Exact solutions of inflected functionally graded nano-beams in integral elasticity

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ABSTRACT

The elastostatic problem of a Bernoulli-Euler functionally graded nanobeam is formulated by adopting stress-driven nonlocal elasticity theory, recently proposed by G. Romano and R. Barretta. According to this model, elastic bending curvature is got by convoluting bending moment interaction with an attenuation function. The stress-driven integral relation is equivalent to a differential problem with higher-order homogeneous constitutive boundary conditions, when the special bi-exponential kernel introduced by Helmholtz is considered. Simple solution procedures, based on integral and differential formulations, are illustrated in detail to establish the exact expressions of nonlocal transverse displacements of inflected nano-beams of technical interest. It is also shown that all the considered nano-beams have no solution if Eringen's strain-driven integral model is adopted. The solutions of the stress-driven integral method indicate that the stiffness of nanobeams increases at smaller scales due to size effects. Local solutions are obtained as limit of the nonlocal ones when the characteristic length tends to zero.

1. Introduction

Nanosize structures have superior mechanical, chemical and electrical properties so that they are widely used in nanoelectronic and nanodevice fields [1–4].

Since it is well known that experiments at the nano-level are difficult to conduct and lattice dynamics or molecular dynamics simulations are computationally expensive, methodologies of continuum mechanics [5–13] can be usefully adopted.

Size effects can be effectively modeled by employing nonlocal theories. Nonlocal models are obtained by suitably defining nonlocal elastic relations, see e.g. Refs. [14–31], since local constitutive laws fail to capture small-scale effects.

The Eringen differential model (EDM) [32–35] is surely one of the most notable constitutive approaches for the analysis of bending, buckling and vibration of nano-beams and it is widely adopted in literature [36–47].

It is worth noting that flaws on the application of the EDM have been called paradoxes and have been discussed in, e.g. [48–51].

Actually, in nonlocal structural problems involving bounded domains and standard kinematic boundary constraints, the elastostatic problem formulated according to the Eringen strain-driven integral model (EIM) admits no solution [52]. Moreover, the equivalence between the elastostatic problem formulated according to the EIM and the EDM equipped with two constitutive boundary conditions (CBCs) is proved in Ref. [53].

Accordingly the improperly claimed paradoxes in nano-beams rise from the inappropriate usual choice of disregarding the CBCs associated with the EDM.

An innovative stress-driven nonlocal integral model (StreDM) has been proposed by G. Romano and R. Barretta in Ref. [54] to study the size-dependent static behavior of nano-beams. Unlike the EIM, the StreDM provides a consistent approach for the design of nano-structures [55]. A general treatment on nonlocal integral theories for elastic nano-beams is contributed in Ref. [56].

The StreDM is resorted to in the present paper to address the size-dependent static behavior of inflected functionally graded (FG) nano-beams of technical interest. No paradox arises and small-scale effects

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are present for any boundary condition and applied load. In particular, closed form solutions for StreDM are provided for cantilever, simply-supported, clamped-pinned and fully-clamped FG nano-beams subject to different load conditions. It is then shown that the EIM admits no solution for all the considered nano-structural problems. Moreover, it is argued that the improper usual choice of modeling nano-beams by the EDM, that is disregarding the CBCs of EIM in solving the differential problem, can lead to physically unacceptable structural solutions.

Finally, numerical analyses are provided as benchmark results for applications and experimental tests on nonlocal bending.

2. Eringen strain-driven integral model for FG nano-beams

Local continuum theory assumes that the stress at a point is a function of the strain at that point. On the other hand, strain-driven nonlocal continuum mechanics considers that the stress at a point is a function of strains at all points of the body or in a neighbourhood of the point. An internal length scale is thus introduced into the constitutive equations as a material parameter in order to model interactions among atoms.

According to Eringen [32], [33], [34], [35], the state of stress $\boldsymbol{\sigma}$ at a point $\mathbf{x}$ in an elastic nonlocal body is given by the convolution between the elastic strain field $\boldsymbol{\varepsilon}$ and a scalar averaging kernel depending on a positive characteristic length L_c

$$\boldsymbol{\sigma}(\mathbf{x}) = \int_{\mathscr{B}} \phi(\mathbf{x}-\mathbf{y}, L_c)\mathbf{E}(\mathbf{y})\boldsymbol{\varepsilon}(\mathbf{y})d\mathbf{y} \tag{1}$$

where $\mathbf{x}$, $\mathbf{y}$ are points of the domain $\mathscr{B}$ of the Euclidean space occupied by the body, the material parameter L_c depends on the internal characteristic length describing the mutual long-range elastic interactions, $\mathbf{E}$ is the elastic stiffness and its inverse $\mathbf{C} = \mathbf{E}^{-1}$ is the elastic compliance. The kernel ϕ represents the space weight function that is non-negative and decays rapidly with the increase of $\|\mathbf{x}-\mathbf{y}\|$. In addition, the kernel function ϕ is that for $L_c \to 0$ we obtain the Dirac unit impulse at $\mathbf{x}$. .

In this paper, we consider a functionally graded (FG) straight beams of length L, with a rectangular cross-section Ω under uniaxial bending. The pair $\{\bar{y}, \bar{z}\}$ are principal axes of cross-sectional geometric inertia. The x-coordinate is taken along the beam axis, while the $\bar{y}$-coordinate along the bending axis (thickness) and the $\bar{z}$-coordinate along the nano-beam width originating at the cross-section geometric centre O. The FG nanobeam is composed of two different elastically isotropic materials, with Euler-Young moduli E_1, E_2, Poisson ratio $\nu = \nu_1 = \nu_2$ and shear moduli $\mu_1 = E_1/2(1+\nu)$, $\mu_2 = E_2/2(1+\nu)$. The effective normal and shear moduli of the FG nanobeam are respectively denoted by E and μ with a continuous change in the thickness direction $\bar{y}$.

Denoting by $V_1(\bar{y})$ and $V_2(\bar{y})$ the volume fractions of the two materials at $\bar{y}$, the effective material properties can be expressed by means of the following rule of mixtures

$$\begin{cases} E(\bar{y}) = E_1V_1(\bar{y}) + E_2V_2(\bar{y}), \\ \mu(\bar{y}) = \mu_1V_1(\bar{y}) + \mu_2V_2(\bar{y}) \end{cases} \tag{2}$$

with $V_1(\bar{y}) + V_2(\bar{y}) = 1$.

The coordinate along the $\bar{y}-$ axis of the elastic centre C of the distribution of Euler-Young moduli, see Eq. $(2)_1$, is then provided by the formula

$$\bar{y}_C = \frac{S_E}{A_E} = \frac{\int_\Omega E(\bar{y})\bar{y}dA}{\int_\Omega E(\bar{y})dA} \tag{3}$$

with A_E elastic area and S_E first moment of elastic area along the $\bar{y}$ axis according the effective Euler-Young modulus E. In the new elastic coordinate reference system Cyz, defined by the elastic centre C, we have $y = \bar{y} - \bar{y}_C$, $z = \bar{z}$. It is apparent that $E = E_1, \mu = \mu_1$ if $y = -h/2 - \bar{y}_C$ and $E = E_2, \mu = \mu_2$ if $y = h/2 - \bar{y}_C$.

Since Young's modulus E is assumed to be FG only along the transverse direction, the elastic bending stiffness I_E, defined by the second moment of elastic area along the y axis, is independent of the nano-beam axial abscissa. Its evaluation is performed by considering the bending abscissa $y = \bar{y} - \bar{y}_C$ originating at the elastic centre C of Euler-Young's moduli distribution E, so that

$$I_E = \int_\Omega E(y)y^2dA. \tag{4}$$

Remark 1. *Usually, the effective material properties [2] of the FG nanobeam are defined by the following power-law form [57] of the volume fractions V_1 and V_2 of the materials*

$$V_1(\bar{y}) = 1 - V_2(\bar{y}), \qquad V_2(\bar{y}) = \left(\frac{\bar{y}}{h} + \frac{1}{2}\right)^k. \tag{5}$$

The power-law exponent k is the non-negative parameter prescribing the material variation through the nanobeam thickness h. Hence, using Eq. (2) and (5)*, the effective Euler-Young modulus E is functionally graded along the thickness h according to the following expression*

$$E(\bar{y}) = E_1 + (E_2 - E_1)\left(\frac{\bar{y}}{h} + \frac{1}{2}\right)^k. \tag{6}$$

Setting $k = 1$, the effective FG Euler-Young modulus in Eq. (6) *has a linear variation along the nanobeam thickness. Explicit evaluations of the elastic centre and of the second moment of elastic area are provided in Ref.* [58].

In Bernoulli-Euler beam theory, applied loads and geometry are such that the displacements (s_x, s_y, s_z) along the coordinates (x, y, z) are functions of the $x-$ and $y-$coordinates and are given by

$$s_x(x, y) = -v^{(1)}(x)y, \qquad s_y(x, y) = v(x), \qquad s_z(x, y) = 0 \tag{7}$$

with v cross-sectional transverse displacement (along the bending axis y) and the apex $\bullet^{(n)}$denotes the $n-$derivative of the function $\bullet$ along the nanobeam axis x. The rotation φ of the nano-beam cross-section is $\varphi(x) = v^{(1)}(x)$.

Accordingly, the nonvanishing kinematically compatible deformation is given by the axial strain

$$\varepsilon_x(x, y) = -v^{(2)}(x)y = -\chi(x)y \tag{8}$$

being $\chi(x) = v^{(2)}(x)$ the bending curvature of the nano-beam.

Elastic and total bending curvature fields of the examined nano-beams are coincident and denoted in the sequel by the same symbol χ.

The Eringen strain-driven integral model (EIM) following from Eq. (1) provides the bending moment M in terms of the elastic curvature χ and the local elastic flexural stiffness I_E

$$M(x) = \int_0^L \phi(x-t, L_c)I_E\chi(t)dt \tag{9}$$

where M is the stress resultant moment

$$M = -\int_\Omega \sigma y dA. \tag{10}$$

Differential and boundary conditions of equilibrium require $M^{(2)}(x) = q(x)$ in $[0, L]$, $T(x) = -M^{(1)}(x) = \mathcal{F}$, $M(x) = \mathcal{M}$ at the nano-beam end point $x = L$ and $T(x) = -\mathcal{F}$, $M(x) = -\mathcal{M}$ at $x = 0$ with T shear force, $(\mathcal{F}, \mathcal{M})$ transverse force and couple respectively and q transverse distributed load along y.

A typical choice for the scalar kernel ϕ is the bi-exponential function

$$\phi(x, L_c) = \frac{1}{2L_c}\exp\left(-\frac{|x|}{L_c}\right) \tag{11}$$

where L_c is a positive characteristic length describing small scale phenomena.

Adopting the kernel in Eq. (11), it is shown in G. Romano et al. (2017) [53] that the convolution in Eq. (9) is equivalent to the following differential equation

$$M(x) - L_c^2M^{(2)}(x) = I_E\chi(x). \tag{12}$$

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with $x \in [0, L]$, subject to the following two constitutive boundary conditions

$$\begin{cases} M^{(1)}(0) = \frac{1}{L_c} M(0) \\ M^{(1)}(L) = -\frac{1}{L_c} M(L). \end{cases} \tag{13}$$

Remark 2. *The bending moment M in statically determinate FG nano-beams is uniquely defined by equilibrium requirements, so that it turns out to be independent of the small scale parameter L_c. On the contrary, in statically indeterminate structures, the bending moment is defined by equilibrium equation, compatibility and constitutive relations so that it is influenced by L_c.*

It is worth noting that the usual approach, commonly adopted in literature, based on Eringen's differential model consists in assuming the differential Eq. (12) without considering the constitutive boundary conditions reported in Eq. (13). As shown in the sequel, this nonlocal strategy, labeled by EDM (Eringen's differential model) provides paradoxical results.

3. Stress-driven nonlocal integral model for FG nano-beams

An innovative nonlocal stress-driven integral model (StreDM) for nano-beams has been recently introduced by G. Romano and R. Barretta (2017) [54].

The bending curvature χ of the FG nano-beam is obtained in terms of the following convolution

$$\chi(x) = \int_0^L \phi(x - t, L_c) \frac{1}{I_E} M(t) dt \tag{14}$$

where the bending moment M fulfils the equilibrium conditions.

Assuming the bi-exponential function ϕ reported in Eq. (11), it is proved in Ref. [54] that the convolution defined in Eq. (14) is equivalent to the following second-order differential equation

$$\frac{1}{L_c^2}\chi(x) - \chi^{(2)}(x) = \frac{1}{L_c^2}\frac{1}{I_E}M(x), \tag{15}$$

with $x \in [0, L]$, subject to the following two constitutive boundary conditions

$$\begin{cases} \chi^{(1)}(0) = \frac{1}{L_c}\chi(0) \\ \chi^{(1)}(L) = -\frac{1}{L_c}\chi(L). \end{cases} \tag{16}$$

Such a model is well-posed and no paradoxical results are obtained in contrast with the nonlocal Eringen theory. Hence in this paper we adopt the nonlocal StreDM to derive analytical solutions for bending problems of Bernoulli-Euler FG nano-beams that are then compared with the ones of the current literature.

4. Bending solutions for various boundary conditions of FG nano-beams

In this section, we derive the exact solutions according to the fully nonlocal StreDM for FG nano-beams, with various boundary conditions, subject to concentrated and distributed applied loads. Effects of the small-scale parameter on the bending results are also investigated.

The solution procedure of the nonlocal StreDM can be obtained from the convolution (14) or, equivalently, from the differential problem (15) equipped with the constitutive boundary conditions (16).

Considering the differential problem (15,16), the nonlocal StreDM can be solved according to the following *PROCEDURE (D)*.

BOX 1

PROCEDURE (D). Solution procedure of the nonlocal StreDM using the differential form.

STEP 1D - Solve the equilibrium equation $M^{(2)}(x) = q(x)$ to get the expression of the bending moment

$$M(x) = \int_0^x (x - s)q(s)ds + A_1 x + A_2 = \Gamma(x) + A_1 x + A_2 \tag{17}$$

in terms of two integration constants (A_1, A_2). If a uniform distributed load q is considered, the function Γ is

$$\Gamma(x) = \frac{1}{2}qx^2. \tag{18}$$

STEP 2D - Solve the second-order differential Eq. (15) in the form

$$\frac{1}{L_c^2}\chi(x) - \chi^{(2)}(x) = \frac{1}{L_c^2}\frac{1}{I_E}[\Gamma(x) + A_1 x + A_2], \tag{19}$$

to get the expression of the bending curvature χ of the nano-beam in terms of four integration constants (A_1, A_2, A_3, A_4) to be determined.

STEP 3D - Solve the second-order differential equation

$$v^{(2)}(x) = \chi(x) \tag{20}$$

in terms of the transverse displacement v of the FG nano-beam to get the expression of v in terms of six integration constants $(A_1, A_2, A_3, A_4, A_5, A_6)$ to be determined.

STEP 4D - Determine the six integration constants by imposing the two higher-order constitutive boundary conditions given by Eq. (16) in terms of $v(x)$, that is

$$\begin{cases} v^{(3)}(0) = \frac{1}{L_c}v^{(2)}(0) \\ v^{(3)}(L) = -\frac{1}{L_c}v^{(2)}(L) \end{cases} \tag{21}$$

and four boundary conditions at the FG nano-beam end points $x \in \{0, L\}$ specifying one element of each of the following two pairs

$$\begin{cases} v \quad \text{or} \quad M^{(1)} \\ v^{(1)} \quad \text{or} \quad M. \end{cases} \tag{22}$$

The nonlocal StreDM can be equivalently solved by means of the convolution reported in Eq. (14) according to the following *PROCEDURE (C)*.

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BOX 2
PROCEDURE (C). Solution procedure of the nonlocal StreDM using the convolution.

STEP 1C - Coincident to STEP 1D.

STEP 2C - Solve the convolution [14] in the form

$$\chi(x) = \int_0^L \phi(x-t, L_c)\frac{\Gamma(t)}{I_E}dt+ \\ + A_1 \int_0^L \phi(x-t, L_c)\frac{t}{I_E}dt + A_2 \int_0^L \phi(x-t, L_c)\frac{1}{I_E}dt, \tag{23}$$

to get the expression of the bending curvature χ of the FG nano-beam in terms of two integration constants (A_1, A_2) to be determined.

STEP 3C - Solve the second-order differential equation

$$v^{(2)}(x) = \chi(x) \tag{24}$$

in terms of the transverse displacement v of the FG nano-beam to get the expression of the transverse displacement v in terms of four integration constants (A_1, A_2, A_3, A_4) to be determined.

STEP 4C - Determine the four integration constants by imposing the boundary conditions at the FG nano-beam end points $x \in \{0, L\}$ specifying one element of each of the following two pairs

$$\begin{cases} v \quad \text{or} \quad M^{(1)} \\ v^{(1)} \quad \text{or} \quad M. \end{cases} \tag{25}$$

If the elastic bending stiffness of the nano-beam is constant along the nanobeam axis, the last two integrals in Eq. (23) involving the kernel ϕ assume the explicit expressions

$$\int_0^L \phi(x-t, L_c)\frac{1}{I_E}dt = \frac{1}{I_E}\left(1 - \frac{1}{2}e^{-\frac{x}{L_c}} - \frac{1}{2}e^{\frac{x-L}{L_c}}\right)$$

$$\int_0^L \phi(x-t, L_c)\frac{t}{I_E}dt= \\ = \frac{1}{2I_E}\left[L_c + L_c\left(e^{-\frac{x}{L_c}} - 1\right) - e^{\frac{x-L}{L_c}}(L_c + L) + 2x\right]. \tag{26}$$

If a statically determinate FG nano-beam is considered, the PROCEDURE (D) and the PROCEDURE (C), reported in the Boxes 1 and 2 above, can be simplified as shown in Appendix A.

4.1. Cantilever FG nano-beams with concentrated load at the free end

Let us consider a FG nanocantilever with length L subject to a transverse applied force F at the free end point.

The solution of the nonlocal StreDM using the differential approach can be provided by following the simplified PROCEDURE (DA), see Box 3 in Appendix A.

The equilibrium equation $M^{(2)}(x) = 0$ and the boundary conditions $M(L) = 0$, $M^{(1)}(L) = 0$ provide $\Gamma(x) = 0$, $A_1 = -F$ and $A_2 = FL$ so that the bending moment is $M(x) = F(L-x)$.

The second-order differential equation reported in Eq. (97) of Box 3 in Appendix A becomes

$$\frac{1}{L_c^2}\chi(x) - \chi^{(2)}(x) = \frac{1}{L_c^2}\frac{1}{I_E}F(L-x), \tag{27}$$

with $x \in [0, L]$, subject to the following two constitutive boundary conditions

$$\begin{cases} \chi^{(1)}(0) = \frac{1}{L_c}\chi(0) \\ \chi^{(1)}(L) = -\frac{1}{L_c}\chi(L). \end{cases} \tag{28}$$

The solution of the differential problem [27] [28] provides the bending curvature for the FG nano-cantilever

$$\chi(x) = \frac{F(L-x)}{I_E} - \frac{FL_c e^{-\frac{x}{L_c}}}{2I_E} + \frac{FL_c e^{\frac{x-L}{L_c}}}{2I_E} - \frac{FLe^{-\frac{x}{L_c}}}{2I_E} \tag{29}$$

which is the sum of the classical (local) curvature and of a nonlocal one depending on the small scale parameter L_c.

According to STEP 3DA and STEP 4DA of Box 3, the nonlocal displacement is obtained by the equation $v^{(2)} = \chi$ with the boundary conditions $v(0) = 0$ and $v^{(1)}(0) = 0$. Hence, using Eq. (29), we get

$$v(x) = v_{\mathcal{L}}(x) + \frac{FL_c e^{-\frac{x+L}{L_c}}}{2I_E}\left[L_c^2\left(e^{\frac{x}{L_c}} - 1\right)\left(e^{\frac{L}{L_c}} + e^{\frac{x}{L_c}}\right)+ \\ - e^{\frac{x+L}{L_c}}Lx + L_c\left(-e^{\frac{L}{L_c}}L + e^{\frac{x+L}{L_c}}(L-x) - e^{\frac{x}{L_c}}x\right)\right] \tag{30}$$

where $v_{\mathcal{L}}$ is the classical (local) displacement of the FG cantilever

$$v_{\mathcal{L}}(x) = \frac{Fx^2}{6I_E}(3L - x). \tag{31}$$

By letting the nonlocal parameter $L_c \to 0$, the StreDM model collapses to the corresponding local solution.

The maximum deflection of the FG nano-beam is attained at the tip $x = L$

$$v_{\max}(L) = \frac{FL^3}{3I_E} + \frac{FL_c e^{-\frac{L}{L_c}}}{2I_E}\left[2L_c^2\left(e^{\frac{L}{L_c}} - 1\right) - 2L_cL+ \\ - e^{\frac{L}{L_c}}L^2\right]. \tag{32}$$

In structural applications, it is useful to get the explicit expression of the nonlocal structural stiffness Σ of the FG nano-cantilever under the applied force which can be easily obtained from Eq. (32) setting $F = 1$

$$\Sigma= \frac{6I_E}{6L_c^3\left(1 - e^{-\frac{L}{L_c}}\right) - 6LL_c^2e^{-\frac{L}{L_c}} - 3L_cL^2 + 2L^3}. \tag{33}$$

The solution of the nonlocal StreDM using the convolution (14) can be provided by following the simplified PROCEDURE (CA), see Box 4 in Appendix A.

Introducing the expression $M(x) = F(L-x)$ of the bending moment in the convolution given by Eq. (101), we have

$$\chi(x) = \frac{1}{I_E}\int_0^L \phi(x-t, L_c)F(L-t)dt \tag{34}$$

and a direct evaluation provides the expression Eq. (29) of the nonlocal bending curvature.

Following the STEP 3CA and STEP 4CA of Box 4, the nonlocal displacement v reported in Eq. (30) is recovered.

BOX 3

PROCEDURE (DA). Solution procedure of the StreDM nonlocal model using the differential form for statically determinate FG nano-beams.

STEP 1DA - The equilibrium equation $M^{(2)}(x) = q(x)$ provides the expression of the bending moment

$$M(x) = \int_0^x (x - s)q(s)ds + A_1 x + A_2 \tag{95}$$

in terms of two integration constants (A_1, A_2). The integration constants (A_1, A_2) can be determined by specifying, at the FG nano-beam end points $x \in \{0, L\}$, two conditions between the following ones

$$M^{(1)}(x) = -\mathcal{F}, \quad M(x) = \mathcal{M}. \tag{96}$$

The explicit expression of the bending moment is thus obtained.

STEP 2DA - Solve the second-order differential Eq. (15) in the form

$$\frac{1}{L_c^2}\chi(x) - \chi^{(2)}(x) = \frac{1}{L_c^2}\frac{M(x)}{I_E}, \tag{97}$$

to get the bending curvature χ of the nano-beam in terms of two integration constants (A_3, A_4). The integration constants (A_3, A_4) can be determined by using the higher-order constitutive boundary conditions

$$\begin{cases} \chi^{(1)}(0) = \frac{1}{L_c}\chi(0) \\ \chi^{(1)}(L) = -\frac{1}{L_c}\chi(L). \end{cases} \tag{98}$$

STEP 3DA - Solve the second-order differential equation

$$v^{(2)}(x) = \chi(x) \tag{99}$$

in terms of the nano-beam transverse displacement v to get the expression of v in terms of two integration constants (A_5, A_6) to be determined.

STEP 4DA - Determine the two integration constants (A_5, A_6) by specifying one element of each of the following two pairs at $x \in \{0, L\}$

$$\begin{cases} v \quad \text{or} \quad M^{(1)} \\ v^{(1)} \quad \text{or} \quad M. \end{cases} \tag{100}$$

If statically determinate nano-beams are considered, the PROCEDURE (C) reported in Box 2 of Section 4 can be simplified as reported hereafter.

4.1.1. Eringen nonlocal model for cantilever FG nano-beams with concentrated load at the free end

The FG nano-cantilever with concentrated load at the free end has no solution using the Eringen nonlocal model.

In fact, due to the equivalence between EIM reported in Eq. (9) and EDM of Eq. (12) equipped with the constitutive boundary conditions (13), we consider the Eringen nonlocal model using the differential problem. Recalling that the bending moment is $M(x) = F(L - x)$ so that $M^{(1)}(x) = -F$, $M^{(2)}(x) = 0$, Eq. (12) becomes

$$F(L - x) = I_E\chi(x), \tag{35}$$

with $x \in [0, L]$, and the two constitutive boundary conditions in Eq. (13) reduce to

$$\begin{cases} -F = \frac{FL}{L_c} \\ -F = 0 \end{cases} \tag{36}$$

being $M^{(1)}(0) = M^{(1)}(L) = -F$, $M(0) = FL$ and $M(L) = 0$.

It is apparent that the boundary conditions [36] are fulfilled if a vanishing applied force is considered. Accordingly, FG cantilever nano-beams with concentrated load (different from zero) at the free end have no solution using EIM.

BOX 4

PROCEDURE (CA). Solution procedure of the StreDM nonlocal model using the convolution for statically determinate FG nano-beams.

STEP 1CA - Coincident to STEP 1DA.

STEP 2CA - Solve the convolution [14] in the form

$$\chi(x) = \int_0^L \phi(x - \xi, L_c)\frac{1}{I_E}M(\xi)d\xi \tag{101}$$

to get the explicit expression of the bending curvature χ of the FG nano-beam.

STEP 3CA - Solve the second-order differential equation

$$v^{(2)}(x) = \chi(x) \tag{102}$$

in terms of the nano-beam transverse displacement v to get the expression of v in terms of two integration constants to be determined.

STEP 4CA - Determine the two integration constants by specifying one element of each of the following two pairs at $x \in \{0, L\}$

$$\begin{cases} v \quad \text{or} \quad M^{(1)} \\ v^{(1)} \quad \text{or} \quad M. \end{cases} \tag{103}$$

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If the CBCs Eq. (36) are disregarded, the EDM provides the classical (local) curvature given by Eq. (35). Such a solution is not consistent with the EIM since the evaluation of the convolution reported in Eq. (9) using Eq. (35) for the bending curvature yields the following expression

$$M_{\mathscr{E}}(x) = I_E \int_0^L \phi(x - t, L_c)\chi(t)dt = M(x) - \frac{1}{2}Fe^{-\frac{L+x}{L_c}}\left[L_c\left(e^{\frac{L}{L_c}} - e^{\frac{2x}{L_c}}\right) + e^{\frac{L}{L_c}}L\right] \tag{37}$$

which is given by the bending moment M plus spurious terms depending on the small scale parameter L_c. A direct evaluation easily shows that the equilibrium equation $M_{\mathscr{E}}^{(2)}(x) = 0$ is not fulfilled.

Once more, EIM of a nano-cantilever subject to a transverse applied force F has no solution so that EDM necessarily leads to paradoxical results.

4.2. Cantilever FG nano-beams with a couple at the free end

Let us consider a FG nano-cantilever with length L subject to an applied couple $\mathcal{M}$ at the free end.

The solution of the nonlocal StreDM using the differential approach can be provided by following the simplified PROCEDURE (DA), see Box 3 in Appendix A.

The equilibrium equation $M^{(2)}(x) = 0$ and the boundary conditions $M(L) = \mathcal{M}$, $M^{(1)}(L) = 0$ provide $\Gamma(x) = 0$, $A_1 = 0$ and $A_2 = \mathcal{M}$ so that the expression of the bending moment is $M(x) = \mathcal{M}$.

The second-order differential equation reported in Eq. (97) of Box 3 in Appendix A becomes

$$\frac{1}{L_c^2}\chi(x) - \chi^{(2)}(x) = \frac{1}{L_c^2}\frac{\mathcal{M}}{I_E}, \tag{38}$$

with $x \in [0, L]$, subject to the following two constitutive boundary conditions

$$\begin{cases} \chi^{(1)}(0) = \frac{1}{L_c}\chi(0) \\ \chi^{(1)}(L) = -\frac{1}{L_c}\chi(L). \end{cases} \tag{39}$$

The solution of the differential problem [38] [39] provides the bending curvature for the nano-beam

$$\chi(x) = \frac{\mathcal{M}}{I_E} - \frac{\mathcal{M}e^{-\frac{x}{L_c}}}{2I_E} - \frac{\mathcal{M}e^{\frac{x-L}{L_c}}}{2I_E} \tag{40}$$

which is the sum of the classical (local) curvature and of a nonlocal one depending on the small scale parameter L_c.

According to STEP 3DA and STEP 4DA of Box 3, the nonlocal displacement is obtained by the equation $\chi = v^{(2)}$ with the boundary conditions $v(0) = 0$ and $v^{(1)}(0) = 0$. Hence, using Eq. (40), we get

$$v(x) = v_{\mathscr{L}}(x) + \frac{\mathcal{M}L_c e^{-\frac{x+L}{L_c}}}{2I_E}\left[L_c\left(e^{\frac{L}{L_c}} - e^{\frac{x}{L_c}}\right)\left(e^{\frac{x}{L_c}} - 1\right) - e^{\frac{x}{L_c}}\left(e^{\frac{L}{L_c}} - 1\right)x\right] \tag{41}$$

where $v_{\mathscr{L}}$ is the classical (local) displacement of the cantilever

$$v_{\mathscr{L}}(x) = \frac{\mathcal{M}x^2}{2I_E}. \tag{42}$$

The maximum deflection of the nano-beam is attained at the tip $x = L$

$$v_{\max}(L) = \frac{\mathcal{M}L^2}{2I_E} - \frac{\mathcal{M}L_cL}{2I_E}\left(1 - e^{-\frac{L}{L_c}}\right). \tag{43}$$

The solution of the nonlocal StreDM using the convolution [14] can be provided by following the simplified PROCEDURE (CA), see Box 4 in Appendix A.

Introducing the expression $M(x) = \mathcal{M}$ of the bending moment in the convolution given by Eq. (101), we have

$$\chi(x) = \frac{\mathcal{M}}{I_E}\int_0^L \phi(x - t, L_c)dt \tag{44}$$

and a direct evaluation, see Eq. $(26)_1$, provides the expression (40) of the nonlocal bending curvature.

Following the STEP 3CA and STEP 4CA of Box 4, the nonlocal displacement v reported in Eq. (41) is recovered.

4.2.1. Eringen nonlocal model for cantilever FG nano-beams with a couple at the free end

Let us now show that the FG nano-cantilever with a couple at the free end has no solution using the Eringen nonlocal model.

Due to the equivalence between EIM reported in Eq. (9) and EDM of Eq. (12) equipped with the CBCs (13), we consider the Eringen nonlocal model using the differential problem. Recalling that the bending moment is $M(x) = \mathcal{M}$ so that $M^{(1)}(x) = M^{(2)}(x) = 0$, Eq. (12) becomes

$$0 = \chi(x), \tag{45}$$

with $x \in [0, L]$, and the two constitutive boundary conditions, see Eq. (13), reduce to

$$\begin{cases} 0 = \frac{\mathcal{M}}{L_c} \\ 0 = -\frac{\mathcal{M}}{L_c} \end{cases} \tag{46}$$

being $M^{(1)}(0) = M^{(1)}(L) = 0$ and $M(0) = M(L) = \mathcal{M}$.

It is apparent that the CBCs Eq. (46) are fulfilled if a vanishing applied couple is considered. Accordingly, FG cantilever nano-beams with concentrated couple (different from zero) at the free end have no solution using EIM.

If the CBCs (36) are disregarded, EDM provides a null curvature, see Eq. (35). Such a solution is not consistent with EIM since the evaluation of the convolution reported in Eq. (9) using Eq. (45) for the bending curvature yields a null bending moment, i.e.

$$M_{\mathscr{E}}(x) = I_E \int_0^L \phi(x - t, L_c)\chi(t)dt = 0 \tag{47}$$

Once more, EIM of a nano-cantilever subject to an applied couple $\mathcal{M}$ has no solution so that EDM necessarily leads to paradoxical results.

4.3. Cantilever FG nano-beams with uniformly distributed load

Let us consider a FG cantilever nano-beam with length L subject to a uniform load q.

The solution of the nonlocal StreDM using the differential approach can be provided by following the simplified PROCEDURE (DA), see Box 3 in Appendix A.

The equilibrium equation $M^{(2)}(x) = q$ and the boundary conditions $M(L) = 0$, $M^{(1)}(L) = 0$ provide $\Gamma(x) = \frac{qx^2}{2}$, $A_1 = -qL$ and $A_2 = \frac{qL^2}{2}$ so that the bending moment is $M(x) = \frac{1}{2}q(L - x)^2$.

The second-order differential equation reported in Eq. (97) of Box 3 in Appendix A becomes

$$\frac{1}{L_c^2}\chi(x) - \chi^{(2)}(x) = \frac{1}{L_c^2}\frac{1}{2I_E}q(L - x)^2, \tag{48}$$

with $x \in [0, L]$, subject to the following two constitutive boundary conditions

$$\begin{cases} \chi^{(1)}(0) = \frac{1}{L_c}\chi(0) \\ \chi^{(1)}(L) = -\frac{1}{L_c}\chi(L). \end{cases} \tag{49}$$

The solution of the differential problem Eq. (48), with the boundary conditions Eq. (49), provides the bending curvature for the nano-beam

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$$\chi(x) = -\frac{2qL_c^2 e^{\frac{x-L}{L_c}} + qe^{-\frac{x}{L_c}}(2L_c^2 + 2L_cL + L^2) - 2q[2L_c^2 + (L-x)^2]}{4I_E}. \quad (50)$$

According to STEP 3DA and STEP 4DA of Box 3, the nonlocal displacement is obtained by using the relation $\chi = v^{(2)}$ with the boundary conditions $v(0) = 0$ and $v^{(1)}(0) = 0$. Hence, using Eq. (50), we get

$$v(x) = v_{\mathcal{L}}(x) + \frac{qL_c e^{-\frac{x+L}{L_c}}}{4I_E}\Bigg[2L_c^3\left(e^{\frac{L}{L_c}} - e^{\frac{x}{L_c}}\right)\left(e^{\frac{x}{L_c}} - 1\right) - e^{\frac{x+L}{L_c}}L^2x + 2L_c^2\left(-Le^{\frac{L}{L_c}} + e^{\frac{x+L}{L_c}}(L-x) + e^{\frac{x}{L_c}}x\right) + L_c e^{\frac{L}{L_c}}\left(\left(e^{\frac{x}{L_c}} - 1\right)L^2 - 2e^{\frac{x}{L_c}}Lx + 2e^{\frac{x}{L_c}}x^2\right)\Bigg] \quad (51)$$

where $v_{\mathcal{L}}$ is the classical (local) displacement of the FG cantilever

$$v_{\mathcal{L}}(x) = \frac{qx^2(6L^2 - 4Lx + x^2)}{24I_E}. \quad (52)$$

The maximum deflection of the FG nano-beam is attained at the tip $x = L$

$$v_{\max}(L) = \frac{qL^4}{8I_E} + \frac{qL^2L_c}{4I_E}\left(L_c - L_ce^{-\frac{L}{L_c}} - L\right). \quad (53)$$

The solution of the nonlocal StreDM using the convolution Eq. (14) can be provided by following the simplified PROCEDURE (CA), see Box 4 in Appendix A.

Using the expression of the bending moment in the convolution given by Eq. (101), we have

$$\chi(x) = \frac{q}{2I_E}\int_0^L \phi(x-t, L_c)(L-t)^2 dt \quad (54)$$

and the expression reported in Eq. (50) is recovered.

Following the STEP 3CA and STEP 4CA of Box 4, the nonlocal displacement v reported in Eq. (51) is recovered.

4.3.1. Eringen nonlocal model for cantilever FG nano-beams with uniformly distributed load

Let us now show that the nano-cantilever with uniformly distributed load has no solution using the Eringen nonlocal model.

Due to the equivalence between the EIM reported in Eq. (9) and the EDM of Eq. (12) equipped with the CBCs (13), we consider the Eringen nonlocal model using the differential problem. Recalling that the bending moment is $M(x) = \frac{1}{2}q(L-x)^2$ so that $M^{(1)}(x) = -q(L-x)$ and $M^{(2)}(x) = q$, Eq. (12) becomes

$$\frac{1}{2}q(L-x)^2 - L_c^2 q = I_E\chi(x), \quad (55)$$

with $x \in [0, L]$, and the two constitutive boundary conditions, see Eq. (13), reduce to

$$-qL = \frac{qL^2}{2L_c} \quad (56)$$

being $M^{(1)}(0) = -qL$, $M^{(1)}(L) = 0$ and $M(0) = \frac{qL^2}{2}$, $M(L) = 0$.

It is apparent that the CBC Eq. (56) cannot be fulfilled unless a vanishing applied load is considered. Accordingly, FG cantilever nano-beams under a uniformly distributed load (different from zero) have no solution using EIM.

If the CBC reported in Eq. (56) is disregarded, the EDM consists in solving Eq. (55). Hence the nonlocal curvature $\chi_{\mathcal{E}}$ of the FG nano-cantilever is given by

$$\chi_{\mathcal{E}}(x) = \frac{q}{2I_E}[(L-x)^2 - 2L_c^2]. \quad (57)$$

The nonlocal displacement $v_{\mathcal{E}}$ can then be obtained by solving the differential equation $v_E^{(2)}(x) = \chi_E(x)$ equipped with the boundary conditions $v_E(0) = 0$ and $v_E^{(1)}(0) = 0$ to get

$$v_{\mathcal{E}}(x) = \frac{qx^2}{24I_E}[6L^2 - 4Lx + x^2 - 12L_c^2]. \quad (58)$$

It is worth noting that the expression Eq. (57) of the nonlocal bending curvature χ_E is not consistent with the EIM. In fact the evaluation of the convolution reported in Eq. (9) using Eq. (57) for the bending curvature yields the following expression

$$M_{\mathcal{E}}(x) = I_E\int_0^L \phi(x-t, L_c)\chi_E(t)dt = M(x) - \frac{1}{4}qLe^{-\frac{x}{L_c}}(2L_c + L) \quad (59)$$

which is given by the bending moment M plus spurious terms depending on the small scale parameter L_c. A direct evaluation easily shows that the equilibrium equation $M_{\mathcal{E}}^{(2)}(x) = q$ is not fulfilled.

Once more, the EIM of a nano-cantilever subject to a transverse uniform load q has no solution so that the EDM necessarily leads to paradoxical results.

4.4. Simply supported FG nano-beam with uniformly distributed load

Let us consider a simply supported FG nano-beam with length L subject to a uniformly load q.

The solution of the nonlocal StreDM using the differential approach can be provided by following the simplified PROCEDURE (DA), see Box 3 in Appendix A.

The equilibrium equation $M^{(2)}(x) = q$ and the boundary conditions $M(L) = 0$, $M(L) = 0$ provide $\Gamma(x) = \frac{qx^2}{2}$, $A_1 = 0$ and $A_2 = -q\frac{L}{2}$ so that the bending moment is $M(x) = \frac{1}{2}qx(x-L)$.

The second-order differential equation reported in Eq. (97) of Box 3 in Appendix A becomes

$$\frac{1}{L_c^2}\chi(x) - \chi^{(2)}(x) = \frac{1}{L_c^2}\frac{1}{2I_E}qx(x-L), \quad (60)$$

with $x \in [0, L]$, subject to the following two constitutive boundary conditions

$$\begin{cases} \chi^{(1)}(0) = \frac{1}{L_c}\chi(0) \\ \chi^{(1)}(L) = -\frac{1}{L_c}\chi(L). \end{cases} \quad (61)$$

The solution of the differential problem (60), with the boundary conditions (61), provides the bending curvature for the simply supported FG nano-beam

$$\chi(x) = -\frac{qL_ce^{-\frac{x}{L_c}}(2L_c + L) + qL_ce^{\frac{x-L}{L_c}}(2L_c + L) - 2q[2L_c^2 + x(x-L)]}{4I_E}. \quad (62)$$

According to STEP 3DA and STEP 4DA of Box 3, the nonlocal displacement is obtained using the relation $\chi = v^{(2)}$ with the boundary conditions $v(0) = 0$ and $v(L) = 0$. Hence, employing Eq. (62), we get

$$v(x) = v_{\mathcal{L}}(x) + \frac{qL_c^2e^{-\frac{x+L}{L_c}}}{4I_E}\Bigg[2L_c^2\left(e^{\frac{L}{L_c}} - e^{\frac{x}{L_c}}\right)\left(e^{\frac{x}{L_c}} - 1\right) + L_cL\left(e^{\frac{L}{L_c}} - e^{\frac{x}{L_c}}\right)\left(e^{\frac{x}{L_c}} - 1\right) - 2e^{\frac{x+L}{L_c}}(L-x)x\Bigg] \quad (63)$$

where $v_{\mathcal{L}}$ is the classical (local) displacement of the nano-beam

$$v_{\mathcal{L}}(x) = \frac{qx(L^3 - 2Lx^2 + x^3)}{24I_E}. \quad (64)$$

The maximum deflection of the nano-beam is attained at the midpoint $x = \frac{L}{2}$

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FIG. 1

$$v_{\max}\left(\frac{L}{2}\right) = \frac{5qL^4}{384I_E} + \frac{qL_c^2e^{-\frac{L}{L_c}}}{8I_E}\left[4L_c^2\left(e^{\frac{L}{2L_c}} - 1\right) + 2L_cL\left(e^{\frac{L}{2L_c}} - 1\right)^2 - e^{\frac{L}{L_c}}L^2\right]. \tag{65}$$

The solution of the nonlocal StreDM using the convolution Eq. (14) can be provided by following the simplified PROCEDURE (CA), see Box 4 in Appendix A.

Using the expression of the bending moment in the convolution given by Eq. (101), we have

$$\chi(x) = \frac{q}{2I_E}\int_0^L \phi(x - t, L_c)t(t - L)dt \tag{66}$$

and the expression reported in Eq. (62) is provided.

Following the STEP 3CA and STEP 4CA of Box 4, the nonlocal displacement v reported in Eq. (63) is recovered.

4.4.1. *Eringen nonlocal model for simply supported FG nano-beams with uniformly distributed load*

Let us now show that the simply supported FG nano-beam with uniformly distributed load has no solution using the Eringen nonlocal model.

Due to the equivalence between the EIM reported in Eq. (9) and the EDM of Eq. (12) equipped with the CBCs (13), we solve the Eringen nonlocal model using the differential problem. Recalling that the bending moment is $M(x) = \frac{1}{2}qx(x - L)$ so that $M^{(1)}(x) = qx - q\frac{L}{2}$ and $M^{(2)}(x) = q$, Eq. (12) becomes

$$\frac{1}{2}qx(x - L) - L_c^2q = I_E\chi(x), \tag{67}$$

with $x \in [0, L]$, and the two constitutive boundary conditions in Eq. (13) reduce to

$$\begin{cases} -q\frac{L}{2} = 0 \\ q\frac{L}{2} = 0 \end{cases} \tag{68}$$

being $M^{(1)}(0) = -q\frac{L}{2}$, $M^{(1)}(L) = q\frac{L}{2}$ and $M(0) = M(L) = 0$.

It is apparent that the CBC (68) cannot be fulfilled unless a vanishing applied load $q = 0$ is considered. Accordingly, FG simply supported nano-beams under uniformly distributed load (different from zero) have no solution using EIM.

If the CBCs in Eq. (68) are disregarded, EDM consists in solving Eq. (67) to obtain the nonlocal bending curvature

$$\chi_{\mathcal{E}}(x) = \frac{qx}{2I_E}(x - L) - \frac{L_c^2}{I_E}q. \tag{69}$$

The nonlocal displacement $v_{\mathcal{E}}$ of the EDM model can then be obtained by the differential equation $v_{\mathcal{E}}^{(2)}(x) = \chi_{\mathcal{E}}(x)$ equipped with the boundary conditions $v_{\mathcal{E}}(0) = 0$ and $v_{\mathcal{E}}(L) = 0$ to get

$$v_{\mathcal{E}}(x) = v_{\mathcal{L}} + \frac{qxL_c^2}{2I_E}(L - x) \tag{70}$$

where $v_{\mathcal{L}}$ is the classical (local) displacement (64) of the simply

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FIG. 2

Table 1
- Comparison of the dimensionless tip deflection $v^*(1)$ in a FG nanocantilever subject to a concentrated load ($C-F$,F), toa concentrated couple ($C-F$,M) and to a uniform load ($C-F$, q).

λ	$C-F, \mathcal{F}$			$C-F, \mathcal{M}$			$C-F, q$		
	StreDM	*EDM*	*LBE*	*StreDM*	*EDM*	*LBE*	*StreDM*	*EDM*	*LBE*
0^+	0.33333	0.33333	0.33333	0.5	0.5	0.5	0.125	0.125	0.125
0.1	0.284333	0.33333	0.33333	0.450002	0.5	0.5	0.1025	0.12	0.125
0.2	0.24101	0.33333	0.33333	0.400674	0.5	0.5	0.0849326	0.105	0.125
0.3	0.206159	0.33333	0.33333	0.355351	0.5	0.5	0.0716973	0.08	0.125
0.4	0.178946	0.33333	0.33333	0.316417	0.5	0.5	0.0617166	0.045	0.125
0.5	0.157583	0.33333	0.33333	0.283834	0.5	0.5	0.0540415	0.000	0.125

Table 2
Comparison of the dimensionless midpoint deflection $v^*(0.5)$in a simply supported FG nano-beam ($S-S$, q) and fully clamped FG nano-beam ($C-C$, q)subject to a uniform load.

λ	$S-S, q$			$C-C, q$		
	StreDM	*EDM*	*LBE*	*StreDM*	*EDM*	*LBE*
0^+	0.0130208	0.0130208	0.0130208	0.00260417	0.00260417	0.00260417
0.1	0.0120668	0.0142708	0.0130208	0.0014832	0.00260417	0.00260417
0.2	0.01038	0.0180208	0.0130208	0.000778448	0.00260417	0.00260417
0.3	0.0088764	0.0242708	0.0130208	0.00045184	0.00260417	0.00260417
0.4	0.0076822	0.0330208	0.0130208	0.000289573	0.00260417	0.00260417
0.5	0.00674436	0.0442708	0.0130208	0.000199887	0.00260417	0.00260417

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supported FG nano-beam.

The expression (69) of the nonlocal bending curvature $\chi_{\mathcal{E}}$ is not consistent with the EIM. In fact the evaluation of the convolution reported in Eq. (9) using Eq. (69) for the bending curvature yields the following expression

$$M_{\mathcal{E}}(x) = I_E \int_0^L \phi(x - t, L_c)\chi_{\mathcal{E}}(t)dt = M(x) - \frac{1}{4}qL_c L e^{-\frac{x+L}{L_c}}\left(e^{\frac{L}{L_c}} + e^{\frac{2x}{L_c}}\right) \tag{71}$$

which is given by the bending moment M plus spurious terms depending on the small scale parameter L_c. A direct evaluation easily shows that the equilibrium equation $M_{\mathcal{E}}^{(2)}(x) = q$ is not fulfilled.

Once more, EIM of a simply supported nano-beam subject to a transverse uniform load q has no solution so that the EDM necessarily leads to paradoxical results.

4.5. Clamped-pinned FG nano-beam with distributed load

Let us consider a FG clamped-pinned nano-beam with length L subject to a uniformly distributed load q.

The solution of the nonlocal StreDM using the differential approach can be provided by following the PROCEDURE (D), see Box 1 in Section 4.

The equilibrium equation $M^{(2)}(x) = q$ provide $\Gamma(x) = \frac{qx^2}{2}$ so that the expression of the bending moment is $M(x) = \frac{qx^2}{2} + A_1 x + A_2$. In statically indeterminate FG nano-beams, the bending moment M is determined in terms of equilibrium equation, kinematic compatibility relation and constitutive law so that the bending moment depends on the small scale parameter.

The second-order differential equation reported in Eq. (19) of Box 1 becomes

$$\frac{1}{L_c^2}\chi(x) - \chi^{(2)}(x) = \frac{1}{L_c^2}\frac{1}{I_E}\left(\frac{qx^2}{2} + A_1 x + A_2\right). \tag{72}$$

with $x \in [0, L]$, subject to the two constitutive boundary conditions, see Eq. (16).

According to STEP 3D and STEP 4D of Box 1, the nonlocal displacement v follows from the differential equation $v^{(2)} = \chi$ in terms of six integration constants which can be determined by means of the two constitutive boundary conditions in Eq. (21) and by the four conditions $v(0) = 0$, $v^{(1)}(0) = 0$, $v(L) = 0$, $M(L) = 0$ following from Eq. (22).

The lengthy expression for the transverse deflection v is omitted in this case for brevity and we shall give the numerical comparisons in the next Section. For completeness we report the expression of the bending moment M

$$M(x) = M_{\mathcal{L}} + \\ + \frac{3qL_c L(L-x)}{8}\frac{6L_c^2\left(e^{\frac{L}{L_c}} - 1\right) - 2L_c\left(1 + 2e^{\frac{L}{L_c}}\right)L + e^{\frac{L}{L_c}}L^2}{6L_c^3\left(e^{\frac{L}{L_c}} - 1\right) - 6L_c^2 L - 3L_c e^{\frac{L}{L_c}}L^2 + 2e^{\frac{L}{L_c}}L^3} \tag{73}$$

where $M_{\mathcal{L}}$ is the classical (local) bending moment of the FG clamped-pinned nano-beam

$$M_{\mathcal{L}}(x) = \frac{q(L^2 - 5Lx + 4x^2)}{8}. \tag{74}$$

The solution of the nonlocal StreDM using the convolution Eq. (14) can be provided by following the PROCEDURE (C), see Box 2 in Section 4. The bending curvature can be evaluated by the convolution Eq. (23) in terms of the bending moment $M(x) = \frac{qx^2}{2} + A_1 x + A_2$

$$\chi(x) = \frac{1}{I_E}\int_0^L \phi(x - t, L_c)\left(\frac{qt^2}{2} + A_1 t + A_2\right)dt. \tag{75}$$

Following the STEP 3C and STEP 4C of Box 2, the nonlocal displacement v can be obtained solving the differential equation $v^{(2)}(x) = \chi(x)$ with the boundary conditions $v(0) = 0$, $v^{(1)}(0) = 0$, $v(L) = 0$, $M(L) = 0$, see Eq. (52). The expression of v coincides to the one obtained from the differential approach of the StreDM.

4.5.1. Eringen nonlocal model for clamped-pinned FG nano-beams with uniformly distributed load

Let us now show that the FG clamped-pinned nano-beam with distributed load has no solution using the Eringen nonlocal model.

Due to the equivalence between EIM reported in Eq. (9) and EDM of Eq. (12) equipped with the CBCs (13), we consider the Eringen nonlocal model using the differential problem. Recalling the expression of the bending moment $M(x) = \frac{qx^2}{2} + A_1 x + A_2$, we have $M^{(1)}(x) = qx + A_1$ and $M^{(2)}(x) = q$. The bending curvature is replaced by the second derivative of the transverse displacement, being $\chi = v^{(2)}$, so that Eq. (12) is written in the form

$$\frac{qx^2}{2} + A_1 x + A_2 - L_c^2 q = I_E v^{(2)}(x), \tag{76}$$

with $x \in [0, L]$, subject to the two CBCs, see Eq. (13), and four boundary constraints $v(0) = 0$, $v^{(1)}(0) = 0$, $v(L) = 0$, $M(L) = 0$ to get

$$\begin{cases} A_1 = \frac{A_2}{L_c} \\ qL + A_1 = -\frac{1}{L_c}\left(\frac{qL^2}{2} + A_1 L + A_2\right) \\ v(0) = 0 \\ v^{(1)}(0) = 0 \\ v(L) = 0 \\ \frac{qL^2}{2} + A_1 L + A_2 = 0 \end{cases} \tag{77}$$

being $M^{(1)}(0) = A_1$, $M^{(1)}(L) = qL + A_1$ and $M(0) = A_2$, $M(L) = \frac{qL^2}{2} + A_1 L + A_2$.

Eq. $(77)_{1,2,6}$ yield thus $A_1 = -qL$ and $A_2 = -qLL_c$ so that the second order differential equation (76) with boundary conditions $(77)_{3,4,5}$ can be solved if $q = 0$ to get $v(x) = 0$. Accordingly, FG clamped-pinned nano-beams under uniformly distributed load (different from zero) have no solution using EIM.

If the CBCs $(77)_{1\text{-}2}$ are disregarded, the EDM is provided by the differential equation (76) equipped with the boundary conditions $(77)_{3\text{-}6}$. The nonlocal displacement $v_{\mathcal{E}}$ of the EDM is then given by

$$v_{\mathcal{E}}(x) = v_{\mathcal{L}} + \frac{qx^2 L_c^2}{4I_E L}(L - x) \tag{78}$$

where $v_{\mathcal{L}}$ is the classical (local) displacement of the clamped-pinned nano-beam

$$v_{\mathcal{L}}(x) = \frac{qx^2(3L^2 - 5Lx + 2x^2)}{48I_E}. \tag{79}$$

The expression (78) of the nonlocal transversal displacement $v_{\mathcal{E}}$ is not consistent with EIM. In fact the evaluation of the convolution reported in Eq. (9) using the bending curvature $\chi_{\mathcal{E}} = v_{\mathcal{E}}^{(2)}$, following from Eq. (78), yields the following expression for the bending moment

$$M_{\mathcal{E}}(x) = I_E \int_0^L \phi(x - t, L_c)\chi_{\mathcal{E}}(t)dt = -\frac{1}{16L}q\Bigg[L_c^3\left(12e^{-\frac{x}{L_c}} - 12e^{\frac{x-L}{L_c}}\right) + \\ + L_c e^{-\frac{x+L}{L_c}}\left(5e^{\frac{L}{L_c}} + 3e^{\frac{2x}{L_c}}\right)L^2 + 12L_c^2\left(-2L + e^{-\frac{x}{L_c}}L + 2x\right) + \\ + L\left(-2L^2 + e^{-\frac{x}{L_c}}L^2 + 10Lx - 8x^2\right)\Bigg]. \tag{80}$$

It is apparent that the second derivative of the bending moment $M_{\mathcal{E}}$ reported in Eq. (80) does not provide the uniform distributed applied load q so that the equilibrium equation $M_{\mathcal{E}}^{(2)}(x) = q$ is not fulfilled.

Once more, the EIM of a clamped-pinned nano-beam subject to a transverse uniform load q has no solution so that the EDM necessarily

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leads to paradoxical results.

4.6. Fully clamped FG nano-beam with distributed load

Let us consider a fully clamped FG nano-beam with length L subject to a uniformly distributed load $= q$.

The solution of the nonlocal StreDM using the differential approach can be provided by following the PROCEDURE (D), see Box 1 in Section 4.

The equilibrium equation $M^{(2)}(x) = q$ provide $\Gamma(x) = \frac{qx^2}{2}$ so that the bending moment is $M(x) = \frac{qx^2}{2} + A_1 x + A_2$. The second-order differential equation reported in Eq. (19) of Box 1 becomes

$$\frac{1}{L_c^2}\chi(x) - \chi^{(2)}(x) = \frac{1}{L_c^2}\frac{1}{I_E}\left(\frac{qx^2}{2} + A_1 x + A_2\right), \tag{81}$$

with $x \in [0, L]$, subject to the following two constitutive boundary conditions

$$\begin{cases} \chi^{(1)}(0) = \frac{1}{L_c}\chi(0) \\ \chi^{(1)}(L) = -\frac{1}{L_c}\chi(L). \end{cases} \tag{82}$$

According to STEP 3D and STEP 4D of Box 1, the nonlocal displacement v follows from the differential equation $v^{(2)} = \chi$ in terms of four integration constants to be determined using the constitutive boundary conditions given by Eq. (82) and the constraint conditions $v(0) = 0$, $v^{(1)}(0) = 0$, $v(L) = 0$, $v^{(1)}(L) = 0$. Hence we have

$$v(x) = v_{\mathcal{L}}(x) + \frac{1}{24I_E\left(L_c - L_c e^{\frac{L}{L_c}} + e^{\frac{L}{L_c}}L\right)}\left[qL_c e^{-\frac{x}{L_c}}L\cdot (6L_c + L)\cdot\left(L_c\left(e^{\frac{L}{L_c}} - e^{\frac{x}{L_c}}\right)\left(e^{\frac{x}{L_c}} - 1\right)L + e^{\frac{x}{L_c}}\left(e^{\frac{L}{L_c}} - 1\right)x(x - L)\right)\right] \tag{83}$$

where $v_{\mathcal{L}}$ is the classical (local) displacement of the fully clamped FG nano-beam

$$v_{\mathcal{L}}(x) = \frac{q(L - x)^2 x^2}{24I_E}. \tag{84}$$

The maximum deflection of the FG nano-beam is attained at the midpoint $x = \frac{L}{2}$

$$v_{\max}\left(\frac{L}{2}\right) = \frac{qL^4}{384I_E} + \frac{qL_c L^2(6L_c + L)}{96I_E\left(L_c - L_c e^{\frac{L}{L_c}} + e^{\frac{L}{L_c}}L\right)}\left[4L_c\left(e^{\frac{L}{2L_c}} - 1\right) - L\left(e^{\frac{L}{L_c}} - 1\right)\right]. \tag{85}$$

The expression of the bending moment M is

$$M(x) = M_{\mathcal{L}} - \frac{qL_c}{12\left[L_c\left(e^{\frac{L}{L_c}} - 1\right) - e^{\frac{L}{L_c}}L\right]}\left[12L_c^2\left(e^{\frac{L}{L_c}} - 1\right) - 6L_c\left(e^{\frac{L}{L_c}} + 1\right)L + \left(e^{\frac{L}{L_c}} - 1\right)L^2\right] \tag{86}$$

where $M_{\mathcal{L}}$ is the classical (local) bending moment of the fully clamped FG nano-beam

$$M_{\mathcal{L}}(x) = \frac{q(L^2 - 6Lx + 6x^2)}{12}. \tag{87}$$

The solution of the nonlocal StreDM using the convolution Eq. (14) can be provided by following the PROCEDURE (C), see Box 2 in Section 4. The bending curvature can be evaluated by the convolution Eq. (23) in terms of the integrations constants A_1, A_2

$$\chi(x) = \frac{1}{I_E}\int_0^L \phi(x - t, L_c)\left(\frac{qt^2}{2} + A_1 t + A_2\right)dt. \tag{88}$$

Following the STEP 3C and STEP 4C of Box 2, the nonlocal displacement v can be obtained solving the differential equation $v^{(2)}(x) = \chi(x)$ with the boundary conditions $v(0) = 0$, $v^{(1)}(0) = 0$ and $v(L) = 0$, $v^{(1)}(L) = 0$. The expression (83) is thus recovered.

Remark 3. *In structural applications, it is useful to get the explicit expression of the structural stiffness Σ of the fully clamped nano-beam under a transverse unit displacement of clamped cross-section at $x = L$. The procedure shown above for the nonlocal StreDM can be easily applied to get the following stiffness*

$$\Sigma = \frac{12I_E e^{\frac{L}{L_c}}}{12L_c^3\left(e^{\frac{L}{L_c}} - 1\right) - 12L_c^2 L - 3L_c\left(1 + e^{\frac{L}{L_c}}\right)L^2 + e^{\frac{L}{L_c}}L^3} \tag{89}$$

4.6.1. Eringen nonlocal model for fully clamped FG nano-beams with uniformly distributed load

The fully clamped FG nano-beam with distributed load has no solution using the Eringen nonlocal model.

Due to the equivalence between the EIM reported in Eq. (9) and the EDM of Eq. (12) equipped with the CBCs (13), we solve the Eringen nonlocal model using the differential problem. Recalling the expression of the bending moment $M(x) = \frac{qx^2}{2} + A_1 x + A_2$, its derivatives are $M^{(1)}(x) = qx + A_1$ and $M^{(2)}(x) = q$. The bending curvature can be replaced by the second derivative of the transverse displacement, being $\chi = v^{(2)}$, so that Eq. (12) is written in the form

$$\frac{qx^2}{2} + A_1 x + A_2 - L_c^2 q = I_E v^{(2)}(x), \tag{90}$$

with $x \in [0, L]$, subject to the two constitutive boundary conditions, see Eq. (13), and four boundary constraints $v(0) = 0$, $v^{(1)}(0) = 0$ and $v(L) = 0$, $v^{(1)}(L) = 0$ to get

$$\begin{cases} A_1 = \frac{A_2}{L_c} \\ qL + A_1 = -\frac{qL^2}{2L_c} - \frac{A_1 L}{L_c} - \frac{A_2}{L_c} \\ v(0) = 0 \\ v^{(1)}(0) = 0 \\ v(L) = 0 \\ v^{(1)}(L) = 0 \end{cases} \tag{91}$$

being $M^{(1)}(0) = A_1$, $M^{(1)}(L) = qL + A_1$ and $M(0) = A_2$, $M(L) = \frac{qL^2}{2} + A_1 L + A_2$.

Hence, Eq. $(91)_{1,2}$ yield $A_1 = -q\frac{L}{2}$ and $A_2 = -q\frac{L}{2}L_c$ so that the second order differential equation (76) with boundary conditions $(77)_{3,4,5,6}$ can be solved if $q = 0$ to get $v(x) = 0$. Accordingly, FG fully clamped nano-beams with uniformly distributed load (different from zero) have no solution using EIM.

If the CBCs $(91)_{1\text{-}2}$ are disregarded, the differential equation (90) can be solved with the boundary conditions reported in Eq. $(91)_{3\text{-}6}$ to obtain the nonlocal transverse displacement of the EDM. As a consequence such a nonlocal displacement has no relationships with the integral Eringen model (9).

In fact the nonlocal displacement $v_{\mathcal{E}}$ of the Eringen model is

$$v_{\mathcal{E}}(x) = \frac{q(L - x)^2 x^2}{24I_E} = v_{\mathcal{L}} \tag{92}$$

and coincides to the classical (local) displacement of the fully clamped FG nano-beam $v_{\mathcal{L}}$.

The expression (92) of the nonlocal transversal displacement $v_{\mathcal{E}}$ is not consistent with the Eringen model. In fact the evaluation of the convolution reported in Eq. (9) using the bending curvature $\chi_{\mathcal{E}} = v_{\mathcal{E}}^{(2)}$, following from Eq. (92), yields the following expression

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Table 3
Comparison of the dimensionless midpoint deflection $v^*(0.5)$ and of the dimensionless maximum deflection $v^*_{\max}$ in a clamped-pinned FG nano-beam $(C-P, q)$subject to a uniform load.

λ		ξ	v^*	$\xi_{\max}$	$v^*_{\max}$
0^+	*StreDM*	0.5	0.00520833	0.578465	0.00541612
	EDM	0.5	0.00520833	0.578465	0.00541612
	LBE	0.5	0.00520833	0.578465	0.00541612
0.1	*StreDM*	0.5	0.00369168	0.59826	0.0039459
	EDM	0.5	0.00552083	0.583702	0.00576976
	LBE	0.5	0.00520833	0.578465	0.00541612
0.2	*StreDM*	0.5	0.00239882	0.611948	0.00261668
	EDM	0.5	0.00645833	0.596506	0.00684052
	LBE	0.5	0.00520833	0.578465	0.00541612
0.3	*StreDM*	0.5	0.00161582	0.620057	0.0017841
	EDM	0.5	0.00802083	0.611228	0.00864675
	LBE	0.5	0.00520833	0.578465	0.00541612
0.4	*StreDM*	0.5	0.00114629	0.62517	0.00127549
	EDM	0.5	0.0102083	0.624071	0.0112
	LBE	0.5	0.00520833	0.578465	0.00541612
0.5	*StreDM*	0.5	0.00085049	0.628641	0.000951361
	EDM	0.5	0.0130208	0.633975	0.0145032
	LBE	0.5	0.00520833	0.578465	0.00541612

$$M_\ell(x) = I_E \int_0^L \phi(x-t, L_c)\chi_\ell(t)dt = \frac{1}{12}q(12L_c^2 - 6Lx + L^2 + 6x^2) - \frac{1}{24}qe^{-\frac{x+L}{L_c}}\left(e^{\frac{L}{L_c}} + e^{\frac{2x}{L_c}}\right)\cdot(12L_c^2 + 6L_cL + L^2). \tag{93}$$

It is apparent that the second derivative of the bending moment M_ℓ reported in Eq. (93) does not provide the uniform distributed applied load q so that the equilibrium equation $M_\ell^{(2)}(x) = q$ is not fulfilled. Once more, EIM of a fully clamped nano-beam subject to a transverse uniform load q has no solution so that EDM necessarily leads to paradoxical results.

5. Discussions of the results

In this section, the analytical solutions obtained for the fully nonlocal StreDM in Section 4 are analysed and compared with the classical

Table 4
Comparison of the dimensionless bending moment M^* at the initial cross-section $\xi = 0$ in a clamped-pinned nano-beam $(C-P, q)$ and fully clamped nano-beam $(C-C, q)$subject to a uniform load.

λ	$C-P$			$C-C$		
	StreDM	*EDM*	*LBE*	*StreDM*	*EDM*	*LBE*
0^+	0.125	0.125	0.125	0.0833333	0.0833333	0.0833333
0.1	0.139507	0.14	0.125	0.0881474	0.0933333	0.0833333
0.2	0.147597	0.185	0.125	0.0887813	0.123333	0.0833333
0.3	0.152224	0.26	0.125	0.0883138	0.173333	0.0833333
0.4	0.155111	0.365	0.125	0.0877214	0.243333	0.0833333
0.5	0.157059	0.5	0.125	0.0871981	0.333333	0.0833333

(local) behavior and with the EDM, i.e. the nonlocal model following from the EIM, with the bi-exponential averaging kernel, by disregarding the CBCs. In the numerical comparisons, for simplicity, we denote by StreDM the solution obtained from the fully nonlocal StreDM, by LBE the solution obtained from the local Euler-Bernoully model and by EDM the solution obtained from the EDM.

Moreover, the following shortenings are used for identifying the constraints of the FG nano-beam and the related applied load: $C-F, \mathcal{F}$ for a clamped-free nano-beam subject to a concentrated load at the free end, $C-F, \mathcal{M}$ for a clamped-free nano-beam subject to a couple at the free end, $C-F, q$ for a clamped-free nano-beam subject to a uniform load, $S-S, q$ for a simply supported nano-beam subject to a uniform load, $C-P, q$ for a clamped-pinned nano-beam subject to a uniform load and $C-C, q$ for a fully clamped nano-beam subject to a uniform load.

Dimensionless variables are used in the plots by introducing the following quantities

$$\xi = \frac{x}{L}, \quad \lambda = \frac{L_c}{L}, \quad \Sigma^* = \Sigma\frac{L^3}{I_E}, \quad M^* = \frac{M}{qL^2}$$
$$v^* = v\frac{I_E}{FL^3} \text{ or } v^* = v\frac{I_E}{ML^2} \text{ or } v^* = v\frac{I_E}{qL^4} \tag{94}$$

where the expression of the dimensionless transverse displacement v^* depends on the considered applied load on the nano-beam.

Fig. 1 report the dimensionless transverse displacement for the nonlocal StreDM and EDM and for the classical LBE model. The

FIG. 3

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FIG. 4

dimensionless small-scale parameter λ ranges in the set {0.1, 0.2, 0.3, 0.4, 0.5} and the classical (local) model is recovered as λ tends to 0^+. The plot of the LBE model is reported with the dotted black line.

Fig. 1 show that the dimensionless small-scale parameter λ has the effect of reducing the deflections of the FG nano-beams evaluated by the nonlocal StreDM with respect to the LBE model, that is nano-beams stiffen for increasing λ. It is worth noting that this effect is independent from the external constraints of the FG nano-beam and the applied load. On the contrary, the dimensionless transverse displacement provided by the nonlocal EDM shows no nonlocal effects for $C-F, \mathcal{F}$ (Fig. 1a), $C-F, \mathcal{M}$ (Fig. 1b) and for $C-C, q$ (Fig. 1f). If a FG nano-beam $C-F, q$ is considered (Fig. 1c), the nonlocal EDM stiffens the nano-beam. The opposite behavior is displayed, that is the EDM softens the nano-beam, if the FG nano-beam $S-S, q$ (Fig. 1d) and $C-P, q$ (Fig. 1e) are considered.

The plots of the dimensionless transverse displacement at $\xi = 1$ or $\xi = \frac{1}{2}$ versus the dimensionless small-scale parameter λ for the nonlocal StreDM and EDM and for the classical LEB model are reported in Fig. 2 for various boundary conditions and applied loads. It is apparent that the nonlocal StreDM always stiffens the nano-beam with respect to the corresponding local one. The nonlocal EDM shows no small-scale effects for $C-F, \mathcal{F}$, $C-F, \mathcal{M}$ and $C-C, q$, the nonlocal EDM softens the FG nano-beam $C-F, q$ and stiffens the FG nano-beam for $S-S, q$ and $C-C, q$. In addition, Fig. 2c shows that the tip displacement of the FG nano-beam $C-F, q$ moves in the opposite direction of the applied load if the dimensionless small-scale parameter λ is greater than 0.5, i.e. $\lambda > 0.5$.

The numerical values of the dimensionless transverse maximum displacement at the tip $\xi = 1$ for FG nanobeams $C-F, \mathcal{F}$, $C-F, \mathcal{M}$ and $C-F, q$ are reported in Table 1 for the nonlocal StreDM and EDM and for the classical LBE model. Analogously, the numerical values of the dimensionless transverse maximum displacement at the midpoint $\xi = \frac{1}{2}$ for FG nano-beams $S-S, q$ and $C-C, q$ are reported in Table 2. Moreover, the dimensionless transverse displacement at the midpoint and the dimensionless transverse maximum displacement, with the related dimensionless position ξ, for the FG nanobeam $C-P, q$ are reported in Table 3.

The dimensionless bending moment M^* for the FG nano-beams $C-P, q$ and $C-C, q$ are plotted in Fig. 3a and (b) respectively. The nonlocal StreDM for FG nanobeams $C-P, q$ and $C-C, q$ shows that the dimensionless bending moment M^* is greater than the corresponding value of the classical LBE model. Moreover, the dimensionless bending moment M^* obtained by the StreDM points out a variation in terms of the dimensionless small-scale parameter λ that is less than the correspondent variation for the EDM.

The dimensionless bending moment M^* at $\xi = 0$ for FG nanobeam $C-P, q$ increases for increasing value of the dimensionless small scale parameter λ. On the contrary, the dimensionless bending moment M^* at $\xi = 0$ for FG nanobeam $C-C, q$ attains the maximum value $M^*(0) = 0.0888029$ for $\lambda = 0.180497$ and then decreases tending to the limit value $M^*(0) = \frac{1}{12} = 0.08333$ of the (local) LEB model (see Fig. 3c).

The values of the dimensionless bending moment M^* at $\xi = 0$ for FG nanobeams $C-P, q$ and $C-C, q$ obtained by the nonlocal StreDM are reported in Table 4. Finally, the dimensionless nano-beam stiffness Σ^* versus the dimensionless small-scale parameter λ for the nonlocal StreDM and the classical LBE model is reported in Fig. 4 for $C-F$ and $C-C$ FG nano-beams.

6. Concluding remarks

In this paper an analytical study is carried out on the bending problem of functionally graded nano-beams using the nonlocal stress-driven fully nonlocal model (StreDM) proposed by G. Romano and R. Barretta in Ref. [54]. Exact solutions for the bending problem of FG nano-beams with the most usual loading conditions and external constraints are established. On the contrary, it is pointed out that Eringen's nonlocal strain-driven fully nonlocal model (EIM) has no solution for all the considered FG nano-beams. In addition, if the higher-order constitutive boundary conditions (CBCs) are disregarded, Eringen's differential model (EDM) can be solved and paradoxical results are obtained.

The StreDM has instead the advantage of being well-posed for any FG nano-beam.

The analytical solutions contributed in this paper provide new benchmark examples for numerical analyses in nonlocal continuum mechanics.

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APPENDIX A

If statically determinate nano-beams are considered, the PROCEDURE (D) reported in Box 1 of Section 4 can be simplified as reported hereafter.

Appendix A. Supplementary data

Supplementary data related to this article can be found at http://dx.doi.org/10.1016/j.compositesb.2017.12.022.

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FIGURES

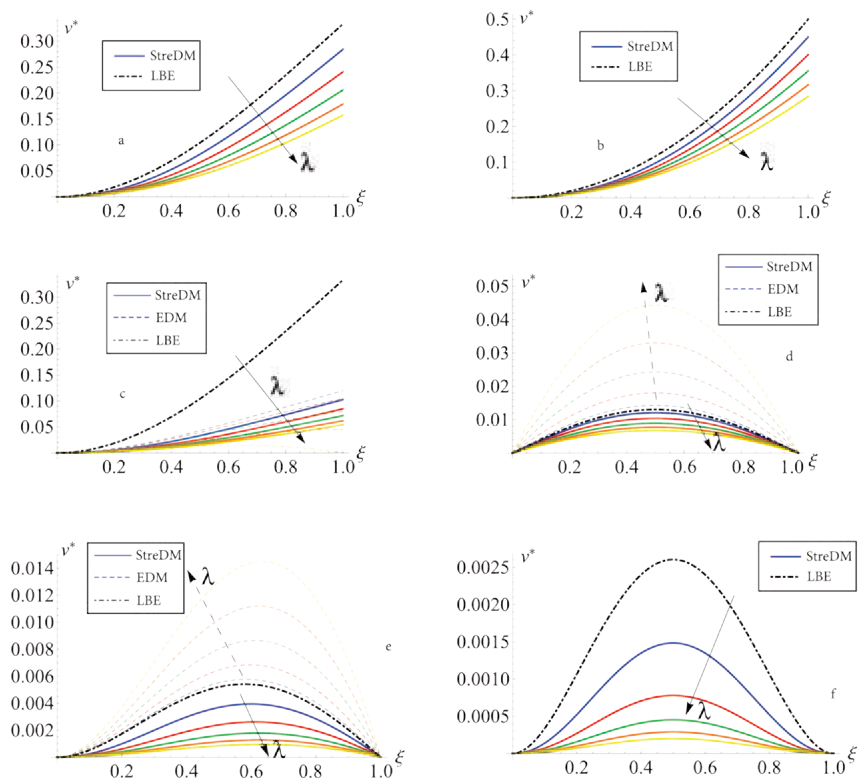

Fig. 1. Plots of the dimensionless transverse displacement v^*, for increasing values of the small-scale parameter λ in the set $\{0.1, 0.2, 0.3, 0.4, 0.5\}$, obtained by StreDM, EDM and LBE model for the following FG nano-beams: (a) $C-F, \mathcal{F}$; (b) $C-F, \mathcal{M}$; (c) $C-F, q$; (d) $S-S, q$; (e) $C-P, q$; (f) $C-C, q$.

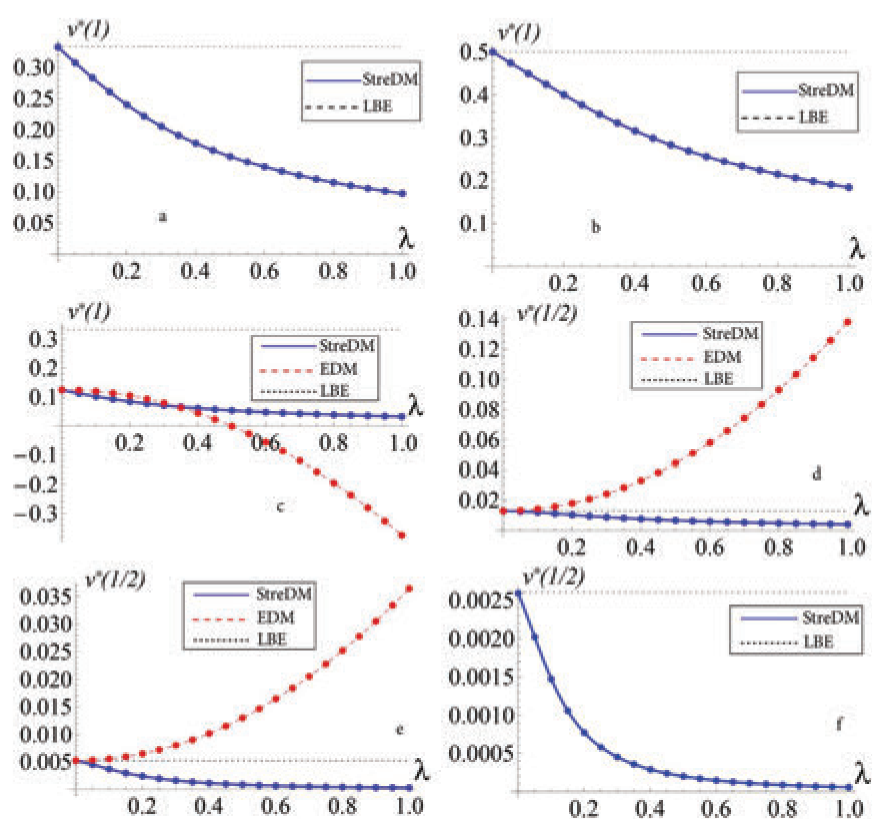

Fig. 2. Plots of the dimensionless transverse maximum displacement v^*_{max}, obtained by StreDM, EDM and LBE method, versus the dimensionless small-scale parameter λ for the following FG nano-beams: (a) $C-F, \mathcal{F}$; (b) $C-F, \mathcal{M}$; (c) $C-F, q$; (d) $S-S, q$; (f) $C-C, q$; (e) Plots of the dimensionless transverse midpoint displacement v^*, obtained by StreDM and LBE method, versus the dimensionless small-scale parameter λ for the FG nano-beam $C-P, q$.

FIGURES

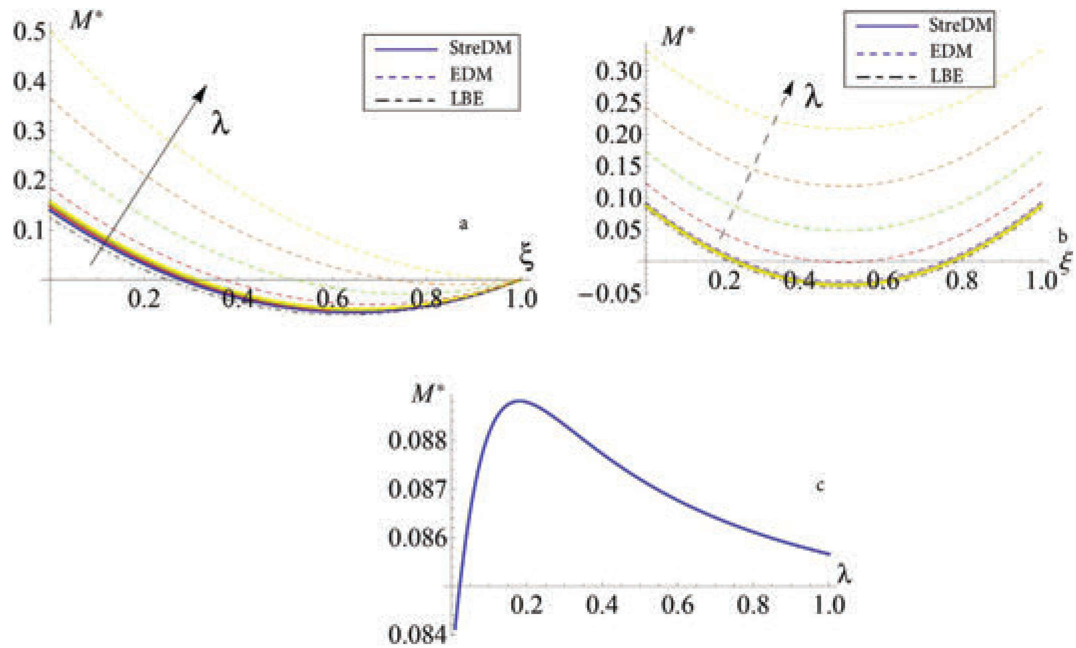

Fig. 3. Plots of the dimensionless bending moment M^* for the FG nano-beams: (a) $C-P$, q; (b) $C-C$, q. (c) Plot of the dimensionless bending moment M^* for the FG nano-beam $C-P$, q versus the dimensionless small scale parameter λ.

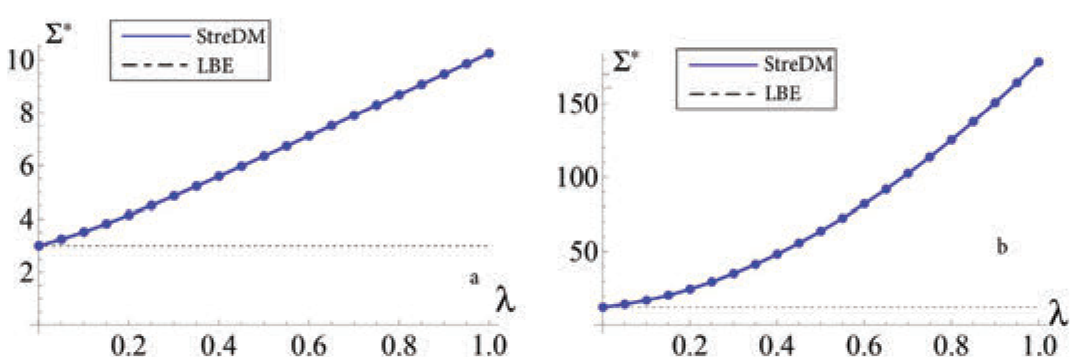

Fig. 4. Plots of the dimensionless stiffness Σ^* versus the dimensionless small-scale parameter λ obtained by the StreDM for the following FG nano-beams: (a) $C-F$; (b) $C-C$.