# Stress-driven two-phase integral elasticity for torsion of nano-beams

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ABSTRACT

Size-dependent structural behavior of nano-beams under torsion is investigated by two-phase integral elasticity. An effective torsional model is proposed by convexly combining the purely nonlocal integral stress-driven relation with a local phase. Unlike Eringen's strain-driven mixture, the projected model does not exhibit singular behaviors and leads to well-posed elastostatic problems in all cases of technical interest. The new theory is illustrated by studying torsional responses of cantilever and doubly-clamped nano-beams under simple loading conditions. Specifically, the integral convolution of the two-phase mixture is done by considering the special bi-exponential kernel. With this choice, the stress-driven two-phase model is shown to be equivalent to a differential problem equipped with higher-order constitutive boundary conditions. Exact solutions are established and comparisons with pertinent results obtained by the Eringen strain-driven two-phase mixture and by the strain gradient theory of elasticity are carried out. The outcomes could be useful for the design and optimization of nano-devices and provide new benchmarks for numerical analyses.

## 1. Introduction

Elastic torsion of cylindrical bars is a classical problem of the theory of elasticity [1]. Notwithstanding this, such a topic is still of interest in literature [2–5]. It is known that the theory of local elasticity is not able to capture scale phenomena in nano-structures [6,7]. Experimental evidences demonstrate that size effects become dominant by decreasing the structural scales [8,9].

Carbon nanotubes and graphene sheets are extensively employed in Nanotechnology to fabricate modern Nano-Electro-Mechanical Systems (NEMS). Design of nano-sensors and nano-actuators can be conveniently carried out by resorting to Continuum Mechanics [10–15]. Sometimes, micro-torsional tests are needed to assess constitutive parameters.

A variety of gradient theories of elasticity based on strain-driven nonlocal models are employed in literature to examine the size dependent torsional behavior of nano-structures including the Eringen differential law of elasticity [16–26], Eringen's differential law of visco-elasticity [27], strain gradient theory [28–30], modified couple stress theory [31–33], enhanced Eringen model [34], unified gradient elasticity theory [35,36] and Eringen strain-driven two-phase integral elasticity [37].

In the strain-driven purely nonlocal integral model, introduced by Eringen [38], the functional dependence of the stress tensor on the elastic strain field, as the source field, is described by an integral convolution and a smoothing positive-decaying kernel dependent on a characteristic length parameter. To study size-effects in nano-structures, the strain-driven integral convolution (equipped with the special averaging kernel proposed by Helmholtz) has been improperly and extensively assumed in literature to be equivalent to a differential relation. While in unbounded structural domains the strain-driven purely nonlocal integral law can be substituted with a suitable differential relation due to the tacit fulfillment of vanishing constitutive boundary conditions at infinity, for nonlocal structural problems involving bounded domains such constitutive boundary conditions are in contrast with equilibrium requirements [39]. The strain-driven integral model cannot thus be adopted to investigate bounded continuous nano-structures due to the fact that the corresponding elastostatic problem admits no solution in all cases of engineering interest [39–41].

The nonlocal differential formulation can lead to inconsistent results regarding flexural and torsional behaviors of nano-beams of applicative interest [39,40]. Specifically, the differential law associated with Eringen's strain-driven purely nonlocal model for torsional analysis provides local structural responses if either uniformly torsional loads

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[34] or only concentrated torques are applied [37]. Unlike Eringen's differential model, in some structural problems, the strain gradient theory of elasticity [42–44] can be used to capture scale effects.

A ground-breaking stress-driven purely nonlocal integral model has been recently proposed by G. Romano and R. Barretta [45,46] to overcome all the inconsistencies of strain-driven formulations. The stress-driven model provides a mathematically and mechanically consistent approach to study nano-structures. Such a theory has been successfully adopted to investigate free transverse and longitudinal vibrations of Bernoulli-Euler nano-beams [47,48] and nano-beams under torsion [49]. Exact solutions of stress-driven inflected functionally graded nano-beams have been recently established in Ref. [50].

An alternative constitutive strategy to the Eringen nonlocal integral theory is in considering a two-phase mixture defined by a convex combination of local and nonlocal contributions. The usual assumption in literature consists in modeling the nonlocal fraction by means of Eringen's strain-driven integral convolution [51–60].

Yet, as shown in Ref. [39], Eringen's strain-driven mixture for bounded structures is only a partial constitutive remedy to the inapplicable Eringen's strain-driven purely nonlocal model. Indeed, in the limit of a vanishing local fraction a singular behavior emerges, and consequently, the structural ill-posedness of Eringen's purely nonlocal theory is not completely eliminated.

An advantageous two-phase stress-driven constitutive mixture has been recently envisaged by Barretta et al. [61] to investigate the flexural problem of functionally graded nano-beams. Such a nonlocal approach is introduced by a convex combination of local and nonlocal phases wherein the nonlocal fraction is based on the well-posed stress-driven model [46].

Motivation of the present study is in formulating a stress-driven local/nonlocal two-phase mixture for nano-beams under torsion, extending the treatment in Ref. [49] conceived in the special context of the purely nonlocal theory of elasticity. The plan is the following. Torsional elastic model based on Eringen strain-driven local/nonlocal two-phase mixture is preliminarily recalled in Sect.2. The corresponding equivalent differential problem equipped with higher-order constitutive boundary conditions is also presented. The new stress-driven two-phase mixture model is illustrated in Sect.3. For comparison sake, the strain gradient theory of elasticity for nano-beams under torsion is formulated by making recourse to Hellinger-Reissner variational principle in Sect.4. The new stress-driven model presented in Sect.3 is adopted in Sect.5 in order to establish exact solutions of cantilever and doubly-clamped nano-beams under uniformly distributed torsional couples.

## 2. Eringen strain-driven mixture of elasticity

A circular nano-beam of length $L$ with cross-section $\Xi$, subjected to a distribution of torsional couples per unit length $m_t$ in the interval $[0, L]$ and concentrated couples $\overline{M}_t$ is considered, as schematically depicted in Fig. 1. The abscissa along the nano-beam axis will be denoted by $x$ and the pair of Cartesian axes $(y, z)$ belongs to the cross-section, originated at the center $O$.

Cartesian components of the displacement field $u_x, u_y, u_z$, up to an inessential rigid body motion, of a nano-beam under torsion are given by

$$u_x = 0, \quad u_y = -\theta(x)z, \quad u_z = \theta(x)y \tag{1}$$

where $\theta$ denotes the cross-sectional torsional rotation function. Shear stress vector $\boldsymbol{\tau}$, shear strain vector $\boldsymbol{\gamma}$, position vector $\mathbf{r}$ and rotation tensor $\mathbf{R}$, respectively, write as

$$\boldsymbol{\tau} = \begin{bmatrix} \tau_{yx} \\ \tau_{zx} \end{bmatrix}, \quad \boldsymbol{\gamma} = \begin{bmatrix} \gamma_{yx} \\ \gamma_{zx} \end{bmatrix}, \quad \mathbf{r} = \begin{bmatrix} y \\ z \end{bmatrix}, \quad \mathbf{R} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \tag{2}$$

where the vector $\mathbf{Rr}$ is the $\pi/2$ counterclockwise rotated of the position vector $\mathbf{r}$.

The ensuing non-vanishing kinematically compatible strain field is expressed by

$$\boldsymbol{\gamma} = \mathbf{Rr}\chi_t \tag{3}$$

where $\chi_t = d\theta/dx$ is the torsional curvature.

Differential and classical boundary conditions of equilibrium write as [62].

$$\begin{aligned} &\frac{\partial M_t}{\partial x} + m_t = 0 \\ &(M_t + \overline{M}_t)\delta\theta|_{x=0} = (M_t - \overline{M}_t)\delta\theta|_{x=L} = 0 \end{aligned} \tag{4}$$

with the twisting moment $M_t$ provided by

$$M_t = \iint_{\Xi} \boldsymbol{\tau} \cdot \mathbf{Rr} dA \tag{5}$$

The symbol "•" stands for inner product between vectors and $J$ is the polar moment of inertia about the center of the circular cross-section

$$J = \iint_{\Xi} \mathbf{r} \cdot \mathbf{r} dA \tag{6}$$

Eringen's strain-driven integral mixture is introduced as a convex combination of local and nonlocal phases [63,64]. According to this model, the torsional nonlocal elastic law for nano-beams provides the twisting moment $M_t$ in terms of the torsional curvature $\chi_t$ and the elastic torsional stiffness $K_t$ as

$$M_t(x) = \alpha K_t \chi_t(x) + (1-\alpha)\int_0^L \varphi(x-\zeta, l_c) K_t \chi_t(\zeta) d\zeta \tag{7}$$

where $0 \le \alpha \le 1$ represents the phase parameter. The purely nonlocal constitutive model can be recovered by setting $\alpha = 0$ and the local law corresponds to $\alpha = 1$.

The scalar averaging kernel $\varphi$ is a space weight function fulfilling symmetry, positivity and limit impulsivity properties [45,46]. Furthermore, the elastic torsional stiffness is given by

$$K_t = \iint_{\Xi} \mu(\mathbf{r} \cdot \mathbf{r}) dA \tag{8}$$

with $\mu$ shear modulus. The special bi-exponential law, proposed by

FIG. 1


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Helmholtz, is a well-known choice for the averaging kernel $\varphi$ [39].

$$\varphi(x, l_c) = \frac{1}{2l_c}\exp\left(-\frac{|x|}{l_c}\right) \tag{9}$$

where $l_c = e_0 L$ is the characteristic length, expressing the amplitude of the range of nonlocal action, along with $e_0$ being a material constant. It can be shown that the aforementioned Eringen's strain-driven mixture model Eq. (7) employing the bi-exponential kernel Eq. (9) is equivalent to the subsequent mixture Eringen differential model [39,65],

$$\frac{1}{l_c^2}M_t(x) - \frac{d^2M_t(x)}{dx^2} = \frac{1}{l_c^2}K_t\chi_t(x) - \alpha K_t\frac{d^2\chi_t(x)}{dx^2} \tag{10}$$

equipped with the homogeneous constitutive boundary conditions, at $x = 0$ and $x = L$, as

$$\begin{aligned}&\tfrac{dM_t(0)}{dx} - \tfrac{1}{l_c}M_t(0) = \alpha K_t\tfrac{d\chi_t(0)}{dx} - \tfrac{1}{l_c}\alpha K_t\chi_t(0)\\&\tfrac{dM_t(L)}{dx} + \tfrac{1}{l_c}M_t(L) = \alpha K_t\tfrac{d\chi_t(L)}{dx} + \tfrac{1}{l_c}\alpha K_t\chi_t(L)\end{aligned} \tag{11}$$

Eringen's purely nonlocal strain-driven integral model and the equivalent differential and constitutive boundary conditions, are respectively recovered from Eq. (7) and Eqs. (10) and (11) setting the phase parameter $\alpha = 0$. Nevertheless, since the elastostatic problem associated with such a model admits no solution for bounded continuous nano-beams, consequently, the Eringen strain-driven mixture model leads to an ill-posed structural problem as the phase parameter $\alpha \to 0$ [40,46].

## 3. Stress-driven mixture of elasticity

Founded on the innovative nonlocal stress-driven integral law introduced by G. Romano and R. Barretta [45,46], an effective stress-driven integral mixture has been recently proposed by Barretta et al. [61] for inflected Bernoulli-Euler nano-beams. Analogously, torsion of nano-beams can be studied by assuming that the torsional curvature $\chi_t$ is provided by the following stress-driven two-phase law

$$\chi_t(x) = \alpha\frac{M_t(x)}{K_t} + (1-\alpha)\int_0^L \varphi(x-\zeta, l_c)\frac{M_t(\zeta)}{K_t}d\zeta \tag{12}$$

where the twisting moment $M_t$ fulfils the equilibrium condition Eq. (4) and $0 \le \alpha \le 1$ is the mixture parameter. As the stress-driven purely nonlocal integral model corresponds to $\alpha = 0$, the local constitutive law can be recovered by setting $\alpha = 1$. Employing the Helmholtz bi-exponential kernel Eq. (9), it may be shown that the abovementioned stress-driven mixture is equivalent to the subsequent second-order differential equation

$$\frac{1}{l_c^2}\chi_t(x) - \frac{d^2\chi_t(x)}{dx^2} = \frac{1}{l_c^2}\frac{M_t(x)}{K_t} - \alpha\frac{d^2}{dx^2}\frac{M_t(x)}{K_t} \tag{13}$$

supplemented with the constitutive boundary conditions, at $x = 0$ and $x = L$, as

$$\begin{aligned}&\frac{d\chi_t(0)}{dx} - \frac{1}{l_c}\chi_t(0) = \frac{\alpha}{K_t}\frac{dM_t(0)}{dx} - \frac{\alpha}{l_c}\frac{M_t(0)}{K_t}\\&\frac{d\chi_t(L)}{dx} + \frac{1}{l_c}\chi_t(L) = \frac{\alpha}{K_t}\frac{dM_t(L)}{dx} + \frac{\alpha}{l_c}\frac{M_t(L)}{K_t}\end{aligned} \tag{14}$$

While the Eringen strain-driven mixture exhibits a singular behavior when the local part vanishes [39], the proposed stress-driven mixture Eq. (12) does not lead to inconsistencies.

## 4. Strain gradient model of elasticity

To compare the results of the two-phase constitutive integral models with the counterpart results of the classical strain gradient theory of elasticity, size-dependent torsion of nano-beams is formulated via Hellinger-Reissner variational principle [66].

Hellinger-Reissner's functional $\mathfrak{R}$ is introduced as

$$\mathfrak{R} = \int_0^L \left[M_t^{(0)}\frac{\partial\theta}{\partial x} + M_t^{(1)}\frac{\partial^2\theta}{\partial x^2} - m_t\theta - \frac{1}{2K_t}\left(\left(M_t^{(0)}\right)^2 + \frac{1}{l_s^2}\left(M_t^{(1)}\right)^2\right)\right]dx - \overline{M}_t\theta|_0^L \tag{15}$$

where the fields $M_t^{(0)}$, $M_t^{(1)}$ are defined as the dual mathematical objects of the elastic torsional curvature and of its derivative $\chi_t^{(0)} = \partial\theta/\partial x$, $\chi_t^{(1)} = \partial^2\theta/\partial x^2$, respectively. The nano-beam is also assumed to be subjected to distributed torsional couples $m_t$, and in addition, $l_s$ is a length-scale parameter.

While the torsional rotation of cross-sections $\theta$ and the fields $M_t^{(0)}$, $M_t^{(1)}$ are selected as the primary variables subject to variation, performing the first-order variation of the Hellinger-Reissner functional, followed by integration by parts, we get

$$\begin{aligned}\delta\mathfrak{R} = \int_0^L &\left[\delta M_t^{(0)}\left(\frac{\partial\theta}{\partial x} - \frac{M_t^{(0)}}{K_t}\right) + \delta M_t^{(1)}\left(\frac{\partial^2\theta}{\partial x^2} - \frac{M_t^{(1)}}{K_t l_s^2}\right)\right.\\&\left.-\left(\frac{\partial M_t^{(0)}}{\partial x} - \frac{\partial^2 M_t^{(1)}}{\partial x^2} + m_t\right)\delta\theta\right]dx + \left(M_t^{(0)} - \frac{\partial M_t^{(1)}}{\partial x} - \overline{M}_t\right)\delta\theta\Big|_0^L\\&+ M_t^{(1)}\delta\frac{\partial\theta}{\partial x}\Big|_0^L\end{aligned} \tag{16}$$

Imposing stationarity of the Hellinger-Reissner functional, differential and boundary conditions of strain gradient nano-beams under torsion take the form

$$\begin{aligned}&\tfrac{\partial M_t^{(0)}}{\partial x} - \tfrac{\partial^2 M_t^{(1)}}{\partial x^2} + m_t = 0\\&\left(M_t^{(0)} - \tfrac{\partial M_t^{(1)}}{\partial x} + \overline{M}_t\right)\delta\theta\Big|_{x=0} = \left(M_t^{(0)} - \tfrac{\partial M_t^{(1)}}{\partial x} - \overline{M}_t\right)\delta\theta\Big|_{x=L} = 0\\&\left(M_t^{(1)}\delta\tfrac{\partial\theta}{\partial x}\right)\Big|_{x=0} = \left(M_t^{(1)}\delta\tfrac{\partial\theta}{\partial x}\right)\Big|_{x=L} = 0\end{aligned} \tag{17}$$

The governing equations can be further simplified introducing the twisting moment field by $M_t = M_t^{(0)} - \partial M_t^{(1)}/\partial x$ and assuming arbitrary variations of torsional curvatures

$$\begin{aligned}&\tfrac{\partial M_t}{\partial x} + m_t = 0\\&(M_t + \overline{M}_t)\delta\theta|_{x=0} = (M_t - \overline{M}_t)\delta\theta|_{x=L} = 0\\&M_t^{(1)}(0) = M_t^{(1)}(L) = 0\end{aligned} \tag{18}$$

As it is expected from the Hellinger-Reissner stationary variational principle, the desired constitutive relations cast as ordinary differential equations and the static fields $M_t$, $M_t^{(0)}$, $M_t^{(1)}$, respectively, are given by

$$M_t^{(0)} = K_t\frac{\partial\theta}{\partial x},\quad M_t^{(1)} = K_t l_s^2\frac{\partial^2\theta}{\partial x^2},\quad M_t = K_t\left(1 - l_s^2\frac{\partial^2}{\partial x^2}\right)\frac{\partial\theta}{\partial x} \tag{19}$$

In the present study the stress-driven mixture model is utilized to derive exact solutions for torsional analysis of nano-beams and the results are then compared with the Eringen strain-driven mixture model as well as the classical strain gradient theory of elasticity.

## 5. Case studies, exact torsional solutions and comparisons

The size-dependent nonlocal models illustrated in the previous sections are here adopted to examine the torsional behavior of cantilever and doubly-clamped nano-beams of length $L$ subjected to a uniform distribution $\overline{m}_t$ of couples per unit length. Hereafter, the acronyms CC and CF stand for clamp-clamp and clamp-free kinematic boundary conditions.


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The acronyms MEIM, MSDM and SGT denote the mixture Eringen strain-driven integral model, the mixture stress-driven integral model and the strain gradient theory of elasticity.

The differential conditions of equilibrium, Eq. (4)$_1$ or Eq. (18)$_1$, can be integrated to get an explicit expression of the twisting moment $M_t$. Accordingly, for a uniform distribution $\overline{m}_t$ of couples, twisting moment is determined in terms of an integration constant $\Upsilon_1$ as

$$M_t(x) = -\overline{m}_t x + \Upsilon_1 \tag{20}$$

As a result of substituting Eq. (20) in Eringen's differential mixture model Eq. (10) and employing the differential condition of kinematic compatibility, the governing equation for elastic torsional rotation function writes as

$$\frac{K_t}{l_c^2}\frac{d\theta(x)}{dx} - \alpha K_t \frac{d^3\theta(x)}{dx^3} = \frac{1}{l_c^2}(-\overline{m}_t x + \Upsilon_1) \tag{21}$$

equipped with the constitutive boundary conditions Eq. (11)

$$\begin{aligned} &-\overline{m}_t - \tfrac{1}{l_c}\Upsilon_1 = \alpha K_t \tfrac{d^2\theta}{dx^2}(0) - \tfrac{1}{l_c}\alpha K_t \tfrac{d\theta}{dx}(0) \\ &-\overline{m}_t + \tfrac{1}{l_c}(-\overline{m}_t L + \Upsilon_1) = \alpha K_t \tfrac{d^2\theta}{dx^2}(L) + \tfrac{1}{l_c}\alpha K_t \tfrac{d\theta}{dx}(L) \end{aligned} \tag{22}$$

The general solution of Eq. (21) is expressed by

$$\begin{aligned} \theta_{\mathrm{MEIM}}(x) = {} & \Upsilon_4 + l_c\sqrt{\alpha}\exp\left(-\frac{x}{l_c\sqrt{\alpha}}\right)\left(-\Upsilon_2 + \Upsilon_3\exp\left(\frac{2x}{l_c\sqrt{\alpha}}\right)\right) \\ & + \frac{1}{K_t}\left(\Upsilon_1 x - \frac{\overline{m}_t}{2}x^2\right) \end{aligned} \tag{23}$$

where the 4 unknown constants ($\Upsilon_1$, $\Upsilon_2$, $\Upsilon_3$, $\Upsilon_4$) can be determined applying the two constitutive boundary conditions Eq. (22) and the two classical boundary conditions given in Eq. (4)$_2$ or Eq. (18)$_2$. The kinematic boundary conditions for doubly-clamped and cantilever nano-beams are expressed by $\theta(0) = \theta(L) = 0$ and $\theta(0) = M_t(L) = 0$, respectively. Therefore, the elastic torsional rotations according to Eringen's strain-driven mixture for doubly-clamped and cantilever nano-beams are given by

$$\begin{aligned} \theta^{\mathrm{CC}}_{\mathrm{MEIM}}(x) = {} & \frac{\overline{m}_t}{2K_t\left(\cosh\frac{L}{2l_c\sqrt{\alpha}} + \sqrt{\alpha}\sinh\frac{L}{2l_c\sqrt{\alpha}}\right)}\Bigg[(L-x)x\Bigg(\cosh\frac{L}{2l_c\sqrt{\alpha}} \\ & + \sqrt{\alpha}\sinh\frac{L}{2l_c\sqrt{\alpha}}\Bigg) \\ & + (\alpha-1)l_c(L+2l_c)\left(-\cosh\frac{L}{2l_c\sqrt{\alpha}} + \cosh\frac{L-2x}{2l_c\sqrt{\alpha}}\right)\Bigg] \\ \theta^{\mathrm{CF}}_{\mathrm{MEIM}}(x) = {} & \frac{\overline{m}_t}{2K_t\left(2\sqrt{\alpha}\cosh\frac{L}{l_c\sqrt{\alpha}} + (\alpha+1)\sinh\frac{L}{l_c\sqrt{\alpha}}\right)}\Bigg[(2L-x)x\Bigg(2\sqrt{\alpha}\cosh\frac{L}{l_c\sqrt{\alpha}} \\ & + (\alpha+1)\sinh\frac{L}{l_c\sqrt{\alpha}}\Bigg) \\ & + 4(\alpha-1)l_c\sinh\frac{x}{2l_c\sqrt{\alpha}}\Bigg(-L\left(\cosh\frac{2L-x}{2l_c\sqrt{\alpha}} + \sqrt{\alpha}\sinh\frac{2L-x}{2l_c\sqrt{\alpha}}\right) \\ & -2l_c\sinh\frac{L-x}{2l_c\sqrt{\alpha}}\left(\sqrt{\alpha}\cosh\frac{L}{2l_c\sqrt{\alpha}} + \sinh\frac{L}{2l_c\sqrt{\alpha}}\right)\Bigg)\Bigg] \end{aligned} \tag{24}$$

Introducing the expression of the twisting moment Eq. (20) in the stress-driven integral mixture model and employing the differential condition of kinematic compatibility, the governing equation on the elastic torsional rotation is given by

$$\frac{1}{l_c^2}\frac{d\theta(x)}{dx} - \frac{d^3\theta(x)}{dx^3} = \frac{-\overline{m}_t x + \Upsilon_1}{l_c^2 K_t} \tag{25}$$

equipped with the constitutive boundary conditions Eq. (14)

$$\begin{aligned} &\tfrac{d^2\theta}{dx^2}(0) - \tfrac{1}{l_c}\tfrac{d\theta}{dx}(0) = -\tfrac{\alpha\overline{m}_t}{K_t} - \tfrac{\alpha\Upsilon_1}{l_c K_t} \\ &\tfrac{d^2\theta}{dx^2}(L) + \tfrac{1}{l_c}\tfrac{d\theta}{dx}(L) = -\tfrac{\alpha\overline{m}_t}{K_t} + \tfrac{\alpha}{l_c K_t}(\Upsilon_1 - L\overline{m}_t) \end{aligned} \tag{26}$$

The general solution to Eq. (25) is given by

$$\theta_{\mathrm{MSDM}}(x) = \Upsilon_4 + l_c\left(-\Upsilon_2\exp\frac{-x}{l_c} + \Upsilon_3\exp\frac{x}{l_c}\right) + \frac{1}{K_t}\left(\Upsilon_1 x - \frac{\overline{m}_t}{2}x^2\right) \tag{27}$$

Once more, the unknown constants ($\Upsilon_1$, $\Upsilon_2$, $\Upsilon_3$, $\Upsilon_4$) can be determined applying the two constitutive boundary conditions Eq. (26) and the two classical boundary conditions given in Eq. (4)$_2$ or Eq. (18)$_2$. Applying the classical boundary conditions for doubly-clamped and cantilever nano-beams, the elastic torsional rotations associated with the stress-driven mixture for CC and CF are provided by

$$\begin{aligned} \theta^{\mathrm{CC}}_{\mathrm{MSDM}}(x) = {} & \tfrac{\overline{m}_t}{4K_t}\Bigg[2(L-x)x + (L+2l_c)l_c(\alpha-1)\left(\exp\tfrac{-x}{l_c} - \exp\tfrac{-L}{l_c}\right)\Bigg(-1 \\ & + \exp\tfrac{x}{l_c}\Bigg)\Bigg] \\ \theta^{\mathrm{CF}}_{\mathrm{MSDM}}(x) = {} & \tfrac{\overline{m}_t}{2K_t}\Bigg[2(L-x)x + l_c(\alpha-1)\left(-l_c\exp\tfrac{-L}{l_c} + (L+l_c)\exp\tfrac{-x}{l_c}\right)\Bigg( \\ & -1 + \exp\tfrac{x}{l_c}\Bigg)\Bigg] \end{aligned} \tag{28}$$

As expected, the elastic torsional rotations associated with the stress–driven purely nonlocal integral model can be recovered by setting $\alpha = 0$ as

$$\begin{aligned} \theta^{\mathrm{CC}}_{\mathrm{SDM}}(x) &= \tfrac{\overline{m}_t}{4K_t}\left[2(L-x)x + (L+2l_c)l_c\left(-\exp\tfrac{-x}{l_c} + \exp\tfrac{-L}{l_c}\right)\left(-1 + \exp\tfrac{x}{l_c}\right)\right] \\ \theta^{\mathrm{CF}}_{\mathrm{SDM}}(x) &= \tfrac{\overline{m}_t}{2K_t}\left[2(L-x)x + l_c\left(l_c\exp\tfrac{-L}{l_c} - (L+l_c)\exp\tfrac{-x}{l_c}\right)\left(-1 + \exp\tfrac{x}{l_c}\right)\right] \end{aligned} \tag{29}$$

where the elastic torsional rotations of doubly-clamped nano-beam corresponding to the stress–driven purely nonlocal integral model $\theta^{\mathrm{CC}}_{\mathrm{SDM}}$ are in full agreement with the results introduced by Barretta et al. [49].

Finally, substitution of the twisting moment Eq. (20) in the constitutive relation of the total, twisting moment field Eq. (19) leads to the governing equation on the torsional rotation associated with the classical strain gradient theory of elasticity as

$$K_t\left(1 - l_s^2\frac{d^2}{dx^2}\right)\frac{d\theta}{dx} = -\overline{m}_t x + \Upsilon_1 \tag{30}$$

equipped with the higher-order natural boundary conditions Eq. (18)$_3$

$$K_t l_s^2\frac{d^2\theta}{dx^2}(0) = K_t l_s^2\frac{d^2\theta}{dx^2}(L) = 0 \tag{31}$$

The general solution of Eq. (30) may be written as

$$\theta_{\mathrm{SGT}}(x) = \Upsilon_4 + l_s\left(-\Upsilon_2\exp\frac{-x}{l_s} + \Upsilon_3\exp\frac{x}{l_s}\right) + \frac{1}{K_t}\left(\Upsilon_1 x - \frac{\overline{m}_t}{2}x^2\right) \tag{32}$$

Due to applying a uniform distribution $\overline{m}_t$ of couples, the general solution of the torsional rotation of nano-beams for both the stress-driven mixture and the strain gradient theory of elasticity are the same provided that the gradient length-scale parameter $l_s$ is substituted with the characteristic length $l_c$. Nevertheless, such an assumption is only feasible for positive values of the characteristic length $l_c$. Indeed, while


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FIG. 2

FIG. 3

in the classical strain gradient theory it is possible to set the gradient length-scale parameter $l_s = 0$, the stress-driven mixture, equipped with the bi-exponential kernel Eq. (9), is not defined when the characteristic length is vanishing. Only a limit evaluation as $l_c$ tends to zero is permissible.

Yet again, the unknown constants $(\Upsilon_1, \Upsilon_2, \Upsilon_3, \Upsilon_4)$ can be determined imposing the two higher-order natural boundary conditions Eq. (31) and the two classical boundary conditions given in Eq. $(4)_2$ or Eq. $(18)_2$. Imposing the kinematic boundary conditions for doubly clamped and cantilever nano-beams, the elastic torsional rotations corresponding to the strain gradient theory of elasticity are expressed by

$$\begin{aligned} \theta_{SGT}^{CC}(x) &= \frac{m_t}{2K_t}\left[(L-x)x + 2l_s^2\left(-1+\cosh\frac{L-2x}{2l_s}\operatorname{sech}\frac{L}{2l_s}\right)\right] \\ \theta_{SGT}^{CF}(x) &= \frac{m_t}{2K_t}\left[(2L-x)x + 2l_s^2\left(-1+\cosh\frac{L-2x}{2l_s}\operatorname{sech}\frac{L}{2l_s}\right)\right] \end{aligned} \tag{33}$$

where the elastic torsional rotations of doubly-clamped nano-beam associated with the strain gradient theory of elasticity $\theta_{SGT}^{CC}$ are in full agreement with the results established in Ref. [49].

The following dimensionless characteristic parameter $\lambda = l/L$ is adopted here, in which the characteristic parameters of strain gradient theory and of nonlocal integral mixtures are assumed to be coincident and denoted by $l$. Such an assumption is only feasible for positive values of the characteristic length, due to the fact that nonlocal integral models are characterized by a positive small-scale parameter.

Also, the dimensionless torsional rotation $\bar{\theta}$ is introduced in all of the illustrative results

$$\bar{\theta}(x) = \frac{K_t}{\overline{m}_t L^2}\theta(x) \tag{34}$$

Plots of the dimensionless maximum torsional rotations $\bar{\theta}_{max}$ corresponding to the mixture Eringen strain-driven integral model, mixture stress-driven integral model and strain gradient theory of elasticity for doubly-clamped and cantilever nano-beams are given in Figs. 2 and 3. As the characteristic parameter $\lambda$ is ranging in the interval $]0,0.1[$, the phase parameter $\alpha$ is assumed to range in the set $\{0.01, 0.1, 0.5, 1\}$. It is noticeably inferred from Figs. 2 and 3 that, for a given value of the phase parameter $\alpha$, while both the mixture stress-driven integral model and strain gradient theory of elasticity exhibit a stiffening behavior on rotation solutions in terms of the dimensionless characteristic parameter $\lambda$, the mixture Eringen strain-driven integral model reveals a softening behavior. Nevertheless, the stiffening effect of the mixture stress-driven integral model is more noticeable in comparison with the one of the strain gradient theory of elasticity. Additionally, while the phase parameter $\alpha$ has the effect of stiffening the structural response obtained by Eringen's strain-driven mixture model, the stress-driven model mixture reveals a softening behavior in terms of the phase parameter. Both mixture models and strain gradient theory provide the local elastic solution for vanishing dimensionless characteristic parameter $\lambda \to 0$, no matter what the value of the phase parameter $\alpha$ is. However, discrepancy between the torsional results obtained by the size-dependent models is enhanced by increasing the characteristic parameter $\lambda$. As the phase parameter $\alpha \to 1$, both mixture nonlocal integral models coincide with the local model independently of the value of $\lambda$. The stress-driven mixture tends to the corresponding stress-driven purely nonlocal integral model for a vanishing phase parameter


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**Table 1**
Dimensionless maximum torsional rotations $\bar{\theta}_{max}$ of doubly clamped nano-beam versus the nonlocal parameter $\lambda$.

| $\bar{\theta}_{max}$ | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| $\lambda$ | MEIM | | | | MSDM | | | | SGT |
| | $\alpha = 0.01$ | $\alpha = 0.1$ | $\alpha = 0.5$ | $\alpha = 1$ | $\alpha = 0.01$ | $\alpha = 0.1$ | $\alpha = 0.5$ | $\alpha = 1$ | |
| $0^+$ | 0.125 | 0.125 | 0.125 | 0.125 | 0.125 | 0.125 | 0.125 | 0.125 | 0.125 |
| 0.01 | 0.12959 | 0.128487 | 0.126494 | 0.125 | 0.122476 | 0.122705 | 0.123725 | 0.125 | 0.1249 |
| 0.02 | 0.13436 | 0.132111 | 0.128046 | 0.125 | 0.119852 | 0.12032 | 0.1224 | 0.125 | 0.1246 |
| 0.03 | 0.13931 | 0.135872 | 0.129657 | 0.125 | 0.11713 | 0.117845 | 0.121025 | 0.125 | 0.1241 |
| 0.04 | 0.14444 | 0.139769 | 0.131326 | 0.125 | 0.114308 | 0.11528 | 0.1196 | 0.125 | 0.1234 |
| 0.05 | 0.14975 | 0.143804 | 0.133055 | 0.125 | 0.111389 | 0.112626 | 0.118126 | 0.125 | 0.1225 |
| 0.06 | 0.15524 | 0.147975 | 0.134841 | 0.125 | 0.108376 | 0.109887 | 0.116604 | 0.125 | 0.121402 |
| 0.07 | 0.16091 | 0.152283 | 0.136685 | 0.125 | 0.105281 | 0.107073 | 0.115041 | 0.125 | 0.120108 |
| 0.08 | 0.16676 | 0.156727 | 0.138586 | 0.125 | 0.102121 | 0.104201 | 0.113445 | 0.125 | 0.118625 |
| 0.09 | 0.17279 | 0.161308 | 0.140541 | 0.125 | 0.098918 | 0.101289 | 0.111827 | 0.125 | 0.116963 |
| 0.1 | 0.179 | 0.166026 | 0.142544 | 0.125 | 0.095699 | 0.098363 | 0.110201 | 0.125 | 0.115135 |

**Table 2**
Dimensionless maximum torsional rotations $\bar{\theta}_{max}$ of cantilever nano-beam versus the nonlocal parameter $\lambda$.

| $\bar{\theta}_{max}$ | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| $\lambda$ | MEIM | | | | MSDM | | | | SGT |
| | $\alpha = 0.01$ | $\alpha = 0.1$ | $\alpha = 0.5$ | $\alpha = 1$ | $\alpha = 0.01$ | $\alpha = 0.1$ | $\alpha = 0.5$ | $\alpha = 1$ | |
| $0^+$ | 0.5 | 0.5 | 0.5 | 0.5 | 0.5 | 0.5 | 0.5 | 0.5 | 0.5 |
| 0.01 | 0.509 | 0.506838 | 0.502929 | 0.5 | 0.49505 | 0.4955 | 0.4975 | 0.5 | 0.5 |
| 0.02 | 0.518 | 0.513675 | 0.505858 | 0.5 | 0.4901 | 0.491 | 0.495 | 0.5 | 0.5 |
| 0.03 | 0.527 | 0.520513 | 0.508787 | 0.5 | 0.48515 | 0.4865 | 0.4925 | 0.5 | 0.5 |
| 0.04 | 0.536 | 0.527351 | 0.511716 | 0.5 | 0.4802 | 0.482 | 0.49 | 0.5 | 0.5 |
| 0.05 | 0.545 | 0.534189 | 0.514645 | 0.5 | 0.47525 | 0.4775 | 0.4875 | 0.5 | 0.5 |
| 0.06 | 0.554 | 0.541026 | 0.517574 | 0.5 | 0.4703 | 0.473 | 0.485 | 0.5 | 0.5 |
| 0.07 | 0.563 | 0.547864 | 0.520503 | 0.5 | 0.46535 | 0.4685 | 0.4825 | 0.5 | 0.5 |
| 0.08 | 0.572 | 0.554702 | 0.523431 | 0.5 | 0.4604 | 0.464 | 0.48 | 0.5 | 0.5 |
| 0.09 | 0.581 | 0.56154 | 0.52636 | 0.5 | 0.455451 | 0.459501 | 0.4775 | 0.5 | 0.5 |
| 0.1 | 0.59 | 0.568377 | 0.529289 | 0.5 | 0.450502 | 0.455002 | 0.475001 | 0.5 | 0.5 |

$\alpha \to 0$ [45,46]. On the contrary, the parameter $\alpha$ cannot vanish in Eringen's strain-driven mixture since the corresponding Eringen's strain-driven purely nonlocal law leads to ill-posed structural problems [40]. Analogous size-dependent phenomena have been also observed for flexural analysis of Bernoulli-Euler nano-beams based on the stress-driven mixture model [61]. As it can be interestingly deduced from Fig. 3, for cantilever nanobeams, the maximum torsional rotations $\bar{\theta}_{max}$ based on the strain gradient theory of elasticity are independent of the characteristic parameter $\lambda$ and coincide with the ones of the local theory. While the demonstrated results in Fig. 3 reveal some drawbacks of the strain gradient theory, such drawbacks are absent in the results associated with the stress-driven integral mixture model. The numerical values of dimensionless maximum torsional rotations $\bar{\theta}_{max}$ evaluated by Eringen strain-driven mixture model, stress-driven mixture theory and strain gradient theory for doubly-clamped and cantilever nano-beams are tabulated in Tables 1 and 2.

## 6. Conclusions

In the present study, torsional static behavior of elastic nano-beams has been investigated utilizing two-phase constitutive mixtures, defined by convex combination of local and nonlocal inputs, associated with the strain-driven and stress-driven purely nonlocal laws. Firstly, the classical strain-driven mixture (MEIM) by Eringen [63] has been recalled. Secondly, the stress-driven purely nonlocal law [46] has been exploited to conceive an innovative mixture model (MSDM) for nano-beams under torsion. Lastly, for comparison sake, the strain gradient model (SGT) has been formulated by Hellinger-Reissner principle. Exact torsional rotation solutions of doubly-clamped and cantilever nano-beams, corresponding to MEIM, MSDM and SGT, have been also established.

The main outcomes of the present study can be enumerated as follows.

- Elastic torsional rotations, evaluated both by the stress-driven two-phase model and the strain gradient theory, expose a stiffening behavior with respect to the scale parameter. On the contrary, Eringen's strain-driven mixture model provides an elastic torsional rotation increasing with the scale parameter.
- The stiffening effect of the stress-driven mixture model is more noticeable in comparison with the one of the strain gradient theory of elasticity.
- The mixture parameter has the effect of decreasing the elastic torsional rotations obtained by Eringen's strain-driven two-phase model.
- The stress-driven two-phase model leads to elastic torsional rotations which increase with the mixture parameter.
- Both the two-phase models and the strain gradient theory provide the local response as the nonlocal parameter tends to zero, independently of the mixture parameter.
- As the mixture parameter tends to 1, both two-phase models provide local structural responses independently of the nonlocal parameter.
- The stress-driven two-phase model tends to the corresponding well-posed stress-driven purely nonlocal model for a vanishing mixture parameter. On the contrary, for bounded beams, the mixture parameter cannot vanish in Eringen's strain-driven two-phase law since


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the ensuing purely nonlocal law leads to ill-posed elastic problems.

- The maximum torsional rotation of a cantilever nano-beam, evaluated by the strain gradient theory of elasticity, is independent of the nonlocal parameter and coincides with the local one. Such a result is not physically acceptable. The proposed stress-driven mixture model does not present such drawbacks.

The contributed results could be also useful for the design and optimization of modern nano-scaled devices, extensively employed in Nano-Electro-Mechanical-Systems (NEMS).

## Appendix A. Supplementary data

Supplementary data related to this article can be found at http://dx.doi.org/10.1016/j.compositesb.2018.02.020.

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# FIGURES

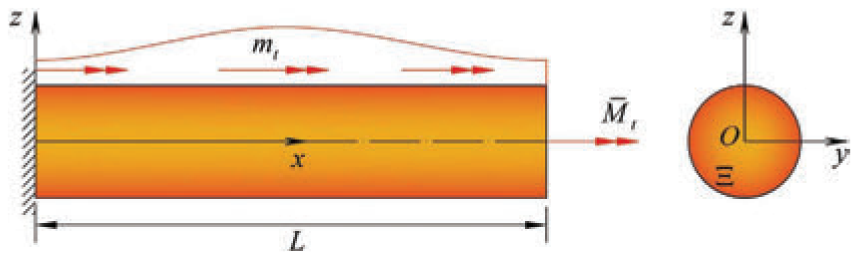


**Fig. 1.** Coordinate system and configuration of a nano-beam under torsion.

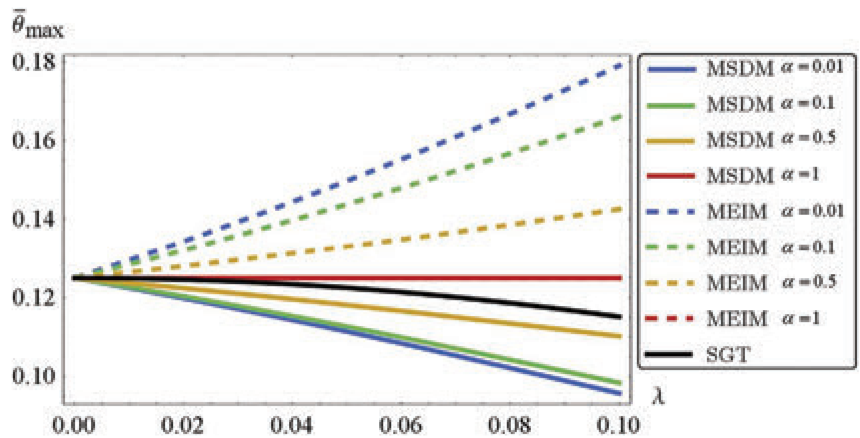


**Fig. 2.** Doubly clamped nano-beam: dimensionless maximum torsional rotations $\bar{\theta}_{max}$ versus the characteristic parameter $\lambda$.

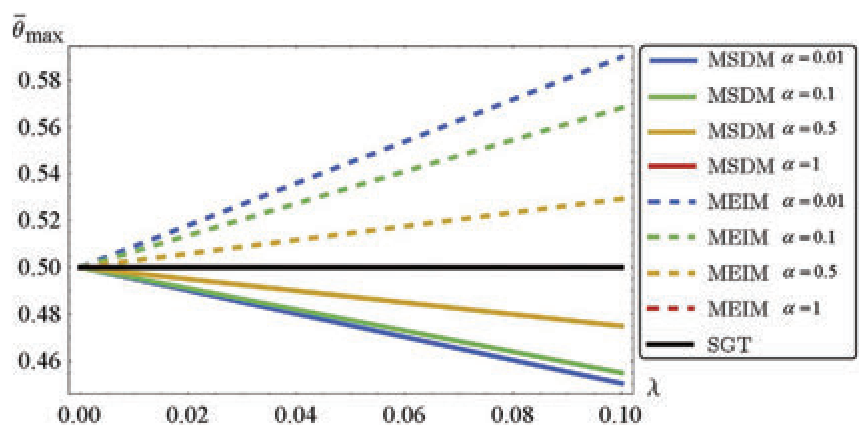


**Fig. 3.** Cantilever nano-beam: dimensionless maximum torsional rotations $\bar{\theta}_{max}$ versus the characteristic parameter $\lambda$.