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A two-weight Sobolev inequality for Carnot-Carathéodory spaces

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Abstract

Let $X = \{X_1, X_2, \ldots, X_m\}$ be a system of smooth vector fields in $\mathbb{R}^n$ satisfying the Hörmander's finite rank condition. We prove the following Sobolev inequality with reciprocal weights in Carnot-Carathéodory space $\mathbb{G}$ associated to system X

$$\left(\frac{1}{\int_{B_R} K(x)\,dx}\int_{B_R}|u|^t K(x)\,dx\right)^{1/t} \leq C\,R\left(\frac{1}{\int_{B_R}\frac{1}{K(x)}\,dx}\int_{B_R}\frac{|Xu|^2}{K(x)}\,dx\right)^{1/2},$$

where Xu denotes the horizontal gradient of u with respect to X. We assume that the weight K belongs to Muckenhoupt's class A_2 and Gehring's class G_τ, where τ is a suitable exponent related to the homogeneous dimension.

Keywords Carnot-Carathéodory spaces · Weighted Sobolev inequalities · Muckenhoupt and Gehring weights

Mathematics Subject Classification 35R03 · 39B62

1 Introduction

This paper is devoted to study some basic functional and geometric properties of general families of vector fields that include the Hörmander's type as a special case.

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Similar to their Euclidean counterparts, such properties play an important role in the analysis of the relevant differential operators (both linear and nonlinear).

We are concerned with a two-weight Sobolev type inequality on $\mathbb{G}$, where $\mathbb{G}$ denotes the Carnot-Carathèodory space (Ω, d) (suitably defined - see Sect. 2.1) associated to a system of smooth vector fields $X = \{X_1, X_2, \dots, X_m\}$ on $\mathbb{R}^n$ satisfying the Hörmander's finite rank condition. This fact introduces a kind of degeneracy different from that Euclidean one. Here, Ω is an open (Euclidean) bounded and connected set of $\mathbb{R}^n$, $n \geq 2$, and d is the metric generated by X.

Let $u \in \mathrm{Lip}\,(\mathbb{G})$. We denote by $Xu = (X_1 u, \dots, X_m u)$ the horizontal gradient of u with respect to the system X, where X_j plays the role of the first order differential operator acting on u given by

$$X_j u(x) = \langle X_j(x), \nabla u(x)\rangle \qquad \text{for} \ \ j = 1, \dots, m.$$

Set

$$|Xu| = \Big(\sum_{j=1}^{m} (X_j u)^2\Big)^{1/2},$$

the length of the horizontal gradient of u. We refer to [5,12] for more details.

In our paper we prove a two-weight Sobolev type inequality where the weights K and K^{-1} form a 2-admissible pair (K^{-1}, K), namely

1) K is locally doubling in Ω and K^{-1} belongs to $A_2(\mathbb{G})$.
2) Given a compact set $V \subset \Omega$ there exist $t > 2$ and $\overline{C} \geq 1$ such that, for every ball B with center in V and $0 < r < 1$, it holds

$$r \left(\frac{\int_{rB} K(x)\,dx}{\int_B K(x)\,dx}\right)^{1/t} \leq \overline{C} \left(\frac{\int_{rB} K^{-1}(x)\,dx}{\int_B K^{-1}(x)\,dx}\right)^{1/2}. \tag{1.1}$$

Note that inequality (1.1) is the Chanillo-Wheeden condition (see [8]), with exponents t and 2, adapted to the Carnot-Carathèodory geometry (see [18]).

Our main result reads as follows.

Theorem 1.1 *Let K be in $A_2(\mathbb{G}) \cap G_\tau(\mathbb{G})$ with $\tau = 1 + \frac{2(Q-1)}{n+2-Q}$. Let $t > 2$. Then, for every $u \in \mathcal{C}_0^1(B_R)$, there exists a constant $C \geq 1$ such that*

$$\left(\frac{1}{\int_{B_R} K(x)\,dx} \int_{B_R} |u|^t K(x)\,dx\right)^{1/t} \leq C\,R \left(\frac{1}{\int_{B_R} \frac{1}{K(x)}\,dx} \int_{B_R} \frac{|Xu|^2}{K(x)}\,dx\right)^{1/2} \tag{1.2}$$

with

$$C = c(Q, n, t, q)\,\overline{C}\,[K^{-1}]_{A_2}^{\frac{1}{2}}\,[K]_{A_2}^{\frac{1}{t'}-\frac{1}{q'}},$$

where $\overline{C}$ is the constant in (1.1), $2 < q < t$, and B_R denotes the ball centered at the origin with radius $R > 0$. Here, $[K^{-1}]_{A_2}$ and $[K]_{A_2}$ stand for A_2 constants of K^{-1} and K, respectively.

By properties of Muchenoupt's class $A_p(\mathbb{G})$, we have that since $K \in A_2(\mathbb{G})$, then $K^{-1} \in A_2(\mathbb{G})$. Moreover, by [12, Theorem 4.8], the assumption that K belongs to $A_2(\mathbb{G}) \cap G_\tau(\mathbb{G})$, with $\tau = 1 + \frac{2(Q-1)}{n+2-Q}$, guarantees that the pair $\left(K^{-1}, K\right)$ satisfies condition (1.1). Thus, one deduces that $\left(K^{-1}, K\right)$ is a 2-admissible pair in Ω. We emphasize that the 2-admissible property of $\left(K^{-1}, K\right)$ will be used in the proof of Theorem 1.1.

The tools used to obtain inequality (1.2) are the classical ones of the Euclidean case. Neverthless, here we deal with a degeneracy into the geometry due to the presence of a differential operator Xu different from the classical gradient ∇u. In particular, this fact causes a change of metric on R^n and consequently some of the results valid for Euclidean metric have been enlarged to Carnot-Carathèodory metric.

Let us emphasize that more general weighted inequalities for Euclidean case have been extensively investigated, and are the subject of a rich literature (see e.g. [1–4,6,8–11,14,15,26]).

In the Euclidean setting, Theorem 1.1 generalizes similar result contained in [2], where the authors prove a weighted Sobolev inequality of the same type as (1.2), with the weight $K(x)$ related to the function $|u|^t$ and the weight $K^{-1}(x)$ to the gradient $|\nabla u|^2$.

Problems of this kind, involving weighted Sobolev inequalities for Carnot-Carathèodory space $\mathbb{G}$, have been systematically studied in the literature (see e.g. [7,12,16,17,19]).

The result of Theorem 1.1 is a particular case of that contained in [12, Corollary 3.4] with $v(x)$ replaced by $K(x)$ and $w(x)$ replaced by $K^{-1}(x)$. In [12] the authors show the following more general weighted Sobolev inequality

$$\left(\frac{1}{\int_{B_R} v(x)\,dx}\int_{B_R} |u|^t v(x)\,dx\right)^{1/t} \le C\,R\left(\frac{1}{\int_{B_R} w(x)\,dx}\int_{B_R} |Xu|^p w(x)\,dx\right)^{1/p}, \tag{1.3}$$

where $1 < p < t < \infty$, $C > 0$ is a constant, and (w, v) is a p-admissible pair in Ω. Herein, we prove inequality (1.2) by using different techniques which rely upon a combination of an estimate for fractional integral of first order with other some properties of $A_2(\mathbb{G})$ and $G_\tau(\mathbb{G})$ classes. Moreover, in contrast with the result in [12, Corollary 3.4], we give the explicit value of constant C in our inequality (1.2).

Our paper is organized as follows. In Sect. 2 we give some preliminary results. Actually, in Sect. 2.1 we recall definition and basic properties of Hörmander vector fields, including Carnot-Carathèodory spaces; in Sect. 2.2 we discuss the theory of Muckenhoupt's and Gehring's weights. In Sect. 3 we present the machinery we need to work with the inequality we are interested in. Finally, we prove our main theorem.

2 Preliminary results

2.1 Carnot Carathéodory spaces

Let Ω be an open (Euclidean) bounded and connected subset in $\mathbb{R}^n$, with $n \geq 2$. Let $X = \{X_1, \dots, X_m\}$ be a system of $\mathcal{C}^\infty$ vector fields on $\mathbb{R}^n$.

We denote by $Lie\,[X_1, \dots, X_m]$ the *Lie algebra* generated by $X_1, \dots, X_m$ and by their commutators of any order. We say that a field Z belongs to $Lie\,[X_1, \dots, X_m]$ if and only if Z is a finite linear combination of terms of this type

$$[X_{i_1}[X_{i_2}, \dots, [X_{i_{k-1}}, X_{i_k}]]]$$

for $k \in \mathbb{N}$, $1 \leq i_h \leq m$, $1 \leq h \leq k$.

We define, for any fixed $x \in \mathbb{R}^n$, the *Lie rank* as

$$rank\ Lie\,[X_1, \dots, X_m] = \dim V(x),$$

where $V(x) = \{Z(x) : Z \in Lie\,[X_1, \dots, X_m]\}$ is a subspace of $\mathbb{R}^n$. Henceforth, we assume that X satisfies the following *Hörmander's finite rank condition* in Ω

$$rank\,Lie\,[X_1, \dots, X_m] = n, \tag{2.1}$$

namely there exist a neighborhood Ω_0 of $\overline{\Omega}$ and $m \in \mathbb{N}$ such that the family of commutators of the vector fields in X up to length m span $\mathbb{R}^n$ at every point of Ω_0.

Let $\mathcal{C}_X$ be the family of absolutely continuous curves $\gamma : [a, b] \to \mathbb{R}^n$ such that there exist measurable functions $c_j : [a, b] \to \mathbb{R}$, with $j = 1, \dots, m$, fulfilling

$$\sum_{j=1}^{m} c_j(t)^2 \leq 1 \qquad \text{and} \qquad \gamma\,'(t) = \sum_{j=1}^{m} c_j(t) X_j(\gamma(t)) \qquad \text{for a.e. } t \in [a, b].$$

We define *Carnot-Carathéodory distance* d as

$$d(x, y) = \inf\{T > 0 : \exists\, \gamma \in \mathcal{C}_X, \gamma(0) = x, \gamma(T) = y\} \qquad \text{for } x, y \in \Omega. \tag{2.2}$$

Note that, owing to Hörmander's finite rank condition (2.1), d is a metric. This fact is not true in general. The *Carnot-Carathéodory space* $\mathbb{G}$ is the pair (Ω, d) associated to a system of $\mathcal{C}^\infty$ vector fields $X = \{X_1, \cdots, X_m\}$ on $\mathbb{R}^n$ fulfilling (2.1).

For $x \in \mathbb{R}^n$ and $R > 0$, set $B(x, R) = \{y \in \mathbb{R}^n : d(x, y) < R\}$. The basic properties of these balls have been obtained by Nagel, Stein and Wainger in [25]. In particular, in the following proposition, the authors prove that the metric d is locally Hölder continuous with respect to the Euclidean metric.

Proposition 2.1 *([25, Proposition 1.1]) Let $X_1, \cdots X_m$ be as above. Then, for any compact set $E \subset\subset \Omega$, there are positive constants c_1, c_2 and $\lambda \in (0, 1]$ such that*

$$c_1|x - y| \leq d(x, y) \leq c_2|x - y|^\lambda$$

for every $x, y \in E$.

Thanks to Proposition 2.1, the topology of Carnot-Carathéodory induced by d on Ω coincides with the Euclidean ones. In the sequel, all the distances will be understood in the sense of the Carnot-Carathéodory metric d. In particular, all the balls will be defined with respect to d.

We denote by $|\cdot|$ the Lebesgue measure in $(\mathbb{R}^n, d)$ and, by $\fint_B f(x)\,dx$, the average of a function f on the ball B, i.e.

$$\fint_B f(x)\,dx = \frac{1}{|B|}\int_B f(x)\,dx.$$

Note that the Lebesgue measure locally satisfies the following *doubling condition* (see e.g. [25]).

Proposition 2.2 *For any compact set* $E \subset\subset \Omega$*, if* $x_0 \in E$*, there exist a constant* $C_d \geq 1$*, called doubling constant, and* $R_0 > 0$ *such that*

$$|B(x_0, 2R)| \leq C_d |B(x_0, R)|$$

for $0 < R < R_0$.

Let $Y_1, \ldots, Y_l$ be the collection of the X_j's and of those commutators which are needed to generate $\mathbb{R}^n$. To each Y_i it is associated a formal "degree" $deg(Y_i) \geq 1$, namely the corresponding order of the commutator. Set $I = (i_1, \ldots, i_n)$, with $1 \leq i_j \leq l$, an n-tuple of integers. We define (see also [25]) the *degree* of I as

$$\tilde{d}(I) = \sum_{j=1}^{n} deg(Y_{i_j}).$$

For a given compact set $E \subset \mathbb{R}^n$, we define Q by

$$Q = \sup\{\tilde{d}(I) : |a_I(x)| \neq 0, x \in E\},$$

the local homogeneous dimension of E with respect to system X, where $a_I(x) = det(Y_{i_1}, \ldots, Y_{i_n})$.

We define by

$$Q(x) = \inf\{\tilde{d}(I) : |a_I(x)| \neq 0\}$$

the homogeneous dimension at $x \in \mathbb{R}^n$ with respect to X. It is obvious that $3 \leq n \leq Q(x) \leq Q$.

Just to give an idea, we consider in $\mathbb{R}^3$ the system (see [13])

$$X = \{X_1, X_2, X_3\} = \left\{\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, x_1\frac{\partial}{\partial x_3}\right\}.$$

It is easy to see that $l = 4$ and

$$\{Y_1, Y_2, Y_3, Y_4\} = \{X_1, X_2, X_3, [X_1, X_3]\}.$$

Moreover, $Q(x) = 3$ for all $x \neq 0$, whereas for any compact set E containing the origin, $Q(0) = Q = 4$.

Let Y be a metric space and μ a Borel measure in Y. Assume μ finite on bounded sets and satisfying the doubling condition on every open, bounded subset Ω in Y. We say that Q is a *homogeneous dimension* relative to Ω, if there exists a positive constant C such that

$$\frac{\mu(B)}{\mu(B_0)} \geq C \left(\frac{R}{R_0} \right)^Q$$

for any ball B_0 having center in Ω and radius $R_0 <$ diam, and any ball B centered in $x_0 \in B_0$ and having radius $R \leq R_0$.

It is well known that the doubling condition implies the existence of the homogeneous dimension Q. However, Q is not unique and it may change with Ω. Obviously, any $Q' \geq Q$ it is also a homogeneous dimension.

For a bounded open set Ω containing a family of vector fields satisfying the Hörmander's finite rank condition, the homogeneous dimension of the Carnot-Carathéodory space $\mathbb{G}$, defined with the Lebesgue measure, is given by $Q = \log_2 C_d$, where C_d is the doubling constant.

2.2 Some properties of A_p and G_q classes

In this section, we recall a few properties of Muckenhoupt's and Gehring's classes (see [22,24,27,28]).

We recall that a weight is a positive function in $L^1_{loc}(\mathbb{R}^n)$. We say that a weight w is doubling in Ω if

$$\int_{2B} w(x)\, dx \leq C \int_B w(x)\, dx,$$

where the constant C is independent by the ball $B \subset \Omega$.

We say that w is locally doubling in Ω if for each compact set $V \subset \Omega$ and $\bar{R} > 0$ there exists $C_{V,\bar{R}}$ such that

$$\int_{2B} w(x)\, dx \leq C_{V,\bar{R}} \int_B w(x)\, dx,$$

where the ball B has center in V and radius $R < \bar{R}$ and $2B$ is the ball concentric with B and having radius 2-times that of B.

We say that a weight w belongs to the class $A_p(\mathbb{G})$ (briefly, $w \in A_p(\mathbb{G})$) for some $p \in (1, +\infty)$ if

$$[w]_{A_p} = \sup_B \left(\fint_B w(x)\, dx \right) \left(\fint_B w(x)^{1-p'}\, dx \right)^{p-1} \tag{2.3}$$

is finite, where the supremum is taken over all balls $B \subset \Omega$. Here, p' denotes the Hölder conjugate of p. The quantity $[w]_{A_p}$ is called the A_p constant of w.

When $p = 1$, we say that $w \in A_1(\mathbb{G})$ if there exists a constant $c \geq 1$ such that, for every ball $B \subset \Omega$,

$$\fint_B w(x)\,dx \leq c \operatorname{ess\,inf}_B w.$$

If a weight belongs to a class A_p, it is called a Muckenhoupt weight.

A weight w is said to belong to the class $G_q(\mathbb{G})$ (briefly, $w \in G_q(\mathbb{G})$) for some $q \in (1, +\infty)$ if

$$[w]_{G_q} = \sup_B \frac{\left(\fint_B w(x)^q\,dx\right)^{\frac{1}{q}}}{\fint_B w(x)\,dx}$$

is finite. The quantity $[w]_{G_q}$ is called the G_q constant of w.

If a weight belongs to a class G_q, it is called a Gehring weight.

Here, we recall some properties of A_p classes with respect to dyadic cubes which we will be used to prove Theorem 1.1.

We use a grid $\mathcal{D}_h$ of dyadic cubes $\mathcal{Q}$, which are "almost balls", where h is a large negative integer which indexes the edgelengths $l(Q)$ of the smallest cubes $\mathcal{Q} \in \mathcal{D}_h$. In other words, the smallest edgelengths are λ^h for an appropriate geometric constant $\lambda > 1$ and each cube in the grid has edgelength λ^k for some $k \geq h$.

In particular, we will make use of a grid of dyadic cubes in the ball B_R in the same spirit of [29], where it is proved that there exists a constant $\lambda > 1$ such that, for every $h \in \mathbb{Z}$, there are points $x_j^k \in B_R$ and a family of cubes $\mathcal{D}_h = \{\mathcal{Q}_j^k\}$ for $j \in \mathbb{N}$ and $k = h, h+1, \ldots$ such that

i) $B(x_j^k, \lambda^k) \subset \mathcal{Q}_j^k \subset B(x_j^k, \lambda^{k+1})$.

ii) For each $k = h, h+1, \ldots,$ the family $\{\mathcal{Q}_j^k\}$ is pairwise disjoint in j and $B_R = \bigcup_j \mathcal{Q}_j^k$.

iii) If $h \leq k < l$, then either $\mathcal{Q}_j^k \cap \mathcal{Q}_j^l = \emptyset$ or $\mathcal{Q}_j^k \subset \mathcal{Q}_j^l$.

We call the family $\mathcal{D} = \bigcup_{h \in \mathbb{Z}} \mathcal{D}_h$ a dyadic cube decomposition of B_R and we refer to its sets as dyadic cubes which will be denoted by $\mathcal{Q}$. We observe explicitly that being $\mathcal{D}$ a decomposition of B_R, then any dyadic cube $\mathcal{Q} \in \mathcal{D}$ is contained in the ball B_R.

By [29], making use of (2.3), one can deduce the following lemma.

Lemma 2.3 *Let $w \in A_2(\mathbb{G})$ and let $\mathcal{Q}$ and $\mathcal{Q}_0$ dyadic cubes in $\mathbb{R}^n$ such that $\mathcal{Q} \subset \mathcal{Q}_0$. If $\beta > 1$, then*

$$\sum_{\mathcal{Q} \subset \mathcal{Q}_0} \left(\int_{\mathcal{Q}} w\,dx\right)^{\beta} \leq \left(c(Q, n)[w]_{A_2}\right)^{\beta - 1} \left(\int_{\mathcal{Q}_0} w\,dx\right)^{\beta}. \tag{2.4}$$

Another important property of $A_p(\mathbb{G})$ classes is given by the following proposition (see [20], [30, Chapter 5, p. 195]).

Proposition 2.4 *If $w \in A_p(\mathbb{G})$, then, for any nonnegative f,*

$$\left(\frac{1}{|Q|}\int_Q f(x)\,dx\right)^p \le [w]_{A_p}\ \frac{1}{\int_Q w(x)\,dx}\int_Q |f(x)|^p w(x)\,dx \qquad \forall Q \subset \mathbb{G}. \tag{2.5}$$

2.3 Some preliminary estimates

In order to prove our main theorem, let us prove some preliminary results.

The first lemma yields an estimate of the fractional integral of order 1 (see e.g. [5]). In general, the fractional integral of order $\alpha \in (0, Q)$ of a locally integrable function g in $\mathbb{R}^n$ is defined as

$$I_\alpha g(x) = \int_{\mathbb{R}^n} \frac{g(y)}{d(x,y)^{Q-\alpha}}\,dy \qquad \text{for } x \in \mathbb{R}^n. \tag{2.6}$$

Lemma 2.5 *Let $g \in L^1_{loc}(\mathbb{G})$ and assume that $g \ge 0$. Then*

$$I_1 g(x) \le c_0 \sum_{Q\in\mathcal{D}} \left(|Q|^{\frac{1}{Q}-1}\int_{3Q} g(y)\,dy\right)\chi_Q(x) \qquad \forall x \in \Omega, \tag{2.7}$$

where c_0 is an absolute constant.

Proof Thanks to a dyadic cube decomposition, we discretize the operator I_1

$$\begin{aligned} I_1 g(x) &= \sum_{k\in Z}\left(\int_{2^{k-1}<d(x,y)\le 2^k} \frac{g(y)}{d(x,y)^{Q-1}}dy\right) \\ &\le c_0 \sum_{k\in Z}\sum_{\substack{Q\in\mathcal{D}\\ l(Q)=2^k}} \left[\left(\frac{1}{l(Q)^{Q-1}}\int_{d(x,y)\le l(Q)} g(y)dy\right)\chi_Q(x)\right] \\ &\le c_0 \sum_{Q\in\mathcal{D}} \left[\left(|Q|^{\frac{1-Q}{Q}}\int_{3Q} g(y)dy\right)\chi_Q(x)\right], \end{aligned}$$

where the last inequality follows by $|Q| = l(Q)^Q$ and, moreover, by $B(x, l(Q)) \subset 3Q$ if $x \in Q$. Hence, inequality (2.7) is proved. □

Let us consider a Dirichlet problem in this form

$$\begin{cases} \Delta_{\mathbb{G}}\varphi = f(x) & \text{in } B_R \\ \varphi = 0 & \text{on } \partial B_R, \end{cases} \tag{2.8}$$

where $\Delta_{\mathbb{G}}$ denotes the canonical sub-Laplacian operator defined as $\Delta_{\mathbb{G}} = \sum_{j=1}^m X_j^2$, with $\{X_1, ..., X_m\}$ the family of smooth vector fields on $\mathbb{R}^n$ satisfying the Hörmander's finite rank condition.

Let $\mathcal{F}_\alpha(B_R)$ be the anisotropic Hölder space, with $\alpha \in (0, 1)$, defined by

$$\mathcal{F}_\alpha(B_R) = \left\{ f : B_R \to \mathbb{R} : \sup_{\substack{x,y \in B_R \\ x \neq y}} \frac{f(x) - f(y)}{d(x, y)^\alpha} < \infty \right\}, \tag{2.9}$$

where d is the Carnot-Carathéodory distance given by (2.2).

In [21, Theorem 3.2], the authors proved that, if $f \in \mathcal{F}_\alpha(B_R)$, then there exists a unique solution $\varphi \in \mathcal{C}^2(B_R) \cap \mathcal{C}^1(\overline{B_R})$ to problem (2.8), represented by the formula

$$\varphi(x) = \int_{B_R} \Delta_{\mathbb{G}} \varphi \, \Gamma_x(y) \, dy. \tag{2.10}$$

Here, $\Gamma_x(y)$ is the fundamental solution of the sub-Laplacian. Thanks to [21, Theorem 2.2], there exists a positive constant c such that

$$\Gamma_x(y) = c \, d(x, y)^{2-Q}. \tag{2.11}$$

Consequently, combining (2.10) and (2.11) yields

$$\varphi(x) = c \int_{B_R} \frac{f(y)}{d(x, y)^{Q-2}} dy. \tag{2.12}$$

The next lemma gives an estimate of the gradient of the solution to problem (2.8) through the fractional integral of order 1.

Lemma 2.6 *Let $f \in \mathcal{F}_\alpha(B_R)$ and let φ be the solution to problem* (2.8)*. Then, there exists a positive constant c such that*

$$|X\varphi(x)| \leq c \, I_1 f(x), \tag{2.13}$$

where $I_1(f)$ denotes the fractional integral of order 1 *of f.*

Proof Owing to (2.12), it follows that

$$X_j \varphi(x) = c \int_{B_R} \frac{f(y)}{d(x, y)^{Q-1}} X_j(d(x, y)) dy. \tag{2.14}$$

Thus,

$$\begin{aligned} |X\varphi(x)| &= \Big(\sum_{j=1}^{n} |X_j \varphi(x)|^2 \Big)^{\frac{1}{2}} \\ &= \Big(\sum_{j=1}^{n} \Big| c \int_{B_R} \frac{f(y)}{d(x, y)^{Q-1}} X_j(d(x, y)) dy \Big|^2 \Big)^{\frac{1}{2}}. \end{aligned} \tag{2.15}$$

Since $\left|X_j(d(x,y))\right| = 1$ (see [23]), by (2.15) and (2.6) one can deduce that

$$
\begin{aligned}
|X\varphi(x)| &\le \left(\sum_{j=1}^{n}\left|c\int_{B_R}\frac{f(y)}{d(x,y)^{Q-1}}dy\right|^2\right)^{\frac{1}{2}} \\
&\le n\,c\int_{B_R}\frac{f(y)}{d(x,y)^{Q-1}}dy = c\,I_1 f(x), \qquad (2.16)
\end{aligned}
$$

where the second inequality is due to the fact that $a^2+b^2 \le (a+b)^2$. □

3 Proof of main result

The following preliminary lemma will be use in the proof of Theorem 1.1.

Lemma 3.1 *If $K \in A_2(\mathbb{G})$ and $u \in \mathcal{C}_0^1(B_R)$, then*

$$
\begin{aligned}
S_1 &= \left[\sum_{\mathcal{Q}\in\mathcal{D}}\left(\int_{\mathcal{Q}}K(x)\,dx\right)^{\frac{q'}{t'}}\left(\frac{1}{\int_{\mathcal{Q}}K(x)\,dx}\int_{3\mathcal{Q}}|u|^{t-1}K(x)\,dx\right)^{q'}\right]^{\frac{1}{q'}} \\
&\le C\left(\int_{B_R}|u|^t K(x)\,dx\right)^{\frac{1}{t'}}, \qquad (3.1)
\end{aligned}
$$

where $2 < q < t$ and $C = c(Q,n,t,q)[K]_{A_2}^{\frac{1}{t'}-\frac{1}{q'}}$.

Proof For each $h \in \mathbb{Z}$, we set

$$
\mathcal{C}^h = \left\{\mathcal{Q}\text{ dyadic cube} : 2^h < \frac{1}{\int_{\mathcal{Q}}K(x)\,dx}\int_{\mathcal{Q}}|u|^{t-1}K(x)\,dx \le 2^{h+1}\right\}. \qquad (3.2)
$$

Note that, if $\mathcal{Q}$ is any dyadic cube such that $|u|^{t-1}K(x)$ is not identically zero on $\mathcal{Q}$, then $\mathcal{Q}$ belongs to only one collection $\mathcal{C}^h$.

For each $h \in \mathbb{Z}$, let us build the collection $\{\mathcal{Q}_j^h\}_j$ of pairwise disjoint maximal dyadic cubes (maximal with respect to inclusion) in $\mathcal{C}^h$. If $\mathcal{Q} \in \mathcal{C}^h$, then there exists $j \in \mathbb{N}$ such that $\mathcal{Q} \subset \mathcal{Q}_j^h$. Note also that for each fixed h, the cubes $\mathcal{Q}_j^h$ are disjoint with respect to j. Nevertheless, they may not be disjoint for different values of h.

By (3.2),

$$
\begin{aligned}
S_1 &\le \left(\sum_{h\in\mathbb{Z}} 2^{(h+1)q'} \sum_{Q\in\mathcal{C}^h} \left(\int_Q K(x)\,dx \right)^{\frac{q'}{t'}} \right)^{\frac{1}{q'}} \\
&\le \left(\sum_{h\in\mathbb{Z}} 2^{(h+1)q'} \sum_{j\in\mathbb{N}} \sum_{Q\subset Q_j^h} \left(\int_Q K(x)\,dx \right)^{\frac{q'}{t'}} \right)^{\frac{1}{q'}}. \qquad (3.3)
\end{aligned}
$$

By Lemma 2.3, since $\dfrac{q'}{t'} > 1$, we have

$$
\sum_{Q\subset Q_j^h} \left(\int_Q K(x)\,dx \right)^{\frac{q'}{t'}} \le \left(c(Q,n)[K]_{A_2} \right)^{\frac{q'}{t'}-1} \left(\int_{Q_j^h} K(x)\,dx \right)^{\frac{q'}{t'}}. \qquad (3.4)
$$

By (3.3) and (3.4), we deduce

$$
S_1 \le \left(\left(c(Q,n)[K]_{A_2} \right)^{\frac{q'}{t'}-1} \sum_{h\in\mathbb{Z}} 2^{(h+1)q'} \sum_{j\in\mathbb{N}} \left(\int_{Q_j^h} K(x)\,dx \right)^{\frac{q'}{t'}} \right)^{\frac{1}{q'}}. \qquad (3.5)
$$

Since $Q_j^h \in \mathcal{C}^h$, by (3.2)

$$
\frac{1}{\int_{Q_j^h} K(x)\,dx} \int_{Q_j^h} |u|^{t-1} K(x)\,dx > 2^h.
$$

Thus,

$$
\frac{1}{\int_{Q_j^h} K(x)\,dx} \int_{Q_j^h \cap \{x\in B_R : |u|>2^{h-10}\}} |u|^{t-1} K(x)\,dx \ge C_1\, 2^h, \qquad (3.6)
$$

where $C_1 = C_1(Q,n)$ is a constant. Consequently,

$$
\int_{Q_j^h} K(x)\,dx \le C_1\, 2^{-h} \int_{Q_j^h \cap \{x\in B_R : |u|>2^{h-10}\}} |u|^{t-1} K(x)\,dx. \qquad (3.7)
$$

Owing to (3.5) and (3.7),

$$
S_1 \le C_2 \left(\sum_{h\in\mathbb{Z}} 2^{(h+1)q'} \sum_{j\in\mathbb{N}} \left(2^{-h} \int_{Q_j^h \cap \{x\in B_R :\ |u|>2^{h-10}\}} |u|^{t-1} K(x)\,dx \right)^{\frac{q'}{t'}} \right)^{\frac{1}{q'}}, \qquad (3.8)
$$

where $C_2 = C_1(Q, n)\ \left(c(Q, n)[K]_{A_2}\right)^{\frac{1}{t'}-\frac{1}{q'}}$.

By (3.8), we have

$$\begin{aligned}
S_1 &\le C_2 \left(\sum_{h\in\mathbb{Z}} 2^{(h+1)-\frac{h}{t'}} \sum_{j\in\mathbb{N}} \int_{Q_j^h \cap \{x\in B_R:\ |u|>2^{h-10}\}} |u|^{t-1} K(x)\ dx \right)^{\frac{1}{t'}} \\
&= 2^{\frac{1}{t'}} C_2 \left(\sum_{h\in\mathbb{Z}} 2^{\frac{h}{t}} \sum_{j\in\mathbb{N}} \int_{Q_j^h \cap \{x\in B_R:\ |u|>2^{h-10}\}} |u|^{t-1} K(x)\ dx \right)^{\frac{1}{t'}} \\
&\le 2^{\frac{1}{t'}} C_2 \left(\sum_{h\in\mathbb{Z}} 2^{h} \int_{\{x\in B_R:\ |u|>2^{h-10}\}} |u|^{t-1} K(x)\ dx \right)^{\frac{1}{t'}} \\
&= 2^{\frac{1}{t'}} C_2 \left(\int_{B_R} |u|^{t-1} K(x) \sum_{\{h\in\mathbb{Z}:\ 2^h<2^{10}|u|\}} 2^h\ dx \right)^{\frac{1}{t'}},
\end{aligned} \tag{3.9}$$

where the first inequality is a consequence of the fact that $\sum_h a_h^{\frac{q'}{t'}} \le \left[\sum_h a_h\right]^{\frac{q'}{t'}}$, the third one holds because, fixed $h \in \mathbb{Z}$, Q_j^h are disjoint in j, the fourth one is due to Fubini's type Theorem. To conclude the proof, we have to evaluate the quantity

$$\sum_{\{h\in\mathbb{Z}:\ 2^h<2^{10}|u|\}} 2^h. \tag{3.10}$$

Set $H = \log_2(2^{10}|u|)$. Thus, (3.10) yields

$$\begin{aligned}
\sum_{h=-\infty}^{H} 2^h &= \sum_{h=-H}^{+\infty} \left(\frac{1}{2}\right)^h = \sum_{h=-H}^{+\infty} \left(\frac{1}{2}\right)^{h+H-H} = \left(\frac{1}{2}\right)^{-H} \sum_{h=-H}^{\infty} \left(\frac{1}{2}\right)^{h+H} \\
&= \left(\frac{1}{2}\right)^{-H} \sum_{m=0}^{+\infty} \left(\frac{1}{2}\right)^{m} = \left(\frac{1}{2}\right)^{-H} 2 = 2^{\log_2(2^{10}|u|)}\, 2 = 2^{11}|u|.
\end{aligned} \tag{3.11}$$

Then, by (3.9) and (3.11), we obtain

$$S_1 \le C_3 \left(\int_{B_R} |u|^t K(x)\ dx \right)^{\frac{1}{t'}},$$

with $C_3 = 2^{\frac{1}{t'}+11} C_1(Q, n)\ \left(c(Q, n)[K]_{A_2}\right)^{\frac{1}{t'}-\frac{1}{q'}}$ and inequality (3.1) is proved. □

Now we are in position to prove our main result.

Proof of Theorem 1.1. By Theorem 3.2 of [21], there exists a solution φ to the following Dirichlet problem for sub-Laplacian

$$\begin{cases} \Delta_{\mathbb{G}}\varphi = |u|^{t-1}K(x) & \text{in } B_R \\ \varphi = 0 & \text{on } \partial B_R, \end{cases}$$

with $u \in \mathcal{C}_0^1(B_R)$. By Lemma 2.6, we get

$$|X\varphi(x)| \leq c\, I_1(|u|^{t-1}K(x)) \qquad \forall x \in B_R, \tag{3.12}$$

where c is a positive constant.

Thanks to Lemma 2.5, it follows that

$$I_1(|u|^{t-1}K)(x) \leq c_0 \sum_{\mathcal{Q}\in\mathcal{D}} \left(|\mathcal{Q}|^{\frac{1}{n}-1} \int_{3\mathcal{Q}} |u(y)|^{t-1}K(y)\, dy\right) \chi_{\mathcal{Q}}(x) \qquad \forall x \in B_R, \tag{3.13}$$

where c_0 is an absolute constant.

Combining (3.12) and (3.13) yields

$$\begin{aligned}
&\int_{B_R} |u(x)|^t K(x)\, dx \\
&= \int_{B_R} |u(x)||u(x)|^{t-1}K(x)\, dx = \int_{B_R} |u(x)|\Delta_{\mathbb{G}}\varphi\, dx \\
&\leq \int_{B_R} |Xu||X\varphi|\, dx \leq c \int_{B_R} |Xu| I_1(|u|^{t-1}K)(x)\, dx \\
&\leq C_6 \int_{B_R} |Xu(x)| \sum_{\mathcal{Q}\in\mathcal{D}} \left(|\mathcal{Q}|^{\frac{1}{n}-1} \int_{3\mathcal{Q}} |u(y)|^{t-1}K(y)\, dy\right) \chi_{\mathcal{Q}}(x)\, dx \\
&= C_6 \int_{B_R} \sum_{\mathcal{Q}\in\mathcal{D}} |\mathcal{Q}|^{\frac{1}{n}-1} |Xu(x)|\, \chi_{\mathcal{Q}}(x) \left(\int_{3\mathcal{Q}} |u(y)|^{t-1}K(y)\, dy\right)\, dx \\
&= C_6 \sum_{\mathcal{Q}\in\mathcal{D}} |\mathcal{Q}|^{\frac{1}{n}-1} \int_{B_R\cap\mathcal{Q}} |Xu(x)|\, dx \left(\int_{3\mathcal{Q}} |u(y)|^{t-1}K(y)\, dy\right) \\
&= C_6 \sum_{\mathcal{Q}\in\mathcal{D}} |\mathcal{Q}|^{\frac{1}{n}} \left(\frac{1}{|\mathcal{Q}|}\int_{\mathcal{Q}} |Xu|\, dx\right) \left(\int_{3\mathcal{Q}} |u(y)|^{t-1}K(y)\, dy\right),
\end{aligned} \tag{3.14}$$

where $C_6 = c\, c_0$. Note that the last inequality is the consequence of the fact that $B_R \cap \mathcal{Q} = \mathcal{Q}$.

By (2.5),

$$\frac{1}{|\mathcal{Q}|}\int_{\mathcal{Q}} |Xu|\, dx \leq \left[K^{-1}\right]_{A_2}^{\frac{1}{2}} \left(\frac{1}{\int_{\mathcal{Q}} \frac{1}{K(x)}\, dx} \int_{\mathcal{Q}} \frac{|Xu|^2}{K(x)}\, dx\right)^{\frac{1}{2}}. \tag{3.15}$$

Coupling inequalities (3.14) and (3.15) tells us that

$$\int_{B_R} |u|^t K(x)\, dx N$$
$$\leq C_6 \left[K^{-1}\right]_{A_2}^{\frac{1}{2}} \sum_{Q\in\mathcal{D}} |Q|^{\frac{1}{n}} \left(\frac{1}{\int_Q \frac{1}{K(x)}\, dx} \int_Q \frac{|Xu|^2}{K(x)}\, dx\right)^{\frac{1}{2}} \left(\int_{3Q} |u|^{t-1} K(y)\, dy\right). \tag{3.16}$$

By (3.16), the following chain of inequality holds

$$\int_{B_R} |u|^t K(x)\, dx$$
$$\leq C_7 |B_R|^{1/n} \frac{\left(\int_{B_R} K(x)\, dx\right)^{1/t}}{\left(\int_{B_R} \frac{1}{K(x)}\, dx\right)^{1/2}} \sum_{Q\in\mathcal{D}} \left(\int_Q K(x)\, dx\right)^{-1/t} \left(\int_Q \frac{1}{K(x)}\, dx\right)^{1/2}$$
$$\times \left(\frac{1}{\int_Q \frac{1}{K(x)}\, dx} \int_Q \frac{|Xu|^2}{K(x)}\, dx\right)^{1/2} \left(\int_{3Q} |u|^{t-1} K(x)\, dx\right)$$
$$= C_7 |B_R|^{1/n} \frac{\left(\int_{B_R} K(x)\, dx\right)^{1/t}}{\left(\int_{B_R} \frac{1}{K(x)}\, dx\right)^{1/2}} \sum_{Q\in\mathcal{D}} \left(\int_Q \frac{1}{K(x)}\, dx\right)^{1/2}$$
$$\left(\frac{1}{\int_Q \frac{1}{K(x)}\, dx} \int_Q \frac{|Xu|^2}{K(x)}\, dx\right)^{1/2} \times \left(\int_Q K(x)\, dx\right)^{1/t'-1} \int_{3Q} |u|^{t-1} K(x)\, dx$$
$$\leq C_7 |B_R|^{1/n} \frac{\left(\int_{B_R} K(x)\, dx\right)^{1/t}}{\left(\int_{B_R} \frac{1}{K(x)}\, dx\right)^{1/2}} \Bigg[\sum_{Q\in\mathcal{D}} \left(\int_Q \frac{1}{K(x)}\, dx\right)^{q/2}$$
$$\left(\frac{1}{\int_Q \frac{1}{K(x)}\, dx} \int_Q \frac{|Xu|^2}{K(x)}\, dx\right)^{q/2}\Bigg]^{1/q}$$
$$\times \left[\sum_{Q\in\mathcal{D}} \left(\int_Q K(x)\, dx\right)^{q'/t'} \left(\frac{1}{\int_Q K(x)\, dx} \int_{3Q} |u|^{t-1} K(x)\, dx\right)^{q'}\right]^{1/q'}$$
$$= C_7 |B_R|^{1/n} \frac{\left(\int_{B_R} K(x)\, dx\right)^{1/t}}{\left(\int_{B_R} \frac{1}{K(x)}\, dx\right)^{1/2}} \left[\sum_{Q\in\mathcal{D}} \left(\int_Q \frac{|Xu|^2}{K(x)}\, dx\right)^{q/2}\right]^{1/q}$$
$$\left[\sum_{Q\in\mathcal{D}} \left(\int_Q K(x)\, dx\right)^{q'/t'} \left(\frac{1}{\int_Q K(x)\, dx} \int_{3Q} |u|^{t-1} K(x)\, dx\right)^{q'}\right]^{1/q'}$$

$$\le C_7|B_R|^{1/n}\frac{\left(\int_{B_R}K(x)\,dx\right)^{1/t}}{\left(\int_{B_R}\frac{1}{K(x)}\,dx\right)^{1/2}}\left(\int_{B_R}\frac{|Xu|^2}{K(x)}\,dx\right)^{1/2}\Bigg[\sum_{Q\in\mathcal{D}}\left(\int_Q K(x)\,dx\right)^{q'/t'}$$

$$\left(\frac{1}{\int_Q K(x)\,dx}\int_{3Q}|u|^{t-1}K(x)\,dx\right)^{q'}\Bigg]^{1/q'}$$

$$= C_7|B_R|^{1/n}\frac{\left(\int_{B_R}K(x)\,dx\right)^{1/t}}{\left(\int_{B_R}\frac{1}{K(x)}\,dx\right)^{1/2}}\left(\int_{B_R}\frac{|Xu|^2}{K(x)}\,dx\right)^{1/2}S_1, \tag{3.17}$$

where the first inequality follows by Chanillo-Wheeden condition (1.1), the second one holds since $1/t = 1 - 1/t'$, the third one is due to Hölder's inequality, for $2 < q < t$, and the fifty one comes from the fact that $\mathcal{D}$ is a decomposition of B_R. Here, constant $C_7 = C_6\,\overline{C}\,\left[K^{-1}\right]_{A_2}^{\frac{1}{2}}$. The quantity S_1 is introduced in Lemma 3.1 above.

Combining (3.17) and (3.1) shows that

$$\left(\int_{B_R}|u|^t K(x)\,dx\right)^{1/t} \le C_8|B_R|^{1/n}\frac{\left(\int_{B_R}K(x)\,dx\right)^{1/t}}{\left(\int_{B_R}\frac{1}{K(x)}\,dx\right)^{1/2}}\left(\int_{B_R}\frac{|Xu|^2}{K(x)}\,dx\right)^{1/2}, \tag{3.18}$$

where $C_8 = c(Q,n,t,q)\,\overline{C}\,[K^{-1}]_{A_2}^{\frac{1}{2}}[K]_{A_2}^{\frac{1}{t'}-\frac{1}{q'}}$. Then, inequality (1.2) follows. □

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Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interest.

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