda+\mu+\mu\, z_2 + \frac{\lambda}{z_2} = \frac{4\,\vartheta\,(1-\vartheta)(\lambda-\mu)^2}{\lambda+\mu+\mu\,(z_1-z_2)} \ge 0.$$

Hence, for $0 < \vartheta < 1$ the process $Y(t)$ does not admit a limit behavior. Instead, for $\vartheta = 0$, $Y(t)$ has a steady-state behavior when $\varrho > 1$ and one has $\zeta_n = \lim_{t\to+\infty} h_{j,n}(t|t_0) = (1-\varrho^{-1})\,\varrho^{-|n|}/(1+\varrho^{-1})$ $(n \in \mathbb{Z})$. Finally, for $\vartheta = 1$ the process $Y(t)$ exhibits a steady-state behavior when $\varrho < 1$ and one obtains $\zeta_n = \lim_{t\to+\infty} h_{j,n}(t|t_0) = (1-\varrho)\,\varrho^{|n|}/(1+\varrho)$ $(n \in \mathbb{Z})$.

For $Y(t)$ with intensity functions (47), making use of (57), we obtain the *k-th* order conditional moment:

$$
\begin{aligned}
\mathrm{E}[Y^k(t)|Y(t_0)=j] &= \frac{e^{-(\lambda+\mu)\,\Phi(t|t_0)}\;\varrho^{-|j|/2}}{\vartheta+(1-\vartheta)\varrho^{-|j|}}\left\{\sum_{n=1}^{+\infty} n^k\varrho^{n/2}\big[\vartheta+(1-\vartheta)\varrho^{-n}\big]\right.\\
&\times\Big[I_{n-j}\big[2\sqrt{\lambda\mu}\;\Phi(t|t_0)\big]+(-1)^k I_{n+j}\big[2\sqrt{\lambda\mu}\;\Phi(t|t_0)\big]\Big]+\big[1+(-1)^k\big]\\
&\times\frac{(\lambda-\mu)\,(2\vartheta-1)}{\mu\,(z_1-z_2)}\sum_{r=1}^{+\infty}(-1)^{r+1}\varrho^{-(r+1)/2}\,I_{r+|j|+1}\big[2\sqrt{\lambda\mu}\;\Phi(t|t_0)\big]\\
&\left.\times\sum_{n=1}^{r} n^k(-\varrho)^n\big[\vartheta+(1-\vartheta)\varrho^{-n}\big]\big(z_1^{r-n+1}-z_2^{r-n+1}\big)\right\},\qquad k=1,2,\ldots \qquad (59)
\end{aligned}
$$

In particular, from (59) it follows that $\mathrm{E}[Y^k(t)|Y(t_0)=0]=0$ for $t \geq t_0$.

Remark 5. *Under the assumptions of Proposition 7, the following results hold:*

- *for $\vartheta = 1/2$, $X(t)$ and $Y(t)$ are characterized by the same BD intensity functions, so that from (33) for $j,n \in \mathbb{Z}$ and $t \geq t_0$ it follows:*

$$
h_{j,n}(t|t_0) = f_{j,n}(t|t_0) = \frac{\varrho^{n/2}+\varrho^{-n/2}}{\varrho^{j/2}+\varrho^{-j/2}}\;e^{-(\lambda+\mu)\Phi(t|t_0)}\,I_{n-j}\big[2\sqrt{\lambda\,\mu}\,\Phi(t|t_0)\big]; \qquad (60)
$$

- *for $\vartheta = 0$, one has $z_1 = 1$ and $z_2 = -\varrho$, so that (57) leads to:*

$$
\begin{aligned}
h_{j,n}(t|t_0) &= e^{-(\lambda+\mu)\Phi(t|t_0)}\left\{\varrho^{-(|n|-|j|)/2}\,I_{n-j}\big[2\sqrt{\lambda\mu}\,\Phi(t|t_0)\big]\right.\\
&\left.-\frac{1-\varrho}{1+\varrho}\sum_{k=|j|+|n|+1}^{+\infty}\varrho^{-|n|+k/2}\big[1-(-\varrho)^{-k+|j|+|n|}\big]\,I_k\big[2\sqrt{\lambda\mu}\,\Phi(t|t_0)\big]\right\},\\
&\qquad j,n\in\mathbb{Z}; \qquad (61)
\end{aligned}
$$

- *for $\vartheta = 1$, one has $z_1 = \varrho$ and $z_2 = -1$, so that (57) becomes:*

$$
\begin{aligned}
h_{j,n}(t|t_0) = e^{-(\lambda+\mu)\Phi(t|t_0)} \Big\{ & \varrho^{(|n|-|j|)/2}\, I_{n-j}\big[2\sqrt{\lambda\mu}\,\Phi(t|t_0)\big] \\
& + \frac{1-\varrho}{1+\varrho} \sum_{k=|j|+|n|+1}^{+\infty} \varrho^{|n|-k/2}\big[1-(-\varrho)^{k-|j|-|n|}\big]\, I_k\big[2\sqrt{\lambda\mu}\,\Phi(t|t_0)\big] \Big\}, \\
& \qquad j,n \in \mathbb{Z}. \qquad (62)
\end{aligned}
$$

In particular, for $\vartheta = 1/2$ the transition probability $h_{0,n}(t|t_0)$ ($n \in \mathbb{Z}$), given in (60), identifies with the transition probability of the process

$$
\widetilde{Y}(t) = \mathcal{B}_t\, X_1(t), \qquad \widetilde{Y}(t_0) = 0,
$$

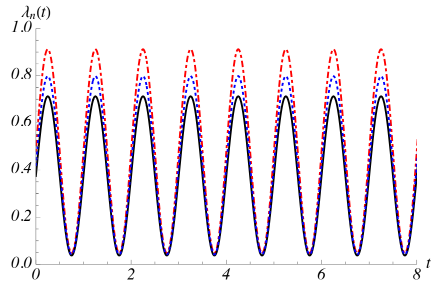
where $X_1(t)$ is the time-inhomogeneous double-ended BD chain with $\lambda_n = \lambda\,\varphi(t)$ and $\mu_n = \mu\,\varphi(t)$ ($n \in \mathbb{Z}$) and $\mathcal{B}_t$ is a Bernoulli random variable taking values ± 1 with symmetric probability $1/2$, indipendent on $X_1(t)$ for all $t \geq t_0$.

Remark 6. *Under the assumptions of Proposition 7, by replacing ϑ with $1-\vartheta$ and λ with μ, the expression of $h_{j,n}(t|t_0)$ does not change, i.e.*

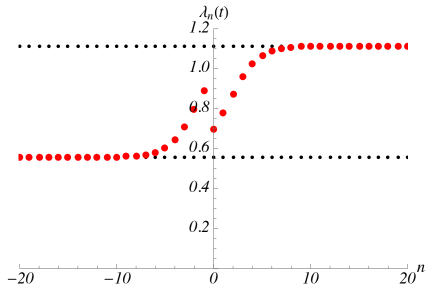
$$
h_{j,n}(t|t_0)\Big|_{\vartheta,\lambda,\mu} = h_{j,n}(t|t_0)\Big|_{1-\vartheta,\mu,\lambda}, \qquad j,n \in \mathbb{Z}, t \geq t_0.
$$

Example 4. We consider the BD chain $Y(t)$, whose intensity functions $\lambda_n(t)$ and $\mu_n(t)$ are given in (47), with $\lambda(t) = \lambda\,\varphi(t)$ and $\mu(t) = \mu\,\varphi(t)$, where $\lambda > 0$, $\mu > 0$ and $\varphi(t)$ is specified in (40). The symmetric process $Y(t)$ is suitable to model the motion of a particle on a straight line with proportional intensity functions.

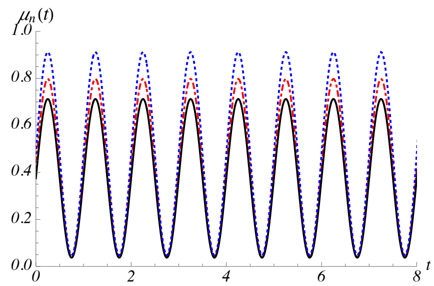
In Figure 10, by choosing $\vartheta = 0.25$, the BD intensities $\lambda_n(t)$ and $\mu_n(t)$ are represented as function of t for $n = -1, 0, 1$ in (a) and (c) and as function of n for $t = 5.2$ in (b) and (d). By comparing (a) with (c) and (b) with (d), we note that the symmetric property of the intensity functions holds, i.e. $\lambda_n(t) = \mu_{-n}(t)$ for $n \in \mathbb{Z}$ and $t \geq 0$. Instead, in Figure 11, the probabilities $h_{j,n}(t|0)$, given in (57), are shown as function of t for $j = 1$, $n = -1, 0, 1$ in (a) and (c) and as function of n for $t = 5.2$ in (b) and (d). Moreover, in Figure 12 the transition probabilities $h_{0,n}(t|0)$ are plotted as function of n for $t = 5.2$ and $\vartheta = 0.25$ (on the left) and $\vartheta = 0.75$ (on the right), by highlighting the symmetry property $h_{0,n}(t|0) = h_{0,-n}(t|0)$. As shown in Figures 11 and 12, the transition probabilities may exhibit different behaviors as function of n, depending on the regulation parameter



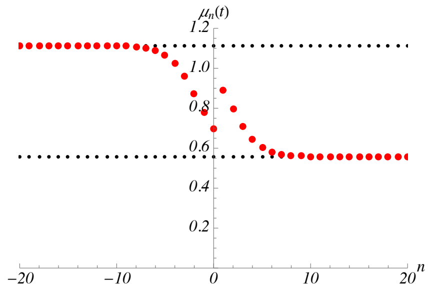
(a) $\vartheta = 0.25, n = -1, 0, 1$



(b) $\vartheta = 0.25, t = 5.2$



(c) $\vartheta = 0.25, n = -1, 0, 1$



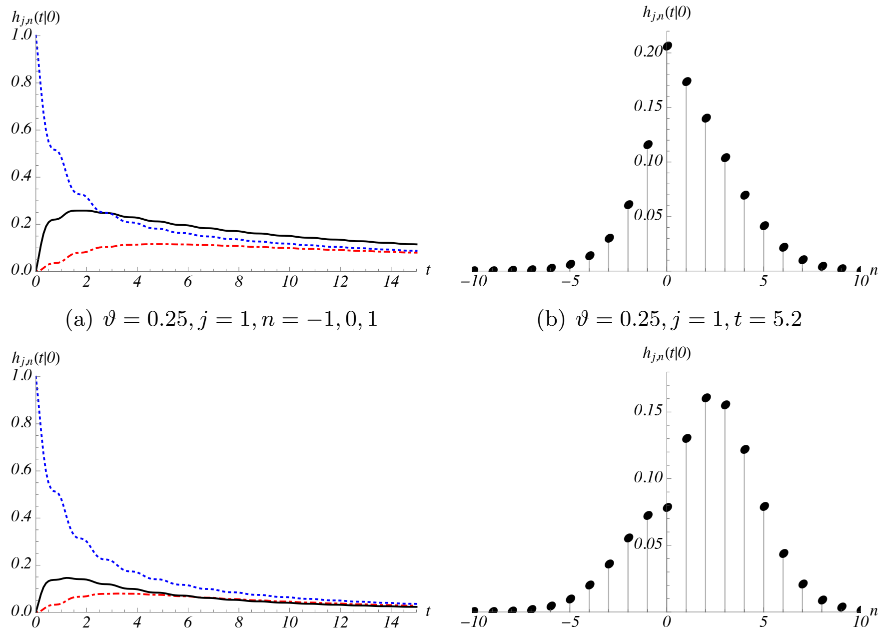
(d) $\vartheta = 0.25, t = 5.2$

Figure 10: For $\lambda = 0.6$, $\mu = 0.3$, $Q = 1$, $a = 0.9$, the BD intensity functions (47), with $\varphi(t)$ given in (40), are drawn as function of t for $n = -1$ (red, dash-dotted curve), $n = 0$ (black, solid curve) and $n = 1$ (blue, dotted curve) in (a) and (c) and as function of n for $t = 0.5$ in (b) and (d).

ϑ and the initial state j. Furthermore, in Figure 13, the conditional mean $E[Y(t)|Y(t_0) = j]$ and the conditional variance $\text{Var}[Y(t)|Y(t_0) = j]$ given in (59), are represented as function of t for $\vartheta = 0.25, 0.5, 0.75$. Thanks to Remark 5, for $\vartheta = 0.5$ the conditional mean and variance coincide with those of Figure 6. Furthermore, differently from Figure 6, now the conditional mean always oscillates around an increases line. $\diamond$

6. Conclusion

Special time-inhomogeneous BD processes, unrestricted or restricted, have been obtained. Our procedure provides stochastic processes suitable to model chemical queues, in which the motion of a molecule can be described by means of a BD chain with time-varying intensity functions. Furthermore, the considered restricted BD process can be usefully applied to model



(a) $\vartheta = 0.25, j = 1, n = -1, 0, 1$

(b) $\vartheta = 0.25, j = 1, t = 5.2$

(c) $\vartheta = 0.75, j = 1, n = -1, 0, 1$

(d) $\vartheta = 0.75, j = 1, t = 5.2$

Figure 11: For $\lambda = 0.6$, $\mu = 0.3$, $Q = 1$, $a = 0.9$ in (47), the transition probabilities (57) are plotted as function of t with $n = -1$ (red, dash-dotted curve), $n = 0$ (black, solid curve) and $n = 1$ (blue, dotted curve) in (a) and (c) and as function of n for $t = 5.2$ in (b) and (d), respectively.

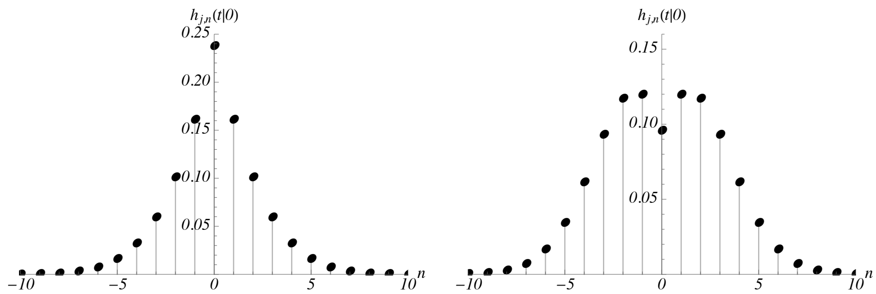
systems, as cancer masses or bacterial populations, characterized by rapid growth. For periodic intensity functions, various numerical evaluations have been developed to highlight the role of the regulation parameter.

Appendix A. Some results on the LT for $X(t)$

When $\varphi(t) = 1$ in (27), the BD chain $X(t)$ is time-homogeneous. Then, under the assumptions of Proposition 2, one has:

$$\begin{aligned} & f_{j,n}(t|t_0) = f^*_{j,n}[\Phi(t|t_0)|0], \qquad j, n \in \mathbb{Z}, t \geq t_0, \\ & \Omega_{j,n}(t|t_0) = \varphi(t)\ \Omega^*_{j,n}[\Phi(t|t_0)|0], \qquad j, n \in \mathbb{Z}, t \geq t_0, \\ & g_{j,n}(t|t_0) = \varphi(t)\ g^*_{j,n}[\Phi(t|t_0)|0], \qquad j, n \in \mathbb{Z}, j \neq n, t \geq t_0, \end{aligned} \tag{A.1}$$

where $f_{j,n}(t|t_0)$, $\Omega_{j,n}(t|t_0)$ and $g_{j,n}(t|t_0)$ are given in (28), (35) and (37) and where $f^*_{j,n}(t|0)$, $\Omega^*_{j,n}(t|0)$ and $g^*_{j,n}(t|0)$ are obtained from them by setting



(a) $\vartheta = 0.25, j = 0, t = 5.2$ (b) $\vartheta = 0.75, j = 0, t = 5.2$

Figure 12: For $\lambda = 0.6$, $\mu = 0.3$, $Q = 1$, $a = 0.9$ in (47), the transition probabilities (57) are plotted as function of n.

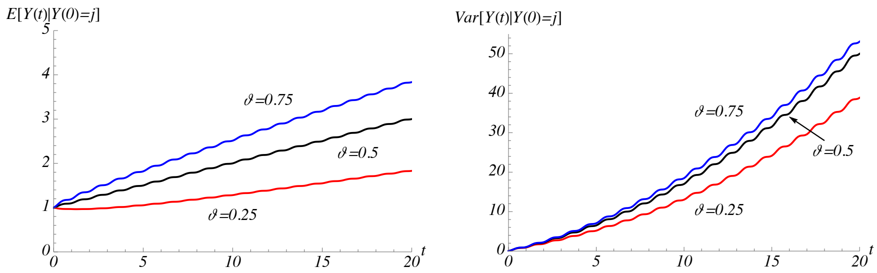


Figure 13: The conditional mean and the conditional variance (59) are drawn as function of t for $j = 1$, with $\lambda = 0.6$, $\mu = 0.3$, $Q = 1$ and $a = 0.9$ in (47).

$\varphi(t) = 1$. The LT of the transition probability $f^*_{j,n}(t|0)$, given in (28) with $\varphi(t) = 1$, is:

$$\widehat{f}^*_{j,n}(s) = \frac{\vartheta + (1-\vartheta)\varrho^{-n}}{\vartheta + (1-\vartheta)\varrho^{-j}} \, \frac{[\psi_\epsilon(s)]^{n-j}}{\mu\,[\psi_1(s) - \psi_2(s)]}, \qquad j, n \in \mathbb{Z}, \tag{A.2}$$

where $\epsilon = 1$ if $n \le j$, $\epsilon = 2$ if $n > j$ and

$$\psi_1(s), \psi_2(s) = \frac{s + \lambda + \mu \pm \sqrt{(s+\lambda+\mu)^2 - 4\lambda\mu}}{2\mu}, \qquad \psi_1(s) > \psi_2(s). \tag{A.3}$$

We note that $\psi_1(s) > 1$, $0 < \psi_2(s) < 1$ and

$$\psi_1(s)\psi_2(s) = \varrho, \qquad [\psi_1(s) - 1]\,[1 - \psi_2(s)] = \frac{s}{\mu}\cdot \tag{A.4}$$

Moreover, the LT of the probability current $\Omega^*_{j,n}(t|0)$, given in (35) with $\varphi(t)=1$, is:

$$\widehat{\Omega}^*_{j,n}(s)=\frac{[\psi_\epsilon(s)]^{n-j}}{\mu\,[\psi_1(s)-\psi_2(s)]}\;\frac{\mu\,\vartheta\,[\psi_{3-\epsilon}(s)-1]+(1-\vartheta)\,\varrho^{-n}\,[\mu\,\psi_{3-\epsilon}(s)-\lambda]}{\vartheta+(1-\vartheta)\,\varrho^{-j}},\qquad j,n\in\mathbb{Z}. \tag{A.5}$$

In particular, if $j=n=0$, from (A.5) one has:

$$\widehat{\Omega}^*_{0,0}(s)=\frac{\mu\,\psi_2(s)-\mu\vartheta-\lambda(1-\vartheta)}{\mu\,[\psi_1(s)-\psi_2(s)]}\cdot \tag{A.6}$$

Furthermore, by virtue of (36), the LT of the FPT density $g^*_{j,n}(t|0)$, given in (37) with $\varphi(t)=1$, is:

$$\widehat{g}^*_{j,n}(s)=\frac{\vartheta+(1-\vartheta)\varrho^{-n}}{\vartheta+(1-\vartheta)\varrho^{-j}}\,\big[\psi_\epsilon(s)\big]^{n-j},\qquad j,n\in\mathbb{Z},n\neq j. \tag{A.7}$$

Appendix B. Proof of Proposition 4 for $R(t)$

When $\varphi(t)=1$ in (41), the restricted BD process $R(t)$ is time-homogeneous; let $p^*_{j,n}(t|0)$ be the related probabilities. Recalling (42), with $\lambda_n(t)$ and $\mu_n(t)$ given in (41), one has:

$$p_{j,n}(t|t_0)=p^*_{j,n}[\Phi(t|t_0)|0],\qquad j,n\in\mathbb{N}_0,t\geq t_0. \tag{B.1}$$

Making use of (A.1) and (B.1) in (43), one obtains:

$$p^*_{j,n}[\Phi(t|t_0)|0]=f^*_{j,n}[\Phi(t|t_0)|0]-\int_0^{\Phi(t|t_0)}\Omega^*_{j,0}(\tau|0)\,p^*_{0,n}[\Phi(t|t_0)-\tau|0]\,d\tau,\qquad j,n\in\mathbb{N}_0,t\geq t_0. \tag{B.2}$$

When $\varphi(t)=1$, by applying in (B.2) the convolution theorem and making use of (A.2), (A.5) and (A.6), one obtains the LT of $p^*_{j,n}(t|0)$ of $R(t)$:

$$\begin{aligned}
\widehat{p}^*_{0,n}(s)&=\frac{\widehat{f}^*_{0,n}(s)}{1+\widehat{\Omega}^*_{0,0}(s)}=\frac{\big[\vartheta+(1-\vartheta)\,\varrho^{-n}\big]\,[\psi_2(s)]^n}{\mu\,\psi_1(s)-\beta},\quad n\in\mathbb{N}_0,\\
\widehat{p}^*_{j,n}(s)&=\widehat{f}^*_{j,n}(s)-\widehat{\Omega}^*_{j,0}(s)\,\widehat{p}^*_{0,n}(s)=\frac{\vartheta+(1-\vartheta)\,\varrho^{-n}}{\vartheta+(1-\vartheta)\,\varrho^{-j}}\bigg\{\frac{[\psi_\epsilon(s)]^{n-j}}{\mu\,[\psi_1(s)-\psi_2(s)]}\\
&\quad-\frac{\varrho^{-j}\,[\psi_\epsilon(s)]^{n+j}}{\mu\,[\psi_1(s)-\psi_2(s)]}+\frac{\varrho^{-j}\,[\psi_2(s)]^{n+j}}{\mu\,\psi_1(s)-\beta}\bigg\},\quad j,n\in\mathbb{N}_0,
\end{aligned} \tag{B.3}$$

where $\epsilon = 1$ if $n \le j$, $\epsilon = 2$ if $n > j$ and $\beta = \mu\vartheta + \lambda(1-\vartheta)$. In order to obtain the inverse LT of (B.3), we note that

$$0 < \frac{\beta}{\mu\,\psi_1(s)} = \frac{\beta}{\lambda}\,\psi_2(s) < 1, \qquad 0 \le \vartheta \le 1, \tag{B.4}$$

so that for $j, n \in \mathbb{N}_0$ one has

$$\frac{[\psi_2(s)]^{n+j}}{\mu\,\psi_1(s) - \beta} = \frac{1}{\lambda}\sum_{k=0}^{+\infty}\Big(\frac{\beta}{\lambda}\Big)^k\,[\psi_2(s)]^{n+j+k+1}, \tag{B.5}$$

with (cf. Eq. (49), pag. 237, of Erdélyi et al. [48]):

$$[\psi_2(s)]^k = \int_0^{+\infty} e^{-s\,t}\Big[e^{-(\lambda+\mu)t}\,\frac{k}{t}\,\varrho^{k/2}\,I_k(2t\sqrt{\lambda\mu})\Big]\,dt, \qquad k \in \mathbb{N}. \tag{B.6}$$

Hence, recalling (A.2), (B.5) and (B.6), taking the inverse LT of (B.3), for $j, n \in \mathbb{N}_0$ one obtains:

$$\begin{aligned} p^*_{j,n}(t|0) &= \frac{\vartheta + (1-\vartheta)\,\varrho^{-n}}{\vartheta + (1-\vartheta)\,\varrho^{-j}} e^{-(\lambda+\mu)t}\Big\{\varrho^{(n-j)/2} I_{n-j}(2t\sqrt{\lambda\mu}) \\ &-\varrho^{(n-j)/2}\,I_{n+j}(2t\sqrt{\lambda\mu}) + \frac{\varrho^{-j}}{\beta\,t}\,\Big(\frac{\lambda}{\beta}\Big)^{n+j}\sum_{r=n+j+1}^{+\infty}\Big(\frac{\beta}{\lambda}\Big)^r r\,\varrho^{r/2}\,I_r(2t\sqrt{\lambda\mu})\Big\}. \end{aligned} \tag{B.7}$$

An alternative expression for (B.7) can be derived. Indeed, since

$$2\,r\,I_r(z) = z\big[I_{r-1}(z) - I_{r+1}(z)\big], \tag{B.8}$$

it results:

$$\begin{aligned} \frac{1}{t}\sum_{r=k}^{+\infty}\Big(\frac{\beta}{\lambda}\Big)^r r\,\varrho^{r/2}\,I_r(2t\sqrt{\lambda\mu}) &= \lambda\,\Big(\frac{\beta}{\lambda}\Big)^k\varrho^{(k-1)/2} I_{k-1}(2t\sqrt{\lambda\mu}) \\ +\beta\,\Big(\frac{\beta}{\lambda}\Big)^k\varrho^{k/2} I_k(2t\sqrt{\lambda\mu}) &+ \beta\Big(1 - \frac{\lambda\mu}{\beta^2}\Big)\sum_{r=k+1}^{+\infty}\Big(\frac{\beta}{\lambda}\Big)^r\varrho^{r/2} I_r(2t\sqrt{\lambda\mu}) \end{aligned} \tag{B.9}$$

for $k \ge 1$. Then, making use of (B.9) in (B.7), we obtain:

$$\begin{aligned} p^*_{j,n}(t|0) = \frac{\vartheta + (1-\vartheta)\,\varrho^{-n}}{\vartheta + (1-\vartheta)\,\varrho^{-j}}\,e^{-(\lambda+\mu)t}\,&\Big\{\varrho^{(n-j)/2}\,I_{n-j}\big(2\,t\sqrt{\lambda\mu}\big) \\ &+\frac{\beta}{\lambda}\,\varrho^{(n-j+1)/2}\,I_{n+j+1}\big(2\,t\sqrt{\lambda\mu}\big) + \varrho^{-j}\Big(\frac{\lambda}{\beta}\Big)^{n+j}\Big(1 - \frac{\lambda\mu}{\beta^2}\Big) \\ &\times\sum_{r=n+j+2}^{+\infty}\Big(\frac{\beta}{\lambda}\Big)^r\varrho^{r/2}\,I_r\big(2\,t\sqrt{\lambda\mu}\big)\Big\}, \qquad j, n \in \mathbb{N}_0. \end{aligned} \tag{B.10}$$

Finally, by changing t with $\Phi(t|t_0)$ in (B.10), by virtue of (B.1), we immediately obtain (44).

Appendix C. Proof of Proposition 5 for $\boldsymbol{Y(t)}$

When $\varphi(t)=1$ in (47), the symmetric BD chain $Y(t)$ is time-homogeneous; in this case, denoting by $h^*_{j,n}(t|0)$ the related transition probabilities, one has

$$h_{j,n}(t|t_0)=h^*_{j,n}[\Phi(t|t_0)|0], \qquad j,n\in\mathbb{Z}, t\geq t_0. \tag{C.1}$$

Furthermore, making use of (47) and (C.1) in (49), for the probability current of $Y(t)$ it follows:

$$\Theta_{j,n}(t|t_0)=\varphi(t)\;\Theta^*_{j,n}[\Phi(t|t_0)|0], \qquad j,n\in\mathbb{Z}, t\geq t_0, \tag{C.2}$$

where $\Theta^*_{j,n}(t|0)$ denotes the probability current of $Y(t)$ with $\varphi(t)=1$ in (47). Making use of (C.1) and (C.2) in the first of (51), one obtains:

$$\begin{aligned} h^*_{j,n}[\Phi(t|t_0)|0] &= h^*_{-j,-n}[\Phi(t|t_0)|0]=p^*_{j,n}[\Phi(t|t_0)|0]\\ &+\int_0^{\Phi(t|t_0)}\Theta^*_{j,0}(\tau|0)\;p^*_{0,n}[\Phi(t|t_0)-\tau|0]\,d\tau, \qquad j,n\in\mathbb{N}_0, t\geq t_0. \end{aligned} \tag{C.3}$$

Equations (C.1), (C.2) and (C.3) allow to derive the transition probabilities (53) by using the LT. Indeed, taking the LT of $\Theta^*_{0,0}(t|0)$, from (50) we obtain:

$$\widehat{\Theta}^*_{0,0}(s)=\frac{-1+s\,\widehat{h}^*_{0,0}(s)}{2}. \tag{C.4}$$

Hence, by setting $j=0$ and $\varphi(t)=1$ in (C.3), by virtue of (C.4) we have:

$$\begin{aligned} &\widehat{h}^*_{0,0}(s)=\frac{\widehat{p}^*_{0,0}(s)}{2-s\,\widehat{p}^*_{0,0}(s)},\\ &\widehat{h}^*_{0,n}(s)=\widehat{h}^*_{0,-n}(s)=\frac{1+s\,\widehat{h}^*_{0,0}(s)}{2}\,\widehat{p}^*_{0,n}(s)=\frac{\widehat{p}^*_{0,n}(s)}{2-s\,\widehat{p}^*_{0,0}(s)}, \quad n\in\mathbb{N}, \end{aligned} \tag{C.5}$$

with $\widehat{p}^*_{0,n}(s)$ given in (B.3). Substituting the first of (B.3) in (C.5), it results:

$$\begin{aligned} \widehat{h}^*_{0,n}(s)=\widehat{h}^*_{0,-n}(s)&=\big[\vartheta+(1-\vartheta)\varrho^{-n}\big]\;\frac{[\psi_2(s)]^n}{2\,\mu\,\psi_1(s)-2\,\beta-s}\\ &=-\,\frac{[\vartheta+(1-\vartheta)\varrho^{-n}]\;[\psi_2(s)]^{n+1}}{\mu\,\psi_2^2(s)+(\lambda-\mu)(1-2\,\vartheta)\,\psi_2(s)-\lambda}, \quad n\in\mathbb{N}_0, \end{aligned} \tag{C.6}$$

with $\beta = \mu\,\vartheta + \lambda\,(1-\vartheta)$ and where the last equality follows by virtue of (A.4). The roots of the second degree equation $\mu\, z^2 + (\lambda-\mu)(1-2\vartheta)\, z - \lambda = 0$ are z_1, z_2, given in (54), so that (C.6) becomes:

$$\widehat{h}^*_{0,n}(s) = \widehat{h}^*_{0,-n}(s) = \big[\vartheta + (1-\vartheta)\varrho^{-n}\big]\, \frac{[\psi_2(s)]^{n+1}}{\mu(z_1-z_2)} \left[\frac{1}{z_1-\psi_2(s)} + \frac{1}{\psi_2(s)-z_2}\right] \tag{C.7}$$

for $n \in \mathbb{N}_0$. Since $z_2 < z_1$, we note that

$$z_2 < 0, \quad z_1 > 0, \quad z_1 z_2 = -\varrho, \quad z_1 + z_2 = \frac{(\lambda-\mu)(2\vartheta-1)}{\mu}. \tag{C.8}$$

Therefore, by choosing s in such a way that $0 < \psi_2(s) < \min(-z_2, z_1)$, one has:

$$\widehat{h}^*_{0,n}(s) = \widehat{h}^*_{0,-n}(s) = \frac{\vartheta + (1-\vartheta)\varrho^{-n}}{\mu(z_1-z_2)}$$
$$\times \left\{ \sum_{k=0}^{+\infty} \Big(\frac{1}{z_1}\Big)^{k+1} [\psi_2(s)]^{n+k+1} - \sum_{k=0}^{+\infty} \Big(\frac{1}{z_2}\Big)^{k+1} [\psi_2(s)]^{n+k+1} \right\}, \quad n \in \mathbb{N}_0. \tag{C.9}$$

Taking the inverse LT in (C.9), by virtue of (B.6) one obtains:

$$h^*_{0,n}(t|0) = h^*_{0,-n}(t|0) = \frac{\vartheta + (1-\vartheta)\varrho^{-n}}{\mu(z_1-z_2)}\; \frac{e^{-(\lambda+\mu)t}}{t}$$
$$\times \sum_{r=n+1}^{+\infty} (-1)^{r-n-1}\, r\, \varrho^{n-r/2} \big(z_1^{r-n} - z_2^{r-n}\big)\, I_r(2\,t\sqrt{\lambda\mu}), \quad n \in \mathbb{N}_0. \tag{C.10}$$

Making use of (B.8) and (C.8) in (C.10), we have the following alternative expression:

$$h^*_{0,n}(t|0) = h^*_{0,-n}(t|0) = \Big[\vartheta + (1-\vartheta)\varrho^{-n}\Big]\, e^{-(\lambda+\mu)\,t}\, \varrho^{n/2}$$
$$\times \bigg\{ I_n\big(2\,t\sqrt{\lambda\mu}\big) + \frac{(\lambda-\mu)\,(2\vartheta-1)}{\mu(z_1-z_2)} \sum_{k=n+1}^{+\infty} (-1)^{k-n} \varrho^{(n-k)/2}$$
$$\times (z_1^{k-n} - z_2^{k-n})\, I_k\big(2\,t\sqrt{\lambda\mu}\big) \bigg\}, \qquad n \in \mathbb{N}_0. \tag{C.11}$$

Finally, by virtue of (C.1), by changing t with $\Phi(t|t_0)$ in (C.11), one obtains (53).

Appendix D. Proof of Proposition 6 for $Y(t)$

We consider the items: *(i)* $n \le 0 < j$ and *(ii)* $j, n \in \mathbb{N}$.

Case (i): $n \leq 0 < j$

Recalling (A.1) and (C.1), from the second of (51) for $t \geq t_0$ one has:

$$h^*_{j,n}[\Phi(t|t_0)|0] = \int_0^{\Phi(t|t_0)} g^*_{j,0}(\tau|0)\; h^*_{0,n}[\Phi(t|t_0) - \tau|0]\; d\tau, \quad n \leq 0 < j. \quad \text{(D.1)}$$

By setting $\varphi(t) = 1$ and taking the LT in (D.1), one has $\widehat{h}^*_{j,n}(s) = \widehat{g}^*_{j,0}(s)\; \widehat{h}^*_{0,n}(s)$, with $\widehat{g}^*_{j,0}(s)$ and $\widehat{h}^*_{0,n}(s)$ given in (A.7) and (C.6), respectively. Hence,

$$\begin{aligned}\widehat{h}^*_{j,n}(s) = \widehat{h}^*_{-j,-n}(s) &= \frac{\vartheta + (1-\vartheta)\varrho^n}{\vartheta + (1-\vartheta)\varrho^{-j}}\; \frac{\varrho^{-j}\; [\psi_2(s)]^{j-n}}{2\,\mu\,\psi_1(s) - 2\,\beta - s} \\ &= \frac{\vartheta + (1-\vartheta)\varrho^n}{\vartheta + (1-\vartheta)\varrho^{-j}}\; \frac{\varrho^{-j}\; \widehat{h}^*_{0,j-n}(s)}{\vartheta + (1-\vartheta)\varrho^{n-j}}, \qquad n \leq 0 < j. \qquad \text{(D.2)}\end{aligned}$$

Then, taking the inverse LT of (D.2), it follows

$$h^*_{j,n}(t|0) = h^*_{-j,-n}(t|0) = \frac{\vartheta + (1-\vartheta)\varrho^n}{\vartheta + (1-\vartheta)\varrho^{-j}}\; \frac{\varrho^{-j}\; h^*_{0,j-n}(t|0)}{\vartheta + (1-\vartheta)\varrho^{n-j}}, \qquad n \leq 0 < j,$$

from which, by changing t with $\Phi(t|t_0)$ and recalling (C.1), one obtains (55).

Case (ii): $j, n \in \mathbb{N}$

By virtue of (A.1) and (C.2), from the first of (52) for $t \geq t_0$ one has:

$$\Theta^*_{j,0}[\Phi(t|t_0)|0] = \int_0^{\Phi(t|t_0)} g^*_{j,0}(\tau|0)\; \Theta^*_{0,0}[\Phi(t|t_0) - \tau|0]\; d\tau, \quad j > 0. \qquad \text{(D.3)}$$

By setting $\varphi(t) = 1$ in (C.3) and in (D.3), making use of the convolution theorem one obtains:

$$\begin{aligned}\widehat{h}^*_{j,n}(s) = \widehat{h}^*_{-j,-n}(s) &= \widehat{p}^*_{j,n}(s) + \widehat{\Theta}^*_{j,0}(s)\; \widehat{p}^*_{0,n}(s) \\ &= \widehat{p}^*_{j,n}(s) + \widehat{g}^*_{j,0}(s)\; \widehat{\Theta}^*_{0,0}(s)\; \widehat{p}^*_{0,n}(s), \quad j, n \in \mathbb{N}, \qquad \text{(D.4)}\end{aligned}$$

where $\widehat{g}^*_{j,0}(s)$, $\widehat{p}^*_{j,n}(s)$, $\widehat{\Theta}^*_{0,0}(s)$ are given in (A.7), (B.3) and (C.4), respectively. From (C.4) and (C.6), for $n = 0$ it follows

$$\widehat{\Theta}^*_{0,0}(s) = \frac{-\mu\,\psi_1(s) + \beta + s}{2\mu\,\psi_1(s) - 2\beta - s} = \frac{\mu\psi_2(s) - \alpha}{2\mu\,\psi_1(s) - 2\beta - s}, \qquad \text{(D.5)}$$

with $\alpha = \lambda\,\vartheta + \mu\,(1-\vartheta)$ and $\beta = \mu\,\vartheta + \lambda\,(1-\vartheta)$. Hence, by virtue of (A.7), (B.3) and (D.5), one has:

$$\widehat{g}^*_{j,0}(s)\; \widehat{\Theta}^*_{0,0}(s)\; \widehat{p}^*_{0,n}(s) = \frac{\vartheta + (1-\vartheta)\varrho^{-n}}{\vartheta + (1-\vartheta)\varrho^{-j}}\; \frac{\mu\,\psi_2(s) - \alpha}{\mu\,\psi_1(s) - \beta}\; \frac{\varrho^{-j}\; [\psi_2(s)]^{n+j}}{2\mu\,\psi_1(s) - 2\beta - s},$$

so that, recalling (B.4), one obtains:

$$\widehat{g}^*_{j,0}(s)\,\widehat{\Theta}^*_{0,0}(s)\,\widehat{p}^*_{0,n}(s)=\frac{\vartheta+(1-\vartheta)\varrho^{-n}}{\vartheta+(1-\vartheta)\varrho^{-j}}\;\varrho^{-j}\bigg\{\frac{\mu}{\lambda}\sum_{k=0}^{+\infty}\Big(\frac{\beta}{\lambda}\Big)^k\frac{[\psi_2(s)]^{k+n+j+2}}{2\mu\,\psi_1(s)-2\beta-s}$$
$$-\frac{\alpha}{\lambda}\sum_{k=0}^{+\infty}\Big(\frac{\beta}{\lambda}\Big)^k\frac{[\psi_2(s)]^{k+n+j+1}}{2\mu\,\psi_1(s)-2\beta-s}\bigg\},\qquad j,n\in\mathbb{N}.\qquad\text{(D.6)}$$

By virtue of (C.6), from (D.6) it results::

$$\widehat{g}^*_{j,0}(s)\,\widehat{\Theta}^*_{0,0}(s)\,\widehat{p}^*_{0,n}(s)=\frac{\vartheta+(1-\vartheta)\varrho^{-n}}{\vartheta+(1-\vartheta)\varrho^{-j}}\;\varrho^{-j}\bigg\{\frac{\mu}{\beta}\sum_{k=1}^{+\infty}\Big(\frac{\beta}{\lambda}\Big)^k\frac{\widehat{h}^*_{0,k+n+j+1}(s)}{\vartheta+(1-\vartheta)\varrho^{-k-n-j-1}}$$
$$-\frac{\alpha}{\lambda}\sum_{k=0}^{+\infty}\Big(\frac{\beta}{\lambda}\Big)^k\frac{\widehat{h}^*_{0,k+n+j+1}(s)}{\vartheta+(1-\vartheta)\varrho^{-k-n-j-1}}\bigg\},\qquad j,n\in\mathbb{N}.\qquad\text{(D.7)}$$

Noting that

$$\frac{\mu}{\beta}-\frac{\alpha}{\lambda}=-\frac{\vartheta\,(1-\vartheta)\,(\lambda-\mu)^2}{\lambda\,\beta},$$

and substituting (D.7) in (D.4), one obtains:

$$\widehat{h}^*_{j,n}(s)=\widehat{h}^*_{-j,-n}(s)=\widehat{p}^*_{j,n}(s)-\frac{\vartheta+(1-\vartheta)\varrho^{-n}}{\vartheta+(1-\vartheta)\varrho^{-j}}\;\varrho^{-j}\bigg\{\frac{\alpha}{\lambda}\,\frac{\widehat{h}^*_{0,n+j+1}(s)}{\vartheta+(1-\vartheta)\varrho^{-n-j-1}}$$
$$+\frac{\vartheta\,(1-\vartheta)\,(\lambda-\mu)^2}{\lambda\,\beta}\sum_{k=1}^{+\infty}\Big(\frac{\beta}{\lambda}\Big)^k\frac{\widehat{h}^*_{0,k+n+j+1}(s)}{\vartheta+(1-\vartheta)\varrho^{-k-n-j-1}}\bigg\},\qquad j,n\in\mathbb{N},\ \text{(D.8)}$$

with $\widehat{p}^*_{j,n}(s)$ given in (B.3). Then, taking the inverse LT of (D.8), one has:

$$h^*_{j,n}(t|0)=h^*_{-j,-n}(t|0)=p^*_{j,n}(t|0)-\frac{\vartheta+(1-\vartheta)\varrho^{-n}}{\vartheta+(1-\vartheta)\varrho^{-j}}\bigg\{\frac{\alpha}{\lambda}\,\frac{\varrho^{-j}\,h^*_{0,n+j+1}(t|0)}{\vartheta+(1-\vartheta)\varrho^{-n-j-1}}$$
$$+\frac{\vartheta\,(1-\vartheta)\,(\lambda-\mu)^2}{\lambda\,\beta}\varrho^{-j}\sum_{k=1}^{+\infty}\Big(\frac{\beta}{\lambda}\Big)^k\frac{h^*_{0,k+n+j+1}(t|0)}{\vartheta+(1-\vartheta)\varrho^{-k-n-j-1}}\bigg\},\qquad j,n\in\mathbb{N},$$

from which by changing t with $\Phi(t|t_0)$ and making use of (B.1) and (C.1), finally (56) follows.

Appendix E. Proof of Proposition 7 for $Y(t)$

By setting $j=0$, it is easy to see that (57) is equivalent to (53). Furthermore, by assuming that $n\le 0<j$ or $j<0\le n$, it is immediate to observe that (57) coincides with (55). We now prove that when $j, n \in \mathbb{N}$ or $-j, -n \in \mathbb{N}$, the expression (56) coincides with (57). First, we focus on the series on the right side of (56) and we make use of (53), so that

$$\begin{aligned}
&\sum_{k=1}^{+\infty}\Big(\frac{\beta}{\lambda}\Big)^k \frac{h_{0,k+n+j+1}(t|t_0)}{\vartheta+(1-\vartheta)\varrho^{-k-n-j-1}} = e^{-(\lambda+\mu)\Phi(t|t_0)}\,\varrho^{(n+j+1)/2} \\
&\quad\times\Bigg\{\sum_{k=1}^{+\infty}\Big(\frac{\beta}{\lambda}\Big)^k \varrho^{k/2} I_{k+n+j+1}\big[2\sqrt{\lambda\mu}\,\Phi(t|t_0)\big] - \frac{(\lambda-\mu)(2\vartheta-1)}{z_1-z_2} \\
&\qquad\times\Bigg[\Big(\frac{z_1}{\beta+\mu\, z_1} - \frac{z_2}{\beta+\mu\, z_2}\Big)\sum_{k=1}^{+\infty}\Big(\frac{\beta}{\mu}\Big)^k \varrho^{-k/2} I_{k+n+j+1}\big[2\sqrt{\lambda\mu}\,\Phi(t|t_0)\big] \\
&\qquad+\beta\sum_{k=1}^{+\infty}(-1)^k\,\varrho^{-k/2}\Big(\frac{z_1^k}{\beta+\mu\, z_1} - \frac{z_2^k}{\beta+\mu\, z_2}\Big) I_{k+n+j+1}\big[2\sqrt{\lambda\mu}\,\Phi(t|t_0)\big]\Bigg]\Bigg\}.
\end{aligned} \tag{E.1}$$

with $\beta = \mu\,\vartheta + \lambda\,(1-\vartheta)$ and z_1, z_2 given in (54). Then, making use of (44), (53), and (E.1) in (56) one obtains:

$$\begin{aligned}
&h_{j,n}(t|t_0) = h_{-j,-n}(t|t_0) = \frac{\vartheta+(1-\vartheta)\varrho^{-n}}{\vartheta+(1-\vartheta)\varrho^{-j}}\; e^{-(\lambda+\mu)\,\Phi(t|t_0)} \\
&\quad\times\Bigg\{\varrho^{(n-j)/2} I_{n-j}\big[2\sqrt{\lambda\mu}\,\Phi(t|t_0)\big] + (\lambda-\mu)\,(2\vartheta-1) \\
&\qquad\times\Bigg[-\frac{\varrho^{-j}}{\beta}\,\Big(\frac{\lambda}{\beta}\Big)^{n+j} \sum_{k=n+j+1}^{+\infty}\Big(\frac{\beta}{\lambda}\Big)^k \varrho^{k/2} I_k\big[2\sqrt{\lambda\mu}\,\Phi(t|t_0)\big] \\
&\qquad+\frac{\alpha}{\lambda\mu}\,\varrho^{(n-j)/2} I_{n+j+2}\big[2\sqrt{\lambda\mu}\,\Phi(t|t_0)\big] + \frac{\vartheta(1-\vartheta)(\lambda-\mu)^2\,\varrho^{n+1}}{\lambda\beta(z_1-z_2)}\,\Big(\frac{\mu}{\beta}\Big)^{n+j+1} \\
&\qquad\times\Big(\frac{z_1}{\beta+\mu\, z_1} - \frac{z_2}{\beta+\mu\, z_2}\Big)\sum_{k=n+j+3}^{+\infty}\Big(\frac{\beta}{\mu}\Big)^k \varrho^{-k/2} I_k\big[2\sqrt{\lambda\mu}\,\Phi(t|t_0)\big] \\
&\qquad-\frac{\varrho^{(n-j+1)/2}}{\lambda\,(z_1-z_2)}\sum_{k=2}^{+\infty}(-1)^k\varrho^{-k/2}\Big(\frac{\alpha\, z_1+\lambda}{\beta+\mu\, z_1}\, z_1^k - \frac{\alpha\, z_2+\lambda}{\beta+\mu\, z_2}\, z_2^k\Big) \\
&\qquad\times I_{k+n+j+1}\big[2\sqrt{\lambda\mu}\,\Phi(t|t_0)\big]\Bigg]\Bigg\},
\end{aligned} \tag{E.2}$$

with $\alpha = \lambda\,\vartheta + \mu\,(1-\vartheta)$. Finally, noting that

$$\begin{aligned} &\frac{z_1}{\beta+\mu\,z_1} - \frac{z_2}{\beta+\mu\,z_2} = \frac{\beta\,(z_1-z_2)}{\vartheta\,(1-\vartheta)\,(\lambda-\mu)^2}, \\ &\frac{\alpha\,z_1+\lambda}{\beta+\mu\,z_1}\,z_1^k - \frac{\alpha\,z_2+\lambda}{\beta+\mu\,z_2}\,z_2^k = z_1^{k+1} - z_2^{k+1}, \end{aligned}$$

from (E.2) one obtains:

$$\begin{aligned} &h_{j,n}(t|t_0) = h_{-j,-n}(t|t_0) = \frac{\vartheta+(1-\vartheta)\varrho^{-n}}{\vartheta+(1-\vartheta)\varrho^{-j}}\; e^{-(\lambda+\mu)\,\Phi(t|t_0)} \\ &\times\Big\{\varrho^{(n-j)/2} I_{n-j}\big[2\sqrt{\lambda\mu}\;\Phi(t|t_0)\big] + \frac{(\lambda-\mu)\,(2\vartheta-1)}{\mu(z_1-z_2)} \sum_{k=n+j+1}^{+\infty} (-1)^{k-n-j}\varrho^{n-k/2} \\ &\times(z_1^{k-n-j} - z_2^{k-n-j})\; I_k\big[2\sqrt{\lambda\mu}\;\Phi(t|t_0)\big]\Big\}, \qquad j,n\in\mathbb{N}, \end{aligned} \tag{E.3}$$

that coincides with (57) for j,n integers of the same sign. In conclusion, (57) holds for all $j,n\in\mathbb{Z}$.

Acknowledgements.
This research is partially supported by MIUR - PRIN 2017, project "Stochastic Models for Complex Systems" no. 2017JFFHSH. The authors are members of the research group GNCS of INdAM. The authors thank two anonymous reviewers for their useful comments and suggestions that helped to improve this paper.

References

[1] B.W. Conolly, On randomized random walks. SIAM Rev. 13(1), 81–99 (1971)

[2] A. Di Crescenzo, B. Martinucci, On a symmetry, nonlinear birth-death process with bimodal transition probabilities. Symmetry 1, 201–214 (2009)

[3] M.O. Hongler, P.R. Parthasarathy, On a super-diffusive, non linear birth and death process. Physics Letters A 372, 3360–3362 (2008)

[4] W.E. Pruitt, Bilateral birth and death processes. Trans. Amer. Math. Soc. 107(3), 508–525 (1963)

[5] A.M.K. Tarabia, A.H. El-Baz, Transient solution of a random walk with chemical rule. Physica A 382, 430-438 (2007)

[6] J.M. Dobbie, Letter to the Editor - A doubled-ended queueing problem of Kendall. Operations Research 9(5), 755–757 (1961)

[7] O.P. Sharma, N.S.K. Nair, Transient behaviour of a double ended Markovian queue. Stoch. Anal. Appl. 9, 71–83 (1991)

[8] A.M.K. Tarabia, On the transient behaviour of a double ended Markovian queue. J. Combin. Inform. System Sci. 26, 125–134 (2001)

[9] B.W. Conolly, P.R. Parthasarathy, N. Selvaraju, Doubled-ended queues with impatience. Comput. Oper. Res. 29, 2053–2072 (2002)

[10] A. Buonocore, A. Di Crescenzo, V. Giorno, A.G. Nobile, L.M. Ricciardi, A Markov chain-based model for actomyosin dynamics. Sci. Math. Jpn. 70, 159–174 (2009)

[11] B.W. Conolly, P.R. Parthasarathy, S. Dharmaraja, A chemical queue. Math. Sci. 22, 83–91 (1997)

[12] F.W. Crawford, M.A. Suchard, Transition probabilities for general birth–death processes with applications in ecology, genetics, and evolution. J. Math. Biol. 65, 553–580 (2012)

[13] L.M. Ricciardi, Stochastic population theory: birth and death processes. In: Hallam T.G. and Levin S.A. (eds.) Mathematical Ecology, Biomathematics 17, 155–190, Springer-Verlag, Berlin Heidelberg (1986)

[14] V. Giorno, C. Negri, A.G. Nobile, A solvable model for a finite-capacity queueing system. J. Appl. Probab. 22(4), 903–911 (1985)

[15] V. Giorno, A.G. Nobile, E. Pirozzi, A state-dependent queueing system with asymptotic logarithmic distribution. J. Math. Anal. Appl. 458, 949–966 (2018)

[16] R.B. Lenin, P.R. Parthasarathy, W.R.W. Scheinhardt, E.A. van Doorn, Families of birth-death processes with similar time-dependent behaviour. J. Appl. Probab. 37, 835–849 (2000)

[17] J. Medhi, Stochastic Models in Queueing Theory, Academic Press, Amsterdam (2003)

[18] O.P. Sharma, Markovian Queues, Ellis Horwood, London (1990)

[19] M. Abundo, On the continuous diffusion approximation of some discrete Markov chain, The Open Applied Mathematics Journal 3, 7–13 (2009)

[20] S. Dharmaraja, A. Di Crescenzo, V. Giorno, A.G. Nobile, A continuous-time Ehrenfest model with catastrophes and its jump-diffusion approximation. J. Stat. Phys. 161, 326–345 (2015)

[21] A. Di Crescenzo, V. Giorno, B. Krishna Kumar, A.G. Nobile, A double-ended queue with catastrophes and repairs, and a jump-diffusion approximation. Method. Comput. Appl. Probab. 14, 937–954 (2012)

[22] A. Economou, D. Fakinos, A continuous-time Markov chain under the influence of a regulating point process and applications in stochastic models with catastrophes. Eur. J. Oper. Res. 149, 625–640 (2003)

[23] A. Economou, D. Fakinos, Alternative approaches for the transient analysis of Markov chains with catastrophes. J. Stat. Theory Pract. 2, 183–197 (2008)

[24] V. Giorno, A.G. Nobile, On a bilateral linear birth and death process in the presence of catastrophes, Lecture Notes in Computer Science 8111, 28–35, Springer-Verlag (2013)

[25] V. Giorno, A.G. Nobile, S. Spina, On some time non-homogeneous queueing systems with catastrophes. Appl. Math. Comp. 245, 220–234 (2014)

[26] S. Kapodistria, T. Phung-Duc, J. Resing, Linear birth/immigration-death process with binomial catastrophes. Prob. Eng. Inf. Sciences 30, 79-111 (2016)

[27] A. Di Crescenzo, V. Giorno, B. Krishna Kumar, A.G. Nobile, A time-non-homogeneous double-ended queue with failures and repairs and its continuous approximation. Mathematics 6, 81 (2018)

[28] A. Di Crescenzo, V. Giorno, B. Krishna Kumar, A.G. Nobile, $M/M/1$ queue in two alternating environments and its heavy traffic approximation. J. Math. Anal. Appl. 458, 973–1001 (2018)

[29] A. Di Crescenzo, V. Giorno, A.G. Nobile, Constructing transient birth-death processes by means of suitable transformations. Appl. Math. Comp. 281, 152–171 (2016)

[30] A. Di Crescenzo, B. Martinucci, A review on symmetry properties of birth-death processes. Lect. Notes Semin. Interdiscip. Mat. 12, 81–96 (2015)

[31] V. Giorno, A.G. Nobile, First-passage times and related moments for continuous-time birth-death chains. Ric. Mat. 68, 629–659 (2019)

[32] R.B. Lenin, P.R. Parthasarathy, A birth-death process suggested by a chain sequence. Comput. Math. Appl. 40, 239–247 (2000)

[33] A. Poskroblo, E. Girejko, Families of ν-similar birth-death processes and limiting conditional distributions. Advances in Difference Equations 251, 1–13 (2015)

[34] V. Giorno, A.G. Nobile, L.M. Ricciardi, On some time-nonhomogeneous diffusion approximations to queueing systems. Adv. Appl. Prob. 19, 974–994 (1987)

[35] S. M. Giveen, A taxicab problem with time-dependent arrival rates. SIAM Review 5(2), 119–127 (1963)

[36] A. Di Crescenzo, A.G. Nobile, Diffusion approximation to a queueing system with time dependent arrival and service rates. QUESTA 19, 41–62 (1995)

[37] Qi Zheng, Note on the non-homogeneous Prendiville process. Mathematical Biosciences 148, 1–5 (1998)

[38] A. Zeifman, Y. Satin, V. Korolev, S. Shorgin, On truncations for weakly ergodic inhomogeneous birth and death processes. Int. J. Appl. Math. Comput. Sci. 24, 503–518 (2014)

[39] F.W. Crawford, L.S.T. Ho, M.A. Suchard, Computational methods for birth-death processes. Wiley Interdiscip. Rev. Comput. Stat. 10(2), doi:10.1002/wics.1423 (2018)

[40] M. Scott, D.L. Isaacson, Proportional intensities and strong ergodicity for Markov processes. J. Appl. Prob. 20, 185–190 (1983)

[41] A.I. Zeifman, D.L. Isaacson, On strong ergodicity for nonhomogeneous continuous-time Markov chains. Stochastic Processes and their Applications 50, 263–273 (1994)

[42] J. Dong, W. Whitt, Using a birth-and-death process to estimate the steady-state distribution of a periodic queue. Naval Research Logistics 62(8), 664–685 (2015)

[43] Y. Satin, A. Zeifman, A. Kryukova, On the rate of convergence and limiting characteristics for a nonstationary queueing model. Mathematics 7, 678 (2019)

[44] W. Whitt, The steady-state distribution of the $M_t/M/\infty$ queue with sinusoidal arrival rate function. Operations Research Letters 42, 311-318 (2014)

[45] H.R. Thieme, Mathematics in Population Biology. Princeton University Press (2003)

[46] V. Giorno, A.G. Nobile, On the construction of a special class of time-inhomogeneous diffusion processes. Journal of Statistical Physics 177(2), 299–323 (2019)

[47] M. Abramowitz, I. Stegun, *Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables*; Dover Publications, Inc., New York (1972)

[48] A. Erdélyi, W. Magnus, F. Oberhettinger, F.G. Tricomi *Tables of integral transforms*. Vol. I. Based, in part, on notes left by Bateman, H. McGraw-Hill, New York (1954)