# Advanced Traveller Information Systems under recurrent traffic conditions: network equilibrium and stability

**Gennaro N. Bifulco [1], Giulio E. Cantarella[2], Fulvio Simonelli[3], Pietro Velonà[4]**
[1] Università degli Studi di Napoli Federico II
[2] Università degli Studi di Salerno
[3] Università degli Studi del Sannio
[4] Università degli Studi Mediterranea di Reggio Calabria

## Abstract

In this paper the stability of traffic equilibrium is analysed by using a framework where advanced traveller information systems (ATIS) are explicitly modelled. The role played by information in traffic networks is discussed, with particular reference to the day-to-day dynamics of the traffic network and to system stability at equilibrium. The perspective adopted is that of transportation planning under recurrent network conditions. The network is considered to be in equilibrium, viewed as a fixed-point state of a day-to-day deterministic process, here modelled as a time-discrete non-linear Markovian dynamic system. In discussing the effects generated by the introduction of ATIS, the paper examines: changes in the fixed point(s) with respect to the absence of ATIS, how the theoretical conditions for fixed-point existence and uniqueness are affected, and the impact on the stability properties and the stability region at equilibrium. Most of the analyses are carried out with explicit theoretical considerations. Moreover, a toy network is also employed to explore numerically the effects of removing some assumptions concerning the accuracy of ATIS.

**Keywords**: Day-to-day Dynamics; Intelligent Transportation Systems; Advanced Traveller Information Systems; Equilibrium Stability; Stability region for ATIS; Traffic Networks; User Equilibrium; Fixed Point; Information Accuracy; Route Guidance; Anticipatory Route Guidance; ATIS Market Penetration.

## 1 Introduction

Advanced traveller information systems (ATIS) are considered a powerful tool to enhance the travellers' experience and avoid wasting travel time (Khattak et al., 2003; Bifulco et al., 2009). They are mainly useful for travellers who are unfamiliar with the network (e.g. tourists), as well as for all travellers should the network be temporarily affected by some significant disruption and/or by unexpected or non-recurrent traffic conditions (Emmerink et al., 1995). Moreover, ATIS are also claimed to be useful under recurrent network congestion as they reduce the uncertainty of travellers with respect to travel times (Ettema and Timmermans, 2006).

The disaggregated effect (not considering network phenomena) of ATIS on traveller's behaviour was analysed by Ben-Elia et al. (2013) using (panel) data collected in a laboratory experiment based on a travel simulator. Both statistical analyses and calibration of a mixed-logit model were performed to show that a major role is played by the reliability of the dispatched information. Tanaka et al. (2014), basing their analysis on a different laboratory experiment and different panel data, confirmed the ability of information services to influence travellers' behaviour; they modelled this influence by means of a mixed-logit model as well. Previous studies by Tsirimpa et al. (2007) were based on field-collected data: both the multinomial-logit and the mixed-logit approaches confirmed that in switching between alternative routes a relevant role is played by pre-trip (vs. en-route) information and by the source and content of the dispatched information. Tseng et al. (2013) also observed revealed preferences for the Dutch A12 motorway with reference to a sample of 340 participants equipped with on-board units. Switching from commuting in the morning on the motorway to alternative travel modes was analysed within a repeated choice experiment; behaviours were compared with respect to pre- and post-adoption of a smartphone-based information service, and a mixed-logit model was specified to reproduce such behaviours. Various approaches to disaggregate modelling of ATIS influence on travel choices not based on the classic random utility theory have been proposed. Amongst others, Chorus et al. (2006a) integrated notions of search theory and Bayesian updating perception within a utilitarian framework, Paz and Peeta (2009) adopted a fuzzy multinomial logit structure, and Dell'Orco and Teodorovic (2009) applied uncertainty-based information theory with a fuzzy approach.

At a network level due account has to be made for the aggregate effect of travellers' behaviour, as well as interaction with network congestion and applied information strategies (which depend on current traffic conditions). The result is a *network assignment model under ATIS*, and several of such models have been developed. One was proposed, for non-recurrent traffic conditions, by Al-Deek and Kanafani (1993) and Al-Deek et al. (1998); a deterministic queuing approach was developed and applied to a two-route network to assess the benefit of providing information to travellers under incident conditions. Hall (1996) considered a dynamic traffic system with two alternative routes; he developed and discussed the traffic simulation model under different hypotheses about the reliability of the information provided and the market penetration of the information system. Interestingly, Hall defended the conjecture that ATIS are not suitable for use as control tools; unfortunately, others have since disregarded this suggestion, and the temptation to propose ATIS as a control/optimisation tool for traffic networks has been recurrent in the scientific literature. Dia (2002) combined a dynamic flow propagation model based on microsimulation with an agent-based route choice model based on the travel behaviours observed in a field experiment in Brisbane; the main aim of the work was to prove the suitability of dispatching traveller information (in simplified network conditions). Cantarella (2013) simulated the presence of the information service by appropriately modifying the value of the parameters of a day-to-day dynamic process, based on the exponential smoothing approach. Lo and Szeto (2004) considered the information services within a framework where network flow propagation is implemented by the Daganzo cell transmission model and route choice is made according to a stochastic dynamic user-optimum model. He et al. (2013) explicitly addressed the problem of real-time adapting an information strategy, rather than simulating the effects of an exogenously given (and fixed) information set; they dealt with a simplified network, explicitly simulated (dynamic) queues and discussed the extension of the proposed strategy to more complex networks.

Overall, the literature on network assignment models under ATIS is very extensive and a comprehensive review lies well beyond the scope of this paper. For a more detailed literature review we suggest Chorus et al. (2006b) and De Palma et al. (2012).

Here our focus is on the mathematical properties of the network assignment model under ATIS for recurrent traffic conditions, with particular emphasis on equilibrium stability, which is a relatively unexplored field. Previous findings, numerically obtained in Bifulco et al. (2014a) are herein extended and generalised towards identifying theoretical properties. In particular, this work shows that ATIS constitute a powerful tool for enhancing the stability of traffic systems, ensuring the stability of network configurations that are otherwise unstable, a matter of crucial importance for traffic systems planning and design. Stability properties are theoretically proved in the case of accurate travellers' information; however, some inaccuracy could affect the dispatched information. This may be due to several different factors, including the well-known anticipatory route guidance problem (Bottom, 2000). Therefore, besides the theoretical work, in discussing the theoretical results (section 4) we also carry out some numerical simulations to show how variable levels of accuracy can affect network stability.

Our analyses are framed within the unified theory of equilibrium and day-to-day deterministic dynamic processes, as first established by Cantarella and Cascetta (1995) and then by Watling and Hazelton (2003). Recurrent conditions are considered, boundary conditions of the day-to-day dynamics are constant while the traffic system evolves, possibly towards the equilibrium. The hypothesis of recurrent traffic condition could be questioned, as both common experience and mobility studies (e.g. Schrank et al., 2012) have shown that relevant fluctuations can be observed with respect to recurrent conditions, as a consequence of incidents, bad weather and other random phenomena. These fluctuations are such that travellers are subject over-time to dispersed actual travel times. The most common approach to deal with these fluctuations is to adopt probabilistic route choice models. Of course, if the fluctuations are too high, then the probabilistic approach is not adequate and non-recurrent conditions have to be explicitly considered. One way to mitigate the costs of non-recurring congestion is to provide travelers with information about actual travel conditions. This is the context in which are framed some recent papers such as Lindsey et al. (2014) and Rapoport et al (2014). In these works a model is developed (and an experiment carried out, that broadly confirm the theory) for a two-link network; as one of the results of these works, it is shown that the way the non-recurrent conditions (e.g. incidents) are probabilistically distributed over the two alternative routes, and the difference in free-flow travel time between the two routes, can modify the effect of the information in terms of (deterministic) route choice behaviour versus system-optimum conditions. Our work, instead, is not limited to a two-link network, and adopts a probabilistic route choice approach. The approach is able to deal with non-recurring random conditions and allows for considering, if requested, covariate distributions of randomness in route travel times.

In our approach the within-day dynamics of the traffic system is neglected; this is a customary assumption made by transport analysts when they conduct planning and designing studies. It is a standard hypothesis also in the case of analyses related to the long-term effects of ATIS adoption, as in Yang (1998) where the network equilibrium is analysed with an endogenous representation of ATIS market penetration, or in Yang and Meng (2001), where the model is extended to the case of day-to-day dynamics. A within-day-static approach was also adopted in Lo and Szeto (2002) when extending the analysis by Yang to take account of different system actors (travellers, information providers and traffic authorities), with different and possibly conflicting objectives. The within-day static approach is also adopted by the already mentioned works by Lindsey et al. (2014) and Rapoport et al. (2014) for the two-link network under different levels of dispatched information. Their results were somehow paradoxical as a full level of traffic information seemed to perform worse than zero information when the network exhibited a tendency to covariate distribution of bad traffic conditions on the two routes; however, these analyses did not explicitly take into account (at least in the theoretical formulation) the effect of information accuracy on travellers' compliance.

It is worth noting that in our work recurrent conditions and within-day-static hypotheses allow a mathematical formalisation that is easier to manage in theoretical terms. This is exploited to show theoretical properties in the event of accurate information, while in the case of inaccuracy some numerical simulations are discussed.

## 2 Notation and model formulation

The notation is that used by Bifulco et al. (2014a), with some minor adaptation. Assume that:

i   is a demand class with given characteristics (e.g. travellers of the same O/D pair).

t   is a simulation day in the day-to-day dynamic process.

$K_i$   is the set of alternative routes for the demand class *i* (e.g. all routes connecting O/D pair *i*).

$\boldsymbol{B}_i$ is the time-independent link-route incidence matrix for all routes of class $i$, with entries $B_{a,k}$ equal to one if link $a$ belongs to route $k \in K_i$, zero otherwise.

$p^t_i$ is the vector of route-choice probabilities for class $i$ at day $t$, with entries $p_k$ $\forall$ $k \in K_i$; these are computed from route travel times by using a time-independent route choice map $p_i(\cdot)$, for instance we could refer to route-choice maps based on a random utility model (RUM) approach, as in Domencich and McFadden (1975) and Ben-Akiva and Lerman (1985).

$d_i$ is the travel demand (e.g. vehic/hour) for class $i$ ($d_i \geq 0$ $\forall$ $i$), considered to be time-independent (recurrent conditions from day to day).

$\boldsymbol{f}^t$ is the link flow vector at day $t$; it belongs to the feasibility set $S_f$, which accounts for non-negativity and ensures both demand conservation between O/D pairs and flow conservation at nodes.

$\mathbf{c}(\cdot)$ is the congestion model, which gives link travel times as a function of link flows.

$\mathbf{x}^t$ are the expected travel times associated to network links at day $t$.

$\eta$ is the time-independent ATIS market penetration, assumed for the sake of simplicity to be the same for all classes $i$; the fraction $(1 - \eta)$ is the percentage of *non-equipped* travellers (without access to the ATIS).

$m^t_i$ is the fraction of *compliant travellers* of class $i$, they are equipped travellers ($m^t_i \leq \eta$) that have access to ATIS and at time $t$ are compliant with the received information; the fraction of *non-compliant* travellers is $\eta - m^t_i$; compliant travellers behave according to the ATIS information, while non-compliant ones make their decisions without taking into account the information system; at day $t$, the fractions of compliant travellers for all classes $i$ can be arranged into a vector $\boldsymbol{m}^t = [...,m^t{}_{,i},...]^T$.

$\boldsymbol{r}^t_i$ is the vector of estimated route travel times at time $t$ for class $i$ (e.g. O/D pair $i$); estimates are made by the ATIS; these can be directly dispatched to the equipped travellers (descriptive ATIS) or employed to compute routes to be suggested (prescriptive ATIS); at day $t$, the estimated travel times for each class $i$ can be arranged into a vector $\boldsymbol{r}^t = [\ldots \mid \boldsymbol{r}^{t\,T}_i \mid \ldots]^T$.

$\boldsymbol{\pi}^t_i$ is the vector of *route-spreading* coefficients for compliant travellers of class $i$ at day $t$; each entry $\pi^t_k$ ($\forall$ $k \in K_i$) is the percentage of compliant travellers of class $i$ choosing route $k$ at time $t$. In the case of prescriptive ATIS, these percentages are computed by the system which suggests that the fraction $\pi^t_k$ of travellers follow route $k$; this allow to spread the travellers among the alternative routes, thereby avoiding concentration and improper oversaturation and trying to be consistent with the estimated travel times ($\boldsymbol{r}^t_i$). In the case of descriptive ATIS, these percentages are the result of the route-choice model applied by the compliant travellers, based on the ATIS-estimated (and dispatched) travel-times. In both cases it can be considered that the vector of route-spreading coefficients is a map with the estimated travel times as argument $\pi_i(\mathbf{r}^t_i)$. This map is the route-choice model for compliant travellers (descriptive ATIS) or the routing-design model (prescriptive ATIS).

As a consequence of the within-day static hypothesis, two linear models can be expressed for the network loading maps of the compliant and non-compliant (or non-equipped) travellers:

$$\mathbf{f}_C(\boldsymbol{r}^t, \boldsymbol{m}^t) = \Sigma_i\, m_i^t\, d_i\, \boldsymbol{B}_i\, \boldsymbol{\pi}_i(\mathbf{r}^t_i) \tag{1}$$

$$\mathbf{f}_U(\mathbf{x}^t, \boldsymbol{m}^t) = \Sigma_i\, (1 - m_i^t)\, d_i\, \boldsymbol{B}_i\, \mathbf{p}_i(\boldsymbol{B}_i^T\, \mathbf{x}^t) \tag{2}$$

For the sake of simplicity, in the above equations, route-choice maps for non-equipped and for non-compliant travellers are considered the same, equal to $\mathbf{p}_i(\cdot)$. The overall network loading map is therefore:

$$\mathbf{f}_{NL}(\boldsymbol{x}^t, \boldsymbol{r}^t, \boldsymbol{m}^t) = \mathbf{f}_C(\boldsymbol{r}^t, \boldsymbol{m}^t) + \mathbf{f}_U(\mathbf{x}^t, \boldsymbol{m}^t) = \Sigma_i\, \boldsymbol{B}_i \left( m_i^t\, \boldsymbol{\pi}_i(\mathbf{r}^t_i) + (1 - m_i^t)\, \mathbf{p}_i(\boldsymbol{B}_i^T\, \mathbf{x}^t) \right) d_i \tag{3}$$

According to the day-to-day dynamic hypothesis, the loading map is included within a simple (but effective) exponential-smoothing approach, whose mathematical structure adapts well to our analytical framework. Under ATIS we not only have to describe the utilities-learning and choice-updating processes but also the process that simulates the evolution over days of both compliance and the ATIS-estimated travel times:

$$\boldsymbol{x}^t = \beta\, \mathbf{c}(\boldsymbol{f}^{t-1}) + (1 - \beta)\, \boldsymbol{x}^{t-1} \quad \text{with} \quad \boldsymbol{x}^{t=0} = \boldsymbol{x}^0 \tag{4.a}$$

$$\boldsymbol{f}^t = \alpha\, \mathbf{f}_{NL}(\boldsymbol{x}^t, \boldsymbol{r}^t, \boldsymbol{m}^t) + (1 - \alpha)\, \boldsymbol{f}^{t-1} \quad \text{with} \quad \boldsymbol{f}^{t=0} = \boldsymbol{f}^0 \tag{4.b}$$

$$\boldsymbol{r}_i^t = \rho\, \boldsymbol{B}_i^T\, c^{t-1} + (1 - \rho)\, \boldsymbol{r}_i^{t-1} \quad \text{with} \quad \boldsymbol{r}_i^{t=0} = \boldsymbol{r}_i^0 \quad \forall\, i \tag{4.c}$$

$$m_i^t = \mu\, \mathrm{m}(In_i^{t-1}, \eta) + (1 - \mu)\, m_i^{t-1} \quad \text{with} \quad m_i^{t=0} = m_i^0 \quad \forall\, i \tag{4.d}$$

where $\beta$ is the utilities-learning dynamic parameter; $\alpha$ is the choice-updating dynamic parameter, $\rho$ is the parameter of the travel-time estimation process, $\mu$ is the parameter of the compliance dynamics, and $\boldsymbol{x}^0$, $\boldsymbol{f}^0$, $\boldsymbol{r}_i^0$ ($\forall\, i$) and $m_i^0$ ($\forall\, i$) are some known initial conditions of the dynamic process. Finally, as better detailed in the following, $In_i^{t-1}$ is information inaccuracy and $m(\cdot)$is a function that identifies the decrease in compliance as a function of information inaccuracy.

The information updating and compliance updating models are discussed in sections 2.1 and 2.2 as they are less extensively covered in the scientific literature and merit greater detail. The utilities-learning and choice-updating mechanisms in this exponential smoothing approach (also called *α-and-β* approach) are well known (see for instance Cantarella and Cascetta, 1995). Alternative updating mechanisms could be applied, such as in Jha et al. (1998), where a Bayesian updating model is adopted, or in Mahmassani and Liu (1999), where a pre-trip route selection and dynamic sequence of en-route switching probabilities are considered. However, these approaches are more complex to deal with analytically and require numerical simulation.

## 2.1 The information strategy

The process of travel time estimation ($\boldsymbol{r}^t$) depends on the actual implementation of the ATIS. The quite general information strategy described by equation 4.c refers to an adjusting mechanism where, based on the travel times actually observed on the network at a previous day ($\mathbf{c}^{t-1}$), the estimated travel times for today are updated. Within this mechanism, as is usual in the case of exponential smoothing, once the system as a whole is attracted towards a fixed point, the link travel times are repeated over days, as are the estimated travel times, which tend to coincide with the actual ones. However, there is no immediate evidence on how this general information strategy impacts on the overall dynamics of the systems, nor on the stability of the fixed point(s), if any and if reached.

It can be argued that, as usually occurs in exponential smoothing, a very small value for the dynamic parameter $\rho$ could reduce the dynamics of the system and help its stability. This seems reasonable if equation 4.c is viewed independently of the others. However, if the information process is carried out with a very small dynamic parameter (as an extreme case, say $\rho$=0), the estimated information ($\mathbf{r}$) tends to be potentially unrelated to (and independent of) the history of experienced travel times ($\mathbf{c}$), which could negatively impact on compliance with the information, thus leading to disruption in the system stability.

Actually, using fixed information ($\boldsymbol{r}_i^t = \boldsymbol{r}_i^0 \ \forall\ t, \forall\ i$) can be viewed as a particular case of equation 4.c, where $\rho$=0. This information strategy is less unusual than expected, given that this type of information is that dispatched by the great majority of existing route navigators, which base their information on static estimates of travel times, related to free-flow or average traffic conditions, possibly varying within-day in a predefined way. Implementation of predetermined (and static) information strategies was numerically proven by Avinieri and Prashker (2006) to be in some cases even more effective than providing adaptive information. However, we are not of the same opinion, at least not for general cases, and we argue that their results were obtained thanks to comparison with poor adaptive information strategies. Others (Oh et al., 2001; Jahn et al., 2005) have suggested dispatching a specific type of fixed information concerning the desired (or objective) traffic conditions. This suggestion is often given in order to support and help the attainment of a system-optimum traffic pattern. Also in this case we would argue against this possibility and have numerically shown (Bifulco et al., 2007) that the objective cannot be obtained, or, if it can, it provides negligible benefits.

A very special case of information strategy is where the estimated information is *fully accurate*: the system at each day and for each class of traveller is able to estimate travel times exactly as they will be experienced on the network once the traffic has been propagated. In analytical terms, this means that $\forall\ i, t\ \ \boldsymbol{r}_i^t = \boldsymbol{B}_i^T\, c(\boldsymbol{f}^t)$. It is intuitive that if the dispatched information is based on estimated travel times that are equal to actual times, then the accuracy is the highest and the traveller compliance with this information is expected to reach its maximum value (market penetration, $\eta$); *full compliance* will be detailed in the next section.

## 2.2 Compliance with information

As expressed in equation 4.d, a dynamic process can also be applied to the phenomenon that describes the evolution over time of travellers' compliance with the dispatched information. A crucial concept in our framework is that compliance is far from an exogenous variable or a parameter of the ATIS model. Rather, it is an endogenous variable and it depends through the *compliance function* $m(In_i^{t-1}, \eta)$ on the inaccuracy of the information and on the market penetration of the ATIS. A compliance function, as the one we adopt for instance in equation 5 below, is a proxy of a more complex phenomenon (see, for instance, Bifulco et al., 2014b), as it is convenient for the analytical tractability of the model; it should be identified so as to be as close as possible to the relationship between inaccuracy and compliance established by using more sophisticated theories, such as Bogers et al. (2006) or Ben-Elia et al. (2013). A recent theoretical review about the effects of information on travellers' behaviour can be found in Ben-Elia and Avineri (2015) ; it includes the impact of phenomena such as reinforced learning, risk and loss aversion, probability weighting, anchoring and ambiguity aversion, and regret aversion

Here, we compute inaccuracy of the information via equation 5 below and the corresponding compliance via the linear equation 6 below.

$$In_i^t = ||\boldsymbol{r}_i^t - \boldsymbol{B}_i^T \mathbf{c}(\boldsymbol{f}^t)|| \,/\, ||\boldsymbol{B}_i^T \mathbf{c}(\boldsymbol{f}^t)|| \tag{5}$$

where ||.|| is the Euclidean Norm of a vector.

The formula computes the (relative) error between the estimated travel times dispatched by the ATIS and the actual travel times. A reasonable compliance function should attain its maximum value (the ATIS market penetration) when the inaccuracy is null, and should present null values starting from a given critical value of inaccuracy $In_{cr}$. For the sake of simplicity, we suggest adopting a linearly decreasing formula for such a compliance function:

$$m(In_i^t, \eta) = max\left(0, \eta\left(1 - \frac{In_i^t}{In_{cr}}\right)\right) \tag{6}$$

It is worth noting that the analytical formulation of the compliance function plays a role only if the information is not fully accurate. Otherwise, in the case of fully accurate information, whatever the compliance function, the value of compliance coincides with market penetration.

## 2.3 Some relevant cases: equilibrium, fully accurate equilibrium and fully accurate dynamics

A particular point of the dynamic process is a fixed point, where the dispatched information assumes a specific value ($\mathbf{r}^\circ$) and the flows are consistent with, say, the user-equilibrium ($\mathbf{f}^*$). The information at the equilibrium and the equilibrium link flow vector imply the equilibrium link travel times ($\mathbf{c}^*$, which is equal to the expected travel times $\mathbf{x}^*$) and the equilibrium compliance ($\mathbf{m}^*$). In analytical terms, adopting the compliance function and substituting in $\mathbf{f}_{NL}(\cdot)$ the explicit formulations in equations 1 and 2 for the loading maps $\mathbf{f}_C(\cdot)$ and $\mathbf{f}_U(\cdot)$, equations from 4.a to 4.d can be rearranged as:

$$\boldsymbol{f}^* = \Sigma_i\, m_i^*\, d_i\, \boldsymbol{B}_i\, \boldsymbol{\pi}_i(\mathbf{r}_i^\circ) + \Sigma_i\, (1 - m_i^*)\, d_i\, \boldsymbol{B}_i\, \mathbf{p}_i(\boldsymbol{B}_i^T\, \mathbf{c}(\boldsymbol{f}^*)) \tag{7.a}$$

$$m_i^* = \mathrm{m}(||\boldsymbol{r}_i^\circ - \boldsymbol{B}_i^T\, \mathbf{c}(\boldsymbol{f}^*)||\, /\, ||\, \boldsymbol{B}_i^T\, \mathbf{c}(\boldsymbol{f}^*)||, \eta) \qquad \forall\, i \tag{7.b}$$

Interestingly, the equilibrium depends on the dispatched information ($\mathbf{r}^\circ$) and thus can be viewed as a combined (non-separable) problem of solving a fixed-point problem and designing appropriate information ($\mathbf{r}_i^\circ$, $\forall\ i$) to be dispatched at equilibrium. Nevertheless, how much the equilibrium under ATIS can actually differ from the equilibrium without ATIS should be briefly discussed. Prior to such discussion, however, we would like to introduce a particular case of equilibrium under ATIS, which is that reached if the dispatched information is fully accurate. In this case the information varies over time in a constrained way; in particular, $\boldsymbol{r}_i^t = \boldsymbol{B}_i^T\, \mathrm{c}(\boldsymbol{f}^t)$ and $m_i^t = \eta$ $\forall\ i$, $\forall\ t$; hence, by substitution in 7.a and 7.b, the equilibrium can be expressed as:

$$\boldsymbol{f}^* = \Sigma_i\, \boldsymbol{B}_i\, \big(\, \eta\, \boldsymbol{\pi}_i(\boldsymbol{B}_i^T\, \mathbf{c}(\boldsymbol{f}^*)) + (1 - \eta)\, \mathbf{p}_i(\boldsymbol{B}_i^T\, \mathbf{c}(\boldsymbol{f}^*))\, \big)\, d_i \tag{8}$$

Let us now introduce the *route-spreading difference* map, which represents the difference in the route-choice behaviour between compliant and non-compliant drivers once subject to the same stimulus:

$$\boldsymbol{\Delta p}_i(\cdot) = \boldsymbol{\pi}_i(\cdot) - \mathbf{p}_i(\cdot) \qquad \forall\, i$$

At equilibrium, in the case of fully accurate information, the route-spreading difference map is:

$$\boldsymbol{\Delta p}_i(\boldsymbol{B}_i^T\, \mathbf{c}(\boldsymbol{f}^*)) = \boldsymbol{\pi}_i(\boldsymbol{B}_i^T\, \mathbf{c}(\boldsymbol{f}^*)) - \mathbf{p}_i(\boldsymbol{B}_i^T\, \mathbf{c}(\boldsymbol{f}^*)) \qquad \forall\, i \tag{9}$$

With some simple algebra, equations 8 and 9 can be further rearranged as:

$$\boldsymbol{f}^* = \Sigma_i\, \boldsymbol{B}_i\, \big(\eta\, \boldsymbol{\Delta p}_i(\boldsymbol{B}_i^T\, \mathbf{c}(\boldsymbol{f}^*)) + \mathbf{p}_i(\boldsymbol{B}_i^T\, \mathbf{c}(\boldsymbol{f}^*))\, \big)\, d_i \tag{10}$$

Now it is useful to note that in the absence of ATIS the stochastic user equilibrium can be computed as:

$$\boldsymbol{f}^* = \Sigma_i\, \boldsymbol{B}_i\, \mathbf{p}_i(\boldsymbol{B}_i^T\, \mathbf{c}(\boldsymbol{f}^*))\, d_i \tag{11}$$

Thus, by comparison of equations 10 and 11, it is evident in the case of fully accurate information that the difference between ATIS and non-ATIS conditions depends on $\eta\boldsymbol{\Delta p}_i(\boldsymbol{B}_i^T\ \mathbf{c}(\boldsymbol{f}^*))$. In practice, in order to have equilibrium conditions significantly influenced by ATIS, both high market penetration and a significant difference between maps $\boldsymbol{\pi}_i(\cdot)$ and $\mathbf{p}_i(\cdot)$, evaluated at equilibrium, are required.

In another very particular case, let us consider what happens if the dispatched information is significantly inaccurate (hence conditions are far from the fully accurate information, discussed above). Compliance decreases, even sharply, depending on the gradient of the compliance function and on the value of the critical inaccuracy. Thus, from equation 7.a, if the compliance tends to a null value, $\boldsymbol{m} \rightarrow \mathbf{0}$, then the equilibrium flows tend to the non-ATIS equilibrium:

$$\boldsymbol{lim}_{\boldsymbol{m}\to 0}\boldsymbol{f}^{*} = \Sigma_{i} \boldsymbol{B}_{i}\, \mathbf{p}_{i}(\boldsymbol{B}_{i}^{T}\, \mathbf{c}(\boldsymbol{f}^{*}))\, d_{i} \tag{12}$$

Some simulations on the difference between the ATIS and non-ATIS equilibria can be found in Bifulco et al. (2007), where the inaccuracy of the dispatched information is oriented towards the (unsuccessful) establishment of system-optimum traffic patterns instead of user-optimum patterns.

Of course, the dispatching of fully accurate information is not only a major issue to be explored at equilibrium, but also an important task for the dynamic process. In the case of fully accurate information, compliance coincides with the market penetration and the dispatched travel times with the actual ones. Thus the network loading function for compliant and non-compliant (or non-equipped) travellers can be computed as $\mathbf{f}_{C}(\boldsymbol{r}^{t}, \boldsymbol{m}^{t}) = \mathbf{f}_{C}(\boldsymbol{B}_{i}^{T}\, \mathbf{c}(\boldsymbol{f}^{t}), \eta)$ and $\mathbf{f}_{U}(\mathbf{x}^{t}, \boldsymbol{m}^{t}) = \mathbf{f}_{U}(\mathbf{x}^{t}, \eta)$. As a result, equations from 4.a to 4.d can be particularised as:

$$\boldsymbol{x}^{t} = \beta\, \mathbf{c}(\boldsymbol{f}^{t-1}) + (1-\beta)\, \boldsymbol{x}^{t-1} \qquad \text{with} \quad \boldsymbol{x}^{t=0} = \boldsymbol{x}^{0} \tag{13.a}$$
$$\boldsymbol{f}^{t} = \alpha\, (\mathbf{f}_{C}(\boldsymbol{B}_{i}^{T}\, \mathbf{c}(\boldsymbol{f}^{t}), \eta) + \mathbf{f}_{U}(\boldsymbol{x}^{t}, \eta)) + (1-\alpha)\, \boldsymbol{f}^{t-1} \qquad \text{with} \quad \boldsymbol{f}^{t=0} = \boldsymbol{f}^{0} \tag{13.b}$$

It is worth noting that in equation 13.b at each day a fixed-point problem has to be solved, given that the flow vector at day $t$ ($\boldsymbol{f}^{t}$) is present in both the value and the argument of the equation (which is not the case for $\boldsymbol{f}^{t}$ in equation 4.b). This is due to the fact that the full accuracy hypothesis gives rise to the anticipatory-route-guidance problem (Bottom, 2000) in which the dispatched information has to be consistent with the travel times it induces, according to both compliance and congestion mechanisms.

It was previously shown that in the case of fully accurate information and if the *route-spreading difference-map* tends to zero (compliant and non-compliant travellers react similarly to stimuli, $\Delta\mathbf{p}_{i}(\cdot)\cong 0\ \forall\ i$), the effect of ATIS tends to zero if evaluated at equilibrium (comparison of equations 10 and 11). Interestingly, the same does not hold for the trajectory of the dynamic process Indeed, in the hypothesis that the information is accurate and $\Delta\mathbf{p}_{i}(\cdot)\cong 0$ ($\forall\ i$), equations 13.a and 13.b can be rewritten as:

$$\boldsymbol{x}^{t} = \beta\, \mathbf{c}(\boldsymbol{f}^{t-1}) + (1-\beta)\, \boldsymbol{x}^{t-1} \qquad \text{with } \boldsymbol{x}^{t=0} = \boldsymbol{x}^{0} \tag{14.a}$$
$$\boldsymbol{f}^{t} = \alpha\, \Sigma_{i}\, d_{i}\, \boldsymbol{B}_{i} \left(\eta\, \mathbf{p}_{i}(\boldsymbol{B}_{i}^{T}\, \mathbf{c}(\boldsymbol{f}^{t})) + (1-\eta)\, \mathbf{p}_{i}(\boldsymbol{B}_{i}^{T}\, \boldsymbol{x}^{t})\right) + (1-\alpha)\, \boldsymbol{f}^{t-1} \qquad \text{with } \boldsymbol{f}^{t=0} = \boldsymbol{f}^{0} \tag{14.b}$$

Finally, given that $\mathbf{c}(\boldsymbol{f}^{t}) \neq \boldsymbol{x}^{t}$ (unless the equilibrium has been reached), the ATIS has an effect.

The previous discussions can be summarised saying that, in the case of fully accurate information, compliant travellers have instantaneous and perfect knowledge of travel times (supplied by the system), while non-compliant travellers are subject to a dynamic learning process. However, in these conditions, the learning process is such that at equilibrium also the non-compliant (or non-equipped) travellers have attained the same (fully accurate) knowledge as compliant travellers.

# 3 Theoretical properties

In this section on theoretical properties, the existence and uniqueness properties of the equilibrium will first be discussed, and then the (deterministic) dynamic process will be investigated with respect to both its dissipative characteristics (that ensure the convergence of the process towards an attractor, possibly a fixed point/equilibrium) and the stability of the equilibrium. As the discussion of theoretical properties moves from existence/uniqueness towards stability, more simplifying hypotheses among these introduced in section2 will be assumed in order to preserve the theoretical tractability of the problem.

## 3.1 Equilibrium under ATIS: existence and uniqueness

The existence of the equilibrium under ATIS is guaranteed if the following hypotheses hold.

**HYP 1:** The route choice, the *route-spreading coefficients*, and the compliance functions $\mathbf{p}_i(\cdot)$, $\boldsymbol{\pi}_i(\cdot)$, $m_i(\cdot)$ $\forall\, i$, are continuous.

**HYP 2:** (The network is connected and) the congestion model $\mathbf{c}(\cdot)$ is continuous.

**THEOREM 1 (existence)**: Hypotheses HYP 1 and HYP 2 above ensure the existence of the equilibrium solution under ATIS as formulated by equations 7a and 7b.
*Proof. Given any value of the market penetration ($\eta$) and any value of the information dispatched at the equilibrium ($\boldsymbol{r}_i^\circ$), all functions composing equation 7.a (and 7.b, that can be substituted in 7.a in order to obtain the fixed-point function) are continuous. The domain set of the fixed-point function is the set of feasible link-flow vectors, identified by demand conservation and flow conservation at nodes; it can be expressed by linear inequalities and is convex, closed and limited. The codomain set of the fixed-point function is obtained by the convex combination of quantities that respect both the condition of demand conservation and flow conservation at nodes; thus it is contained within the domain set. Since all conditions of Brouwer's theorem (Brouwer, 1911) can be applied, the fixed-point solution exists.*

Note that HYP 1 holds for commonly applied models. Moreover, Theorem 1 does not require a fully accurate ATIS and, apart from continuity, the compliance function (and the information strategy) are not required to satisfy any particular condition.

In order to discuss the uniqueness of the equilibrium, some further hypotheses have to be introduced. Generally, different equilibrium flows determine different levels of congestion, hence different actual travel times and finally different inaccuracies with respect to the (fixed) information dispatched by the ATIS. So far, the level of compliance depends both on the provided information and the flows, which in turn depend on the information. However, we introduce the following simplifying assumption.

**HYP 3:** for a given item of information $\boldsymbol{r}_i°$ ($\forall\ i$), a level of compliance ($m_i°$, $\forall\ i$) is attained and the effect of the actual equilibrium link flows on the compliance is neglected.

This hypothesis leads to a special (and simplified) case for the system of equations 7.a and 7.b in section 2.3, where equation 7.b is neglected and only equation 7.a is considered, rewritten with a given exogenously computed compliance vector ($m_i°$, $\forall\ i$):

$$\boldsymbol{f}^* = \Sigma_i\, m_i^\circ\, d_i\, \boldsymbol{B}_i\, \boldsymbol{\pi}_i(\mathbf{r}_i^\circ) + \Sigma_i\, (1 - m_i^\circ)\, d_i\, \boldsymbol{B}_i\, \mathbf{p}_i(\boldsymbol{B}_i^{\mathrm{T}}\, \mathbf{c}(\boldsymbol{f}^*))$$

Then, adapting the loading map from equation 3 to the particular case of given **r** and **m**, and considering that at equilibrium **x** = **c**:

$$\mathbf{f}_{\mathrm{NL}}(\boldsymbol{c} \,/\, \boldsymbol{r}^\circ, \boldsymbol{m}^\circ) = \mathbf{f}(\boldsymbol{c}) = \Sigma_i\, m_i^\circ\, d_i\, \boldsymbol{B}_i\, \boldsymbol{\pi}_i(\mathbf{r}_i^\circ) + \Sigma_i\, (1 - m_i^\circ)\, d_i\, \boldsymbol{B}_i\, \mathbf{p}_i(\boldsymbol{B}_i^{\mathrm{T}}\, \mathbf{c}) \qquad (3')$$

we get:

$$\boldsymbol{f}^* = \boldsymbol{f}\big(\boldsymbol{c}(\boldsymbol{f}^*)\big)$$

If the congestion model $\boldsymbol{c}(\boldsymbol{f})$ and the loading map $\boldsymbol{f}(\boldsymbol{c})$ are continuously differentiable, as is commonly the case, their Jacobian matrices $\mathbf{J}_c(\boldsymbol{f})$ = Jac[$\boldsymbol{c}(\boldsymbol{f})$] and $\mathbf{J}_f(\boldsymbol{c})$ = Jac[$\boldsymbol{f}(\boldsymbol{c})$] are well defined. Now we introduce equation 15 below which can be proved (Theorem 2) to be the *general uniqueness* condition.

$$|\, \mathbf{I} - \mathbf{J}_f\, \mathbf{J}_c \,| \neq 0 \qquad (15)$$

**THEOREM 2 (uniqueness)**: Under the hypotheses of continuously differentiable congestion model $\boldsymbol{c}(\boldsymbol{f})$ and loading map $\boldsymbol{f}(\boldsymbol{c})$, condition 15 ensures the uniqueness of the equilibrium.

<u>*Proof*</u>. *Let us note that* $|\, \mathbf{I} - \mathbf{J}_f\ \mathbf{J}_c \,|$ *is the Jacobian determinant of the function* $\phi(\mathbf{f}) = \mathbf{f} - \mathbf{f}\big(\mathbf{c}(\boldsymbol{f})\big)$. *From condition 15 the Jacobian of* $\phi(\boldsymbol{f})$ *is non-singular. Thus the function is invertible as a consequence of the global inverse function theorem (Hadamard, 1906). Hence any equilibrium is given by* $\mathbf{f}^* = \phi^{-1}(\mathbf{0})$. *Provided that* $\phi(\cdot)$ *is invertible, it is also injective, and the inverse* $\phi^{-1}(\cdot)$ *is injective as well. This means that there exists at most one value that corresponds to the equilibrium* ($\mathbf{f}^* = \phi^{-1}(\mathbf{0})$).

As a consequence of Theorem 2, Corollary 1 below can be proved, showing that condition 16 below is a sufficient condition for uniqueness.

$$\mathrm{Re}[\mathrm{eigenvalue}(\mathbf{J}_f\, \mathbf{J}_c)] < 1 \qquad (16)$$

**COROLLARY 1**: Inequality 16 is a uniqueness condition since it is a sufficient condition for 15.

<u>*Proof*</u>.

$$\mathrm{Re}[\mathrm{eigenvalue}(\mathbf{J}_f\, \mathbf{J}_c)] < 1 \Rightarrow 1 - \mathrm{Re}[\mathrm{eigenvalue}(\mathbf{J}_f\, \mathbf{J}_c)] > 0$$

*Hence:*

$$|\, \mathbf{I} - \mathbf{J}_f\, \mathbf{J}_c \,| > 0 \;\Rightarrow\; |\, \mathbf{I} - \mathbf{J}_f\, \mathbf{J}_c \,| \neq 0$$

Conditions 15 and 16 above are more general than common uniqueness conditions

(HYP 4 and HYP 5 below) as shown by Theorem 3.

**HYP 4:** the congestion model (is continuously differentiable and) has a positive definite Jacobian matrix $\mathbf{J}_c(\boldsymbol{f}) = \text{Jac}[\boldsymbol{c}(\boldsymbol{f})]$, which is not necessarily symmetric.

**HYP 5:** Function $\boldsymbol{f}(\boldsymbol{c})$ (is continuously differentiable and) has a symmetric negative semi-definite Jacobian matrix $\mathbf{J}_f(\boldsymbol{c}) = \text{Jac}[\boldsymbol{f}(\boldsymbol{c})]$.

**THEOREM 3**: Hypotheses HYP 3, HYP 4 and HYP 5 together are sufficient conditions for both the uniqueness conditions 15 and 16.
*Proof. From HYP 4 and HYP 5, the real part of the eigenvalues of the product matrix $\boldsymbol{J}_f(c)$ $\boldsymbol{J}_c(f)$ is non-positive:* $\text{Re}[\text{eigenvalue}(\mathbf{J}_f\,\mathbf{J}_c)] \leq 0$

$$\text{Re}[\text{eigenvalue}(\mathbf{J}_f\,\mathbf{J}_c)] \leq 0 \;\Rightarrow\; \text{Re}[\text{eigenvalue}(\mathbf{J}_f\,\mathbf{J}_c)] < 1$$

*Thus, condition 16 holds and, since COROLLARY 1,condition 15 holds too.*

It is worth noting that if HYP 3 holds, HYP 5 is actually a mild assumption, as it is ensured if all route choice models ($\mathbf{p}_i(\cdot)\ \forall\, i$) are based on invariant RUM (random utility model) specifications. Indeed, the Jacobian of the overall loading map, as evident from equation 3', is a linear combination with non-negative coefficients $(1 - m°_i)$ of the Jacobian matrices $\mathbf{J}^i_p = \text{Jac}[\mathbf{p}_i(\cdot)]\ \forall\, i$. For further details, see Cantarella (1997).

### 3.2 Dynamic process under ATIS: convergence properties

For all demonstrations in this section we assume that HYP 1 and HYP 2 are always met. Moreover, it is assumed that the information is fully accurate ($\boldsymbol{r}_i^t = \boldsymbol{B}_i^T\, c(\boldsymbol{f}^t)\ \forall\, i, \forall\, t$). As full accuracy is a special case of that discussed in section 3.1, all the results obtained in that section also hold. Moreover, it should be noted that in the case of fully accurate information HYP 3 is not a simplifying assumption. It holds in an exact way since, in the case of fully accurate information, compliance is fixed and equal to market penetration ($\eta$).

The dynamic process is identified by equations 13.a and 13.b, here rewritten by making the loading maps of compliant and non-compliant travellers explicit:

$$\boldsymbol{x}^t = \beta\, \mathbf{c}(\boldsymbol{f}^{t-1}) + (1 - \beta)\, \boldsymbol{x}^{t-1} \qquad \text{with} \quad \boldsymbol{x}^{t=0} = \boldsymbol{x}^0 \tag{17.a}$$

$$\boldsymbol{f}^t = \alpha\, \Sigma_i\, d_i\, \boldsymbol{B}_i \left( \eta\, \boldsymbol{\pi}_i(\boldsymbol{B}_i^T\, \mathbf{c}(\boldsymbol{f}^t)) + (1-\eta)\, \mathbf{p}_i(\boldsymbol{B}_i^T\, \boldsymbol{x}^t) \right) + (1 - \alpha)\, \boldsymbol{f}^{t-1} \qquad \text{with} \quad \boldsymbol{f}^{t=0} = \boldsymbol{f}^0 \tag{17.b}$$

To analyse model 17 we need a proper transition function in which today's state variables ($\boldsymbol{f}^t$, $\boldsymbol{x}^t$) only appear as such on the left side. To this aim, consider a function $\boldsymbol{\psi}(\mathbf{f}) = \mathbf{f} - \alpha\, \eta\, \Sigma_i\, d_i\, \boldsymbol{B}_i\, (\boldsymbol{\pi}_i(\boldsymbol{B}_i^T\, \mathbf{c}(\boldsymbol{f})))$ and assume the inverse function $\boldsymbol{\psi}^{-1}$ exists (conditions in Theorem 4). At any given point $\mathbf{y}$, the inverse function provides the flow $\mathbf{f} = \boldsymbol{\psi}^{-1}(\mathbf{y})$ that ensures $\mathbf{f} - \alpha\, \eta\, \Sigma_i\, d_i\, \boldsymbol{B}_i\, (\mathbf{p}_i(\boldsymbol{B}_i^T\, \mathbf{c}(\boldsymbol{f}))) = \mathbf{y}$. Thus the transition functions of the dynamic process can be written as:

$$\boldsymbol{x}^t = (1 - \beta)\, \boldsymbol{x}^{t-1} + \beta \mathbf{c}(\boldsymbol{f}^{t-1}) \tag{18.a}$$

$$\boldsymbol{f}^t = \boldsymbol{\psi}^{-1}\big(\mathbf{y}(\boldsymbol{f}^{t-1}, \boldsymbol{x}^{t-1})\big) \tag{18.b}$$

where $\mathbf{y}(\boldsymbol{f}^{t-1}, \boldsymbol{x}^{t-1}) = \alpha\,(1-\eta)\,\Sigma_i\, d_i\, \boldsymbol{B}_i\, \mathbf{p}_i\big(\boldsymbol{B}_i^{T}\,(\,(1-\beta)\,\boldsymbol{x}^{t-1} + \beta\mathbf{c}(\boldsymbol{f}^{t-1})\,)\,\big) + (1-\alpha)\,\boldsymbol{f}^{t-1}$

Equations 18.a and 18.b can be considered together in order to write the transition function $\boldsymbol{\varphi}(\cdot)$:

$$(\mathbf{x}^{t}, \mathbf{f}^{t}) = \boldsymbol{\varphi}(\mathbf{x}^{t-1}, \mathbf{f}^{t-1}) \tag{19}$$

The existence of the inverse function $\boldsymbol{\psi}^{-1}$ can be proved under the hypothesis that:

$$\mathrm{Re}[\mathrm{eigenvalue}(\mathbf{J}_{fc}\,\mathbf{J}_{c})] < 1 \tag{16'}$$

where $\mathbf{J}_{fc} = \mathbf{Jac}[\Sigma_i\, d_i\, \boldsymbol{B}_i\,(\boldsymbol{\pi}_i(\boldsymbol{B}_i^{T}\,\mathbf{c}(\boldsymbol{f})))]$ and plays the role of $\mathbf{J}_f$ in equation 16.

**THEOREM 4**: Under condition 16' the inverse function $\boldsymbol{\psi}^{-1}$ exists. Hence the transition function, as identified by equations 18.a and 18.b (or by equation 19), can be actually expressed.

*Proof. The Jacobian determinant of function $\boldsymbol{\psi}(\boldsymbol{f})$ can be computed as:*

$$|\,\mathrm{Jac}[\boldsymbol{\psi}(\mathbf{f})]\,| = |\,\mathbf{I} - \alpha\,\eta\,\mathbf{J}_{fc}\,\mathbf{J}_{c}\,|$$

*The real part of the eigenvalues of matrix* $[\mathbf{I} - \alpha\,\eta\,\boldsymbol{J}_{fc}\boldsymbol{J}_c\,]$ *can be written as:*

$$\mathrm{Re}[\mathrm{eigenvalue}(\mathbf{I} - \alpha\,\eta\,\boldsymbol{J}_{fc}\boldsymbol{J}_c)] = 1 - \alpha\,\eta\;\mathrm{Re}[\mathrm{eigenvalue}(\mathbf{J}_{fc}\,\mathbf{J}_c)]$$

*Given condition 16'* and *that by definition* $0 < \alpha\,\eta \le 1$*:*

$1 - \alpha\,\eta\;\mathrm{Re}[\mathrm{eigenvalue}(\mathbf{J}_{fc}\,\mathbf{J}_c)] > 0$ *and* $\mathrm{Re}[\mathrm{eigenvalue}(\mathbf{I} - \alpha\,\eta\,\boldsymbol{J}_{fc}\boldsymbol{J}_c)] > 0$

*Then:*

$$|\,\mathbf{I} - \alpha\,\eta\,\boldsymbol{J}_{fc}\boldsymbol{J}_c\,| > 0 \tag{20}$$

*Previous condition (20) ensures that the Jacobian of function* $\boldsymbol{\psi}(\mathbf{f})$ *is everywhere non-singular. As a consequence, thanks to the global inverse function theorem, the inverse function* $\boldsymbol{\psi}^{-1}(\cdot)$ *exists*, with

$$\mathrm{Jac}[\boldsymbol{\psi}^{-1}(\mathbf{y})]_{y=y(f^{t-1})=y(f^{t})} = [\,\mathrm{Jac}[\boldsymbol{\psi}(\boldsymbol{f}^{t})]\,]^{-1} = [\mathrm{I} - \alpha\,\eta\,J_{fc}\;J_c\,]^{-1} \tag{21}$$

It can be shown that condition 16' is more general than the usually applied sufficient conditions HYP 4 above and HYP 6 below:

**HYP 6:** The *route-spreading coefficients* map $\pi_i(\cdot)$ is a continuously differentiable function with respect to ATIS information, with a negative semi-definite Jacobian matrix $\mathbf{J}^{i}_{\pi}(\boldsymbol{r}_i) = \mathrm{Jac}[\pi_i(\boldsymbol{r}_i)]\ \forall\, i$.

Note that from HYP 6, the fully accurate ATIS being ($\boldsymbol{r}_i = \boldsymbol{B}_i^{T}\,c\ \forall\, i$), the function $\pi_i(\mathbf{B}_i^{T}c)$ also has a negative semi-definite Jacobian with respect to link costs. Hence, similarly to what was stated in the previous section, it is trivial to show that if HYP 4 and HYP 6 hold, then 16' is satisfied.

It is also worth noting that HYP 6 is satisfied in the case of descriptive ATIS if the route-choice function for compliant travellers is based on a RUM. In the case of prescriptive ATIS this also holds if the *route-spreading coefficients* are computed as a function of estimated travel costs by means of a RUM.

Now, using $J_{fp}$ to indicate the Jacobian of the loading function of the uncompliant travellers ($\Sigma_i\ d_i\ \boldsymbol{B}_i\ \mathbf{p}_i(\boldsymbol{B}_i^T\ \boldsymbol{x}^t)$), the Jacobian of the transition function φ(·) can be calculated, also employing equation 21 for Jac[$\boldsymbol{\psi}^{-1}(\mathbf{y})$], as:

$$Jac[\varphi(.)] = \begin{bmatrix} (1-\beta)I & \beta J_c \\ \alpha(1-\eta)(1-\beta)[I-\alpha\,\eta\,J_{fc}J_c]^{-1}J_{fp} & [I-\alpha\,\eta\,J_{fc}J_c]^{-1}[\alpha(1-\eta)\beta\,J_{fp}J_c+(1-\alpha)I] \end{bmatrix} \tag{22}$$

The Jacobian of the transition function can be analysed in order to check whether the dynamic process 13 (or rewritten as 17) is dissipative; if so, the dynamic process converges towards some attractor (possibly a fixed point). Such a dissipative characteristic is proven in Theorem 5 below.

**THEOREM 5**: Under condition 16', the dynamic process 13, is dissipative.
*<u>Proof</u>. The determinant of any block matrix (say, matrix L) can be computed as:*

$$|L| = \begin{vmatrix} A & B \\ C & D \end{vmatrix} = |A|\ |D - C\,A^{-1}\,B|$$

*Thus, the Jacobian determinant of* |Jac[φ(·)]| *identified by equation 22 can be computed as:*

$$\begin{aligned} |Jac[\varphi(\cdot)]| &= (1-\beta)^n\,|\,[\mathbf{I}-\alpha\,\eta\,\boldsymbol{J}_{fc}\boldsymbol{J}_c]^{-1}\,[\alpha\,\beta\,(1-\eta)\,\boldsymbol{J}_{fp}\boldsymbol{J}_c + (1-\alpha)\,\mathbf{I}\ + \\ &\quad - (\alpha\,\beta\,(1-\eta)\,(1-\beta))/(1-\beta)\ \boldsymbol{J}_{fp}\boldsymbol{J}_c]\,| = \\ &= (1-\beta)^n\,|\ [\mathbf{I}-\alpha\,\eta\,\boldsymbol{J}_{fc}\boldsymbol{J}_c]^{-1}\,(1-\alpha)\,\mathbf{I}\,| = (1-\alpha)^n\,(1-\beta)^n\ |\,[\mathbf{I}-\alpha\,\eta\,\boldsymbol{J}_{fc}\boldsymbol{J}_c]^{-1}\,| \end{aligned} \tag{23}$$

*where n is the dimension of the link flows vector.*
*In order to prove the theorem, the Jacobian determinant of the transition function should be less than one. Hence, given equation 23:*

$$(1-\alpha)^n\,(1-\beta)^n\ |\,[\mathbf{I}-\alpha\,\eta\,\boldsymbol{J}_{fc}\boldsymbol{J}_c]^{-1}\,| < 1$$

*By simple algebra, the equation becomes:*

$$(1-\alpha)^n\,(1-\beta)^n < |\,[\mathbf{I}-\alpha\,\eta\,\boldsymbol{J}_{fc}\boldsymbol{J}_c]\,| \tag{24}$$

*When η = 0 equation (24) becomes* $(1-\alpha)^n\ (1-\beta)^n < 1$ *which always holds and thus the theorem is proved.*
*When η > 0, consider the eigenvalues of* $[\mathbf{I} - \alpha\ \eta\ \boldsymbol{J}_{fc}\ \boldsymbol{J}_c]$*, that are given by* $1 - \alpha\ \eta$ *[*eigenvalue($\boldsymbol{J}_{fc}\boldsymbol{J}_c$)]. Because of condition 16', *Re[*eigenvalue($\boldsymbol{J}_{fc}\boldsymbol{J}_c$)] < 1*, then* $1 - \alpha\,\eta$ *Re[*eigenvalue($\boldsymbol{J}_{fc}\boldsymbol{J}_c$)] > $(1-\alpha\,\eta)$.
*Given that* $(1-\alpha\,\eta) < 1$, *then* $(1-\alpha\,\eta) > (1-\alpha\,\eta)^n$ *and thus it results a fortiori that:*
$1 - \alpha\,\eta$ *Re[*eigenvalue($\boldsymbol{J}_{fc}\boldsymbol{J}_c$)] > $(1-\alpha\,\eta)^n$
*Thus* $|\,[\mathbf{I}-\alpha\,\eta\,\boldsymbol{J}_{fc}\boldsymbol{J}_c]\,| > (1-\alpha\,\eta)^n$. *Given that* $(1-\alpha\,\eta)^n > (1-\alpha)^n$, *it holds a fortiori that* $|\,[\mathbf{I}-\alpha\,\eta\,\boldsymbol{J}_{fc}\boldsymbol{J}_c]\,| > (1-\alpha)^n$.
*Finally, given that* $(1-\beta)^n < 1$ *and so* $(1-\alpha)^n \geq (1-\alpha)^n\,(1-\beta)^n$, *then a fortiori*
$|\,[\mathbf{I}-\alpha\,\eta\,\boldsymbol{J}_{fc}\boldsymbol{J}_c]\,| > (1-\alpha)^n\,(1-\beta)^n$. *Thus condition 24 holds and the theorem is proved also for η > 0.*

## 3.3 Equilibrium stability under ATIS

The stability of the equilibrium of the dynamic process could be investigated through the analysis of eigenvalues (λ) of the Jacobian of the transition function **φ**(·) (equation

19) at the equilibrium point. These are the solutions of the equation:

$$|\boldsymbol{J}_{\varphi}-\lambda\,\boldsymbol{I}|=\begin{bmatrix}(1-\beta-\lambda)\boldsymbol{I} & \beta\boldsymbol{J}_{\mathrm{c}}\\ \alpha(1-\eta)(1\text{-}\beta)[\boldsymbol{I}-\alpha\,\eta\,\boldsymbol{J}_{fc}\boldsymbol{J}_{\mathrm{c}}]^{-1}\,\boldsymbol{J}_{fp} & [\boldsymbol{I}-\alpha\,\eta\,\boldsymbol{J}_{fc}\boldsymbol{J}_{\mathrm{c}}]^{-1}[\alpha(1-\eta)\beta\,\boldsymbol{J}_{fp}\boldsymbol{J}_{c}+(1-\alpha)\boldsymbol{I}]-\lambda\boldsymbol{I}\end{bmatrix}=0$$

The previous determinant can easily be calculated by considering the block structure as:

$$|\boldsymbol{J}_{\varphi}-\lambda\,\boldsymbol{I}|=(1-\beta-\lambda)^{n}\,|[\boldsymbol{I}-\alpha\,\eta\,\boldsymbol{J}_{fC}\boldsymbol{J}_{\mathrm{c}}]^{-1}[\alpha(1-\eta)\beta\,\boldsymbol{J}_{fp}\boldsymbol{J}_{c}+(1-\alpha)\boldsymbol{I}-\frac{\alpha\beta(1-\eta)(1\text{-}\beta)}{(1-\beta-\lambda)}\boldsymbol{J}_{fp}\boldsymbol{J}_{\mathrm{c}}\,]-\lambda\boldsymbol{I}\,|=$$

$$=(1-\beta-\lambda)^{n}\,|[\boldsymbol{I}-\alpha\,\eta\,\boldsymbol{J}_{fC}\boldsymbol{J}_{\mathrm{c}}]^{-1}[(1-\alpha)\boldsymbol{I}+\alpha\beta(1-\eta)\,\boldsymbol{J}_{fp}\boldsymbol{J}_{c}-\frac{\alpha\beta(1-\eta)(1\text{-}\beta)}{(1-\beta-\lambda)}\boldsymbol{J}_{fp}\boldsymbol{J}_{c}-\lambda\,(\boldsymbol{I}-\alpha\,\eta\,\boldsymbol{J}_{fC}\boldsymbol{J}_{\mathrm{c}})\,\boldsymbol{I}\,|=$$

$$=(1-\beta-\lambda)^{n}\,|\frac{1}{(1-\beta-\lambda)}[\boldsymbol{I}-\alpha\,\eta\,\boldsymbol{J}_{fC}\boldsymbol{J}_{\mathrm{c}}]^{-1}[(1-\alpha-\lambda)(1-\beta-\lambda)\boldsymbol{I}-\lambda\alpha\beta(1-\eta)\,\boldsymbol{J}_{fp}\boldsymbol{J}_{c}+(1-\beta-\lambda)\lambda\,\alpha\,\eta\,\boldsymbol{J}_{fc}\boldsymbol{J}_{c}]\,|=$$

$$=(1-\beta-\lambda)^{n}\,|\frac{1}{(1-\beta-\lambda)}[\boldsymbol{I}-\alpha\,\eta\,\boldsymbol{J}_{fC}\boldsymbol{J}_{\mathrm{c}}]^{-1}[(1-\alpha-\lambda)(1-\beta-\lambda)\boldsymbol{I}-\lambda\alpha\beta(1-\eta)\,\boldsymbol{J}_{fp}\boldsymbol{J}_{c}+(1-\beta-\lambda)\lambda\,\alpha\,\eta\,\boldsymbol{J}_{fc}\boldsymbol{J}_{c}]\,|=$$

$$=\frac{1}{|\boldsymbol{I}-\alpha\,\eta\,\boldsymbol{J}_{fC}\;\boldsymbol{J}_{\mathrm{c}}\,|}\,|(1-\alpha-\lambda)(1-\beta-\lambda)\boldsymbol{I}-[\lambda\alpha\beta(1-\eta)\boldsymbol{J}_{fp}\boldsymbol{J}_{c}-(1-\beta-\lambda)\lambda\,\alpha\,\eta\;\;\boldsymbol{J}_{fc}\boldsymbol{J}_{c}]\,|$$

In Theorem 4, equation 21 was proved, ensuring that $|\,\mathbf{I}-\alpha\,\eta\,\boldsymbol{J}_{fc}\boldsymbol{J}_{c}\,|$ is strictly positive. Thus the eigenvalues λ are the solution of the equation:

$$|(1-\alpha-\lambda)(1-\beta-\lambda)\boldsymbol{I}-[\lambda\alpha\beta(1-\eta)\boldsymbol{J}_{fp}\boldsymbol{J}_{c}-(1-\beta-\lambda)\lambda\,\alpha\,\eta\;\;\boldsymbol{J}_{fc}\boldsymbol{J}_{c}]\,|=0 \qquad (25)$$

Here a further hypothesis is introduced.

**HYP 7:** The difference between how compliant and non-compliant (or non-equipped) travellers react to travel times (supplied by the systems or learned from the network) is negligible: $\mathbf{p}_i(\cdot) \cong \pi_i(\cdot)$ or $\boldsymbol{\Delta p}_i \cong \mathbf{0}\ \forall\, i$.

As a direct consequence of HYP 7, the equilibrium can be expressed by equation 10 with $\boldsymbol{\Delta p}_i \cong \mathbf{0}\ \forall\, i$, that is by equation 11 and thus it is a standard stochastic user equilibrium (SUE). Moreover, the dynamic process refers to equations 14.a and 14.b and Theorem 6 below can be proved.

**THEOREM 6**: Under HYP 7, the eigenvalues ($\lambda$) of the Jacobian of the transition function of the dynamic process, as identified by equations 14.a and 14.b, can be related to the eigenvalues ($\gamma$) of the product matrix $\mathbf{J}_{fc}\mathbf{J}_{c}$. In particular, for each eigenvalue $\gamma_k$ two eigenvalues ( $\lambda_{k1}\,\lambda_{k2}$ ) can be obtained, according to equation 26 below:

$$(1-\alpha-\lambda)\,(1-\beta-\lambda)=[\lambda\,\alpha\,\beta\,(1-\eta)-(1-\beta-\lambda)\,\lambda\,\alpha\,\eta\,]\,\gamma_k \qquad (26)$$

*<u>Proof</u>. If HYP 7 holds and the dynamic process is identified by equations 14.a and 14.b,* $\mathbf{J}_{fc}=\mathbf{J}_{fp}$ *and thus we can substitute one with the other in equation 25 above. Assume that equation 26 holds and substitute it into equation 25:*

$$|\,(1-\alpha-\lambda)\,(1-\beta-\lambda)\,\mathbf{I}-[\lambda\,\alpha\,\beta\,(1-\eta)-(1-\beta-\lambda)\,\lambda\,\alpha\,\eta\,]\,\mathbf{J}_{fc}\mathbf{J}_{c}\,|=$$
$$=\;[\lambda\,\alpha\,\beta\,(1-\eta)-(1-\beta-\lambda)\,\lambda\,\alpha\,\eta\,]^{n}\,|\,\gamma_k\,\mathbf{I}-\mathbf{J}_{fc}\mathbf{J}_{c}\,|$$

*However,* $|\,\gamma_k\,\mathbf{I}-\mathbf{J}_{fc}\mathbf{J}_{c}\,|$ *is null by definition of eigenvalue γ. Then the searched eigenvalues λ can be obtained by solving equation 26. As the equation is quadratic, for each γ, two λ can be obtained. Formally, they can be represented by the functions in equation 27 below:*

$$\lambda_{k1}=L_1\,(\alpha,\beta,\eta,\gamma_k) \qquad \lambda_{k2}=L_2\,(\alpha,\beta,\eta,\gamma_k) \qquad (27)$$

*where α, β and η are real-valued scalars, while* $\gamma_k$ *is complex.*

The stability region of the dynamic process can be identified by imposing proper conditions to be respected by the eigenvalues $\lambda$. These conditions can be solved by considering the relation between $\lambda$ and eigenvalues $\gamma$, as established by Theorem 6 above. This procedure is applied by Theorem 7 below.

**THEOREM 7**: The stability region of the fixed point attractor of the dynamic process 14 is described by a quartic identified by the dynamic process parameters ($\alpha$ and $\beta$) and by ATIS market penetration ($\eta$).

*<u>Proof</u>. The stability condition to be imposed is that the maximum modulus of the λs (expressed as a function of the eigenvalues γ as in equation 27 above in Theorem 6) is less than 1:*

$$\max_k \{\max\{|\lambda_{k1}, \lambda_{k2}|\}\} < 1 \qquad (28)$$

*Decomposing $\gamma_k$ in its real and imaginary parts ($\gamma^R_k$, $\gamma^I_k$), and making functions $L_1(\alpha, \beta, \eta, \gamma_k)$ and $L_2(\alpha, \beta, \eta, \gamma_k)$ explicit, condition 28 can be written as:*

$$\frac{\left[\frac{\vartheta}{\alpha\cdot\beta}\gamma^R_k - \left(1-\left(\frac{1+(1-\alpha)\cdot(1-\beta)}{\alpha\cdot\beta}\right)\right)\right]^2}{\left(\frac{1+(1-\alpha)\cdot(1-\beta)}{\alpha\cdot\beta}\right)^2} + \frac{\left[\frac{\vartheta}{\alpha\cdot\beta}\gamma^I_k\right]^2}{\left(\frac{1-(1-\alpha)\cdot(1-\beta)}{\alpha\cdot\beta}\right)^2} + \frac{\eta}{[1-(1-\alpha)^2\cdot(1-\beta)^2]^2}\cdot\psi\left(\gamma^R_k, \gamma^I_k, \alpha, \beta, \eta\right) < 1 \qquad (29)$$

*where*

$$\psi\left(\gamma^R_k, \gamma^I_k, \alpha, \beta, \eta\right) = \psi_1\left(\gamma^R_k, \gamma^I_k, \alpha, \beta, \eta\right)\cdot\psi_2\left(\gamma^R_k, \gamma^I_k, \alpha, \beta, \eta\right) + \psi_3\left(\gamma^R_k, \gamma^I_k, \alpha, \beta, \eta\right)$$

$$\psi_1\left(\gamma^R_k, \gamma^I_k, \alpha, \beta, \eta\right) = \alpha^2\cdot\eta\cdot\left({\gamma^R_k}^2 + {\gamma^I_k}^2\right) - 2\cdot\alpha\cdot\gamma^R_k$$

$$\psi_2\left(\gamma^R_k, \gamma^I_k, \alpha, \beta, \eta\right) = -\alpha^2\cdot\eta^2\cdot\left({\gamma^R_k}^2 + {\gamma^I_k}^2\right) + 2\cdot\alpha\cdot\eta\cdot\gamma^R_k - 2\cdot[1-(1-\alpha)^2\cdot(1-\beta)^2]^2 +$$
$$+\left(2-\alpha-\beta+\vartheta\cdot\gamma^R_k\right)^2 + \vartheta^2\cdot{\gamma^I_k}^2$$

$$\psi_3\left(\gamma^R_k, \gamma^I_k, \alpha, \beta, \eta\right) = \alpha\cdot(1-\alpha)\cdot(1-\beta)\cdot$$
$$\cdot\left\{2\cdot\left[\left(2-\alpha-\beta+\vartheta\cdot\gamma^R_k\right)^2 - \vartheta^2\cdot{\gamma^I_k}^2\right]\cdot\gamma^R_k + 4\cdot\left(2-\alpha-\beta+\vartheta\cdot\gamma^R_k\right)\cdot\vartheta\cdot{\gamma^I_k}^2\right\}$$

$$\theta = \alpha\cdot\beta\cdot(1-\eta) - \alpha\cdot\eta\cdot(1-\beta)$$

*By equation 29, the points of coordinates $\gamma^R_k$, $\gamma^I_k$ , where represented in the complex plane (or Argand plane), have to be within a quartic whose size depends on parameters α and β of the dynamic process and on ATIS market penetration (η). This quartic identifies/bounds the stability region of the attractor of the dynamic process.*

Importantly, in equation 29 of Theorem 7 in the absence of ATIS ($\eta = 0$), the quartic is an ellipse, as expected and described in the literature (e.g. Cantarella and Cascetta, 1995). The quartic can be explored with respect to ATIS market penetration ($\eta$) once $\alpha$ and $\beta$ have been fixed. For instance, assuming $\alpha = 0.5$ and $\beta = 0.6$, four different patterns can be viewed in Figures 1.a to 1.d below for $\eta = \{0.2, 0.29, 0.32, 0.57\}$, where the non-ATIS ellipse is also shown in red. From figure 1.a to 1.c the stability region grows and rapidly includes the two quadrants with a negative real part. These two quadrants are the actually interesting part, given that under HYP 4 and HYP 6 the eigenvalues $\gamma_k$ of a negative semi-definite matrix cannot have a positive real part. At $\eta$=0.57 the stability region covers almost all the Argand plane (Figure 1.d).

Unfortunately, a direct interpretation for the variables $\gamma^R$ and $\gamma^I$ does not exist; however, some general rules can be given for the absolute value of each eigenvalue $|\gamma|$. It increases as demand flows increase, as route choice dispersion decreases (toward Wardrop choice behavior), and as arc cost function sensitivity to arc flow increases. It can be analytically shown for the peculiar (and extreme) case that $\alpha = \beta = 1$, which represents a critical (likely unreal) case for stability in absence of ATIS, that any value $\eta \geq 0.5$ ensures the stability region covers the whole two quadrants with negative real parts and thus ensures the stability of the equilibrium.

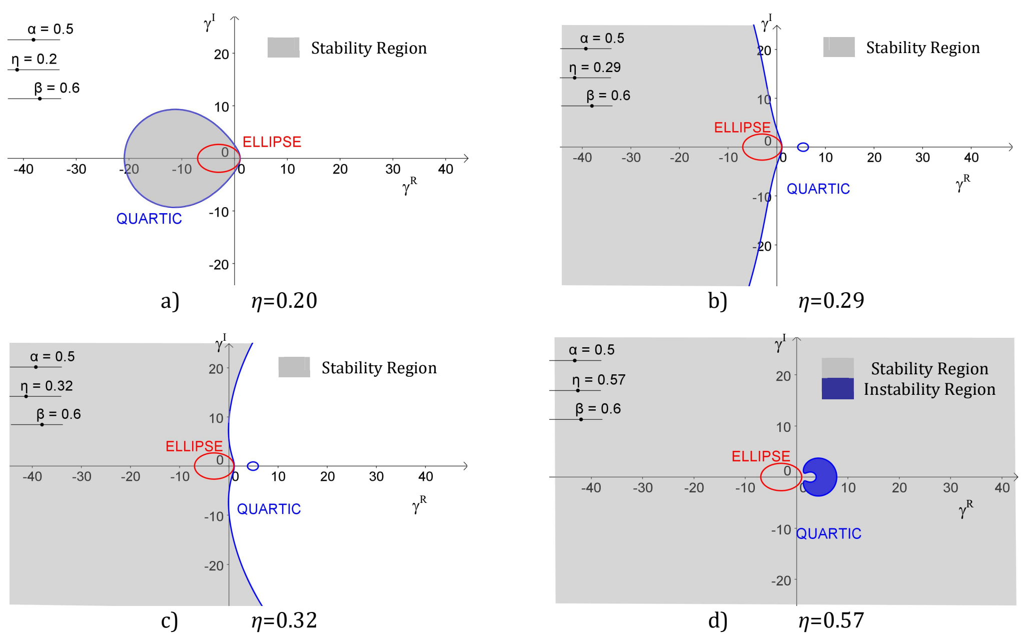


*Figure 1 – Stability domain depending on ATIS market penetration ($\alpha$ = 0.5 and $\beta$=0.6)*

## 4 Discussion

In the previous sections a dynamic process under ATIS was formalised, as well as its fixed-point (equilibrium) attractors and their theoretical properties. Importantly, a fully accurate ATIS tends, at equilibrium, to a non-ATIS configuration and the difference between ATIS and non-ATIS equilibria is due to the value of ATIS market penetration ($\eta$) and on the route-spreading difference. Similarly, if compliance falls short (inaccurate information, $m \cong 0$), the ATIS equilibrium collapses to that of non-ATIS. Things go differently for the dynamic process. In the case of very inaccurate information ($m \cong 0$) the dynamic process still coincides with that of non-ATIS, while in the case of fully accurate information the dynamic process differs greatly from the non-informed one, even in the case of a null route-spreading difference map. In this latest case, the most evident effect is towards the stability of the equilibrium, and the quartic which represents the stability region rapidly increases with market penetration. This effect allows optimal signal setting conditions to be both attained and maintained, otherwise unstable without ATIS (Bifulco et al. 2014a). On the other hand, ATIS does not appear suitable (Bifulco et al. 2007) to force traffic patterns towards conditions (i.e. the system optimum) different from user equilibria. In this context, the full accuracy of the information plays a crucial role, as it can provide maximum compliance. In a matter of days, this leads to increasing market penetration of the system. Furthermore, the compliance attained in recurrent conditions can be useful to manage non-recurrent conditions. Indeed, in case of non-recurrent conditions, it is not possible for travellers to confirm the accuracy of the received information. As a consequence, the compliance is not modified by non-recurrent conditions and the one reached in recurrent condition is applied by travellers. If the reached compliance is high, then the information can be effectively applied (for the time the system is in non-recurrent conditions with modified and unexpected – by travellers – boundary conditions) in order to drive the network toward a system-optimum. For this reason, fully-accurate information in recurrent conditions can be viewed as a target strategy; there is no guarantee that in practical applications such a target can be reached, and the actual information can be viewed as fully accurate with some errors.

Even if the theoretical analysis of non-fully accurate information is not addressed in this paper, a few numerical examples can be given. We assume that the estimated information is a random variable which, on average, is equal to the accurate information:

$$\boldsymbol{r}_i^t = \boldsymbol{B}_i^T c^t + \boldsymbol{err}_i \qquad \forall\, i, \forall\, t$$

where $\boldsymbol{err}_i$ are (multivariate) normal random variables.

In this case a regular stochastic process is obtained, if the mean converges to a fixed point. In the following the corresponding (deterministic) mean process is discussed.

We assume that compliance is related to the variation coefficient (standard deviation on mean ratio) of $\boldsymbol{r}_i^t$. For the sake of simplicity and without loss of generality with respect to our aims, we use a linear function.

$$\mathrm{m}(cv_r^i, \eta) = \eta - \frac{\eta}{cv_{cr}} cv_r^i$$

where $cv_{cr}$ is the value of the coefficient of variation of the information that makes the compliance null (assumed to be 0.5 in our numerical analysis below).

Assuming that the *route-spreading coefficients* map is based on a random utility model where the perceived utility is $\boldsymbol{U}_i^t = \boldsymbol{r}_i^t + \boldsymbol{\tau}_i$ , with $\boldsymbol{\tau}_i$ distributed according to the multinomial-logit approach, then $\boldsymbol{\pi}$ can be computed as a function of $\mathbf{c}$ by using a mixed-logit approach.

The model was adopted to numerically test the equilibrium and its stability on a simple test network. The network in question is the same as in Bifulco et al. (2014a).

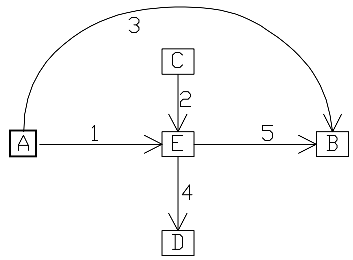


*Figure 2 – The toy network*

The origin/destination flows are constant over time and equal for both O/D pairs (d1 = d2 = d = 500 vehic/h). Congestion is simulated by a polynomial cost function:

$$c_a(\boldsymbol{f}) = tr_a(\boldsymbol{f}) + tw_a(\boldsymbol{f})$$

The running time $tr_a$ is computed by using a Davidson function (Davidson, 1966), while the waiting time $tw_a$ with a Webster function (Webster and Cobbe;1966).

*Table 1 – Network characteristics*

| Link | Nodes | Saturation flow (vehic/h) | running+waiting free-flow time (s) |
|---|---|---|---|
| 1 | AE | 900 | 100 |
| 2 | CE | 3000 | 10 |
| 3 | AB | 1000 | 350 |
| 4 | ED | 5000 | 10 |
| 5 | EB | 1000 | 100 |

A desired network configuration was selected; it corresponds to the optimal signal-setting pattern that minimises the network travel time. The desired configuration proves unstable. The minimum market penetration required for stabilising the network was calculated for different values of $cv_r$. Figure 3 below represents the maximum modulus of the eigenvalues (the spectral radius) of the Jacobian of the transition function at equilibrium as a function of market penetration and for different accuracy levels ($cv_r$).

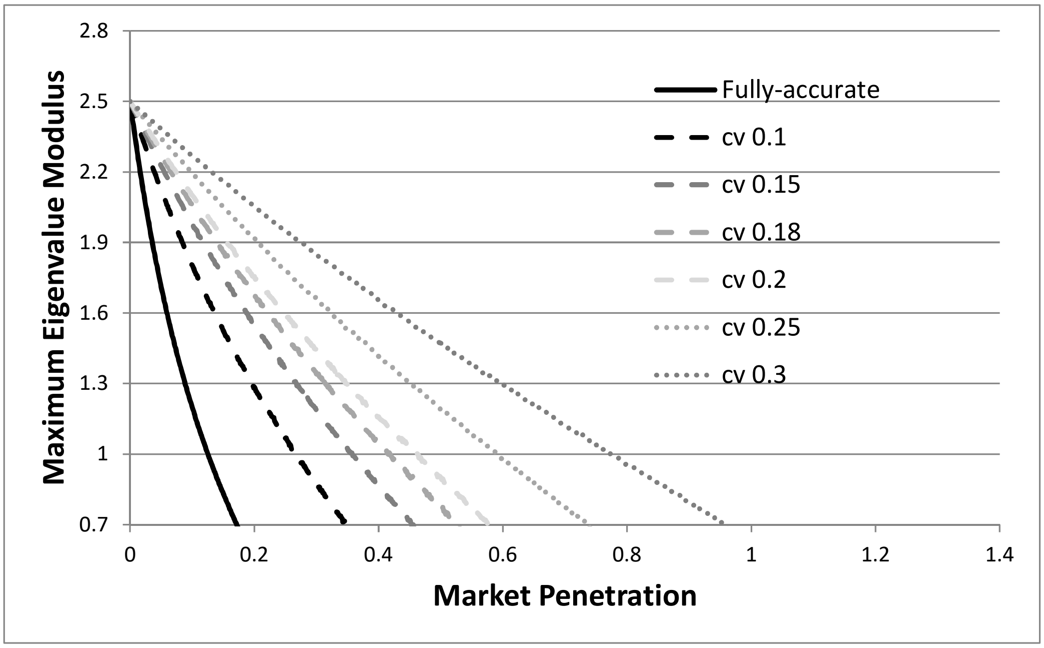


*Figure 3 – Variation of the spectral radius as a function of the ATIS market penetration*

The value of market penetration that corresponds to the value 1 of the maximum eigenvalue modulus is the minimum required market penetration for stability. It is shown in figure 4 below as a function of the information (in)accuracy $cv_r$.

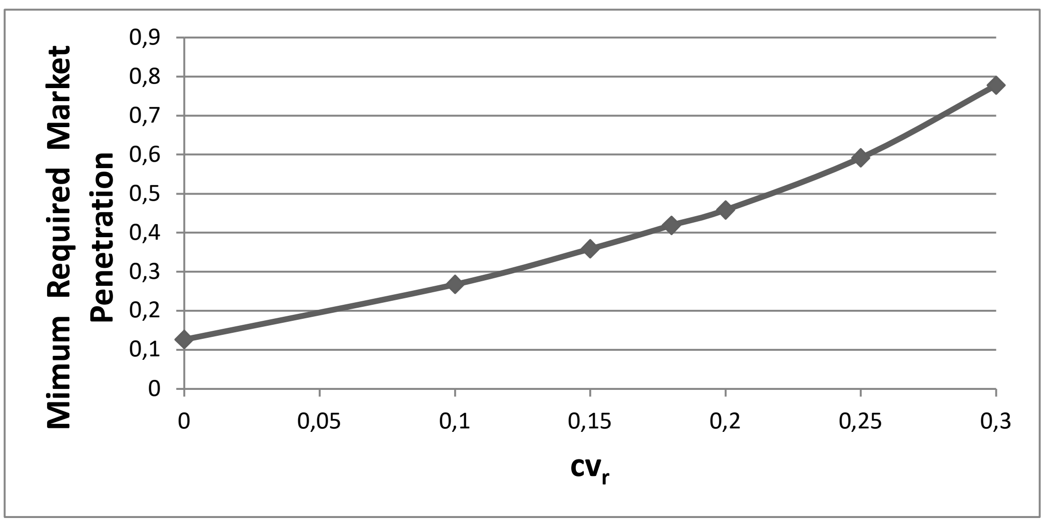


*Figure 4 – Minimal required market penetration for stability as a function of the information accuracy*

Note that the marginal increment required in market penetration increases more than linearly with the (in)accuracy of the information. With reference to this numerical example, some mild inaccuracy does not disrupt the stabilization effect, at least if the market penetration is significant. On the other hand, in the case of fully accurate (or almost fully accurate) information, a very small value of market penetration is required. These results are not generalizable but they are, of course, in accord with expectations.

# 5 Conclusions

A (deterministic) dynamic process in the presence of ATIS is formalised, as well as the corresponding equilibrium model. The model allows for considering various possibilities for the routing strategy. The theoretical properties of the model are shown and discussed, with particular reference to the existence and uniqueness of the equilibrium, the convergence properties of the dynamic process, and the stability of the equilibrium. Various hypotheses are made: the properties of the equilibrium can be theoretically shown under mild sufficient conditions that involve the monotonicity of the cost functions and the use of a RUM for route choices and the ATIS *route-spreading coefficients* map. Weaker hypotheses can also be adopted, involving the eigenvalues of the Jacobian of the route choice map, the *route-spreading coefficients* map and the congestion model. The hypothesis of a fully accurate ATIS is required for theoretically addressing convergence. Theoretical stability analysis for a general network can be performed only under both the hypothesis of accurate ATIS and the hypothesis of a null (or negligible) *route-spreading difference map*, which represents the difference of the route-choice percentages between compliant and non-compliant travellers once subject to the same stimulus.

The most evident effect of ATIS (in the case of fully accurate information) is on the stability of the equilibrium. The stability region was shown to be a quartic, its dimension rapidly increasing with ATIS market penetration. Even for relatively low levels of market penetration the system is always stable. Hence (accurate) information is able to stabilise systems that may be otherwise unstable. Of course the stability we refer to is the local one (around the equilibrium). However, a common experience for analysts is that in the event of local stability the simulation of the process always converges to the equilibrium, even if different starting points are adopted.

The theoretical properties of the dynamic process under the more general hypothesis of non-constant and non-fully accurate information will be tackled in a future paper. Here a few numerical simulations are carried out in these conditions using a small test network. The simulations confirm expectation that a small (in)accuracy can be accepted without disrupting the stabilising properties of the ATIS. However, as the inaccuracy increases, the required market penetration (for stability) rapidly increases. Such a requirement is not likely to be satisfied in the real world, where inaccurate information is expected to lead to a decreasing actual market penetration, the information system being perceived by travellers as not really useful.

## Acknowledgments

Authors wish to thank the anonymous reviewers for their helpful comments. This research has been partially supported by the University of Salerno (IT) under local grants n. ORSA132945 (financial year 2013) and ORSA145550 (financial year 2014), as well as by the Unversity of Naples Federico II (IT) under national grants B61H11000400005 and B68F12001150005 (Italian Ministry of University and

Research). An earlier version of this research has been presented at 5th International Symposium on Dynamic Traffic Assignment (DTA 2014) held in Salerno (IT) in June 2014.

## References

Al-Deek, H.M., and Kanafani, A. (1993) Modeling the benefit of advanced traveler information systems in corridors with incidents. *Transportation Research Part C: Emerging Technologies*, **1**(4), 303-324.

Al-Deek, H.M., Khattak, A.J, and Thananjeyan, P. (1998) A combined traveller behaviour and system performance model with advanced traveller information systems. *Transportation Research Part A: Policy and Practice*, **32**(7), 479-493.

Avineri, E., Prashker, J.N. (2006) The impact of travel time information on travelers' learning under uncertainty, *Transportation*, **33**(4), 393-408

Ben-Akiva, M., and Lerman, S.R. (1985) *Discrete choice analysis*, MIT Press Cambridge, MA (USA)

Ben-Elia, E., Di Pace, R., Bifulco, G.N., and Shiftan, Y. (2013) The impact of travel information's accuracy on route-choice. *Transportation Research Part C: Emerging Technologies*, **26**, 146-159

Ben-Elia, E., Avineri,E. (2015) Response to Travel Information: a behavioural review. *Transport Reviews*, **35**(3), 352-377

Bifulco, G.N., Simonelli, F., Di Pace, R. (2007) Endogenous Driver Compliance and network performances under ATIS. *IEEE Intelligent Transportation Systems Conference 2007*, 1028-1033

Bifulco G.N., Simonelli, F., Di Pace, R. (2009) The role of the uncertainty in ATIS applications. *Advances in Soft Computing*, **52**, 230-239

Bifulco, G.N., Cantarella, G.E., and Simonelli, F. (2014a) Design of Signal Setting and Advanced Traveler Information Systems. *Journal of Intelligent Transportation Systems*, **18**(1), 30-40

Bifulco, G.N., Di Pace, R., Viti, F. (2014b) Evaluating the effects of information reliability on travellers' route choice. *European Transport Research Review*, **6**(1), 61-70

Bogers, E., Viti, F., Hoogendoorn, S. (2006) Joint modeling of advanced travel information service, habit and learning impacts on route choice by laboratory simulator experiments. *Transportation research Record*, **1926**, 189-197.

Bottom, J.A., (2000) *Consistent anticipatory route guidance*. PhD thesis. Department of Civil and Environmental Engineering, MIT

Brouwer, L.E.J. (1911) Uber Abbildung von Mannigfaltigkeiten, *Mathematische Annalen*, **71**(1), 97-115

Cantarella, G.E. (1997) Fixed-point stability and bifuracation in dynamic processes for traffic assignment. In *Advances in Intelligent Systems*, ed. F.C. Morabito, IOS-Press, Amsterdam (NL), 254-261

Cantarella, G.E. (2013) Day-to-day dynamic models for Intelligent Transportation System design and appraisal. *Transportation Research Part C: Emerging Technologies*, **29**, 117-130

Cantarella, G.E., Cascetta, E. (1995) Dynamic process and equilibrium in transportation networks: towards a unifying theory, *Transportation Science*, **31**, 107-128

Chorus, C. G., Arentze, T. A., Molin, E. J. E., Timmermans, H. J. P., and Van Mee, B. (2006a) The value of travel information: Decision strategy-specific conceptualizations and numerical examples. *Transportation Research Part B: Methodological*, **40**(6), 504-519

Chorus, C. G., Molin, E. J. E., Van Wee, B. (2006b) Use and effects of Advanced Traveller Information Services (ATIS): A Review of the literature. *Transport Reviews*, **26**(2), 127-149

Davidson, K.B. (1966) A flow-travel time relationship for use in transportation planning, *Proceedings of the 3rd Australian Road Research Board Conference*, 3 (1), 183-194.

Dell'Orco, M., Teodorovic, D. (2009) Data fusion for updating information in modelling drivers' choice behaviour. *Transportmetrica*, **5**(2), 107-123

De Palma, A., Lindsey, R., and Picard, N. (2012) Aversion, the value of the information, and traffic equilibrium. *Transportation Science*, **46**(1), 1-26

Dia, H. (2002) An agent-based approach to modelling driver route choice behavior under the influence of real-time information. *Transportation research Part C: Emerging Technologies*, 10(5-6), 331-349.

Domencich, T.A. and McFadden, D. (1975) *Urban travel demand: a behavioural analysis*. North Holland, Amsterdam (NL)

Emmerink, R. H. M., Axhausen, K. W., Nijkamp, P., and Rietveld, P. (1995) The potential of information provision in a simulated road transport network with non-recurrent congestion. *Transportation Research C: Emerging Technologies*, **3**(5), 293-309

Ettema, D., and Timmermans, H (2006) Cost of travel time uncertainty and benefit of travel time information: conceptual model and numerical examples. *Transportation Research Part C: Emerging Technologies*, **14**(5), 335-350

Hadamard, J. (1906) Sur les transformations ponctuelles. *Bull. Soc. Math. France*, **34**, 71-84

Hall, R.W. (1996) Route choice and advanced traveller information systems on a capacitated and dynamic network. *Transportation Research Part C: Emerging Technologies*, **4**(5), 289-306

He, Z., Guan, W., and Ma, S. (2013) A traffic-condition-based route guidance strategy for a single destination road network. *Transportation research Part C: Emerging Technologies*, **32**, 89-102.

Jahn, O., Mohring, R.H., Schulz, A.S., Stier-Moses, N.E. (2005) System-optimal routing of traffic flows with user constraints in networks with congestion. *Operations Research*, **53**(4), 600-616

Jha, M., Madanat, S., and Peeta, S. (1998) Perception updating and day-to-day travel choice dynamics in traffic networks with information provision. *Transportation Research Part C: Emerging Technologies*, **6**(3), 189-212

Khattak, A., Yim, Y., and Stalker, L. (2003) Willingness to pay for travel information, *Transportation Research Part C: Emerging Technologies*, **11**(2), 137-159

Lindsey, R., Daniel, T., Gisches, E., and Rapoport, A. (2014) Pre-trip information and route-choice decision with stochastic travel conditions: theory. *Transportation Research Part B: Methodological*, **67**(3), 187-207

Lo, H.K., and Szeto, W.Y. (2002) A methodology for sustainable traveler information services. *Transportation Research Part B: Methodological*, **36**(3), 113-130

Lo, H.K., and Szeto, W.Y. (2004) Modeling advanced traveler information services: static versus dynamic paradigms. *Transportation Research Part B: Methodological*, **38**(6), 495-515

Mahmassani, H. and Liu, Y. (1999). Dynamics of commuting decision behavior under advanced traveler information systems. *Transportation Research Part C: Emerging Technologies*, **2**(2-3), 91-107

Oh, J., Jayakrishnan, R, Chen, A., Yang, H. (2001) Parametric evaluation for route guidance systems with analysis of sustainable driver compliance. *Transportation Research Record*, **1771**, 18-27

Paz, A., and Peeta, S. (2009) On-line calibration of behavior parameters for behavior-consistent route guidance. *Transportation Research Part B: Methodological*, **43**(4), 403-421

Rapoport, A., Gisches, E. J., Daniel, T., and Lindsey, R., (2014) Pre-trip information and route-choice decisions with stochastic travel conditions: Experiment. *Transportation Research Part B: Methodological*, **68**, 154-172.

Schrank, D., Eisele, B. and Lomax, T., (2012) The 2012 Urban Mobility Report, Powered by INRIX Traffic Data, College Station: Texas Transportation Institute, Texas A&M University, December http://s3.documentcloud.org/documents/566377/2012-urban-mobility-report.pdf (accessed 29.07.15)

Tanaka, M., Uno, N., Shiomi, Y. And Ahn, Y. (2014) Experimental study of effects of travel time distribution information on dynamic route choice behaviour. *Journal of Intelligent Transportation Systems*, **18**(2), 215-226

Tseng, Y., Knockaert, J., and Verhoef, E.T. (2013) A revealed-preference study of behavioural impacts of real-time traffic information. *Transportation Research C: Emerging Technologies*, **30**, 196-209

Tsirimpa, A., Polydoropoulou, A., and Antoniou, C. (2007) Development of a mixed multi-nomial logit model to capture the impact of information systems on traveler's switching behaviour. *Journal of Intelligent Transportation Systems*, **11**(2), 79-89

Watling, D., and Hazelton, M.L. (2003) The dynamics and equilibria of day-to-day assignment models. *Network and Spatial Economics*, **3**, 349-370

Webster, F.V., Cobbe, B.M. (1966) Traffic Signals, *Road Research Technical Paper 56*, Road Research Laboratory, London, UK.

Yang, H. (1998) Multiple equilibrium behaviors and advanced traveler information systems with en-dogenous market penetration. *Transportation Research Part B: Methodological*, **32**(3), 205-218

Yang, H., and Meng, Q. (2001) Modeling user adoption of advanced traveler information systems: dynamic evolution and stationary equilibrium. *Transportation Research Part A: Policy and Practice*, **35**, 895-912

*Corresponding author: Gennaro Nicola Bifulco, Università degli Studi di Napoli Federico II, Dipartimento di Ingegneria Civile Edile ed Ambientale, 80125 Napoli, Italy, phone: +39 081 7683883, e-mail: gennaro.bifulco@unina.it*

*This is a post-peer-review, pre-copyedit version of an article published in Transportation Research – part B.*

*The final authenticated version is available online at: URL http://dx.doi.org/10.1016/j.trb.2015.12.008*