# "This is a post-peer-review, pre-copyedit version of an article published in "Computers and Structures". <br> The final authenticated version is available online at: http://dx.doi.org/10.1016/j.compstruct.2014.09.007" 

# Static analysis of a Guastavino helical stair as a layered masonry shell 

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## A R T I C L E I N F O

## 1. Introduction

Guastavino vaulting is a technique for constructing robust, selfsupporting thin shells, using interlocking terracotta tiles and layers of mortar, with the tiles following the curve of the roof.

The tiles are standardized: 25 mm thick, and approximately 150 mm by 300 mm across. They are usually set in three herring-bone-pattern courses with a sandwich of thin layers of cement; then timbrel structures are a sort of low-tech layered structures.

The Guastavinos constructed their structures without scaffolding. The masons, working at the free edge of the previous day's construction, would lean over and move the edge forward by adding new tiles and gluing them one on top of the other with rapidlyhardening gypsum plaster and hydraulic cement.

This kind of construction is also known as timbrel vaulting, because of its likeness to the skin of a timbrel (see Guastavino [1]). An extensive and complete description of Guastavino vaulting technique can be found in the book edited by Huerta in [2].

The Guastavinos believed that these timbrel vaults represented an innovation in structural engineering, because the tile system provided solutions that were impossible with traditional masonry arches and vaults, but mainly because of their supposed strong cohesion. Subsequent research has shown the timbrel vault is simply a thinner masonry vault producing a horizontal thrust also,


#### Abstract

An analytical structural study of general helicoidal timbrel shells is presented. The study is concerned in particular with Guastavino staircases based on a circular planform. Such stairs are composite masonry structures formed by a two or three layers of tiles disposed in a herringbone-pattern. The analysis is based on the assumption that the material is unilateral, namely a No-Tension material in the sense of Heyman; in particular the safe theorem of Limit Analysis is employed. In the spirit of the safe theorem the structure is stable if a statically admissible stress field ean be constructed; for the unilateral material here employed, singular stress fields, that is stress concentrated on surfaces (membranes) or lines (arches) are allowed. The statically admissible stress fields that are constructed, combining membrane stresses and 3d diffuse uniaxial stresses, are purely compressive and balance transverse loads either uniformly distributed or localized. A simple order-of-magnitude calculation confirm that bending and torsion resistance is small compared to the structural demand, and that a purely compressive membrane equilibrium stress field is required, the leyel of compressive stresses required to balance the load is below the limit compressive threshold.


simply to a lesser degree than traditional vaults due to its lighter weight (see Ochsendorf and Freeman [3]). This concept is expressed clearly by Huerta in [4]: ". . .timbrel vaults are masonry vaults. Like any other masonry they have little resistance to tension, they crack and thrust. They are neither monolithic nor cohesive. They can and should be calculated with the same methods used for a vault of masonry".

In the present paper by following this line of thinking (that is on assuming that the material has no cohesion), i.e. it is No-Tension in the sense of Heyman in [6], I shall attempt to study the equilibrium of Guastavino spiral staircases of circular planform. Such stairs were built inward from a groove in a cylindrical outer wall, and had a free inner edge. The steps, apparently as non-structural elements, were finally added (see Huerta [4]).

For the purpose of the structural analysis the staircase is modeled as a uniform thin helicoidal shell made of Rigid No-Tension material. The scope of the paper is not to study in detail and quantitatively any particular stair, rather I shall try to give an admissible equilibrium solution (compatible with the unilateral material assumption) for the two cases shown in (Fig. 1(a) and (b)), namely the case of uniform and localized load.

The existence of such equilibrium solutions is a proof that, in the spirit of the safe theorem of Limit Analysis, the structure can stand without resorting to any cohesion.

In Section 3, an extensive reference on the No-Tension (NT) model and on the application of Limit Analysis to structures composed of such ideal material is given.


Fig. 1. (a) Guastavino spiral staircase, First National Bank, Paterson, New Jersey, ca. (1890), (b) A Guastavino helical stair with circular planform, locally over-loaded.

## 2. Tile vaults as layered shell composites

Traditional tile vaults originated in the Catalan Region of Spain, yet features of this vaulting technique can be found in other cultures (see Guastavino [1]). The main difference between Catalan vaults and traditional masonry vaults is that the tiles are laid flat, with a substantial mortar thickness between layers. They can have two or three layers: the first, at least, is built with Plaster of Paris, (which can be mixed in small batches to set in about 10 s ) and executed without centering; the consecutive ones are joined with weather-proof Portland cement mortar. Their slenderness (i.e. ratio of radius of curvature and span) is often about 100 , but can be higher. The construction process is simple and inexpensive in the context of pre-industrial techniques: a vault without centering and quick execution that could be built easily. Tile vaults are layered composite structures, typically, composed of two sheets, about 10 cm of thickness overall, including the intemediate layer of mortar and coatings. The main reason for this arrangement is to facilitate construction but the combination of the different materials and the way they are set together, improves quality, homogenizes strength and stiffness and smear out the possible defects of the individual tiles. Guastavino himself in [2] states the structural importance of the layered composite construction, on comparing the different performance of single layer and double layer arches (see Fig. 2(a)), also if he believes that the main effect of layering is to give bending strength to the vault. The more frequent pattern in which the tiles are arranged in a single ply is shown in (Fig. 2(b)),
through a number of different and more complex patterns were possible. The usual through thickness arrangement of plies in the two and three sheet settings are 0/90 and 0/90/0.

The main assumption of classical layered composite structures is that, under deformation, straight normals remain straight, which enforces full composite behavior between layers. In the present paper the assumed presence of this rigid through thickness connection is used with a different purpose. The main idea of equilibrium for a masonry vault is to think that purely compressive membrane equilibrium can be realized within the masonry, namely a thin membrane carrying the load and transmitting it to the boundary (the abutments, the walls or the supporting arches) can be fitted inside the masonry. The thickness of this ideal membrane can be a small fraction of the whole thickness, the compressive stress required to carry the transverse load still remaining largely safe. The aforesaid rigid through thickness connections, typical of the classical lamination theory, are used here to transfer the load, either due to the self load or to the over load, to the membrane structure. This load transfer requires that both compressive and tensile uniaxial stresses be transmitted along such elements.

## 3. The NT model

As a first approximation to the behavior of the masonry shell that we are studying, the Rigid No-Tension model is adopted. This crude unilateral model material that idealizes the real material as indefinitely strong in compression but incapable of sustaining


Fig. 2. (a) Comparison, (b) pattern.
tensile stresses, is rigid in compression and can elongate freely, a positive deformation of the material being interpreted as a measure of fracture into the material (either smeared or concentrated). It must be observed that, though this ideal material has a limited repertoire of admissible stresses and strains and can exhibit fractures, its uniaxial behavior in elongation is elastic in the sense that strain determines stress and there are no residual strains upon reloading in compression.

The unilateral model for masonry, that, though in a mathematically unconscious way, has been around since the nineteenth century (see Moseley, in [5]), first rationally introduced by Heyman [6], was divulgated and extended in Italy, thanks to the effort of Salvatore Di Pasquale [7] and other members of the Italian School of Structural Mechanics, such as Del Piero [8,9], and Como [10].

A research group of the University of Salerno (of which the present author have been a leading part) has contributed to the debate on the unilateral models for masonry, with a number of papers, since the early works of Angelillo [11], till the more recent articles of Fortunato [12] and Angelillo et al. [13-16].

The formulation of the BVP for unilateral masonry materials, that is Rigid No-Tension materials for which the latent strains (fractures) satisfy a normality condition with respect to the admissible stresses, can be found in [13].

In particular, in the application I present here, I focus on the statical approach, namely on the safe theorem of Limit Analysis. By working with the safe theorem, one can admit also singular stresses representing concentrated compressive stresses with support on surfaces or curves located inside the masonry.

Indeed the more efficient tool that can be introduced for applying the unilateral No-Tension model to masonry structures is the systematic use of singular stress and strain fields, within the framework defined by the two theorems of Limit Analysis (see Angelillo et al. [16] for applications of the safe theorem and Angelillo et al. [17] for applications of the kinematic theorem to walls).

The concept of compatible loads and distortions, and the valit ${ }^{-}$ ity of the two theorems of Limit Analysis, admitting singular stress and discontinuous displacements, are discussed in Angelillo et al. [16]. The use of singular equilibrated stresses for approximating plane equilibrium problems can be traced to the work of Fraternali et al. [18], and for vaults to the more recent paper by Block and Ochsendorf [19]. In recent papers (see [23,24]) the issue of giving to curved masonry structures the otherwise missing resistance to traction, by adding FRP composite reinforcements, is addressed.

## 4. Analysis

### 4.1. Geometry

The structure of the stair is assimilated to a spiral shell surface $S$ of thickness $t$. A schematic 3d view of such structure is depicted in (Fig. 3(a)) to which we refer for notations. A schematic plan of the stair is depicted in (Fig. 3(b)).

In particular the relevant geometrical dimensions of the stair are the internal radius $R^{\circ}$, the external radius $R$, the height of one complete landing $h$ and the thickness $t$.

By denoting $\{., .,$.$\} the components of vector quantities with$ respect to the Cartesian frame shown in Fig. 3(a), whose base vectors are $\left\{\hat{\mathbf{e}}_{1}, \hat{\mathbf{e}}_{2}, \hat{\mathbf{e}}_{3}\right\}$, the parametric description of the surface $S$ in the Cartesian frame shown in Fig. 3(a) is
$\mathbf{x}(r, \theta)=\{r \cos \theta, r \sin \theta, c \theta\}$,
where the constant $c$ is
$c=\frac{h}{2 \pi}$.

The natural base vectors convected to the surface $S$ are defined as follows
$\mathbf{g}_{1}=\{\cos \theta, \sin \theta, 0\}$,
$\mathbf{g}_{2}=\{-r \sin \theta, r \cos \theta, c\}$,
$\mathbf{g}_{3}=\frac{1}{\sqrt{r^{2}+c^{2}}}\{-c \sin \theta, c \cos \theta,-r\}$,
and the covariant components of the curvature tensor, in this curvilinear reference, are
$\chi_{11}=0, \quad \chi_{12}=\chi_{21}=\frac{c}{\sqrt{r^{2}+c^{2}}}, \quad \chi_{22}=0$.

### 4.2. Membrane equilibrium

The most simple way of thinking about membrane equilibrium of a thin shell under purely vertical loading (defined per unit projected area)
$\mathbf{q}=\left\{0,0, p_{z}\right\}$,
is to adopt Pucher's approach (see [20]). By adopting a Monge description of the surface $S$ based on polar coordinates $\left\{\theta^{1}=r, \theta^{2}=\theta\right\}$ defined in the planform of the membrane $\mathbf{x}(r, \theta)=\{r \cos \theta, r \sin \theta, f(r, \theta)\}$,
with Pucher's transformation the generalized stresses $\hat{\mathbf{T}}$ on the surface are transformed into projected stresses $\mathbf{S}=J \hat{\mathbf{T}}$ in the planform ( $J$ being the ratio between the surface areas on the surface and on the planform). On introducing in the plane of the planform the polar reference $\left\{\theta^{1}=r, \theta^{2}=\theta\right\}$, the associated covariant bases
$\mathbf{b}_{1}=\{\cos \theta, \sin \theta\}$,
$\mathbf{b}_{2}=\{-r \sin \theta, r \cos \theta\}$,
and the variabale Cartesian base
$\mathbf{k}_{1}=\{\cos \theta, \sin \theta\}$,
$\mathbf{k}_{2}=\{-\sin \theta, \cos \theta\}$,
the projected stress $\mathbf{S}$ can be represented as follows
$\mathbf{S}=S^{\alpha \beta} \mathbf{b}_{\alpha} \otimes \mathbf{b}_{\beta}=\sigma_{\alpha \beta} \mathbf{k}_{\alpha} \otimes \mathbf{k}_{\beta}$,
where summation convention over repeated indices has been adopted (with $\alpha, \beta=1,2$ ) and $S^{\alpha \beta}, \sigma_{11}=\sigma_{r r}, \sigma_{22}=\sigma_{\theta \theta}, \sigma_{12}=\sigma_{r \theta}$, denote the contravariant components and the physical components of the projected stress in the polar reference $\{r, \theta\}$.

The equilibrium equations for such projected stresses in the plane of the planform are identical to those of the plane problem and, on considering the general case of distributed loading defined per unit projected area
$\mathbf{q}=p^{1} \hat{\mathbf{b}}_{1}+p^{2} \hat{\mathbf{b}}_{2}+p_{z} \hat{\mathbf{e}}_{3}=p_{r} \hat{\mathbf{k}}_{1}+p_{\theta} \hat{\mathbf{k}}_{2}+p_{z} \hat{\mathbf{e}}_{3}$,
read
$S_{/ 1}^{11}+S_{/ 2}^{12}+p^{1}=0$,
$S_{/ 1}^{21}+S_{/ 2}^{22}+p^{2}=0$,
where ${ }_{. / \alpha}$ stands for covariant derivative with respect to $\theta^{\alpha}$. In the case of pure vertical loading such equations may be solved with the use of an Airy stress function.

The forces acting transversely to the surface $S$ (defined per unit projected area of the planform) are balanced by the scalar product of the matrix of the contravariant components of the projected stress times the matrix of the covariant components of the curvature:
$S^{\alpha \beta} f_{/ \alpha \beta}-f_{, \alpha} p^{\alpha}+p_{z}=0$
where ${ }_{,, \alpha}$ denotes differentiation with respect to $\theta^{\alpha}$.


Fig. 3. Schematic 3d view of the stair: (a). Rlan: (b).

The above equilibrium equations can be rewritten in terms of physical stress components $\sigma_{r r}, \sigma_{r \theta}, \sigma_{\theta \theta}$ on considering that, for the case at hand:
$S^{11}=\sigma_{r r}, \quad S^{22}=\frac{1}{r^{2}} \sigma_{\theta \theta}, \quad S^{12}=\frac{1}{r} \sigma_{r \theta}$,
and
$f_{/ 11}=f_{, 11}, \quad f_{/ 22}=r^{2}\left(\frac{1}{r} f_{, 1}+\frac{1}{r^{2}} f_{, 22}\right), \quad f_{/ 12}=r\left(\frac{1}{r} f_{, 2}\right)$,
By substituting into the above equations, after some algebra, one obtains
$\frac{1}{r} \frac{\partial\left(r \sigma_{r r}\right)}{\partial r}+\frac{1}{r} \frac{\partial \sigma_{r \theta}}{\partial \theta}-\frac{\sigma_{\theta \theta}}{r}+p_{r}=0$,
$\frac{1}{r^{2}} \frac{\partial\left(r^{2} \sigma_{\theta r}\right)}{\partial r}+\frac{1}{r} \frac{\partial \sigma_{\theta \theta}}{\partial \theta}+p_{\theta}=0$,
$\sigma_{r r} \frac{\partial^{2} f}{\partial r^{2}}+\sigma_{\theta \theta}\left(\frac{1}{r} \frac{\partial f}{\partial r}+\frac{1}{r^{2}} \frac{\partial^{2} f}{\partial \theta^{2}}\right)+2 \sigma_{r \theta} \frac{\partial\left(\frac{\partial f}{\partial r}\right)}{\partial r}-\frac{\partial f}{\partial r} p_{r}-\frac{1}{r} \frac{\partial f}{\partial \theta} p_{\theta}+p_{z}=0$.
In the present case, by considering a pure vertical load directed downward, that is $p_{z}=-q$, and assuming (see Section 4.1)
$f(r, \theta)=c \theta$,
the equation of equilibrium transverse to the membrane reduces to the form
$-2 \sigma_{r \theta} \frac{c}{r^{2}}-q=0$.
If the membrane is made of NT material the surface stress tensor must be negative semi-definite and the matrix itself of the projected stresses must be negative semi-definite. The first to consider membrane equilibrium with Puher's transformation for NT materials were Angelillo and Fortunato in [21]. In a more recent paper on masonry vaults (Angelillo et al. [22]) a number of equilibrium solutions for domes and vaults is presented.

With the description of the surface corresponding to $f=c \theta$, only the curvature components $\chi_{12}, \chi_{21}$ are different from zero. Then in the case of a pure vertical load $-q$ uniformly distributed
all over $S$ (such as, approximately, the effect of the self load), the load must be balanced by the stress component $\sigma_{r \theta}$, that is by a shear stress in the planform
$\sigma_{r 0}=\frac{q r^{2}}{2 c}$.
To enforce equilibrium in the planform and the unilateral assumption on stress one should add to this stress field of pure polar shear
$\mathbf{S}=-\frac{q r^{2}}{2 c}\left(\mathbf{k}_{1} \otimes \mathbf{k}_{2}+\mathbf{k}_{2} \otimes \mathbf{k}_{1}\right)$,
balanced with the transverse load, another balanced stress field of the type
$\mathbf{S}=\sigma_{r r} \mathbf{k}_{1} \otimes \mathbf{k}_{1}+\sigma_{\theta \theta} \mathbf{k}_{2} \otimes \mathbf{k}_{2}$,
not affecting the transverse equilibrium.
Given the homogeneous traction boundary condition at the free internal edge of the surface, a string of concentrated stress along the edge is required, carrying distributed loads acting both in the tangent and in the normal direction. Such singular stresses are often useful in the equilibrium analysis of NT materials (see [16]), but, though acceptable for idealizing zones of high stress level inside thick walls, are less likely for the case at hand, given the thickness of the structure. Another reason for excluding here such edge tendon is the difficulty to make this 1d structure a purely compressed member.

Then if one wishes to obtain a statically admissible membrane equilibrium, the shape of the surface must be changed. In [22] an equilibrium solution for a spiral membrane made of NT material is also presented. The shape of the surface must be a double curvature spiral: a shape of that sort that fits into the masonry for the case at hand is given by
$f=-\frac{r\left(r+2 R^{\circ}\right) t}{R^{2}+2 R R^{\circ}-3 R^{\circ^{2}}}+c \theta$.
The equation of equilibrium transverse to the surface becomes
$-\sigma_{r r} \frac{2 t}{R^{2}+2 R R^{\circ}-3 R^{0^{2}}}-\sigma_{\theta \theta} \frac{2 t\left(r+R^{\circ}\right)}{r\left(R^{2}+2 R R^{\circ}-3 R^{\circ}\right)}-2 \sigma_{r \theta} \frac{c}{r^{2}}-q=0$.
and the projected compressive stress field in equilibrium in the planform and balancing the transverse load proposed in [22] is
$\sigma_{r r}=-q \frac{R^{2}+2 R R^{\circ}-3 R^{\rho^{2}}}{4 t r}\left(r-R^{\circ}\right)$,
$\sigma_{\theta \theta}=-q \frac{R^{2}+2 R R^{\circ}-3 R^{\rho^{2}}}{4 t}$,
with $\sigma_{r \theta}=0$.
Due to the thickness of the stair we are considering the curvatures in the radial and circumferential directions are small, and the equilibrium solution proposed in [22] produces a large uniform circumferential stress. How large this stress can be depends on the ratio between $\left(r-R^{\circ}\right) / t$, and I will come to the evaluation of such a value, for a particular case, in the final section. Moreover this kind of solution requires a perfect axial symmetry of the load and is hardly adaptable to the case of a Guastavino stair loaded locally as in Fig. 1(b).

Therefore we are forced to look for a 3d equilibrium solution, or, maybe, a combination of a 3d and a membrane balanced stress field.

### 4.3. 3d Equilibrium solution for a uniform transverse load

Then, in the case of a vertical load uniformly distributed all over $S$ or on a sector $S_{\alpha}$ of span $\alpha$, one should resort to a 3d equilibrium solution. The 3d equilibrium solution that I construct is based on the curvilinear system $\left\{\theta^{1}=\rho, \theta^{2}=\theta, \theta^{3}=y\right\}$ defined as follows
$\mathbf{x}(\rho, \theta, y)=\left\{R\left(1-\rho \frac{y}{t}\right) \cos \theta, R\left(1-\rho \frac{y}{t}\right) \sin \theta, c \theta+y\right\}$.
The corresponding curvilinear lines $\left\{\theta^{1}=\rho, \theta^{3}=y\right\} \quad$ at $\theta^{2}=\theta=$ const are shown in Fig. 4(a).

The natural base vectors associated to this curvilinear system are
$\mathbf{a}_{1}=\left\{-R \frac{y}{t} \cos \theta,-R \frac{y}{t} \sin \theta, 0\right\}$,
$\mathbf{a}_{2}=\left\{-R\left(1-\rho \frac{y}{t}\right) \sin \theta, R\left(1-\rho \frac{y}{t}\right) \cos \theta, c\right\}$,

$\mathbf{a}_{3}=\left\{-R \frac{\rho}{t} \cos \theta,-R \frac{\rho}{t} \sin \theta, 1\right\}$,


The stress field that I consider consists of a fan of self-balanced uniaxial stresses laying along compression rays directed as the base vectors $\mathbf{a}_{3}$ :
$\mathbf{T}=\sigma \hat{\mathbf{a}}_{3} \otimes \hat{\mathbf{a}}_{3}$,
( $\hat{\mathbf{a}}_{3}$ being the unit vector in the direction of $\mathbf{a}_{3}$ ) and of a surface generalized stress $\hat{\mathbf{T}}$ concentrated on the membrane $S$, that is the surface $y=t$. The membrane on which the stress is concentrated is loaded by a distributed traction load $f_{r}$, directed radially. The radial load represents the vector sum of the unbalanced stresses transmitted by the uniaxial fan and by the vertical load, such a sum being assumed tangent to the membrane itself, as shown in the graphic of Fig. 4(b). Taking into account that $R \rho=R-r$, and calling
$\hat{\mathbf{n}} \equiv \mathbf{g}_{3}=\frac{1}{\sqrt{c^{2}+r^{2}}}\{-c \sin \theta,-c \cos \theta,-r\}$,
the unit normal to the surface $S$, from Fig. 4(b) one obtains
$f_{r}=q \frac{R-r}{t},\left.\quad \sigma\right|_{y=t}\left(\hat{\mathbf{a}}_{3} \cdot \hat{\mathbf{n}}\right)=q \sqrt{1+\frac{(R-r)^{2}}{t^{2}}}$,
from which
$\left.\sigma\right|_{y=t}=-q \sqrt{c^{2}+r^{2}} \frac{(R-r)^{2}+t^{2}}{r t^{2}}$.
The equilibrium equations for the projected stress $\mathbf{S}$ for the membrane $S$, are then
$\frac{1}{r} \frac{\partial\left(r \sigma_{r r}\right)}{\partial r}+\frac{1}{r} \frac{\partial \sigma_{r \theta}}{\partial \theta}-\frac{\sigma_{\theta \theta}}{r}-f_{r}=0$,
$\frac{1}{r^{2}} \frac{\partial\left(r^{2} \sigma_{\theta r}\right)}{\partial r}+\frac{1}{r} \frac{\partial \sigma_{\theta \theta}}{\partial \theta}=0$,
$-\frac{2 c}{r^{2}} \sigma_{r \theta}=0$.
The simplest statically admissible solution one can think of, valid if $f_{r}$ (that is $q$ ) is independent of $\theta$ is
$\sigma_{\theta \theta}=-r q \frac{R-r}{t}$,

4.4. 3d Equilibrium solution for a uniform transverse load applied locally

If the star is loaded by a vertical uniform load of intensity $q_{1}$, but only in a sector $S_{\alpha}$ of arbitrary amplitude $\alpha<2 \pi$, we may still adopt the previous solution with
$\sigma_{\theta \theta}=-r q_{1} \frac{R-r}{t}, \quad \sigma_{r r}=\sigma_{r \theta}=0$.
in the sector of span $S_{\alpha}$, whilst outside such a sector, where the transverse load is zero, we may assume
$\sigma_{\theta \theta}=-r q_{1} \frac{R-r}{t} \sigma_{r r}=q_{1} \frac{\left(3 R-2 R^{\circ}\right) R^{\rho^{2}}-3 R r^{2}+2 r^{3}}{6 r t}, \quad S_{r \theta}=0$,
a stress field which is statically admissible, since it is compressive and satisfies the homogeneous equilibrium equations in the planform and transverse to the surface $S$. A plot of these stress components versus the radius $r$, in the interval $\left[R^{\circ}, R\right]$, is shown in Fig. 5.

Notice that the emerging stress at the free edge are zero for both the statically admissible stress fields considered above. The first stress field is uniform and is statically admissible with a transverse load uniformly distributed per unit projected area. The second stress field is statically admissible with a uniform transverse load applied only on a sector of amplitude $\alpha$. Combining linearly such stress field with the previous one, we obtain a statically admissible stress field for the case of Fig. 1(b).

## 5. Discussion

In the previous analysis I have shown that a transverse load, either uniformly distributed or localized on a small sector of the stair, can be balanced by compressive stresses if the material is assumed as Rigid No-Tension.

(a)

(b)

Fig. 4. Curvilinear lines at $\theta=$ const: (a). Relation between $q, f_{r}$ and $\sigma$ along a compression ray: (b).

 $q_{1}=10 \mathrm{KN} / \mathrm{m}^{2}(r$ in m , generalized stress in $\mathrm{KN} / \mathrm{m})$.

The assumption that the material is unilateral (No-Tension) must not be a matter of faith, rather should be a rational consequence of a comparison between the order of magnitude of the tensile stresses required to maintain equilibrium and the allowable level of tensile stress that the material can sustain.

Indeed, if the mean tensile stresses can be, wherever, at least two orders of magnitude less that the tensile strength of the masonry material, even cautiously considering the brittleness of the material, would be reasonable to accept them. For example the tensile stresses necessary to bring the self-load from the interior of an arch to its extrados are very low and can be safely exploited to this effect.

Moreover there can be monolithic stone elements behaving as beams in flexure and torsion, for which the tensile and shear stresses that build up inside the material in these stress regimes, though relatively high, can be low with respect to the tensile strength of the stone itself. Therefore insisting to treat these stone elements as composed of unilateral material would be dull.

In the case at hand, in which the tensile strength of the masonry assembly, rather than that of a single monolithic block, should be exploited, a very simple analysis shows that the tensile stress level required to balance the loads is excessively high.

Consider for example the case of the localized load (Fig. 1(b)), and, for simplicity, assume that the transverse load is concentrated on the segment $A B$ (Fig. 6(a)). To make some simple calculations assume that $R^{\circ}=1 \mathrm{~m}, R=2 \mathrm{~m}, t=0.1 \mathrm{~m}, h=5 \mathrm{~m}$ and that the load is $P=10 \mathrm{KN} / \mathrm{m}$.

If one assumes that the stair behaves as a shell $S$ resisting to bending and torsion, the generalized stress can have a shear component transverse to the surface $S$. Calling $q_{2}$, the vertical shear component relative to the surface of normal $\hat{\mathbf{e}}_{2}$, a possible equilibrium solution (excluding membrane stresses) is
$q_{2}=\left\{\begin{array}{l}Q / 2 \text { for } x_{1}>R^{\circ}, x_{2}>0 \\ -Q / 2 \text { for } x_{1}>R^{\circ}, x_{2}<0 \\ 0 \text { for } x_{1}<R^{\circ} .\end{array}\right.$
With this solution one should have tensile stresses in the upper part of the shell (Fig. 6(a)), and, in particular, a tensile normal stress component at the point $A$ of value
$\sigma=\frac{c}{\sqrt{c^{2}+R^{\mathrm{o}^{2}}}=\frac{10}{2} \frac{5}{\sqrt{26}}} \mathrm{KN} / \mathrm{m} \approx 5 \mathrm{KN} / \mathrm{m}$.
Given the thickness $t=0.1 \mathrm{~m}$ of the shell, this corresponds approximately to a value of stress of 0.05 MPa , a value close to the tensile strength of a brick masonry material.

Therefore one should assume safely that the entire load is carried by the lower part of the shell (grey area in Fig. 6(a)). In such a case one has
$q_{2}=\left\{\begin{array}{l}-Q \text { for } x_{1}>R^{\circ}, x_{2}<0 \\ 0 \text { otherwise. }\end{array}\right.$
This shear must be balanced by bending and torsional moments per unit length $m_{r \theta}, m_{r r}, m_{\theta \theta}$ arising inside the shell. A possible equilibrium solution that exploits the bending and torsion resistance of the radial segments only is
$m_{r \theta}=-q \cos \theta\left(r-\frac{R}{\cos \theta}\right)$,
$m_{\theta \theta}=-q \sin \theta\left(r-\frac{R^{\circ}}{\cos \theta}\right)$.
A simple computation shows that the maximum moment is
$m_{r \theta}=q\left(R-R^{\circ}\right)=-10(2-1) \mathrm{KN} / \mathrm{m}=-10 \mathrm{KN} / \mathrm{m}$,
Therefore an estimate for the corresponding shear stress is
$\tau=\frac{6 m_{r \theta}}{h^{2}}=\frac{6(10)}{(0.1)^{2}} \mathrm{KN} / \mathrm{m}=6 \mathrm{MPa}$,
a value well above the admissible shear strength for brick masonry.
This is the main motivation to look for unilateral equilibrium solutions.

The 3d equilibrium solution we have given above has two aspects that require a discussion. The first one is that the stair transmits radial forces to the wall and circumferential membrane stresses to its extremities. Proper constraints must be assumed both at the wall and at the ends.

For a very long stair of a few landings, the propagation of the circumferential stress all the way to the extremities, in the case of a localized transverse load, would appear artificial and one could exploit the possibility of a stress field that is confined to a neighborhood of the loaded sector. In Fig. 6(b) a possible equilibrium state having the same form of the second equilibrium solution in the sector $A$, and a similar form in the region $B$, but in a slightly different curvilinear reference (see Fig. 6(b)) could be considered. This change of reference would require a slight change in the shape of the surface in order to keep the center of the region $B$ close to the surface $S$. Depending on the given slope $c$ the lateral parts of such modified surface based on the region $B$, could be outside the masonry and would imply some small bending moments in the shell.

The second aspect is the fact that concentrated compressive stresses build up inside the material, and a proper evaluation of the amount of these concentrated stresses and on the possible distribution area must be performed.

In particular, with the localized load we have a circumferential membrane stress concentrated on the surface $S$, of value
$\sigma_{\theta \theta}=-r q_{1} \frac{R-r}{t}$.


Fig. 6. Concentrated transverse load applied on the segment $A B$, upper and lowershear areas: (a). Uniform transverse load applied on the sector $A$, alternative equilibrium solution: (b).

For the particular case here considered, this corresponds to a maximal value of circumferential stress
$\sigma_{\theta \theta}=-(1) 10 \frac{2-1}{0.1}=-100 \mathrm{KN} / \mathrm{m}$.
To keep the compressive stress within reasonable limits (say less than 5 MPa ), this would require to smear the stress on a thickness of 2 cm , that is to consider the surface $S$ at 1 cm below the extrados of the stair, with a small (100/90 increase) of the membrane stress.

Finally I wish to compare the level of stress corresponding to the 3d solution with the membrane solution proposed in [22]. The balanced membrane stress field given in [22] consists of radial and circumferential stress components only. In particular the circumferential stress component is constant, and, for the particular case considered above it amounts to
$\sigma_{\theta \theta}=-q_{1} \frac{R^{2}+2 R R^{\circ}-3 R^{\mathrm{o}^{2}}}{4 t}=10 \frac{4+4-3}{4(0.1)} \mathrm{KN} / \mathrm{m}=-125 \mathrm{KN} / \mathrm{m}$,
that is a value every where greater than the maximum value derived above for the 3d solution.

## Acknowledgments

I am grateful to Santiago Huerta for the inspiring lectures given at the University of Salerno during his recent visit, for many conversations about structural masonry, and for information on the Guastavinos and their construction system.

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