The weak Maximum Principle for degenerate elliptic operators in unbounded domains *

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Abstract

We investigate and prove the validity of the maximum principle in narrow, possibly unbounded domains for very degenerate elliptic operators, just requiring a strict ellipticity in one direction and moreover establishing related Phragmén-Lindelöf principles.

1 Introduction and presentation of the results

In this article, we investigate the validity of various forms of the weak Maximum Principle for degenerate elliptic operators F which are strictly elliptic at least in one direction ν in domains Ω which are bounded in the direction ν and possibly unbounded in different directions.

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More precisely, let $F = F(x, s, p, M)$ be defined in $\Omega \times \mathbb{R} \times \mathbb{R}^n \times \mathcal{S}^n$ where $\mathcal{S}^n$ is the set of the $n \times n$ real symmetric matrices with the usual partial ordering $M \leq N$, meaning that $N - M$ is positive semidefinite. We shall always assume that

$$F \text{ is continuous from } \Omega \times \mathbb{R} \times \mathbb{R}^n \times \mathcal{S}^n \text{ into } \mathbb{R} \tag{1}$$

The mapping F is *degenerate elliptic* if the following monotonicity property holds:

$$F(x, s, p, M) \leq F(x, s, p, N) \quad \text{if } M \leq N \tag{2}$$

On the other hand, F is *strictly elliptic in the direction* ν if

$$F(x, 0, p, M + t\nu \otimes \nu) - F(x, 0, p, M) > 0 \quad \forall t \in \mathbb{R}_+, \tag{3}$$

for all $(x, p, M) \in \Omega \times \mathbb{R}^n \times \mathcal{S}^n$. Here ν is a unit vector in $\mathbb{R}^n$ and Ω is an open connected subset (a domain) of $\mathbb{R}^n$.

Denoting by $USC(\overline{\Omega})$ the set of upper semicontinuous functions on $\overline{\Omega}$, by weak Maximum Principle for F in Ω, MP in short, we mean that the following *sign propagation* property holds:

$$u \in USC(\overline{\Omega}),\ F(x, u, Du, D^2u) \geq 0 \ in\ \Omega,\ u \ \leq 0 \text{ on } \partial\Omega \quad \text{implies} \quad u \leq 0 \ in\ \Omega \tag{MP}$$

For $u \in C^2(\Omega)$, Du and D^2u denote, respectively, the gradient and the Hessian of the function u and the differential inequality in MP has the classical pointwise meaning. On the other hand, for nonsmooth u the partial differential inequality is understood in the viscosity sense, see [6, 10].

For smooth F, it is known from the work of Caffarelli-Li-Nirenberg [7] that the following condition

$$\sum_{i,j=1}^{n} F_{M_{ij}}(x, s, p, M)\, \nu_i\, \nu_j > 0\,, \tag{4}$$

a differential version of (3), together with $F(x, 0, 0, 0) = 0$ and

$$F_s(x, s, p, M) \leq 0 \tag{5}$$

imply the validity of the weak Maximum Principle (MP) in a *bounded* domain Ω. Let us mention also the work of P. Mannucci [12] where a comparison principle is established for viscosity solutions of some fully nonlinear subelliptic equations satisfying a non-degeneracy condition in a fixed direction.

The aim of the present article, which takes the move from [7] and previous work of the authors [5, 8, 9, 15, 16], is to single out coupled structure conditions on F and on the geometry of the domain Ω in order to enforce the validity of MP in different settings such as:

(i) unbounded domains contained in slabs,

(ii) narrow unbounded domains,

(iii) Phragmén-Lindelöf principles with exponential growth.

In the aforementioned works of the present authors such topics have been addressed with the perspective of extending maximum (see [5, 9, 15]) and Phragmén-Lindelöf type principles (see [8, 16]) to non regular unbounded domains satisfying generalized measure-geometric conditions and operators F including notably lower order terms, but always in the framework of uniform ellipticity. Here, following [7], we focus instead on fully nonlinear operators which are uniform elliptic just in one direction.

We introduce the following set of assumptions, to which we refer collectively as the structure condition $(\mathrm{SC})_\nu$, to be satisfied for all $(x, p, M) \in \Omega \times \mathbb{R}^n \times \mathcal{S}^n$:

$$\exists\, \lambda > 0 \ :\ F(x, 0, p, M + t\nu \otimes \nu) - F(x, 0, p, M) \geq \lambda t \quad \forall\, t \in \mathbb{R}_+ \,, \tag{6}$$

$$\exists\, \gamma > 0 \ :\ F(x, 0, p + q, M) - F(x, 0, p, M) \leq \gamma |q| \quad \forall\, q \in \mathbb{R}^n \,, \tag{7}$$

$$F(x, s, p, M) \geq F(x, r, p, M) \quad \text{if } r > s \,, \tag{8}$$

$$F(x, 0, 0, 0) = 0 \quad \forall\, x \in \Omega \tag{9}$$

Observe that (6) is a slightly strengthened form of (3).

In what follows, when F is evaluated at $s = u(x)$, $p = Du(x)$ and $M = D^2u(x)$ we will also use the notations $F[u]$ for $F(x, u, Du, D^2u)$ and, occasionally, Lu in the case of linear operators.

Example 1.1. Linear operators of the form

$$Lu = k(x)\,\frac{\partial^2 u}{\partial x_1^2} + \sum_{i=1}^{n} b_i(x)\,\frac{\partial u}{\partial x_i} + c(x)u \tag{10}$$

satisfy the structural condition $(\mathrm{SC})_\nu$ in $\mathbb{R}^n$ with respect to the direction $\nu = (1, 0, \ldots, 0)$, provided $k(x) \geq \lambda$, $\left|\sum_i b_i^2(x)\right|^{1/2} \leq \gamma$ and $c(x) \leq 0$.
As further examples we consider the fully nonlinear operators of Bellman-Isaacs type such as

$$F[u] = \sup_\alpha \inf_\beta L^{\alpha\beta} u, \tag{11}$$

where

$$L^{\alpha\beta} u = \sum_{i,j=1}^{n} a_{ij}^{\alpha\beta}\,\frac{\partial^2 u}{\partial x_i x_j} + \sum_{i=1}^{n} b_i^{\alpha\beta}\,\frac{\partial u}{\partial x_i} + c^{\alpha\beta} u$$

with constant coefficients depending α and β running in some sets of indexes $\mathcal{A}, \mathcal{B}$.
If $A^{\alpha\beta} = [a_{ij}^{\alpha\beta}]$ is positive semidefinite for all α, β and

$$\sum_{i,j=1}^{n} a_{ij}^{\alpha\beta} \nu_i \nu_j \geq \lambda, \quad |b_i^{\alpha\beta}| \leq \gamma, \quad c^{\alpha\beta} \leq 0\,,$$

then F satisfies the structure condition $(\mathrm{SC})_\nu$.

Corresponding to condition (6) on F, we will require that the possibly unbounded domain Ω is bounded in the direction ν, namely

$$\Omega \subseteq \{x \in \mathbb{R}^n : a \leq x \cdot \nu \leq a + d\} := S \quad \text{for some } a \in \mathbb{R}, d > 0\,. \tag{12}$$

Domains as S will be referred as *slabs* of thickness d. Such domains are typically unbounded but satisfy the measure-geometric (G) condition considered by Cabré who obtained an Alexandrov-Bakelman-Pucci (ABP) estimate and, as a consequence, the validity of MP in the case of linear uniformly elliptic operators, see [4]. These type of results have been generalized to viscosity solutions of fully nonlinear uniformly elliptic equations in [9]. It is also worth to mention that in a joint work with Birindelli [2] the authors have proved a generalization of MP in such domains for a different class of degenerate elliptic and also singular operators, based on the ABP estimate obtained by C. Imbert [11].

In [4] it is observed that the validity of the Maximum Principle in such domains, even for classical solutions of the Laplace equation, requires some restrictions on the growth of the solution $u(x)$ at infinity. The same occurs for the degenerate elliptic operators considered here.

Example 1.2. The function $u(x_1, x_2) = e^{x_2} \sin x_1$ with $(x_1, x_2) \in \Omega = (0, \pi) \times \mathbb{R}$ satisfies

$$\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial u}{\partial x_2} = 0 \quad in \quad \Omega, \quad u(x_1, x_2) = 0 \quad on \quad \partial\Omega,$$

This example shows that MP fails since $u(x_1, x_2) > 0$ in Ω.

In our results below we restrict the attention to functions $u \in USC(\overline{\Omega})$ having sub-linear growth at infinity, namely

$$\limsup_{\substack{|x| \to \infty \\ x \in \Omega}} \frac{u(x)}{|x|} \leq 0\ .$$

Letting

$$u_0^+(x) = \begin{cases} u^+(x) \equiv \max(u(x), 0) & \text{if } x \in \overline{\Omega} \\ 0 & \text{if } x \notin \overline{\Omega} \end{cases}$$

we will use equivalently the notation $u_0^+(x) = o(|x|)$ as $|x| \to \infty$. The introduction of function u_0^+ is useful for a unified statements of our results which, with the obvious exception of Theorem 6 on Phragmèn-Lindelöf principles, are valid both for bounded and unbounded domains.

Our first result is for the general case where Ω is contained in a slab S and F depends explicitly on x. In order to deal with this case, we need to require some convergence rate of F to 0, recall that we are assuming (9), as the matrix variable X tends to 0, uniformly when $|x| \to \infty$. More precisely, we assume that for matrix increments along $Q = I - \nu \times \nu$, which represents the projection operator on the orthogonal subspace to ν (the bounded direction of Ω), the following holds:

$$\liminf_{\varepsilon \to 0^+} F(x_\varepsilon, 0, 0, \tfrac{\varepsilon}{|x_\varepsilon|}\, Q) = 0 \tag{13}$$

for any sequence $x_\varepsilon \in \Omega$ such that $|x_\varepsilon| \to \infty$ as $\varepsilon \to 0^+$.

Example 1.3. Condition (13) is of course satisfied when F is independent of x. It is also satisfied, for instance, in the case of quasi-linear operators $F[u] = \sum_{i,j=1}^n a_{ij}(x) D_{ij} u + b(x, u, Du)$, assuming $a_{ij}(x) = O(|x|)$ as $|x| \to \infty$ and $b(x, 0, 0) = 0$.

Theorem 1.4. *Let Ω be a domain contained in a slab S of $\mathbb{R}^n$ as in* (12) *and assume that F satisfies (1), (2), the structure condition $(SC)_\nu$ and* (13)*. Then (MP) holds for any u such that $u_0^+(x) = o(|x|)$ as $|x| \to \infty$.*

The technical assumption (13) is needed in the proof in order to obtain a non-negative supersolution growing at least linearly at infinity for the possibly both side infinite domains considered in the statement. For the one-side unbounded domains considered in Theorem 1.6 the proof does not require indeed condition (13).

A comparison principle between an upper semicontinuous subsolution u and a C^2 supersolution v follows at once from the theorem above when applied to the operator

$$G(x, s, p, M) = F(x, s + v(x), p + Dv(x), M + D^2v(x)) - F(x, v(x), Dv(x), D^2v(x)).$$

Here is the result:

Corollary 1.5. *Assume on F and Ω the same conditions as in Theorem 1.4. Assume also that (13) holds for G. If $u \in USC(\overline{\Omega})$ and $v \in C^2(\Omega) \cap LSC(\overline{\Omega})$ satisfy*

$$F(x, v, Dv, D^2v) \leq F(x, u, Du, D^2u) \quad \text{in } \Omega$$

in the viscosity sense and $(u - v)^+(x) = o(|x|)$ as $|x| \to \infty$, then

$$u \leq v \quad \text{on } \partial\Omega \quad \text{implies} \quad u \leq v \quad \text{in } \Omega\,.$$

For special classes of unbounded domains the technical condition (13) can be avoided. This is the case of *semi-infinite slabs* S_+ of thickness d in the direction ν

$$S_+ = \{x \in \mathbb{R}^n : a_1 \leq x \cdot \nu \leq a_1 + d, \quad x \cdot \mu_i \geq a_i,\ i = 2, \ldots, n\} \tag{14}$$

for a set of real numbers a_i, $i = 1, \ldots, n$, and an orthonormal basis $\{\mu_2, \ldots, \mu_n\}$ of $\{\nu\}^\perp$, the orthogonal subspace to the direction ν.

Another important case is that of *cylinders* C of axis ν and thickness $d \in \mathbb{R}_+$ in all directions μ orthogonal to ν, namely

$$C = \{x \in \mathbb{R}^n : |(x - x^0) \cdot \nu| < d/2\ , \text{ for all } \mu \text{ such that } \mu \cdot \nu = 0\} \tag{15}$$

for some $x^0 \in \mathbb{R}^n$.

Theorem 1.6. *Let Ω be a domain of $\mathbb{R}^n$ contained either in a semi-infinite slab S_+ as in (14) or in a cylinder C as in* (15) *and assume that F satisfies (1), (2) and the structure condition $(SC)_\nu$. Then (MP) holds for any u such that $u_0^+(x) = o(|x|)$ as $|x| \to \infty$.*

To deal with (MP) in narrow domains, announced as item (ii) above, we need the following L^∞ estimate which can be deduced from the previous results.

Proposition 1.7. *Suppose that F satisfies (1), (2) and the structure condition $(SC)_\nu$. Assume that Ω satisfies one of the alternative assumptions of Theorems* 1.4 *and* 1.6*.*
If $u \in USC(\overline{\Omega})$ is such that $u_0^+(x) = o(|x|)$ as $|x| \to \infty$ and

$$F(x, u, Du, D^2u) \geq f(x) \quad in\, \Omega\,,$$

where f is continuous and bounded from below, then

$$\sup_{\overline{\Omega}} u \leq \sup_{\partial\Omega} u^+ + \frac{e^{1+\frac{\gamma d}{\lambda}}}{1+\frac{\gamma d}{\lambda}} \frac{\|f^-\|_\infty}{\lambda} d^2 \tag{16}$$

where $f^-(x) = -\min(f(x), 0)$.

Caffarelli *et al.* proved in [7] the validity of (MP) in a bounded domain Ω of $\mathbb{R}^n$ for a smooth F, assuming the structure condition $(\mathrm{SC})_\nu$ with (3) instead of (6). They point out that their result is obtained in under a very weak ellipticity assumption, with no limitation on the size of derivatives $F_{M_{ij}}$.

Conversely, they show with a counterexample that, assuming the one-directional ellipticity (4) but not the monotonicity (5), then (MP) fails to hold however small the diameter of the domain is taken, differently from the case of uniformly elliptic operators (see for instance [3, 8, 9, 16]).

We will see instead that condition (6) allow to obtain MP in a domain Ω which is bounded in a direction ν even relaxing the monotonicity condition, provided that the thickness of Ω in that direction is small enough.

At this purpose we introduce the weaker structural condition $(SC^-)_\nu$, corresponding to $(\mathrm{SC})_\nu$ with the monotonicity condition (8) replaced by

$$F(x, s, p, M) - F(x, r, q, M) \leq c(s - r) \ \text{ if } r < s \tag{17}$$

for a positive constant c. The next result shows that MP continues to hold in slabs or cylinders of thickness d, provided the product cd^2 is sufficiently small.

Theorem 1.8. *Let Ω be a domain of $\mathbb{R}^n$ contained either in a semi-infinite slab S_+ as in (14) or in a cylinder C as in* (15) *and assume that F satisfies (1), (2) and the structure condition $(SC^-)_\nu$. Then there exists a number $\delta = \delta(\gamma, \lambda) > 0$ such that if $cd^2 < \delta$ then MP holds for $u \in USC(\overline{\Omega})$, u bounded above.*

For a fixed $c > 0$, this result yields MP in the so called *narrow domains*, characterized by a sufficiently small thickness d. Conversely, for a fixed $d > 0$, MP holds true under the structure condition $(SC^-)_\nu$, provided c is a sufficiently small positive number.

The above result can be used as an intermediate step, as we will do to prove Theorem 1.10 below, to obtain Phragmèn-Lindelöf principles in unbounded domains, that is the weak MP for subsolutions with the maximal admissible growth at infinity corresponding to the geometry of the domain, which is expected in the uniform elliptic case.

For instance, in the case of conical domains, it is well known that (MP) holds for subsolutions having at most a polynomial growth, see [13] and the references therein for second order linear uniformly ellitptic operators: we refer to the more recent papers [14] for operators with unbounded coefficients and [1] for Pucci operators.

In the case of slabs and cylinders, we expect that MP holds true for subsolutions having a suitable exponential growth, see [16] for linear operators and [8] in the fully nonlinear viscosity setting.

However, in the generality of the structure assumptions of Theorems 1.4 and 1.6, we cannot go above a polynomial growth, as the following counterexample shows.

Example 1.9. The function $u(x_1, x_2) = x_2^2 \sin x_1$ is a solution of the differential equation

$$\frac{\partial^2 u}{\partial x_1^2} + \frac{1}{2}\, x_2^2\, \frac{\partial^2 u}{\partial x_2^2} = 0$$

in the slab $S = (0, \pi) \times \mathbb{R}$. Nonetheless, both in the case of S and the semi-infinite slab $(0, \pi) \times \mathbb{R}_+$, we have $u > 0$ in Ω even if $u = 0$ on $\partial\Omega$, contradicting (MP).

Indeed, condition $(\mathrm{SC})_\nu$ prescribes a control only from below (6) on the difference quotients of F with respect to the Hessian matrix. In order to obtain a Phragmèn-Lindelöf principle with exponential growth of subsolutions, we also need some control from above, at least for increments along the orthogonal projection Q on the subspace orthogonal to ν.

At this purpose, we introduce the structure condition $(SC^+)_\nu$, which consists of $(\mathrm{SC})_\nu$ complemented by

$$\text{there exists } \Lambda > 0 : F(x,0,p,M+tQ) - F(x,0,p,M) \le \Lambda t \quad \forall t \in \mathbb{R}_+ \tag{18}$$

for all $(x,p,M) \in \Omega \times \mathbb{R}^n \times \mathcal{S}^n$. It is worth to observe that this condition implies (13): in fact, for any sequence of points $x_\varepsilon \in \Omega$ such that $|x_\varepsilon| \to \infty$ we have

$$F(x,0,0,\tfrac{\varepsilon}{|x_\varepsilon|}Q) = F(x,0,0,\tfrac{\varepsilon}{|x_\varepsilon|}Q) - F(x,0,0,0) \le \Lambda \tfrac{\varepsilon}{|x_\varepsilon|} \to 0 \quad \text{as } \varepsilon \to 0^+.$$

On the other hand, conditions (6) and (18), requiring, respectively, a control from below only with respect to the direction ν and a control from above only in the orthogonal directions, comprise a much weaker condition on F than uniform ellipticity. This requires indeed a uniform control of the difference quotients both from below and from above with respect to all possible positive matrix increments.

The proof of our Phragmèn-Lindelöf type result below relies on the use of a comparison technique using exponential barrier functions related on the geometry of the domain.

Theorem 1.10. *Assume that F satisfy (1), (2) and the structure condition $(SC^+)_\nu$. Let Ω be contained in a slab S as in* (12)*.*

For any fixed $\beta_0 > 0$ there exists a positive constant $d = d(n,\lambda,\Lambda,\gamma,\beta_0)$ such that if S has thickness d, then MP holds for functions u such that $u^+(x) = O(e^{\beta_0|x|})$ as $|x| \to \infty$.

Conversely, for any fixed $d_0 > 0$ there exists a positive constant $\beta = \beta(n,\lambda,\Lambda,\gamma,d_0)$ such that (MP) holds for functions u such that $u^+(x) = O(e^{\beta|x|})$ as $|x| \to \infty$.

The paper is organized as follows. In Section 2 we collect some useful facts about viscosity solutions and ellipticity. In Section 3 we prove Theorems 1.4 and 1.6, based on the one-directional strict ellipticity (6) and the monotonicity condition (8). In Section 4 we prove MP in narrow domains of Theorem 1.8 when (8) is relaxed, replacing it with (17), via the uniform estimate of Theorem 1.7. Finally, in Section 5 we construct suitable exponential barrier functions and show the proof of the Phragmèn-Lindelöf principle of Theorem 1.10 using the additional condition (18).

2 Some useful facts

Let $F : \Omega \times \mathbb{R} \times \mathbb{R}^n \times \mathcal{S}^n \to \mathbb{R}$ be a continuous degenerate elliptic operator, see (2), and f be a continuous function on the domain Ω of $\mathbb{R}^n$.
We recall that a viscosity subsolution of equation $F(x, u, Du, D^2u) = f$ in Ω, that is a viscosity solution of $F(x, u, Du, D^2u) \geq f$, is a function $u \in USC(\Omega)$, that is, an upper semicontinuous function on Ω, such that

$$F(x, \varphi(x_0), D\varphi(x_0), D^2\varphi(x_0)) \geq f(x_0)$$

for all $x_0 \in \Omega$ and all test functions $\varphi \in C^2(\Omega)$ touching u from above at x_0, that is, $\varphi(x) \geq u(x)$ and $\varphi(x_0) = u(x_0)$.

Analogously, $u \in LSC(\Omega)$, a lower semicontinuous function on Ω, is a viscosity solution of $F(x, u, Du, D^2u) \leq f$, that is a viscosity supersolution of equation $F(x, u, Du, D^2u) = f$, if

$$F(x, \varphi(x_0), D\varphi(x_0), D^2\varphi(x_0)) \leq f(x_0)$$

for all $x_0 \in \Omega$ and all test functions test functions φ touching from below. A continuous function u will be called viscosity solution if it is both a subsolution and a supersolution.

The mapping F is uniformly elliptic if there exists in addition $\Lambda \geq \lambda$ such that

$$\lambda Tr(N) \leq F(x, s, p, M + N) - F(x, s, p, M) \leq \Lambda Tr(N) \quad \forall N \geq 0 \tag{19}$$

for all $(x, s, p, M) \in \Omega \times \mathbb{R} \times \mathbb{R}^n \times \mathcal{S}^n$, where $Tr(N)$ is the trace of matrix N.

If V is a linear subspace of $\mathbb{R}^n$, let P be the projection operator on V. The mapping F is strictly elliptic, respectively bounded, with respect to the directions of V if there exists a positive number λ such that

$$F(x, s, p, M + tP) - F(x, s, p, M) \geq \lambda t \quad \forall t \in \mathbb{R}_+, \tag{20}$$

respectively there exists a positive number Λ such that

$$F(x, s, p, M + tP) - F(x, s, p, M) \leq \Lambda t \quad \forall t \in \mathbb{R}_+\,, \tag{21}$$

for all $(x, s, p, M) \in \Omega \times \mathbb{R} \times \mathbb{R}^n \times \mathcal{S}^n$.

If V is a one-dimensional linear subspace, generated by direction ν, then $P = \nu \otimes \nu$ and our assumption (6) can be viewed a sort of strict ellipticity in one direction ν. If Q is the projection operator on the linear subspace orthogonal to V, i.e.

$$I = P + Q, \quad PQ = 0 = QP,$$

condition (18) of Theorem 1.10 means that F is bounded with respect to the $n-1$ orthogonal directions with respect to the one-dimensional subspace generated by ν.

For a detailed account on viscosity solutions of second order fully nonlinear elliptic equations we refer to [6, 10].

Here we only observe that under assumption (9), if $u \in USC(\Omega)$ is a viscosity solution of the differential inequality $F(x, u, Du, D^2u) \geq f$ then $F(x, u^+, Du^+, D^2u^+) \geq -f^-$, where $u^+ = \max(u, 0)$ and $f^- = -\min(f, 0)$. Moreover, if (8) also holds true, then

$$F(x, 0, Du^+, D^2u^+) \geq -f^- . \tag{22}$$

3 Maximum principles via strict ellipticity

We start proving Theorem 1.4, then we show Theorem 1.6, case (1), and finally case (2).

Proof. (Theorem 1.4) Arguing by contradiction, we suppose that there exists $x^o \in \Omega$ such that $u(x^o) = k > 0$. We may assume that ν is the unit vector along the positive x_1-axis and $-\frac{d}{2} \leq x_1 \leq \frac{d}{2}$ for $x = (x_1, \dots, x_n) \in \Omega$. So the orthogonal linear subspace will be generated by the positive directions along $x_2, \dots, x_n$ and

$$P = \begin{pmatrix} 1 & 0 \\ 0 & 0_{n-1} \end{pmatrix}, \qquad Q = \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{I}_{n-1} \end{pmatrix},$$

0_{n-1} and $\mathbb{I}_{n-1}$ being the $(n-1) \times (n-1)$ zero and identity matrix, respectively.

For $\varepsilon > 0$ we consider the function

$$u_\varepsilon(x) = u(x) - \varepsilon\varphi(x)$$

where $\varphi(x)=\sqrt{x_2^2+\cdots+x_n^2+1}$ so that, since $u(x)=o(|x|)$ at infinity, we have $u_\varepsilon(x)\to 0$ as $|x|\to\infty$, if Ω is unbounded. Then u_ε satisfies the differential inequality

$$F(x,u_\varepsilon+\varepsilon\varphi(x),Du_\varepsilon+\varepsilon D\varphi(x),D^2u_\varepsilon+\varepsilon D^2\varphi(x))\geq 0\,.$$

and for sufficiently small $\varepsilon>0$ we have $k_\varepsilon\equiv\sup_\Omega u_\varepsilon\geq\frac{k}{2}$.

Since $u\leq 0$ on $\partial\Omega$, and $u_\varepsilon(x)\to-\infty$ as $|x|\to\infty$ if Ω is unbounded, there exists $x_\varepsilon\in\Omega$ such that $u_\varepsilon(x_\varepsilon)=k_\varepsilon$.
Thus we can consider, for $\alpha>\frac{\gamma}{\lambda}$, the function

$$h_\varepsilon(x)=k_\varepsilon(1+e^{-\alpha d})-k_\varepsilon e^{\alpha(x_1-x_{\varepsilon,1})-\alpha d}$$

such that $h_\varepsilon(x_\varepsilon)=k_\varepsilon=u_\varepsilon(x_\varepsilon)$ and $h_\varepsilon\geq k_\varepsilon e^{-2\alpha d}\geq\frac{k}{2}\,e^{-2\alpha d}$ on $\overline{\Omega}$.

Eventually raising h_ε, we find $c_\varepsilon\geq 0$ such that $h_\varepsilon+c_\varepsilon$ touches u_ε from above at a point $x^*_\varepsilon\in\Omega$. Therefore $h_\varepsilon(x)+c_\varepsilon$ is a test function and will satisfy the inequality for the subsolution $u_\varepsilon(x)$, namely

$$F(x^*_\varepsilon,h_\varepsilon(x^*_\varepsilon)+c_\varepsilon+\varepsilon\varphi(x^*_\varepsilon),Dh_\varepsilon(x^*_\varepsilon)+\varepsilon D\varphi(x^*_\varepsilon),D^2h_\varepsilon(x^*_\varepsilon)+\varepsilon D^2\varphi(x^*_\varepsilon))\geq 0.$$

Since $h_\varepsilon(x^*_\varepsilon)+c_\varepsilon+\varepsilon\varphi(x^*_\varepsilon)>0$ then by monotonicity condition (8)

$$F(x^*_\varepsilon,0,Dh_\varepsilon(x^*_\varepsilon)+\varepsilon D\varphi(x^*_\varepsilon),D^2h_\varepsilon(x^*_\varepsilon)+\varepsilon D^2\varphi(x^*_\varepsilon))\geq 0\,. \tag{23}$$

On the other hand, computing the derivatives

$$\begin{aligned}Dh_\varepsilon(x^*_\varepsilon)&=-\alpha k_\varepsilon e^{\alpha(x^*_{\varepsilon,1}-x_{\varepsilon,1})-\alpha d}\,\nu,\\ D^2h_\varepsilon(x^*_\varepsilon)&=-\alpha^2k_\varepsilon e^{\alpha(x^*_{\varepsilon,1}-x_{\varepsilon,1})-\alpha d}\,P\,,\end{aligned}$$

taking into account that

$$k_\varepsilon e^{\alpha(x_1-x_{\varepsilon,1})-\alpha d}\geq\frac{k}{2}\,e^{-2\alpha d},$$

and using (6), (7), from the choice $\alpha>\frac{\gamma}{\lambda}$ we have

$$\begin{aligned}&F(x^*_\varepsilon,0,Dh_\varepsilon(x^*_\varepsilon)+\varepsilon D\varphi(x^*_\varepsilon),D^2h_\varepsilon(x^*_\varepsilon)+\varepsilon D^2\varphi(x^*_\varepsilon))\\ \leq&F(x^*_\varepsilon,0,\varepsilon D\varphi(x^*_\varepsilon),\varepsilon D^2\varphi(x^*_\varepsilon))-\alpha\,k_\varepsilon e^{\alpha(x_1-x_{\varepsilon,1})-\alpha d}(\lambda\alpha-\gamma)\\ \leq&F(x^*_\varepsilon,0,\varepsilon D\varphi(x^*_\varepsilon),\varepsilon D^2\varphi(x^*_\varepsilon))-\alpha\,\frac{k}{2}\,e^{-2\alpha d}(\lambda\alpha-\gamma)\end{aligned} \tag{24}$$

Collecting (23) and (24), we obtain

$$0 \leq F(x_\varepsilon^*, 0, \varepsilon D\varphi(x_\varepsilon^*), \varepsilon D^2\varphi(x_\varepsilon^*)) - \alpha\,\frac{k}{2}\,e^{-2\alpha d}(\lambda\alpha - \gamma) \tag{25}$$

If x_ε^* is bounded, then we can extract a subsequence converging to $x^* \in \overline{\Omega}$. Actually $x^* \in \Omega$, because by construction $\frac{k}{2} \leq u_\varepsilon(x_\varepsilon^*) \to u(x^*)$ whereas $u \leq 0$ on $\partial\Omega$.

Then taking the limit as $\varepsilon \to 0$ in (25), using the continuity of F and (9) we get a contradiction:

$$0 < \alpha\,\frac{k}{2}\,e^{-2\alpha d}(\lambda\alpha - \gamma) \leq 0\,. \tag{26}$$

If x_ε^* is unbounded, then we take a subsequence such that $|x_\varepsilon^*| \to \infty$ as $\varepsilon \to 0$. Computing the derivatives of $\varphi(x)$, we get

$$|D\varphi(x)| \leq 1, \quad D^2\varphi(x) \leq \frac{Q}{\varphi(x)}.$$

From this, using (2) and (7), we get

$$F(x_\varepsilon^*, 0, \varepsilon D\varphi(x_\varepsilon^*), \varepsilon D^2\varphi(x_\varepsilon^*)) \leq F(x_\varepsilon^*, 0, 0, \varepsilon\,\tfrac{Q}{\varphi(x_\varepsilon^*)}) + \gamma\varepsilon \tag{27}$$

Estimating (25) with (27) and taking the liminf as $\varepsilon \to 0^+$, we get again contradiction (26), which concludes the proof. □

Proof. (Theorem 1.6, case 1) The proof of Theorem 1.6, case (1), closely follows the previous one. In this case we suppose in addition by semi-boundedness

$$x_2 > 0, \ldots, x_n > 0$$

for $x = (x_1, x_2, \ldots, x_n) \in \Omega$. Arguing by contradiction as before, we can use the same argument but choosing $\varphi(x) = \frac{1}{\sqrt{n-1}}\,(x_2 + \cdots + x_n)$, so that

$$|D\varphi(x)| = 1, \quad D^2\varphi(x) = 0.$$

In this case, using (7) and (9), we get directly

$$F(x_\varepsilon^*, 0, \varepsilon D\varphi(x_\varepsilon^*), \varepsilon D^2\varphi(x_\varepsilon^*)) \leq \gamma\varepsilon \tag{28}$$

instead of (27) and conclude as at the end the proof of Theorem 1.4. □

The proof of Theorem 1.6, case (2), needs two preliminary lemmas. The first one is a very simple consequence of degenerate ellipticity.

Lemma 3.1. *Let Ω be a domain of $\mathbb{R}^n$ and F be a continuous degenerate elliptic operator satisfying* (8) *and* (9)*. If $u \in USC(\Omega)$ satisfies the differential inequality*

$$F(x, u, Du, D^2u) > 0 \tag{29}$$

in Ω, then u cannot have a real non-negative maximum in Ω.

Proof. Arguing by contradiction, suppose that u has a maximum $k \geq 0$ at some $x_0 \in \Omega$. Then $\varphi(x) \equiv k$ touches above $u(x)$ at x_0 and therefore $F(x_0, k, 0, 0) > 0$, whereas (8) and (9) imply on the other hand $F(x_0, k, 0, 0) \leq F(x_0, 0, 0, 0) = 0$, a contradiction that proves the assertion. □

Combined with Theorem 1.6, case (1), the above Lemma yields a MP for strict subsolutions in cylinders.

Lemma 3.2. *Let Ω be a domain of $\mathbb{R}^n$ contained in a cylinder C of axis ν and thickness d as* (15)*. Suppose F is a continuous degenerate elliptic operator satisfying the structure condition $(SC)_\nu$. Then (MP) holds true in Ω for $u \in USC(\overline{\Omega})$ satisfying the strict inequality (29), such that $u(x) = o(|x|)$ at infinity.*

Proof. We may assume that ν is the positive direction of the axis x_n, and that Ω is contained in the cylinder $C = B_r \times \mathbb{R}$, where B_r is the ball centered at the origin of radius $r = \frac{d}{2}$.

If there is no point $x \in \Omega$ with $x_n = 0$, then Ω is contained in a semi-infinite slab as (14) and the result is obtained by means of Theorem 1.6, case (1).

So we are left with the case

$$\Omega_0 \equiv \{x \in \Omega : x_n = 0\} \neq \emptyset.$$

If $u \leq 0$ in Ω_0, we are done, again by Theorem 1.6, case (1), applied in both domains

$$\Omega_+ \equiv \{x \in \Omega : x_n > 0\} \quad \text{and} \quad \Omega_- \equiv \{x \in \Omega : x_n < 0\},$$

contained in the semi-infinite slabs $S_\pm \equiv B_r \times \mathbb{R}_\pm$.

To conclude we will show that u cannot be positive on Ω_0.

Arguing by contradiction, suppose indeed that $u > 0$ in some point of Ω_0. Since $u \leq 0$ on $\partial\Omega$, then there exists $x_0 \in \Omega_0$ such that $u(x_0) = \sup_{\Omega_0} u > 0$.
Setting $v = u - u(x_0)$, we have $v \leq 0$ on $\partial\Omega_\pm$. Moreover by (9)

$$F(x, v, Dv, D^2 v) = F(x, u - u(x_0), Du, D^2 u) \geq F(x, u, Du, D^2 u) > 0$$

and therefore, again by Theorem 1.6, case 1, applied to $\Omega_\pm$, we get $v \leq 0$ in Ω, namely

$$u(x) \leq u(x_0) \quad \forall x \in \Omega\,.$$

But this means that $u(x)$ would have a positive interior maximum in Ω, in contrast with Lemma 3.1, finishing the proof. □

We are ready to prove the second part of Theorem 1.6.

Proof. (Theorem 1.6, case 2) We will use the setting of Lemma 3.2. For the viscosity subsolution $u \in USC(\overline{\Omega})$ of equation $F(x, u, Du, D^2 u) = 0$ we recall that $F(x, 0, Du^+, D^2 u^+) \geq 0$ by (22) and introduce the function

$$w(x) = u^+(x) - \varepsilon(e^{\alpha d} - e^{\alpha x_1})$$

where α is a positive constant to be chosen in the sequel and ε is any positive real number. Using (6) and (7), we get

$$\begin{aligned} F(x, 0, Dw, D^2 w) \geq & F(x, 0, Dw, D^2 w) - F(x, 0, Du^+, D^2 u^+) \\ \geq & \varepsilon\lambda\alpha^2 e^{\alpha x_1} - \varepsilon\gamma\alpha e^{\alpha x_1} = \varepsilon\alpha e^{\alpha x_1}(\alpha\lambda - \gamma) \end{aligned}$$

so that, choosing $\alpha > \frac{\gamma}{\lambda}$, we get

$$F(x, 0, Dw, D^2 w) \geq \varepsilon\alpha e^{-\alpha d}(\alpha\lambda - \gamma) > 0\,.$$

Assuming $w \leq 0$ on $\partial\Omega$, then Lemma 3.2 yields $w \leq 0$ in Ω so that

$$u(x) \leq \varepsilon(e^{\alpha d} - e^{\alpha x_1})$$

in Ω. Letting $\varepsilon \to 0^+$, we get then $u(x) \leq 0$ for all $x \in \Omega$, and the proof is complete. □

4 MPs in narrow domains

The results of the previous section continue to hold if we relax monotonicity condition (8) assuming instead (17), i.e substituting the structure condition $(\mathrm{SC})_\nu$ with the weaker one $(SC^-)_\nu$, provided the domain is assumed of sufficiently small thickness.

This is obtained through the uniform estimate of Theorem 1.7, which we prove below, for a viscosity solution $u \in USC(\overline{\Omega})$ of the non-homogeneous differential inequality $F[u] \geq f$ in a slab or a cylinder Ω of $\mathbb{R}^n$ under the respective assumptions of Theorems 1.4 and 1.6.

Proof. (Theorem 1.7) Let $u \in USC(\overline{\Omega})$ be a viscosity subsolution, bounded above, of equation $F(x, u, Du, D^2u) = f(x)$ in a domain Ω of the types considered in Theorems 1.4 and 1.6 with the respective assumptions of the operator F. Again by (22) we have

$$F(x, 0, Du^+, D^2u^+) \geq -f^-(x)\,.$$

We may assume that ν is the positive direction of axis x_1 and $0 \leq x_1 \leq d$.

Suppose for the moment $d = 1$ and set

$$w(x) = u^+(x) + C_1 e^{\alpha x_1} - k\,,$$

where C_1 and $\alpha > \frac{\gamma}{\lambda}$ are positive constants to be chosen in the sequel and

$$k = \sup_{\partial\Omega} u^+ + C_1 e^{\alpha}\,.$$

Using the structure condition $(\mathrm{SC})_\nu$, we get

$$\begin{aligned} F(x, 0, Dw, D^2w) &\geq F(x, 0, Du^+, D^2u^+) - \gamma C_1 \alpha e^{\alpha x_1} + \lambda C_1 \alpha^2 e^{\alpha x_1} \\ &\geq -f^-(x) + \alpha C_1 e^{\alpha x_1}(\alpha\lambda - \gamma) \\ &\geq -\|f^-\|_\infty + \alpha C_1(\alpha\lambda - \gamma) \quad \textit{in } \Omega\,, \end{aligned}$$

where $\|f^-\|_\infty = \sup_\Omega f^-$. Choosing

$$\alpha = 1 + \frac{\gamma}{\lambda}, \quad C_1 = \frac{\|f^-\|_\infty}{\lambda\alpha}$$

we obtain

$$F(x,0,Dw,D^2w) \geq 0 \ \ in\ \Omega, \quad w \leq 0 \ \ on\ \partial\Omega\,.$$

Since the mapping $(x,s,p,M) \to F(x,0,p,M)$ satisfies the conditions required to F in the respective Theorems 1.4 and 1.6, then we conclude that $w \leq 0$ in Ω, which implies

$$u(x) \leq u^+(x) + C_1 e^{\alpha x_1} \leq k = \sup_{\partial\Omega} u^+ + \frac{e^{1+\frac{\gamma}{\lambda}}}{1+\frac{\gamma}{\lambda}} \frac{\|f^-\|_\infty}{\lambda}$$

proving (16) in this case, $d = 1$.

For an arbitrary $d > 0$, rescaling we are led to the elliptic operator

$$G(y,s,p,M) = d^2 F(dy, s, d^{-1}p, d^{-2}M)$$

and to the subsolution $v(y) = u(dy)$, $y \in d^{-1}\Omega \subset (0,1) \times \mathbb{R}^{n-1}$, of the equation

$$G(y, v(y), Dv(y), D^2v(y)) = f(dy)d^2.$$

For all $(y,s,p,M) \in (d^{-1}\Omega \times \mathbb{R} \times \mathbb{R}^n \times \mathcal{S}^n)$, $q \in \mathbb{R}^n$ and $t \in \mathbb{R}_+$

$$G(y,s,p+q,M+tP) - G(y,s,p,M) \geq \lambda t - \gamma d|q|\,,$$

then we may apply the result of case $d = 1$ to obtain

$$\begin{aligned} u(x) \equiv v(y) &\leq \sup_{\partial(d^{-1}\Omega)} v^+ + \frac{e^{1+\frac{\gamma d}{\lambda}}}{1+\frac{\gamma d}{\lambda}} \frac{\|f^-\|_\infty}{\lambda}\, d^2 \\ &= \sup_{\partial\Omega} u^+ + \frac{e^{1+\frac{\gamma d}{\lambda}}}{1+\frac{\gamma d}{\lambda}} \frac{\|f^-\|_\infty}{\lambda}\, d^2 \end{aligned}$$

concluding the proof. □

We are in position to relax monotonicity condition (8), showing that the results of Theorems 1.4 and 1.6 can be restated replacing (8) with condition (17) both for a sufficiently small positive coefficient c, depending on the thickness d of the domain, and for narrow domains of sufficiently small thickness d, depending on the positive coefficient c.

The proof is based on the uniform estimate of Theorem 1.7.

Proof. (Theorem 1.8) We observe that by assumptions F satisfies all the properties required by Theorems 1.4 and 1.6, respectively, except for monotonicity condition (8). This condition can be restored, without loosing the other ones, considering the mapping $(x, s, p, M) \to F(x, 0, p, M)$.

In doing this, we recall that $F(x, u^+, Du^+, D^2u^+) \geq 0$, as already observed in Section 2, and by (17) we obtain the following differential inequality:

$$F(x, 0, Du^+, D^2u^+) \geq F(x, u^+, Du^+, D^2u^+) - cu^+ \geq -cu^+$$

From this, using the above Theorem 1.7, then we get

$$\sup_{\Omega} u \leq \frac{e^{1+\frac{\gamma d}{\lambda}}}{1+\frac{\gamma d}{\lambda}} \frac{c}{\lambda} \sup_{\Omega} u^+ \, d^2$$

whence the assertion follows making $c\, d^2$ small enough. □

5 Phragmèn-Lindelöf principles

In this Section, we prove the Phragmèn-Lindelöf principle stated in Theorem 1.10 using the method of barrier functions. By virtue of the monotonicity condition (8), as already observed in (22) and in many proofs, we will make use of the fact that from a subsolution u of equation $F(x, u, Du, D^2u) = 0$ we can pass to $u^+ = \max(u, 0)$, which is in turn a viscosity subsolution of equation $F(x, 0, Du^+, D^2u^+) = 0$.

Proof. (Theorem 1.10) We show the proof of the first part, the second being similar. Hence we assume that β_0 is a fixed positive real number and $u^+(x) = O(e^{\beta_0|x|})$ as $|x| \to \infty$.

Of course, we may suppose ν is the positive direction along axis x_1 so that the orthogonal subspace is generated by the positive directions along $x_2, \ldots, x_n$, and P, Q are as in the proof of Theorem 1.4.

We will consider therefore a domain Ω contained in a slab S of thickness d in direction x_1.

Our aim is to prove that for sufficiently small $d > 0$ and viscosity solutions $u \in USC(\overline{\Omega})$ of the differential inequality

$$F(x, 0, Du^+, D^2u^+) \geq 0 \quad in\ \Omega$$

such that $u(x) \leq 0$ on $\partial\Omega$ and $u(x) = O(e^{\beta_0 |x|})$, we have in turn $u \leq 0$ in Ω.

Let $x = (x_1, x') \in \mathbb{R}^n$, $r = |x'| = (x_2^2 + \cdots + x_n^2)^{\frac{1}{2}}$ and set

$$v(x) = \sin \alpha x_1 \, e^{\beta\varphi(r)}, \quad \varphi(r) = \sqrt{r^2 + 1}$$

where $\beta > \beta_0$ and $\alpha > \beta$ will be chosen as follows. Computing $Dv(x)$ we get

$$Dv(x) = \begin{pmatrix} \alpha \cos \alpha x_1 \; e^{\beta\varphi(r)} \\ \\ \sin \alpha x_1 \; D_{x'} e^{\beta\varphi(r)} \end{pmatrix}$$

where

$$e^{-\beta\varphi(r)} \left| \frac{\partial}{\partial x'} \, e^{\beta\varphi(r)} \right| = \beta\varphi'(r) = \beta \, \frac{r}{\varphi(r)},$$

from which

$$e^{-\beta\varphi} |Dv(x)| \leq \alpha + \beta\,. \tag{30}$$

Computing $D^2v(x^c)$, where $x^c = (\frac{\pi}{2\alpha}, x')$, we find

$$e^{-\beta\varphi(r)} D^2 v(x^c) = \begin{pmatrix} -\alpha^2 & 0 \\ 0 & e^{-\beta\varphi(r)} D_{x'x'} e^{\beta\varphi(r)} \end{pmatrix},$$

where $e^{-\beta\varphi(r)} D_{x'x'} e^{\beta\varphi(r)}$ is a positive definite $(n-1) \times (n-1)$ real matrix, having eigenvalues

$$\beta \left(\beta(\varphi'(r))^2 + \varphi''(r) \right) = \beta \left(\beta \, \frac{r^2}{\varphi^2(r)} + \frac{1}{\varphi^3(r)} \right)$$

of multiplicity 1 and

$$\frac{\beta\varphi'(r)}{r} = \frac{\beta}{\varphi(r)}$$

of multiplicity $n - 2$. It follows that

$$e^{-\beta\varphi} D^2 v(x^c) \leq -\alpha^2 P + \beta(\beta + 1) Q \tag{31}$$

in the sense of the usual partial ordering of matrices and we will choose $\alpha > 0$ big enough in order that

$$-\frac{1}{2}\,\lambda\,\alpha^2 + 2\Lambda\beta(\beta+1) + \gamma(\alpha+\beta) \leq 0\,. \tag{32}$$

From (31) by continuity there exists $d_0 \in (0, \frac{\pi}{\alpha})$ such that for $x_1 \in (\frac{\pi}{2\alpha} - \frac{d}{2}, \frac{\pi}{2\alpha} + \frac{d}{2})$ we have

$$e^{-\beta\varphi(r)} D^2 v(x) \leq -\frac{1}{2}\,\alpha^2 P + 2\beta(\beta+1)Q \tag{33}$$

for all $x \in S \equiv (\frac{\pi}{2\alpha} - \frac{d}{2}, \frac{\pi}{2\alpha} + \frac{d}{2}) \times \mathbb{R}^{n-1}$.
Now, assuming $\Omega \subset S$, we set

$$w(x) = u^+(x) - c_R v(x), \quad x \in \Omega_R = \Omega \cap B_R(0)$$

where

$$c_R = \frac{\sup_{x\in\partial\Omega_R} u^+(x)}{e^{\beta R} cos(\alpha\frac{d}{2})}.$$

In this way, assuming $u \leq 0$ on $\partial\Omega$, we have $w \leq 0$ on $\partial\Omega_R$.

On the other side, using (2), $(SC^+)_\nu$ and (33), from (32) we get

$$\begin{aligned} F(x,0,Dw,D^2w) =\ & F(x,0,Du^+ - c_R Dv, D^2u^+ - c_R D^2 v) \\ \geq\ & F(x,0,Du^+ - c_R Dv, D^2u^+ + c_R e^{\beta\varphi}(\tfrac{1}{2}\,\alpha^2 P - 2\beta(\beta+1)Q)) \\ \geq\ & F(x,0,Du^+, D^2u^+) \\ +\ & c_R e^{\beta\varphi}\left(\tfrac{1}{2}\,\lambda\alpha^2 - 2\Lambda\beta(\beta+1) - \gamma(\alpha+\beta)\right) \geq 0\,. \end{aligned}$$

Hence Theorem 1.4 yields $w \leq 0$ in the bounded domain Ω_R, namely

$$u(x) \leq c_R v(x) = \frac{\sup_{\partial\Omega_R} u^+}{e^{\beta R} cos(\alpha\frac{d}{2})}\, v(x)\,.$$

Finally, consider an arbitrary $x \in \Omega$, and choose $R > 0$ big enough in order that $x \in \Omega_R$. Letting $R \to \infty$ in the above, since $\sup_{\partial\Omega_R} u^+ = O(e^{\beta_0 R})$ and $\beta > \beta_0$, we obtain $u(x) \leq 0$. □

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