Published on Nonlinear Analysis
doi: 10.1016/j.na.2017.07.008

Noncoercive elliptic equations with discontinuous coefficients in unbounded domains

Sara Monsurrò *

Dipartimento di Matematica, Università di Salerno,
via Giovanni Paolo II, 132, I - 84084, Fisciano (SA), Italy

Maria Transirico †

Dipartimento di Matematica, Università di Salerno,
via Giovanni Paolo II, 132, I - 84084, Fisciano (SA), Italy

Abstract

In this paper we study Dirichlet problems for noncoercive linear elliptic equations with discontinuous coefficients in unbounded domains. Exploiting a nonlinear approach, we achieve existence, uniqueness and regularity results.

Keywords: Noncoercive elliptic equations, discontinuous coefficients, unbounded domains.

*Electronic address: `smonsurro@unisa.it`; Corresponding author
†Electronic address: `mtransirico@unisa.it`
Sara Monsurrò and Maria Transirico are members of the Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM)

1 Introduction

Let Ω be an open subset of $\mathbb{R}^N$, $N > 2$. We consider the Dirichlet problem

$$(1)\qquad \begin{cases} -\mathrm{div}(M(x)\nabla u) + \mu u = -\mathrm{div}(u\, E(x)) + f(x) & \text{in } \Omega, \\ \\ u \in W_0^{1,2}(\Omega), \end{cases}$$

where $M : \Omega \to \mathbb{R}^{N^2}$ is a measurable matrix field such that there exist α, $\beta \in \mathbb{R}_+$ such that

$$(2)\qquad \alpha|\xi|^2 \le M(x)\,\xi\cdot\xi, \quad |M(x)| \le \beta, \quad \text{a.e. } x \in \Omega, \quad \forall\, \xi \in \mathbb{R}^N,$$

$$(3)\qquad \mu > 0,$$

$E : \Omega \to \mathbb{R}^N$ is a vector field and $f : \Omega \to \mathbb{R}$ is a real function.

Guido Stampacchia proved that if Ω is bounded, $|E| \in L^N(\Omega)$, $f \in L^{\frac{2N}{N+2}}(\Omega)$ and if μ is large enough, problem (1) admits a unique weak solution u.
Moreover, he also proved that
- if $|E| \in L^N(\Omega)$ and $f \in L^m(\Omega)$, $\frac{2N}{N+2} \le m < \frac{N}{2}$, then the solution u of (1) is in $L^{m^{**}}(\Omega)$, with

$$(4)\qquad m^{**} = (m^*)^* = \frac{Nm}{N-2m},$$

m^* being the Sobolev conjugate of m;
- if $|E| \in L^N(\Omega)$ and $f \in L^m(\Omega)$, $m > \frac{N}{2}$, then the solution u of (1) is in $L^\infty(\Omega)$.

Successively, Lucio Boccardo, in [1], studied the case $\mu = 0$, obtaining the same results. We point out that the main difficulty here relies on the noncoercitivity of the operator $-\mathrm{div}(M(x)\nabla u) + \mathrm{div}(u\, E(x))$ due to the presence of the second term on which no smallness assumptions are done.

In this paper we generalize these existence, uniqueness and regularity results to the case when Ω is unbounded. We explicitly observe that, since the domain is unbounded, we need to assume hypothesis (3). Nevertheless, we still have the noncoercitivity of the operator since we do not require that μ is large enough.
More precisely, we prove the unique solvability of problem (1) under the assumptions

$$(5)\qquad |E| \in L^2(\Omega) \cap M_0^N(\Omega)$$

and

$$(6)\qquad f \in L^1(\Omega) \cap L^{\frac{2N}{N+2}}(\Omega),$$

where $M^N_0(\Omega)$ is a functional space strictly containing $L^N(\Omega)$, described in Section 2.
Furthermore, we also generalize to the case of unbounded domains the regularity results proving that
- if $|E| \in L^2(\Omega) \cap M^N_0(\Omega)$ and $f \in L^1(\Omega) \cap L^m(\Omega)$, $\frac{2N}{N+2} \le m < \frac{N}{2}$, then the solution u of (1) is in $L^{m^{**}}(\Omega)$, with m^{**} given by (4);
- if $|E| \in L^2(\Omega) \cap L^r(\Omega)$, $r > N$, and $f \in L^1(\Omega) \cap L^m(\Omega)$, $m > \frac{N}{2}$, then the solution u of (1) is in $L^\infty(\Omega)$.

The techniques used to achieve these results issue from an idea of [1], inspired by the papers of Guido Stampacchia [23, 24], and by [7, 8, 10], where certain nonlinear problems are treated. In [1] the author approximates the noncoercive problem by coercive nonlinear problems and then passes to the limit. Here, due to the assumption (5) on the coefficient appearing in the noncoercive term, one can pass to the limit thanks to a compactness result in $M^N_0(\Omega)$ proved in [26] (see also Lemma 2.2).

For similar problems on bounded domains we refer the reader also to [2, 3, 5, 6, 11, 21, 27]. Linear coercive problems on unbounded domains are studied in [15, 16, 17, 18, 19, 20].

2 The spaces $M^p(\Omega)$ and $M^p_0(\Omega)$

From now on, let Ω be an unbounded subset of $\mathbb{R}^N$, $N > 2$. We start recalling the definitions and some properties of a class of spaces that were introduced for the first time in [25].

Let us give some notation. The σ-algebra of all Lebesgue measurable subsets of Ω is denoted by $\Sigma(\Omega)$. Given $O \in \Sigma(\Omega)$, $|O|$ is its Lebesgue measure, χ_O is its characteristic function, and $O(x, r)$ is the intersection $O \cap B(x, r)$ ($x \in \mathbb{R}^N$, $r \in \mathbb{R}_+$), where $B(x, r)$ is the open ball with center in x and radius r. The class of restrictions to $\bar{\Omega}$ of functions $\zeta \in C^\infty_o(\mathbb{R}^n)$ is denoted by $\mathcal{D}(\bar{\Omega})$. For $p \in [1, +\infty[$, $L^p_{loc}(\bar{\Omega})$ is the class of all functions $g : \Omega \to \mathbb{R}$ such that $\zeta\, g \in L^p(\Omega)$ for any $\zeta \in \mathcal{D}(\bar{\Omega})$.

For $p \in [1, +\infty[$, the space $M^p(\Omega)$ is the set of all the functions g in $L^p_{loc}(\bar{\Omega})$ such that

$$||g||_{M^p(\Omega)} = \sup_{x\in\Omega} ||g||_{L^p(\Omega(x,1))} < +\infty, \tag{7}$$

endowed with the norm defined in (7). Moreover $M^p_0(\Omega)$ denotes the subspace of $M^p(\Omega)$ made up of functions $g \in M^p(\Omega)$ such that

$$\lim_{x\to+\infty} ||g||_{L^p(\Omega(x,1))} = 0.$$

We point out that

$$L^p(\Omega) \subseteq M^p_0(\Omega), \tag{8}$$

the inclusion being strict. Indeed, as shown in [26], the function

$$\frac{1}{1+|x|^\alpha} \in M^p_0(\Omega), \forall\, p>1 \text{ and } \forall\, \alpha \in \mathbb{R}_+,$$

while for $p>1$ and $0<\alpha<N/p$

$$\frac{1}{1+|x|^\alpha} \notin L^p(\Omega).$$

Furthermore (see [26] for details) one has

$$M^q(\Omega) \subseteq M^p(\Omega), \quad M^q_0(\Omega) \subseteq M^p_0(\Omega) \quad \text{ if } 1 \le p \le q < +\infty. \tag{9}$$

As already observed in [26], if $g \in M^p(\Omega)$ the following three properties are equivalent:
i) $g \in M^p_0(\Omega)$,
ii) for any $\varepsilon \in \mathbb{R}_+$ there exist $\nu_\varepsilon, \sigma_\varepsilon \in \mathbb{R}_+$ such that

$$O \in \Sigma(\Omega),\ |O(0,\sigma_\varepsilon)| \le \nu_\varepsilon \Rightarrow ||g\chi_O||_{M^p(\Omega)} \le \varepsilon, \tag{10}$$

iii) for any $\varepsilon \in \mathbb{R}_+$ there exist $h_\varepsilon, k_\varepsilon \in \mathbb{R}_+$ such that

$$\|(1-\zeta_{h_\varepsilon})\, g\|_{M^p(\Omega)} \le \varepsilon, \quad O \in \Sigma(\Omega), \quad \sup_{x\in O} |O(x,1)| \le k_\varepsilon \Rightarrow \|g\,\chi_O\|_{M^p(\Omega)} \le \varepsilon, \tag{11}$$

where, for $h \in \mathbb{R}_+$, ζ_h is a function of class $C^\infty_o(\mathbb{R}^n)$ such that

$$0 \le \zeta_h \le 1\,, \qquad \zeta_{h|_{\overline{B(0,h)}}} = 1\,, \qquad \operatorname{supp} \zeta_h \subset B(0,2h). \tag{12}$$

If g belongs to $M^p_0(\Omega)$, a modulus of continuity of g in $M^p_0(\Omega)$ is an application $\sigma^p_o[g] : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\|(1-\zeta_h)g\|_{M^p(\Omega)} + \sup_{\substack{O\in\Sigma(\Omega) \\ \sup\limits_{x\in\Omega} |O(x,1)| \le \frac{1}{h}}} \|g\,\chi_O\|_{M^p(\Omega)} \le \sigma^p_o[g](h), \text{ with} \lim_{h\to+\infty} \sigma^p_o[g](h) = 0\,. \tag{13}$$

Let us remind some results proved in Lemma 3.1 of [26], see also [13], adapted here to our needs, that allow us to approximate functions in $M^p_0(\Omega)$ by means of sequences of functions in $L^1(\Omega) \cap L^p(\Omega)$.

Lemma 2.1. *If $g \in M_0^p(\Omega)$, with $p > 1$, then there exists a sequence g_h, with $g_h \in L^1(\Omega) \cap L^p(\Omega)$, $h \in \mathbb{N}$, such that*

$$g_h \to g \quad \text{in } M^p(\Omega), \tag{14}$$

$$|g_h(x)| \le |g(x)|, \quad \text{a.e. in } \Omega, \forall\, h \in \mathbb{N}, \tag{15}$$

$$\sigma_o^p[g_h] = \sigma_o^p[g], \forall\, h \in \mathbb{N}. \tag{16}$$

For the reader's convenience, in next lemma we recall some results of [25] concerning the multiplication operator

$$u \in W_0^{1,2}(\Omega) \longrightarrow g\, u \in L^2(\Omega), \tag{17}$$

where the function g belongs to $M^N(\Omega)$.

Lemma 2.2. *If $g \in M^N(\Omega)$, then the operator in* (17) *is bounded and there exists a positive constant c such that*

$$\|g\, u\|_{L^2(\Omega)} \le c\, \|g\|_{M^N(\Omega)}\, \|u\|_{W^{1,2}(\Omega)} \quad \forall\, u \in W_0^{1,2}(\Omega), \tag{18}$$

with $c = c\,(N)$.
Moreover, if $g \in M_0^N(\Omega)$, then the operator in (17) *is also compact.*

3 Preliminary results

Let $k \in \mathbb{R}_+$. Recall Stampacchia's definition of truncate:

$$T_k(t) = \begin{cases} t, & \text{if } |t| \le k, \\ k\frac{t}{|t|}, & \text{if } |t| > k, \end{cases} \tag{19}$$

and let

$$G_k(t) = t - T_k(t). \tag{20}$$

Given $u \in W_0^{1,2}(\Omega)$, we put

$$A_k = \{x \in \Omega \,:\, |u(x)| > k\}. \tag{21}$$

Let us recall a known result proved in [24] and generalized to the case of unbounded domains in [12].

Lemma 3.1. *Let G be a uniformly Lipschitz function such that $G(0) = 0$ and $u \in W_0^{1,2}(\Omega)$. Then $G \circ u \in W_0^{1,2}(\Omega)$.*

The next lemma collects some useful properties of the composition of T_k with G_k and $u \in W_0^{1,2}(\Omega)$, needed in the sequel.

Lemma 3.2. *For every $u \in W_0^{1,2}(\Omega)$ and $k \in \mathbb{R}_+$ one has*

$$T_k(u) = T_k \circ u \in W_0^{1,2}(\Omega), \tag{22}$$

$$\nabla u \cdot \nabla T_k(u) = |\nabla T_k(u)|^2, \ \textit{a.e. in } \Omega, \tag{23}$$

$$u\, T_k(u) \geq |T_k(u)|^2, \ \textit{a.e. in } \Omega, \tag{24}$$

$$u\, \nabla T_k(u) = T_k(u) \nabla T_k(u), \ \textit{a.e. in } \Omega, \tag{25}$$

$$G_k(u) = G_k \circ u \in W_0^{1,2}(\Omega), \tag{26}$$

$$|G_k(u)| \leq |u|, \ \textit{a.e. in } \Omega, \tag{27}$$

$$|u| \leq |G_k(u)| + k, \ \textit{a.e. in } \Omega, \tag{28}$$

$$\nabla u \cdot \nabla G_k(u) = |\nabla G_k(u)|^2, \ \textit{a.e. in } \Omega, \tag{29}$$

$$u\, G_k(u) \geq |G_k(u)|^2, \ \textit{a.e. in } \Omega, \tag{30}$$

$$\operatorname{supp} G_k(u) \subseteq \overline{A}_k, \tag{31}$$

$$(G_k(u))_{x_i} = \begin{cases} u_{x_i} & \textit{a.e. in } A_k, \\ 0 & \textit{a.e. in } \Omega \setminus A_k, \ i = 1 \ldots n. \end{cases} \tag{32}$$

Proof. The staments in (22) and (26) can be obtained by Lemma 3.1, the other properties are straightforward consequence of the definitions of T_k, G_k and A_k. □

Let us now recall Lemma 4.1 of [24] by Stampacchia.

Lemma 3.3. *Let $k_0 > 0$ and $\varphi : [k_0, +\infty[\to \mathbb{R}$ be a non negative and non increasing function such that*

$$\varphi(h) \leq \frac{C}{(h-k)^\gamma} [\varphi(k)]^\delta \quad \forall\, h > k \geq k_0, \tag{33}$$

where C, γ and δ are positive constants, with $\delta > 1$. Then there exists

$$d = 2^{\frac{\delta}{\delta-1}} C^{1/\gamma} \left[\varphi(k_0)\right]^{\frac{\delta-1}{\gamma}} \tag{34}$$

such that

$$\varphi(k_0 + d) = 0. \tag{35}$$

Lemma 3.3 allows us to prove the next result obtained following some techniques used in [14, 23].

LEMMA 3.4. *Assume (2), (3), $|F| \in L^2(\Omega)$ and $f \in L^{\frac{2N}{N+2}}(\Omega)$. Then there exists a unique solution u of the problem*

$$\begin{cases} -\operatorname{div}(M(x)\nabla u) + \mu u = -\operatorname{div}(F(x)) + f(x) \quad \text{in } \Omega, \\ \\ u \in W_0^{1,2}(\Omega). \end{cases} \tag{36}$$

If in addition $|F| \in L^p(\Omega)$ and $f \in L^{\frac{p}{2}}(\Omega)$, $p > N$, then the solution u is of class $L^\infty(\Omega)$.

PROOF. The Lax-Milgram Lemma gives the existence and uniqueness of the solution.

Let us prove the boundedness. Take $G_k(u)$ as test function in the variational formulation of (36) (this is allowed by (26)). Then by (2), (21), (29), (30), (31), Hölder and Sobolev inequalities one gets

$$\alpha \int_\Omega |\nabla G_k(u)|^2 + \mu \int_\Omega |G_k(u)|^2 \leq \int_{A_k} |F||\nabla G_k(u)| + \int_{A_k} |f||G_k(u)|$$

$$\leq \Big(\|F\|_{L^p(\Omega)} |A_k|^{\frac{1}{2}-\frac{1}{p}} + \frac{1}{S} \|f\|_{L^{\frac{p}{2}}(\Omega)} |A_k|^{1-\frac{1}{2^*}-\frac{2}{p}} \Big) \|\nabla G_k(u)\|_{L^2(\Omega)}.$$

Whence

$$\|G_k(u)\|_{L^{2^*}(\Omega)} \leq C(|A_k|^{\frac{1}{2}-\frac{1}{p}} + |A_k|^{1-\frac{1}{2^*}-\frac{2}{p}}),$$

with $C = C(\alpha, S, \|F\|_{L^p(\Omega)}, \|f\|_{L^{\frac{p}{2}}(\Omega)})$ and where S is the Sobolev constant as in Theorem 3.17 of [4].

Now, observe that since $|A_k| \to 0$, as $k \to +\infty$, we can assume that there exists $k_0 \in \mathbb{R}_+$ such that $|A_k| \leq 1$, for $k \geq k_0$. Moreover, since $p > N$, then $\frac{1}{2} - \frac{1}{p} < 1 - \frac{1}{2^*} - \frac{2}{p}$, therefore

$$\|G_k(u)\|_{L^{2^*}(\Omega)} \leq C' |A_k|^{\frac{1}{2}-\frac{1}{p}}, \forall k \geq k_0, \tag{37}$$

with $C' = C'(\alpha, S, \|F\|_{L^p(\Omega)}, \|f\|_{L^{\frac{p}{2}}(\Omega)})$.

On the other hand, by (21) and (28)

$$h|A_h|^{\frac{1}{2^*}} = \Big(\int_{A_h} |h|^{2^*} \Big)^{\frac{1}{2^*}} \leq \|u\|_{L^{2^*}(A_h)} \leq \|G_k(u)\|_{L^{2^*}(A_h)} + k|A_h|^{\frac{1}{2^*}}.$$

Thus

$$(h-k)|A_h|^{\frac{1}{2^*}} \le \|G_k(u)\|_{L^{2^*}(A_h)}, \quad \forall h > k. \tag{38}$$

Putting together (37) and (38), we obtain

$$|A_h| \le C'' \frac{|A_k|^{\frac{2^*}{2}-\frac{2^*}{p}}}{(h-k)^{2^*}}, \quad \forall h > k \ge k_0,$$

with $C'' = C''(\alpha, S, \|F\|_{L^p(\Omega)}, \|f\|_{L^{\frac{p}{2}}(\Omega)})$.

Finally, again as a consequence of the fact that $N < p$, one gets that $\frac{2^*}{2} - \frac{2^*}{p} > 1$, hence Lemma 3.3 applies and therefore there exists $d \in \mathbb{R}_+$ such that $|A_{k_0+d}| = 0$, thus $u \in L^\infty(\Omega)$. □

4 Existence result

Here, we want to prove the existence of a weak solution of (1). Following an idea of [1], that issues from the papers of Guido Stampacchia and from [7, 8, 10], we use a nonlinear approach to our linear problem.

Indeed, we consider the following nonlinear approximate problems

$$\begin{cases} -\mathrm{div}(M(x)\nabla u_n) + \mu u_n = -\mathrm{div}\Big(\dfrac{u_n}{1+\frac{1}{n}|u_n|}\,\dfrac{E(x)}{1+\frac{1}{n}|E(x)|}\Big) + \dfrac{f}{1+\frac{1}{n}|f|}, \\ u_n \in W_0^{1,2}(\Omega). \end{cases} \tag{39}$$

We start proving that, for every fixed n, a bounded weak solution u_n of (39) exists. This is done in Theorem 4.1 for $n = 1$ and it can be analogously proved for $n \ge 2$. Successively, in Theorem 4.6, we show that the sequence u_n of the solutions of problems (39) is bounded in $W_0^{1,2}(\Omega)$. To this aim, some preliminary results are needed. Namely, in Lemma 4.2, we obtain that for any $k \in \mathbb{R}_+$, the sequence $T_k(u_n)$ is bounded in $W_0^{1,2}(\Omega)$ and successively, in Lemma 4.5, we get that the sequence $G_k(u_n)$ is bounded in $W_0^{1,2}(\Omega)$ too, for sufficiently large k. Thus, fixed k sufficiently large, in view (20), we get the boundedness of u_n in $W_0^{1,2}(\Omega)$. Finally, in Theorem 4.7, by approximation, we get the existence result of a weak solution of problem (1).

Theorem 4.1. *Assume (2), (3), $|E| \in L^2(\Omega) \cap M_0^N(\Omega)$ and $f \in L^1(\Omega)$. Then there exists a weak solution u of class $L^\infty(\Omega)$ of the following problem*

$$\begin{cases} -\mathrm{div}(M(x)\nabla u) + \mu u = -\mathrm{div}\Big(\dfrac{u}{1+|u|}\,\dfrac{E(x)}{1+|E(x)|}\Big) + \dfrac{f}{1+|f|}, \\ u \in W_0^{1,2}(\Omega). \end{cases} \tag{40}$$

Proof. Let $w \in W_0^{1,2}(\Omega)$. By Lemma 3.4 there exists a unique and bounded solution u of the following problem

$$
(41) \qquad \begin{cases} -\mathrm{div}(M(x)\nabla u) + \mu u = -\mathrm{div}\Big(\dfrac{w}{1+|w|}\,\dfrac{E(x)}{1+|E(x)|}\Big) + \dfrac{f}{1+|f|}, \\ fu \in W_0^{1,2}(\Omega). \end{cases}
$$

Consider then the operator

$$
(42) \qquad P : w \in W_0^{1,2}(\Omega) \to u = Pw \in W_0^{1,2}(\Omega).
$$

In order to prove our claim it is enough to show that P has a fixed point. To do that we make use of the Schauder fixed point Theorem (in its formulation given, for instance, in Theorem 1.11 of [4]).

Let us therefore show that the following two hypotheses are satisfied:

1. P admits a bounded and closed invariant convex set.
2. P is completely continuous.

1. Take u as test function in the variational formulation of (41). We have

$$
\int_\Omega M(x)\nabla u\cdot\nabla u + \int_\Omega \mu u^2 = \int_\Omega \frac{w}{1+|w|}\frac{E(x)}{1+|E(x)|}\cdot\nabla u + \int_\Omega \frac{f\,u}{1+|f|}\,.
$$

Hence, by (2), (3) and by the Hölder and Sobolev inequalities we obtain that there exist two positive constants $C_0 = C_0(\alpha,\mu)$ and $C = C(\alpha,\mu,\|E\|_{L^2(\Omega)},\|\frac{f}{1+|f|}\|_{L^{\frac{2N}{N+2}}(\Omega)},S)$ such that

$$
\|u\|^2_{W^{1,2}(\Omega)} \le C_0(\|E\|_{L^2(\Omega)}\|\nabla u\|_{L^2(\Omega)} + \|\frac{f}{1+|f|}\|_{L^{\frac{2N}{N+2}}(\Omega)}\|u\|_{L^{2^*}(\Omega)}) \le C\|u\|_{W^{1,2}(\Omega)}.
$$

Therefore, if we consider the closed ball $\|w\|_{W^{1,2}(\Omega)} \le C$, we obtain that $\|Pw\|_{W^{1,2}(\Omega)} = \|u\|_{W^{1,2}(\Omega)} \le C$. This concludes the proof of the first point.

2. Let $w_n \rightharpoonup \bar{w}$ weakly in $W_0^{1,2}(\Omega)$, we must show that $Pw_n \to P\bar{w}$ in $W_0^{1,2}(\Omega)$.
Let $u_n = Pw_n$ and $\bar{u} = P\bar{w}$. Take $u_n - \bar{u}$ as test function in the variational formulations of (41) written in correspondence of $w = w_n$ and $w = \bar{w}$, respectively. We get

$$
\int_\Omega M(x)\nabla u_n\cdot\nabla(u_n-\bar{u}) + \int_\Omega \mu u_n(u_n-\bar{u}) = \int_\Omega \frac{w_n}{1+|w_n|}\frac{E(x)}{1+|E(x)|}\cdot\nabla(u_n-\bar{u}) + \int_\Omega \frac{f\,(u_n-\bar{u})}{1+|f|}
$$

and

$$\int_\Omega M(x)\nabla\bar{u}\cdot\nabla(u_n-\bar{u})+\int_\Omega \mu\bar{u}(u_n-\bar{u}) = \int_\Omega \frac{\bar{w}}{1+|\bar{w}|}\frac{E(x)}{1+|E(x)|}\cdot\nabla(u_n-\bar{u})+\int_\Omega \frac{f\,(u_n-\bar{u})}{1+|f|}\,.$$

Subtracting the second equality from the first one we obtain

$$\int_\Omega M(x)[\nabla(u_n-\bar{u})]^2+\int_\Omega \mu(u_n-\bar{u})^2 = \int_\Omega \Big(\frac{w_n}{1+|w_n|}-\frac{\bar{w}}{1+|\bar{w}|}\Big)\frac{E(x)}{1+|E(x)|}\cdot\nabla(u_n-\bar{u}).$$

Hence, by (2), (3) and by the Hölder inequality there exists a positive constant $C=C(\alpha,\mu)$ such that

$$\|u_n-\bar{u}\|^2_{W^{1,2}(\Omega)} \le C\Big\|\Big(\frac{w_n}{1+|w_n|}-\frac{\bar{w}}{1+|\bar{w}|}\Big)|E|\Big\|_{L^2(\Omega)}\|\nabla(u_n-\bar{u})\|_{L^2(\Omega)},$$

whence

$$\|u_n-\bar{u}\|_{W^{1,2}(\Omega)} \le \Big\|\Big(\frac{w_n}{1+|w_n|}-\frac{\bar{w}}{1+|\bar{w}|}\Big)|E|\Big\|_{L^2(\Omega)}.$$

Now, by the compactness of the operator $u\in W_0^{1,2}(\Omega)\to|E|u\in L^2(\Omega)$, stated in Lemma 2.2, since $w_n\rightharpoonup\bar{w}$ weakly in $W_0^{1,2}(\Omega)$, we obtain $|E|w_n\to|E|\bar{w}$ in $L^2(\Omega)$, and therefore, up to a subsequence, w_n converges to $\bar{w}$ a.e. in Ω. Thus the Lebesgue dominated convergence Theorem applies and we get that

$$\Big\|\Big(\frac{w_n}{1+|w_n|}-\frac{\bar{w}}{1+|\bar{w}|}\Big)|E|\Big\|_{L^2(\Omega)}\to 0.$$

This conclude our proof. □

The estimates contained in the following Lemmas 4.2, 4.3 and 4.5 allow us to prove the a priori bounds on $\{u_n\}$ of Theorem 4.6.

Lemma 4.2. *Assume (2), (3), $|E|\in L^2(\Omega)\cap M_0^N(\Omega)$ and $f\in L^1(\Omega)\cap L^{\frac{2N}{N+2}}(\Omega)$. Then, for any $k\in\mathbb{R}_+$, the sequence $\{T_k(u_n)\}$ is bounded in $W_0^{1,2}(\Omega)$. More precisely we have:*

$$\frac{\alpha}{2}\int_\Omega|\nabla T_k(u_n)|^2+\mu\int_\Omega|T_k(u_n)|^2\le\frac{k^2}{2\alpha}\int_\Omega|E|^2+k\int_\Omega|f|. \tag{43}$$

PROOF. Let us take $T_k(u_n)$ as test function in the variational formulation of (39), this can be done in view of (22). We have

$$\int_\Omega M(x)\nabla u_n \cdot \nabla T_k(u_n) + \mu \int_\Omega u_n T_k(u_n)$$

$$= \int_\Omega \frac{u_n}{1+\frac{1}{n}|u_n|}\frac{E(x)}{1+\frac{1}{n}|E(x)|} \cdot \nabla T_k(u_n) + \int_\Omega \frac{f}{1+\frac{1}{n}|f|} T_k(u_n)\,.$$

In view of (2), (23), (24), (25) and by Young inequality we get

$$\alpha \int_\Omega |\nabla T_k(u_n)|^2 + \mu \int_\Omega |T_k(u_n)|^2$$

$$\le \int_\Omega |T_k(u_n)|\;|E|\;|\nabla T_k(u_n)| + \int_\Omega |f|\;|T_k(u_n)|$$
$$\le \frac{\alpha}{2}\int_\Omega |\nabla T_k(u_n)|^2 + \frac{1}{2\alpha}\int_\Omega |T_k(u_n)|^2\;|E|^2 + \int_\Omega |f|\;|T_k(u_n)|.$$

Therefore (43) follows.

□

LEMMA 4.3. *Assume (2), (3), $|E| \in L^2(\Omega) \cap M_0^N(\Omega)$ and $f \in L^1(\Omega) \cap L^{\frac{2N}{N+2}}(\Omega)$. Then the solutions u_n of (39) satisfy*

$$\int_\Omega |u_n| \le \frac{1}{\mu}\int_\Omega |f|. \tag{44}$$

PROOF. Let $\varepsilon > 0$ and take $\dfrac{u_n}{\varepsilon + |u_n|}$ as test function in (39). We have

$$\varepsilon \int_\Omega \frac{M(x)\nabla u_n \cdot \nabla u_n}{(\varepsilon+|u_n|)^2} + \mu \int_\Omega \frac{|u_n|^2}{\varepsilon+|u_n|} = \varepsilon \int_\Omega \frac{u_n}{1+\frac{1}{n}|u_n|}\frac{E(x)}{1+\frac{1}{n}|E(x)|}\cdot\frac{\nabla u_n}{(\varepsilon+|u_n|)^2} + \int_\Omega \frac{f_n\, u_n}{\varepsilon+|u_n|}\,.$$

Since $\frac{|u_n|}{\varepsilon+|u_n|} \le 1$ we have, using (2) and the fact that $|f_n| \le |f|$,

$$\alpha\varepsilon \int_\Omega \frac{|\nabla u_n|^2}{(\varepsilon+|u_n|)^2} + \mu \int_\Omega \frac{|u_n|^2}{\varepsilon+|u_n|} \le \varepsilon \int_\Omega \frac{E\cdot\nabla u_n}{\varepsilon+|u_n|} + \int_\Omega |f|\,.$$

Now, observe that Young inequality gives

$$\frac{\alpha\varepsilon}{2}\int_\Omega \frac{|\nabla u_n|^2}{(\varepsilon+|u_n|)^2}+\mu\int_\Omega \frac{|u_n|^2}{\varepsilon+|u_n|}\le \frac{\varepsilon}{2\alpha}\int_\Omega |E|^2+\int_\Omega |f|,$$

which concludes the proof letting $\varepsilon\to 0$. □

Remark 4.4. *Remark that thanks to the estimate (44), one has*

$$\text{meas}\Big\{x\in\Omega : |u_n(x)|>k\Big\}\le \frac{1}{k\mu}\int_\Omega |f|. \tag{45}$$

Thus, for any $\epsilon>0$, it is possible to choose k_ϵ such that

$$\text{meas}\Big\{x\in\Omega : |u_n(x)|>k\Big\}\le \epsilon,\ \forall\, k>k_\epsilon, \forall\, n\in\mathbb{N}. \tag{46}$$

□

Lemma 4.5. *Assume (2), (3), $|E|\in L^2(\Omega)\cap M_0^N(\Omega)$ and $f\in L^1(\Omega)\cap L^{\frac{2N}{N+2}}(\Omega)$. Then there exists $k^*\in\mathbb{R}_+$, with $k^*=k^*(N,\sigma_o^N[E])$, such that the sequence $\{G_k(u_n)\}$ is bounded in $W_0^{1,2}(\Omega)$, for every $k>k^*$. More precisely we have:*

$$\frac{\alpha}{2}\int_\Omega |\nabla G_k(u_n)|^2+\frac{\mu}{2}\int_\Omega |G_k(u_n)|^2\le \frac{2k^2}{\alpha}\int_\Omega |E|^2+\frac{2}{\alpha S^2}\Big[\int_\Omega |f|^{\frac{2N}{N+2}}\Big]^{\frac{N+2}{N}}, \tag{47}$$

where S is the Sobolev constant as in Theorem 3.17 of [4].

Proof. Let $k\in\mathbb{R}_+$ and $n\in\mathbb{N}$, define

$$A_n(k)=\{x\in\Omega : k<|u_n(x)|\}.$$

The use of $G_k(u_n)$ as test function in the variational formulation of (39) (that can be done in view of (26)), (2), (28), (29) and (30) give that

$$\begin{aligned}&\alpha\int_\Omega |\nabla G_k(u_n)|^2+\mu\int_\Omega |G_k(u_n)|^2\\ &\le \int_\Omega |G_k(u_n)||E||\nabla G_k(u_n)|+k\int_\Omega |E||\nabla G_k(u_n)|+\int_\Omega |G_k(u_n)||f|.\end{aligned} \tag{48}$$

By (31), Hölder inequality and (18) of Lemma 2.2, we get that

$$
\begin{aligned}
\int_\Omega |G_k(u_n)||E||\nabla G_k(u_n)| &\le \Big(\int_{A_n(k)} |E|^2|G_k(u_n)|^2\Big)^{\frac12}\Big(\int_\Omega |\nabla G_k(u_n)|^2\Big)^{\frac12} \\
&\le c\|E\|_{M^N(A_n(k))}\|G_k(u_n)\|_{W^{1,2}(\Omega)}\Big(\int_\Omega |\nabla G_k(u_n)|^2\Big)^{\frac12}.
\end{aligned} \tag{49}
$$

Therefore, by (48), (49) and Young inequality one has that, for $\epsilon > 0$,

$$
\alpha \int_\Omega |\nabla G_k(u_n)|^2 + \mu \int_\Omega |G_k(u_n)|^2
$$

$$
\le c\|E\|_{M^N(A_n(k))}\Big(\int_\Omega |\nabla G_k(u_n)|^2 + \int_\Omega |G_k(u_n)|^2\Big) + \epsilon \int_\Omega |\nabla G_k(u_n)|^2 + \tfrac{k^2}{4\epsilon}\int_{A_n(k)} |E|^2
$$

$$
+\epsilon \int_\Omega |\nabla G_k(u_n)|^2 + \frac{1}{4\epsilon S^2}\Big[\int_{A_n(k)} |f|^{\frac{2N}{N+2}}\Big]^{\frac{N+2}{N}}.
$$

Thus it results

$$
\Big[\alpha - c\|E\|_{M^N(A_n(k))} - 2\epsilon\Big]\int_\Omega |\nabla G_k(u_n)|^2 + \Big[\mu - c\|E\|_{M^N(A_n(k))}\Big]\int_\Omega |G_k(u_n)|^2
$$

$$
\le \tfrac{k^2}{4\epsilon}\int_{A_n(k)} |E|^2 + \tfrac{1}{4\epsilon S^2}\Big[\int_{A_n(k)} |f|^{\frac{2N}{N+2}}\Big]^{\frac{N+2}{N}}.
$$

Fix ϵ so that $2\epsilon = \frac{\alpha}{4}$. Then (10) and (46) imply that there exists $k^* \in \mathbb{R}_+$, such that

$$
c\|E\|_{M^N(A_n(k))} \le \min\Big\{\frac{\alpha}{4}, \frac{\mu}{2}\Big\}, \quad \forall k > k^*. \tag{50}
$$

Let us explicitly observe that, in view of (10), (11) and by the definition (13) of $\sigma_o^N[E]$, one has $k^* = k^*(N, \sigma_o^N[E])$. This concludes our proof. □

THEOREM 4.6. *Assume (2), (3), $|E| \in L^2(\Omega) \cap M_0^N(\Omega)$ and $f \in L^1(\Omega) \cap L^{\frac{2N}{N+2}}(\Omega)$. Then the sequence $\{u_n\}$ of the solutions of problems (39) is bounded in $W_0^{1,2}(\Omega)$. More precisely, there exists a positive constant $C = C(N, \alpha, \mu, S, \sigma_o^N[E])$ such that*

$$
\|u_n\|^2_{W^{1,2}(\Omega)} \le C(\|E\|^2_{L^2(\Omega)} + \|f\|_{L^1(\Omega)} + \|f\|^2_{L^{\frac{2N}{N+2}}(\Omega)}). \tag{51}
$$

Proof. Let k^* be given by Lemma 4.5. Definition (20) together with the estimates (43) and (47) imply that for any $k > k^*$ there exists a positive constant $C' = C'(\alpha, \mu, S)$ such that

$$\int_\Omega |\nabla u_n|^2 + \int_\Omega |u_n|^2 \le C'\Big(k^2 \int_\Omega |E|^2 + k \int_\Omega |f| + \Big[\int_\Omega |f|^{\frac{2N}{N+2}}\Big]^{\frac{N+2}{N}}\Big). \tag{52}$$

This gives (51). □

Finally, let us prove the existence result.

Theorem 4.7. *Assume (2), (3), $|E| \in L^2(\Omega) \cap M_0^N(\Omega)$ and $f \in L^1(\Omega) \cap L^{\frac{2N}{N+2}}(\Omega)$. Then there exists $u \in W_0^{1,2}(\Omega)$ weak solution of (1), that is*

$$\int_\Omega M(x)\nabla u \cdot \nabla v + \mu \int_\Omega uv = \int_\Omega u\, E(x) \cdot \nabla v + \int_\Omega f\, v, \qquad \forall\ v \in W_0^{1,2}(\Omega). \tag{53}$$

Moreover, there exists a positive constant $C = C(N, \alpha, \mu, S, \sigma_o^N[E])$ such that

$$\|u\|^2_{W^{1,2}(\Omega)} \le C(\|E\|^2_{L^2(\Omega)} + \|f\|_{L^1(\Omega)} + \|f\|^2_{L^{\frac{2N}{N+2}}(\Omega)}). \tag{54}$$

Proof. The sequence $\{u_n\}$ is bounded in $W_0^{1,2}(\Omega)$ by Theorem 4.6. Then, up to a subsequence, u_n converges weakly in $W_0^{1,2}(\Omega)$ to a function u.
Since u_n is a solution of (39), one has that

$$\begin{aligned} &\int_\Omega M(x)\nabla u_n \cdot \nabla v + \mu \int_\Omega u_n v \\ &= \int_\Omega \frac{u_n}{1+\frac{1}{n}|u_n|} \frac{E(x)}{1+\frac{1}{n}|E(x)|} \cdot \nabla v + \int_\Omega \frac{f}{1+\frac{1}{n}|f|} v\,, \end{aligned} \tag{55}$$

for every $v \in W_0^{1,2}(\Omega)$. Let us to pass to the limit, as $n \to +\infty$, in (55).
Clearly

$$\int_\Omega M(x)\nabla u_n \cdot \nabla v \to \int_\Omega M(x)\nabla u \cdot \nabla v$$

and

$$\mu \int_\Omega u_n v \to \mu \int_\Omega uv.$$

Moreover

$$\frac{f}{1+\frac{1}{n}|f|}v \to fv \ \text{ a.e. in } \Omega$$

and

$$\Big|\frac{f}{1+\frac{1}{n}|f|}v\Big| \leq |fv| \in L^1(\Omega).$$

Thus, by the Lebesgue dominated convergence Theorem one has

$$\int\limits_{\Omega} \frac{f}{1+\frac{1}{n}|f|}v \to \int\limits_{\Omega} fv.$$

It remains to pass to the limit in

$$\int\limits_{\Omega} \frac{u_n}{1+\frac{1}{n}|u_n|}\frac{E(x)}{1+\frac{1}{n}|E(x)|}\cdot \nabla v.$$

Since u_n converges weakly to u in $W_0^{1,2}(\Omega)$, by Lemma 2.2 we obtain that $|E|u_n$ converges strongly to $|E|u$ in $L^2(\Omega)$. Hence, by the Vitali Theorem (see, for instance, [22]) one has that for any $\varepsilon > 0$ there exists $\Omega_\varepsilon \subset \Omega$ with $|\Omega_\varepsilon| < +\infty$ such that

$$\int_{\Omega\setminus\Omega_\varepsilon} |u_n|^2|E|^2 < \varepsilon, \text{ uniformly with respect to } n,$$

and there exists $\delta > 0$ such that for every $A \subset \Omega$ with $|A| < \delta$, one has

$$\int_A |u_n|^2|E|^2 < \varepsilon, \text{ uniformly with respect to } n.$$

Now,

$$\int_{\Omega\setminus\Omega_\varepsilon} \frac{|u_n|^2}{(1+\frac{1}{n}|u_n|)^2}\frac{|E(x)|^2}{(1+\frac{1}{n}|E(x)|)^2} \leq \int_{\Omega\setminus\Omega_\varepsilon} |u_n|^2|E|^2 < \varepsilon,$$

and

$$\int_A \frac{|u_n|^2}{(1+\frac{1}{n}|u_n|)^2}\frac{|E(x)|^2}{(1+\frac{1}{n}|E(x)|)^2} \leq \int_A |u_n|^2|E|^2 < \varepsilon,$$

uniformly with respect to n and moreover, since u_n converges a.e. to u, one gets

$$\frac{u_n}{1+\frac{1}{n}|u_n|}\frac{|E(x)|}{1+\frac{1}{n}|E(x)|} \to u|E| \text{ a.e. in } \Omega.$$

Thus, using again the Vitali Theorem, in the reverse sense, we obtain that

$$\frac{u_n}{1+\frac{1}{n}|u_n|}\frac{|E(x)|}{1+\frac{1}{n}|E(x)|} \to u_n\,|E| \text{ in } L^2(\Omega).$$

Passing to the limit, as $n \to +\infty$, in (55) we obtain (53).
Estimate (54) follows then by (51). □

5 Uniqueness result

In this section we prove the uniqueness of the solution of problem (1). To achieve this result we follow some ideas of [1, 9].

Theorem 5.1. *Assume (2), (3), $|E| \in L^2(\Omega) \cap M_0^N(\Omega)$ and $f \in L^1(\Omega) \cap L^{\frac{2N}{N+2}}(\Omega)$. Then the weak solution u of (1) is unique.*

Proof. Let u, w be weak solutions of (1) and let $\delta \in \mathbb{R}_+$ and $\epsilon \in]0, \delta[$. We use $T_\epsilon(u-w)$ as test function in the variational formulation of problem (1), written in correspondence of the solutions u and w respectively. This can be done in view of (22). By subtracting we obtain

$$\int_\Omega M(x)\nabla(u-w)\nabla T_\epsilon(u-w) + \mu\int_\Omega (u-w)T_\epsilon(u-w)$$

$$= \int_\Omega (u-w)E(x)\nabla T_\epsilon(u-w).$$

By (2), (23), (24) and (25) we have

$$\alpha\int_\Omega |\nabla T_\epsilon(u-w)|^2 + \mu\int_\Omega |T_\epsilon(u-w)|^2 \le \epsilon \int_{0<|u(x)-w(x)|<\epsilon} |E(x)||\nabla T_\epsilon(u-w)|$$

and the Hölder inequality gives then

$$\min\{\alpha,\mu\}\Big(\int_\Omega |\nabla T_\epsilon(u-w)|^2 + \int_\Omega |T_\epsilon(u-w)|^2\Big)$$

$$\le \epsilon\Big(\int_{0<|u(x)-w(x)|<\epsilon} |E(x)|^2\Big)^{\frac{1}{2}}\Big(\int_\Omega |\nabla T_\epsilon(u-w)|^2\Big)^{\frac{1}{2}}.$$

Therefore

$$(\min\{\alpha,\mu\})^2\Big(\int_\Omega |\nabla T_\epsilon(u-w)|^2+\int_\Omega |T_\epsilon(u-w)|^2\Big)\le \epsilon^2\int_{0<|u(x)-w(x)|<\epsilon}|E|^2.$$

Then

$$\int_{\delta<|u(x)-w(x)|}|T_\epsilon(u-w)|^2\le\int_\Omega |T_\epsilon(u-w)|^2\le\frac{\epsilon^2}{(\min\{\alpha,\mu\})^2}\int_{0<|u(x)-w(x)|<\epsilon}|E|^2.$$

Thus

$$\epsilon^2\,\mathrm{meas}(\{\delta<|u(x)-w(x)|\})\le\frac{\epsilon^2}{(\min\{\alpha,\mu\})^2}\int_{0<|u(x)-w(x)|<\epsilon}|E|^2.$$

Since

$$\bigcap_{\varepsilon>0}\{0<|u(x)-w(x)|<\epsilon\}=\{0<|u(x)-w(x)|\le 0\}=\emptyset,$$

the continuity of the measure with respect to intersection then implies that

$$\mathrm{meas}(\{0<|u(x)-w(x)|<\epsilon\})\to 0,\ \text{as } \varepsilon\to 0.$$

Then

$$\int_{0<|u(x)-w(x)|<\epsilon}|E|^2\to 0,$$

and so $\mathrm{meas}\{\delta<|u(x)-w(x)|\}=0$ for any $\delta>0$, that is $u(x)=w(x)$ almost everywhere. □

6 Regularity results

This section is devoted to the proof of two regularity results for the weak solution $u\in W_0^{1,2}(\Omega)$ of problem (1).

More precisely, in Theorem 6.5 we show that if $f\in L^1(\Omega)\cap L^m(\Omega)$, with $\frac{2N}{N+2}\le m<\frac{N}{2}$, then $u\in L^{m^{**}}(\Omega)$, where m^{**} is given by (4).

In Theorem 6.6, we prove that if we require stronger assumptions on E and f, namely if $|E|\in L^2(\Omega)\cap L^r(\Omega)$, $r>N$, and $f\in L^1(\Omega)\cap L^m(\Omega)$, $m>\frac{N}{2}$, then $u\in L^\infty(\Omega)$.

To show Theorem 6.5, as done to obtain the existence of the weak solution of problem (1), some preliminary results for the sequences $T_k(u_n)$ and $G_k(u_n)$ are needed.

Namely, in Lemma 6.1, we obtain that if (2) and (3) hold, $|E| \in L^2(\Omega) \cap M_0^N(\Omega)$ and $f \in L^1(\Omega) \cap L^{\frac{2N}{N+2}}(\Omega)$, then for any $k \in \mathbb{R}_+$, the sequence $T_k(u_n)$ is bounded in $L^{m^{**}}(\Omega)$, for every $\frac{2N}{N+2} \leq m < \frac{N}{2}$. An analogous result for the $G_k(u_n)$, with $|E| \in L^2(\Omega) \cap M_0^N(\Omega)$, cannot be obtained. Hence, in Lemma 6.2, we prove that the sequence $G_k(u_n)$ is bounded in $L^{m^{**}}(\Omega)$, for sufficiently large k, but under the stronger assumption $|E| \in L^2(\Omega) \cap L^N(\Omega)$ and if $f \in L^1(\Omega) \cap L^m(\Omega)$, with $\frac{2N}{N+2} \leq m < \frac{N}{2}$. Thus, if $|E| \in L^2(\Omega) \cap L^N(\Omega)$ and $f \in L^1(\Omega) \cap L^m(\Omega)$, with $\frac{2N}{N+2} \leq m < \frac{N}{2}$, fixed k sufficiently large, in view of (20), we get the boundedness of u_n in $L^{m^{**}}(\Omega)$.

This allows to obtain, in Corollary 6.4, that under the same hypotheses, the weak solution u of problem (1) is in $L^{m^{**}}(\Omega)$.

Finally, in Theorem 6.5, we get the claimed regularity result for u, by approximation, assuming $|E| \in L^2(\Omega) \cap M_0^N(\Omega)$ and making use of Lemma 2.1.

Lemma 6.1. *Assume (2), (3), $|E| \in L^2(\Omega) \cap M_0^N(\Omega)$ and $f \in L^1(\Omega) \cap L^{\frac{2N}{N+2}}(\Omega)$. If $\frac{2N}{N+2} \leq m < \frac{N}{2}$, then, for any $k \in \mathbb{R}_+$, the sequence $\{T_k(u_n)\}$ is bounded in $L^{m^{**}}(\Omega)$. More precisely, there exists a positive constant $C = C(N, m, \alpha, S)$ such that*

$$\Big[\int_\Omega |T_k(u_n)|^{m^{**}}\Big]^{\frac{2}{2^*}} \leq C\Big(k^{\frac{2m^{**}}{2^*}} \int_\Omega |E|^2 + k^{\frac{2m^{**}}{2^*}-1} \int_\Omega |f|\Big). \tag{56}$$

Proof. Observe that the function $|t|^{2(\lambda-1)}t$, with $\lambda > 1$, satisfies the hypotheses of Lemma 3.1, provided that $|t| \leq M$, for some $M > 0$. Thus, since by Theorem 4.1 the function $u_n \in L^\infty(\Omega)$, we can take $\dfrac{|T_k(u_n)|^{2(\lambda-1)}T_k(u_n)}{2\lambda-1}$, with $\lambda = \dfrac{m^{**}}{2^*}$, as test function in the variational formulation of problem (39).
Thus, by (2), (23), (25) and the Young inequality we get

$$\alpha \int_\Omega |T_k(u_n)|^{2(\lambda-1)}|\nabla T_k(u_n)|^2$$

$$\leq \int_\Omega |T_k(u_n)|^{2\lambda-1}|E||\nabla T_k(u_n)| + \frac{1}{2\lambda-1}\int_\Omega |f||T_k(u_n)|^{2\lambda-1}$$

$$\leq \frac{\alpha}{2}\int_\Omega |T_k(u_n)|^{2(\lambda-1)}|\nabla T_k(u_n)|^2 + \frac{1}{2\alpha}\int_\Omega |T_k(u_n)|^{2\lambda}|E|^2$$

$$+\frac{1}{2\lambda-1}\int_\Omega |f||T_k(u_n)|^{2\lambda-1}.$$

Hence,

$$\frac{\alpha}{2}\int_\Omega |T_k(u_n)|^{2(\lambda-1)}|\nabla T_k(u_n)|^2 \le \frac{k^{2\lambda}}{2\alpha}\int_\Omega |E|^2 + \frac{k^{2\lambda-1}}{2\lambda-1}\int_\Omega |f|.$$

Thanks to Sobolev inequality, we obtain

$$\begin{aligned}\frac{\alpha}{2}\Big[\int_\Omega |T_k(u_n)|^{\lambda 2^*}\Big]^{\frac{2}{2^*}} &\le \frac{\alpha}{2S^2}\int_\Omega |\nabla(|T_k(u_n)|^\lambda)|^2 \\ &= \frac{\alpha\lambda^2}{2S^2}\int_\Omega |T_k(u_n)|^{2(\lambda-1)}|\nabla T_k(u_n)|^2.\end{aligned} \tag{57}$$

Therefore,

$$\frac{\alpha}{2}\Big[\int_\Omega |T_k(u_n)|^{\lambda 2^*}\Big]^{\frac{2}{2^*}} \le \frac{\lambda^2}{S^2}\Big[\frac{k^{2\lambda}}{2\alpha}\int_\Omega |E|^2 + \frac{k^{2\lambda-1}}{2\lambda-1}\int_\Omega |f|\Big].$$

The choice of λ gives then the result. □

Lemma 6.2. *Assume (2), (3), $|E| \in L^2(\Omega)\cap L^N(\Omega)$ and $f \in L^1(\Omega)\cap L^m(\Omega)$. If $\frac{2N}{N+2} \le m < \frac{N}{2}$, then there exists a $\tilde{k} \in \mathbb{R}_+$, with $\tilde{k} = \tilde{k}(N, \sigma_o{}^N[E])$, such that the sequence $\{G_k(u_n)\}$ is bounded in $L^{m^{**}}(\Omega)$, for every $k > \tilde{k}$. More precisely, there exists a positive constant $C = C(N, m, \alpha, S, \sigma_o^N[E])$ such that*

$$\Big[\int_\Omega |G_k(u_n)|^{m^{**}}\Big]^{\frac{2}{2^*}-\frac{1}{m'}} \le C\Big(k^2 + \|f\|_{L^m(\Omega)}\Big). \tag{58}$$

Proof. Arguing as in the previous lemma, we observe that $\dfrac{|G_k(u_n)|^{2(\lambda-1)}G_k(u_n)}{2\lambda-1}$, with $\lambda = \frac{m^{**}}{2^*}$, can be taken as test function in the variational formulation of (39). Then, following along the lines the proof of Lemma 5.4 in [1], with suitable modifications, we obtain the desired result. □

Theorem 6.3. *Assume (2), (3), $|E| \in L^2(\Omega)\cap L^N(\Omega)$ and $f \in L^1(\Omega)\cap L^m(\Omega)$. If $\frac{2N}{N+2} \le m < \frac{N}{2}$, then the sequence $\{u_n\}$ is bounded in $L^{m^{**}}(\Omega)$. More precisely, there exists a positive constant $C = C(N, m, \alpha, S, \sigma_o^N[E])$ such that*

$$\|u_n\|^{m^{**}}_{L^{m^{**}}(\Omega)} \le C(\|E\|^{\frac{2}{2^*}}_{L^2(\Omega)} + \|f\|^{\frac{2}{2^*}}_{L^1(\Omega)} + \|f\|^{\frac{2^*}{2-2^*/m'}}_{L^m(\Omega)} + 1). \tag{59}$$

Proof. The proof easily follows by (20), Lemma 6.1 and Lemma 6.2, once fixed $k > \tilde{k}$. □

Corollary 6.4. *Assume (2), (3), $|E| \in L^2(\Omega) \cap L^N(\Omega)$ and $f \in L^1(\Omega) \cap L^m(\Omega)$. If $\frac{2N}{N+2} \leq m < \frac{N}{2}$, then the weak solution u of (1) belongs to $W_0^{1,2}(\Omega) \cap L^{m^{**}}(\Omega)$. More precisely, there exists a positive constant $C = C(N, m, \alpha, S, \sigma_o^N[E])$ such that*

$$\|u\|_{L^{m^{**}}(\Omega)}^{m^{**}} \leq C(\|E\|_{L^2(\Omega)}^{\frac{2}{2^*}} + \|f\|_{L^1(\Omega)}^{\frac{2}{2^*}} + \|f\|_{L^m(\Omega)}^{\frac{2^*}{2-2^*/m'}} + 1). \tag{60}$$

Proof. By Theorem 6.3 we know that the sequence $\{u_n\}$ is bounded in $L^{m^{**}}(\Omega)$. Hence, up to a subsequence, u_n converges weakly to some function v in $L^{m^{**}}(\Omega)$. On the other hand, in view of Theorems 4.6 and 4.7, up to a subsequence, u_n converges weakly to u in $L^2(\Omega)$, where u is the solution of (53). Thus

$$\int_\Omega u_n \varphi \to \int_\Omega u\varphi$$

and

$$\int_\Omega u_n \varphi \to \int_\Omega v\varphi,$$

for every $\varphi \in C_0^\infty(\Omega)$.
Hence

$$\int_\Omega (u - v)\varphi = 0,$$

for every $\varphi \in C_0^\infty(\Omega)$. This gives $u = v$ a.e. in Ω. Estimate (60) follows then by (59). □

Theorem 6.5. *Assume (2), (3), $|E| \in L^2(\Omega) \cap M_0^N(\Omega)$ and $f \in L^1(\Omega) \cap L^m(\Omega)$. If $\frac{2N}{N+2} \leq m < \frac{N}{2}$, then the weak solution u of (1) belongs to $W_0^{1,2}(\Omega) \cap L^{m^{**}}(\Omega)$. More precisely, there exists a positive constant $C = C(N, m, \alpha, S, \sigma_o^N[E])$ such that*

$$\|u\|_{L^{m^{**}}(\Omega)}^{m^{**}} \leq C(\|E\|_{L^2(\Omega)}^{\frac{2}{2^*}} + \|f\|_{L^1(\Omega)}^{\frac{2}{2^*}} + \|f\|_{L^m(\Omega)}^{\frac{2^*}{2-2^*/m'}} + 1). \tag{61}$$

Proof. Observe that since $|E| \in L^2(\Omega) \cap M_0^N(\Omega)$ by Lemma 2.1 we obtain that there exists a sequence $\{E_h\}$ with $|E_h| \in L^2(\Omega) \cap L^N(\Omega)$, $h \in \mathbb{N}$, such that

$$\lim_{h \to +\infty} \|E - E_h\|_{M^N(\Omega)} \to 0, \tag{62}$$

$$\|E_h\|_{L^2(\Omega)} \le \|E\|_{L^2(\Omega)}, \forall\, h \in \mathbb{N}, \tag{63}$$

$$\sigma_o^N[E_h] = \sigma_o^N[E], \forall\, h \in \mathbb{N}. \tag{64}$$

Let, now, u_h, $h \in \mathbb{N}$, be the solutions of the following problems:

$$\begin{cases} -\mathrm{div}(M(x)\nabla u_h) + \mu u_h = -\mathrm{div}(u_h\, E_h(x)) + f(x) \quad \text{in } \Omega, \\ \\ u_h \in W_0^{1,2}(\Omega). \end{cases} \tag{65}$$

In view of Theorems 4.7 and 6.4 and by (63) and (64) one has

$$\|u_h\|^2_{W^{1,2}(\Omega)} \le C'(\|E\|^2_{L^2(\Omega)} + \|f\|_{L^1(\Omega)} + \|f\|^2 L^{\frac{2N}{N+2}}(\Omega)), \tag{66}$$

with $C' = C'(N, \alpha, \mu, S, \sigma_o^N[E])$, and

$$\|u_h\|^{m^{**}}_{L^{m^{**}}(\Omega)} \le C''(\|E\|^{\frac{2}{2^*}}_{L^2(\Omega)} + \|f\|^{\frac{2}{2^*}}_{L^1(\Omega)} + \|f\|^{\frac{2^*}{2-2^*/m'}}_{L^m(\Omega)} + 1), \tag{67}$$

with $C'' = C''(N, m, \alpha, S, \sigma_o^N[E])$.
Therefore there exist u' and u'' such that, up to subsequences,

$$\begin{aligned} u_h &\rightharpoonup u' \text{ weakly in } W^{1,2}(\Omega), \\ u_h &\rightharpoonup u'' \text{ weakly in } L^{m^{**}}(\Omega). \end{aligned} \tag{68}$$

Hence $u' = u'' \in W_0^{1,2}(\Omega) \cap L^{m^{**}}(\Omega)$.
Furthermore,

$$\|u'\|^{m^{**}}_{L^{m^{**}}(\Omega)} \le C''(\|E\|^{\frac{2}{2^*}}_{L^2(\Omega)} + \|f\|^{\frac{2}{2^*}}_{L^1(\Omega)} + \|f\|^{\frac{2^*}{2-2^*/m'}}_{L^m(\Omega)} + 1). \tag{69}$$

Now set

$$a_h(w, v) = \int_\Omega M(x)\nabla w \cdot \nabla v + \mu \int_\Omega wv - \int_\Omega w\, E_h(x) \cdot \nabla v,$$

$$a(w, v) = \int_\Omega M(x)\nabla w \cdot \nabla v + \mu \int_\Omega wv - \int_\Omega w\, E(x) \cdot \nabla v,$$

$w, v \in W_0^{1,2}(\Omega)$.
One has

$$a_h(w, v) = a(w, v) - \int_\Omega w\,(E_h - E) \cdot \nabla v \tag{70}$$

$w, v \in W_0^{1,2}(\Omega)$.
Then, since u is the solution of problem (1) and u_h of (65), by (70) we get

$$
\begin{aligned}
a(u,v) &= \textstyle\int_\Omega fv\,dx = a_h(u_h,v) \\
&= a(u_h,v) - \textstyle\int_\Omega u_h\,(E_h - E)\cdot\nabla v.
\end{aligned}
\tag{71}
$$

Therefore, passing to the limit as $h \to +\infty$, since the first convergence in (68) takes place in $W_0^{1,2}(\Omega)$ and taking into account the Hölder inequality, the embedding results of Lemma 2.2 and the convergence in (62) one gets

$$
a(u,v) = a(u',v). \tag{72}
$$

Thus $u = u'$ and therefore u belongs to $L^{m^{**}}(\Omega)$ and satisfies estimate (61).

□

Now, let us finally prove that if $|E| \in L^2(\Omega) \cap L^r(\Omega)$, $r > N$, and $f \in L^1(\Omega) \cap L^m(\Omega)$, $m > \frac{N}{2}$, then the solution u is bounded.

We follow Stampacchia's method ([23], see also [1]) that relies on the boundedness of the function $\log(1 + |u|)$.

THEOREM 6.6. *Assume (2), (3). If $|E| \in L^2(\Omega) \cap L^r(\Omega)$, $r > N$, and $f \in L^1(\Omega) \cap L^m(\Omega)$, $m > \frac{N}{2}$, then the weak solution u of (1) belongs to $W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$.*

PROOF. Let us define the function

$$
G(t) = \begin{cases} 0, & \text{if } |t| \le l, \\ \frac{t}{1+t} - \frac{l}{1+l}, & \text{if } t > l, \\ \frac{t}{1-t} + \frac{l}{1+l}, & \text{if } t < -l, \end{cases}
$$

with $l \in \mathbb{R}_+$.

Let $u \in W_0^{1,2}(\Omega)$ be the solution of (1). In view of Lemma 3.1, we can take $G(u)$ as test function in the variational formulation of (1) obtaining that

$$
\begin{aligned}
&\int\limits_{u>l} M(x)\nabla u\cdot\nabla\Big(\frac{u}{1+u}\Big) + \int\limits_{u<-l} M(x)\nabla u\cdot\nabla\Big(\frac{u}{1-u}\Big) \\
&+ \mu\int\limits_{u>l} u\Big(\frac{u}{1+u} - \frac{l}{1+l}\Big) + \mu\int\limits_{u<-l} u\Big(\frac{u}{1-u} + \frac{l}{1+l}\Big) \\
&= \int\limits_{u>l} uE(x)\ \cdot\nabla\Big(\frac{u}{1+u}\Big) + \int\limits_{u<-l} uE(x)\ \cdot\nabla\Big(\frac{u}{1-u}\Big)
\end{aligned}
$$

$$+\int\limits_{u>l} f\Big(\frac{u}{1+u}-\frac{l}{1+l}\Big)+\int\limits_{u<-l} f\Big(\frac{u}{1-u}+\frac{l}{1+l}\Big).$$

Whence, taking into account that the third and fourth integrals on the left-hand side of the previous equality are non negative, by simple calculation and since $|G(u)|\le 1$, we get

$$\int\limits_{|u|>l} M(x)\frac{|\nabla u|^2}{(1+|u|)^2}\le \int\limits_{|u|>l}\frac{|u|}{1+|u|}|E|\frac{|\nabla u|}{1+|u|}+\int\limits_{|u|>l}|f|.$$

Now, by (2) and Young inequality we have

$$\frac{\alpha}{2}\int\limits_{|u|>l}\frac{|\nabla u|^2}{(1+|u|)^2}\le\frac{1}{2\alpha}\int\limits_{|u|>l}|E|^2+\int\limits_{|u|>l}|f|,$$

which implies (with $l=\mathrm{e}^k-1$)

$$\frac{\alpha}{2}\int\limits_{\log(1+|u|)>k}|\nabla\log(1+|u|)|^2\le\int\limits_{\log(1+|u|)>k}\Big[\frac{1}{2\alpha}|E|^2+|f|\Big].$$

Now, set $v=\log(1+|u|)$ and $g=\frac{|E|^2}{2\alpha}+|f|$, the previous inequality can be rewritten as

$$\frac{\alpha}{2}\int\limits_{|v|>k}|\nabla v|^2\le\int\limits_{|v|>k}|g|.$$

Since g belongs to $L^q(\Omega)$, for some $q>\frac{N}{2}$, Sobolev and Hölder inequalities give

$$\|G_k(v)\|_{L^{2^*}(\Omega)}\le C\|g\|^{\frac12}_{L^q(\Omega)}|A_k|^{\frac12-\frac{1}{2q}},\tag{73}$$

with $C=C(\alpha,S)$ positive constant, and where G_k is defined in (20) and

$$A_k=\{x\in\Omega\,:\,|v(x)|>k\}.$$

By (28) one has

$$h|A_h|^{\frac{1}{2^*}}=\Big(\int_{A_h}|h|^{2^*}\Big)^{\frac{1}{2^*}}\le\|v\|_{L^{2^*}(A_h)}\le\|G_k(v)\|_{L^{2^*}(A_h)}+k|A_h|^{\frac{1}{2^*}}.$$

Thus

$$(h-k)|A_h|^{\frac{1}{2^*}}\le\|G_k(v)\|_{L^{2^*}(A_h)},\quad\forall h>k.\tag{74}$$

For a fixed $k_0 > 0$, combining (73) and (74), we get then

$$|A_h| \leq C^{2^*} \|g\|_{L^q(\Omega)}^{\frac{2^*}{2}} \frac{|A_k|^{\frac{2^*}{2} - \frac{2^*}{2q}}}{(h-k)^{2^*}}, \quad \forall h > k \geq k_0.$$

Finally, since $q > \frac{N}{2}$, one gets that $\frac{2^*}{2} - \frac{2^*}{2q} > 1$, hence Lemma 3.3 applies and therefore there exists $d \in \mathbb{R}_+$ such that $|A_{k_0+d}| = 0$. This gives $v \in L^\infty(\Omega)$ and

$$\|v\|_{L^\infty(\Omega)} \leq k_0 + d,$$

where

$$d = 2^{\frac{2^*(q-1)}{2^*(q-1)-2q}} C \|g\|_{L^q(\Omega)}^{\frac{1}{2}} |A_{k_0}|^{\frac{1}{2} - \frac{1}{2q} - \frac{1}{2^*}}$$

and $C = C(\alpha, S)$.

In conclusion, we have proved that there exists a positive constant L such that

$$\|\log(1 + |u|)\|_{L^\infty(\Omega)} \leq L,$$

and so

$$\|u\|_{L^\infty(\Omega)} \leq \mathrm{e}^L - 1.$$

□

Acknowledgements

The authors warmly thank Lucio Boccardo for the fruitful discussions and suggestions and the referee for the valuable comments.

Competing interests

The authors declare that they have no competing interests.

Author's contributions

The authors conceived and wrote this article in collaboration and with same responsibility. All of them read and approved the final manuscript.

References

[1] L. Boccardo: Some developments on Dirichlet problems with discontinuous coefficients; Boll. Unione Mat. Ital., **2** (2009), 285-297.

[2] L. Boccardo: Finite energy solutions of nonlinear Dirichlet problems with discontinuous coefficients; Boll. Unione Mat. Ital., **5** (2012), 357-368.

[3] L. Boccardo: Dirichlet problems with singular convection terms and applications; J. Diff. Eq., **258** (2015), 2290-2314.

[4] L. Boccardo, G. Croce: Esistenza e regolarità di soluzioni di alcuni problemi ellittici; Pitagora Ed., 2010.

[5] L. Boccardo, A. Dall'Aglio, L. Orsina: Existence and regularity results for some elliptic equations with degenerate coercivity, dedicated to Prof. C. Vinti (Perugia, 1996); Atti Sem. Mat. Fis. Univ. Modena, **46** (1998), 51-81.

[6] L. Boccardo, J.I. Diaz, D. Giachetti, F. Murat: Existence and regularity of renormalized solutions for some elliptic problems involving derivatives of nonlinear terms; *J. Diff. Eq.*, **106** (1993), 215-237.

[7] L. Boccardo, T. Gallouët: Nonlinear elliptic and parabolic equations involving measure data; *J. Funct. Anal.*, **87** (1989), 149-169.

[8] L. Boccardo, T. Gallouët: Nonlinear elliptic equations with right hand side measures; *Comm. Partial Differential Equations*, **17** (1992), 641-655.

[9] L. Boccardo, T. Gallouët, F. Murat: Unicité de la solution pour des equations elliptiques non linéaires; *C. R. Acad. Sc. Paris*, **315** (1992), 1159-1164.

[10] L. Boccardo, D. Giachetti: Existence results via regularity for some nonlinear elliptic problems; *Comm. Partial Differential Equations*, **14** (1989), 663-680.

[11] L. Boccardo, L. Orsina, A. Porretta: Some noncoercive parabolic equations with lower order terms in divergence form. Dedicated to Philippe Bénilan; J. Evol. Equ., **3** (2003), 407-418.

[12] G. Bottaro, M.E. Marina: Problema di Dirichlet per equazioni ellittiche di tipo variazionale su insiemi non limitati; Boll. Un. Mat. Ital., **4** (1973), 46-56.

[13] L. Caso, R. D'Ambrosio, S. Monsurrò: Some remarks on spaces of Morrey type; Abstr. Appl. Anal. **vol. 2010** (2010), 22 pages.

[14] C. Miranda: Alcune osservazioni sulla maggiorazione in L^ν delle soluzioni deboli delle equazioni ellittiche del secondo ordine; *Ann. Mat. Pura Appl.*, **61** (1963), 151-169.

[15] S. Monsurrò, M. Salvato, M. Transirico: $W^{2,2}$-a priori bounds for a class of elliptic operators; *Int. J. Differ. Equ.,* **vol. 2011** (2011), 17 pages.

[16] S. Monsurrò, M.Transirico: A L^p-estimate for weak solutions of elliptic equations; *Abstr. Appl. Anal.,* **vol. 2012** (2012), 15 pages.

[17] S. Monsurrò, M.Transirico: Dirichlet problem for divergence form elliptic equations with discontinuous coefficients; *Bound. Value Probl.,* **vol. 2012** (2012), 20 pages.

[18] S. Monsurrò, M.Transirico: A priori bounds in L^p for solutions of elliptic equations in divergence form; *Bull. Sci. Math.,* **137** (2013), 851-866.

[19] S. Monsurrò, M.Transirico: A $W^{2,p}$-estimate for a class of elliptic operators; *Int. J. Pure Appl. Math.,* **83** (2013), 489-499.

[20] S. Monsurrò, M.Transirico: A weighted $W^{2,p}$-bound for a class of elliptic operators; *J. Inequal. Appl.,* **263** (2013), 11 pages.

[21] G. Moscariello: Existence and uniqueness for elliptic equations with lower-order terms; *Adv. Calc. Var.* **4** (2011), 421-444.

[22] D. Pascali, S. Sburlan: Nonlinear mappings of monotone type; Martinus Nijhoff Publishers, The Hague, Sijthoff and Noordhoff International Publishers, Alphen aan den Rijn, 1978.

[23] G. Stampacchia: Le probléme de Dirichlet pour les équations elliptiques du second ordre à coefficients discontinus; *Ann. Inst. Fourier* (Grenoble), **15** (1965), 189-258.

[24] G. Stampacchia: Equations elliptiques du second ordre à coefficients discontinus; Séminaire de mathématiques supérieures, Université de Montréal, 4^e session, été 1965, Les presses de l'Université de Montréal (1966).

[25] M. Transirico, M. Troisi: Equazioni ellittiche del secondo ordine a coefficienti discontinui e di tipo variazionale in aperti non limitati; *Boll. Un. Mat. Ital.*, **7** (1988), 385-398.

[26] M. Transirico, M. Troisi: Equazioni ellittiche del secondo ordine di tipo non variazionale in aperti non limitati; *Ann. Mat. Pura Appl.*, **4** (1988), 209-226.

[27] G. Zecca: Existence and uniqueness for nonlinear elliptic equations with lower-order terms; *Nonlinear Anal.*, **75** (2012), 899-912.