COMPACTNESS FOR SOBOLEV-TYPE TRACE OPERATORS

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Abstract. Compactness of arbitrary-order Sobolev type embeddings for traces of n-dimensional functions on lower dimensional subspaces is investigated. Sobolev spaces built upon any rearrangement-invariant norm are allowed. In particular, we characterize compactness of trace embeddings for classical Sobolev, Lorentz-Sobolev and Orlicz-Sobolev type spaces.

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1. Introduction and main results

A general form of Sobolev embedding theorem states that, if Ω is a bounded open set in $\mathbb{R}^n$, $n \geq 2$, satisfying the cone property, and Ω_d is the (nonempty) intersection of Ω with a d-dimensional affine subspace of $\mathbb{R}^n$, $1 \leq d \leq n$, then any function u from the standard Sobolev space $W^{m,p}(\Omega)$, with $m \in \mathbb{N}$ and $p \in [1, \infty]$, admits a trace $\operatorname{Tr} u$ on Ω_d. The linear operator Tr which associates u with its trace $\operatorname{Tr} u$ fulfills

$$\operatorname{Tr}\colon W^{m,p}(\Omega) \hookrightarrow \begin{cases} L^{\frac{pd}{n-mp}}(\Omega_d) & \text{if } m < n \text{ and } p \in \left[1, \frac{n}{m}\right), \\ \exp L^{\frac{n}{n-m}}(\Omega_d) & \text{if } m < n \text{ and } p = \frac{n}{m}, \\ L^\infty(\Omega_d) & \text{otherwise,} \end{cases} \tag{1.1}$$

provided that $d \in (n - mp, n]$ when $1 < p < \frac{n}{m}$, or

$$d \geq n - m \tag{1.2}$$

when $p = 1$ (see, e.g. [2, Theorem 4.12]). Here, the arrow '$\hookrightarrow$' stands for a continuous operator, $L^q(\Omega_d)$, $q \in [1, \infty]$, denotes the space of summable functions on Ω_d with respect to the d-dimensional Hausdorff measure $\mathcal{H}^d$ on $\mathbb{R}^n$ and $\exp L^{\frac{n}{n-m}}(\Omega_d)$ is the Orlicz space on Ω_d associated with the Young function $\exp(t^{\frac{n}{n-m}} - 1)$. When $d = n$, clearly $\Omega_n = \Omega$, and Tr is the identity operator.
Let us observe that (1.1) collects classical embedding theorems due to Gagliardo [16] ($1 \leq p < \frac{n}{m}$ or $p > \frac{n}{m}$), Nirenberg [26] ($p = 1$, $d = n$), Sobolev [33] ($d = n$, and $1 < p < \frac{n}{m}$ or $p > \frac{n}{m}$), Pohozaev [29], Trudinger [34], Yudovich [35] ($d = n$, $p = \frac{n}{m}$), Adams [1] and Maz'ya [23] ($p = \frac{n}{m}$).

Even though the classical Rellich and Sobolev-Kondrashov theorems provide compactness for Sobolev trace embeddings in some cases, their compactness is usually lost in extreme cases. Namely, the embedding (1.1) is not compact when $p \in [1, \frac{n}{m}]$ if $m < n$, and when $p = 1$ if $m = n$. Such compactness results turn out to be of crucial use in the analysis of solvability of partial differential equations and in spectral theory for linear and nonlinear partial differential operators.

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Starting with these classical results the analysis of boundedness and compactness for Sobolev embeddings has been the subject of a wide number of investigations, along various directions. Among the first references in a nowadays extremely vast literature, see e.g. the monographs [2, 13, 18, 19, 23, 24].

In the recent paper [10], trace embeddings have been studied in the situation when the class of spaces which describe the integrability degree of functions and of their weak derivatives is enlarged to include general rearrangement-invariant spaces. Sobolev spaces built upon general rearrangement-invariant norms are taken into account, i.e. norms which only depend on the "size" of functions, or, more precisely, on the measure of their level sets. Customary examples of rearrangement-invariant spaces are Lebesgue spaces, Orlicz spaces and Lorentz spaces.

In [10], arbitrary-order Sobolev type embeddings for traces of n-dimensional functions on lower dimensional subspaces are established. A natural question of finding out whether they are compact then arises. In the present paper we address this issue by providing compactness criteria of diverse nature.

To be more specific, let $\|\cdot\|_{X(0,1)}$ and $\|\cdot\|_{Y(0,1)}$ be rearrangement invariant function norms on the Riesz space $L^0(0,1)$ of all (equivalence classes of) Lebesgue measurable functions on $(0,1)$ taking values in $[-\infty,\infty]$. Then $X(\Omega)$ denotes the rearrangement-invariant space on Ω associated with $\|\cdot\|_{X(0,1)}$ and $X(0,1)$ is called its representation space (see Sect. 2 for precise definitions). For $m\in\mathbb{N}$,

$$W^mX(\Omega)=\big\{u\colon u \text{ is } m\text{-times weakly differentiable in } \Omega, \text{ and } |\nabla^k u|\in X(\Omega) \text{ for } k=0,\dots,m\big\},$$

equipped with the norm

$$\|u\|_{W^mX(\Omega)}=\sum_{k=0}^{m}\|\,|\nabla^k u|\,\|_{X(\Omega)},$$

is the Sobolev type Banach space built upon the rearrangement-invariant norm $\|\cdot\|_{X(0,1)}$. Here, $\nabla^m u$ stands for the vector of all m-th order weak derivatives of u. We simply denote $\nabla^1 u$ by ∇u; also, $\nabla^0 u$ is nothing but u. In the case when $X(\Omega)=L^p(\Omega)$, $p\in[1,\infty]$, one clearly has $W^mX(\Omega)=W^{m,p}(\Omega)$.

Throughout, we assume that condition (1.2) is in force. This guarantees a well-defined trace operator Tr from $W^mX(\Omega)$ into (at least) $L^1(\Omega_d)$, whatever $m\in\mathbb{N}$ and $\|\cdot\|_{X(0,1)}$ are. This is true since $W^mX(\Omega)$ is continuously embedded in $W^{m,1}(\Omega)$, as Ω has finite measure, and classically $\mathrm{Tr}\colon W^{m,1}(\Omega)\hookrightarrow L^1(\Omega_d)$. Thus, one may deal with the Sobolev type trace embedding

$$\mathrm{Tr}\colon W^mX(\Omega)\hookrightarrow Y(\Omega_d), \tag{1.3}$$

where $Y(\Omega_d)$ is the rearrangement-invariant space on Ω_d with respect to the d-dimensional Hausdorff measure $\mathcal{H}^d$ restricted to Ω_d associated with the norm $\|\cdot\|_{Y(0,1)}$. Note that $\mathcal{H}^d(\Omega_d)<\infty$, since Ω_d is isometric to a bounded subset of $\mathbb{R}^d$. We stress that the condition (1.2) does not require any restriction on the rearrangement-invariant space $X(\Omega)$. In fact, in the case when $m<n$, it allows the norm $\|\cdot\|_{X(0,1)}$ to be 'close' to $\|\cdot\|_{L^1(0,1)}$, which would not be possible if (1.2) was omitted.

In [10], the optimal target space (i.e. the smallest possible in the class of rearrangement-invariant Banach function spaces on Ω) for embedding (1.3) is explicitely determined. This is the space $X^m_{d,n}(\Omega_d)$, whose associate norm is defined by

$$\|f\|_{(X^m_{d,n})'(0,1)}=\big\|\mathrm{R}^m_{d,n}f\big\|_{X'(0,1)}\qquad\text{for } f\in L^0(0,1). \tag{1.4}$$

Here, $\mathrm{R}^m_{d,n}\colon L^0(0,1)\to L^0(0,1)$ is the one-dimensional Hardy type operator, defined for each f in $L^0(0,1)$ as

$$\mathrm{R}^m_{d,n}f(s)=s^{-1+\frac{m}{n}}\int_0^{s^{\frac{d}{n}}}|f(t)|\,dt\qquad\text{for } s\in(0,1). \tag{1.5}$$

Moreover, embedding (1.3) is characterized by the boundedness of the one-dimensional Hardy type operator $\mathrm{H}^m_{d,n}\colon L^0(0,1)\to L^0(0,1)$ from $X(0,1)$ into $Y(0,1)$, defined by

$$\mathrm{H}^m_{d,n}f(s)=\int_{s^{\frac{n}{d}}}^1|f(t)|\,t^{-1+\frac{m}{n}}\,dt\qquad\text{for } s\in(0,1), \tag{1.6}$$

for every f in $L^0(0,1)$.

Our compactness results for the Sobolev type trace operator (1.3) are stated in the next two theorems. We distinguish the cases when $Y(\Omega_d)\neq L^\infty(\Omega_d)$ and $Y(\Omega_d)=L^\infty(\Omega_d)$ because they are essentially different and their treatment requires different techniques to be used.

In the most interesting case, namely, when $m < n$, we give equivalent characterizations of compactness for (1.3) involving Hardy type operators and optimal target spaces mentioned above. The last named characterization is formulated in terms of the 'almost-compact' embedding [28, Section 7.11] of the representation space of the optimal target space $X^m_{d,n}(\Omega_d)$ for (1.3) in $Y(\Omega_d)$, denoted by $X^m_{d,n}(0,1) \overset{*}{\hookrightarrow} Y(0,1)$. It thus turns out that the unit ball of $X^m_{d,n}(0,1)$ has uniformly absolutely continuous norm in $Y(0,1)$ if and only if the trace operator in question is compact.

Theorem 1.1. *Let Ω be a bounded open set in $\mathbb{R}^n$, $n \geq 2$, satisfying the cone property, and let $\|\cdot\|_{X(0,1)}$ and $\|\cdot\|_{Y(0,1)}$ be rearrangement-invariant function norms. Assume that $m, d \in \mathbb{N}$, $1 \leq d \leq n$, and d fulfills* (1.2). *Suppose, in addition, that*

$$Y(0,1) \neq L^\infty(0,1). \tag{1.7}$$

Then,

- *for $m < n$, the following statements are equivalent:*
  - (i) *the Sobolev type trace operator* $\mathrm{Tr}\colon W^m X(\Omega) \to Y(\Omega_d)$ *is compact;*
  - (ii) *the Hardy type operator* $\mathrm{H}^m_{d,n}\colon X(0,1) \to Y(0,1)$ *is compact;*
  - (iii) $\displaystyle\lim_{r\to 0^+} \sup_{\|f\|_{X(0,1)}\leq 1} \|\mathrm{H}^m_{d,n}(f\chi_{(0,r)})\|_{Y(0,1)} = 0;$
  - (iv) $\displaystyle\lim_{r\to 0^+} \sup_{\|f\|_{X(0,1)}\leq 1} \|\chi_{(0,r)}\mathrm{H}^m_{d,n} f\|_{Y(0,1)} = 0;$
  - (v) $X^m_{d,n}(0,1) \overset{*}{\hookrightarrow} Y(0,1);$
- *for $m \geq n$, the Sobolev type trace operator* $\mathrm{Tr}\colon W^m X(\Omega) \to Y(\Omega_d)$ *is compact.*

In the case when $d = n$ and Ω has Lipschitz boundary, the equivalence between (i) and (ii) has been established in [17]. Let us also mention that relationships among properties (ii)-(iv) for similar classes of integral operators have been investigated in [21], and their intimate connection with the almost compact embedding described in (v) is stated in [31]. Unlike [21], the approach of [31] allows us to express conditions (ii) and (iv) without any explicit reference to absolute continuity of the norms in question. Indeed, an alternative formulation of statement (iii) is that the set $\mathrm{R}^m_{d,n}(B_{Y'(0,1)})$ has uniformly absolutely continuous norm in $X'(0,1)$. Here, $B_{Y'(0,1)}$ denotes the unit ball in the associated space $Y'(0,1)$ of $Y(0,1)$. Analogously, assertion (iv) means that the set $\mathrm{H}^m_{d,n}(B_{X(0,1)})$ is of uniformly absolutely continuous norm in $X(0,1)$ (see [32, Remarks 4.9 and 5.2]).

Theorem 1.2. *Assume that Ω, n, m, d and $\|\cdot\|_{X(0,1)}$ are as in Theorem 1.1. Then,*

- *for $m < n$, the following statements are equivalent:*
  - (i) *the Sobolev type trace operator* $\mathrm{Tr}\colon W^m X(\Omega) \to L^\infty(\Omega_d)$ *is compact;*
  - (ii) *the Hardy type operator* $\mathrm{H}^m_{d,n}\colon X(0,1) \to L^\infty(0,1)$ *is compact;*
  - (iii) $\displaystyle\lim_{r\to 0^+} \sup_{\|f\|_{X(0,1)}\leq 1} \int_0^r f^*(s)\, s^{\frac{m}{n}-1}\, ds = 0;$
  - (iv) $\displaystyle\lim_{r\to 0^+} \|s^{-1+\frac{m}{n}}\, \chi_{(0,r)}(s)\|_{X'(0,1)} = 0;$
  - (v) $X(0,1) \overset{*}{\hookrightarrow} L^{\frac{n}{m},1}(0,1);$
- *for $m = n$, the Sobolev type trace operator* $\mathrm{Tr}\colon W^m X(\Omega) \to L^\infty(\Omega_d)$ *is compact if and only if* $X(0,1) \neq L^1(0,1)$*;*
- *for $m > n$, the Sobolev type trace operator* $\mathrm{Tr}\colon W^m X(\Omega) \to L^\infty(\Omega_d)$ *is always compact.*

As a consequence of Theorems 1.1 – 1.2 we obtain

Corollary 1.3. *Assume that Ω, n, m and d are as in Theorem 1.1, and let $\|\cdot\|_{X(0,1)}$, $\|\cdot\|_{Y(0,1)}$, $\|\cdot\|_{Z(0,1)}$ be rearrangement-invariant norms. If $X(0,1) \overset{*}{\hookrightarrow} Y(0,1)$ and $\mathrm{Tr}\colon W^m Y(\Omega) \hookrightarrow Z(\Omega_d)$, then the Sobolev type trace operator $\mathrm{Tr}\colon W^m X(\Omega) \hookrightarrow Z(\Omega_d)$ is compact.*

The structure of the paper is as follows. We recall in Sect. 2 a few definitions and properties of function spaces and one-dimensional operators playing a role in what follows. In Sect. 3, we state and prove key tools employed later for the proofs of our main results, which are the content of Sect. 4. Compactness of trace operator on customary Sobolev type spaces, as Lorentz-Sobolev spaces and Orlicz-Sobolev spaces, are discussed in Sect. 5.

2. Background

We collect in this section some definitions, notations and properties about functions, function spaces and one-dimensional operators involved in our discussion. For more details and proofs, we refer to [6, 28].

2.1. ***Rearrangement-invariant function spaces.*** Let (S,Σ,μ) be a σ-finite measure space. For any $E\in\Sigma$, we denote by $L^0(E,\mu)$ the Riesz space of all (equivalence classes of) μ-measurable functions on E whose values belong to $[-\infty,\infty]$. Then $L^0_+(E,\mu)$ stands for the positive cone of $L^0(E,\mu)$, i.e. $L^0_+(E,\mu)=\{u\in L^0(E,\mu)\colon u\geq 0\ \mu\text{-a.e. in}\, E\}$, and $L^0_0(E,\mu)=\{u\in L^0(E,\mu)\colon u\,\text{is finite}\,\mu\text{-a.e. in}\, E\}$. In the special case when (S,Σ,μ) is the Lebegue measure space $(\mathbb{R}^n,\mathcal{L}_n,\lambda_n)$, we simply write $\mathbb{R}^n$ in place of $(\mathbb{R}^n,\mathcal{L}_n,\lambda_n)$, $|E|$ instead of $\lambda_n(E)$, and we omit the explicit reference to λ_n in the above notations.

Given a function $u\in L^0(E,\mu)$, we denote by $\eta_u\colon\mathbb{R}\to[0,\infty]$ its distribution function defined as

$$\eta_u(t)=\mu\left(\{x\in E\colon u(x)>t\}\right)\quad\text{for } t\in\mathbb{R}.$$

Two functions $u,v\in L^0(E,\mu)$ are equimeasurable, writing $u\sim v$, if $\eta_{|u|}=\eta_{|v|}$.
The *non-increasing rearrangement* $u^*\colon[0,\infty)\to[0,\infty]$ of a u is defined as

$$u^*(s)=\inf\{t\geq 0\colon \mu_{|u|}(t)\leq s\}\quad\text{for } s\in[0,\infty).$$

The function u^* is a (unique) right-continuous, non-increasing function in $[0,\infty)$ fulfilling the property $u^*\sim u$. Clearly, $u^*(s)=0$ if $s\geq\mu(E)$. The map $u\mapsto u^*$ is monotone in the sense that

$$|u|\leq|v|\qquad \mu\text{-a.e. on } E\qquad\Longrightarrow\qquad u^*\leq v^*\qquad\text{a.e. on }[0,\infty).$$

The *Hardy-Littlewood inequality*, which tells us that

$$\int_E |uv|d\mu\ \leq\ \int_0^{\mu(E)} u^*(s)v^*(s)\,ds \tag{2.1}$$

for every $u,v\in L^0(E,\mu)$, is a basic property of non-increasing rearrangements.
As u^* is non-increasing, the function $u^{**}\colon(0,\infty)\to[0,\infty]$, defined by

$$u^{**}(s)\ =\ \frac{1}{s}\int_0^s u^*(t)dt\quad\text{for } s\in(0,\infty)\,,$$

is also non-increasing, and $u^*\leq u^{**}$ on $(0,\infty)$. Moreover,

$$(u+v)^{**}\ \leq\ u^{**}+v^{**}$$

for every $u,v\in L^0_+(E,\mu)$.

A *rearrangement-invariant Banach (extended) function norm* is a functional $\|\cdot\|_{X(0,1)}\colon L^0(0,1)\to[0,\infty]$ such that

$$\|f+g\|_{X(0,1)}\ \leq\ \|f\|_{X(0,1)}+\|g\|_{X(0,1)}\quad\text{for all } f,g\in L^0_+(0,1); \tag{2.2}$$

$$\|\alpha f\|_{X(0,1)}\ =\ \alpha\,\|f\|_{X(0,1)}\quad\text{for all } \alpha\in[0,\infty),\ f\in L^0_+(0,1)\ (\text{counting } 0\cdot\infty=0,\text{ as usual}); \tag{2.3}$$

$$\|f\|_{X(0,1)}\ >\ 0\quad\text{for every non-zero } f\in L^0_+(0,1); \tag{2.4}$$

$$\|f\|_{X(0,1)}\ \leq\ \|g\|_{X(0,1)}\quad\text{whenever } f\ \leq\ g \text{ in } L^0_+(0,1); \tag{2.5}$$

$$\lim_k\|f_k\|_{X(0,1)}\ =\ \|f\|_{X(0,1)}\quad\text{whenever } f_k\nearrow f \text{ in } L^0_+(0,1); \tag{2.6}$$

$$\|1\|_{X(0,1)}\ <\ \infty; \tag{2.7}$$

$$\text{there is a positive constant } C \text{ such that } \|f\|_{L^1(0,1)}\leq C\,\|f\|_{X(0,1)} \text{ for all } f\in L^0_+(0,1); \tag{2.8}$$

$$\|u\|_{X(0,1)}\ =\ \|v\|_{X(0,1)}\quad\text{for all } u,v\in L^0(0,1)\text{ such that } u\sim v. \tag{2.9}$$

Given a rearrangement-invariant Banach function norm $\|\cdot\|_{X(0,1)}$, the functional on $L^0(0,1)$, denoted by $\|\cdot\|_{X'(0,1)}$, and defined as

$$\|g\|_{X'(0,1)}=\sup_{\|f\|_{X(0,1)}\leq 1}\int_0^1|fg|(t)dt\qquad\text{for } g\in L^0(0,1), \tag{2.10}$$

is a rearrangement-invariant function norm, called the *associate function norm of* $\|\cdot\|_{X(0,1)}$. The functional on $L^0(0,1)$, denoted by $\|\cdot\|_{X'_D(0,1)}$, and defined as

$$\|g\|_{X'_D(0,1)} = \sup_{\|f\|_{X(0,1)}\leq 1} \int_0^1 f^*(t)|g(t)|dt \qquad \text{for } g \in L^0(0,1), \tag{2.11}$$

is still a rearrangement-invariant function norm, called the *down associate function norm of* $\|\cdot\|_{X(0,1)}$. For any $g \in L^0(0,1)$, $\|g\|_{X'_D(0,1)} \leq \|g\|_{X'(0,1)}$; moreover, via (2.1),

$$\|g\|_{X'_D(0,1)} = \|g\|_{X'(0,1)} \qquad \text{for non-increasing } g \in L^0_+(0,1). \tag{2.12}$$

The *dilation operator* $\mathcal{D}_\lambda$, with $\lambda > 0$, is defined for $f \in L^0(0,1)$ by

$$(\mathcal{D}_\lambda f)(s) = \begin{cases} f(\lambda^{-1}s) & \text{if } 0 < s \leq \lambda \\ 0 & \text{if } \lambda < s < 1, \end{cases} \tag{2.13}$$

and is bounded on any rearrangement-invariant space $X(0,1)$, with norm not exceeding $\max\{1, \lambda^{-1}\}$ [6, Chapter 3, Proposition 5.11].

Given a rearrangement-invariant function norm $\|\cdot\|_{X(0,1)}$ and any $E \in \Sigma$ of finite measure, the space $X(E,\mu)$ is defined as the collection of all functions $u \in L^0(E,\mu)$ such that the quantity

$$\|u\|_{X(E,\mu)} = \|u^*(\mu(E)\,\cdot)\|_{X(0,1)} \tag{2.14}$$

is finite. This quantity defines a norm on $X(E,\mu)$, called a *rearrangement-invariant norm*, which makes $X(E,\mu)$ a Banach space, called a *rearrangement-invariant space*. Notice that

$$\|u\|_{X(E,\mu)} = \|\,|u|\,\|_{X(E,\mu)} \qquad \text{for } u \in X(E,\mu), \tag{2.15}$$

and

$$X(E,\mu) \subset L^0_0(E,\mu). \tag{2.16}$$

Let us warn the reader that, when $\mu(E) < \infty$, the norm $\|\cdot\|_{X(E,\mu)}$ is just equivalent to (but possibly different from) more customary norms, because the measure of E may exceed 1.

The space $X(0,1)$ is then called the *representation space* of $X(E,\mu)$. The rearrangement-invariant space $X'(E,\mu)$ built upon the function norm $\|\cdot\|_{X'(0,1)}$ is called the *associate space* of $X(E,\mu)$. As

$$\|\cdot\|_{(X')'(0,1)} = \|\cdot\|_{X(0,1)},$$

it holds that $(X')'(E,\mu) = X''(E,\mu) = X(E,\mu)$. Therefore, any rearrangement-invariant space $X(E,\mu)$ is always the associate space of another rearrangement-invariant space, namely $X'(E,\mu)$.

Since

$$\int_0^1 |fg|(t)\,dt \leq \|f\|_{X(0,1)}\|g\|_{X'(0,1)},$$

holds for every $f, g \in L^0(0,1)$, the *Hölder inequality*

$$\int_E |uv|\,d\mu \ \leq\ \mu(E)\|u\|_{X(E,\mu)}\|v\|_{X'(E,\mu)} \tag{2.17}$$

holds for every u, v in $L^0(E,\mu)$.

Given two rearrangement-invariant function norms $\|\cdot\|_{X(0,1)}$ and $\|\cdot\|_{Y(0,1)}$, for any $E \in \Sigma$ we write $X(E,\mu) \hookrightarrow Y(E,\mu)$ to denote that $X(E,\mu)$ is continuously embedded into $Y(E,\mu)$. Observe that $X(E,\mu) \hookrightarrow Y(E,\mu)$ if and only if there exists a positive constant C such that $\|g\|_{Y(0,1)} \leq C\|g\|_{X(0,1)}$ for every $g \in L^0(0,1)$. Moreover,

$$X(E,\mu) \hookrightarrow Y(E,\mu) \qquad \Longleftrightarrow \qquad X(E,\mu) \subseteq Y(E,\mu),$$

and

$$X(E,\mu) \hookrightarrow Y(E,\mu) \qquad \Longleftrightarrow \qquad Y'(E,\mu) \hookrightarrow X'(E,\mu), \tag{2.18}$$

with the same embedding constants.

Whenever $\mu(E) < \infty$,

$$L^\infty(E,\mu) \hookrightarrow X(E,\mu) \hookrightarrow L^1(E,\mu). \tag{2.19}$$

We write $X(E,\mu) \overset{*}{\hookrightarrow} Y(E,\mu)$ to denote that $X(E,\mu)$ is almost compactly embedded into $Y(E,\mu)$, namely

$$\lim_{r\to 0^+} \sup_{\|u\|_{X(E,\mu)}\le 1} \|u\,\chi_{F_k}\|_{Y(E,\mu)} = 0$$

for every sequence $(F_k)_{k\in\mathbb{N}}$ of measurable subsets of E with $(\chi_{F_k})_{k\in\mathbb{N}}$ converging to 0 μ-a.e. in E.

Note that

$$X(E,\mu) \hookrightarrow\hookrightarrow Y(E,\mu) \quad\Longrightarrow\quad X(E,\mu) \overset{*}{\hookrightarrow} Y(E,\mu) \quad\Longrightarrow\quad X(E,\mu) \hookrightarrow Y(E,\mu), \tag{2.20}$$

where $\hookrightarrow\hookrightarrow$ stands for a compact embedding [28, Theorem 7.11.6 and Theorem 7.11.5], and

$$X(E,\mu) \overset{*}{\hookrightarrow} Y(E,\mu) \quad\Longleftrightarrow\quad Y'(E,\mu) \overset{*}{\hookrightarrow} X'(E,\mu) \tag{2.21}$$

[28, Theorem 7.11.3].

If $\mu(E) < \infty$, then

$$X(E,\mu) \overset{*}{\hookrightarrow} Y(E,\mu) \quad\Longleftrightarrow\quad \lim_{r\to 0^+} \sup_{\|f\|_{X(E,\mu)}\le 1} \|f^*\,\chi_{(0,r)}\|_{Y(0,1)} = 0 \tag{2.22}$$

[28, Lemma 7.11.15]; moreover

$$X(E,\mu) \neq L^\infty(E,\mu) \quad\Longleftrightarrow\quad L^\infty(E,\mu) \overset{*}{\hookrightarrow} X(E,\mu), \tag{2.23}$$

$$X(E,\mu) \neq L^1(E,\mu) \quad\Longleftrightarrow\quad X(E,\mu) \overset{*}{\hookrightarrow} L^1(E,\mu) \tag{2.24}$$

[28, Theorem 7.11.13 and Theorem 7.11.14].

We recall now the definition of some customary, and also some less standard, instances of rearrangement-invariant function norms which shall be used in our applications. In what follows, we set $p' = \frac{p}{p-1}$ for $p \in [1,\infty]$, and adopt the the convention that $1/\infty = 0$.

Prototypal examples of rearrangement-invariant function norms are the classical Lebesgue norms. Indeed, $\|f\|_{L^p(0,1)} = \|f^*\|_{L^p(0,1)}$ if $p \in [1,\infty]$; in particular, $\|f\|_{L^\infty(0,1)} = f^*(0^+)$. By (2.19), $L^\infty(E,\mu)$ and $L^1(E,\mu)$ are the smallest and the largest, respectively, rearrangement-invariant spaces on $E \in \Sigma$.

Let A be a Young function, namely a left-continuous convex function from $[0,\infty)$ into $[0,\infty]$ vanishing at 0, which is neither identically equal to 0, nor to ∞. Thus, A has the form

$$A(t) = \int_0^t a(s)\,ds \qquad \text{for } t \ge 0\,, \tag{2.25}$$

for some (non-trivial) non-decreasing left-continuous function $a\colon [0,\infty) \to [0,\infty]$. The *Luxemburg rearrangement-invariant function norm* associated with A is defined as

$$\|f\|_{L^A(0,1)} = \inf\left\{\lambda > 0\colon \int_0^1 A\left(\frac{|f(t)|}{\lambda}\right) dt \;\le\; 1\right\} \tag{2.26}$$

for $f \in L^0(0,1)$. The space $L^A(0,1)$ is called an *Orlicz space*. In particular, $L^A(0,1) = L^p(0,1)$ if $A(t) = t^p$ for $p \in [1,\infty)$, and $L^A(0,1) = L^\infty(0,1)$ if $A(t) = 0$ for $t \in [0,1]$ and $A(t) = \infty$ for $t > 1$.

Recall that A is said to be *equivalent near infinity* to another Young function B if there exist positive constants c, C and t_0 such that

$$A(ct) \le B(t) \le A(Ct) \qquad \text{for } t \ge t_0\,. \tag{2.27}$$

Note that

(2.28) $L^A(E,\mu) = L^B(E,\mu)$ up to equivalent norms if and only if A is equivalent to B near infinity.

Let $p,q \in [1,\infty]$. Assume that either $1 < p < \infty$ and $1 \le q \le \infty$, or $p = q = 1$, or $p = q = \infty$. Then the functional defined as

$$\|f\|_{L^{p,q}(0,1)} = \|s^{\frac{1}{p}-\frac{1}{q}}\; f^*(s)\|_{L^q(0,1)} \tag{2.29}$$

for $f \in L^0(0,1)$ is equivalent to a rearrangement-invariant function norm, and

$$(L^{p,q})'(0,1) = L^{p',q'}(0,1).$$

For $p, q \in (0, \infty]$ the functional $\|\cdot\|_{L^{(p,q)}(0,1)}$, defined as

$$\|f\|_{L^{(p,q)}(0,1)} = \left\| s^{\frac{1}{p}-\frac{1}{q}} f^{**}(s) \right\|_{L^q(0,1)}$$

for $f \in L^0(0,1)$, is a rearrangement-invariant function norm (see e.g. [28, Theorem 9.7.5]) if either $0 < p < \infty$ and $1 \le q \le \infty$, or $p = q = \infty$. The norms $\|\cdot\|_{L^{p,q}(0,1)}$ and $\|\cdot\|_{L^{(p,q)}(0,1)}$ are called *Lorentz function norms*, and the corresponding spaces $L^{p,q}(0,1)$ and $L^{(p,q)}(0,1)$ are called *Lorentz spaces.* The following inclusion relations between Lorentz spaces hold:

(2.30) $L^{p,p}(0,1) = L^p(0,1) \quad \text{for } p \in [1, \infty];$

(2.31) $L^{p,q}(0,1) \hookrightarrow L^{p,r}(0,1) \quad \text{for } 1 \le q \le r \le \infty;$

(2.32) $L^{(p,q)}(0,1) \hookrightarrow L^{p,q}(0,1) \quad \text{for } p, q \in [1, \infty];$

(2.33) if either $p \in (1, \infty)$ and $1 \le q \le \infty$, or $p = q = \infty$, then $L^{(p,q)}(0,1) = L^{p,q}(0,1)$ up to equivalent norms.

Assume now that either $1 < p < \infty$, $1 \le q \le \infty$ and $\alpha \in \mathbb{R}$, or $p = 1$, $q = 1$ and $\alpha \ge 0$, or $p = \infty$, $q = \infty$ and $\alpha \le 0$, or $p = \infty$, $1 \le q < \infty$ and $\alpha + \frac{1}{q} < 0$. Then also the functional given by

$$\|f\|_{L^{p,q}(\log L)^\alpha(0,1)} = \|s^{\frac{1}{p}-\frac{1}{q}} \; (1 + \log \tfrac{1}{s})^\alpha f^*(s)\|_{L^q(0,1)} \tag{2.34}$$

for $f \in L^0(0,1)$ is equivalent to a rearrangement-invariant function norm, and $\|\cdot\|_{L^{p,q}(\log L)^\alpha(0,1)}$ is called a *Lorentz–Zygmund function norm.* The corresponding space $L^{p,q}(\log L)^\alpha(0,1)$ is called a *Lorentz–Zygmund space.* A detailed study of (generalized) Lorentz-Zygmund spaces can be found in [27] or [15] – see also [28, Chapter 9].

Let us notice that, besides including the Lebesgue spaces ($p = q$, $\alpha = 0$) and the Lorentz spaces ($\alpha = 0$), the class of Lorentz-Zygmund spaces overlaps with that of the Orlicz spaces. Actually, it reproduces (up to equivalent norms) the Orlicz spaces $L^p(\log L)^\beta(0,1)$ ($1 < p = q$, $\alpha = \beta/p$) associated with any Young function equivalent to $t^p(\log t)^\beta$ near infinity, and the Orlicz spaces $\exp L^\beta(0,1)$ ($p = q = \infty$, $\alpha = -1/\beta$) associated with any Young function equivalent to $\exp(t^\beta)$ near infinity.

2.2. ***Sublinear one-dimensional operators.*** We refer to the monographs [2, 4] for unexplained terminology from nonlinear operator theory and Banach lattices. Here, we just recall that, given an operator $T\colon L^0(0,1) \to L^0(0,1)$, we say that T is *sublinear* if

$$|T(f+g)| \le |Tf| + |Tg|, \qquad \text{and} \qquad |T(af)| = |a|\,|Tf|, \tag{2.35}$$

for every $f, g \in L^0(0,1)$ and $a \in \mathbb{R}$; T is *positive* when

$$T(L^0_+(0,1)) \subseteq L^0_+(0,1); \tag{2.36}$$

T is *monotone* when

$$Tf \le Tg \qquad \text{for } f \le g \text{ in } L^0_+(0,1); \tag{2.37}$$

and T is *determined completely by its action on the positive cone* $L^0_+(0,1)$ when

$$Tf = T|f| \qquad \text{for } f \in L^0(0,1), \tag{2.38}$$

where the above relations are understood a.e. on $(0,1)$.

Let $T\colon L^0(0,1) \to L^0(0,1)$ be a sublinear operator, and let $\|\cdot\|_{X(0,1)}$ and $\|\cdot\|_{Y(0,1)}$ be two rearrangement-invariant norms. Then T is *bounded* from $X(0,1)$ to $Y(0,1)$, briefly

$$T\colon X(0,1) \to Y(0,1) \quad \text{is bounded} \tag{2.39}$$

if the quantity

$$\|T\| = \sup_{\substack{f \in X(0,1)\\ \|f\|_{X(0,1)} \le 1}} \|\,Tf\,\|_{Y(0,1)}$$

is finite. Such a quantity is called the *norm of the operator* T. Moreover, $T\colon X(0,1) \to Y(0,1)$ is compact if $T(M)$ is relatively compact in $Y(0,1)$ for every bounded set $M \subset X(0,1)$.

The space $Y(0,1)$ is called *optimal* in (2.39) within a certain class, if, whenever $Z(0,1)$ is another rearrangement-invariant space from the same class fulfilling the property that T is bounded from $X(0,1)$

to $Z(0,1)$, then $Y(0,1) \hookrightarrow Z(0,1)$. Equivalently, the corresponding function norm $\|\cdot\|_{Y(0,1)}$ is said to be optimal in (2.39) in the relevant class.

If $T\colon L^0(0,1) \to L^0(0,1)$ is a sublinear operator fulfilling condition (2.38), then the *associate operator* of T is the sublinear operator $T'\colon L^0(0,1) \to L^0(0,1)$ fulfilling property (2.38) and defined via the identity

$$\int_0^1 Tf(s)|g(s)|\,ds = \int_0^1 |f(s)|T'g(s)\,ds \qquad \text{for } f,g \in L^0(0,1).$$

Note that

$$T\colon X(0,1) \to Y(0,1) \quad \text{is bounded} \qquad \Longleftrightarrow \qquad T'\colon Y'(0,1) \to X'(0,1) \quad \text{is bounded} \tag{2.40}$$

and $\|T\| = \|T'\|$. This follows by a simple argument involving Fubini's theorem and the definition of the associate norm.

As hinted above, given $m \in \mathbb{N}$ and $d \in \mathbb{N}$ such that $1 \le d \le n$ and $d \ge n-m$, we deal with the Hardy type operators $R^m_{d,n}$ and $H^m_{d,n}$ defined by (1.5) and (1.6), respectively. Both of them are sublinear operators satisfying properties (2.36)–(2.38), and they are mutually associated, namely

$$\int_0^1 \left(\mathrm{H}^m_{d,n} f\right)(s)\,|g(s)|\,ds = \int_0^1 |f(s)|\left(\mathrm{R}^m_{d,n} g\right)(s)\,ds \qquad \text{for } f,g \in L^0(0,1). \tag{2.41}$$

Note that

$$\mathrm{H}^m_{d,n} f \quad \text{is non-increasing on } (0,1) \tag{2.42}$$

for every $f \in L^0(0,1)$.

Let us conclude this section with a few observations.

Given any rearrangement-invariant norms $\|\cdot\|_{X(0,1)}$ and $\|\cdot\|_{Y(0,1)}$, for a sublinear (not linear) operator $T\colon L^0(0,1) \to L^0(0,1)$ fulfilling condition (2.38), the fact that $T\colon X(0,1) \to Y(0,1)$ is bounded is just equivalent to its continuity at 0, and T may fail to be (uniformly) continuous. However, when T also fulfills the additional properties (2.36)–(2.37), the equivalence between boundedness and uniform continuity for T still holds; moreover, the well-known result about the continuity of positive linear operators between Banach lattice [5] (see, e.g., [4, Theorem 4.3]) pertains within such a sublinear setting. This is the content of our following result, which explains why we do not adhere to the convention of using "$\to$" in the notation $T\colon X(0,1) \to Y(0,1)$ to indicate boundedness for T, as in [10].

Proposition 2.1. *Let $\|\cdot\|_{X(0,1)}$ and $\|\cdot\|_{Y(0,1)}$ be rearrangement-invariant norms. For a sublinear operator $T\colon L^0(0,1) \to L^0(0,1)$ satisfying* (2.36)–(2.38)*, the following three properties are mutually equivalent:*

(i) $T\colon X(0,1) \to Y(0,1)$*;*
(ii) $T\colon X(0,1) \to Y(0,1)$ *is bounded;*
(iii) $T\colon X(0,1) \to Y(0,1)$ *is uniformly continuous.*

Proof. (i) *implies* (ii) Suppose, by contradiction, that $T\colon X(0,1) \to Y(0,1)$ is unbounded. Since T satisfies (2.38), then one may select a sequence $(f_k)_{k\in\mathbb{N}}$ in $X_+(0,1)$, with

$$\|f_k\|_{X(0,1)} \le 1 \qquad \text{and} \qquad \|Tf_k\|_{Y(0,1)} \ge k^3 \qquad \text{for each } k \in \mathbb{N}. \tag{2.43}$$

Since the series $\sum_k k^{-2} f_k$ is absolutely convergent in the Banach space $X(0,1)$, then it converges to some function $f \in X_+(0,1)$. Clearly, $0 \le k^{-2} f_k \le f$ a.e. on $(0,1)$. Thus, by (2.35)–(2.37),

$$0 \le \frac{1}{k^2}\,Tf_k \le Tf \qquad \text{a.e. on } (0,1).$$

Hence, via (2.3) and (2.5), the second estimate in (2.43) provides that

$$\|Tf\|_{Y(0,1)} \ge k^{-2}\|Tf_k\|_{Y(0,1)} \ge k \qquad \text{for all } k \in \mathbb{N},$$

namely, a contradiction to assumption (i). Consequently, $T\colon X(0,1) \to Y(0,1)$ is bounded.

(ii) *implies* (iii) Assume that (ii) holds. Equivalently, $T\colon X(0,1) \to Y(0,1)$ is continuous at 0, since T fulfills (2.38). For any $f,g \in X(0,1)$, then $|Tf - Tg| \in Y(0,1)$, according to (2.15); moreover, thanks to (2.36)–(2.38) and to the inequality in (2.35),

$$|Tf - Tg| \le T(f-g) \qquad \text{a.e. on } (0,1).$$

Hence, from (2.5) and (2.15) one infers that

$$\|Tf - Tg\|_{Y(0,1)} \leq \|T(f-g)\|_{Y(0,1)} \qquad \text{for } f,g \in X(0,1).$$

Hence, the conclusion follows by the continuity of T at 0.

(iii) *implies* (i) Obvious. □

Coupling Proposition 2.1 with (2.40) yields

$$\mathrm{H}^m_{d,n}\colon X(0,1) \to Y(0,1) \qquad \Longleftrightarrow \qquad \mathrm{R}^m_{d,n}\colon Y'(0,1) \to X'(0,1), \tag{2.44}$$

with $\|\mathrm{H}^m_{d,n}\| = \|\mathrm{R}^m_{d,n}\|$ finite.

3. Key auxilary results

We collect here results to be exploited in the proofs of Theorems 1.1 – 1.2. We start by focusing on Hardy type operators $\mathrm{H}^m_{d,n}$ defined by (1.6). According to Proposition 2.1, the next result clarifies that these operators are bounded on any rearrangement-invariant space and their behavior on bounded sequences is also shown.

Proposition 3.1. *Assume that n, m, d and $\|\cdot\|_{X(0,1)}$ are as in Theorem 1.1. Then*

$$\mathrm{H}^m_{d,n}\colon X(0,1) \to X(0,1). \tag{3.1}$$

Moreover, if $(f_k)_{k\in\mathbb{N}}$ is a bounded sequence in $X(0,1)$, then there exist a subsequence $(f_{k_l})_{l\in\mathbb{N}}$ of $(f_k)_{k\in\mathbb{N}}$ and some function $g \in L^0(0,1)$ such that

$$\mathrm{H}^m_{d,n} f_{k_l} \to g \qquad \text{a.e. in } (0,1).$$

Proof. We first observe that

$$\mathrm{H}^m_{d,n}\colon L^1(0,1) \to L^1(0,1), \qquad \text{and} \qquad \mathrm{H}^m_{d,n}\colon L^\infty(0,1) \to L^\infty(0,1). \tag{3.2}$$

Indeed, the use of (2.11), (2.42) and (2.41) yields that

$$\left\|\mathrm{H}^m_{d,n} f\right\|_{L^1(0,1)} = \sup_{\|g\|_{L^\infty(0,1)}\leq 1} \int_0^1 g^*(s)\, \mathrm{H}^m_{d,n} f(s)\, ds \leq \int_0^1 |f(s)|\, s^{\frac{m+d-n}{n}}\, ds \leq \|f\|_{L^1(0,1)}$$

for all$f \in L^1(0,1)$, because $m+d-n \geq 0$. Moreover, owing to (2.42),

$$\left\|\mathrm{H}^m_{d,n} f\right\|_{L^\infty(0,1)} = \int_0^1 t^{-1+\frac{m}{n}}\, |f(t)|\, dt \leq \frac{n}{m}\, \|f\|_{L^\infty(0,1)}$$

for all $f \in L^\infty(0,1)$.

Next, we define the linear operator

$$\hat{\mathrm{H}}^m_{d,n} f(s) = \int_{s^{\frac{n}{d}}}^1 t^{\frac{m}{n}-1} f(t)\, dt \qquad \text{for } s \in (0,1), \tag{3.3}$$

for $f \in L^1(0,1)$. Since

$$\hat{\mathrm{H}}^m_{d,n} = \mathrm{H}^m_{d,n} \qquad \text{on } X_+(0,1), \tag{3.4}$$

where $X_+(0,1) = X(0,1) \cap L^0_+(0,1)$, through Proposition 2.1, (3.2) and the use of (2.5), the linear operator $\hat{\mathrm{H}}^m_{d,n}$ turns out to be bounded on both $L^1(0,1)$ and $L^\infty(0,1)$, with norm depending only on n and m. By an interpolation theorem of Calderón [6, Chapter 3, Theorem 2.12], $\hat{\mathrm{H}}^m_{d,n}\colon X(0,1) \to X(0,1)$ is thus bounded on any rearrangement–invariant space $X(0,1)$, with norm depending only on n and m. Hence, conclusion (3.1) follows via property (2.38) of $\mathrm{H}^m_{d,n}$ and (3.4).

Now, let $(f_k)_{k\in\mathbb{N}}$ be a bounded sequence in $X(0,1)$. Then, by (3.1), the sequence $(\mathrm{H}^n_{d,m} f_k)_{k\in\mathbb{N}}$ is bounded on $X(0,1)$, and on $L^1(0,1)$ as well, owing to (2.19).

Fix any $j \in \mathbb{N}$, and define

$$g_{j,k} = (\mathrm{H}^n_{d,m} f_k)_{|(\frac{1}{j+1},1)} \qquad \text{for } k \in \mathbb{N}.$$

The sequence $(g_{j,k})_{k\in\mathbb{N}}$ is bounded in $W^{1,1}\left(\frac{1}{j},1\right)$. Indeed, it is bounded in $L^1\left(\frac{1}{j+1},1\right)$; moreover, exploiting (2.42), for each $k\in\mathbb{N}$ one infers that

$$g'_{j,k}(t)=-\frac{n}{d}\ \left|f_k(t^{\frac{n}{d}})\right|\ t^{-1+\frac{m}{d}}\qquad\text{for a.e. } t\in\left(\tfrac{1}{j+1},1\right),$$

and

$$\left\|g'_{j,k}\right\|_{L^1\left(\frac{1}{j+1},1\right)}=\frac{n}{d}\int_{\frac{1}{j+1}}^{1}\left|f_k(t^{\frac{n}{d}})\right|t^{-1+\frac{m}{d}}\,dt=\int_{\left(\frac{1}{j+1}\right)^{\frac{n}{d}}}^{1}\left|f_k(s)\right|s^{\frac{m}{n}-1}\,ds\le\max\left\{(j+1)^{1-\frac{m}{n}},1\right\}\left\|f_k\right\|_{L^1(0,1)}.$$

The compactness of the embedding $W^{1,1}\left(\frac{1}{j+1},1\right)\hookrightarrow L^1\left(\frac{1}{j+1},1\right)$ thus provides the existence of some function $g_j\in L^1\left(\frac{1}{j+1},1\right)$ such that

$$g_{j,k}\to g_j\qquad\text{a.e. in}\quad\left(\frac{1}{j+1},1\right),$$

up to subsequences. Hence, the conclusion follows via a diagonal argument, because $\left(\frac{1}{j+1},1\right)\nearrow(0,1)$. □

The previous result enables us to show that – roughly speaking – the optimal rearrangement-invariant norm in the Sobolev trace embedding cannot be worse than the rearrangement-invariant norm on the domain.

Proposition 3.2. *Assume that Ω, n,m,d and $\|\cdot\|_{X(0,1)}$ are as in Theorem 1.1. Then the Sobolev trace operator* $\mathrm{Tr}\colon W^mX(\Omega)\to X(\Omega_d)$ *is continuous. Moreover,*

$$X^m_{d,n}(\Omega_d)\hookrightarrow X(\Omega_d).\tag{3.5}$$

Proof. The first part of the statement follows from combining (3.1), Proposition 2.1 and [10, Theorem 1.3]. Hence, [10, Theorem 1.1] provides the latter. □

In light of Proposition 3.1, we now analyse optimal domain spaces for our Hardy type operators.

Proposition 3.3. *Assume that n,m and d are as in Theorem 1.1, and let $\|\cdot\|_{Y(0,1)}$ be a rearrangement-invariant norm. Then, the functional $\|\cdot\|_{Y^m_{d,n}(0,1)}\colon L^0(0,1)\to[0,\infty]$, defined as*

$$\|g\|_{Y^m_{d,n}(0,1)}=\sup_{L^0(0,1)\ni f\sim g}\left\|\mathrm{H}^m_{d,n}f\right\|_{Y(0,1)}\qquad\textit{for } g\in L^0(0,1),$$

is a rearrangement-invariant norm. Moreover,

$$\mathrm{H}^m_{d,n}\colon Y^m_{d,n}(0,1)\to Y(0,1),\tag{3.6}$$

and $Y^m_{d,n}(0,1)$ is the largest rearrangement-invariant space for which (3.6) holds.

In particular, when

$$m\le n\quad\textit{and}\quad Y(0,1)=L^\infty(0,1),\tag{3.7}$$

then

$$(L^\infty)^m_{d,n}(0,1)=L^{\frac{n}{m},1}(0,1).\tag{3.8}$$

Proof. The first part follows along the same lines as in [32, Proposition 4.5], taking into account Proposition 2.1. For the case $m\le n$, needed computations are just minor modifications of [32, Lemma 5.6] □

Remark 3.4. When (3.7) is in force, the optimal domain space for (3.6) is explicitly described by (3.8), and, interestingly enough, it is independent of the dimension d of the affine subspace.

An intimate connection between the optimal rearrangement-invariant domain space for embedding (3.6) and the compactness of the Hardy operator $\mathrm{H}^m_{d,n}$ is displayed by the following proposition.

Proposition 3.5. *Assume that n,m and d are as in Theorem 1.1, and let $\|\cdot\|_{X(0,1)}$, $\|\cdot\|_{Y(0,1)}$ be rearrangement-invariant norms. If*

$$X(0,1)\neq L^1(0,1),\tag{3.9}$$

then the following assertions are equivalent:

(i) *the Hardy type operator* $\mathrm{H}^m_{d,n}\colon X(0,1)\to Y(0,1)$ *is compact;*

(ii) $X(0,1) \overset{*}{\hookrightarrow} Y^m_{d,n}(0,1)$.

Proof. It follows the outline of [32, Theorem 4.6]. □

Next result exhibits sufficient conditions for weak sequential compactness in $L^1(\Omega)$ and $W^{1,1}(\Omega)$, respectively.

Theorem 3.6. *Let Ω be a open set in $\mathbb{R}^n$, with $n\geq 1$, of finite measure, and let $\|\cdot\|_{X(0,1)}$ be a rearrangement-invariant function norm fulfilling condition* (3.9)*.*

(i) *If M is a bounded set in $X(\Omega)$, then M is bounded and relatively weakly (sequentially) compact in $L^1(\Omega)$.*

(ii) *If $(u_k)_{k\in\mathbb{N}}$ is a bounded sequence in $W^1X(\Omega)$, then $(u_k)_{k\in\mathbb{N}}$ is bounded in $W^{1,1}(\Omega)$, and there exist a subsequence $(u_{k_l})_{l\in\mathbb{N}}$ of $(u_k)_{k\in\mathbb{N}}$ and a function $u\in W^{1,1}(\Omega)$ such that*

$$u_{k_l}\rightharpoonup u \qquad \text{in } \ W^{1,1}(\Omega).$$

Proof. (i) Let M be a bounded set in $X(\Omega)$. Then, since $|\Omega|<\infty$, the second embedding in (2.19) tells us that M is bounded in $L^1(\Omega)$. On account of Dunford and Pettis theorem (see, e.g., [3, Theorem 5.2.8]), it thus suffices to prove that M is equi-integrable in $L^1(\Omega)$, i.e.

$$\lim_{|E|\to 0^+}\sup_{u\in M}\|u\|_{L^1(E)}=0. \tag{3.10}$$

To see this, note that

$$\sup_{u\in M}\|u\|_{L^1(E)} \ \leq \ \left(|\Omega| \ \sup_{u\in M}\|u\|_{X(\Omega)}\right) \ \|\chi_E\|_{X'(\Omega)} \tag{3.11}$$

for every Lebesgue measurable set $E\subseteq\Omega$, via Hölder inequality (2.17). Now assumption (3.9) tells us that $X'(\Omega)\neq L^\infty(\Omega)$. So, [31, Theorem 5.2] assures that

$$\lim_{|E|\to 0^+}\|\chi_E\|_{X'(\Omega)}=0. \tag{3.12}$$

Hence, (3.10) follows from (3.11) and (3.12).

(ii) Assume that $(u_k)_{k\in\mathbb{N}}$ is a bounded sequence in $W^1X(\Omega)$. In view of (i), $(u_k)_{k\in\mathbb{N}}$ is bounded in $W^{1,1}(\Omega)$ and there exist $n+1$ functions $u,w_1,\dots,w_n$ in $L^1(\Omega)$ such that

$$u_k\rightharpoonup u \quad \text{in } L^1(\Omega), \qquad \text{and} \qquad \frac{\partial u_k}{\partial x_i}\rightharpoonup w_i \quad \text{in } L^1(\Omega) \quad \text{for } i=1,\dots,n,$$

up to subsequences.

Fix $i=1,\dots,n$. For any $\varphi\in C^\infty_c(\Omega)$,

$$\int_\Omega w_i\ \varphi\,dx=\lim_{k\to\infty}\int_\Omega\frac{\partial u_k}{\partial x_i}\ \varphi\,dx=\lim_{k\to\infty}\ -\int_\Omega u_k\ \frac{\partial\varphi}{\partial x_i}\,dx= \ -\int_\Omega u\frac{\partial\varphi}{\partial x_i}\,dx.$$

Hence, $w_i=\frac{\partial u}{\partial x_i}$. This proves that $u\in W^{1,1}(\Omega)$, and $u_k\rightharpoonup u$ in $W^{1,1}(\Omega)$, up to subsequences. □

In the next proposition, we point out the property, which is of independent interest, that any rearrangement-invariant space fulfilling condition (3.9) is always contained in an Orlicz space different from $L^1(0,1)$.

Proposition 3.7. *Let Ω be a open set in $\mathbb{R}^n$, with $n\geq 1$, of finite measure. If $\|\cdot\|_{X(0,1)}$ is rearrangement-invariant function norm fulfilling condition* (3.9)*, then there exists an Orlicz norm $\|\cdot\|_{L^A(0,1)}$ such that*

$$X(\Omega)\subseteq L^A(\Omega)\subsetneq L^1(\Omega). \tag{3.13}$$

Proof. Let $\overline{B}_{X(\Omega)}(0,1)$ be the closed unit ball of $X(\Omega)$. Then condition (3.10) actually holds with M replaced by $\overline{B}_{X(\Omega)}(0,1)$, namely, $\overline{B}_{X(\Omega)}(0,1)$ is equi-integrable in $L^1(\Omega)$. By La Vallée Poussin theorem [25, Theorem T22], this is equivalent to the existence of a function $A\colon[0,\infty)\to[0,\infty]$, $A(0)=0$, which is non-decreasing, convex, and such that

$$\sup_{\|u\|_{X(0,1)}\leq 1}\int_\Omega A\left(|u(x)|\right)\ dx\leq 1, \qquad \text{and} \qquad \lim_{s\to\infty}\frac{A(s)}{s}=\infty. \tag{3.14}$$

Hence, the first and second inclusion in (3.13) follows from the first and second condition, respectively, described in (3.14). □

We conclude the section dealing with traces of functions from higher-order Sobolev type spaces built upon a rearrangement-invariant function norm. We preliminarily highlight the behaviour of the trace operator on $W^{1,1}(\Omega)$ with respect to the superposition mapping

$$v \in W^{1,1}(\Omega) \mapsto f \circ v \in W^{1,1}(\Omega) \tag{3.15}$$

associated with any Lipschitz function $f\colon \mathbb{R} \to \mathbb{R}$. As usual, u^+ denotes below the positive part of a function u.

Lemma 3.8. *Let Ω be a bounded open set in $\mathbb{R}^n$, $n \geq 2$, satisfying the cone property. If $u \in W^{1,1}(\Omega)$, then*

$$\operatorname{Tr}(f \circ u) = f \circ (\operatorname{Tr} u) \qquad \mathcal{H}^{n-1}\text{-a.e. on } \Omega_{n-1} \tag{3.16}$$

for every Lipschitz function $f\colon \mathbb{R} \to \mathbb{R}$. In particular,

(i) *$\operatorname{Tr}(|u|) = |\operatorname{Tr} u|$ and $\operatorname{Tr}(u^+) = (\operatorname{Tr} u)^+$ $\mathcal{H}^{n-1}$-a.e. on Ω_{n-1};*
(ii) *if $u \leq M$ a.e. in Ω for some $M \in \mathbb{R}$, then $\operatorname{Tr} u \leq M$ $\mathcal{H}^{n-1}$-a.e. on Ω_{n-1}.*

Proof. Let $u \in W^{1,1}(\Omega)$. Assume that $f\colon \mathbb{R} \to \mathbb{R}$ is a Lipschitz function. Then $f \circ u \in W^{1,1}(\Omega)$ and

$$\frac{\partial (f \circ u)}{\partial x_i}(x) = f'(u(x))\, \frac{\partial u}{\partial x_i}(x) \qquad \text{for a.e. } x \in \Omega,$$

for each $i \in \{1, \dots, n\}$, where $f'(u(x))\, \frac{\partial u}{\partial x_i}(x)$ is interpreted to be zero whenever $\frac{\partial u}{\partial x_i}(x) = 0$ (see, e.g., [36, Theorem 2.1.11]).
Now, via Meyers-Serrin theorem (see, e.g., [2, Theorem 3.17]), pick a sequence $(\phi_k)_{k\in\mathbb{N}}$ in $C^1(\Omega) \cap W^{1,1}(\Omega)$ such that

$$\phi_k \to u \qquad \text{in } W^{1,1}(\Omega). \tag{3.17}$$

Then

$$f \circ \phi_k \to f \circ u \qquad \text{in } W^{1,1}(\Omega),$$

because of the continuity of the superposition mapping (3.15) associated with f [22, Theorem 1].
Hence,

$$\operatorname{Tr}(f \circ \phi_k) \to \operatorname{Tr}(f \circ u) \qquad \text{in } L^1(\Omega_{n-1}), \tag{3.18}$$

since $\operatorname{Tr}\colon W^{1,1}(\Omega) \hookrightarrow L^1(\Omega_{n-1})$.
Since ϕ_k and $f \circ \phi_k$ are continuous for each $k \in \mathbb{N}$, one infers that

$$\operatorname{Tr} \phi_k = \phi_k \quad \text{and} \quad \operatorname{Tr}(f \circ \phi_k) = f \circ \phi_k \qquad \text{on } \Omega_{n-1}. \tag{3.19}$$

Thus, coupling (3.18) with (3.19) tells us that

$$\operatorname{Tr}(f \circ \phi_k) = f \circ (\operatorname{Tr} \phi_k) \qquad \mathcal{H}^{n-1}\text{-a.e. on } \Omega_{n-1}. \tag{3.20}$$

By the continuity of the mapping $v \in L^1(\Omega_{n-1}) \mapsto f \circ v \in L^1(\Omega_{n-1})$ and of the trace embedding $\operatorname{Tr}\colon W^{1,1}(\Omega) \hookrightarrow L^1(\Omega_{n-1})$, via (3.17) and (3.20), one thus entails that

$$f \circ (\operatorname{Tr} \phi_k) \to f \circ (\operatorname{Tr} u) \qquad \text{in } L^1(\Omega_{n-1}). \tag{3.21}$$

Hence, combining (3.18), (3.20) and (3.21) yields (3.16).
Then, (i) follows from (3.16) applied with $f(x) = |x|$ and $f(x) = \max\{x, 0\}$, $x \in \mathbb{R}$, whereas for (ii) use $f(x) = \min\{x, M\}$, $x \in \mathbb{R}$. □

Last result of this section is a key step towards the proof of Theorem 1.1 in the case when $m < n$. Throughout, we will adopt the usual convention that, for $1 \leq k < n$, a point $x \in \mathbb{R}^n$, $n \geq 2$, is labelled by (x', x''), where $x' = (x_1, \dots, x_{n-k}) \in \mathbb{R}^{n-k}$ and $x'' = (x_{n-k+1}, \dots, x_n) \in \mathbb{R}^k$. Given a Lebesgue measurable set $E \subset \mathbb{R}^n$, for $x' \in \mathbb{R}^{n-k}$, we set

$$E_{x'} = \{x'' \in \mathbb{R}^k \colon (x', x'') \in E\}.$$

For $v \in L^1(E)$, with a slight abuse of notation, we set

$$\int_{E_{x'}} v(x', x'')\, dx'' = 0$$

when $E_{x'}$ is empty, so that, by Fubini's theorem,

$$\int_E v(x)\, dx = \int_{\mathbb{R}^{n-k}} \int_{E_{x'}} v(x', x'')\, dx'' dx'.$$

Finally,

$$\pi_k(E) = \{x' \in \mathbb{R}^{n-k} \colon (x', 0) \in E\}.$$

Theorem 3.9. *Let Ω be a bounded open set in $\mathbb{R}^n$, $n \geq 2$, satisfying the cone property, and let $\|\cdot\|_{X(0,1)}$ be rearrangement-invariant function norm fulfilling* (3.9). *Assume that $m \in \mathbb{N}$, with $1 \leq m < n$.*

If $(u_k)_{k\in\mathbb{N}}$ is a bounded sequence in $W^m X(\Omega)$, then there exist a subsequence $(u_{k_l})_{l\in\mathbb{N}}$ of $(u_k)_{k\in\mathbb{N}}$ and a function $g \in L^0(\Omega_{n-m})$ such that

$$\operatorname{Tr} u_{k_l} \to g \qquad \mathcal{H}^{n-m}\text{-a.e in } \Omega_{n-m}. \tag{3.22}$$

Proof. On taking, if necessary, a rigid motion, we may assume that $0 \in \Omega_{n-m}$, and

$$\Omega_{n-m} = \Omega \cap \{x = (x', 0) \colon x' \in \mathbb{R}^{n-m}, 0 \in \mathbb{R}^m\}. \tag{3.23}$$

We distinguish the cases when $m = 1$ and $m \geq 2$.

Case $m = 1$. Let $(u_k)_{k\in\mathbb{N}}$ be a bounded sequence in $W^1X(\Omega)$. By Proposition 3.2, applied with $m = 1$ and $d = n-1$, the sequence $(\operatorname{Tr} u_k)_{k\in\mathbb{N}}$ is bounded in $X^1_{n-1,n}(\Omega_{n-1})$. Note that $X^1_{n-1,n}(\Omega_{n-1}) \neq L^1(\Omega_{n-1})$, owing to (3.5) and (3.9). Thus, Theorem 3.6 (i) ensures that the sequence $(\operatorname{Tr} u_k)_{k\in\mathbb{N}}$ converges weakly in $L^1(\Omega_{n-1})$ to some $g \in L^1(\Omega_{n-1})$, up to subsequences.

We only need to show that $g = \operatorname{Tr} u$ for some $u \in W^1X(\Omega)$, and $(\operatorname{Tr} u_k)_{k\in\mathbb{N}}$ converges to $\operatorname{Tr} u$ in $L^1(\Omega_{n-1})$, up to subsequences. Indeed, this will imply (3.22), thanks to (2.16).
For that, we are going to use the fact that for any function $w \in W^{1,1}(\Omega)$, there exist a function $\bar{w} \colon \Omega \to \mathbb{R}$ and a measurable subset N_w of Ω_{n-1}, with $\mathcal{H}^{n-1}(N_w) = 0$, such that

(3.24) $w = \bar{w}$ a.e on Ω_{n-1};

(3.25) the function $\bar{w}(x', \cdot)$ is absolutely continuous on the connected component of 0 in $\Omega_{x'}$, for all $x' \in \pi_1(\Omega_{n-1} \setminus N_w)$.

(see e.g. [36, Theorem 2.1.4]).

We first observe that, by Theorem 3.6 (ii), the sequence $(u_k)_{k\in\mathbb{N}}$ is bounded in $W^{1,1}(\Omega)$, and there exists some $u \in W^{1,1}(\Omega)$ such that

$$\bar{u}_k \rightharpoonup \bar{u} \qquad \text{in } \ W^{1,1}(\Omega), \tag{3.26}$$

up to subsequences. On the other hand, since Ω satisfies the cone property, then $\operatorname{Tr} \colon W^{1,1}(\Omega) \hookrightarrow L^1(\Omega_{n-1})$ [2, Theorem 4.11] and, clearly,

$$\operatorname{Tr} u_k = \operatorname{Tr} \bar{u}_k \quad \text{for each } k \in \mathbb{N}, \quad \text{and} \quad \operatorname{Tr} u = \operatorname{Tr} \bar{u} \qquad \mathcal{H}^{n-1}\text{-a.e on } \Omega_{n-1}.$$

Therefore, our conclusion will follow once it is verified that

$$\operatorname{Tr} \bar{u}_k \to \operatorname{Tr} \bar{u} \qquad \mathcal{H}^{n-1}\text{-a.e in } \Omega_{n-1}, \tag{3.27}$$

up to subsequences. Note that coupling (3.26) with the compactness of the embedding $W^{1,1}(\Omega) \hookrightarrow L^1(\Omega)$ implies that

$$\bar{u}_k \to \bar{u} \quad \text{in } L^1(\Omega), \qquad \frac{\partial \bar{u}_k}{\partial x_i} \rightharpoonup \frac{\partial \bar{u}}{\partial x_i} \quad \text{in } L^1(\Omega) \quad \text{for } i = 1, \dots, n,$$

up to subsequences. Hence, on setting

$$v_k = \bar{u}_k - \bar{u} \qquad \text{for each } k \in \mathbb{N},$$

claim (3.27) can be rewritten as

$$\operatorname{Tr} v_k \to 0 \qquad \mathcal{H}^{n-1}\text{-a.e in } \Omega_{n-1}, \tag{3.28}$$

up to subsequences.

Clearly, the sequence $(v_k)_{k\in\mathbb{N}}$ is bounded in $W^{1,1}(\Omega)$, and

$$v_k \to 0 \quad \text{in } L^1(\Omega), \quad \text{and} \quad \frac{\partial v_k}{\partial x_i} \rightharpoonup 0 \quad \text{in } L^1(\Omega) \quad \text{for } i = 1, \dots, n, \tag{3.29}$$

up to subsequences. Thus, in particular, the set $\left\{ \frac{\partial v_k}{\partial x_n} : k \in \mathbb{N} \right\}$ is bounded and relatively weakly compact in $L^1(\Omega)$. Hence, coupling Dunford and Pettis theorem and a criterion for equi-integrablity in $L^1(\Omega)$ (see, e.g., [3, Theorem 5.2.8, and Lemma 5.2.5]) yields that

$$\lim_{M\to\infty} \sup_k \int_{\left\{x\in\Omega:\, \left|\frac{\partial v_k}{\partial x_n}(x)\right| > M\right\}} \left| \frac{\partial v_k}{\partial x_n}(x) \right| dx = 0. \tag{3.30}$$

Moreover, for each $k \in \mathbb{N}$, there exists a measurable subset N_{v_k} of Ω_{n-1}, with $\mathcal{H}^{n-1}(N_{v_k}) = 0$, such that

(3.31) $v_k(x', \cdot)$ is absolutely continuous on the connected component of 0 in $\Omega_{x'}$, for every $x' \in \pi_1(\Omega_{n-1} \setminus N_{v_k})$.

Let us define $N = \bigcup_{k\in\mathbb{N}} N_{v_k}$. Then, $\mathcal{H}^{n-1}(N) = 0$, and $\mathcal{H}^{n-1}(\Omega_{n-1} \setminus N) < \infty$. For each $x \in \Omega_{n-1} \setminus N$, choose $r_x > 0$ so that the closed cube $\overline{Q}(x, r_x)$ in $\mathbb{R}^n$, with center at x and edges of length $2r_x$ parallel to the coordinate axes, is contained in Ω. Then the collection

$$\mathcal{R} = \{\overline{Q}(x, r) : x \in \Omega_{n-1} \setminus N, r \in (0, r_x]\}$$

of closed cube covers $\Omega_{n-1} \setminus N$, and Vitali-Besicovitch theorem (see, e.g., [14, Theorem 1.28]) provides a countable family $\mathcal{G} \subseteq \mathcal{R}$ consisting of disjoint cube and such that

$$\mathcal{H}^{n-1}\Big((\Omega_{n-1} \setminus N) \setminus \bigcup_{\overline{Q}\in\mathcal{G}} \overline{Q} \Big) = 0. \tag{3.32}$$

We claim that

$$\operatorname{Tr}(v_k \chi_{\overline{Q}}) \to 0 \qquad \mathcal{H}^{n-1}\text{-a.e. in } \Omega_{n-1}, \tag{3.33}$$

up to subsequences, for every cube $\overline{Q} \in \mathcal{G}$.
To begin with, notice that if $\overline{Q} \in \mathcal{G}$, then

$$\overline{Q} = \overline{Q'}(x', r) \times [-r, r] \subset \Omega, \tag{3.34}$$

for some $x' \in \pi_1(\Omega_{n-1} \setminus N)$ and $r > 0$. Here, $\overline{Q'} = \overline{Q'}(x', r)$ stands for the closed cube in $\mathbb{R}^{n-1}$, with center at x' and edges of length $2r$ parallel to the coordinate axes. Moreover,

$$\overline{Q'}(x', r) \subset \pi_1(\Omega_{n-1}).$$

For each $k \in \mathbb{N}$, property (3.31) implies that $\operatorname{Tr}(v_k \chi_{\overline{Q}})$ and v_k agree $\mathcal{H}^{n-1}$-a.e. in $\Omega_{n-1} \cap \overline{Q}$, or, equivalently,

$$\operatorname{Tr}(v_k \chi_{\overline{Q}}) = v_k(\cdot, 0) \qquad \text{a.e. on } \overline{Q'}(x', r), \tag{3.35}$$

as well as that

$$v_k(x', h) = v_k(x', 0) + \int_0^h \frac{\partial v_k}{\partial x_n}(x_n)\, dx_n \quad \text{for every } h \in (-r, r). \tag{3.36}$$

Since $\pi_1(N)$ is a negligible subset of $\pi_1(\Omega_{n-1})$, from (3.36) and Tonelli's theorem one infers that

$$\begin{aligned} \|v_k(\cdot, 0) - v_k(\cdot, h)\|_{L^1(\overline{Q'})} &\le \int_{\overline{Q'}} \left| \int_0^h \left| \frac{\partial v_k}{\partial x_n}(x', x_n) \right| dx_n \right| dx' \\ &= \int_{\overline{Q'}} \left| \int_0^h \left| \frac{\partial v_k}{\partial x_n}(x', x_n) \right| \chi_{S_M}(x', x_n)\, dx_n + \int_0^h \left| \frac{\partial v_k}{\partial x_n}(x', x_n) \right| \chi_{\overline{Q}\setminus S_M}(x', x_n)\, dx_n \right| dx' \\ &\le \int_{S_M} \left| \frac{\partial v_k}{\partial x_n}(x) \right| dx + M\, |h|\, |Q'|, \end{aligned} \tag{3.37}$$

for every $k \in \mathbb{N}$, $h \in (-r, r)$ and $M \in \mathbb{R}_+$. Here, $S_M = \left\{ x \in \overline{Q} : \left| \frac{\partial v_k}{\partial x_n}(x) \right| > M \right\}$.

We also note that

$$\liminf_{k\to\infty} \|v_k(\cdot,h)\|_{L^1(\overline{Q'})} = 0 \qquad \text{for a.e. } h \in (-r,r), \tag{3.38}$$

up to subsequences.
Indeed, from Fatou's lemma and the first convergence in (3.29) it follows that

$$\int_{-r}^{r} \liminf_{k\to\infty} \int_{\overline{Q'}} |v_k(x',x_n)|\, dx'\, dx_n \le \lim_{k\to\infty} \int_{-r}^{r} \int_{\overline{Q'}} |v_k(x',x_n)|\, dx'\, dx_n = \lim_{k\to\infty} \|v_k\|_{L^1(\overline{Q})} = 0.$$

Owing to (3.30), (3.34) and (3.37), fixing any $\epsilon > 0$, there exists some $M_0 \in \mathbb{R}_+$ such that

$$\|v_k(\cdot,0) - v_k(\cdot,h)\|_{L^1(\overline{Q'})} \le \epsilon + M_0\, |h|\, |\overline{Q'}| \quad \text{for every } h \in (-r,r),$$

up to subsequence. On the other hand, the first convergence in (3.29) and (3.38) allow us to pick some $h_0 \in \mathbb{R}$, with $0 < |h_0| < \min\left\{\frac{\epsilon}{M_0\,|Q'|}, r\right\}$, such that

$$\|v_k\|_{L^1(\overline{Q'}\times\{h_0\})} \to 0, \tag{3.39}$$

up to subsequences. Consequently,

$$\|v_k(\cdot,0) - v_k(\cdot,h_0)\|_{L^1(\overline{Q'})} \le\ 2\epsilon,$$

and

$$\|v_k(\cdot,0)\|_{L^1(\overline{Q'})} \le \|v_k(\cdot,0) - v_k(\cdot,h_0)\|_{L^1(\overline{Q'})} + \int_{\overline{Q'}} |v_k(x',h_0)|\, dx' \le 2\varepsilon + \|v_k\|_{L^1(\overline{Q'}\times\{h_0\})},$$

up to subsequences. Thus, by (3.39) and the arbitrariness of ε, one concludes that

$$v_k(\cdot,0) \to 0 \qquad \text{in } \ L^1(\overline{Q'}(x',r)),$$

up to subsequences, which ends the proof of the claim (3.33), by (3.35).

Hence, conclusion (3.28) follows from (3.32), (3.33) and the disjointness of the countable elements of $\mathcal{G}$. For $n = 2$, this also ends the proof.

Case $m \ge 2$. We first observe that, by [10, Theorem 1.5], for each $j \in \{0,\cdots,m\}$

$$X^m_{n-m,n}(\Omega_{n-m}) = \left(X^{m-j}_{n-m+j,n}\right)^j_{n-m,n-m+j}(\Omega_{n-m})$$

and

$$\mathrm{Tr} : W^{m-j}X(\Omega) \hookrightarrow X^{m-j}_{n-m+j,n}(\Omega_{n-m+j}). \tag{3.40}$$

Clearly,

$$\Omega_{n-m+j} \subseteq \Omega_{n-m+j+1} \qquad \text{for } j \in \{0,\cdots,m-1\},$$

where, according to (3.23), $\Omega_{n-m+j} = \Omega \cap \{x = (x',0) \colon x' \in \mathbb{R}^{n-m+j}, 0 \in \mathbb{R}^{m-j}\}$.

Because (3.9) is in force, via (3.40), Proposition 3.2 assures that

$$X^{m-j}_{n-m+j,n}(0,1) \ne L^1(0,1) \qquad \text{for } j \in \{0,\cdots,m\}. \tag{3.41}$$

Let $(u_k)_{k\in\mathbb{N}}$ be a bounded sequence in $W^mX(\Omega)$. Then, by the trace embedding (3.40) with $j=1$, the sequence $(\mathrm{Tr}\, u_k)_{k\in\mathbb{N}}$ is bounded in $W^1X^{m-1}_{n-m+1,n}(\Omega_{n-m+1})$, with $\|\cdot\|_{X^{m-1}_{n-m+1,n}}$ fulfilling condition (3.9) thanks to (3.41).

For each $k \in \mathbb{N}$, let

$$w_k(x',t) = \mathrm{Tr}\, u_k(x',t,0) \qquad \text{for } \ \mathcal{H}^{n-m+1}\text{-a.e} \quad (x',t) \in \pi_{m-1}(\Omega_{n-m+1}). \tag{3.42}$$

Clearly, the sequence $(w_k)_{k\in\mathbb{N}}$ is bounded in $W^1X(\pi_{m-1}(\Omega_{n-m+1}))$. The set $\pi_{m-1}(\Omega_{n-m+1})$ is open and bounded in $\mathbb{R}^{n-m+1}$, but it may fail to satisfy the cone property. Moreover, the set $\pi_m(\Omega_{n-m})$ is (open and) bounded in $\mathbb{R}^{n-m}$, and

$$\pi_m(\Omega_{n-m}) \times \{0\} \subseteq \pi_{m-1}(\Omega_{n-m+1}). \tag{3.43}$$

Let

$$\mathcal{R} = \{Q((x',0),r_{x'}) \colon x' \in \pi_m(\Omega_{n-m}), r_{x'} > 0\}$$

be a covering of $\pi_m(\Omega_{n-m})\times\{0\}$ consisting of open cubes in $\mathbb{R}^{n-m+1}$ such that

$$\overline{Q}((x',0),r_{x'})=\overline{Q'}(x',r_{x'})\times[-r_{x'},r_{x'}]\subset\pi_{m-1}(\Omega_{n-m+1}), \tag{3.44}$$

$$\overline{Q'}(x',r_{x'})\subset\pi_m(\Omega_{n-m}). \tag{3.45}$$

Here, $Q'(x',r_{x'})$ is the open cube in $\mathbb{R}^{n-m}$, with center at x' and edges of length $2r_{x'}$ parallel to the coordinate axes.

By a Besicovitch type theorem [12, Theorem 1.1], there exist some $N\in\mathbb{N}\cup\{\infty\}$, $\theta=\theta(n-m+1)\in\mathbb{R}_+$, sequences $(x_i')_{i=1}^N$, $(r_i)_{i=1}^N$, in $\pi_m(\Omega_{n-m})$ and in $\mathbb{R}_+$, respectively, such that

$$\pi_m(\Omega_{n-m})\times\{0\}\ \subset\bigcup_{i=1}^N\overline{Q}((x_i',0),r_i)\subset\pi_{m-1}(\Omega_{n-m+1}) \tag{3.46}$$

and

$$\sum_{i=1}^N\chi_{Q((x_i',0),r_i)}\le\theta\qquad\text{a.e on }\mathbb{R}^{n-m+1}. \tag{3.47}$$

Thanks to (3.44) and (3.45),

$$\pi_m(\Omega_{n-m})=\bigcup_{i=1}^N Q'(x_i',r_i)\cup\bigcup_{i=1}^N\partial Q'(x_i',r_i), \tag{3.48}$$

where $\mathcal{H}^{n-m}(\bigcup_{i=1}^N\partial Q'(x_i',r_i))=0$.

Fix any $i\in\mathbb{N}$, with $i\le N$. The restriction of each w_k to $Q_i=Q((x_i',0),r_i)$ belongs to $W^1X^{m-1}_{n-m+1,n}(Q_i)$, and, with a slight abuse of notation, the sequence $(w_k)_{k\in\mathbb{N}}$ is bounded in $W^1X^{m-1}_{n-m+1,n}(Q_i)$. Since $\|\cdot\|_{X^{m-1}_{n-m+1,n}}$ fulfills (3.9), from the case $m=1$ applied with Ω replaced by Q_i, we deduce that

$$\operatorname{Tr}w_k\to g_i\qquad\mathcal{H}^{n-m}\text{-a.e in }Q'(x_i',r_i) \tag{3.49}$$

for some function $g_i\in L^1(Q_i')$, up to subsequences. Set $(g_i)_0$ to be the zero extension of g_i on $\pi_m(\Omega_{n-m})$.

Hence, according to (3.48), we infer from (3.42), (3.49) and the fact that the sum is locally finite owing to (3.47), via the well-known diagonal method, that (3.22) holds with

$$g=\sum_{i=1}^N(g_i)_0\chi_{Q_i'\setminus(\bigcup_{j<i}Q_j')}. \qquad\square$$

4. Proofs of the main results

Proof of Theorem 1.1. Case $m<n$. (i) *implies* (ii). On taking, if necessary, a rigid motion, we may assume that $0\in\Omega_d$, and $\Omega_d=\Omega\cap\{x=(x',0)\colon x'\in\mathbb{R}^d\,,0\in\mathbb{R}^{n-d}\}$. Then, choose $R>0$ so that $B_R(0)\subset\subset\Omega$, and $|B_R(0)|=\omega_nR^n\le1$.

Assume that $(f_k)_{k\in\mathbb{N}}$ is a bounded sequence in $X(0,1)$, and set $g_k=\mathcal{D}_{\omega_nR^n}f_k$ for each $k\in\mathbb{N}$, where $\mathcal{D}_{\omega_nR^n}$ is the dilation operator (2.13). Then the sequence $(u_k)_{k\in\mathbb{N}}$, where

$$u_k(x)=\begin{cases}\int_{\omega_n|x|^n}^{\omega_nR^n}\int_{r_1}^{\omega_nR^n}\cdots\int_{r_{m-1}}^{\omega_nR^n}g_k(r_m)r_m^{-m+\frac{m}{n}}\,dr_m\,\dots\,dr_1 & \text{if }x\in B_R(0),\\ 0 & \text{if }x\in\Omega\setminus B_R(0),\end{cases}$$

is well-defined, each u_k belongs to $W^mX(\Omega)$, and

$$\|u_k\|_{W^mX(\Omega)}\le C\|g_k(\omega_nR^n\,\cdot)\|_{X(0,1)}=C\|f_k\|_{X(0,1)}$$

for some positive constant $C=C(m,n,R,|\Omega|)$ (see [10, proof of Theorem 1.3, and (4.20)]). Thus, the sequence $(u_k)_{k\in\mathbb{N}}$ is bounded in $W^mX(\Omega)$. By (i), the sequence $(\operatorname{Tr}u_k)_{k\in\mathbb{N}}$ converges (up to subsequences) to some function in $Y(\Omega_d)$, and therefore it is (up to subsequences) a Cauchy sequence in $Y(\Omega_d)$.

Since, by [10, proof of Theorem 1.3, and (4.21)],

$$\|\operatorname{Tr}u_k\|_{Y(\Omega_d)}\ge C'\left\|\int_{s^{\frac{n}{d}}}^1 g_k(\omega_nR^nr)\,r^{-1+\frac{m}{n}}\,dr\right\|_{Y(0,1)}=C'\left\|\mathrm{H}^m_{d,n}f_k\right\|_{Y(0,1)}$$

for every $k \in \mathbb{N}$, where $C' = C'(\Omega)$ is a positive constant, then $(\mathrm{H}^m_{d,n} f_k)_{k\in\mathbb{N}}$ is (up to subsequences) a Cauchy sequence in $Y(0,1)$. Hence, the completeness of $Y(0,1)$ ensures the existence of some function $g \in Y(0,1)$ such that

$$\mathrm{H}^m_{d,n} f_k \to g \qquad \text{in } Y(0,1),$$

up to subsequences.

(ii) *implies* (iii). The proof proceeds along the same lines as those of [32, Lemma 4.10, Step (i)⇒(ii)] since the assumption (1.7) is not exploited to infer the result in this step.

(iii) *implies* (iv). Let us preliminarily observe that, by [31, Theorem 5.2], assumption (1.7) is equivalent to

$$\lim_{r\to 0^+} \|\chi_{(0,r)}\|_{Y(0,1)} = 0. \tag{4.1}$$

Take any $f \in X(0,1)$ and $r \in (0,1)$. Then

$$\chi_{(0,r)} \mathrm{H}^m_{d,n} f \le \mathrm{H}^m_{d,n}(f\chi_{(0,a)}) + \chi_{(0,r)}\, \mathrm{H}^m_{d,n}(f\chi_{(a,1)}) \tag{4.2}$$

for every $a \in (0,1)$.

Note that, for any $a \in (0,1)$, the function $\mathrm{H}^m_{d,n}(f\chi_{(a,1)}) \in L^\infty(0,1)$.
In fact,

$$\mathrm{H}^m_{d,n}(f\chi_{(a,1)}) \le a^{\frac{m}{n}-1}\|f\|_{L^1(0,1)} \le C_X a^{\frac{m}{n}-1}\|f\|_{X(0,1)} \qquad \text{on } (0,1), \tag{4.3}$$

where C_X is the embedding constant of $X(0,1)$ into $L^1(0,1)$.

Combining (2.2), (2.5), (4.2) and (4.3) yields that

$$\|\chi_{(0,r)}\, \mathrm{H}^m_{d,n} f\|_{Y(0,1)} \le \|\mathrm{H}^m_{d,n}(f\chi_{(0,a)})\|_{Y(0,1)} + C_X a^{\frac{m}{n}-1}\|f\|_{X(0,1)}\|\chi_{(0,r)}\|_{Y(0,1)}$$

for $a \in (0,1)$. Hence, owing to (4.1), one entails that

$$\begin{aligned}\lim_{r\to 0^+} \sup_{\|f\|_{X(0,1)}\le 1} \|\chi_{(0,r)}\, \mathrm{H}^m_{d,n} f\|_{Y(0,1)} &\le \sup_{\|f\|_{X(0,1)}\le 1} \|\mathrm{H}^m_{d,n}(f\chi_{(0,a)})\|_{Y(0,1)} + C_X a^{\frac{m}{n}-1} \lim_{r\to 0^+} \|\chi_{(0,r)}\|_{Y(0,1)} \\ &= \sup_{\|f\|_{X(0,1)}\le 1} \|\mathrm{H}^m_{d,n}(f\chi_{(0,a)})\|_{Y(0,1)}\end{aligned}$$

for $a \in (0,1)$. The conclusion thus follows from assumption (iii).

(iv) *is equivalent to* (v). On account of (2.21) and (2.22), assertion (v) may be rewritten as

$$\lim_{r\to 0^+} \sup_{\|g\|_{Y'(0,1)}\le 1} \left\|g^*\chi_{(0,r)}\right\|_{(X^m_{d,n})'(0,1)} = 0.$$

It thus suffices to show that

$$\lim_{r\to 0^+} \sup_{\|g\|_{Y'(0,1)}\le 1} \left\|g^*\chi_{(0,r)}\right\|_{(X^m_{d,n})'(0,1)} = \lim_{r\to 0^+} \sup_{\|f\|_{X(0,1)}\le 1} \|\chi_{(0,r)}\, \mathrm{H}^m_{d,n} f\|_{Y(0,1)}. \tag{4.4}$$

To this purpose, we note that

$$\begin{aligned}\sup_{\|g\|_{Y'(0,1)}\le 1} \left\|g^*\chi_{(0,r)}\right\|_{(X^m_{d,n})'(0,1)} &= \sup_{\|g\|_{Y'(0,1)}\le 1} \left\|\mathrm{R}^m_{d,n}\left(g^*\chi_{(0,r)}\right)\right\|_{X'(0,1)} && \text{(by (1.4))} \\ &= \sup_{\|g\|_{Y'(0,1)}\le 1} \sup_{\|f\|_{X(0,1)}\le 1} \int_0^1 \mathrm{R}^m_{d,n}(g^*\chi_{(0,r)})(s)\ |f(s)|\ ds && \text{(by (2.10))} \\ &= \sup_{\|g\|_{Y'(0,1)}\le 1} \sup_{\|f\|_{X(0,1)}\le 1} \int_0^1 g^*(t)\, \chi_{(0,r)}(t)\, \mathrm{H}^m_{d,n} f(t)\, dt && \text{(by (2.41))} \\ &= \sup_{\|f\|_{X(0,1)}\le 1} \sup_{\|g\|_{Y'(0,1)}\le 1} \int_0^1 \chi_{(0,r)}(t)\, \mathrm{H}^m_{d,n} f(t)\ g^*(t)\, dt \\ &= \sup_{\|f\|_{X(0,1)}\le 1} \|\chi_{(0,r)}\, \mathrm{H}^m_{d,n} f\|_{Y_D(0,1)} && \text{(by (2.11) and (2.42))} \\ &= \sup_{\|f\|_{X(0,1)}\le 1} \|\chi_{(0,r)}\, \mathrm{H}^m_{d,n} f\|_{Y(0,1)} && \text{(by (2.12) and (2.42))}\end{aligned}$$

for every $r \in (0,1)$.

Hence, equality (4.4) holds.

(v) *implies* (i). If (v) is in force, then

$$X^m_{d,n}(0,1) \hookrightarrow Y(0,1),$$

by the second implication in (2.20). Then [10, Theorem 1.1] tells us that

$$\operatorname{Tr}\colon W^m X(\Omega) \hookrightarrow X^m_{d,n}(\Omega_d). \tag{4.5}$$

Let $(u_k)_{k\in\mathbb{N}}$ be a bounded sequence in $W^m X(\Omega)$. By (4.5), the sequence $(\operatorname{Tr} u_k)_{k\in\mathbb{N}}$ is bounded in $X^m_{d,n}(\Omega_d)$.

We claim that there exists a function $g \in L^0(\Omega_d)$ such that

$$\operatorname{Tr} u_k \to g \qquad \mathcal{H}^d\text{-a.e. in } \ \Omega_d, \tag{4.6}$$

up to subsequences.
To verify our claim, we observe that, in the case when $d > n-m$, it follows just from coupling the second embedding in (2.19) with the compactness of the trace embedding $\operatorname{Tr}\colon W^{m,1}(\Omega) \hookrightarrow L^1(\Omega_d)$ (see, e.g., [2, Theorem 6.3]).
For $d = n-m$, we deduce from (v) that condition (3.9) is satisfied. Indeed, if $X(0,1) = L^1(0,1)$, then [10, Theorem 5.1] yields that

$$\left(L^1\right)^m_{n-m,n}(0,1) = L^1(0,1),$$

which leads to a contradiction of [31, Theorem 5.3]. Thus, the claim is actually the assertion of Theorem 3.9.

It remains to show that

$$g \in X^m_{d,n}(\Omega_d), \tag{4.7}$$

and

$$\operatorname{Tr} u_k \to g \qquad \text{in } Y(\Omega_d), \tag{4.8}$$

up to subsequences.

Condition (4.7) follows from Fatou's lemma in $X^m_{d,n}(\Omega_d)$, via (4.6) and the boundedness of $(\operatorname{Tr} u_k)_{k\in\mathbb{N}}$ in $X^m_{d,n}(\Omega_d)$. Hence, we establish (4.8) by combining (v) with [31, Theorem 3.1].

Case $m \geq n$. Let $(u_k)_{k\in\mathbb{N}}$ be a bounded sequence in $W^m X(\Omega)$. Then, from coupling the second embedding in (2.19) with the compactness of the trace embedding $\operatorname{Tr}\colon W^{m,1}(\Omega) \hookrightarrow L^1(\Omega_d)$, there exists a function $g \in L^1(\Omega_d)$ such that

$$\operatorname{Tr} u_k \to g \qquad \mathcal{H}^d\text{-a.e. in } \Omega_d, \tag{4.9}$$

up to subsequences.

On the other hand, [10, Corollary 1.2] assures that $\operatorname{Tr}\colon W^m X(\Omega) \hookrightarrow L^\infty(\Omega_d)$. So, $g \in L^\infty(\Omega_d)$ by Fatou's lemma in $L^\infty(\Omega_d)$, via (4.9) and the boundedness of $(\operatorname{Tr} u_k)_{k\in\mathbb{N}}$ in $L^\infty(\Omega_d)$. Besides, assumption (1.7), owing to (2.23), tells us that $L^\infty(\Omega_d) \overset{*}{\hookrightarrow} Y(\Omega_d)$. By [31, Theorem 3.1], this implies that

$$\operatorname{Tr} u_k \to g \qquad \text{in } Y(\Omega_d),$$

up to subsequences. □

Proof of Theorem 1.2. Case $m < n$. (i) *implies* (ii). It follows via the arguments of (i) *implies* (ii) in the proof of Theorem 1.1 since the assumption (1.7) is not exploited in this implication.

(ii) *implies* (iii). One may argue as in the proof of (ii) *implies* (iii) of Theorem 1.1. Then it suffices to observe that

$$\sup_{\|f\|_{X(0,1)}\leq 1} \left\| \mathrm{H}^m_{d,n}(f\chi_{(0,r)}) \right\|_{L^\infty(0,1)} = \sup_{\|f\|_{X(0,1)}\leq 1} \int_0^r |f(s)| s^{-1+\frac{m}{n}}\, ds = \sup_{\|f\|_{X(0,1)}\leq 1} \int_0^r f^*(s) s^{-1+\frac{m}{n}}\, ds$$

for every $r \in (0,1)$, where the last equality is due to (2.1).

(iii) *is equivalent to* (iv). It is a consequence of (2.11) and (2.12). Indeed, for any $r \in (0,1)$, the function $g(s) = s^{-1+\frac{m}{n}}\, \chi_{(0,r)}(s)$, $s \in (0,1)$, is non-increasing, and

$$\left\| s^{-1+\frac{m}{n}}\, \chi_{(0,r)}(s) \right\|_{X'(0,1)} = \left\| s^{-1+\frac{m}{n}}\, \chi_{(0,r)}(s) \right\|_{X'_D(0,1)} = \sup_{\|f\|_{X(0,1)}\leq 1} \int_0^r f^*(t) t^{-1+\frac{m}{n}}\, dt. \tag{4.10}$$

(iii) *implies* (ii). Fixed any $a \in (0,1)$, define the operator

$$\mathrm{H}_a f = \mathrm{H}^m_{d,n}(f\,\chi_{(a,1)}) \quad \text{for } f \in L^1(0,1). \tag{4.11}$$

By (4.3),

$$\mathrm{H}_a \colon X(0,1) \to L^\infty(0,1). \tag{4.12}$$

This together with assumption (iii) ensures that (3.9) holds.
Indeed, assume that (3.9) fails, namely, $X(0,1) = L^1(0,1)$. Passing to the limit as $r \to 0^+$ in equality (4.10) yields

$$\lim_{r\to 0^+} \sup_{\|f\|_{L^1(0,1)}\le 1} \int_0^r f^*(t) t^{-1+\frac{m}{n}}\, dt = \lim_{r\to 0^+} \left\| s^{-1+\frac{m}{n}}\, \chi_{(0,r)}(s)\right\|_{L^\infty(0,1)} = \infty,$$

which contradicts the (equivalent) condition (iv).

We claim that $\mathrm{H}_a \colon X(0,1) \to L^\infty(0,1)$ is compact.
To verify this, we first observe that

$$\mathrm{H}_a(X(0,1)) \subseteq C^0[0,1]. \tag{4.13}$$

In fact, if $f \in X(0,1)$, then $f \in L^1(0,1)$, and the absolutely continuity of the indefinite integral of f leads to

$$|\mathrm{H}_a f(s_1) - \mathrm{H}_a f(s_2)| \le a^{-1+\frac{m}{n}} \int_{s_1^{\frac{n}{d}}}^{s_2^{\frac{n}{d}}} |f(t)|\, dt \;\to\; 0 \qquad \text{as } |s_1 - s_2| \to 0^+$$

for $s_1, s_2 \in [0,1]$, with $s_1 < s_2$.
Setting $M = \{\mathrm{H}_a f \colon f \in X(0,1), \|f\|_{X(0,1)} \le 1\}$, we shall prove that M is bounded and equicontinuous in $C^0([0,1])$. Hence, by Arzelà-Ascoli theorem, M is relatively compact in $C^0([0,1])$.
The boundedness of M in $C^0([0,1])$ follows, via Proposition 2.1, from (4.12) and (4.13). The equicontinuity of M in $C^0([0,1])$ can be deduced via (3.9), owing to (2.24). Indeed,

$$\sup_{\|f\|_{X(0,1)}\le 1} |\mathrm{H}_a f(s_1) - \mathrm{H}_a f(s_2)| \le a^{-1+\frac{m}{n}} \sup_{\|f\|_{X(0,1)}\le 1} \left\| f\,\chi_{(s_1^{\frac{n}{d}}, s_2^{\frac{n}{d}})}\right\|_{L^1(0,1)}$$

for $s_1, s_2 \in [0,1]$, with $s_1 < s_2$. Since $X(0,1) \overset{*}{\hookrightarrow} L^1(0,1)$, the right-hand side of the last inequality tends to zero as $|s_1 - s_2|$ goes to zero by [31, Lemma 5.1].
Owing to (2.5) and (3.4), for any $a \in (0,1)$, the operator

$$\hat{\mathrm{H}}_a f = \hat{\mathrm{H}}^m_{d,n}(f\,\chi_{(a,1)}) \quad \text{for } f \in L^1(0,1),$$

where $\hat{\mathrm{H}}^m_{d,n}$ is defined by (3.3), is a linear compact operator from $X(0,1)$ to $L^\infty(0,1)$. Since

$$\begin{aligned}\lim_{a\to 0^+} \sup_{\|f\|_{X(0,1)}\le 1} \left\| \hat{\mathrm{H}}^m_{d,n} f - \hat{\mathrm{H}}_a f \right\|_{L^\infty(0,1)} &= \lim_{a\to 0^+} \sup_{\|f\|_{X(0,1)}\le 1} \left\| \hat{\mathrm{H}}^m_{d,n}(f\,\chi_{(0,a)})\right\|_{L^\infty(0,1)} \\ &\le \lim_{a\to 0^+} \sup_{\|f\|_{X(0,1)}\le 1} \int_0^a f^*(s)\, s^{\frac{m}{n}-1}\, ds = 0\end{aligned}$$

it follows that $\hat{\mathrm{H}}^m_{d,n} \colon X(0,1) \to L^\infty(0,1)$ is linear compact operator as well (see, e.g., [4, Theorem 5.1]). Hence, (ii) follows from property (2.38) for $\mathrm{H}^m_{d,n}$ and (3.3).

(ii) *implies* (i). Assume that (ii) is in force. This ensures that (the equivalent) condition (iii) holds, and the operator H_a defined by (4.11) fulfills (4.12) for every $a \in (0,1)$. Thus, as observed above, (3.9) also holds. We distinguish two cases, namely, when $m = 1$ and $m \ge 2$.

Case $m = 1$. Propositions 3.3 and 3.5 tell us that

$$X(0,1) \overset{*}{\hookrightarrow} L^{n,1}(0,1), \tag{4.14}$$

and

$$\mathrm{H}^1_{d,n} \colon L^{n,1}(0,1) \to L^\infty(0,1). \tag{4.15}$$

This, via [10, Theorem 1.3], provides us with

$$\mathrm{Tr} \colon W^1 L^{n,1}(\Omega) \;\hookrightarrow\; L^\infty(\Omega_d). \tag{4.16}$$

Now, let $(u_k)_{k\in\mathbb{N}}$ be a bounded sequence in $W^1X(\Omega)$. By (4.14) and the second implication in (2.20), $(u_k)_{k\in\mathbb{N}}$ is bounded in $W^1L^{n,1}(\Omega)$. Therefore, passing to a subsequence if necessary, we may assume that $\|u_k\|_{W^1L^{n,1}(\Omega)} \leq 1$ for each $k \in \mathbb{N}$ and, via the second embedding in (2.19) and Rellich-Kondrachov theorem (see, e.g., [2, Theorem 6.3]),

$$u_k \to u \qquad \text{in measure on } \Omega, \tag{4.17}$$

for some function $u \in L^1(\Omega)$. Thus, in particular, $(u_k)_{k\in\mathbb{N}}$ is a Cauchy sequence in measure on Ω.

Fix any $\varepsilon > 0$. For every $k, l \in \mathbb{N}$,

$$|u_k - u_l| = \min\left\{|u_k - u_l|, \frac{\varepsilon}{2}\right\} + \max\left\{|u_k - u_l| - \frac{\varepsilon}{2}, 0\right\},$$

where both $\min\left\{|u_k - u_l|, \frac{\varepsilon}{2}\right\}$ and $\max\left\{|u_k - u_l| - \frac{\varepsilon}{2}, 0\right\}$ belong to $W^1L^{n,1}(\Omega)$. In particular,

$$\nabla \max\left\{|u_k - u_l| - \frac{\varepsilon}{2}, 0\right\} = \operatorname{sign}(u_k - u_l)\,\nabla(u_k - u_l)\,\chi_{\left\{|u_k-u_l|>\frac{\varepsilon}{2}\right\}}. \tag{4.18}$$

By Lemma 3.8 and embedding (4.16), we infer that

$$\begin{aligned}
\|\operatorname{Tr} u_k - \operatorname{Tr} u_l\|_{L^\infty(\Omega_d)} &= \|\operatorname{Tr}(|u_l - u_k|)\|_{L^\infty(\Omega_d)} \\
&\leq \left\|\operatorname{Tr}\left(\min\left\{|u_k - u_l|, \frac{\varepsilon}{2}\right\}\right)\right\|_{L^\infty(\Omega_d)} + \left\|\operatorname{Tr}\left(\max\left\{|u_k - u_l| - \frac{\varepsilon}{2}, 0\right\}\right)\right\|_{L^\infty(\Omega_d)} \\
&\leq \frac{\varepsilon}{2} + C\left\|\max\left\{|u_k - u_l| - \frac{\varepsilon}{2}, 0\right\}\right\|_{W^1L^{n,1}(\Omega)},
\end{aligned}$$

where C is the embedding constant for (4.16).
Thanks to (4.18),

$$\begin{aligned}
\left\|\max\left\{|u_k - u_l| - \frac{\varepsilon}{2}, 0\right\}\right\|_{W^1L^{n,1}(\Omega)} &= \left\||u_k - u_l|\,\chi_{\left\{|u_k-u_l|>\frac{\varepsilon}{2}\right\}}\right\|_{L^{n,1}(\Omega)} + \left\||\nabla(u_k - u_l)|\,\chi_{\left\{|u_k-u_l|>\frac{\varepsilon}{2}\right\}}\right\|_{L^{n,1}(\Omega)} \\
&\leq \left\|(|u_k|\,\chi_{\left\{|u_k-u_l|>\frac{\varepsilon}{2}\right\}})^*\right\|_{L^{n,1}(0,1)} + \left\|(|u_l|\,\chi_{\left\{|u_k-u_l|>\frac{\varepsilon}{2}\right\}})^*\right\|_{L^{n,1}(0,1)} \\
&\quad + \left\|(\nabla|u_k|\,\chi_{\left\{|u_k-u_l|>\frac{\varepsilon}{2}\right\}})^*\right\|_{L^{n,1}(0,1)} + \left\|(\nabla|u_l|\,\chi_{\left\{|u_k-u_l|>\frac{\varepsilon}{2}\right\}})^*\right\|_{L^{n,1}(0,1)} \\
&\leq 4 \sup_{\|f\|_{X(0,1)}\leq 1} \left\|f^*\chi_{(0,|\{|u_k-u_l|>\frac{\varepsilon}{2}\}|)}\right\|_{L^{n,1}(0,1)}.
\end{aligned}$$

Consequently, for any $k, l \in \mathbb{N}$,

$$\|\operatorname{Tr} u_k - \operatorname{Tr} u_l\|_{L^\infty(\Omega_d)} \leq \frac{\varepsilon}{2} + 4C \sup_{\|f\|_{X(0,1)}\leq 1} \left\|f^*\chi_{(0,|\{|u_k-u_l|>\frac{\varepsilon}{2}\}|)}\right\|_{L^{n,1}(0,1)}. \tag{4.19}$$

Thanks to (4.14), via (2.22), there exists some $\delta > 0$ such that

$$\sup_{\|f\|_{X(0,1)}\leq 1} \left\|f^*\chi_{(0,\delta)}\right\|_{L^{n,1}(0,1)} < \frac{\varepsilon}{8C}. \tag{4.20}$$

On account of (4.17), there exits some $k_0 \in \mathbb{N}$ such that

$$\left|\left\{|u_k - u| > \frac{\varepsilon}{4}\right\}\right| < \frac{\delta}{2} \qquad \text{for } k \geq k_0.$$

Because $\left\{|u_k - u_l| > \frac{\varepsilon}{2}\right\} \subseteq \left\{|u_k - u| > \frac{\varepsilon}{4}\right\} \cup \left\{|u_l - u| > \frac{\varepsilon}{4}\right\}$, one has

$$\left|\left\{|u_k - u_l| > \frac{\varepsilon}{2}\right\}\right| < \delta \qquad \text{for } k, l \geq k_0. \tag{4.21}$$

Thus, combining (4.19)–(4.21) yields that

$$\|\operatorname{Tr} u_k - \operatorname{Tr} u_l\|_{L^\infty(\Omega_d)} < \varepsilon \qquad \text{for } k, l \geq k_0,$$

namely, $(\operatorname{Tr} u_k)_{k\in\mathbb{N}}$ is a Cauchy sequence in $L^\infty(\Omega_d)$. Hence, the completeness of $L^\infty(\Omega_d)$ concludes the proof for $m = 1$.

Case $m > 1$. Propositions 3.3 and 3.5 tell us that

$$X(0,1) \overset{*}{\hookrightarrow} L^{\frac{n}{m},1}(0,1).$$

This together with the embedding $W^1X(\Omega) \hookrightarrow X(\Omega)$ provides that

$$W^1X(\Omega) \hookrightarrow\hookrightarrow L^{\frac{n}{m},1}(\Omega), \tag{4.22}$$

by [31, Theorem 3.2].

On the other hand, [11, Theorem 6.8] states that

$$W^1L^{\frac{n}{m},1}(\Omega) \hookrightarrow L^{\frac{n}{m-1},1}(\Omega), \tag{4.23}$$

where $L^{\frac{n}{m-1},1}(\Omega)$ is the optimal target space among all rearrangement-invariant spaces for (4.23). As $m > 1$, coupling (4.22) and (4.23) implies that

$$W^mL^{\frac{n}{m},1}(\Omega) \hookrightarrow\hookrightarrow W^{m-1}L^{\frac{n}{m-1},1}(\Omega).$$

Our conclusion will thus follows once the continuity of the operator

$$\mathrm{Tr}\colon W^{m-1}L^{\frac{n}{m-1},1}(\Omega) \to L^\infty(\Omega_d) \tag{4.24}$$

is verified. For this, it suffices to note that

$$W^1L^{n,1}(\Omega) \hookrightarrow C^0_B(\Omega),$$

where $C^0_B(\Omega)$ is the Banach space of bounded continuous functions on Ω equipped with the supremum norm (see, e.g., [8, Remark 3.10]), and

$$W^{m-1}L^{\frac{n}{m-1},1}(\Omega) \hookrightarrow W^1L^{n,1}(\Omega),$$

on account of [10, Theorem 5.1] and (2.19).
Hence,

$$\mathrm{Tr}\, u = \tilde{u}\big|_{\Omega_d}$$

for every $u \in W^{m-1}L^{\frac{n}{m-1},1}(\Omega)$, where $\tilde{u}$ is the continuous representative of u, and $\mathrm{Tr}\colon W^{m-1}L^{\frac{n}{m-1},1}(\Omega) \hookrightarrow L^\infty(\Omega_d)$.

(ii) *is equivalent to* (v). It follows from coupling Proposition 3.3 with Proposition 3.5. Note that condition (3.9) turns out to be in force in both implications.

Case $m = n$. Assume that the trace operator $\mathrm{Tr}\colon W^nX(\Omega) \hookrightarrow L^\infty(\Omega_d)$ is compact. If we had $X(0,1) = L^1(0,1)$, then the Hardy type operator $\mathrm{H}^n_{d,n}\colon L^1(0,1) \to L^\infty(0,1)$ would be compact, by the same arguments of the implication (i) implies (ii) in the case $m < n$. This would lead to a contradiction to the assertion (iv), which is still a necessary condition for the compactness of $\mathrm{H}^n_{d,n}\colon L^1(0,1) \to L^\infty(0,1)$. Conversely, suppose that $X(0,1) \neq L^1(0,1)$. By (2.24), $X(0,1) \overset{*}{\hookrightarrow} L^1(0,1)$. Finally, in order to conclude the proof it suffices to mimic the argument in the proof of (ii) implies (i) in the case $m \in \{1,\dots,n-1\}$, which also holds for $m = n$.

Case $m > n$. If follows from coupling (2.19) with the compactness of the trace operator $\mathrm{Tr}\colon W^{m,1}(\Omega) \hookrightarrow L^\infty(\Omega_d)$ (see e.g.[2, Theorem 6.3]). □

Proof of Corollary 1.3. If $\mathrm{Tr}\colon W^mY(\Omega) \hookrightarrow Z(\Omega_d)$, then [10, Theorem 1.3] and Proposition 2.1 tell us that $\mathrm{H}^m_{d,n}\colon Y(0,1) \to Z(0,1)$ and

$$\sup_{\|f\|_{X(0,1)}\le 1} \left\|\mathrm{H}^m_{d,n}(f\chi_{(0,r)})\right\|_{Z(0,1)} \le \left\|\mathrm{H}^m_{d,n}\right\| \sup_{\|f\|_{X(0,1)}\le 1} \left\|f^*\chi_{(0,r)}\right\|_{Y(0,1)}.$$

On account of (2.22), assumption $X(0,1) \overset{*}{\hookrightarrow} Y(0,1)$ implies that

$$\lim_{r\to 0^+} \sup_{\|f\|_{X(0,1)}\le 1} \left\|\mathrm{H}^m_{d,n}(f\chi_{(0,r)})\right\|_{Z(0,1)} = 0.$$

Hence, the compactness of the trace embedding $\mathrm{Tr}\colon W^mX(\Omega) \hookrightarrow Z(\Omega_d)$ follows from Theorem 1.1 in the case when $Z(0,1) \neq L^\infty(0,1)$, and from Theorem 1.2 for $Z(0,1) = L^\infty(0,1)$, respectively, once observed that

$$\sup_{\|f\|_{X(0,1)}\le 1} \left\|\mathrm{H}^m_{d,n}(f\chi_{(0,r)})\right\|_{L^\infty(0,1)} = \sup_{\|f\|_{X(0,1)}\le 1} \int_0^r |f(t)|\, t^{-1+\frac{m}{n}}\, dt = \sup_{\|f\|_{X(0,1)}\le 1} \int_0^r f^*(t)\, t^{-1+\frac{m}{n}}\, dt.$$

□

5. Compactness of trace embeddings for customary Sobolev type spaces

Theorems 1.1 – 1.2 enable us to characterize compactness of trace embedding operator on customary Sobolev type spaces. The following proposition deals with Lorentz-Sobolev spaces. In particular, it covers classical results as special cases.

Proposition 5.1 (Trace embeddings in Lorentz-Sobolev spaces)**.** *Let Ω, n, m, and $d \in \mathbb{N}$ be as in Theorem 1.1, with $m < n$. Assume that either $p \in (1, \infty)$ and $q \in [1, \infty]$, or $p = q = 1$, or $p = q = \infty$, and let $r, s \in [1, \infty]$ and $\beta \in \mathbb{R}$ fulfill one of the following conditions:*

(i) $r = s = 1$ *and* $\beta \geq 0$*;*
(ii) $r \in (1, \infty)$*;*
(iii) $r = \infty$ *and* $s \in [1, \infty)$ *and* $\beta + \frac{1}{s} < 0$*;*
(iv) $r = s = \infty$ *and* $\beta \leq 0$.

Case $p \in [1, \frac{n}{m})$: *The trace embedding*

$$\operatorname{Tr}\colon W^m L^{p,q}(\Omega) \hookrightarrow L^{r,s;\beta}(\Omega_d)$$

is compact if and only if one of the following conditions holds:

(a) $r \in [1, \frac{pd}{n-mp})$*;*
(b) $r = \frac{pd}{n-mp}$, $q \leq s$ *and* $\beta < 0$*;*
(c) $r = \frac{pd}{n-mp}$, $q > s$ *and* $\beta < \frac{1}{q} - \frac{1}{s}$.

Case $p = \frac{n}{m}$: *The trace embedding*

$$\operatorname{Tr}\colon W^m L^{\frac{n}{m},1}(\Omega) \hookrightarrow Y(\Omega_d)$$

is compact for any rearrangement-invariant space $Y(\Omega_d)$, except for $Y(\Omega_d) = L^\infty(\Omega_d)$.

The trace embedding

$$\operatorname{Tr}\colon W^m L^{\frac{n}{m},q}(\Omega) \hookrightarrow L^{r,s;\beta}(\Omega_d),$$

with $q \in (1, \infty]$ is compact if and only if either $r \in [1, \infty)$ or $r = \infty$ and $\beta + \frac{1}{s} < \frac{1}{q} - 1$.

Case $p \in (\frac{n}{m}, \infty]$: *The trace embedding*

$$\operatorname{Tr}\colon W^m L^{p,q}(\Omega) \hookrightarrow L^\infty(\Omega_d)$$

is compact.

In particular, the trace embedding $\operatorname{Tr}\colon W^m L^{p,q}(\Omega) \hookrightarrow L^\infty(\Omega_d)$ is compact if and only if $p \in (\frac{n}{m}, \infty]$.

Proof. It suffices to combine Theorems 1.1 – 1.2 with [10, Theorem 5.1] and [32, Proposition 7.12]. □

The following examples deal with an important type of Orlicz-Sobolev space, namely the Sobolev spaces built upon logarithmic Zygmund classes $L^p (\log L)^\alpha$.

Proposition 5.2 (Trace embeddings in logarithmic Zygmund spaces)**.** *Let Ω, n, m, and $d \in \mathbb{N}$ be as in Theorem 1.1, with $m < n$. Assume that $p \in [1, \infty)$ and $\alpha \in \mathbb{R}$, and let $r, s \in [1, \infty]$ and $\beta \in \mathbb{R}$ fulfill either one of the conditions of Proposition 5.1, or $p \in (1, \infty)$, or $p = 1$ and $\alpha \geq 0$.*

Case $p \in [1, \frac{n}{m})$: *The trace embedding*

$$\operatorname{Tr}\colon W^m L^p (\log L)^\alpha (\Omega) \hookrightarrow L^{r,s;\beta}(\Omega_d)$$

is compact if and only if one of the following conditions holds:

(a) $r \in [1, \frac{pd}{n-mp})$*;*
(b) $r = \frac{pd}{n-mp}$, $p \leq s$ *and* $\frac{\alpha}{p} > \beta$*;*
(c) $r = \frac{pd}{n-mp}$, $p > s$ *and* $\frac{\alpha+1}{p} > \beta + \frac{1}{s}$.

Case $p = \frac{n}{m}$: *The trace embedding*

$$\operatorname{Tr}\colon W^m L^{\frac{n}{m}} (\log L)^\alpha (\Omega) \hookrightarrow L^{r,s;\beta}(\Omega_d),$$

(d) *with $\alpha < \frac{n-m}{m}$, is compact if and only if either $r \in [1, \infty)$ or $r = \infty$ and $\frac{m(\alpha+1)}{n} - 1 > \beta + \frac{1}{s}$;*
(e) *with $\alpha = \frac{n-m}{m}$, is compact if and only if either $r \in [1, \infty)$ or $r = \infty$ and $\beta + \frac{1}{s} < 0$;*

(f) *with* $\alpha > \frac{n-m}{m}$, *is compact also with* $L^\infty(\Omega_d)$ *in place of* $L^{r,s;\beta}(\Omega_d)$.

Case $p \in (\frac{n}{m}, \infty]$: *The trace embedding*

$$\mathrm{Tr}\colon W^m L^p (\log L)^\alpha (\Omega) \hookrightarrow L^\infty(\Omega_d)$$

is compact.

In particular, the trace embedding $\mathrm{Tr}\colon W^m L^p (\log L)^\alpha (\Omega) \hookrightarrow L^\infty(\Omega_d)$ *is compact if and only if either* $p = \frac{n}{m}$ *and* $\alpha > \frac{n-m}{m}$, *or* $p > \frac{n}{m}$.

Proof. It suffices to combine Theorems 1.1 – 1.2 with [10, Example 5.6] and [32, Proposition 7.12]. □

In applications, one often needs to substitute L^∞ with exponential Zygmund classes $\exp L^\alpha$. Therefore, we give some examples here, which are just special cases of Propositions 5.1-5.2 since $\exp L^\beta = L^{\infty,\infty;-\frac{1}{\beta}}$, involving these function spaces.

Examples 5.3. Let Ω, n, m, d, p and q be as in Proposition 5.1, and let $\beta \in (0, \infty)$.

If $p \in [1, \frac{n}{m})$, then the trace embedding $\mathrm{Tr}\colon W^m L^{p,q}(\Omega) \hookrightarrow \exp L^\beta(\Omega_d)$ is not compact.

If $p = \frac{n}{m}$, then the trace embedding $\mathrm{Tr}\colon W^m L^{\frac{n}{m},q}(\Omega) \hookrightarrow \exp L^\beta(\Omega_d)$,

- with $q = 1$, is compact for all β;
- with $q \in (1, \infty]$, is compact if and only if $\beta \in (0, q')$.

Here, q' is the Hölder conjugate of q.

Examples 5.4. Let Ω, n, m, d, p and α be as in Proposition 5.2, and let $\beta \in (0, \infty)$.

If $p \in [1, \frac{n}{m})$, then the trace embedding $\mathrm{Tr}\colon W^m L^p (\log L)^\alpha (\Omega) \hookrightarrow \exp L^\beta(\Omega_d)$ is not compact.

If $p = \frac{n}{m}$, then the trace embedding $\mathrm{Tr}\colon W^m L^{\frac{n}{m}} (\log L)^\alpha (\Omega) \hookrightarrow \exp L^\beta(\Omega_d)$,

- with $\alpha < \frac{n-m}{m}$, is compact if and only if $\beta \in \left(0, \frac{n}{n-m(\alpha+1)}\right)$;
- with $\alpha \geq \frac{n-m}{m}$, is compact.

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