Removable singularities for degenerate elliptic equations without conditions on the growth of the solution*

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Abstract: The aim of the paper is to state removable singularities results for solutions of fully nonlinear degenerate elliptic equations without any knowledge of the behaviour of the solution approaching the singular set and to obtain unconditional results of Brezis-Veron type for operators defined as the partial sum of the eigenvalues of the Hessian matrix.

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1 Introduction and principal results

We will consider a class of second-order elliptic equations including

$$\mathcal{P}_p^+(D^2u) - |u|^{s-1}u = f(x) \tag{1.1}$$

and its dual

$$\mathcal{P}_p^-(D^2u) - |u|^{s-1}u = f(x) \tag{1.2}$$

in a domain of $\mathbb{R}^n$ with $n > 2$ for a positive integer $p \in [3, n]$ and a real number $s > 1$. Here

$$\begin{aligned} \mathcal{P}_p^+(X) &= \lambda_{n-p+1}(X) + \cdots + \lambda_n(X), \\ \mathcal{P}_p^-(X) &= \lambda_1(X) + \cdots + \lambda_p(X)\,, \end{aligned} \tag{1.3}$$

where $\lambda_i(X)$, $i = 1, \ldots, n$, are the eigenvalues of $X \in \mathcal{S}^n$, the set of the $n \times n$ real symmetric matrix, arranged in increasing order: $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$. Note that $\mathcal{P}_p^\pm(D^2u)$ are degenerate elliptic, according to the definition which will be given below in Section 2, but not uniformly elliptic, except for the case $p = n$, when $\mathcal{P}_n^\pm(D^2u) = \Delta u$ is the Laplace operator.

Throughout the paper, for a C^2-function u in an open set of $\mathbb{R}^n$, we will denote by $Du = (D_iu)$ its gradient and by $D^2u = [D_{ij}u]$ its Hessian matrix.

More generally, if u is a continuous function, equations (1.1) and (1.2) will be intended in the viscosity sense, which will be specified in the sequel.

The operators $\mathcal{P}_p^\pm$ arise in the characterizations of manifolds with partially positive curvature, see Wu [56] and Sha [53], [54], and have been largely studied by Harvey and Lawson [28], [29], [30], [31], [32], [34] in the framework of their theory of subequations with respect to existence, uniqueness, removable and prescribed singularities, which have also been considered by Caffarelli, Y.Y.Li and Nirenberg [11], [12], [13], and recently with respect to the existence of entire subsolutions related to Keller-Osserman conditions by Capuzzo Dolcetta, Leoni and Vitolo [14], [15].

It is well known that, also in the case of uniformly elliptic operators and for an isolated singularity, we need conditions on the growth of the solution, related to fundamental solutions; see for instance Gilbarg and Serrin [26] and the subsequent papers by Serrin [48], [50], [51], [52]. In the case of the Laplace equation $\Delta u = 0$, the fundamental solution

$$\mathcal{E}(x) = \left(\frac{1}{|x|}\right)^{n-2} \tag{1.4}$$

provides a smooth positive superharmonic function in $(\mathbb{R}^n)^* = \{x \in \mathbb{R}^n : x \neq 0\}$ such that $\mathcal{E}(x) \to \infty$ as $x \to 0$.

Consequently, if u is any harmonic function in the punctured ball $B_\rho^* = \{0 < |x| < \rho\}$ for some $\rho > 0$, and $u(x) = o(\mathcal{E}(x))$ as $x \to 0$, then $u_\varepsilon(x) \equiv u(x) - \varepsilon\mathcal{E}(x)$ is in turn a subharmonic function in B_ρ^* such that $u_\varepsilon(x) \to -\infty$ as $x \to 0$. Therefore, if v is the solution of the Dirichlet problem

$$\begin{cases} \Delta v = 0 & in \;\; B_r \\ v = u & on \;\; \partial B_r \end{cases} \tag{1.5}$$

for any $r < \rho$ by the comparison principle we have $u \leq v + \varepsilon\mathcal{E}$ in B_r^* and therefore, letting $\varepsilon \to 0^+$, we get $u \leq v$ in B_ρ^*. In a similar manner, by comparison we also get $u \geq v$ in B_ρ^* so that $v(x) = u(x)$ in B_ρ^* provides the harmonic extension of u to B_ρ; in other words, the origin is a removable singularity for the Laplace equation.

The same argument can be carried out in the case of fully nonlinear uniformly elliptic operators with ellipticity constants $0 < \lambda \leq \Lambda$ such that $n > 1+\frac{\Lambda}{\lambda}$, using the corresponding fundamental solutions

$$\mathcal{E}(x) = \left(\frac{1}{|x|}\right)^{\frac{\lambda}{\Lambda}(n-1)-1} . \tag{1.6}$$

for which we refer to Labutin [39]. For Hessian and curvature equations see Labutin [41], Takimoto [55] and the references therein.

For isolated singularities of nonnegative solutions of the p-Laplace operator and ∞-Laplace operator we refer to Serrin [49], Manfredi [45] and Savin, Wang and Yu [47], respectively. See also Cirstea and Du [17], Brandolini, Chiacchio, Cirstea and Trombetti [7], Cirstea [16] for nonnegative solutions of semilinear equations, and Y.Y.Li [43] for conformally invariant fully nonlinear equations.

Here we are interested to unconditional removability results, which do not require any condition on the solution when approaching the singularity.

The issue under consideration goes back to the well known result of Brezis and Veron [9], who proved that equation

$$\Delta u - |u|^{s-1}u = 0 \tag{1.7}$$

with $s \geq \frac{n}{n-2}$ has the property that any isolated singularity is removable, provided $n \geq 3$. This result was already known as a consequence of a theorem of Loewner and Nirenberg [44] in the case $s = \frac{n+2}{n-2}$ and has been generalized by Labutin [38] to general fully nonlinear uniformly elliptic equations

$$F(D^2u) - |u|^{s-1}u = 0 \tag{1.8}$$

with $s \geq \frac{\lambda(n-1)+\Lambda}{\lambda(n-1)-\Lambda}$. If $F(D^2u)$ in (1.8) is the Laplace operator $\Delta u = \mathcal{P}_n^+(D^2u)$, then $\lambda = \Lambda = 1$ and we recover the aforementioned removability condition $s \geq \frac{n}{n-2}$ for equation (1.7).

Note that no condition is assumed on the solution u. That $u(x) = o(\mathcal{E}(x))$ as $x \to 0$ can be deduced from the fact that u is a solution of the equation, and a function $u(x) >> \mathcal{E}(x)$ cannot be a solution of equation (1.7) as soon as $s = \frac{n}{n-2}$. Results of this kind for other classes of fully nonlinear uniformly elliptic operators are obtained by Felmer and Quaas [22].

Our aim is to extend this result to equations (1.8) having a degenerate elliptic principal part $F_p(D^2u)$ such that

$$\mathcal{P}_p^-(X) \leq F_p(X) \leq \mathcal{P}_p^+(X), \quad X \in \mathcal{S}^n\,. \tag{1.9}$$

For this purpose we prove an extended comparison principle in punctured domains between upper semicontinuous (usc) subsolutions and lower semicontinuous (lsc) supersolutions of equation

$$F_p(D^2u) - |u|^{s-1}u = f(x) \tag{1.10}$$

in viscosity sense, also admitting a moderate singularity of $f(x)$.

The natural functions to be compared with are the so-called fundamental solutions of the operator F_p. Here we use the fundamental solutions $u = \mathcal{E}_p$, for $2 < p \leq n$, of the equations $\mathcal{P}_p^+(D^2u) = 0$, corresponding to the maximal operator in the considered class:

$$\mathcal{E}_p(x) = \left(\frac{1}{|x|}\right)^{p-2}, \quad x \neq 0. \tag{1.11}$$

For $p = n$, the definition (1.11) returns the harmonic function $\mathcal{E}_n$, see (1.6), which is the fundamental solution of the Laplace operator.

Let us recall that the fundamental solutions of the Laplace originate from the solution of the Poisson equation

$$\Delta u = \delta$$

in distributional sense with the Dirac unit mass distribution concentrated at the origin such that $u(x) \to 0$ as $|x| \to \infty$, with the physical meaning that the effects from a point disappear far away from it. By linearity, the knowledge of $\mathcal{E}$ allow to construct solutions with a different distribution f by superposition (convolution).

Since the Bôcher theorem [6], every solution u, bounded from below, of the Laplace equation $\Delta u = 0$ in the punctured ball $B_r = \{x \in \mathbb{R}^n : 0 < |x| < r\}$, satisfies the inequalities

$$\gamma\mathcal{E}_n - C \leq u \leq \gamma\mathcal{E}_n + C$$

with $\gamma > 0$, if u cannot be extended to a harmonic function in B_r.

In the fully nonlinear setting, due to by Labutin [39], the same conclusion holds true when the Laplace equation is replaced by the maximal Pucci equation with ellipticity constants $\lambda > 0$ and $\Lambda \geq \lambda$,

$$\mathcal{M}_{\lambda,\Lambda}(D^2u) := \Lambda\sum_{i=1}^{n}\lambda_i^+ - \lambda\sum_{i=1}^{n}\lambda_i^- = 0,$$

and the fundamental solution $\mathcal{E}_n$ by $\mathcal{E}_{(n-1)\frac{\lambda}{\Lambda}+1}$. For general uniformly elliptic equations we refer to Armstrong, Sirakov and Smart [4] for a detailed discussion and more recent results about fundamental solutions. See also [22].

Roughly speaking, since the Bôcher theorem, the fundamental solutions go to infinity at the origin with a typical growth order, which is in turn a limiting growth for the removability of isolated singularities, as in the previous examples.

In this sense, since $\mathcal{P}^+(D^2\mathcal{E}_p(x)) = 0$ for $x \neq 0$, the functions $\mathcal{E}_p$ defined in (1.11) are the fundamental solutions for the operator $\mathcal{P}_p^+$.

We will set, for $s > 1$,

$$p_s = \frac{2s}{s-1} \quad (> 2) \tag{1.12}$$

observing that $p_s \leq p$ if and only if $s \geq \frac{p}{p-2}$, as in the assumption of the following result.

Theorem 1.1. *Let n and p be positive integers such that $2 < p \leq n$, and $s \geq \frac{p}{p-2}$. Suppose f is a continuous function in some punctured ball of $\mathbb{R}^n$, say B_ρ^* with $\rho > 0$, such that*

$$\lim_{x\to 0} |x|^{(p-2)s} f(x) = 0\,. \tag{1.13}$$

If u is both a viscosity subsolution of equation (1.1) *and a viscosity supersolution of equation* (1.2) *in B_ρ^*, then*

$$\lim_{x\to 0} \frac{u(x)}{\mathcal{E}_p(x)} = 0\,. \tag{1.14}$$

Remark 1.2. If instead $1 < s < \frac{p}{p-2}$, i.e. $p < p_s$, then $u(x) = C|x|^{-(p_s-2)}$, with $C^{s-1} = (p_s - 2)\,(p_s - p)$, is a solution of equation $\mathcal{P}_p^+(D^2u) - |u|^{s-1}u = 0$ in $R^n\backslash\{0\}$. Hence the conclusion of Theorem 1.1 fails to hold.

Note also that condition (1.13) on the growth of $f(x)$ cannot be relaxed. In fact, if $s = \frac{p}{p-2}$, i.e. $p = p_s = (p-2)s$, then $u(x) = \mathcal{E}_p(x)$ is a solution of equation

$$\mathcal{P}_p^+(D^2u) - |u|^{s-1}u = -\left(\frac{1}{|x|}\right)^p . \qquad \square$$

Since $p > 2$, the conclusion of above Theorem 1.1 is obviously true when $f(x)$ is bounded. If this is the case, the following corollary shows that the solution u is bounded.

Corollary 1.3. *Suppose that assumptions of Theorem 1.1 are fulfilled with $f(x)$ bounded in B_ρ^*. If $u \in C(B_\rho^*)$ is both a subsolution of equation* (1.1) *and a supersolution of equation* (1.2) *in viscosity sense, then u is bounded in B_r^* for all $r \in (0, \rho)$.*

As a consequence of Corollary 1.3, we get the following result of unconditional removability.

Theorem 1.4. *Let n and p be positive integers such that $2 < p \le n$. Let Ω be a domain* (*open connected set*) *of $\mathbb{R}^n$, and set $\Omega^* = \{x \in \Omega : x \ne x_0\}$. Suppose F_p is a continuous degenerate elliptic operator satisfying*

$$\mathcal{P}_p^-(Y) \le F_p(X+Y) - F_p(X) \le \mathcal{P}_p^+(Y), \quad X \in \mathcal{S}^n , \tag{1.9$'$}$$

and f is a continuous function in Ω.

If u is a continuous viscosity solution of equation (1.10) *in Ω^* with $s \ge \frac{p}{p-2}$, then u can be extended to a solution in all Ω.*

Remark 1.5. As in [38], the result can be furthermore generalized to equation

$$F_p(D^2u) - g(u) = f(x), \tag{1.15}$$

where g is a continuous real function such that

$$\limsup_{t\to-\infty} \frac{g(t)}{|t|^{\frac{p}{p-2}}} < 0 < \liminf_{t\to\infty} \frac{g(t)}{|t|^{\frac{p}{p-2}}} . \tag{1.16}$$

In fact, by the first one, we have $g(u^+(x)) \ge \varepsilon(u^+)^{s-1} - C$ for $s = \frac{p}{p-2}$ and positive constants ε, C so that $u = u^+$ satisfies the differential inequality

$$\mathcal{P}_p^+(D^2u) - \varepsilon|u|^{s-1}u \ge f(x) - C$$

and therefore u^+ satisfies assumptions (4.1) and (4.2), which imply (4.3).

On the other side, we also have $g(-u^-(x)) \leq -\varepsilon(u^-)^{\frac{p}{p-2}} + C$, from which $u = -u^-$ satisfies the differential inequality

$$\mathcal{P}_p^-(D^2(u) - \varepsilon|u|^{s-1}u \leq f^+(x) + C$$

and therefore $-u^-$ satisfies assumptions (4.4) and (4.5), which imply (4.6).

This shows that Lemma 4.1 and therefore Theorem 1.1, which is the basic result, continues to hold when $|u|^{s-1}u$ is replaced with a function $g(u)$ satisfying (1.16).

Example 1.6. Theorem 1.1 and Corollary 1.3 hold true, for instance, for all operators which are partial sums of p eigenvalues such as

$$F_p(X) = \lambda_{i_1}(X) + \cdots + \lambda_{i_p}(X), \tag{1.17}$$

for every choice of p positive integers less than n. Theorem 1.4 holds in particular in the extremal cases $(i_1, \ldots, i_p) = (1, \ldots, p)$ and $(i_1, \ldots, i_p) = (n-p+1, \ldots, n)$, corresponding to $\mathcal{P}_p^-(X)$ and to $\mathcal{P}_p^+(X)$, respectively.

Unconditional results of this kind hold true, for instance, in the case of isolated singularities of minimal surface equation, see Bers [5] in the two-dimensional case, De Giorgi and Stampacchia [20] in higher dimensions.

On the other hand, we can equally find in literature many results about nonisolated removable singular sets, generally assuming that the solutions are bounded. For instance, it is well known that the sets E such that all *bounded* harmonic functions outside E can be extended across E are characterized by having zero capacity; see [35]. For generalizations of this result we refer to Brezis and Nirenberg [8], Labutin [39], [40], [41] and to recent works of Caffarelli, Y.Y.Li and Nirenberg [11], [12], [13], as well as of Harvey and Lawson [32]; see also Amendola, Galise and Vitolo [3], Galise and Vitolo [25].

There are also cases in the literature of unconditional removable singularities which are not isolated. For instance, see [8], where Brezis and Nirenberg show that sets of Newtonian capacity $C_{n-2}(E) = 0$ are removable for a class of equations including $\Delta u - u|Du|^2 = f(x)$ with a smooth $f(x)$. More recently, in Section 6 of [32] Harvey and Lawson, using the restriction theorem of [33], proved that sets E with a suitable Hausdorff measure equal to zero, are removable singularities for subsolutions of the p-th branch of Monge-Ampère equation $\lambda_p(D^2u) = 0$.

In spit of thise, we attack the problem to find how large the removable singular sets can be for our equation (1.10) in the viscosity sense assuming no condition on the size of the solution.

To deal with this issue, we consider a compact subset $E \subset \Omega$ and set $d_E(x) = dist(x, E)$. For sufficiently small $r > 0$ the set $\{d_E(x) \leq r\}$ is still contained in Ω and, for a function $f(x)$ defined in $\Omega \backslash E$, we set

$$\limsup_{x \to E} f(x) = \lim_{r \to 0} \sup_{0 < d_E(x) < r} f(x); \quad \liminf_{x \to E} f(x) = \lim_{r \to 0} \inf_{0 < d_E(x) < r} f(x). \tag{1.18}$$

Using the estimates on the distance function by Ambrosio and Soner [2], we can state the following

Theorem 1.7. *Let n and p be positive integers such that $3 \leq p \leq n$. Let $k \in \mathbb{N}$ be such that $n - k < p - 2$ and define*

$$\alpha := (p - 2) - (n - k) > 0\,. \tag{1.19}$$

Suppose that Γ is a smooth embedded manifold in $\mathbb{R}^n$ of codimension $k < n$, and set

$$\delta(x) = dist(x, \Gamma)\,. \tag{1.20}$$

Let s be a real number such that $s \geq \frac{\alpha+2}{\alpha}$ and f be a continuous function in $\Omega \backslash E$, where E is a compact subset of Ω such that $E \Subset \Gamma$ in the relative topology and all points of E are limit points for $\Omega \backslash \Gamma$. Suppose also that

$$\limsup_{x \to E} d_E^{\alpha s}(x) |f(x)| = 0\,. \tag{1.21}$$

If u is both a viscosity subsolution of equation (1.1) *and a viscosity supersolution of equation* (1.2)*, then*

$$\limsup_{x \to E} \frac{u(x)}{\delta^{-\alpha}(x)} = 0 = \liminf_{x \to E} \frac{u(x)}{\delta^{-\alpha}(x)}\,. \tag{1.22}$$

Since $p > n + 2 - k$, in Theorem 1.7 we can plainly take f bounded in a neighbourhood of E. In this the case we can show the following corollary.

Corollary 1.8. *Suppose that assumptions of Theorem 1.7 are fulfilled with $f(x)$ bounded in a neighborhood of E. If $u \in C(\Omega \backslash E)$ is both a viscosity subsolution of equation* (1.1) *and a viscosity supersolution of equation* (1.2) *in $\Omega \backslash E$, then u is bounded across E, namely in all domains $\Omega' \backslash E$ such that $\Omega' \Subset \Omega$.*

As a consequence of Corollary 1.8, we get the following generalization of Theorem 1.4.

Theorem 1.9. *Suppose that assumptions of Theorem 1.7 are fulfilled with F_p satisfying* $(1.9)'$ *and f continuous in Ω.*
If u is a continuous viscosity solution of equation (1.10) *in $\Omega \backslash E$ with*

$$s \geq \frac{p-(n-k)}{p-2-(n-k)}, \tag{1.23}$$

then u can be extended to a solution in all Ω.

We point out that the results of Theorem 1.7, Corollary 1.8 and Theorem 1.9 for $k=n$ return Theorem 1.1, Corollary 1.3 and Theorem 1.4, respectively.

The paper is organized as follows. In Section 2 we recall the principal notions about elliptic operators and viscosity solutions, introducing the degenerate elliptic operator $\mathcal{P}_p^+$ and its dual $\mathcal{P}_p^-$, defined as partial sum of eigenvalues; in Section 3 we deduce from the comparison principles of Section 2 useful bounds for subsolutions and supersolutions; Sections 4 and 5 contain the proof of the main results of the paper for isolated and not isolated singularities, respectively, starting from the necessary conditions of Lemma 4.1 and Subsection 5.2. We also add for the convenience of the reader Section 6, devoted to existence and uniqueness of equations involving degenerate elliptic operators $F_p \in (\mathcal{P}_p^-, \mathcal{P}_p^+)$.

2 Notations and preliminary results

In this Section we briefly recall the notions of ellipticity and viscosity solutions with the properties mostly used in the paper. For a deeper knowledge we refer to [19], [10], [18], [37].

Let Ω be a domain (open connected set) of $\mathbb{R}^n$. We denote by $\mathcal{S}^n$ the linear spaces of $n \times n$ real symmetric matrices with the partial ordering induced by the semidefinite positiveness. Let $F : \mathcal{S}^n \to \mathbb{R}$ be a continuous map. We say that F is degenerate elliptic if

$$F(X) \leq F(Y) \tag{2.1}$$

for all $X, Y \in \mathcal{S}^n$ such that $X \leq Y$. The operators $\mathcal{P}_p^+$ and $\mathcal{P}_p^-$ defined in (1.3) are degenerate elliptic for all positive integers $p \leq n$. They can also be represented in the form

$$\begin{aligned} \mathcal{P}_p^+(X) &= \sup_{S \in \mathcal{G}_p} Tr_S(X), \\ \mathcal{P}_p^-(X) &= \inf_{S \in \mathcal{G}_p} Tr_S(X), \end{aligned} \qquad X \in \mathcal{S}^n, \tag{2.2}$$

where $\mathcal{G}_p$ is the Grassmanian of all linear p-dimensional subspaces S of $\mathbb{R}^n$ and Tr_S is the trace of the quadratic form associated to X restricted to S. In particular $\mathcal{P}_p^+$ is subadditive and $\mathcal{P}_p^-$ is superadditive:

$$\begin{aligned} \mathcal{P}_p^+(X+Y) \leq& \mathcal{P}_p^+(X) + \mathcal{P}_p^+(Y); \\ \mathcal{P}_p^-(X+Y) \geq& \mathcal{P}_p^-(X) + \mathcal{P}_p^-(Y)\,. \end{aligned} \tag{2.3}$$

Moreover,

$$\mathcal{P}_p^+(-X) = -\mathcal{P}_p^-(X)\,. \tag{2.4}$$

Let λ and $\Lambda \geq \lambda$ be positive real numbers. We say that $F : \mathcal{S}^n \to \mathbb{R}$ is uniformly elliptic with ellipticity constants λ and Λ if

$$X \leq Y \;\Rightarrow\; \lambda Tr(Y-X) \leq F(Y) - F(X) \leq \Lambda Tr(Y-X) \tag{2.5}$$

If $p = n$, then $\mathcal{P}_n^\pm(X) = Tr_{\mathbb{R}^n}(X) \equiv Tr(X)$ is uniformly elliptic with ellipticity constants $\lambda = 1 = \Lambda$, and, acting on Hessian matrices, yields the Laplace operator $\Delta u = Tr(D^2u)$.

Let F be a degenerate elliptic operator and $f(x)$ be a continuous function in a domain Ω of $\mathbb{R}^n$. We recall that an usc (upper semicontinuous) function u in Ω, for short $u \in USC(\Omega)$, is a (viscosity) subsolution of equation $F(D^2u) = f(x)$ in Ω, equivalently u is a solution of the elliptic differential inequality $F(D^2u) \geq f(x)$ if: for all $x_0 \in \Omega$ and all C^2 (test) functions φ such that $\varphi(x_0) = u(x_0)$ and $\varphi(x) \geq u(x)$ in a neighbourhood of x_0, we have

$$F(D^2\varphi(x_0)) \geq f(x_0)\,. \tag{2.6}$$

Similarly, a function $v \in LSC(\Omega)$, i.e. a lsc (lower semicontinuous) function v in Ω, is a (viscosity) supersolution of equation $F(D^2v) = f(x)$ in Ω, equivalently v is a solution of the elliptic differential inequality $F(D^2v) \leq f(x)$ if: for all $x_0 \in \Omega$ and all C^2 (test) functions φ such that $\varphi(x_0) = u(x_0)$ and $\varphi(x) \leq u(x)$ in a neighbourhood of x_0, we have

$$F(D^2\varphi(x_0)) \leq f(x_0)\,. \tag{2.7}$$

In both cases the test is intended to be satisfied if there is no test function.

A continuous function u is a (viscosity) solution of $F(D^2u) = f(x)$ if it is both a subsolution and a supersolution.

It is worth to recall that the supremum (resp. infimum) of a finite family of subsolutions (resp. supersolutions) is still a subsolution (resp. a supersolution).

In particular, supposing $F(0) = 0$ and setting $u^{\pm} = \max(\pm u, 0)$, we have:
i) if $F(D^2u) \geq f(x)$, then $F(D^2u^+) \geq -f^-(x)$;
ii) if $F(D^2u) \leq f(x)$, then $\tilde{F}(D^2u^-) \geq -f^+(x)$, where $\tilde{F}(X) = -F(-X)$.

If instead $\{u_j\}$ is an arbitrary family of subsolutions (resp. supersolutions), we can apply Lemma 4.2 of [19] to infer that the usc envelope of $\overline{u} = \sup_j u_j$ (resp. the lsc envelope of $\underline{u} = \inf_j u_j$) is still a subsolution (resp. a supersolution); see also Theorem 2.6 (E) of [28].

In this respect, we recall that the usc envelope of u in Ω, i.e. the smallest usc function above u, and the lsc envelope of u, i.e. the largest lsc function below u, are given respectively by

$$\begin{aligned} u^*(x) &= \lim_{r\to 0^+} \sup\{u(y) : y \in \Omega, |y-x| < r\}, \\ u_*(x) &= \lim_{r\to 0^+} \inf\{u(y) : y \in \Omega, |y-x| < r\} \end{aligned} \tag{2.8}$$

Suppose now that u is an usc function in $\Omega\backslash E$, which is locally bounded above at points of E, a closed subset of Ω. Following [32, Section 3], such function u has a canonical usc extension U across E to all of Ω defined as follows: if E has interior $Int(E) = \emptyset$, then we set

$$U(x) = \limsup_{z\to x,\, z\notin E} u(z) \equiv \lim_{r\to 0^+} \sup_{z\in B_r(x)\backslash E} u(z); \tag{2.9}$$

if $Int(E) \neq \emptyset$, we put $U(x) = -\infty$ on $Int(E)$; then $U(x) = \widetilde{u}^*(x)$, the usc envelope of the function

$$\widetilde{u}(x) = u(x),\ x \in \Omega\backslash E; \quad \tilde{u}(x) = -\infty,\ x \in E\,. \tag{2.10}$$

Analogously, we can consider a lsc function v in $\Omega\backslash E$, which is locally bounded below at points of E, and define the canonical lsc extension V across E to all of Ω as follows: if $Int(E) = \emptyset$, then we set

$$V(x) = \liminf_{z\to x,\, z\notin E} v(z) \equiv \lim_{r\to 0^+} \inf_{z\in B_r(x)\backslash E} v(z); \tag{2.11}$$

if $Int(E) \neq \emptyset$, we put $V(x) = +\infty$ on $Int(E)$ and $V(x) = \widetilde{v}_*(x)$, the lsc envelope $\widetilde{v}_*$ of the function

$$\widetilde{v}(x) = v(x),\ x \in \Omega \backslash E; \quad \widetilde{v}(x) = \infty,\ x \in E\,. \tag{2.12}$$

The solutions (resp., subsolutions, supersolutions) of equation $\mathcal{P}_p^{\pm}(D^2u) = 0$ will be called $p^{\pm}$-harmonic (resp. subharmonic, superharmonic) functions.

We notice that the maximum principle holds true for p^+-subharmonic functions in all bounded domains $\Omega \subset \mathbb{R}^n$, as checked in [13] and [3], and this allows, by subadditivity of $\mathcal{P}_p^+$ and viscosity notion, to compare p^+-subharmonic and p^+-superharmonic functions when at least one of them is C^2. The same can be said comparing p^--subharmonic and p^--superharmonic functions.

More generally, to compare usc subsolutions and lsc supersolutions of equation (1.10), we will use Theorem 3.3 of [19] considering the operator

$$F(x,t,X) = F_p(X) - |t|^{s-1}t - f(x), \quad (x,t,X) \in \Omega \times \mathbb{R} \times \mathcal{S}^n\,. \tag{2.13}$$

Lemma 2.1. *Let Ω be a bounded domain of $\mathbb{R}^n$, let F_p be a degenerate elliptic operator satisfying* (1.9) *for a positive integer $p \leq n$, and $f \in C(\overline{\Omega})$. If $u \in USC(\overline{\Omega})$ and $v \in LSC(\overline{\Omega})$ are viscosity solutions in Ω of the differential inequalities*

$$F_p(D^2v) - |v|^{s-1}v \leq f(x) \leq F_p(D^2u) - |u|^{s-1}u \tag{2.14}$$

for $s \geq 1$, such that $u \leq v$ on $\partial\Omega$, then $u \leq v$ in Ω.

Proof. We check the assumptions of Theorem 3.3 of [19] with $F(x,t,X)$ as in (2.13). First of all, the degenerate ellipticity assumption is satisfied, being F_p degenerate elliptic.

Next, since

$$u \geq v \quad \Rightarrow \quad |u|^{s-1}u - |v|^{s-1}v \geq \gamma(u-v)^s, \tag{2.15}$$

with $\gamma = \gamma(s) > 0$, we get for any $\delta > 0$

$$u - v \geq \delta \quad \Rightarrow \quad F(x,u,X) - F(x,v,X) \leq -\gamma\,\delta^s, \tag{2.16}$$

which plays the role of (3.13) of [19], being just that for $s = 1$. Moreover, by the degenerate ellipticity of F_p we also get

$$\begin{aligned} X \leq Y \ \Rightarrow F(x,t,X) - F(y,t,Y) &= F_p(X) - F_p(Y) - f(x) + f(y) \\ &\leq -\,f(x) + f(y) \leq \omega(|x-y|), \end{aligned} \tag{2.17}$$

where ω is the continuity modulus of f, which plays the role of (3.14) of [19] (see also Example 3.6 therein).

Following the proof of Theorem 3.3 of [19], assume $u \le v$ on $\partial\Omega$ but suppose by contradiction that

$$\max_{x\in\overline{\Omega}}(u(x)-v(x)) = \delta > 0. \tag{2.18}$$

In view of (3.11) and (3.12) of [19], from this we deduce that there exist sequences of points $x_\alpha, y_\alpha \in \Omega$ such that $|x_\alpha - y_\alpha| \to 0$ as $\alpha \to \infty$ but

$$u(x_\alpha) - v(y_\alpha) \ge \delta, \tag{2.19}$$

and sequences of matrices $X_\alpha, Y_\alpha \in \mathcal{S}^n$ such that $X_\alpha \le Y_\alpha$ such that

$$F(y_\alpha, v(y_\alpha), Y_\alpha) \le 0 \le F(x_\alpha, u(x_\alpha), X_\alpha). \tag{2.20}$$

From this, using (2.16) and (2.17), we get

$$\begin{aligned} 0 &\le F(x_\alpha, u(x_\alpha), X_\alpha) - F(y_\alpha, v(y_\alpha), Y_\alpha) \\ &\le F(x_\alpha, u(x_\alpha), X_\alpha) - F(x_\alpha, v(x_\alpha), X_\alpha) \\ &+ F(x_\alpha, v(x_\alpha), X_\alpha) - F(y_\alpha, v(y_\alpha), Y_\alpha) \\ &\le -\gamma\,\delta^s + \omega(|x_\alpha - y_\alpha|), \end{aligned} \tag{2.21}$$

a contradiction, since $\omega(|x_\alpha - y_\alpha|)$ as $\alpha \to \infty$, and we conclude that $u \le v$ in Ω. □

We will also make use, in the sequel, of the following result on the sum of supersolutions.

Lemma 2.2. *Let $v_i \in LSC(\Omega)$, $i = 1, 2$, be non-negative viscosity solutions in a domain Ω of $\mathbb{R}^n$ of the differential inequalities*

$$\mathcal{P}_p^+(D^2 v_i) - |v_i|^{s-1} v_i \le f_i(x) \tag{2.22}$$

for a positive integer $p \le n$ and a real number $s \ge 1$ and $f_i(x)$, $i = 1, 2$, are continuous functions. Suppose at least one between v_i, $i = 1, 2$, is a C^2-function in Ω.
Then $v = v_1 + v_2$ is a viscosity supersolution of equation

$$\mathcal{P}_p^+(D^2 v) - |v|^{s-1} v = f(x) \tag{2.23}$$

with $f(x) = f_1(x) + f_2(x)$.

The proof is based on subadditivity of $\mathcal{P}_p^+$, inequality

$$v_1^s + v_2^s \leq (v_1 + v_2)^s \tag{2.24}$$

for $v_i \geq 0$ and on the fact that we may handle equations using formally the classical derivatives when at least one of the functions v_1 and v_2 is C^2.

3 Basic estimates

Here we deduce a basic estimate on the behaviour of a solution when approaching the singular set. This will be done by comparison with supersolutions v of Osserman type, see [9] (or also [21], [24] and [23]). We recall that $p_s = \frac{2s}{s-1} > 2$. Actually, we will search for a supersolution v of the form

$$v(x) = c_1 v_1(x) + c_2, \tag{3.1}$$

where

$$v_1(x) = \frac{\rho^{p_s-2}}{(\rho^2 - |x-x_0|^2)^{p_s-2}} \tag{3.2}$$

is a positive function in $B_\rho(x_0)$ and c_i, $i = 1, 2$, are positive constants.

Let $f^\pm = \max(\pm f, 0)$. Taking positive numbers c_i large enough in order that

$$\begin{aligned} c_1^{s-1} &\geq 4\,(p_s-1)\,(p_s-2) + 2\,p\,(p_s-2), \\ c_2^s &\geq \max_{|x-x_0|\leq\rho} f^-(x)\,, \end{aligned} \tag{3.3}$$

we obtain positive C^2 and constant supersolutions, $c_1 v_1$ and c_2, respectively, such that

$$\begin{aligned} \mathcal{P}_p^+(D^2 c_1 v_1) - (c_1 v_1)^s &\leq 0, \\ \mathcal{P}_p^+(D^2 c_2) - (c_2)^s &\leq -f^-(x) \end{aligned} \tag{3.4}$$

in the ball of radius ρ centered at x_0. Using Lemma 2.2 we conclude that $v = c_1 v_1 + c_2$ is a C^2 supersolution of (2.23) in $B_\rho(x_0)$, as claimed.

From this we can deduce an upper bound for subsolutions around the singular set. Suppose that Ω is a domain in $\mathbb{R}^n$ and E is a compact subset of Ω. Recall that $p_s = \frac{2s}{s-1}$.

Lemma 3.1. *Let $s > 1$ and $f \in C(\Omega \backslash E)$. Suppose that*

$$\mathcal{P}_p^+(D^2 u) - |u|^{s-1} u \geq f(x) \quad in\ \Omega \backslash E\,. \tag{3.5}$$

There exists a positive constant $A = A(p, s)$ such that

$$u(x) \leq \frac{A}{d_E^{p_s - 2}(x)} + \max_{|z-x| \leq \frac{1}{2} d_E(x)} \{f^-(z)\}^{\frac{1}{s}} \quad in \quad \Omega \backslash E\,. \tag{3.6}$$

Analogously, if

$$\mathcal{P}_p^-(D^2 u) - |u|^{s-1} u \leq f(x) \quad in\ \Omega \backslash E\,, \tag{3.7}$$

then

$$u(x) \geq -\frac{A}{d_E^{p_s - 2}(x)} - \max_{|z-x| \leq \frac{1}{2} d_E(x)} \{f^+(z)\}^{\frac{1}{s}} \quad in \quad \Omega \backslash E\,. \tag{3.8}$$

Proof. Consider the case of subsolutions (3.5). Let us fix $x_0 \in \Omega \backslash E$, and set $\rho = \frac{1}{2}\, d_E(x_0)$. Using (3.1), (3.2) and (3.3), we construct a supersolution $v(x) = c_1 v(x) + c_2$ of equation $\mathcal{P}_p^+(D^2 v) - |v|^{s-1} v = -f^-(x)$ in $B_\rho(x_0)$. Using the comparison principle of Lemma 2.1, since $v(x) \to \infty$ as $|x| \to \rho^-$, we get $u(x) \leq v(x)$ in $B_\rho(x_0)$ and in particular $u(x_0) \leq v(x_0)$, which yields (3.6). The case (3.7) of supersolutions can be treated applying the result just proved for subsolutions replacing u and f with $-u$ and $-f$ in (3.5), respectively. $\square$

4 Removability of isolated singularities

In this Section, using the comparison principles and the estimates of previous Sections, we will show that a solution u of equation (1.1) in $B_\rho^* = B_\rho \backslash \{0\}$ with $s \geq \frac{p}{p-2}$ must have growth of order strictly less than the fundamental solution $\mathcal{E}_p(x)$ as $x \to 0$. In order to show this, we borrow some ideas from [38] arguing by contradiction and using a sequence of Dirichlet problems approaching the singularity together the scale invariance of the equation to compare the solutions with the fundamental solution: see ii) and iii) below. The ending parts iv) and v) are developed in a pure viscosity setting, since a regularity theory is actually not available for the degenerate elliptic equations under consideration.

Lemma 4.1. *Let n and p be positive integers such that $2 < p \le n$, and $s \ge \frac{p}{p-2}$.*
A) *Suppose that $f(x)$ is a function in B^*_ρ, for some $\rho > 0$, such that $f^-(x)$ is continuous and*

$$\limsup_{x\to 0} |x|^{(p-2)s} f^-(x) = 0 \tag{4.1}$$

If $u \in USC(B^*_\rho)$ satisfies

$$\mathcal{P}^+_p(D^2u) - |u|^{s-1}u \ge f(x) \tag{4.2}$$

in B^*_ρ, then

$$\limsup_{x\to 0} \frac{u^+(x)}{\mathcal{E}_p(x)} = 0. \tag{4.3}$$

B) *Analogously, assuming $f^+(x)$ to be a continuous function such that*

$$\limsup_{x\to 0} |x|^{(p-2)s} f^+(x) = 0, \tag{4.4}$$

if $u \in LSC(B^*_\rho)$ satisfies

$$\mathcal{P}^-_p(D^2u) - |u|^{s-1}u \le f(x)\,, \tag{4.5}$$

then

$$\limsup_{x\to 0} \frac{u^-(x)}{\mathcal{E}_p(x)} = 0. \tag{4.6}$$

Proof. As in Lemma 3.1, it is sufficient to consider the case A) of subsolutions. In fact, if u is a supersolution, which satisfies (4.5), then (4.2) holds true by substituting u with $-u$ and f with $-f$. Then we get (4.3) with $-u$ instead of u, namely (4.6).

Therefore, focusing on subsolutions, we observe that by viscosity

$$\mathcal{P}^+_p(D^2u^+) - |u^+|^{s-1}u^+ \ge -f^-(x)\,. \tag{4.7}$$

We note that the proof is immediate if $s > \frac{p}{p-2}$, that is $p_s < p$. If this is the case, indeed, using estimate (3.6) and recalling that $\mathcal{E}_p(x) = |x|^{-(p-2)}$, we get

$$\begin{aligned}\lim_{x\to 0} \frac{u^+(x)}{\mathcal{E}_p(x)} &\le A \lim_{x\to 0} |x|^{p-p_s} + \lim_{x\to 0} |x|^{p-2} \max_{|z-x|\le \frac{1}{2}|x|} \{f^-(z)\}^{\frac{1}{s}} \\ &\le 2^{p-2} \lim_{x\to 0} \max_{|z-x|\le \frac{1}{2}|x|} \left\{|z|^{(p-2)s} f^-(z)\right\}^{\frac{1}{s}} = 0,\end{aligned}$$

by assumption (4.1), and we are done.
Now we consider the remaining case $s = \frac{p}{p-2}$, in which $p_s \equiv \frac{2s}{s-1} = p$. Here we argue by contradiction, supposing that

$$\limsup_{x\to 0} \frac{u^+(x)}{\mathcal{E}_p(x)} = l > 0 \tag{4.8}$$

and noticing that, since $p_s = p$, Lemma 3.1 implies $l < \infty$.

i) Firstly, we show that then there exists $r_0 \in (0,1)$ such that

$$u^+(x) \leq M + l\,\mathcal{E}_p(x) \quad as \quad 0 < |x| \leq r_0 \tag{4.9}$$

for some positive constant M. It is sufficient, by virtue of assumption (4.1), to take $r_0 \in (0, \rho)$ such that

$$f^-(x) \leq \frac{l^s}{|x|^{(p-2)s}} \quad in \quad B^*_{r_0} \tag{4.10}$$

and to set

$$M = \max_{|x|=r_0} u^+(x)\,. \tag{4.11}$$

From (4.8) we find decreasing sequences of positive numbers $r_j \to 0$ and $\varepsilon_j \to 0$ as $j \to \infty$ such that

$$l\,(1-\varepsilon_j) \leq \frac{u^+(x_j)}{\mathcal{E}_p(x_j)} = \max_{|x|=r_j} \frac{u^+(x)}{\mathcal{E}_p(x)} \leq l\,(1+\varepsilon_j) \tag{4.12}$$

By (4.11) and (4.12), therefore we get inequality

$$u^+(x) \leq M + l\,(1+\varepsilon_j)\,\mathcal{E}_p(x) \tag{4.13}$$

on the boundary of the annular region $\{r_j < |x| < r_0\}$. The latter inequality (4.13) can be extended to all the annular region by using the comparison principle between $u^+(x)$, which is a subsolution by (4.7), and $v = M + l\,(1+\varepsilon_j)\,\mathcal{E}_p$, which is a supersolution by the following computation, based on (4.10):

$$\mathcal{P}^+_p\left(D^2 v\right) - v^s \leq -\frac{l^s(1+\varepsilon_j)^s}{|x|^{(p-2)s}} \leq -f^-(x) \quad in \quad B^*_{r_0}\,. \tag{4.14}$$

Finally, each fixed $x \in B^*_{r_0}$ can be included in any annular region $\{r_j < |x| < r_0\}$ for j large enough and therefore (4.13) still holds at x in the limit as $j \to \infty$, which shows (4.9).

ii) Next, we construct a sequence of functions $u_j(x)$ such that

$$u^+(x) \le u_j(x) \le M + l\,\mathcal{E}_p(x) \quad in \quad B_{\boldsymbol{\rho}_j}(x_j)\,, \tag{4.15}$$

where x_j are the maximum points of (4.12) and $\boldsymbol{\rho}_j = r_j(1-\varepsilon_j)$. This is obtained by solving the Dirichlet problem (see Section 6 below)

$$\begin{cases} \mathcal{P}_p^+(D^2u_j) - |u_j|^{s-1}u_j = -f^-(x) & in \ B_{\boldsymbol{\rho}_j}(x_j) \\ u_j = u^+ & on \ \partial B_{\boldsymbol{\rho}_j}(x_j) \end{cases} . \tag{DP}_j$$

The left-hand inequality of (4.15) follows by comparing $u_j(x)$ with the subsolution $u^+(x)$, while the right-hand inequality is deduced by comparing it with the supersolution $v = M + l\,\mathcal{E}_p$ by virtue of (4.9).

iii) Let $\nu_j = \frac{x_j}{r_j}$ be the direction of x_j and $y_j = r_0\nu_j$. Using the linear mapping

$$y = r_0\left(\nu_j + \frac{x - x_j}{\boldsymbol{\rho}_j}\right), \quad x \in B_{\boldsymbol{\rho}_j}(x_j)\,, \tag{4.16}$$

where $\rho_j = \frac{\boldsymbol{\rho}_j}{r_0}$, and we construct the rescaled function

$$w_j(y) := \rho_j^{p-2}u_j(x) = \rho_j^{p-2}u_j(x_j + \boldsymbol{\rho}_j\,(\tfrac{y}{r_0} - \nu_j)), \quad y \in B_{r_0}(y_j)\,, \tag{4.17}$$

which is by (DP)$_j$ a solution of equation

$$\mathcal{P}_p^+(D^2w_j(y)) - |w_j(y)|^{s-1}w_j(y) = -\rho_j^p f_j^-(y) \quad in \quad B_{r_0}(y_j) \tag{4.18}$$

with $f_j^-(y) = f^-(x_j + \boldsymbol{\rho}_j\,(\frac{y}{r_0} - \nu_j))$.

Since $|y_j| = r_0$ for all $j \in \mathbb{N}$, we may suppose, up to a subsequence, that $y_j \to y_0$, where $|y_0| = r_0$, and also that $B_{\frac{r_0}{2}}(y_0) \subset B_{r_0}(y_j)$, taking $j \in \mathbb{N}$ large enough. We infer that there exists a sequence $\eta_j \searrow 0$ such that

$$\begin{aligned} &\frac{w_j(y)}{1+\eta_j} \le l\,\mathcal{E}_p(y) \quad in \quad B_{\frac{r_0}{2}}(y_0)\,; \\ &\frac{w_j(y_j)}{1+\eta_j} \ge \frac{l\,\mathcal{E}_p(y_0)}{(1+\eta_j)^2}\,. \end{aligned} \tag{4.19}$$

To show this, from the right-hand inequality of (4.15) we get

$$\begin{aligned} w_j(y) &= \rho_j^{p-2} u_j(x_j + \boldsymbol{\rho}_j\,(\tfrac{y}{r_0} - \nu_j)) \\ &\le \rho_j^{p-2}(M + l\,\mathcal{E}_p(x_j + \boldsymbol{\rho}_j\,(\tfrac{y}{r_0} - \nu_j))) \\ &= \rho_j^{p-2} M + r_0^{p-2}\,\frac{l}{|\frac{x_j}{\boldsymbol{\rho}_j} + \frac{y}{r_0} - \nu_j|^{p-2}}\,; \end{aligned} \tag{4.20}$$

Since $\boldsymbol{\rho}_j = r_j(1-\varepsilon_j)$ with $\varepsilon_j \to 0$, then $\frac{x_j}{\boldsymbol{\rho}_j} - \nu_j \to 0$ and the latter sequence in (4.20) converges uniformly for $y \in B_{\frac{r_0}{2}}(y_0)$ as $j \to \infty$, and thus the first inequality in (4.19) is proved.

On the other side, using the left-hand inequality in (4.15), we obtain

$$\begin{aligned} w_j(y_j) &= \rho_j^{p-2} u_j(x_j) \\ &\ge \rho_j^{p-2} u^+(x_j) = \frac{u^+(x_j)}{\mathcal{E}_p(x_j)} \left(\frac{\rho_j}{r_j}\right)^{p-2} \\ &\ge l\,(1-\varepsilon_j)\left(\frac{\rho_j}{r_j}\right)^{p-2} = l\, r_0^{p-2}\,\mathcal{E}_p(y_0)\,(1-\varepsilon_j)\left(\frac{\rho_j}{r_j}\right)^{p-2}, \end{aligned} \tag{4.21}$$

and this also proves the second inequality in (4.19).

Moreover,

$$\mathcal{P}_p^+\left(D^2 \frac{w_j(y)}{1+\eta_j}\right) - \left(\frac{w_j(y)}{1+\eta_j}\right)^s \ge -\rho_j^p f_j^-(y) \quad in \quad B_{\frac{r_0}{2}}(y_0)\,. \tag{4.22}$$

In fact, by (4.18) we have

$$\begin{aligned} &(1+\eta_j)\left(\mathcal{P}_p^+\left(D^2 \frac{w_j(y)}{1+\eta_j}\right) - \left(\frac{w_j(y)}{1+\eta_j}\right)^s\right) \\ \ge\, &(1+\eta_j)\left(\mathcal{P}_p^+\left(D^2 \frac{w_j(y)}{1+\eta_j}\right) - (1+\eta_j)^{s-1}\left(\frac{w_j(y)}{1+\eta_j}\right)^s\right) \\ =\, &\mathcal{P}_p^+\left(D^2 w_j(y)\right) - w_j^s(y) = -\rho_j^p f_j^-(y) \\ \ge &- (1+\eta_j)\,\rho_j^p f_j^-(y) \end{aligned} \tag{4.23}$$

iv) Hence, for all $j_0 \in \mathbb{N}$ the usc envelope w^* (see Section 2) of the function

$$w(y) = \sup_{j \ge j_0} \frac{w_j(y)}{1+\eta_j}, \quad y \in B_{\frac{r_0}{2}}(y_0)\,, \tag{4.24}$$

is a subsolution of equation

$$\mathcal{P}_p^+(D^2 w^*(y)) - |w^*(y)|^{s-1} w^*(y) = -\sup_{j\geq j_0} \rho_j^p f_j^-(y) \quad in \quad B_{\frac{r_0}{2}}(y_0) \tag{4.25}$$

Moreover, from inequalities (4.19) it follows that

$$w^*(y) \leq l\, \mathcal{E}_p(y), \quad y \in B_{\frac{r_0}{2}}(y_0)\,. \tag{4.26}$$

and

$$w^*(y_0) \geq \limsup_{j\to\infty} \frac{w_j(y_j)}{1+\eta_j} = l\, \mathcal{E}_p(y_0)\,. \tag{4.27}$$

v) Conclusion. By (4.26) and (4.27) the function defined as $\varphi(y) = l\, \mathcal{E}_p(y)$ touches from above $w^*(y)$ at y_0 and can be used as a test function for equation (4.25) at $y = y_0$ obtaining

$$\mathcal{P}_p^+(D^2\varphi(y_0)) - (\varphi(y_0))^s \geq -\sup_{j\geq j_0} \rho_j^p f^-(x_j'), \tag{4.28}$$

from which

$$l^s \leq \sup_{j\geq j_0} \rho_j^p f^-(x_j'), \tag{4.29}$$

where $x_j' = x_j + (\frac{y_0}{r_0} - \nu_j) \to 0$ and $\frac{|x_j'|}{\rho_j} \to 1$ as $j \to \infty$.

But letting $j_0 \to \infty$, since $(p-2)s = p$ and therefore $|x_j'|^p f^-(x_j') \to 0$ as $j \to \infty$ by assumption, we should have $l = 0$. This yields a contradiction with our starting assumption $l > 0$ and proves the assertion. □

Proof of Theorem 1.1. The proof of Theorem 1.1 follows at once gathering together (4.3) and (4.6) of Lemma 4.1. □

Let $2 < p \leq n$ and $s \geq \frac{p}{p-2}$. The next corollary shows that, when f is bounded below, the subsolutions u of equation $\mathcal{P}_p^+(D^2u) - |u|^{s-1}u = f(x)$ in the punctured ball B_ρ^* are bounded above. Similarly, when f is bounded above, supersolutions of equation $\mathcal{P}_p^-(D^2u) - |u|^{s-1}u = f(x)$ are bounded from below.

Corollary 4.2. A) *Suppose that assumptions of Lemma 4.1* (A) *are fullfilled with $f(x)$ bounded below. If $u \in C(B_\rho^*)$ is a viscosity subsolution of equation* (1.1)*, then u is bounded*

above in B_r^* for all $r < \rho$.
B) *On the other side, if the assumptions of Lemma 4.1* (B) *are fullfilled with $f(x)$ bounded above and $u \in C(B_\rho^*)$ is a viscosity supersolution of equation* (1.2)*, then u is bounded below in B_r^* for all $r < \rho$.*

Proof. We treat case A), since case B) is similar.

Suppose $f \geq -F^-$ with $F^- \geq 0$ and set $\varphi(x) = \varepsilon \mathcal{E}_p(x) + K(r^2 - |x|^2)$. Then for all $\varepsilon > 0$

$$\mathcal{P}_p^+(D^2\varphi) - |\varphi|^{s-1}\varphi \leq -2Kp \leq -f^-(x) \tag{4.30}$$

in the punctured ball B_r^*, provided $K \geq \frac{F^-}{2p}$.
Next, set $u_0(x) = u^+(x) - \max_{\partial B_r} u^+$. We have

$$\mathcal{P}_p^+(D^2 u_0) - |u_0|^{s-1}\, u_0 \geq -f^-(x), \tag{4.31}$$

so that we can compare u_0 and φ with Lemma 2.1 in any annular region B_r^*. Since $u^+(x) = o(\varphi(x))$ as $x \to 0$ by (4.3), then $u_0(x) \leq \varphi(x)$ in a sufficiently small neighbourhood of the origin; moreover $u_0(x) \leq 0 \leq \varphi(x)$ on ∂B_r. So by comparison $u_0(x) \leq \varphi(x)$ in B_r^*, namely

$$u(x) \leq \varepsilon \mathcal{E}_p(x) + Kr^2 + \max_{\partial B_r} u^+(x) \tag{4.32}$$

in B_r^*. Letting $\varepsilon \to 0^+$, we conclude that u is bounded above in B_r^*, as claimed. $\square$

Proof of Corollary 1.3. This is an immediate consequence of Corollary 4.2. $\square$

Proof of Theorem 1.4. Let u be a viscosity solution of equation $F_p(D^2u) - |u|^{s-1}u = f(x)$ in Ω^*, and suppose that $x_0 = 0$. Then, by condition (1.9) on F_p, the solution u satisfies the assumptions of Corollary 1.3 and therefore is bounded in a punctured ball $B_r^* \subset \overline{B}_r \subset \Omega$ with a sufficiently small radius $r > 0$.
We consider the viscosity solution $w(x)$ of the Dirichlet problem (see Section 6)

$$\begin{cases} F_p(D^2w) - |w|^{s-1}w = f(x) & \text{in } B_r \\ w = u & \text{on } \partial B_r \end{cases} . \tag{4.33}$$

Note that for all $\varepsilon > 0$, using the structure condition (1.9)', the fundamental solution $\mathcal{E}_p$

and the increasing monotonicity of the function $g(t) = |t|^{s-1}t$, we have

$$\begin{aligned} & F_p(D^2(w+\varepsilon\mathcal{E}_p)) - |w+\varepsilon\mathcal{E}_p|^{s-1}(w+\varepsilon\mathcal{E}_p) \\ \le\, & F_p(D^2 w) - |w|^{s-1}w \\ \le\, & f(x) \\ \le\, & F_p(D^2 u) - |u|^{s-1}u. \end{aligned} \tag{4.34}$$

Since u is bounded in B_r^*, we have $u(x) \le w(x) + \varepsilon\mathcal{E}_p(x)$ in a neighbourhhod the origin as well as on ∂B_r. Therefore the comparison principle of Lemma 2.1 yields

$$u(x) \le w(x) + \varepsilon\mathcal{E}_p(x) \tag{4.35}$$

in B_r. Letting $\varepsilon \to 0^+$, we obtain $u(x) \le w(x)$ in B_r^*. Since $w(x)$ is bounded, too, interchanging the role of u and w, we also get the reverse inequality $w(x) \le u(x)$ in B_r^*, so that w is a continuous extension of u to B_r, concluding the proof. □

5 Removability of nonisolated singular sets

In this Section, we suppose that the singular set E is a compact subset of a domain Ω of $\mathbb{R}^n$ and is contained in a smooth (C^2) embedded manifold Γ of codimension k such that $n-p+2 < k < n$ with $3 \le p \le n$.

As already anticipated in Section 1, we will consider the distance function $\delta(x) = dist(x,\Gamma)$ and we refer to Ambrosio and Soner [2] for the properties that will be used here; see also Ambrosio and Mantegazza [1].

We will follow the track of Section 4 with suitable modifications, substituting p and $|x|$, respectively, with the integer $p-(n-k) > 2$ and the function $\delta(x)$. This essentially amounts to substituting the fundamental solution $\mathcal{E}_p(x)$ with the function

$$V_p(x) = \delta^{-\alpha}(x)\,, \quad \delta(x) = dist(x,\Gamma)\,, \tag{5.1}$$

where $\alpha = (p-2)-(n-k)$ is a positive integer, as in the Introduction.

However, differently from $\mathcal{E}_p(x)$, which is p^+-harmonic, the function $V_p(x) = \delta^{-\alpha}(x)$ is neither p^+-harmonic nor p^+-superharmonic, in general. Nonetheless, this will be seen as not invalidating the argument of the proofs.

We also notice that the case that E is a point can be assimilated to co-dimension $k = n$.

5.1 *Supersolutions via distance function*

It will be convenient to use the function $\eta(x) = \frac{1}{2}\,\delta^2(x)$, which, by Theorem 3.1 of [2], is a smooth function in the tubular neighbourhood $\mathcal{T}_\sigma(\Gamma) = \{x \in \mathbb{R}^n : \delta(x) \equiv \mathrm{dist}(x, \Gamma) < \sigma\}$ for some $\sigma > 0$. If $x \in \mathcal{T}_\sigma(\Gamma)$, then $D\eta(x) = \delta(x)\nu_P$, where ν_P is the unit normal vector to Γ from the point $P \in \Gamma$ such that $|x - P| = \delta(x)$. Moreover, by Theorem 3.2 of [2], the Hessian matrix $D^2\eta(x)$ represents the orthogonal projections on the normal space N_P to Γ at P and has k eigenvalues equal to 1 with the remaining $\lambda_1 \le \cdots \le \lambda_{n-k} < 1$ such that

$$|\lambda_i(D^2\eta(x))| \le C\delta(x), \quad i = 1, \ldots, n-k\,, \tag{5.2}$$

where $C = C(\sigma)$ is a positive constant. Next, we compute

$$\begin{aligned} D^2 V_p &= 2^{-\frac{\alpha}{2}} D^2 \eta^{-\frac{\alpha}{2}} = -2^{-\frac{1}{\alpha}2}\frac{\alpha}{2}\, D(\eta^{-(\frac{\alpha}{2}+1)} D\eta) \\ &= \alpha\,\delta^{-(\alpha+2)}\left[(\alpha+2)\nu_P \otimes \nu_P - D^2\eta\right]\,. \end{aligned} \tag{5.3}$$

We notice that $\nu_P \otimes \nu_P$ is a one-rank matrix with non-zero eigenvalue 1 associated to the eigenvector $\nu_P \in N_P$, so that

$$\begin{aligned} \mathcal{P}_p^+((\alpha+2)\nu_P \otimes \nu_P - D^2\eta) &\le (\alpha+2)\mathcal{P}_p^+(\nu_P \otimes \nu_P) - \mathcal{P}_p^-(D^2\eta) \\ &\le (\alpha+2) - \sum_{i=1}^{n-k} \lambda_i(D^2\eta) - p + n - k \\ &\le \alpha - (p-2) + (n-k) + (n-k)C\delta \end{aligned} \tag{5.4}$$

and hence, choosing $\alpha = (p-2) - (n-k)$, we have

$$\mathcal{P}_p^+(D^2 V_p) \le C_1\,\delta^{-(\alpha+1)} \tag{5.5}$$

in the tubular neighbourhood $\mathcal{T}_\sigma(\Gamma)\backslash\Gamma$, for some positive constant C_1.

As already observed above, in this case the function $V_p(x)$ will play the role of the fundamental solution $\mathcal{E}_p(x)$ for punctured domains. But the right term of (5.5) cannot be in general taken to be equal to zero, unless Γ is flat, i.e. $\Gamma = \{x_1 = \cdots = x_k = 0\}$.

Nevertheless, if l is a positive constant and $s = \frac{\alpha+2}{\alpha}$, we have

$$\mathcal{P}_p^+(D^2 lV_p) - (lV_p)^s \le l\,\delta^{-(\alpha+2)}(C_1\delta - l^{s-1}) \le 0 \tag{5.6}$$

if we take $\delta(x) \le C_1^{-1} l^{s-1}$, namely in a suitable tubular neighbourhood $\mathcal{T}_{\sigma_l}(\Gamma)\backslash\Gamma$, and this will be seen as sufficient to show that solutions u of Lemma 3.1 are in fact $o(V_p(x))$ as $x \to E$.

5.2 *Behaviour near singular sets*

We are going to establish the counterpart of Lemma 4.1.

Lemma 5.1. *Let n and p be positive integers such that $3 \le p \le n$. Let $k \in \mathbb{N}$ be such that $\alpha := (p-2)-(n-k) > 0$. Suppose that Γ is a smooth embedded manifold in $\mathbb{R}^n$ of codimension $k < n$, and that E is a compact subset of Ω such that $E \Subset \Gamma$ in the relative topology and all points of E are limit points for $\Omega \backslash \Gamma$. Let s be a real number such that*

$$s \ge \frac{\alpha+2}{\alpha} \equiv \frac{p-(n-k)}{(p-2)-(n-k)}, \tag{5.7}$$

i.e. $p \ge p_s + (n-k)$.

A) *Suppose f^- is a continuous function in $\Omega \backslash E$ such that*

$$\limsup_{x \to E} d_E^{\alpha s}(x)\, f^-(x) = 0\,, \tag{5.8}$$

If $u \in USC(\Omega \backslash E)$ is a viscosity subsolution of equation

$$\mathcal{P}_p^+(D^2 u) - |u|^{s-1}u = f(x) \tag{5.9}$$

in $\Omega \backslash E$, then

$$\limsup_{x \to E} \frac{u^+(x)}{V_p(x)} = 0\,. \tag{5.10}$$

B) *Suppose that f^+ is a continuous function in $\Omega \backslash E$ such that*

$$\limsup_{x \to E} d_E^{\alpha s}(x)\, f^+(x) = 0\,. \tag{5.11}$$

If $u \in LSC(\Omega \backslash E)$ is a viscosity supersolution of equation

$$\mathcal{P}_p^-(D^2 u) - |u|^{s-1}u = f(x) \tag{5.12}$$

in $\Omega \backslash E$, then

$$\limsup_{x \to E} \frac{u^-(x)}{V_p(x)} = 0\,. \tag{5.13}$$

Proof. As in the case of punctured domains of Lemma 4.1, part B) for a supersolution u can be deduced from the part A) passing to the subsolution $-u$, and therefore it is enough to prove part A).

In order to do this, we argue as in the proof of Lemma 4.1, recalling that $V_p(x) = \delta^{-\alpha}(x)$, where $\delta(x) = \mathrm{dist}(x, \Gamma)$ and $\alpha = (p-2) - (n-k) \in \mathbb{N}$.

Firstly, supposing $s > \frac{\alpha+2}{\alpha}$, i.e. $p > p_s + (n-k)$, and observing that $dist(x, \Gamma) \leq dist(x, E) \equiv d_E(x)$, then estimate (3.6) yields, for a viscosity subsolution of equation (5.9),

$$\frac{u^+(x)}{V_p(x)} \leq A\, d_E(x)^{p-(p_s+n-k)}$$
$$+ 2^{p-2} \max_{|z-x| \leq \frac{1}{2}\delta(x)} d_E^\alpha(z) \left\{ f^-(z) \right\}^{\frac{1}{s}}$$

which, by assumption on f^-, proves (5.10) in the present case, letting $x \to E$.

We are left with the case $s = \frac{\alpha+2}{\alpha} \equiv \frac{p-(n-k)}{(p-2)-(n-k)}$, i.e. $p = p_s + n - k \equiv \frac{2s}{s-1} + (n-k)$.

Arguing by contradiction, we suppose

$$\limsup_{x \to E} \frac{u^+(x)}{V_p(x)} = l > 0 \tag{5.14}$$

and notice that, since $p = p_s + n - k$, Lemma 3.1 implies $l < \infty$.

Then we will adapt the proof of Lemma 4.1.

i) Firstly, we find $d_0 > 0$ and a neighbourhood $\Omega_0 = \{x \in \Omega : d_E(x) < d_0\}$ of E such that $\overline{\Omega}_0 \subset \Omega$ and

$$u^+(x) \leq M + l\, V_p(x) \quad in \quad \Omega_0^* \equiv \Omega_0 \backslash E \tag{5.15}$$

with

$$M = \max_{d_E(x)=d_0} u^+(x)\,, \tag{5.16}$$

and

$$f^-(x) \leq \frac{l^s}{2\delta^{\alpha s}(x)} \quad in \quad \Omega_0^*\,, \tag{5.17}$$

using assumption (5.8). Note that Ω_0^* contains points $x \in \Gamma$ where $V_p(x) = \infty$, and we still denote by $V_p(x)$ the canonical lsc extension $(V_p)_*(x)$ of $V_p(x)$ across Γ, namely

$$(V_p)_*(x) = \begin{cases} \delta^{-\alpha}(x) & x \notin \Gamma \\ \infty & x \in \Gamma \end{cases} \tag{5.18}$$

We will also suppose that $\Omega_0 \subset \mathcal{T}_\sigma$, the tubular neighbourhood of Γ where the distance function $\delta(x)$ is smooth.

To show (5.15), from (5.14) we take decreasing sequences of positive numbers $d_j \to 0$ and $\varepsilon_j \to 0$ as $j \to \infty$, and points $x_j \in \Omega_0 \backslash \Gamma$ such that

$$l\,(1-\varepsilon_j) \leq \frac{u^+(x_j)}{V_p(x_j)} = \max_{d_E(x)=d_j} \frac{u^+(x)}{V_p(x)} \leq l\,(1+\varepsilon_j) \tag{5.19}$$

By (5.16) and (5.19), we obtain, on the boundary of the "annular region" $\{d_j < d_E(x) < d_0\}$, the inequality

$$u^+(x) \leq M + l\,(1+\varepsilon_j)\, V_p(x)\,, \tag{5.20}$$

which can be extended to all the annular region comparing the subsolution $u^+(x)$ with the supersolution $M + l\,(1+\varepsilon_j)\, V_p(x)$ of equation $\mathcal{P}_p^+(D^2u) - |u|^{s-1}u = -f^-(x)$ in Ω_0^*. That $M + l\,(1+\varepsilon_j)\, V_p(x)$ is a supersolution deserves some explanation: we suppose, as we may, that Ω_0^* is contained in the tubular neighbourhood $\mathcal{T}_{\sigma_l}$ and $\delta(x) \leq \frac{l^{s-1}}{2C_1}$ in Ω_0^*, where C_1 is the constant in (5.6), which we can use at points $x \in \Omega_0 \backslash \Gamma$ under the assumption (5.17), while there are no test functions at points $x \in \Gamma$.
Then we obtain (5.20) in the annular region $\{d_j < d_E(x) < d_0\}$. As in the proof of Lemma 4.1 (i), each fixed $x \in \Omega_0^*$ will be included in any annular region $\{d_j < d_E(x) < d_0\}$ for j large enough, and therefore (5.20) still holds at x in the limit as $j \to \infty$, yielding (5.15).

ii) Taking the sequence of maximum points $x_j \in \Omega \backslash \Gamma$, $j \in \mathbb{N}$, of (5.19), we put

$$r_j = \delta(x_j) \equiv dist(x_j, \Gamma) \tag{5.21}$$

and we solve the Dirichlet problem $(\mathrm{DP})_j$ as in the proof of Lemma 4.1 (ii) with $\boldsymbol{\rho}_j = r_j\,(1-\varepsilon_j)$. Thus we construct a sequence of functions $u_j(x)$ such that

$$u^+(x) \leq u_j(x) \leq M + l\, V_p(x) \;\; in \;\; B_{\rho_j}(x_j)\,. \tag{5.22}$$

The left-hand inequality follows comparing the solution u_j with the subsolution $u = u^+$ of equation $\mathcal{P}_p^+(D^2u) - |u|^{s-1}u = -f^-(x)$. Concerning the right-hand inequality, we notice that $B_{\rho_j} \subset \Omega_0 \backslash \Gamma$ and use (5.15) to compare u_j with the supersolution $M + l\, V_p(x)$ on the boundary of B_{ρ_j}.

iii) Next, let $P_j \in \Gamma$ such that $|x_j - P_j| = r_j \equiv \delta(x_j)$ and ν_j be the unit normal vector to Γ from P_j to x_j, so that $x_j = P_j + \delta(x_j)\, \nu_j$.

Setting $y_j = P_j + r_0\nu_j$ for a suitable small $r_0 > 0$, independent of $j \in \mathbb{N}$, the distance of y_j from Γ is still realized along ν_j: $\delta(y_j) = |y_j - P_j|$. Then we consider the linear mapping (4.16), which sends x_j into $y_j = P_j + r_0\nu_j$, and $B_{\boldsymbol{\rho}_j}(x_j)$ into $B_{r_0}(y_j)$. Following the proof of Lemma 4.1, with $\rho_j = \frac{\boldsymbol{\rho}_j}{r_0}$ we construct the rescaled function

$$w_j(y) = \rho_j^\alpha u_j(x) = \rho_j^\alpha u_j(x_j + \boldsymbol{\rho}_j \tfrac{y-y_j}{r_0}) \tag{5.23}$$

in $B_{r_0}(y_j)$, which satisfies equation

$$\mathcal{P}_p^+(D^2 w_j(y)) - |w_j(y)|^{s-1} w_j(y) = -\rho_j^{\alpha+2} f_j^-(y) \quad in \quad B_{r_0}(y_j)\,. \tag{5.24}$$

where $f_j^-(y) = f^-(x_j + \boldsymbol{\rho}_j \frac{y-y_j}{r_0})$.

Since the y_j's are bounded, we may suppose, up to a subsequence, that $y_j \to y_0$, and, up to a translation, that the distance of y_0 from Γ is realized at the origin $0 \in E$; therefore $y_0 = r_0\nu_0$. Moreover, $B_{\frac{r_0}{2}}(y_0) \subset B_{r_0}(y_j)$ for $j \in \mathbb{N}$ large enough. Then we can find a sequence $\eta_j \searrow 0$ such that

$$\begin{aligned} &\frac{w_j(y)}{1+\eta_j} \le l\,V_p(y) \quad in \quad B_{\frac{r_0}{2}}(y_0)\,; \\ &\frac{w_j(y_j)}{1+\eta_j} \ge \frac{l\,V_p(y_0)}{(1+\eta_j)^2}\,. \end{aligned} \tag{5.25}$$

To show this, we observe that from the right-hand inequality of (5.22) we obtain

$$\begin{aligned} w_j(y) &= \rho_j^\alpha u_j(x_j + \boldsymbol{\rho}_j \tfrac{y-y_j}{r_0}) \\ &\le \rho_j^\alpha (M + l\,V_p(x_j + \boldsymbol{\rho}_j \tfrac{y-y_j}{r_0})) \\ &= \rho_j^\alpha M + \frac{\rho_j^\alpha\, l}{\delta^\alpha(\boldsymbol{\rho}_j\,(\frac{y}{r_0} + \frac{x_j}{\boldsymbol{\rho}_j} - \frac{y_j}{r_0}))}\,; \end{aligned} \tag{5.26}$$

Since $\boldsymbol{\rho}_j = r_j(1-\varepsilon_j) < 1$, as we may suppose, $P_j \to 0$ and $\nu_j \to \nu_0$, then $\frac{x_j}{\boldsymbol{\rho}_j} - \frac{y_j}{r_0} \to 0$, and the latter sequence in (5.26) converges to $l\,\delta^{-\alpha}(y)$ uniformly for $y \in B_{\frac{r_0}{2}}(y_0)$ as $j \to \infty$. Thus the first inequality in (5.25) is proved.

On the other side, using the left-hand inequality in (5.22), we get

$$\begin{aligned} w_j(y_j) &= \rho_j^\alpha u_j(x_j) \\ &\ge \rho_j^\alpha u^+(x_j) = \frac{u^+(x_j)}{V_p(x_j)} \left(\frac{\boldsymbol{\rho}_j}{r_j}\right)^\alpha \\ &\ge l\,(1-\varepsilon_j) \left(\frac{\boldsymbol{\rho}_j}{r_j}\right)^\alpha = l\,r_0^\alpha\, V_p(y_0)\,(1-\varepsilon_j) \left(\frac{\boldsymbol{\rho}_j}{r_j}\right)^\alpha, \end{aligned} \tag{5.27}$$

and this also proves the second inequality in (5.25).

Moreover, starting from (5.24) and reasoning as in the proof of Lemma 4.1, we also get

$$\mathcal{P}_p^+\left(D^2\frac{w_j(y)}{1+\eta_j}\right)-\left(\frac{w_j(y)}{1+\eta_j}\right)^s\geq-\rho_j^{\alpha+2}f_j^-(y)\ \ in\ \ B_{\frac{r_0}{2}}(y_0)\,, \tag{5.28}$$

where $f_j^-(y)=f^-(x_j+\boldsymbol{\rho}_j\,\frac{y-y_j}{r_0})$.

iv) Observe, again as in the proof of Lemma 4.1, that for all $j_0\in\mathbb{N}$ the usc envelope w^* of function $w(y)=\sup\limits_{j\geq j_0}\dfrac{w_j(y)}{1+\eta_j}$ is a subsolution of equation

$$\mathcal{P}_p^+(D^2w^*(y))-|w^*(y)|^{s-1}w^*(y)=-\sup_{j\geq j_0}\rho_j^{\alpha+2}f_j^-(y)\ \ in\ \ \ B_{\frac{r_0}{2}}(y_0)\,. \tag{5.29}$$

Moreover, from inequalities (5.25) it follows that

$$w^*(y)\leq l\,V_p(y),\quad y\in B_{\frac{r_0}{2}}(y_0)\,. \tag{5.30}$$

and

$$w^*(y_0)\geq l\,V_p(y_0)\,. \tag{5.31}$$

v) Conclusion. By (5.30) and (5.31) the function $\varphi(y)=l\,V_p(y)$ touches from above $w^*(y)$ at y_0 and can be used as a test function in equation (5.29) obtaining

$$\mathcal{P}_p^+(D^2\varphi(y_0))-(\varphi(y_0))^s\geq-\sup_{j\geq j_0}\rho_j^{\alpha+2}f^-(x'_j), \tag{5.32}$$

where $x'_j=x_j+\boldsymbol{\rho}_j\,\frac{y_0-y_j}{r_0}$.

We will get a contradiction, but we have to be a little bit more careful, with respect to Lemma 4.1. By (5.5), it follows from (5.32) that

$$l\,C_1\,r_0^{-\alpha-1}-l^s\ r_0^{-\alpha s}\geq-\sup_{j\geq j_0}\rho_j^{\alpha+2}f^-(x_j+\boldsymbol{\rho}_j\,\tfrac{y_0-y_j}{r_0}). \tag{5.33}$$

Now, letting $j_0\to\infty$, since $\alpha s=\alpha+2$ and $\delta^{\alpha+2}(x'_j)f^-(x'_j)\to 0$ by assumption, as well as $\frac{\rho_j}{\delta(x'_j)}\to 1$ when $j\to\infty$, we should have

$$l^s\ r_0^{-2}-l\,C_1\,r_0^{-1}\leq 0\,. \tag{5.34}$$

Here we observe that the argument works with any sufficiently small $r_0 > 0$, and (5.34) yields a contradiction for r_0 small enough, thereby proving the assertion. $\square$

Proof of Theorem 1.7. The proof of Theorem 1.7 follows at once gathering together (5.10) and (5.13) of the above Lemma 5.1 . $\square$

5.3 *Boundedness across the singular set*

As in the case of punctured domains (Corollary 1.3), we will see that solutions are bounded across Γ if we assume that $f(x)$ is bounded.

Lemma 5.2. A) *Suppose that assumptions of Lemma* 5.1*, part A), are fulfilled with $f(x)$ bounded below. If $u \in USC(\Omega\backslash E)$ is a viscosity subsolution of equation* (1.1)*, then u is bounded above across E, namely bounded above in any open set $\Omega'\backslash E$ such that $\Omega' \Subset \Omega$.*
B) *Suppose that assumptions of Lemma* 5.1*, part B), are fulfilled with $f(x)$ bounded above. If $u \in LSC(\Omega\backslash E)$ is a viscosity supersolution of equation* (1.1)*, then u is bounded below across E, namely bounded below in any open set $\Omega'\backslash E$ such that $\Omega' \Subset \Omega$.*

Proof. We will give the proof for subsolutions of part A), the counterpart for supersolutions of part B) being similar, based on part B) of Lemma 5.1.

We follow the lines of the proof Corollary 4.2 (A), but we cannot simply use $V_p(x)$ in the place of $\mathcal{E}_p(x)$, because in general $V_p(x) \equiv \delta^{-\alpha}(x)$ is not p^+-superharmonic in any tubular neighbourhood of Γ, unless Γ is flat. We take instead $v_p(x)$, the lsc canonical extension across Γ of

$$V_p(x) + \delta^{-\alpha+\frac{1}{2}}(x) \equiv \delta^{-\alpha}(x) + \delta^{-\alpha+\frac{1}{2}}(x)\,, \tag{5.35}$$

such that $v_p(x) \to \infty$ as $x \to \Gamma$ and the boundary limits (5.10) and (5.13) continue to hold with $v_p(x)$ instead of $V_p(x)$. Following the computations leading to (5.5), we get

$$\begin{aligned}\mathcal{P}_p^+(D^2 v_p(x)) &\leq \mathcal{P}_p^+(D^2 V_p(x)) + \mathcal{P}_p^+(D^2 \delta^{-\alpha+\frac{1}{2}}(x)) \\ &\leq C_1\, \delta^{-\alpha-1}(x) - C_2\, \delta^{-\alpha-\frac{3}{2}}(x) + C_3\, \delta^{-\alpha-\frac{1}{2}}(x)\,,\end{aligned} \tag{5.36}$$

where C_i, $i = 1, 2, 3$, are positive constants, and so we can find a tubular neighborhood $\mathcal{T}_d$ of Γ such that

$$\mathcal{P}_p^+(D^2 v_p(x)) \leq 0 \;\; in \;\; \mathcal{T}_d\,; \tag{5.37}$$

then we take a neighbourhood Ω_0 of E such that $E \subset \Omega_0 \Subset \mathcal{T}_d$.

Now, we argue as in the proof of Corollary 4.2, taking $F^- \geq 0$ such that $f \geq -F^-$, $K \geq \frac{F^-}{2p}$ and $r > 0$ such that $\Omega_0 \Subset B_r$. For all $\varepsilon > 0$ we construct the function $\varphi(x) = \varepsilon\, v_p(x) + K(r^2 - |x|^2)$, in order to have a supersolution in Ω_0, namely

$$\mathcal{P}_p^+(D^2\varphi) - |\varphi|^{s-1}\varphi \leq -f^-(x)\,. \tag{5.38}$$

Taking $u_0(x) = u^+(x) - \max_{\partial\Omega_0} u^+$, we also have

$$\mathcal{P}_p^+(D^2u_0) - |u_0|^{s-1}u_0 \geq -f^-(x)\,, \tag{5.39}$$

and we can compare u_0 and φ in $\Omega_0^* = \Omega_0 \backslash E$, observing that $u^+ \leq \varphi$ in a suitably small neighbourhood of E by (5.10); moreover $u_0(x) \leq 0 \leq \varphi(x)$ on $\partial\Omega_0$. Then by comparison we have

$$u(x) \leq \varepsilon v_p(x) + Kr^2 + \max_{\partial\Omega_0} u^+ \tag{5.40}$$

in Ω_0^*. Letting $\varepsilon \to 0$, we conclude that u is bounded above in $\Omega_0 \backslash \Gamma$. Therefore the canonical usc extension U of u across Γ is bounded above, but $u(x) = U(x)$ in $\Omega \backslash E$, and therefere u is bounded above in $\Omega_0 \backslash E$. $\square$

Proof of Corollary 1.8. Corollary 1.8 is an immediate consequence of part A) and B) of this Subsection. $\square$

5.4 *Removability of the singular set*

To show the removability result we will use Theorem 6.1 of [32], showing that a compact subset E with Hausdorff measure $H^{p-2}(E) < \infty$ is a polar set for the operator $\mathcal{P}_p^+$. This means that there exists a p^+-superharmonic function v of equation $\mathcal{P}_p^+(D^2v) = 0$ in a neighbourhood Ω_0 of E, which is smooth in $\Omega_0 \backslash E$, such that

$$v(x) = \infty \quad if \quad x \in E, \quad 0 \leq v(x) < \infty \quad if \quad x \notin E\,. \tag{5.41}$$

In fact, since $H^{p-2}(E) < \infty$, then E has Riesz capacity $C_{p-2}(E) = 0$. Then (see for instance [3]) there exists a unit positive Borel measure μ on E such that the potential

$$v(x) = \int_E \mathcal{E}_p(y - x)\, d\mu, \tag{5.42}$$

satisfies (5.41) in $\Omega_0 = \mathbb{R}^n$ and is C^∞ in $\mathbb{R}^n \backslash E$.

Moreover, as in [3], recalling that $\mathcal{E}_p(x) = \frac{1}{|x|^{p-2}}$, by direct computation we get

$$\mathcal{P}_p^+(D^2 v) \leq \int_E \mathcal{P}_p^+(D^2 \mathcal{E}_p(y-x))\, d\mu \leq 0\,, \quad x \notin E; \tag{5.43}$$

on the other hand, since $v(x) = \infty$ on E, there are no test functions at points $x \in E$ and therefore v is p^+-superharmonic in $\mathbb{R}^n$.

Proof of Theorem 1.9. As a consequence of the above argument, the proof can be obtained as a straightforward application of Theorem 6.1 of [32]. Since we have used a different terminology, we make it explicit for sake of completeness.

Since $E \Subset \Gamma$ and $m < p-2$, then $H^{p-2}(E) = 0$, and from the above it follows that E is a polar set for the operator $\mathcal{P}_p^+$, so there exists a p^+-superharmonic function satisfying (5.41).

Let U be the canonical usc extension of u across E, which is bounded above in Ω from the previous subsection; then we consider the family $\{U - \varepsilon\, v : \varepsilon > 0\}$. Using the structure condition (1.9)' and the nondecreasing monotonicity of the function $g(t) = |t|^{s-1}t$, we get

$$\begin{aligned}
&F_p(D^2(U-\varepsilon v)) - |U-\varepsilon v|^{s-1}(U-\varepsilon v) \\
&\geq F_p(D^2 U) - |U|^{s-1}U \\
&+ (|U|^{s-1}U - |U-\varepsilon v|^{s-1}(U-\varepsilon v)) \geq f(x) \quad in \quad \Omega^*\,,
\end{aligned} \tag{5.44}$$

while there are no test functions at points $x \in E$, where $U(x) - \varepsilon\, v(x) = -\infty$.

Hence the usc envelope $w(x) = \overline{u}^*(x)$ of the function

$$\overline{u}(x) = \sup_{\varepsilon>0} (U(x) - \varepsilon v(x)) = \begin{cases} u(x) & if\ x \in \Omega^* \\ -\infty & if\ x \in E \end{cases} \tag{5.45}$$

is in turn a subsolution (see Section 2) of equation $F_p(D^2 w) - |w|^{s-1}w = f(x)$ in Ω.

Analogously, using the canonical lsc extension V of u across E, the lsc envelope of $\underline{u}_*(x)$ of the function

$$\underline{u}(x) = \inf_{\varepsilon>0} (V(x) + \varepsilon v(x)) = \begin{cases} u(x) & if\ x \in \Omega^* \\ +\infty & if\ x \in E \end{cases} \tag{5.46}$$

will be a supersolution of the same equation in Ω.

Finally, at $x_0 \in E$

$$\underline{u}_*(x_0) \le \liminf_{x\to x_0} u(x) \le \limsup_{x\to x_0} u(x) \le \overline{u}^*(x_0)\,. \tag{5.47}$$

On the other side, since $\overline{u}^*(x)$ is a subsolution and $\underline{u}_*(x)$ a supersolution in Ω such that $\overline{u}^*(x) = u(x) = \underline{u}_*(x)$ for $x \notin E$, we also get by comparison also the opposite inequality $\overline{u}^*(x_0) \le \underline{u}_*(x_0)$. Therefore $\overline{u}^*(x_0) = \underline{u}_*(x_0) = \tilde{u}(x_0)$, say, for $x_0 \in E$ and actually the function

$$w(x) = \begin{cases} u(x) & if\ x \in \Omega^* \\ \tilde{u}(x) & if\ x \in E \end{cases}\,, \tag{5.48}$$

yields a continuous extension of u to a solution in all of Ω. □

Remark 5.3. Note that the smoothness assumption on the manifold $\Gamma \Supset E$ in Theorem 1.9, and hence on the distance function $\delta(x) = dist(x,\Gamma)$ is used to show that a solution $u(x)$ is $o(\delta(x))$ as $x \to E$ and ultimately that u is bounded in $\Omega\backslash E$. If u is assumed to be bounded, then the argument of the above proof shows that a sufficient condition in order that E be a removable singularity for equation $F(D^2u) - |u|^{s-1}u = f(x)$ with $f(x)$ bounded is that $C_{p-2}(E) = 0$.

6 Appendix: existence of solutions

In this Section we recall the existence via the Perron's method of viscosity solutions of the Dirichlet problem, provided by Theorem 4.1 of Crandall, Ishii and Lions (see [27] for the classical Perron method and Ishii [36] for other applications to fully nonlinear second order elliptic equations). Existence results for operators involving $\mathcal{P}_p^+$ can be found in [30]. Here we briefly sketch the proof.

We wish to find viscosity solutions of the Dirichlet problem

$$\begin{cases} F_p(D^2u) - |u|^{s-1}u = f(x) & in\ B \\ u = g & on\ \partial B \end{cases} \tag{DP}$$

in a ball B.

According to Theorem 4.1 of [19], we need in particular to find a supersolution $\overline{u}$ and a subsolution $\underline{u}$ of the equation in (DP) such that

$$\underline{u}_* = g = \overline{u}^* \quad on\ \partial B, \tag{6.1}$$

Lemma 6.1. *Suppose that B is a ball in $\mathbb{R}^n$, $1 < p \leq n$, $s > 1$ and $f \in C(\overline{B})$. There exists a function $\overline{u} \in lsc(\overline{B})$ such that*

$$\begin{cases} \mathcal{P}_p^+(D^2\overline{u}) - |\overline{u}|^{s-1}\overline{u} \leq f(x) & in\ B \\ \overline{u}^* = g & on\ \partial B \end{cases} \tag{6.2}$$

Sketch of the proof. Following the scheme of Section 9 of [18], it is sufficient to find, for all points x_b on the boundary ∂B, an equicontinuous family of solutions $G_b \in C(\overline{B})$ of equation

$$\mathcal{P}_p^+(D^2G_b) \leq -\kappa \quad in \quad B \tag{6.3}$$

with $\kappa > 0$, such that $G_b(x_b) = 0$ and $G_b(x) > 0$ for $x \in \overline{B}\backslash\{x_b\}$.

Indeed, suppose that this has been done. If $g = 0$, choosing $M = \max|f|/\kappa$ we get

$$\mathcal{P}_p^+(D^2(MG_b)) - |MG_b|^{s-1}MG_b \leq -\kappa M \leq f(x) \quad in \quad B$$

and we can choose

$$\overline{u}(x) = \inf_{x_b \in \partial B} MG_b(x),$$

which is continuous by equicontinuity and, since the infimum of supersolutions is a supersolution, satisfies

$$\begin{cases} \mathcal{P}_p^+(D^2\overline{u}) - |\overline{u}|^{s-1}\overline{u} \leq f(x) & in\ B \\ \overline{u} = g & on\ \partial B \end{cases}$$

In the general case (see Hint. 9.3 of [18], Section 7.2 of [46]) we can use as $\overline{u}$ the lsc envelope (see Lemma 4.2 of [19]) of the function

$$u(x) = \inf_{\substack{x_b \in \partial B_1 \\ \varepsilon > 0}} \left(g(x_b) + \varepsilon + M_\varepsilon G_b(x)\right)$$

with suitable constants $M_\varepsilon > 0$.

Hence we are left with the task of finding the functions G_b for each $x_b \in \partial B$. We notice that unfortunately, by lack of uniform ellipticity, we may not proceed taking a radial symmetric function. So we need a different construction.
We may assume

$$B = \{x \in \mathbb{R}^n : x_1^2 + \cdots + x_{n-1}^2 + (x_n - 1)^2 < 1\}.$$

Setting

$$G_0(x) = x_n - (x_1^2 + \cdots + x_{n-1}^2),$$

we have $G_0(0) = 0$, $G_0(x) > 0$ in $B\backslash\{0\}$ and

$$\mathcal{P}_p^+(D^2 G_0) = -2(p-1) < 0\,.$$

For a general point $x_b \in \partial B$ then we can define

$$G_b(x) = G_0(R_b(x - x_b)),$$

where R_b is a suitable rotation matrix, and we notice that the functions $G_b(x)$ have all required properties. $\square$

We are in position to prove the following existence and uniqueness result.

Theorem 6.2. *Let B be a ball of $\mathbb{R}^n$, $1 < p \leq n$, $s > 1$, $f \in C(\overline{B})$ and $g \in C(\partial B)$. There exists a unique continuous viscosity solution of Dirichlet problem (DP).*

Proof. According to Theorem 4.1 of [19], we need:
i) the comparison principle between subsolutions and supersolutions;
ii) the existence of subsolutions and supersolutions with continuous boundary values.

The comparison principle is proved above in Lemma 2.1 and a supersolution $\overline{u}$ in the ball B such that $\overline{u}^* = g$ on ∂B is provided by the above Lemma 6.1, since $F_p(X) \leq \mathcal{P}_p^+(X)$, by (1.9). To find a subsolution, it is enough to find with Lemma 6.1 a solution of the problem

$$\begin{cases} \mathcal{P}_p^+(D^2 w) - |w|^{s-1}v \leq -f(x) & \text{in } B \\ w^* = -g & \text{on } \partial B \end{cases} \tag{6.4}$$

in order that $v = -w$ satisfies

$$\begin{cases} \mathcal{P}_p^-(D^2 v) - |v|^{s-1}w \geq f(x) & \text{in } B \\ v_* = g & \text{on } \partial B \end{cases} \tag{6.5}$$

and therefore v can be used as the subsolution that we needed, since $F_p(X) \geq \mathcal{P}_p^-(X)$. In conclusion, the assumptions of Theorem 4.1 of [19] are fulfilled, and we are done. $\square$

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