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Homogenization of a class of singular elliptic problems in perforated domains

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Abstract

In this work we study the asympototic behaviour of a class of quasilinear elliptic problems posed in a domain perforated by ε-periodic holes of size ε. The quasilinear equations present a nonlinear singular lower order term $f\zeta(u_\varepsilon)$, where u_ε is the solution of the problem at ε-level, ζ is a continuous function singular in zero and f a function whose summability depends on the growth of ζ near its singularity. We prescribe a nonlinear Robin condition on the boundary of the holes contained in Ω and a homogeneous Dirichlet condition on the exterior boundary. The particular case of a Neumann boundary condition on the holes is already new.

The main tool in the homogenization process consists in proving a suitable convergence result, which shows that the gradient of u_ε behaves like that of the solution of a suitable linear problem associated with a weak cluster point of the sequence $\{u_\varepsilon\}$, as $\varepsilon \to 0$. This allows us not only to pass to the limit in the quasilinear term, but also to study the singular term near its singularity, via an accurate a priori estimate. We also get a corrector result for our problem.

The main novelty of this work is that for the first time the unfolding method is used to treat a singular term as $f\zeta(u_\varepsilon)$. This plays an essential role in order to get an almost everywhere convergence of the solution u_ε, needed in the study the asymptotic behavior of the problem.

Keywords: perforated domains, homogenization, periodic unfolding method, quasilinear elliptic equations, singular equations, nonlinear boundary conditions.
MSC: 35B27, 35J62, 35J66, 35J75

1 Introduction

In this paper we deal with the homogenization of a class of quasilinear elliptic problems with singular lower order terms posed in periodically perforated domains. In our study, we use the periodic unfolding method, originally introduced in [13] and [14]. The perforated domain Ω^*_ε is obtained by removing from a connected bounded open set Ω of $\mathbb{R}^N$, $N \geq 2$, a set of ε-periodic holes of size ε. The boundary of Ω^*_ε is decomposed into Γ^ε_1 and Γ^ε_0, which denote the boundary of the holes well contained in Ω and the remaining exterior boundary, respectively (see Section 2 for details). We prescribe a nonlinear Robin condition on Γ^ε_1 and a homogeneous Dirichlet

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condition on Γ_0^ε.
More precisely, we study the asymptotic behavior, as ε goes to zero, of the following problem:

$$\begin{cases} -div(A^\varepsilon(x,u_\varepsilon)\nabla u_\varepsilon) = f\zeta(u_\varepsilon) & \text{in } \Omega_\varepsilon^*, \\ u_\varepsilon = 0 & \text{on } \Gamma_0^\varepsilon, \\ (A^\varepsilon(x,u_\varepsilon)\nabla u_\varepsilon)\nu + \varepsilon^\gamma \rho_\varepsilon(x) h(u_\varepsilon) = g_\varepsilon & \text{on } \Gamma_1^\varepsilon, \end{cases}$$

where ν is the unit outward normal to the holes.
The oscillating coefficients' matrix field A^ε in the quasilinear diffusion term is defined by $A^\varepsilon(x,t) = A\left(\frac{x}{\varepsilon}, t\right)$, where the matrix field A is uniformly elliptic, bounded, periodic in the first variable and Carathéodory. The nonlinear real function $\zeta(s)$ is nonnegative and singular at $s=0$, while f is a nonnegative datum whose summability depends on the growth of ζ near its singularity. Concerning the Robin boundary condition, $\rho_\varepsilon(x) = \rho\left(\frac{x}{\varepsilon}\right)$ where the function ρ is assumed to be periodic, nonnegative and bounded on ∂T, the nonlinear boundary term h is an increasing and continuously differentiable function whose derivative satisfies suitable growth assumptions, and $g_\varepsilon(x) = \varepsilon g\left(\frac{x}{\varepsilon}\right)$, where g is a periodic nonnegative function with prescribed summability.
From the physical point of view, the quasilinear diffusion term describes the behavior of some materials, like glass or wood, in which the heat diffusion depends on the temperature (see [26] for more details). A source term depending on the solution itself and becoming infinite when the solution vanishes may model an electrical conductor, where each point becomes a source of heat as a current flows in it (see [23, Section 3]). Moreover, nonlinear Robin boundary conditions describe certain chemical reactions at the boundaries of perforations (e.g. [21]).
The existence and uniqueness of the weak solution of the problem, for every fixed ε, have already been proved by the authors in [25]. Uniform a priori estimates of the solution u_ε are obtained by adapting to our case some arguments introduced in [25]. Let us mention that the third bound (cf. Proposition 4.4) provides an estimate of the integral of the singular term close to the singular set $\{u_\varepsilon = 0\}$, in terms of the quasilinear one.
In the homogenization process we have to pass to the limit in the quasilinear term, in the singular one and in the nonlinear Robin condition. In order to study the quasilinear term, we prove a crucial convergence, given by Theorem 5.5, which is the main tool when proving our results. It shows that the gradient of u_ε behaves like that of the solution of a suitable linear problem associated with a weak cluster point of the sequence $\{u_\varepsilon\}$, as $\varepsilon \to 0$. This idea was originally introduced in [5] (see also [4]) where some nonlinear problems with quadratic growth are considered. The main difference with respect to [5] consists in the choice of the test functions used in the proof. Indeed, in the quadratic growth case the authors take exponential test functions, while here we have to use appropriate test functions that allow us to treat the singular term. To construct these functions, taking into account the homogenization results of [10], we adapt to our needs some techniques from [23].
As far as it concerns the singular term, as done in [23] and [25], we split it into the sum of two integrals: one on the set where the solution is close to the singularity and one where is it far from it, which results not singular. Near the singularity, we make use of the estimate given by Proposition 4.4. In this way, we shift the study of the singular term to that of the quasilinear one, for which we can use the previous result.
Let us point out that for $h \equiv 0$ we obtain, as a particular case, homogenization and corrector results for the problem when a (homogeneous or not) Neumann boundary condition on Γ_1^ε is prescribed, which are already new in the literature.
Concerning the nonlinear Robin boundary condition, the arguments used in [10] to study the

boundary terms still apply to our case, with appropriate considerations, the hypotheses being slightly different.
Finally, Theorem 5.5 states that the correctors for the nonsingular quasilinear problem in perforated domains studied in [10] are also correctors for our problem.
As already mentioned, all the results of this paper are proved by means of the periodic unfolding method. It was introduced by D. Cioranescu, A. Damlamian and G. Griso in [13] and [14] for fixed domains and extended to perforated ones in [16] and [17].
The main novelty of this work is that, for the first time, the unfolding method is used to treat a singular term as $f\zeta(u_\varepsilon)$. This is essential in order to get the almost everywhere convergences needed to study the asymptotic behavior of the problem. Although in [23] the problem presents the same singular term, it is posed through rough surface. Whence, since the measure of the interface is zero, a compactness result in $L^2(\Omega)$ from [24] gives the almost everywhere convergences. In our case, due to the presence of the holes, this argument does not apply, and the almost everywhere convergences can be obtained only by using the unfolding method.
We refer to [4], [5] for the first results on the periodic homogenization of quadratic nonlinear elliptic problems, and to [2], [3] for that of quasilinear problems, in the case of fixed domains. The homogenization of quasilinear problems in periodically perforated domains with nonlinear Robin boundary conditions has been considered in [8], [10], and in [11], where is also considered a nonlinear term which is quadratic with respect to the gradient. The homogenization of various singular elliptic problems has been studied, for instance, in [6], [23] and [27].
The paper is organized as follows:
In Section 2 we present the setting of the problem and we state the main results.
In Section 3 we give a short presentation of the periodic unfolding operator for perforated domains.
In Section 4 we prove two a priori estimates uniform with respect to ε and we give an estimate of the integral of the singular term close to the singular set $\{u_\varepsilon = 0\}$.
In Section 5 we state and prove the crucial auxiliary convergence result given by Theorem 5.5.
In Section 6 we prove the homogenization theorem.

2 Setting of the problem and main results

Throughout all the paper, we use the notation introduced in [12] and [14] for the periodic unfolding method in perforated domains.
For $N \in \mathbb{N}, N \geq 2$, we suppose that Ω is a connected bounded open set in $\mathbb{R}^N$ whose boundary $\partial\Omega$ is Lipschitz-continuous. Let $b = \{b_1, ..., b_N\}$ be a basis of $\mathbb{R}^N$ and define by Y the following reference cell:

$$Y \doteq \left\{ y \in \mathbb{R}^N : y = \sum_{i=1}^{N} y_i b_i, (y_1, ..., y_N) \in (0,1)^N \right\}. \tag{2.1}$$

Also, T denotes the reference hole, which is a (nonempty) open subset of $\mathbb{R}^N$ such that $\overline{T} \subset Y$, and ∂T is Lipschitz-continuous with a finite number of connected components.
The perforated cell, that is the part of the cell occupied by the material, is defined by $Y^* \doteq Y \setminus \overline{T}$.
Let $\{\varepsilon\}_{\varepsilon>0}$ be a positive parameter taking values in a sequence converging to zero and set

$$G \doteq \left\{ \xi \in \mathbb{R}^N : \xi = \sum_{i=1}^{N} k_i b_i, (k_1, ..., k_N) \in \mathbb{Z}^N \right\} \qquad T_\varepsilon \doteq \bigcup_{\xi \in G} \varepsilon(\xi + T),$$

where $\varepsilon(\xi + T)$ are disjoint traslated sets of T rescaled by ε.
The perforated domain Ω_ε^* is defined by removing from Ω the closure of holes T_ε, that is (see Figure 1)

$$\Omega_\varepsilon^* \doteq \Omega \setminus \overline{T}_\varepsilon.$$

Moreover, we define by $\widehat{\Omega}_\varepsilon$ the interior of the largest union of cells $\varepsilon(\xi + \overline{Y})$ completely included in Ω and by Λ_ε its complement with respect to Ω that contains the cells $\varepsilon(\xi + \overline{Y})$ intersecting $\partial\Omega$, i.e.

$$\widehat{\Omega}_\varepsilon \doteq \text{interior} \left\{ \bigcup_{\xi \in \Xi_\varepsilon} \varepsilon(\xi + \overline{Y}) \right\} \quad \text{and} \quad \Lambda_\varepsilon \doteq \Omega \setminus \widehat{\Omega}_\varepsilon,$$

where

$$\Xi_\varepsilon \doteq \{\xi \in G, \varepsilon(\xi + Y) \subset \Omega\}.$$

The corresponding perforated sets are

$$\widehat{\Omega}_\varepsilon^* \doteq \widehat{\Omega}_\varepsilon \setminus \overline{T}_\varepsilon \quad \text{and} \quad \Lambda_\varepsilon^* \doteq \Omega_\varepsilon^* \setminus \widehat{\Omega}_\varepsilon^*.$$

We decompose the boundary of the perforated domain Ω_ε^* as done, for instance, in [17],[18], [8] and [10], that is

$$\partial\Omega_\varepsilon^* = \Gamma_0^\varepsilon \cup \Gamma_1^\varepsilon \quad \text{where} \;\; \Gamma_1^\varepsilon \doteq \partial\widehat{\Omega}_\varepsilon^* \cap \partial T_\varepsilon \quad \text{and} \quad \Gamma_0^\varepsilon \doteq \partial\Omega_\varepsilon^* \setminus \Gamma_1^\varepsilon.$$

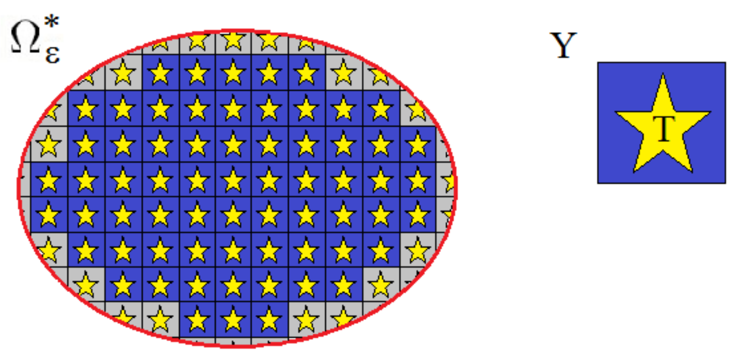

Figure 1: The perforated domain Ω_ε^*, $\widehat{\Omega}_\varepsilon^*$, Λ_ε^* and the reference cell Y

In the sequel, we also denote by:

- $\mathcal{M}_{Y^*}(v) \doteq \frac{1}{|Y^*|}\int_{Y^*} v(y)dy$ the average of any function $v \in L^1(Y^*)$,
- $\mathcal{M}_{\partial T}(v) \doteq \frac{1}{|\partial T|}\int_{\partial T} v(y)d\sigma_y$ the average of any function $v \in L^1(\partial T)$,
- $\sim$ the zero extension to the whole of Ω of functions defined on Ω_ε^*,
- ν the unit outward normal to Ω or Y^*,
- $\theta \doteq \frac{|Y^*|}{|Y|}$ the proportion of the material,
- c different positive constants independent of ε,

- $\mathcal{M}(\alpha, \beta, Y)$ the set of matrix fields $A = (a_{i,j})_{1 \leq i,j \leq N} \in (L^\infty(Y))^{N \times N}$ such that $(A(x)\lambda, \lambda) \geq \alpha|\lambda|^2$ and $|A(x)\lambda| \leq \beta|\lambda|, \ \forall \lambda \in \mathbb{R}^N$ and a.e. in Y, with $\alpha, \beta \in \mathbb{R}$, $0 < \alpha < \beta$,
- $\{e_1, ..., e_N\}$ the canonical basis of $\mathbb{R}^N$.

Furthermore let us recall the classical decomposition for every real function u

$$u = u^+ - u^-, \qquad u^+ \doteq \max\{u, 0\} \ \text{ and } \ u^- \doteq -\min\{u, 0\} \ \text{ a.e. in } \Omega, \tag{2.2}$$

where u^+ and u^- are both nonnegative. We also recall that

$$\chi_{\Omega^*_\varepsilon} \rightharpoonup \theta \quad \text{weakly}^* \text{ in } L^\infty(\Omega), \quad \text{as } \varepsilon \to 0.$$

Definition 2.1. Let f be a function defined almost everywhere in $\mathbb{R}^N$ and Y given by (2.1). The function f is called $Y-$*periodic* if and only if

$$f(x + ky_i b_i) = f(x) \text{ a.e. in } \mathbb{R}^N, \ \ \forall k \in \mathbb{Z} \quad \text{and} \quad \forall i = 1, ..., N.$$

The problem

Our aim is to study the asymptotic behavior, as ε goes to zero, of the following problem:

$$\begin{cases} -div(A^\varepsilon(x, u_\varepsilon)\nabla u_\varepsilon) = f\zeta(u_\varepsilon) & \text{in } \Omega^*_\varepsilon, \\ u_\varepsilon = 0 & \text{on } \Gamma^\varepsilon_0, \\ (A^\varepsilon(x, u_\varepsilon)\nabla u_\varepsilon)\nu + \varepsilon^\gamma \rho_\varepsilon(x) h(u_\varepsilon) = g_\varepsilon & \text{on } \Gamma^\varepsilon_1, \end{cases} \tag{2.3}$$

where a nonlinear Robin condition on the boundary of the holes Γ^ε_1 and a homogeneous Dirichlet condition on the exterior boundary Γ^ε_0 are prescribed. The matrix field A^ε is defined in (2.4), and the functions ρ_ε and g_ε in (2.5).

Assumptions on the data

Throughout this paper, we make the following hypotheses, where the Y-periodicity is taken in the sense of Definition 2.1:

H$_1$) The real $N \times N$ matrix field $A : (y, t) \in Y \times \mathbb{R} \mapsto A(y, t) = (a_{i,j}(y, t))_{i,j=1,...,N} \in \mathbb{R}^{N^2}$ satisfies the following conditions:

$$\begin{cases} \text{i) } A(\cdot, t) \text{ is } Y\text{-periodic for every } t; \\ \text{ii) } A \text{ is a Carathéodory function, i.e.} \\ \quad - A(y, \cdot) \text{ is continuous for a.e. } y \in Y, \\ \quad - A(\cdot, t) \text{ is measurable for every } t \in \mathbb{R}; \\ \text{iii) } A(\cdot, t) \in \mathcal{M}(\alpha, \beta, Y), \text{ for every } t \in \mathbb{R}; \\ \text{iv) there exists a real function } \omega : \mathbb{R} \to \mathbb{R} \text{ satisfying the following conditions:} \\ \quad - \omega \text{ is continuous and non decreasing, with } \omega(t) > 0 \ \forall t > 0, \\ \quad - |A(y, t_1) - A(y, t_2)| \leq \omega(|t_1 - t_2|) \ \text{ for a.e. } y \in Y, \forall t_1 \neq t_2, \\ \quad - \forall s > 0, \ \lim\limits_{y \to 0^+} \displaystyle\int_y^s \frac{dt}{\omega(t)} = +\infty. \end{cases}$$

H$_2$) The functions ζ and f verify

$$\begin{cases} \text{i) } \zeta : [0,+\infty[\to [0,+\infty] \text{ is a function such that} \\ \quad \zeta \in \mathcal{C}^0([0,+\infty[),\ 0 \leq \zeta(s) \leq \dfrac{1}{s^k} \text{ for every } s \in]0,+\infty[,\ \text{with } 0 < k \leq 1; \\ \text{ii) } \zeta \text{ is non increasing;} \\ \text{iii) } f \geq 0 \text{ a.e. in } \Omega,\ \text{with } f \in L^l(\Omega),\ \text{for } l \geq \dfrac{2}{1+k} (\geq 1). \end{cases}$$

H$_3$) Either

$$\text{i) } 0 \leq g \in L^s(\partial T),\ \text{with } \mathcal{M}_{\partial T}(g) \neq 0 \text{ and } \begin{cases} s \geq \dfrac{2(N-1)}{N} & \text{if } N > 2, \\ s > 1 & \text{if } N = 2; \end{cases}$$

or

ii) $g \equiv 0$.

H$_4$) Either $f \not\equiv 0$ or $g \not\equiv 0$.

H$_5$) $\gamma \geq 1$, and ρ is a nonnegative Y-periodic function in $L^\infty(\partial T)$.

H$_6$) The function h is an increasing and continuously differentiable function such that for some positive constant C and an exponent q one has

$$\begin{cases} h(0) = 0, \\ |h'(s)| \leq C(1+|s|^{q-1}), \forall s \in \mathbb{R}, \\ \text{with } 1 \leq q < \infty \text{ if } N = 2,\ \text{and } 1 \leq q \leq \dfrac{N}{N-2} \text{ if } N > 2. \end{cases}$$

◇

Under the above assumptions we set, for almost every $x \in \Omega$ and every $t \in \mathbb{R}$,

$$A^\varepsilon(x,t) \doteq A\left(\frac{x}{\varepsilon}, t\right), \tag{2.4}$$

and, for almost every $x \in \Gamma_1^\varepsilon$,

$$g_\varepsilon(x) \doteq \varepsilon g\left(\frac{x}{\varepsilon}\right), \qquad \rho_\varepsilon(x) \doteq \rho\left(\frac{x}{\varepsilon}\right). \tag{2.5}$$

Now we introduce the natural framework for the study of problem (2.3), that is the space

$$V_\varepsilon \doteq \{v \in H^1(\Omega_\varepsilon^*) : v = 0 \text{ on } \Gamma_0^\varepsilon\}.$$

Remark 2.2. It is known (see for instance [19, Lemma 1], [20]) that the Poincaré inequality in V_ε holds with a constant c_P independent of ε, that is

$$\|v\|_{L^2(\Omega_\varepsilon^*)} \leq c_P \|\nabla v\|_{L^2(\Omega_\varepsilon^*)} \quad \forall v \in V_\varepsilon. \tag{2.6}$$

Consequently, the space V_ε can be equipped by the norm

$$\|v\|_{V_\varepsilon} \doteq \|\nabla v\|_{L^2(\Omega_\varepsilon^*)} \quad \forall v \in V_\varepsilon,$$

which is equivalent to the H^1-norm via a constant independent on ε.
Moreover, the Sobolev embedding theorems also apply to V_ε independently on ε.

The variational formulation associated with problem (2.3) reads

$$\begin{cases} \text{Find } u_\varepsilon \in V_\varepsilon \text{ such that } u_\varepsilon > 0 \ \text{ a.e. in } \Omega_\varepsilon^*, \\ \displaystyle\int_{\Omega_\varepsilon^*} f\zeta(u_\varepsilon)\varphi dx < +\infty \text{ and} \\ \displaystyle\int_{\Omega_\varepsilon^*} A^\varepsilon(x, u_\varepsilon)\nabla u_\varepsilon \nabla\varphi dx + \varepsilon^\gamma \int_{\Gamma_1^\varepsilon} \rho_\varepsilon h(u_\varepsilon)\varphi d\sigma \\ \displaystyle= \int_{\Omega_\varepsilon^*} f\zeta(u_\varepsilon)\varphi dx + \int_{\Gamma_1^\varepsilon} g_\varepsilon \varphi d\sigma, \quad \forall \varphi \in V_\varepsilon. \end{cases} \tag{2.7}$$

Let us point out that the Lipschitz-continuous assumption on ∂T is necessary in order to write here the surface integrals on the boundary of the holes.
We recall the following result:

Theorem 2.3 ([25])**.** *Under assumptions H_1)- H_6), problem* (2.7) *admits a unique solution.*

Remark 2.4. It can be seen from [25, proofs of Theorems 5.2 and 6.2]), that hypotheses $\mathrm{H}_1)_{i,ii,iii}$, $\mathrm{H}_2)_{i,iii}$, H_3-H_6) provide the existence of at least a solution of our problem. To ensure the uniqueness of the solution, as usual in the literature, some additional hypotheses on the matrix field A and on the nonlinear function ζ are required. This is why in [25] the uniqueness is proved assuming that the quasilinear term verifies hypothesis $\mathrm{H}_1)_{iv}$, which was originally introduced in [9] by M. Chipot for quasilinear nonsingular problems (see also [7]). Concerning the function ζ the monotonicity hypothesis $\mathrm{H}_2)_{ii}$ is assumed, as done in [23] and [28].
Let us mention that if A is uniformly Lipschitz-continuous in t with constant L, then $w(t) \doteq Lt$ satisfies the assumption $\mathrm{H}_1)_{iv}$.

Let us introduce here, for every fixed $t \in \mathbb{R}$, the homogenized matrix field $A^0(t)$, defined by

$$A^0(t)\lambda \doteq \frac{1}{|Y|}\int_{Y^*} A(y,t)\nabla_y \widehat{\omega}_\lambda(y,t)dy \quad \forall \lambda \in \mathbb{R}^N, \tag{2.8}$$

where, for every $t \in \mathbb{R}$ and $\lambda \in \mathbb{R}^N$,

$$\widehat{\omega}_\lambda(y,t) = \lambda \cdot y - \widehat{\chi}_\lambda(y,t)$$

and $\widehat{\chi}_\lambda(y,t)$ is solution of the following problem:

$$\begin{cases} -div(A(\cdot,t)\nabla_y \widehat{\chi}_\lambda(\cdot,t)) = -div(A(\cdot,t)\lambda) & \text{in } Y^*, \\ A(\cdot,t)(\lambda - \nabla_y \widehat{\chi}_\lambda(\cdot,t))\nu = 0 & \text{on } \partial T, \\ \widehat{\chi}_\lambda(\cdot,t) \text{ Y-periodic}, \\ \displaystyle\frac{1}{|Y^*|}\int_{Y^*} \widehat{\chi}_\lambda(y,t)dy = 0. \end{cases} \tag{2.9}$$

The homogenized matrix A^0 is that originally introduced in [19] for linear problems with Neumann conditions in perforated domain, successively extended to quasilinear ones in [2], [3].

We recall (see [8] and [19]) that the matrix A^0 satisfies the following properties:

$$
\begin{aligned}
&\text{i) } A^0 \text{ is continuous and } A^0(t) \in M\left(\alpha, \frac{\beta^2}{\alpha}, \Omega\right) \text{ for every } t \in \mathbb{R};\\
&\text{ii) there exists a positive constant C, depending only on } \alpha, \beta, Y \text{ and } T \text{ such that}\\
&\quad |A^0(t_1) - A^0(t_2)| \leq C\omega(|t_1 - t_2|)\\
&\quad \text{for every } t_1, t_2 \in \mathbb{R}, \text{ with } t_1 \neq t_2, \text{ where } \omega \text{ is the function given in H}_1).
\end{aligned}
\tag{2.10}
$$

Now we state the main results of this paper. The following one is proved in Section 4:

Proposition 2.5. *Under assumptions H_1)- H_6), let $u_\varepsilon \in V_\varepsilon$ be the unique solution of problem (2.7). Then, there exist a subsequence (still denoted by ε) and two functions $u_0 \in H_0^1(\Omega)$ and $\widehat{u} \in L^2(\Omega; H^1_{per}(Y^*))$ with $\mathcal{M}_{Y^*}(\widehat{u}) = 0$, such that*

$$
\begin{cases}
i)\ \mathcal{T}^*_\varepsilon(u_\varepsilon) \to u_0 & \text{strongly in } L^2(\Omega; H^1(Y^*)) \text{ and a.e. in } \Omega \times Y^*,\\
ii)\ \mathcal{T}^*_\varepsilon(\zeta(u_\varepsilon)) \to \zeta(u_0) & \text{a.e. in } \Omega \times Y^*,\\
iii)\ \widetilde{u_\varepsilon} \rightharpoonup \theta u_0 & \text{weakly in } L^2(\Omega),\\
iv)\ \mathcal{T}^*_\varepsilon(\nabla u_\varepsilon) \rightharpoonup \nabla u_0 + \nabla_y \widehat{u} & \text{weakly in } L^2(\Omega \times Y^*),\\
v)\ \mathcal{T}^b_\varepsilon(h(u_\varepsilon)) \to h(u_0) & \text{strongly in } L^t(\Omega; W^{1-\frac{1}{t},t}(\partial T)),
\end{cases}
\tag{2.11}
$$

where

$$
t \doteq \begin{cases} \in (1;2) & \text{if } N = 2 \text{ and } q > 1,\\ \dfrac{2N}{q(N-2)+2} & \text{otherwise;} \end{cases}
\tag{2.12}
$$

and q is given by H_6). Moreover,

$$
u_0 \geq 0 \text{ a.e. in } \Omega \quad \text{and} \quad \int_\Omega f\zeta(u_0)\varphi dx < +\infty, \quad \forall \varphi \in H^1_0(\Omega) \cap L^\infty(\Omega).
\tag{2.13}
$$

We also have the result below, which is proved at the end of Section 5.

Proposition 2.6. *Under assumptions H_1)- H_6), let $(u_0, \widehat{u})$ be given by Proposition 2.5. Then*

$$
\widehat{u}(y,x) = -\sum_{i=1}^N \widehat{\chi}_{e_i}(y, u_0(x)) \frac{\partial u_0}{\partial x_i}(x) \in L^2(\Omega; H^1_{per}(Y^*)).
$$

Remark 2.7. The proof of Proposition 2.6 makes use of some results from [10] and [11], where is also considered the case $g \not\equiv 0$ with $\mathcal{M}_{\partial T}(g) = 0$. Here, since we need to assume $g \geq 0$, we can only have the cases $\mathcal{M}_{\partial T}(g) \neq 0$ or $g \equiv 0$.

The homogenization result for problem (2.7) is given by the following theorem, whose proof is presented in Section 6:

Theorem 2.8. *Under assumptions H_1)- H_6), let $u_\varepsilon \in V_\varepsilon$ be the unique solution of problem* (2.7) *and* $(u_0, \widehat{u})$ *be given by Propositions 2.5-2.6. Then the pair* $(u_0, \widehat{u})$ *is the unique solution of the limit equation*

$$\begin{cases} \forall \varphi \in H_0^1(\Omega) \cap L^\infty(\Omega) \ \text{and} \ \forall \psi \in L^2(\Omega; H^1_{per}(Y^*)) \\ \displaystyle\int_{\Omega\times Y^*} A(y,u_0)(\nabla u_0 + \nabla_y \widehat{u})(\nabla\varphi + \nabla_y\psi)dxdy + |Y| c_\gamma \int_\Omega h(u_0)\varphi dx \\ = |Y^*| \displaystyle\int_\Omega f\zeta(u_0)\varphi dx + |\partial T| \mathcal{M}_{\partial T}(g) \int_\Omega \varphi dx, \end{cases} \tag{2.14}$$

where c_γ is defined by

$$c_\gamma \doteq \begin{cases} \dfrac{|\partial T|}{|Y|}\mathcal{M}_{\partial T}(\rho) & \text{if } \gamma = 1, \\ 0 & \text{if } \gamma > 1. \end{cases} \tag{2.15}$$

Finally, $u_0 > 0$ almost everywhere in Ω and u_0 is the unique solution of the following singular limit problem:

$$\begin{cases} -div(A^0(u_0)\nabla u_0) + c_\gamma h(u_0) = \theta f \zeta(u_0) + \dfrac{|\partial T|}{|Y|}\mathcal{M}_{\partial T}(g) & \text{in } \Omega, \\ u_0 = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.16}$$

where the homogenized matrix $A^0(t)$ is given by (2.8) *and verifies*

$$A^0(u_0)\nabla u_0 = \frac{1}{|Y|}\int_{Y^*} A(y,u_0)(\nabla u_0 + \nabla_y \widehat{u})dy. \tag{2.17}$$

Consequently, convergences (2.11) *hold for the whole sequence.*

Also let us introduce the usual corrector matrix field for perforated domains C^ε (see [22]), defined by

$$C^\varepsilon(\cdot,t) = \left(C^\varepsilon_{i,j}(\cdot,t)\right)_{i,j=1,\dots,N} \in \mathbb{R}^{N^2}, \quad \text{for every } t \in \mathbb{R},$$

where

$$\begin{cases} C^\varepsilon(x,t) = C\left(\dfrac{x}{\varepsilon}, t\right) & \text{a.e. in } \Omega^*_\varepsilon, \\ C_{i,j}(y,t) = \dfrac{\partial \widehat{\omega}_j}{\partial y_i}(y,t), \ i,j = 1,\dots,N & \text{a.e. on } Y^*. \end{cases} \tag{2.18}$$

Finally, we prove the following corrector result, which shows that the corrector of a suitable associated non-singular problem introduced in Section 5 (see (5.3)) is also a corrector for our problem:

Theorem 2.9. *Under assumptions of Theorem 2.8, let C^ε be defined by* (2.18)*. Then*

$$\lim_{\varepsilon\to 0} \|\nabla u_\varepsilon - C^\varepsilon(\cdot, u_\varepsilon)\nabla u_0\|_{L^1(\Omega^*_\varepsilon)} = 0.$$

Proof. This result is a direct consequence of Theorem 5.5 in Section 5 and the corrector result for the non-singular case given by [10, Theorem 4.11]. □

3 The periodic unfolding method

As already mentioned in the introduction, the periodic unfolding method was introduced by D. Cioranescu, A. Damlamian and G. Griso in [13] and [14] for fixed domains and extended to perforated ones in [17]-[16]. In this section, we give a short presentation of the periodic unfolding operator for perforated domains. We refer to [12] and [14] for detailed proofs of the results presented here.
Let $z \in \mathbb{R}^N$, we denote by $[z]_Y$ its integer part such that $z - [z]_Y$ belongs to Y and set $\{z\}_Y \doteq z - [z]_Y$. Then, for every positive ε,

$$x = \varepsilon \left(\left[\frac{x}{\varepsilon} \right]_Y + \left\{ \frac{x}{\varepsilon} \right\}_Y \right) \quad \forall x \in \mathbb{R}^N.$$

Definition 3.1. For any Lebesgue-measurable function ϕ on Ω_ε^*, the unfolding operator $\mathcal{T}_\varepsilon^*$ is defined as follows:

$$\mathcal{T}_\varepsilon^*(\phi)(x,y) \doteq \begin{cases} \phi\left(\varepsilon \left[\frac{x}{\varepsilon}\right]_Y + \varepsilon y\right) & \text{a.e. for } (x,y) \in \widehat{\Omega}_\varepsilon \times Y^*, \\ 0 & \text{a.e. for } (x,y) \in \Lambda_\varepsilon \times Y^*. \end{cases} \tag{3.1}$$

This definition makes sense even if ϕ is Lebesgue-measurable on $\widehat{\Omega}_\varepsilon^*$, since we extend it by zero in Λ_ε^*. Observe that, by definition, one has

$$\mathcal{T}_\varepsilon^*(\zeta(\phi))(x,y) = \begin{cases} \zeta(\mathcal{T}_\varepsilon^*(\phi)(x,y)) & \text{a.e. in } \widehat{\Omega}_\varepsilon \times Y^*, \\ 0 & \text{a.e. in } \Lambda_\varepsilon \times Y^*, \end{cases} \tag{3.2}$$

for all measurable fucntions ϕ on Ω_ε^* and any function ζ satisfying $\mathrm{H}_2)_i$. Moreover, the unfolding operator $\mathcal{T}_\varepsilon^*$ has the properties listed below:

Proposition 3.2 ([12][14][17]). *Let $p \in [1, +\infty)$.*

1. *$\mathcal{T}_\varepsilon^*$ is a linear and continuous operator from $L^p(\Omega_\varepsilon^*)$ to $L^p(\Omega \times Y^*)$.*
2. *$\mathcal{T}_\varepsilon^*(\phi\psi) = \mathcal{T}_\varepsilon^*(\phi)\mathcal{T}_\varepsilon^*(\psi)$ for every $\phi, \psi \in L^p(\Omega_\varepsilon^*)$.*
3. *Let $\phi \in L^p(Y^*)$ be a Y-periodic function and set $\phi_\varepsilon(x) = \phi(\frac{x}{\varepsilon})$. Then*
$$\mathcal{T}_\varepsilon^*(\phi_\varepsilon)(x,y) = \phi(y) \quad \text{a.e. in } \widehat{\Omega}_\varepsilon \times Y^*.$$
4. *For all $\phi \in L^1(\Omega_\varepsilon^*)$, the following **integration formula** holds:*
$$\int_{\widehat{\Omega}_\varepsilon^*} \phi(x)dx = \int_{\Omega_\varepsilon^*} \phi(x)dx - \int_{\Lambda_\varepsilon^*} \phi(x)dx = \frac{1}{|Y|} \int_{\Omega \times Y^*} \mathcal{T}_\varepsilon^*(\phi)(x,y)dxdy.$$
5. *$\|\mathcal{T}_\varepsilon^*(\phi)\|_{L^p(\Omega \times Y^*)} \leq |Y|^{\frac{1}{p}} \|\phi\|_{L^p(\Omega_\varepsilon^*)}$ for every $\phi \in L^p(\Omega_\varepsilon^*)$.*
6. *For $\phi \in L^p(\Omega)$,*
$$\mathcal{T}_\varepsilon^*(\phi) \to \phi \quad \text{strongly in } L^p(\Omega \times Y^*).$$
7. *Let $\{\phi_\varepsilon\}$ be a sequence in $L^p(\Omega)$ such that $\phi_\varepsilon \to \phi$ strongly in $L^p(\Omega)$. Then*
$$\mathcal{T}_\varepsilon^*(\phi_\varepsilon) \to \phi \quad \text{strongly in } L^p(\Omega \times Y^*).$$

Corollary 3.3 ([14]) **(Unfolding criterion for integrals (u.c.i.)).** *Let ϕ_ε be in $L^1(\Omega_\varepsilon^*)$. If*

$$\int_{\Lambda_\varepsilon^*} |\phi_\varepsilon| dx \to 0,$$

then

$$\int_{\Omega_\varepsilon^*} \phi_\varepsilon dx - \frac{1}{|Y|} \int_{\Omega\times Y^*} \mathcal{T}_\varepsilon^*(\phi_\varepsilon) dxdy \to 0.$$

In the sequel, if $\{\phi_\varepsilon\}$ is a sequence satisfying u.c.i., we write

$$\int_{\Omega_\varepsilon^*} \phi_\varepsilon dx \simeq \frac{1}{|Y|} \int_{\Omega\times Y^*} \mathcal{T}_\varepsilon^*(\phi_\varepsilon) dxdy.$$

Proposition 3.4 ([14]). *The following properties hold:*

- *Let $\{\phi_\varepsilon\} \subset L^p(\Omega_\varepsilon^*)$ such that $\|\phi_\varepsilon\|_{L^p(\Omega_\varepsilon^*)} \leq c$, with $p \in (1,+\infty)$, and $\psi \in L^{p'}(\Omega_\varepsilon^*)$, then*

$$\int_{\Omega_\varepsilon^*} \phi_\varepsilon \psi dx \simeq \frac{1}{|Y|} \int_{\Omega\times Y^*} \mathcal{T}_\varepsilon^*(\phi_\varepsilon)\mathcal{T}_\varepsilon^*(\psi) dxdy.$$

- *Let $\{\phi_\varepsilon\} \subset L^p(\Omega_\varepsilon^*)$ such that $\|\phi_\varepsilon\|_{L^p(\Omega_\varepsilon^*)} \leq c$, with $p \in (1,+\infty)$, and $\{\psi_\varepsilon\}$ a bounded sequence in $L^{p_0}(\Omega_\varepsilon^*)$ with $\frac{1}{p} + \frac{1}{p_0} < 1$. Then*

$$\int_{\Omega_\varepsilon^*} \phi_\varepsilon \psi_\varepsilon dx \simeq \frac{1}{|Y|} \int_{\Omega\times Y^*} \mathcal{T}_\varepsilon^*(\phi_\varepsilon)\mathcal{T}_\varepsilon^*(\psi_\varepsilon) dxdy.$$

As a consequence of the previous proposition, the following convergence result holds:

Proposition 3.5 ([12][14][17]). *Let $\{\phi_\varepsilon\} \subset L^p(\Omega_\varepsilon^*)$ such that $\|\phi_\varepsilon\|_{L^p(\Omega_\varepsilon^*)} \leq c$, with $p \in [1,+\infty)$. If*

$$\mathcal{T}_\varepsilon^*(\phi_\varepsilon) \rightharpoonup \phi \quad \text{weakly in } L^p(\Omega\times Y^*),$$

then

$$\widetilde{\phi_\varepsilon} \rightharpoonup \theta \mathcal{M}_{Y^*}(\phi) \quad \text{weakly in } L^p(\Omega).$$

We also recall the definition and some properties of the boundary unfolding operator.

Definition 3.6. For any Lebesgue-measurable function ϕ on $\partial\widehat{\Omega}_\varepsilon^* \cap \partial T_\varepsilon$, the boundary unfolding operator $\mathcal{T}_\varepsilon^b$ is defined as follows:

$$\mathcal{T}_\varepsilon^b(\phi)(x,y) \doteq \begin{cases} \phi\left(\varepsilon\left[\frac{x}{\varepsilon}\right]_Y + \varepsilon y\right) & \text{a.e. for } (x,y) \in \widehat{\Omega}_\varepsilon \times \partial T, \\ 0 & \text{a.e. for } (x,y) \in \Lambda_\varepsilon \times \partial T. \end{cases} \tag{3.3}$$

Proposition 3.7 ([12][14][17]). *Let $p \in (1,+\infty)$.*

1. *$\mathcal{T}_\varepsilon^b$ is a linear and continuous operator from $L^p(\partial\widehat{\Omega}_\varepsilon^* \cap \partial T_\varepsilon)$ to $L^p(\Omega\times\partial T)$.*
2. *$\mathcal{T}_\varepsilon^b(\phi\psi) = \mathcal{T}_\varepsilon^b(\phi)\mathcal{T}_\varepsilon^b(\psi)$ for every $\phi,\psi \in L^p(\partial\widehat{\Omega}_\varepsilon^* \cap \partial T_\varepsilon)$.*
3. *Let $\phi \in L^p(\partial T)$ be a Y-periodic function and set $\phi_\varepsilon(x) = \phi(\frac{x}{\varepsilon})$. Then*

$$\mathcal{T}_\varepsilon^b(\phi_\varepsilon)(x,y) = \phi(y) \quad \text{a.e. in } \widehat{\Omega}_\varepsilon \times \partial T.$$

4. *For all $\phi \in L^1(\partial\widehat{\Omega}^*_\varepsilon \cap \partial T_\varepsilon)$, the **integration formula** is given by*

$$\int_{\Gamma_1^\varepsilon} \phi(x) d\sigma_x = \frac{1}{\varepsilon|Y|} \int_{\Omega\times\partial T} \mathcal{T}^b_\varepsilon(\phi)(x,y) dx d\sigma_y.$$

5. $\|\mathcal{T}^b_\varepsilon(\phi)\|_{L^p(\Omega\times\partial T)} \leq \varepsilon^{\frac{1}{p}} |Y|^{\frac{1}{p}} \|\phi\|_{L^p(\partial T_\varepsilon)}$ *for every* $\phi \in L^p(\partial\widehat{\Omega}^*_\varepsilon \cap \partial T_\varepsilon)$.

6. *For $\phi \in L^p(\partial\widehat{\Omega}^*_\varepsilon \cap \partial T_\varepsilon)$,*

$$\mathcal{T}^b_\varepsilon(\phi) \to \phi \quad \text{strongly in } L^p(\Omega \times \partial T).$$

We summarize in the proposition below some results that have been obtained in the literature (we refer to [16], [17] as well to [12] for convergences $i) - iv)$, and to [8] for $v)$).

Proposition 3.8. *Let $\{w_\varepsilon\}$ be a sequence such that $w_\varepsilon \in V_\varepsilon$ and $\|w_\varepsilon\|_{V_\varepsilon} \leq c$. Then, there exist a subsequence (still denoted by ε) and two functions $(w_0, \widehat{w}) \in H^1_0(\Omega) \times L^2(\Omega; H^1_{per}(Y^*))$ with $\mathcal{M}_{Y^*}(\widehat{w}) = 0$, such that*

$$\begin{cases} i)\ \mathcal{T}^*_\varepsilon(w_\varepsilon) \to w_0 & \text{strongly in } L^2(\Omega; H^1(Y^*)), \\ ii)\ \mathcal{T}^*_\varepsilon(\nabla w_\varepsilon) \rightharpoonup \nabla w_0 + \nabla_y \widehat{w} & \text{weakly in } L^2(\Omega \times Y^*), \\ iii)\ \widetilde{w_\varepsilon} \rightharpoonup \theta w_0 & \text{weakly in } L^2(\Omega), \\ iv)\ \mathcal{T}^b_\varepsilon(h(w_\varepsilon)) \to h(w_0) & \text{strongly in } L^t(\Omega; W^{1-\frac{1}{t},t}(\partial T)), \\ v)\ \mathcal{T}^*_\varepsilon(A^\varepsilon(x, w_\varepsilon)) \to A(y, w_0) & \text{a.e. in } \Omega \times Y^*, \end{cases} \tag{3.4}$$

where

$$t \doteq \begin{cases} \in (1; 2) & \text{if } N = 2 \text{ and } q > 1, \\ \dfrac{2N}{q(N-2)+2} & \text{otherwise}; \end{cases} \tag{3.5}$$

and q given by H_6).

Here, let us just point out that

$$\mathcal{T}^*_\varepsilon(A^\varepsilon(x, u_\varepsilon(x))) = \begin{cases} A(y, \mathcal{T}^*_\varepsilon(u_\varepsilon)(x,y)) & \text{a.e. for } (x,y) \in \widehat{\Omega}_\varepsilon \times Y^*, \\ 0 & \text{a.e. for } (x,y) \in \Lambda_\varepsilon \times Y^*. \end{cases} \tag{3.6}$$

We recall now an important result, which plays an essential role when treating surface integrals containing periodic functions. It was originally proved in [15], then revisited and improved by unfolding in [12] and [17] (see also [10, Propositions 3.7-3.8] for the case $g \in L^s$).

Proposition 3.9 ([10])**.** *Let g be a function satisfying hypothesis $H_3)_i$. Then, for every $\varphi \in V_\varepsilon$, the following inequality holds:*

$$\left| \int_{\Gamma_1^\varepsilon} g\left(\frac{x}{\varepsilon}\right) \varphi(x) d\sigma \right| \leq \frac{c}{\varepsilon} (|\mathcal{M}_{\partial T}(g)| + \varepsilon) \|\nabla\varphi\|_{L^2(\Omega^*_\varepsilon)}.$$

Furthermore, if $\mathcal{M}_{\partial T}(g) = 0$, then for every $\varphi \in V_\varepsilon$, one has

$$\lim_{\varepsilon \to 0} \int_{\Gamma_1^\varepsilon} g\left(\frac{x}{\varepsilon}\right) \varphi(x) d\sigma = 0.$$

Moreover, if $\{w_\varepsilon\} \subset V_\varepsilon$ is a sequence such that $(3.4)_i$ holds, then

$$\lim_{\varepsilon \to 0} \varepsilon \int_{\Gamma_1^\varepsilon} g\left(\frac{x}{\varepsilon}\right) w_\varepsilon d\sigma = \frac{|\partial T|}{|Y|} \mathcal{M}_{\partial T}(g) \int_\Omega w_0 dx.$$

4 A priori estimates

In the following propositions we establish two a priori estimates for a solution of problem (2.7), which are uniform with respect to ε. Let us point out that in the sequel we can suppose $\varepsilon < 1$ without loss of generality, since ε will tend to zero.

Proposition 4.1. *Under assumptions H_1)- H_6), let $u_\varepsilon \in V_\varepsilon$ be the solution of problem (2.7). The following a priori estimate holds:*

$$\|u_\varepsilon\|_{V_\varepsilon} \leq c, \tag{4.1}$$

where c depends on α, c_P, $\|f\|_{L^l(\Omega)}$ and $\mathcal{M}_{\partial T}(g)$.

Proof. Let $u_\varepsilon \in V_\varepsilon$ be the solution of problem (2.3) and let us choose u_ε as test function in its variational formulation (2.7). The same computation made in the proof of the [25, Proposition 3.1], together with the nonnegativity of ε and γ, gives

$$\alpha \|\nabla u_\varepsilon\|^2_{L^2(\Omega_\varepsilon^*)} \leq \int_{\Omega_\varepsilon^*} A^\varepsilon(x, u_\varepsilon) \nabla u_\varepsilon \nabla u_\varepsilon dx + \varepsilon^\gamma \int_{\Gamma_1^\varepsilon} \rho_\varepsilon h(u_\varepsilon) u_\varepsilon d\sigma$$

$$\leq \int_{\Omega_\varepsilon^*} f u_\varepsilon^{1-k} dx + \int_{\Gamma_1^\varepsilon} g_\varepsilon u_\varepsilon d\sigma.$$

Applying the Young inequality with exponents $\frac{2}{1-k}$ and $\frac{2}{1+k}$, we have for every $\eta_1 > 0$,

$$f u_\varepsilon^{1-k} \leq c(\eta_1) f^{\frac{2}{1+k}} + \eta_1 u_\varepsilon^2.$$

This implies, in view of the Poincaré inequality (2.6), that

$$\alpha \|\nabla u_\varepsilon\|^2_{L^2(\Omega_\varepsilon^*)} \leq \eta_1 c_P^2 \|\nabla u_\varepsilon\|^2_{L^2(\Omega_\varepsilon^*)} + c(\eta_1) \|f\|^{\frac{2}{1+k}}_{L^{\frac{2}{1+k}}(\Omega)} + \int_{\Gamma_1^\varepsilon} g_\varepsilon u_\varepsilon d\sigma,$$

where c_P is the Poincaré constant in V_ε. Moreover, in view of the definition of g_ε and Proposition 3.9, for the Young inequality we have

$$\int_{\Gamma_1^\varepsilon} g_\varepsilon u_\varepsilon d\sigma \leq c(|\mathcal{M}_{\partial T}(g)| + \varepsilon) \|u_\varepsilon\|_{V_\varepsilon} \leq \eta_2 \|u_\varepsilon\|^2_{V_\varepsilon} + c(\eta_2),$$

independently on ε. Whence, choosing η_1 and η_2 sufficiently small so that $\alpha - \eta_1 c_P^2 - \eta_2 > 0$, we deduce that there exists a constant c, independent on ε, such that

$$\|u_\varepsilon\|_{V_\varepsilon} \leq c,$$

where c depends on α, c_P, $\|f\|_{L^l(\Omega)}$ and $\mathcal{M}_{\partial T}(g)$. □

In the next a priori estimate, we will use the following property that we prove for the reader's convenience:

Lemma 4.2. *Let Ω_1 be a bounded open set in $\mathbb{R}^N$ with $\partial\Omega_1$ Lipschitz-continuous. If u belongs to $H^1(\Omega_1) \cap L^\infty(\Omega_1)$, then $u \in L^\infty(\partial\Omega_1)$.*

Proof. Set $k := \|u\|_{L^\infty(\Omega_1)}$ and take $u_n \in \mathcal{D}(\Omega_1)$ such that

$$u_n \to u \quad \text{strongly in } H^1(\Omega_1).$$

Then, by known results, if T_k denotes the truncation function at level k, we have

$$T_k(u_n) \to T_k(u) = u \quad \text{strongly in } H^1(\Omega_1),$$

as $n \to +\infty$. Hence

$$T_k(u_n) \to u \quad \text{strongly in } L^2(\partial\Omega_1),$$

and, up to a subsequence, almost everywhere on $\partial\Omega_1$. Since $|T_k(u_n)| \leq k$, this gives

$$u \leq k \ \text{ a.e. on } \partial\Omega_1,$$

which implies the result. □

Proposition 4.3. *Under assumptions H_1)- H_6), let $u_\varepsilon \in V_\varepsilon$ be the solution of problem* (2.7). *Then, up to a subsequence,*

$$\|f\zeta(u_\varepsilon)\varphi\|_{L^1(\Omega_\varepsilon^*)} \leq c, \tag{4.2}$$

for every nonnegative $\varphi \in H_0^1(\Omega) \cap L^\infty(\Omega)$ with c depending on α, β, c_P, $|Y|$, $\|\rho\|_{L^\infty(\partial T)}$, $\|\nabla\varphi\|_{L^2(\Omega)}$, $\|f\|_{L^l(\Omega)}$ and $\mathcal{M}_{\partial T}(g)$.

Proof. Let $u_\varepsilon \in V_\varepsilon$ be the solution of problem (2.7) and let us choose a nonnegative function $\varphi \in H_0^1(\Omega) \cap L^\infty(\Omega)$ as test function. Since f, ζ, g and φ are nonnegative, using H$_1$) together with the Hölder inequality, we have

$$\begin{aligned} 0 \leq \int_{\Omega_\varepsilon^*} f\zeta(u_\varepsilon)\varphi dx &= \int_{\Omega_\varepsilon^*} A^\varepsilon(x,u_\varepsilon)\nabla u_\varepsilon \nabla\varphi dx + \varepsilon^\gamma \int_{\Gamma_1^\varepsilon} \rho_\varepsilon h(u_\varepsilon)\varphi d\sigma - \int_{\Gamma_1^\varepsilon} g_\varepsilon \varphi d\sigma \\ &\leq \beta\|\nabla u_\varepsilon\|_{L^2(\Omega_\varepsilon^*)}\|\nabla\varphi\|_{L^2(\Omega_\varepsilon^*)} + \varepsilon^\gamma \int_{\Gamma_1^\varepsilon} \rho_\varepsilon h(u_\varepsilon)\varphi d\sigma. \end{aligned}$$

As far as it concerns the surface integral, thanks to the properties of the boundary unfolding operator (see Proposition 3.7) and H$_5$), we get

$$\begin{aligned} \varepsilon^\gamma \int_{\Gamma_1^\varepsilon} \rho_\varepsilon h(u_\varepsilon)\varphi d\sigma &= \frac{\varepsilon^{\gamma-1}}{|Y|} \int_{\Omega\times\partial T} \mathcal{T}_\varepsilon^b(\rho_\varepsilon)\mathcal{T}_\varepsilon^b(h(u_\varepsilon))\mathcal{T}_\varepsilon^b(\varphi) dx d\sigma_y \\ &= \frac{\varepsilon^{\gamma-1}}{|Y|} \int_{\Omega\times\partial T} \rho(y)\mathcal{T}_\varepsilon^b(h(u_\varepsilon))\mathcal{T}_\varepsilon^b(\varphi) dx d\sigma_y \\ &\leq \frac{\varepsilon^{\gamma-1}}{|Y|} \|\rho\|_{L^\infty(\partial T)} \int_{\Omega\times\partial T} \mathcal{T}_\varepsilon^b(h(u_\varepsilon))\mathcal{T}_\varepsilon^b(\varphi) dx d\sigma_y. \end{aligned} \tag{4.3}$$

In order to estimate the right-hand side of (4.3), let t be a real number veryfing (3.5). Thanks to a priori estimate (4.1), Proposition 3.8 holds true; so that, up to a subsequence, $\mathcal{T}_\varepsilon^b(h(u_\varepsilon))$ is bounded in $L^t(\Omega \times \partial T)$. In addition, by using the assumption on φ, Lemma 4.2 and Proposition 3.7_6 one has the boundedness of $\mathcal{T}_\varepsilon^b(\varphi)$ in $L^{t'}(\Omega \times \partial T)$, where t' is the coniugate of t. Applying the Hölder inequality we obtain

$$\frac{\varepsilon^{\gamma-1}}{|Y|}\|\rho\|_{L^\infty(\partial T)}\int_{\Omega\times\partial T}\mathcal{T}_\varepsilon^b(h(u_\varepsilon))\mathcal{T}_\varepsilon^b(\varphi)dxd\sigma_y$$
$$\leq \frac{\varepsilon^{\gamma-1}}{|Y|}\|\rho\|_{L^\infty(\partial T)}\|\mathcal{T}_\varepsilon^b(h(u_\varepsilon))\|_{L^t(\Omega\times\partial T)}\|\mathcal{T}_\varepsilon^b(\varphi)\|_{L^{t'}(\Omega\times\partial T)}.$$

This, together with (4.1) and hypothesis H_5), allows to state that

$$0 \leq \int_{\Omega_\varepsilon^*} f\zeta(u_\varepsilon)\varphi dx \leq c,$$

where the constant c depends on α, β, c_P, $|Y|$, $\|\rho\|_{L^\infty(\partial T)}$, $\|\nabla\varphi\|_{L^2(\Omega)}$, $\|f\|_{L^l(\Omega)}$ and $\mathcal{M}_{\partial T}(g)$. □

The following result gives an estimate of the integral of the singular term close to the singular set $\{u_\varepsilon = 0\}$. It is inspired by arguments used in [27], [28] and [23], which present a similar singular term. The estimate follows immediately from the a priori estimate [25, Proposition 3.3] in view of [30, Lemma 2.7].

Proposition 4.4 ([25]). *Under assumptions H_1)- H_6), let $u_\varepsilon \in V_\varepsilon$ be the solution of problem (2.7) and δ a fixed positive real number. Then,*

$$\int_{\{0<u_\varepsilon\leq\delta\}} f\zeta(u_\varepsilon)\varphi dx \leq \int_{\Omega_\varepsilon^*} A^\varepsilon(x,u_\varepsilon)\nabla u_\varepsilon\nabla\varphi Z_\delta(u_\varepsilon)dx + c\varepsilon^{\gamma-\frac{1}{2}}h(2\delta)\|\rho\|_{L^\infty(\partial T)}\|\varphi\|_{V_\varepsilon},$$

for every $\varphi \in V_\varepsilon$, $\varphi \geq 0$ and c independent on ε, where Z_δ is an auxiliary function defined by

$$Z_\delta(s) = \begin{cases} 1, & \text{if } 0 \leq s \leq \delta, \\ -\dfrac{s}{\delta} + 2, & \text{if } \delta \leq s \leq 2\delta, \\ 0, & \text{if } s \geq 2\delta. \end{cases}$$

The estimate given in Proposition 4.3 allows us to prove Proposition 2.5.

Proof of Proposition 2.5. Let u_ε be the solution of problem (2.7). Proposition 4.1 allows us to apply Proposition 3.8 which provides the existence of $u_0 \in H_0^1(\Omega)$ and $\widehat{u} \in L^2(\Omega; H^1_{per}(Y^*))$ with $\mathcal{M}_{Y^*}(\widehat{u}) = 0$ such that, up to a subsequence, one has convergences $(2.11)_i$, $(2.11)_{iii}$-$(2.11)_v$, where $\mathcal{T}_\varepsilon^*$ is the unfolding operator defined by (3.1).
To prove $(2.11)_{ii}$, observe that, by construction, for every $x \in \Omega$ there exists $\varepsilon_x > 0$ such that

$$x \in \widehat{\Omega}_\varepsilon, \quad \forall\varepsilon \leq \varepsilon_x.$$

Consequently, since (3.2) holds true, for almost every $(x, y) \in \Omega \times Y^*$, there exists $\varepsilon_x > 0$ such that

$$\mathcal{T}_\varepsilon^*(\zeta(u_\varepsilon)) = \zeta(\mathcal{T}_\varepsilon^*(u_\varepsilon)), \quad \forall\varepsilon \leq \varepsilon_x. \tag{4.4}$$

On the other hand, using the continuity of ζ and $(2.11)_i$, we have

$$\zeta(\mathcal{T}_\varepsilon^*(u_\varepsilon)) \to \zeta(u_0) \quad \text{a.e. in } \Omega \times Y^*. \tag{4.5}$$

This, together with (4.4), gives $(2.11)_{ii}$.

To show that u_0 is nonnegative almost everywhere in Ω, we note that every solution u_ε is positive almost everywhere in Ω_ε^*, by Theorem 2.3. Then, the definition of the unfolding operator implies $\mathcal{T}_\varepsilon^*(u_\varepsilon) \geq 0$ almost everywhere in $\Omega \times Y^*$ so that, in view of $(2.11)_i$,

$$u_0 \geq 0 \quad \text{a.e. in } \Omega.$$

It remains to prove the second condition in (2.13). Let us choose first a nonnegative $\varphi \in H_0^1(\Omega) \cap L^\infty(\Omega)$. Propositions $3.2_{2,4}$ and 4.3, for the subsequence mentioned before, lead to

$$\liminf_{\varepsilon \to 0} \frac{1}{|Y|} \int_{\Omega \times Y^*} \mathcal{T}_\varepsilon^*(f) \mathcal{T}_\varepsilon^*(\zeta(u_\varepsilon)) \mathcal{T}_\varepsilon^*(\varphi) dx dy \leq \liminf_{\varepsilon \to 0} \int_{\Omega_\varepsilon^*} f \zeta(u_\varepsilon) \varphi dx < +\infty. \tag{4.6}$$

Now, from Proposition 3.2_6, $\mathcal{T}_\varepsilon^*(f)$ and $\mathcal{T}_\varepsilon^*(\varphi)$ converge to f and φ, respectively, almost everywhere in $\Omega \times Y^*$, up to a subsequence. Thus, by $(2.11)_{ii}$,

$$\mathcal{T}_\varepsilon^*(f) \mathcal{T}_\varepsilon^*(\zeta(u_\varepsilon)) \mathcal{T}_\varepsilon^*(\varphi) \to f \zeta(u_0) \varphi \quad \text{a.e. in } \Omega \times Y^*.$$

Since $\mathcal{T}_\varepsilon^*(f)$, $\mathcal{T}_\varepsilon^*(\zeta(u_\varepsilon))$ and $\mathcal{T}_\varepsilon^*(\varphi)$ are nonnegative functions, we can use Fatou's lemma and (4.6) to obtain

$$\frac{1}{|Y|} \int_{\Omega \times Y^*} f \zeta(u_0) \varphi dx dy \leq \liminf_{\varepsilon \to 0} \frac{1}{|Y|} \int_{\Omega \times Y^*} \mathcal{T}_\varepsilon^*(f) \mathcal{T}_\varepsilon^*(\zeta(u_\varepsilon)) \mathcal{T}_\varepsilon^*(\varphi) dx dy < +\infty.$$

Being the functions f and u_0 independent on y, this implies in particular that

$$\theta \int_\Omega f \zeta(u_0) \varphi dx < +\infty$$

and ends the proof for $\varphi \geq 0$. For φ with any sign, it suffices to decompose it as in (2.2). □

5 A crucial auxiliary result

In this section we state and prove the convergence result given in Theorem 5.5, which is our main tool when proving Theorem 2.8. Let u_0 a weak cluster point of the sequence $\{u_\varepsilon\}$. Then, Theorem 5.5 shows that the gradient of our solution u_ε is equivalent to the gradient of the solution v_ε of a suitable linear problem, (5.5), associated with u_0. This idea was originally introduced in [5] (see also [4]) in the homogenization of some nonlinear problem with quadratic growth. We refer to [22] and [11] for the case of perforated domains. We also refer to [23] for the case of a singular nonlinearity verifying $\mathrm{H}_2)_i$, in a two-component domain with a oscillating interface. Here we adapt some techniques from [11] and [23], taking into account the homogenization results proved in [10].

In order to prove Theorem 5.5, the following preliminary result is needed (see [10] and [11, Remark 3.1]).

Proposition 5.1 ([11]). *There exists a linear operator $\mathcal{L}_\varepsilon : H^{-1}(\Omega) \to V'_\varepsilon$ satisfying the following condition:*
If $\{\varphi_\varepsilon\}$ is a sequence such that

$$\|\varphi_\varepsilon\|_{V_\varepsilon} \le c \quad \text{and} \quad \widetilde{\varphi_\varepsilon} \rightharpoonup \theta\varphi_0 \quad \text{weakly in } L^2(\Omega), \tag{5.1}$$

then

$$\lim_{\varepsilon\to 0}\langle \mathcal{L}_\varepsilon(Z), \varphi_\varepsilon\rangle_{V'_\varepsilon, V_\varepsilon} = \langle Z, \varphi_0\rangle_{H^{-1}(\Omega), H^1_0(\Omega)}. \tag{5.2}$$

Notice that, see [11, Remark 3.1], if (5.1) holds, one has $\varphi_0 \in H^1_0(\Omega)$.

Remark 5.2. Let us point out that our sequence $\{u_\varepsilon\}$ satisfies (5.1), thanks to Proposition 4.1 and $(2.11)_{iii}$.

Here, as in [11], the suitable linear problem associated with problem (2.3) is

$$\begin{cases} -div(A^\varepsilon(x,u_\varepsilon)\nabla v_\varepsilon) = \mathcal{L}_\varepsilon(-div(A^0(u_0)\nabla u_0) + c_\gamma h(u_0) - \dfrac{|\partial T|}{|Y|}\mathcal{M}_{\partial T}(g)) & \text{in } \Omega^*_\varepsilon, \\ v_\varepsilon = 0 & \text{on } \Gamma^\varepsilon_0, \\ (A^\varepsilon(x,u_\varepsilon)\nabla v_\varepsilon)\nu + \varepsilon^\gamma \rho_\varepsilon(x) h(u_\varepsilon) = g_\varepsilon & \text{on } \Gamma^\varepsilon_1, \end{cases} \tag{5.3}$$

where c_γ is defined by (2.15) and whose homogenization has been studied in [10]. One can easily check (see [11]), using this homogenization result, that

$$\begin{cases} -div(A^0(u_0)\nabla v_0) = -div(A^0(u_0)\nabla u_0) & \text{in } \Omega, \\ v_0 = 0 & \text{on } \partial\Omega. \end{cases}$$

Then, the uniqueness of this problem gives the following result stated in [11]:

Lemma 5.3 ([11]). *Under the assumptions of Proposition 2.5, let v_ε be the solution of problem* (5.5). *Then, v_ε satisfies*

$$\|v_\varepsilon\|_{V_\varepsilon} \le c \quad \text{and} \quad \widetilde{v_\varepsilon} \rightharpoonup \theta u_0 \quad \text{weakly in } L^2(\Omega), \tag{5.4}$$

with u_0 given by Proposition 2.5.

Hence, the gradient of the solution v_ε of problem (5.3) appears to be a natural candidate in order to show its equivalence (in the L^2-norm) to the gradient of our solution u_ε.
The variational formulation of problem (5.3) is

$$\begin{aligned} &\int_{\Omega^*_\varepsilon} A^\varepsilon(x,u_\varepsilon)\nabla v_\varepsilon \nabla\varphi dx + \varepsilon^\gamma \int_{\Gamma^\varepsilon_1} \rho_\varepsilon h(u_\varepsilon)\varphi d\sigma \\ &\quad = \int_{\Gamma^\varepsilon_1} g_\varepsilon \varphi d\sigma + \langle \mathcal{L}_\varepsilon(-div(A^0(u_0)\nabla u_0) + c_\gamma h(u_0) - \frac{|\partial T|}{|Y|}\mathcal{M}_{\partial T}(g)), \varphi\rangle_{V'_\varepsilon, V_\varepsilon}, \quad \forall \varphi \in V_\varepsilon. \end{aligned} \tag{5.5}$$

Unlike the problem treated in [11], under our assumptions, the functions v_ε are not necessarily bounded. Hence, as in [23], we define the following auxiliary functions u_m:

$$\forall m \in \mathbb{N}, m \ge 1, \quad u_m \doteq T_m(u_0),$$

where T_m is the usual truncation function at level m, so that

$$0 \leq u_m \leq u_0 \quad \text{ and } \quad u_m \to u_0 \quad \text{strongly in } H_0^1(\Omega), \quad \text{as } m \to +\infty. \tag{5.6}$$

Then, we define by v_ε^m the solution of the following problem:

$$\begin{cases} -div(A^\varepsilon(x, u_\varepsilon)\nabla v_\varepsilon^m) = \mathcal{L}_\varepsilon(-div(A^0(u_0)\nabla u_m) + c_\gamma h(u_0) - \dfrac{|\partial T|}{|Y|}\mathcal{M}_{\partial T}(g)) & \text{in } \Omega_\varepsilon^*, \\ v_\varepsilon^m = 0 & \text{on } \Gamma_0^\varepsilon, \\ (A^\varepsilon(x, u_\varepsilon)\nabla v_\varepsilon^m)\nu + \varepsilon^\gamma \rho_\varepsilon(x) h(u_\varepsilon) = g_\varepsilon & \text{on } \Gamma_1^\varepsilon, \end{cases}$$

whose variational formulation is

$$\begin{aligned} &\int_{\Omega_\varepsilon^*} A^\varepsilon(x, u_\varepsilon)\nabla v_\varepsilon^m \nabla\varphi dx + \varepsilon^\gamma \int_{\Gamma_1^\varepsilon} \rho_\varepsilon h(u_\varepsilon)\varphi d\sigma \\ &= \int_{\Gamma_1^\varepsilon} g_\varepsilon \varphi d\sigma + \langle \mathcal{L}_\varepsilon(-div(A^0(u_0)\nabla u_m) + c_\gamma h(u_0) - \frac{|\partial T|}{|Y|}\mathcal{M}_{\partial T}(g)), \varphi\rangle_{V_\varepsilon', V_\varepsilon}, \quad \forall \varphi \in V_\varepsilon. \end{aligned} \tag{5.7}$$

The existence and the uniqueness of such a solution $v_\varepsilon^m \in V_\varepsilon$ is straightforward proved by using the Lax-Milgram theorem. Also, from [10, Corollary 4.6] written for $z_\varepsilon = u_\varepsilon$ and for $Z = -div(A^0(u_0)\nabla u_m) + c_\gamma h(u_0) - \dfrac{|\partial T|}{|Y|}\mathcal{M}_{\partial T}(g)$, one has, for every fixed m,

$$\|v_\varepsilon^m\|_{V_\varepsilon} \leq c \quad \text{ and } \quad \widetilde{v_\varepsilon^m} \rightharpoonup \theta u_m \quad \text{weakly in } L^2(\Omega), \tag{5.8}$$

with c independent on m and ε. Hence, v_ε^m satisfies (5.1) and, from Proposition 3.8, the following convergence holds true, up to a subsequence:

$$\mathcal{T}_\varepsilon^*(v_\varepsilon^m) \to u_m \quad \text{ strongly in } L^2(\Omega; H^1(Y^*)). \tag{5.9}$$

Moreover, by classical results from [31], we have that for every fixed m

$$\|v_\varepsilon^m\|_{L^\infty(\Omega_\varepsilon^*)} \leq c_m, \quad \text{for every } \varepsilon. \tag{5.10}$$

Lemma 5.4. *The sequence* $\{(v_\varepsilon^m)^-\}$ *satisfies conditions* (5.1). *Moreover*

$$\mathcal{T}_\varepsilon^b((v_\varepsilon^m)^-) \rightharpoonup 0 \quad \textit{ weakly in } L^r(\Omega \times \partial T), \ \forall r \geq 1. \tag{5.11}$$

Proof. In view of the estimate in (5.8), one has

$$\|(v_\varepsilon^m)^-\|_{V_\varepsilon} \leq \|v_\varepsilon^m\|_{V_\varepsilon} \leq c.$$

Hence, Proposition 3.8 implies

$$\mathcal{T}_\varepsilon^*((v_\varepsilon^m)^-) \to u_m^- = 0 \quad \text{ strongly in } L^2(\Omega; H^1(Y^*)), \tag{5.12}$$

since $u_m \geq 0$ and (5.9) implies convergence (5.12) in $L^2(\Omega \times Y^*)$. Also, by using Proposition 3.5, one has

$$\widetilde{(v_\varepsilon^m)^-} \rightharpoonup \theta \mathcal{M}_{Y^*}(u_m^-) = 0 \quad \text{weakly in } L^2(\Omega).$$

Moreover, from (5.12) we get in particular

$$\mathcal{T}_\varepsilon^*((v_\varepsilon^m)^-) \rightharpoonup u_m^- = 0 \quad \text{ weakly in } L^2(\Omega \times \partial T),$$

and the convergence (5.11) derives from (5.10) and Lemma 4.2. □

We can now state the main result of this section.

Theorem 5.5. *Under assumptions H_1)- H_6), let u_ε and v_ε be solutions of problems* (2.7) *and* (5.5)*, respectively. Then, for the subsequence verifying* (2.11)*,*

$$\lim_{\varepsilon\to 0} \|\nabla u_\varepsilon - \nabla v_\varepsilon\|_{L^2(\Omega_\varepsilon^*)} = 0.$$

Proof. The proof is done in 3 steps. It makes use of some ideas from [23] concerning the singular term, splitted into two terms: one near the singularity and one far from it which results not singular.

Step 1. Let us prove that

$$\lim_{\varepsilon\to 0} \int_{\Omega_\varepsilon^*} [(v_\varepsilon^m)^-]^2 dx = 0. \tag{5.13}$$

Observe that, from the Poincaré inequality (2.6),

$$0 \le \int_{\Omega_\varepsilon^*} [(v_\varepsilon^m)^-]^2 dx \le c_P^2 \int_{\Omega_\varepsilon^*} |\nabla (v_\varepsilon^m)^-|^2 dx, \tag{5.14}$$

for any fixed $m \ge 1$. Then it suffices to prove that

$$\lim_{\varepsilon\to 0} \int_{\Omega_\varepsilon^*} |\nabla (v_\varepsilon^m)^-|^2 dx = 0 \qquad \forall m \ge 1. \tag{5.15}$$

Let us choose $-(v_\varepsilon^m)^- \in V_\varepsilon$ as test fuction in (5.7) getting

$$\begin{aligned}
&-\int_{\Omega_\varepsilon^*} A^\varepsilon(x,u_\varepsilon)\nabla v_\varepsilon^m \nabla (v_\varepsilon^m)^- dx - \varepsilon^\gamma \int_{\Gamma_1^\varepsilon} \rho_\varepsilon h(u_\varepsilon)(v_\varepsilon^m)^- d\sigma \\
&\quad = -\int_{\Gamma_1^\varepsilon} g_\varepsilon (v_\varepsilon^m)^- d\sigma + \langle \mathcal{L}_\varepsilon(-div(A^0(u_0)\nabla u_m) + c_\gamma h(u_0) - \frac{|\partial T|}{|Y|}\mathcal{M}_{\partial T}(g)), -(v_\varepsilon^m)^-\rangle_{V_\varepsilon', V_\varepsilon}.
\end{aligned}$$

From the ellipticity of A and observing that $\nabla v_\varepsilon^m = \nabla (v_\varepsilon^m)^+ - \nabla (v_\varepsilon^m)^-$ and

$$\int_{\Omega_\varepsilon^*} A^\varepsilon(x,u_\varepsilon)\nabla (v_\varepsilon^m)^+ \nabla (v_\varepsilon^m)^- dx = 0,$$

we have

$$\begin{aligned}
&\alpha \|\nabla (v_\varepsilon^m)^-\|^2_{L^2(\Omega_\varepsilon^*)} \le \int_{\Omega_\varepsilon^*} A^\varepsilon(x,u_\varepsilon)\nabla (v_\varepsilon^m)^- \nabla (v_\varepsilon^m)^- dx + \int_{\Gamma_1^\varepsilon} g_\varepsilon (v_\varepsilon^m)^- d\sigma \\
&= \varepsilon^\gamma \int_{\Gamma_1^\varepsilon} \rho_\varepsilon h(u_\varepsilon)(v_\varepsilon^m)^- d\sigma + \langle \mathcal{L}_\varepsilon(-div(A^0(u_0)\nabla u_m) + c_\gamma h(u_0) - \frac{|\partial T|}{|Y|}\mathcal{M}_{\partial T}(g)), -(v_\varepsilon^m)^-\rangle_{V_\varepsilon', V_\varepsilon}.
\end{aligned} \tag{5.16}$$

From Lemma 5.4 and Proposition 5.1, we get

$$\begin{aligned}
&\lim_{\varepsilon\to 0} \langle \mathcal{L}_\varepsilon(-div(A^0(u_0)\nabla u_m) + c_\gamma h(u_0) - \frac{|\partial T|}{|Y|}\mathcal{M}_{\partial T}(g)), -(v_\varepsilon^m)^-\rangle_{V_\varepsilon', V_\varepsilon} \\
&\qquad = \langle -div(A^0(u_0)\nabla u_m) + c_\gamma h(u_0) - \frac{|\partial T|}{|Y|}\mathcal{M}_{\partial T}(g)), 0\rangle_{H^{-1}(\Omega), H_0^1(\Omega)} = 0.
\end{aligned} \tag{5.17}$$

Also, by using Proposition $3.7_{2,3,4}$, $(2.11)_v$ and (5.11) of Lemma 5.4, we get

$$0 \le \lim_{\varepsilon\to 0} \varepsilon^\gamma \int_{\Gamma_1^\varepsilon} \rho_\varepsilon h(u_\varepsilon)(v_\varepsilon^m)^- d\sigma = \lim_{\varepsilon\to 0} \frac{\varepsilon^{\gamma-1}}{|Y|} \int_{\Omega\times\partial T} \rho(y) \mathcal{T}_\varepsilon^b(h(u_\varepsilon)) \mathcal{T}_\varepsilon^b((v_\varepsilon^m)^-) dx d\sigma_y$$

$$\le \lim_{\varepsilon\to 0} \frac{\varepsilon^{\gamma-1}}{|Y|} \|\rho\|_{L^\infty(\partial T)} \int_{\Omega\times\partial T} \mathcal{T}_\varepsilon^b(h(u_\varepsilon)) \mathcal{T}_\varepsilon^b((v_\varepsilon^m)^-) dx d\sigma_y = 0,$$

which, together with (5.16)-(5.17), gives (5.15).

Step 2. In this step we show that

$$\lim_{m\to+\infty} \lim_{\varepsilon\to 0} \int_{\Omega_\varepsilon^*} |\nabla(u_\varepsilon - v_\varepsilon^m)|^2 dx = 0. \tag{5.18}$$

To do that, let us choose $u_\varepsilon - v_\varepsilon^m \in V_\varepsilon$ as test function in (2.7) and (5.7). By subtraction and $H_1)_{iii}$, one has

$$\begin{aligned} \alpha \|\nabla(u_\varepsilon - v_\varepsilon^m)\|^2_{L^2(\Omega_\varepsilon^*)} \le \int_{\Omega_\varepsilon^*} A^\varepsilon(x, u_\varepsilon) \nabla(u_\varepsilon - v_\varepsilon^m) \nabla(u_\varepsilon - v_\varepsilon^m) dx = \int_{\Omega_\varepsilon^*} f\zeta(u_\varepsilon)(u_\varepsilon - v_\varepsilon^m) dx \\ - \langle \mathcal{L}_\varepsilon(-div(A^0(u_0)\nabla u_m) + c_\gamma h(u_0) - \frac{|\partial T|}{|Y|} \mathcal{M}_{\partial T}(g)), u_\varepsilon - v_\varepsilon^m \rangle_{V_\varepsilon', V_\varepsilon}. \end{aligned} \tag{5.19}$$

First we pass to the limit as $\varepsilon \to 0$, and then as $m \to +\infty$. Let us prove that, as $\varepsilon \to 0$, we have

$$\begin{aligned} \alpha \limsup_{\varepsilon\to 0} \|\nabla(u_\varepsilon - v_\varepsilon^m)\|^2_{L^2(\Omega_\varepsilon^*)} \le \theta \int_\Omega f\zeta(u_0)(u_0 - u_m) \chi_{\{u_0>0\}} dx \\ - \langle -div(A^0(u_0)\nabla u_m) + c_\gamma h(u_0) - \frac{|\partial T|}{|Y|} \mathcal{M}_{\partial T}(g)), u_0 - u_m \rangle_{H^{-1}(\Omega), H_0^1(\Omega)}. \end{aligned} \tag{5.20}$$

Concerning the second term in the right-hand side of (5.19), let us observe that $u_\varepsilon - v_\varepsilon^m$ satisfies (5.1), from Remark 5.2 and (5.8). So that, by Proposition 5.1

$$\begin{aligned} \lim_{\varepsilon\to 0} \langle \mathcal{L}_\varepsilon(-div(A^0(u_0)\nabla u_m) + c_\gamma h(u_0) - \frac{|\partial T|}{|Y|} \mathcal{M}_{\partial T}(g)), u_\varepsilon - v_\varepsilon^m \rangle_{V_\varepsilon', V_\varepsilon} \\ = \langle -div(A^0(u_0)\nabla u_m) + c_\gamma h(u_0) - \frac{|\partial T|}{|Y|} \mathcal{M}_{\partial T}(g)), u_0 - u_m \rangle_{H^{-1}(\Omega), H_0^1(\Omega)}. \end{aligned} \tag{5.21}$$

Let $\delta > 0$. We split the integral of the singular term in (5.19) into two terms and write

$$\begin{aligned} \int_{\Omega_\varepsilon^*} f\zeta(u_\varepsilon)(u_\varepsilon - v_\varepsilon^m) dx = \int_{\Omega_\varepsilon^*} f\zeta(u_\varepsilon)(u_\varepsilon - (v_\varepsilon^m)^+) dx + \int_{\Omega_\varepsilon^*} f\zeta(u_\varepsilon)(v_\varepsilon^m)^- dx \\ \le I_\varepsilon^\delta + J_\varepsilon^\delta + K_\varepsilon, \end{aligned} \tag{5.22}$$

where

$$I_\varepsilon^\delta \doteq \int_{\Omega_\varepsilon^* \cap \{0 < u_\varepsilon \le \delta\}} f\zeta(u_\varepsilon) u_\varepsilon dx, \qquad J_\varepsilon^\delta \doteq \int_{\Omega_\varepsilon^* \cap \{u_\varepsilon > \delta\}} f\zeta(u_\varepsilon)(u_\varepsilon - (v_\varepsilon^m)^+) dx$$

$$\text{and} \qquad K_\varepsilon \doteq \int_{\Omega_\varepsilon^*} f\zeta(u_\varepsilon)(v_\varepsilon^m)^- dx.$$

We will compute separately these three limits.

Limit of I_ε^δ: When $k<1$ we get

$$0 \le I_\varepsilon^\delta \le \int_{\Omega_\varepsilon^* \cap \{0<u_\varepsilon \le \delta\}} f u_\varepsilon^{1-k} dx \le \delta^{1-k} \int_\Omega f\chi_{\{0<u_\varepsilon\le\delta\}} dx \le c\delta^{1-k},$$

since the functions involved are all nonnegative, $f \in L^1(\Omega)$ and for the growth conditions on ζ near its singularity (see $\mathrm{H}_2)_i$). Consequently,

$$\lim_{\delta\to 0}\lim_{\varepsilon\to 0} I_\varepsilon^\delta = 0. \tag{5.23}$$

While in the case $k=1$, one has

$$0 \le I_\varepsilon^\delta \le \int_{\Omega_\varepsilon^* \cap \{0<u_\varepsilon \le \delta\}} f\frac{1}{u_\varepsilon} u_\varepsilon dx = \int_{\Omega_\varepsilon^*} f\chi_{\{0<u_\varepsilon\le\delta\}}\chi_{\{u_0\neq\delta\}} dx + \int_{\Omega_\varepsilon^*} f\chi_{\{0<u_\varepsilon\le\delta\}}\chi_{\{u_0=\delta\}} dx. \tag{5.24}$$

Observe that

$$\int_{\Omega_\varepsilon^*} f\chi_{\{0<u_\varepsilon\le\delta\}}\chi_{\{u_0=\delta\}} dx = 0, \tag{5.25}$$

for every $\delta \in \mathbb{R}^+ \backslash D$, where D is the countable set given by

$$D = \{\delta \in \mathbb{R}^+ : |\{(x,y) \in \Omega \times Y^* : u_0(x) = \delta\}| > 0\} \tag{5.26}$$

(see for instance [29], [23] and [25]).

In what follows, we take $\delta \in \mathbb{R}^+ \setminus D$. In order to study the first term in the right-hand side, we make use of the periodic unfolding method. From Proposition $3.2_{2,4}$ one has

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon^*} f\chi_{\{0<u_\varepsilon\le\delta\}}\chi_{\{u_0\neq\delta\}} dx \\
&= \lim_{\varepsilon\to 0}\frac{1}{|Y|}\int_{\Omega\times Y^*} \mathcal{T}_\varepsilon^*(f)\mathcal{T}_\varepsilon^*(\chi_{\{0<u_\varepsilon\le\delta\}})\mathcal{T}_\varepsilon^*(\chi_{\{u_0\neq\delta\}}) dxdy + \lim_{\varepsilon\to 0}\int_{\Lambda_\varepsilon^*} f\chi_{\{0<u_\varepsilon\le\delta\}}\chi_{\{u_0\neq\delta\}} dx \\
&= \lim_{\varepsilon\to 0}\frac{1}{|Y|}\int_{\Omega\times Y^*} \mathcal{T}_\varepsilon^*(f)\mathcal{T}_\varepsilon^*(\chi_{\{0<u_\varepsilon\le\delta\}})\mathcal{T}_\varepsilon^*(\chi_{\{u_0\neq\delta\}}) dxdy,
\end{aligned} \tag{5.27}$$

since $f \in L^1(\Omega)$ and $|\Lambda_\varepsilon| = 0$ when $\varepsilon \to 0$. Moreover, by definition of the unfolding operator, one has

$$\mathcal{T}_\varepsilon^*\left(\chi_{\{0<u_\varepsilon\le\delta\}}\right)(x,y) = \begin{cases} \chi_{\{0<\mathcal{T}_\varepsilon^*(u_\varepsilon)(x,y)\le\delta\}} & \text{a.e. in } \widehat{\Omega}_\varepsilon \times Y^*, \\ 0 & \text{a.e. in } \Lambda_\varepsilon \times Y^*, \end{cases}$$

and, in view of $(2.11)_i$, up to a subsequence,

$$\mathcal{T}_\varepsilon^*(\chi_{\{0<u_\varepsilon\le\delta\}}) \to \chi_{\{0<u_0\le\delta\}} \qquad \text{a.e. in } \Omega \times Y^*. \tag{5.28}$$

Consequently, by Proposition 3.2_6, up to a subsequence,

$$\mathcal{T}_\varepsilon^*(f)\mathcal{T}_\varepsilon^*(\chi_{\{0<u_\varepsilon\le\delta\}})\mathcal{T}_\varepsilon^*(\chi_{\{u_0\neq\delta\}}) \to f\chi_{\{0<u_0\le\delta\}}\chi_{\{u_0\neq\delta\}} = f\chi_{\{0<u_0<\delta\}} \qquad \text{a.e. in } \Omega \times Y^*. \tag{5.29}$$

Moreover from Proposition 3.2$_5$ we have

$$\int_{E\times Y^*}\mathcal{T}^*_\varepsilon(f)\mathcal{T}^*_\varepsilon(\chi_{\{0<u_\varepsilon\le\delta\}})\mathcal{T}^*_\varepsilon(\chi_{\{u_0\neq\delta\}})dxdy\le\int_{E\times Y^*}\mathcal{T}^*_\varepsilon(f)dxdy\le|Y|\,\|f\|_{L^1(E)},$$

for any measurable set $E\times Y^*\subset\Omega\times Y^*$. Via the absolute continuity of the Lebesgue integral and applying the Vitali theorem, from (5.29) we obtain

$$\lim_{\varepsilon\to0}\frac{1}{|Y|}\int_{\Omega\times Y^*}\mathcal{T}^*_\varepsilon(f)\mathcal{T}^*_\varepsilon(\chi_{\{0<u_\varepsilon\le\delta\}})\mathcal{T}^*_\varepsilon(\chi_{\{u_0\neq\delta\}})dxdy=\frac{1}{|Y|}\int_{\Omega\times Y^*}f\chi_{\{0<u_0<\delta\}}dxdy.$$

Then, since $\delta\in\mathbb{R}^+\backslash D$, by (5.24), (5.25) and (5.27),

$$0\le\lim_{\varepsilon\to0}I^\delta_\varepsilon\le\lim_{\varepsilon\to0}\int_{\Omega^*_\varepsilon}f\chi_{\{0<u_\varepsilon\le\delta\}}\chi_{\{u_0\neq\delta\}}dx=\theta\int_\Omega f\chi_{\{0<u_0<\delta\}}dx.$$

Using here the Lebesgue theorem,

$$0\le\lim_{\delta\to0}\lim_{\varepsilon\to0}I^\delta_\varepsilon\le\lim_{\delta\to0}\theta\int_\Omega f\chi_{\{0<u_0<\delta\}}dx=\theta\int_\Omega f\chi_{\{u_0=0\}}dx.$$

Now observe that, since $\zeta(u_0)=+\infty$ when $u_0=0$, (2.13) implies that

$$meas(\{x\in\Omega\mid u_0=0\ \text{ and } f>0\})=0.$$

Hence, $f\chi_{\{u_0=0\}}=0$ a.e. in Ω and also

$$\lim_{\delta\to0}\lim_{\varepsilon\to0}I^\delta_\varepsilon=0. \tag{5.30}$$

Limit of J^δ_ε: We write

$$J^\delta_\varepsilon=\int_{\Omega^*_\varepsilon}f\zeta(u_\varepsilon)(u_\varepsilon-(v^m_\varepsilon)^+)\chi_{\{u_\varepsilon>\delta\}}\chi_{\{u_0\neq\delta\}}dx+\int_{\Omega^*_\varepsilon}f\zeta(u_\varepsilon)(u_\varepsilon-(v^m_\varepsilon)^+)\chi_{\{u_\varepsilon>\delta\}}\chi_{\{u_0=\delta\}}dx, \tag{5.31}$$

where, as before, $\int_{\Omega^*_\varepsilon}f\zeta(u_\varepsilon)(u_\varepsilon-(v^m_\varepsilon)^+)\chi_{\{u_\varepsilon>\delta\}}\chi_{\{u_0=\delta\}}dx=0$ except for a countable set of values of δ which will be excluded. Hence, by unfolding

$$\begin{aligned}J^\delta_\varepsilon&=\int_{\Omega^*_\varepsilon}f\zeta(u_\varepsilon)(u_\varepsilon-(v^m_\varepsilon)^+)\chi_{\{u_\varepsilon>\delta\}}\chi_{\{u_0\neq\delta\}}dx\\&=\frac{1}{|Y|}\int_{\Omega\times Y^*}\mathcal{T}^*_\varepsilon(f)\mathcal{T}^*_\varepsilon(\zeta(u_\varepsilon))\mathcal{T}^*_\varepsilon(u_\varepsilon-(v^m_\varepsilon)^+)\mathcal{T}^*_\varepsilon(\chi_{\{u_\varepsilon>\delta\}})\mathcal{T}^*_\varepsilon(\chi_{\{u_0\neq\delta\}})dxdy\\&\quad+\int_{\Lambda^*_\varepsilon}f\zeta(u_\varepsilon)(u_\varepsilon-(v^m_\varepsilon)^+)\chi_{\{u_\varepsilon>\delta\}}\chi_{\{u_0\neq\delta\}}dx.\end{aligned} \tag{5.32}$$

By using the fact that $u_\varepsilon-(v^m_\varepsilon)^+\le u_\varepsilon$, the Hölder inequality with exponents $\frac{2}{1+k}$ and $\frac{2}{1-k}$, Remark 5.2 and H$_2$), since $l\ge\frac{2}{1+k}$, we have, as $\varepsilon\to0$,

$$\begin{aligned}&\int_{\Lambda^*_\varepsilon}f\zeta(u_\varepsilon)(u_\varepsilon-(v^m_\varepsilon)^+)\chi_{\{u_\varepsilon>\delta\}}\chi_{\{u_0\neq\delta\}}dx\le\int_{\Lambda^*_\varepsilon}f\zeta(u_\varepsilon)u_\varepsilon dx\\&\qquad\le\int_{\Lambda^*_\varepsilon}fu_\varepsilon^{1-k}dx\le\|f\|_{L^{\frac{2}{1+k}}(\Lambda^*_\varepsilon)}\|u_\varepsilon\|^{1-k}_{L^2(\Omega^*_\varepsilon)}\le\|f\|_{L^l(\Lambda^*_\varepsilon)}\|u_\varepsilon\|^{1-k}_{L^2(\Omega^*_\varepsilon)}\to0.\end{aligned} \tag{5.33}$$

On the other hand, in order to apply the Vitali theorem to the first integral on the right-hand side of (5.32), let us first observe that $(2.11)_i$ and (5.9) yield, up to a subsequence,

$$\mathcal{T}_\varepsilon^*(u_\varepsilon - (v_\varepsilon^m)^+) \to u_0 - u_m \quad \text{a.e. in } \Omega \times Y^*.$$

Hence, by $(2.11)_{ii}$ and the same arguments used to prove (5.29), we have that, almost everywhere in $\Omega \times Y^*$,

$$\mathcal{T}_\varepsilon^*(f)\mathcal{T}_\varepsilon^*(\zeta(u_\varepsilon))\mathcal{T}_\varepsilon^*(u_\varepsilon - (v_\varepsilon^m)^+)\mathcal{T}_\varepsilon^*(\chi_{\{u_\varepsilon>\delta\}})\mathcal{T}_\varepsilon^*(\chi_{\{u_0\neq\delta\}}) \to f\zeta(u_0)(u_0 - u_m)\chi_{\{u_0>\delta\}}. \tag{5.34}$$

Moreover, from the growth condition on ζ one has

$$\mathcal{T}_\varepsilon^*(\zeta(u_\varepsilon)) < \frac{1}{\delta^k} \quad \text{on the set } \{u_\varepsilon > \delta\}, \tag{5.35}$$

so that

$$\begin{aligned} &\int_{E\times Y^*} |\mathcal{T}_\varepsilon^*(f)\mathcal{T}_\varepsilon^*(\zeta(u_\varepsilon))\mathcal{T}_\varepsilon^*(u_\varepsilon - (v_\varepsilon^m)^+)\mathcal{T}_\varepsilon^*(\chi_{\{u_\varepsilon>\delta\}})\mathcal{T}_\varepsilon^*(\chi_{\{u_0\neq\delta\}})|dxdy \\ &\leq \int_{E\times Y^*} \mathcal{T}_\varepsilon^*(f)\mathcal{T}_\varepsilon^*(\zeta(u_\varepsilon))\mathcal{T}_\varepsilon^*(u_\varepsilon)dxdy + \frac{1}{\delta^k}\int_{E\times Y^*} \mathcal{T}_\varepsilon^*(f)\mathcal{T}_\varepsilon^*((v_\varepsilon^m)^+)dxdy, \end{aligned} \tag{5.36}$$

for any measurable set $E \times Y^* \subset \Omega \times Y^*$. By using (3.2), $\mathrm{H}_2)_i$, the Hölder inequality with exponents $\frac{2}{1+k}$ and $\frac{2}{1-k}$, Proposition 3.2_5, (2.6) and estimate (4.1), we obtain

$$\int_{E\times Y^*} \mathcal{T}_\varepsilon^*(f)\mathcal{T}_\varepsilon^*(\zeta(u_\varepsilon))\mathcal{T}_\varepsilon^*(u_\varepsilon)dxdy \leq \int_{E\times Y^*} \mathcal{T}_\varepsilon^*(f)[\mathcal{T}_\varepsilon^*(u_\varepsilon)]^{1-k}dxdy \leq c\|f\|_{L^{\frac{2}{1+k}}(E)}. \tag{5.37}$$

Next, in view of Proposition 3.2_5 and (5.10),

$$\frac{1}{\delta^k}\int_{E\times Y^*} \mathcal{T}_\varepsilon^*(f)\mathcal{T}_\varepsilon^*((v_\varepsilon^m)^+)dxdy \leq \frac{|Y|}{\delta^k}c_m\|f\|_{L^1(E)}. \tag{5.38}$$

Hence, from (5.32), using (5.33) and the Vitali theorem (in view of (5.34) and (5.36)-(5.38)), we obtain

$$\lim_{\varepsilon\to 0} J_\varepsilon^\delta = \theta\int_\Omega f\zeta(u_0)(u_0 - u_m)\chi_{\{u_0>\delta\}}dx.$$

On the other hand, assumption $\mathrm{H}_2)$ and the Hölder inequality imply that

$$\int_\Omega f u_0^{1-k}dx \leq \|f\|_{L^{\frac{2}{1+k}}(\Omega)}\|u_0^{1-k}\|_{L^{\frac{2}{1-k}}(\Omega)} \leq \|f\|_{L^l(\Omega)}\|u_0\|_{L^2(\Omega)}^{1-k} < +\infty. \tag{5.39}$$

Consequently, from $\mathrm{H}_2)_i$, (5.39), the boundedness of u_m and (2.13) of Propositon 2.5,

$$0 \leq f\zeta(u_0)(u_0 - u_m)\chi_{\{u_0>\delta\}} \leq f u_0^{1-k} + f\zeta(u_0)u_m \in L^1(\Omega),$$

which, applying the Lebesgue dominated convergence theorem, provides

$$\lim_{\delta\to 0}\lim_{\varepsilon\to 0} J_\varepsilon^\delta = \theta\int_\Omega f\zeta(u_0)(u_0 - u_m)\chi_{\{u_0>0\}}dx. \tag{5.40}$$

Limit of K_ε: Using similar arguments as in [23], we prove now that

$$\lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon^*} f\zeta(u_\varepsilon)(v_\varepsilon^m)^- dx = 0. \tag{5.41}$$

Let us observe that for every $\delta \in \mathbb{R}^+ \setminus D$, where D is given by (5.26), we have

$$\int_{\Omega_\varepsilon^*\cap\{u_\varepsilon>\delta\}} f\zeta(u_\varepsilon)(v_\varepsilon^m)^-\chi_{\{u_0=\delta\}} dx = 0.$$

Hence, for $\delta_0 \in \mathbb{R}^+ \setminus D$, we can write

$$\begin{aligned}\int_{\Omega_\varepsilon^*} f\zeta(u_\varepsilon)(v_\varepsilon^m)^- dx = &\int_{\Omega_\varepsilon^*\cap\{0<u_\varepsilon\le\delta_0\}} f\zeta(u_\varepsilon)(v_\varepsilon^m)^- dx\\ &+\int_{\Omega_\varepsilon^*\cap\{u_\varepsilon>\delta_0\}} f\zeta(u_\varepsilon)(v_\varepsilon^m)^-\chi_{\{u_0\neq\delta_0\}} dx \doteq A_\varepsilon + B_\varepsilon.\end{aligned} \tag{5.42}$$

From Proposition 4.4 written for $\delta = \delta_0$ we get

$$0 \le A_\varepsilon \le \int_{\Omega_\varepsilon^*} A^\varepsilon(x,u_\varepsilon)\nabla u_\varepsilon \nabla (v_\varepsilon^m)^- Z_{\delta_0}(u_\varepsilon)dx + c\varepsilon^{\gamma-\frac{1}{2}}h(2\delta_0)\|\rho\|_{L^\infty(\partial T)}\|(v_\varepsilon^m)^-\|_{V_\varepsilon}.$$

This implies, by using $\mathrm{H}_1)_{iii}$, $\mathrm{H}_5)$, $\mathrm{H}_6)$, the Hölder inequality and (4.1),

$$\begin{aligned}0 \le A_\varepsilon &\le \beta\|\nabla u_\varepsilon\|_{L^2(\Omega_\varepsilon^*)}\|Z_{\delta_0}(u_\varepsilon)\nabla(v_\varepsilon^m)^-\|_{L^2(\Omega_\varepsilon^*)} + c\varepsilon^{\gamma-\frac{1}{2}}h(2\delta_0)\|\rho\|_{L^\infty(\partial T)}\|(v_\varepsilon^m)^-\|_{V_\varepsilon}\\ &\le (c+\varepsilon^{\gamma-\frac{1}{2}}c_1)\|\nabla(v_\varepsilon^m)^-\|_{L^2(\Omega_\varepsilon^*)},\end{aligned}$$

which gives, via (5.15) and $\mathrm{H}_5)$,

$$\lim_{\varepsilon\to 0} A_\varepsilon = 0. \tag{5.43}$$

In order to prove that also

$$\lim_{\varepsilon\to 0} B_\varepsilon = 0, \tag{5.44}$$

by using the integration formula (see Proposition 3.2), we write B_ε as follows:

$$\begin{aligned}B_\varepsilon =& \frac{1}{|Y|}\int_{\Omega\times Y^*} \mathcal{T}_\varepsilon^*(f)\mathcal{T}_\varepsilon^*(\zeta(u_\varepsilon))\mathcal{T}_\varepsilon^*((v_\varepsilon^m)^-)\mathcal{T}_\varepsilon^*(\chi_{\{u_\varepsilon>\delta_0\}})\mathcal{T}_\varepsilon^*(\chi_{\{u_0\neq\delta_0\}})dxdy\\ &+\int_{\Lambda_\varepsilon^*} f\zeta(u_\varepsilon)(v_\varepsilon^m)^-\chi_{\{u_\varepsilon>\delta_0\}}\chi_{\{u_0\neq\delta_0\}}dx.\end{aligned}$$

Concerning the second term in the right-hand side of the previous equation, from the growth condition on ζ on the set $\{u_\varepsilon > \delta_0\}$ and (5.10) it results

$$\begin{aligned}\int_{\Lambda_\varepsilon^*} f\zeta(u_\varepsilon)(v_\varepsilon^m)^-\chi_{\{u_\varepsilon>\delta_0\}}\chi_{\{u_0\neq\delta_0\}}dx &\le \frac{1}{\delta_0^k}\int_{\Lambda_\varepsilon^*} f(v_\varepsilon^m)^- dx\\ &\le \frac{1}{\delta_0^k}\|(v_\varepsilon^m)^-\|_{L^\infty(\Omega_\varepsilon^*)}\|f\|_{L^1(\Lambda_\varepsilon^*)} \to 0, \quad \text{as } \varepsilon\to 0;\end{aligned}$$

while for treating the first one we can use the Vitali theorem. Indeed, from (5.35), Proposition 3.2_5, the Hölder inequality and (5.10), we have

$$\int_{E\times Y^*} \mathcal{T}_\varepsilon^*(f)\mathcal{T}_\varepsilon^*(\zeta(u_\varepsilon))\mathcal{T}_\varepsilon^*((v_\varepsilon^m)^-)\mathcal{T}_\varepsilon^*(\chi_{\{u_\varepsilon>\delta_0\}})\mathcal{T}_\varepsilon^*(\chi_{\{u_0\neq\delta_0\}})dxdy$$

$$\leq |Y|\frac{1}{\delta_0^k}\int_{\Omega_\varepsilon^*\cap E} f(v_\varepsilon^m)^- dx \leq \frac{|Y|}{\delta_0^k}c_m\|f\|_{L^1(E)},$$

for any measurable set $E\times Y^*\subset\Omega\times Y^*$. Also, by Lemma 5.4 and the same arguments used before to prove (5.34), we get, up to a subsequence,

$$\mathcal{T}_\varepsilon^*(f)\mathcal{T}_\varepsilon^*(\zeta(u_\varepsilon))\mathcal{T}_\varepsilon^*((v_\varepsilon^m)^-)\mathcal{T}_\varepsilon^*(\chi_{\{u_\varepsilon>\delta_0\}})\mathcal{T}_\varepsilon^*(\chi_{\{u_0\neq\delta_0\}}) \to f\zeta(u_0)u_m^-\chi_{\{u_0>\delta_0\}} = 0 \ \text{ a.e. in } \Omega\times Y^*,$$

since u_m is nonnegative by definition. Hence, putting together (5.42)-(5.44), we get (5.41).

At present, in order to prove (5.18), we collect (5.19), (5.21)-(5.23), (5.30), (5.40)-(5.41), getting (5.20).

It remains now to pass to the limit as $m\to+\infty$ in (5.20). In this inequality the second term in the right-hand side goes to zero in view of (5.6). The first one also goes to zero via the Lebesgue theorem, since

$$0\leq f\zeta(u_0)(u_0-u_m)\chi_{\{u_0>0\}}\leq f\zeta(u_0)u_0\chi_{\{u_0>0\}}\leq fu_0^{1-k}\in L^1(\Omega),$$

due to (5.39). Thus we obtain (5.18).

Step 3. The final step is to show that

$$\lim_{m\to+\infty}\lim_{\varepsilon\to0}\int_{\Omega_\varepsilon^*}|\nabla(v_\varepsilon^m-v_\varepsilon)|^2dx=0. \tag{5.45}$$

We take $v_\varepsilon^m - v_\varepsilon$ as test function in the variational formulations (5.7) and (5.5). By subtraction, we have

$$\int_{\Omega_\varepsilon^*} A^\varepsilon(x,u_\varepsilon)\nabla(v_\varepsilon^m-v_\varepsilon)\nabla(v_\varepsilon^m-v_\varepsilon)dx = \langle\mathcal{L}_\varepsilon(-div(A^0(u_0)\nabla(u_m-u_0))), v_\varepsilon^m-v_\varepsilon\rangle_{V_\varepsilon',V_\varepsilon}.$$

Passing to the limit on ε, for H_1) and Proposition 5.1 (whose assumptions are satisfied both by v_ε and v_ε^m thanks to (5.4) and (5.8)), we get

$$0\leq\alpha\lim_{\varepsilon\to0}\|\nabla(v_\varepsilon^m-v_\varepsilon)\|^2_{L^2(\Omega_\varepsilon^*)}\leq\lim_{\varepsilon\to0}\langle\mathcal{L}_\varepsilon(-div(A^0(u_0)\nabla(u_m-u_0))), v_\varepsilon^m-v_\varepsilon\rangle_{V_\varepsilon',V_\varepsilon}$$

$$=\langle -div(A^0(u_0)\nabla(u_m-u_0)), u_m-u_0\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}.$$

This together with (5.6) gives

$$0\leq\lim_{m\to+\infty}\lim_{\varepsilon\to0}\alpha\|\nabla(v_\varepsilon^m-v_\varepsilon)\|^2_{L^2(\Omega_\varepsilon^*)}\leq\lim_{m\to+\infty}\int_\Omega A^0(u_0)\nabla(u_m-u_0)\nabla(u_m-u_0)dx=0.$$

Finally combining convergences (5.18) and (5.45), we obtain the desired result. □

As an important consequence of Theorem 5.5, we are able now to prove Proposition 2.6.

Proof of Proposition 2.6. Let v_ε the solution of problem (5.5). The homogenization result given in [10, Theorem 4.5] for $v_0 = u_0$ gives, for the subsequence verifying (2.11),

$$\mathcal{T}_\varepsilon^*(\nabla v_\varepsilon) \rightharpoonup \nabla u_0 + \nabla_y \widehat{v} \quad \text{weakly in } L^2(\Omega \times Y^*), \tag{5.46}$$

where, in our case $\mathcal{M}_{\partial T}(g) \neq 0$ or $g = 0$ (see [10, (5.22)]), $\widehat{v}$ is a function in $L^2(\Omega; H^1_{per}(Y^*))$ with $\mathcal{M}_{Y^*}(\widehat{v}) = 0$, such that

$$\widehat{v}(x,y) = -\sum_{i=1}^{N} \widehat{\chi}_{e_i}(y, u_0(x)) \frac{\partial u_0}{\partial x_i}(x),$$

with $\widehat{\chi}_{e_i}$ solution of (2.9), written for $\lambda = e_i$. Now let us observe that, from Theorem 5.5 and Proposition 3.2_7, we get

$$\mathcal{T}_\varepsilon^*(\nabla u_\varepsilon - \nabla v_\varepsilon) \to 0 \quad \text{strongly in } L^2(\Omega \times Y^*).$$

This, together with $(2.11)_{iv}$ and (5.46), leads to

$$\nabla_y \widehat{v} = \nabla_y \widehat{u} \quad \text{a.e. in } \Omega \times Y^*, \tag{5.47}$$

which implies $\widehat{v} = \widehat{u} + w(x)$, for some function w only depending on x. Since $\mathcal{M}_{Y^*}(\widehat{v}) = \mathcal{M}_{Y^*}(\widehat{u}) = 0$ and

$$\mathcal{M}_{Y^*}(\widehat{v}) = \mathcal{M}_{Y^*}(\widehat{u}) + \mathcal{M}_{Y^*}(w),$$

we derive $w = 0$ and

$$\widehat{v} = \widehat{u}.$$

Whence we obtain the convergence

$$\mathcal{T}_\varepsilon^*(\nabla v_\varepsilon) \rightharpoonup \nabla u_0 + \nabla_y \widehat{u} \quad \text{weakly in } L^2(\Omega \times Y^*), \tag{5.48}$$

and the claimed expression of $\widehat{u}$. □

6 Proof of Theorem 2.8

First, let us observe that, under our assumptions, convergences (2.11) hold true for a subsequence of u_ε (still denoted by ε), where $u_0 \in H^1_0(\Omega)$ is a nonnegative function and $\widehat{u} \in L^2(\Omega; H^1_{per}(Y^*))$ is given by Proposition 2.6. Also we have the validity of (2.13).
Then, we now identify the limit problem satisfied by $(u_0, \widehat{u})$. To do that, we take $\varphi \in H^1_0(\Omega) \cap L^\infty(\Omega)$, $\phi \in \mathcal{D}(\Omega)$ and $\xi \in \mathcal{C}^1_{per}(Y^*)$, and use

$$\psi_\varepsilon(x) = \varphi(x) + \varepsilon \phi(x) \xi\left(\frac{x}{\varepsilon}\right) \in V_\varepsilon$$

as test function in (2.7), obtaining

$$\int_{\Omega_\varepsilon^*} A^\varepsilon(x, u_\varepsilon) \nabla u_\varepsilon \nabla \psi_\varepsilon dx + \varepsilon^\gamma \int_{\Gamma_1^\varepsilon} \rho_\varepsilon h(u_\varepsilon) \psi_\varepsilon d\sigma = \int_{\Omega_\varepsilon^*} f \zeta(u_\varepsilon) \psi_\varepsilon dx + \int_{\Gamma_1^\varepsilon} g_\varepsilon \psi_\varepsilon d\sigma. \tag{6.1}$$

Let us consider the solution v_ε of problem (5.5) and write

$$\lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon^*} A^\varepsilon(x,u_\varepsilon)\nabla u_\varepsilon \nabla\psi_\varepsilon dx = \lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon^*} A^\varepsilon(x,u_\varepsilon)\nabla v_\varepsilon \nabla\psi_\varepsilon dx \\ + \lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon^*} A^\varepsilon(x,u_\varepsilon)\nabla(u_\varepsilon - v_\varepsilon)\nabla\psi_\varepsilon dx. \tag{6.2}$$

By using assumption H_1), the Hölder inequality and Theorem 5.5, one has

$$\lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon^*} A^\varepsilon(x,u_\varepsilon)\nabla(u_\varepsilon - v_\varepsilon)\nabla\psi_\varepsilon dx = 0, \tag{6.3}$$

taking into account that the norm of ψ_ε is bounded in V_ε.
Moreover, by the same arguments used in the proof of Theorem 4.5 of [10] when showing (5.5) and (5.16) (written for $z_\varepsilon = u_\varepsilon$ and $Z = -div(A^0(u_0)\nabla u_0) + c_\gamma h(u_0) - \dfrac{|\partial T|}{|Y|}\mathcal{M}_{\partial T}(g)$), we obtain

$$\lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon^*} A^\varepsilon(x,u_\varepsilon)\nabla v_\varepsilon \nabla\psi_\varepsilon dx = \frac{1}{|Y|}\int_{\Omega\times Y^*} A(y,u_0)(\nabla u_0 + \nabla_y \widehat{u})(\nabla\varphi + \phi\nabla_y\xi)dxdy, \tag{6.4}$$

in view of (5.48). Let us notice that in [10] the function φ is taken in $\mathcal{D}(\Omega)$ and then a density argument is used. Here this is not possible due to the presence of the singular term, so that we need to choose $\varphi \in H_0^1(\Omega)\cap L^\infty(\Omega)$. The only difference with respect to [10] is that now we have to compute the following additional limit:

$$\int_{\Lambda_\varepsilon^*} A^\varepsilon(x,u_\varepsilon)\nabla v_\varepsilon \nabla\varphi dx \le \beta\|\nabla v_\varepsilon\|_{L^2(\Omega_\varepsilon^*)}\|\nabla\varphi\|_{L^2(\Lambda_\varepsilon^*)} \to 0, \quad \text{ as } \varepsilon\to 0.$$

The main difficulty is then to pass to the limit in the singular term. Let us show that

$$\lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon^*} f\zeta(u_\varepsilon)\psi_\varepsilon dx = \theta\int_\Omega f\zeta(u_0)\varphi\chi_{\{u_0>0\}}dx. \tag{6.5}$$

Let us define

$$\mu_\varepsilon(x) \doteq \varepsilon\phi(x)\xi\left(\frac{x}{\varepsilon}\right), \quad \text{that is} \quad \psi_\varepsilon = \varphi + \mu_\varepsilon. \tag{6.6}$$

We remark that, from Propositions $3.2_{2,3,6}$-$3.7_{2,3}$, one has

$$\mathcal{T}_\varepsilon^*(\mu_\varepsilon) = \varepsilon\mathcal{T}_\varepsilon^*(\phi)\xi, \quad \mathcal{T}_\varepsilon^b(\mu_\varepsilon) = \varepsilon\mathcal{T}_\varepsilon^b(\phi)\xi \quad \text{ and } \quad \nabla\mu_\varepsilon = \varepsilon\nabla\phi\xi\left(\frac{\cdot}{\varepsilon}\right) + \phi\nabla_y\xi\left(\frac{\cdot}{\varepsilon}\right),$$

and

$$\begin{cases} \text{i) } \mathcal{T}_\varepsilon^*(\mu_\varepsilon) \to 0 & \text{strongly in } L^2(\Omega\times Y^*), \\ \text{ii) } \mathcal{T}_\varepsilon^*(\nabla\mu_\varepsilon) \to \phi\nabla_y\xi & \text{strongly in } L^2(\Omega\times Y^*). \end{cases} \tag{6.7}$$

From now on, without loss of generality we can assume $\varphi \ge 0$ and $\mu_\varepsilon \ge 0$ in (6.6). Indeed we can decompose the functions in their positive and negative parts as in (2.2). As in the proof of Theorem 5.5, we split the singular integral into two terms: one near the singularity and one far from it. Then, for every positive δ we write

$$0 \le \int_{\Omega_\varepsilon^*} f\zeta(u_\varepsilon)\psi_\varepsilon dx = \int_{\{0<u_\varepsilon\le\delta\}} f\zeta(u_\varepsilon)\psi_\varepsilon dx + \int_{\{u_\varepsilon>\delta\}} f\zeta(u_\varepsilon)\psi_\varepsilon dx \doteq I_\varepsilon^\delta + J_\varepsilon^\delta. \tag{6.8}$$

In view of Proposition 4.4 written for $\varphi, \mu_\varepsilon \geq 0$ we have

$$0 \leq \limsup_{\varepsilon\to 0} I_\varepsilon^\delta \leq \limsup_{\varepsilon\to 0} \left[\int_{\Omega_\varepsilon^*} A^\varepsilon(x,u_\varepsilon)\nabla u_\varepsilon \nabla\psi_\varepsilon Z_\delta(u_\varepsilon)dx + c\varepsilon^{\gamma-\frac{1}{2}} h(2\delta)\|\rho\|_{L^\infty(\partial T)}\|\psi_\varepsilon\|_{V_\varepsilon}\right].$$

From H$_5$), H$_6$) and the boundedness of the norm of ψ_ε in V_ε, the second term in the right-hand side of the above inequality vanishes as ε approches to zero, for every positive δ.
Then, by Proposition 3.2$_{2,4}$ we have

$$\limsup_{\varepsilon\to 0} I_\varepsilon^\delta \leq \limsup_{\varepsilon\to 0} \frac{1}{|Y|}\int_{\Omega\times Y^*} \mathcal{T}_\varepsilon^*(A^\varepsilon(x,u_\varepsilon))\mathcal{T}_\varepsilon^*(\nabla u_\varepsilon)\mathcal{T}_\varepsilon^*(\nabla\psi_\varepsilon Z_\delta(u_\varepsilon))dxdy$$

$$+ \limsup_{\varepsilon\to 0}\int_{\Lambda_\varepsilon^*} A^\varepsilon(x,u_\varepsilon)\nabla u_\varepsilon\nabla\psi_\varepsilon Z_\delta(u_\varepsilon)dx.$$

In view of assumption H$_1$), the Hölder inequality, Proposition 4.1 and the construction of ψ_ε we get

$$\limsup_{\varepsilon\to 0}\int_{\Lambda_\varepsilon^*} A^\varepsilon(x,u_\varepsilon)\nabla u_\varepsilon\nabla\psi_\varepsilon Z_\delta(u_\varepsilon)dx = 0.$$

Then, if ${}^tA$ denotes the transposed matrix field of A,

$$\begin{aligned} 0 \leq \limsup_{\varepsilon\to 0} I_\varepsilon^\delta &\leq \limsup_{\varepsilon\to 0} \frac{1}{|Y|}\int_{\Omega\times Y^*} \mathcal{T}_\varepsilon^*(A^\varepsilon(x,u_\varepsilon))\mathcal{T}_\varepsilon^*(\nabla u_\varepsilon)\mathcal{T}_\varepsilon^*(\nabla\psi_\varepsilon Z_\delta(u_\varepsilon))dxdy \\ &= \limsup_{\varepsilon\to 0} \frac{1}{|Y|}\int_{\Omega\times Y^*} \mathcal{T}_\varepsilon^*({}^tA^\varepsilon(x,u_\varepsilon))\mathcal{T}_\varepsilon^*(\nabla\psi_\varepsilon Z_\delta(u_\varepsilon))\mathcal{T}_\varepsilon^*(\nabla u_\varepsilon)dxdy. \end{aligned} \tag{6.9}$$

We now apply the Vitali theorem to prove that

$$\mathcal{T}_\varepsilon^*({}^tA^\varepsilon(x,u_\varepsilon))\mathcal{T}_\varepsilon^*(\nabla\psi_\varepsilon Z_\delta(u_\varepsilon)) \to {}^tA(y,u_0)(\nabla\varphi+\phi\nabla_y\xi)Z_\delta(u_0) \quad \text{strongly in } L^2(\Omega\times Y^*). \tag{6.10}$$

By the same arguments used to prove $(2.11)_{ii}$ (see proof of Proposition 2.5), we obtain, up to a subsequence,

$$\mathcal{T}_\varepsilon^*(Z_\delta(u_\varepsilon)) \to Z_\delta(u_0) \quad \text{a.e. in } \Omega\times Y^*. \tag{6.11}$$

From $(3.4)_v$, Proposition 3.2$_6$, $(6.7)_{ii}$ and (6.11) follows that, up to a subsequence,

$$\mathcal{T}_\varepsilon^*({}^tA^\varepsilon(x,u_\varepsilon))\mathcal{T}_\varepsilon^*(\nabla\psi_\varepsilon)\mathcal{T}_\varepsilon^*(Z_\delta(u_\varepsilon)) \to {}^tA(y,u_0)(\nabla\varphi+\phi\,\nabla_y\xi)Z_\delta(u_0) \quad \text{a.e. in } \Omega\times Y^*. \tag{6.12}$$

Moreover, from (3.6), H$_1$), Proposition 3.2$_{2,5}$ and the definitions of ψ_ε and Z_δ, we get

$$\int_{E\times Y^*} |\mathcal{T}_\varepsilon^*({}^tA^\varepsilon(x,u_\varepsilon))\mathcal{T}_\varepsilon^*(\nabla\psi_\varepsilon)\mathcal{T}_\varepsilon^*(Z_\delta(u_\varepsilon))|^2dxdy \leq \beta^2\int_{E\times Y^*}|\mathcal{T}_\varepsilon^*(\nabla\psi_\varepsilon)|^2dxdy$$

$$\leq \beta^2|Y|\int_{E\cap\Omega_\varepsilon^*}|\nabla\psi_\varepsilon|^2dx \leq c\int_E|\nabla\varphi|^2dx + c\int_E|\nabla\Phi\,\xi|^2dx + c\int_E|\Phi\nabla\xi|^2dx,$$

for any measurable set $E\times Y^*\subset\Omega\times Y^*$. This, together with (6.12), allows us to apply the Vitali theorem to get (6.10). Using this convergence in (6.9), together with $(2.11)_{iv}$, we have

$$0 \leq \limsup_{\varepsilon\to 0} I_\varepsilon^\delta \leq \frac{1}{|Y|}\int_{\Omega\times Y^*} A(y,u_0)(\nabla u_0+\nabla_y\widehat{u})(\nabla\varphi+\phi\nabla_y\xi)Z_\delta(u_0)dxdy.$$

In order to pass to the limit as δ goes to zero in this inequality, let us observe that

$$Z_\delta(u_0) \to \chi_{\{u_0=0\}} \quad \text{a.e. in } \Omega \quad \text{and} \quad Z_\delta(u_0) \le 1.$$

Hence, using the Lebesgue dominated convergence theorem,

$$0 \le \lim_{\delta\to 0} \limsup_{\varepsilon\to 0} I_\varepsilon^\delta \le \frac{1}{|Y|} \int_{\Omega\times Y^*} A(y,u_0)(\nabla u_0 + \nabla_y \widehat{u})(\nabla\varphi + \phi\nabla_y \xi)\chi_{\{u_0=0\}} dxdy = 0, \tag{6.13}$$

since, due to the expression of $\widehat{u}$ given by Proposition 2.6, the functions ∇u_0 and $\nabla_y \widehat{u}$ vanish where u_0 is equal to 0.
On the other hand, in order to study the limit behaviour of the term J_ε^δ defined in (6.8), observe first that

$$\int_{\Lambda_\varepsilon^*} f\zeta(u_\varepsilon)\psi_\varepsilon \chi_{\{u_\varepsilon>\delta\}} dx \le \frac{1}{\delta^k}\int_{\Lambda_\varepsilon^*} f\psi_\varepsilon dx \le \frac{c}{\delta^k}\|f\|_{L^1(\Lambda_\varepsilon^*)} \to 0, \qquad \text{as } \varepsilon\to 0,$$

since ψ_ε is, in particular, in $L^\infty(\Omega)$. Hence, arguing as in (5.31), and using the integration formula from Proposition 3.2, we can write:

$$\begin{aligned} 0 \le \lim_{\varepsilon\to 0} J_\varepsilon^\delta &= \lim_{\varepsilon\to 0}\frac{1}{|Y|}\int_{\Omega\times Y^*} \mathcal{T}_\varepsilon^*(f)\mathcal{T}_\varepsilon^*(\zeta(u_\varepsilon))\mathcal{T}_\varepsilon^*(\psi_\varepsilon)\mathcal{T}_\varepsilon^*(\chi_{\{u_\varepsilon>\delta\}})dxdy \\ &= \lim_{\varepsilon\to 0}\frac{1}{|Y|}\int_{\Omega\times Y^*} \mathcal{T}_\varepsilon^*(f)\mathcal{T}_\varepsilon^*(\zeta(u_\varepsilon))\mathcal{T}_\varepsilon^*(\psi_\varepsilon)\mathcal{T}_\varepsilon^*(\chi_{\{u_\varepsilon>\delta\}})\mathcal{T}_\varepsilon^*(\chi_{\{u_0\neq\delta\}})dxdy \qquad (6.14) \\ &+ \lim_{\varepsilon\to 0}\frac{1}{|Y|}\int_{\Omega\times Y^*} \mathcal{T}_\varepsilon^*(f)\mathcal{T}_\varepsilon^*(\zeta(u_\varepsilon))\mathcal{T}_\varepsilon^*(\psi_\varepsilon)\mathcal{T}_\varepsilon^*(\chi_{\{u_\varepsilon>\delta\}})\mathcal{T}_\varepsilon^*(\chi_{\{u_0=\delta\}})dxdy. \end{aligned}$$

As in the proof of (6.10), we apply the Vitali theorem to the first term in the right-hand side of (6.14). From (5.35), Proposition $3.2_{2,5}$ and the fact that $\psi_\varepsilon \in L^\infty(\Omega)$, one has

$$\begin{aligned} &\int_{E\times Y^*} |\mathcal{T}_\varepsilon^*(f)\mathcal{T}_\varepsilon^*(\zeta(u_\varepsilon))\mathcal{T}_\varepsilon^*(\psi_\varepsilon)\mathcal{T}_\varepsilon^*(\chi_{\{u_\varepsilon>\delta\}})\mathcal{T}_\varepsilon^*(\chi_{\{u_0\neq\delta\}})|dxdy \\ &\qquad\qquad \le \frac{|Y|}{\delta^k}\int_{\Omega_\varepsilon^*\cap E} |f\psi_\varepsilon|dx \le \frac{c}{\delta^k}\|f\|_{L^1(E)}, \end{aligned}$$

for any measurable set $E\times Y^* \subset \Omega\times Y^*$. Since, by Proposition 3.2_6 and $(6.7)_i$, up to a subsequence,

$$\mathcal{T}_\varepsilon^*(f)\mathcal{T}_\varepsilon^*(\zeta(u_\varepsilon))\mathcal{T}_\varepsilon^*(\psi_\varepsilon)\mathcal{T}_\varepsilon^*(\chi_{\{u_\varepsilon>\delta\}})\mathcal{T}_\varepsilon^*(\chi_{\{u_0\neq\delta\}}) \to f\zeta(u_0)\varphi\chi_{\{u_0>\delta\}} \qquad \text{a.e. in } \Omega\times Y^*,$$

the Vitali theorem gives

$$\begin{aligned} &\lim_{\varepsilon\to 0}\frac{1}{|Y|}\int_{\Omega\times Y^*} \mathcal{T}_\varepsilon^*(f)\mathcal{T}_\varepsilon^*(\zeta(u_\varepsilon))\mathcal{T}_\varepsilon^*(\psi_\varepsilon)\mathcal{T}_\varepsilon^*(\chi_{\{u_\varepsilon>\delta\}})\mathcal{T}_\varepsilon^*(\chi_{\{u_0\neq\delta\}})dxdy \\ &\qquad = \frac{1}{|Y|}\int_{\Omega\times Y^*} f\zeta(u_0)\varphi\chi_{\{u_0>\delta\}}dxdy = \theta\int_\Omega f\zeta(u_0)\varphi\chi_{\{u_0>\delta\}}dx. \end{aligned}$$

Finally, in order to pass to the limit in the last equality, as δ goes to zero, we use again the Lebesgue theorem. Since

$$f\zeta(u_0)\varphi\chi_{\{u_0>\delta\}} \to f\zeta(u_0)\varphi\chi_{\{u_0>0\}} \quad \text{a.e. in } \Omega$$

and by the estimate in (2.13)

$$0 \le f\zeta(u_0)\varphi\chi_{\{u_0>\delta\}} \le f\zeta(u_0)\varphi \in L^1(\Omega),$$

we obtain

$$\begin{aligned}\lim_{\delta\to 0}\lim_{\varepsilon\to 0}\frac{1}{|Y|}\int_{\Omega\times Y^*}\mathcal{T}^*_\varepsilon(f)\mathcal{T}^*_\varepsilon(\zeta(u_\varepsilon))\mathcal{T}^*_\varepsilon(\psi_\varepsilon)\mathcal{T}^*_\varepsilon(\chi_{\{u_\varepsilon>\delta\}})\mathcal{T}^*_\varepsilon(\chi_{\{u_0\neq\delta\}})dxdy \\ = \theta\int_\Omega f\zeta(u_0)\varphi\chi_{\{u_0>0\}}dx.\end{aligned} \tag{6.15}$$

As for the second term in the right-hand side of (6.14), for every $\delta \in \mathbb{R}^+\backslash D$, with D given by (5.26), it results

$$\frac{1}{|Y|}\int_{\Omega\times Y^*}\mathcal{T}^*_\varepsilon(f)\mathcal{T}^*_\varepsilon(\zeta(u_\varepsilon))\mathcal{T}^*_\varepsilon(\psi_\varepsilon)\mathcal{T}^*_\varepsilon(\chi_{\{u_\varepsilon>\delta\}})\mathcal{T}^*_\varepsilon(\chi_{\{u_0=\delta\}})dxdy = 0. \tag{6.16}$$

Consequently, we collect (6.14)-(6.16) and get

$$\lim_{\delta\to 0}\lim_{\varepsilon\to 0} J^\delta_\varepsilon = \theta\int_\Omega f\zeta(u_0)\varphi\chi_{\{u_0>0\}}dx, \tag{6.17}$$

which implies, by (6.8) and (6.13), that (6.5) holds true.
Observe now that the arguments used in the proof of [10, Theorem 4.5], when $\varphi \in \mathcal{D}(\Omega)$, still apply to the case $\varphi \in H^1_0(\Omega)\cap L^\infty(\Omega)$, giving

$$\lim_{\varepsilon\to 0}\varepsilon^\gamma\int_{\Gamma^\varepsilon_1}\rho_\varepsilon h(u_\varepsilon)\psi_\varepsilon d\sigma = c_\gamma\int_\Omega h(u_0)\varphi dx, \tag{6.18}$$

with c_γ defined by (2.15). On the other hand, as a consequence of convergences (6.7),

$$\mathcal{T}^*_\varepsilon(\mu_\varepsilon) \to 0 \quad \text{strongly in } L^2(\Omega; H^1(Y^*)),$$

so that

$$\lim_{\varepsilon\to 0}\int_{\Gamma^\varepsilon_1} g_\varepsilon\psi_\varepsilon d\sigma = \lim_{\varepsilon\to 0}\varepsilon\int_{\Gamma^\varepsilon_1} g\left(\frac{x}{\varepsilon}\right)(\varphi+\mu_\varepsilon)d\sigma = \frac{|\partial T|}{|Y|}\mathcal{M}_{\partial T}(g)\int_\Omega\varphi dx, \tag{6.19}$$

by using Proposition 3.9 written first for $\omega_\varepsilon = \varphi$ and then for $\omega_\varepsilon = \mu_\varepsilon$.
Then, when passing to the limit in (6.1), we combine (6.2)-(6.5), (6.18)-(6.19) and have

$$\begin{aligned}\int_{\Omega\times Y^*} A(y,u_0)(\nabla u_0 + \nabla_y\widehat{u})(\nabla\varphi + \phi\nabla_y\xi)dxdy + |Y|c_\gamma\int_\Omega h(u_0)\varphi dx \\ = |Y^*|\int_\Omega f\zeta(u_0)\varphi\chi_{\{u_0>0\}}dx + |\partial T|\mathcal{M}_{\partial T}(g)\int_\Omega\varphi dx,\end{aligned}$$

for every $\varphi \in H_0^1(\Omega) \cap L^\infty(\Omega)$, $\phi \in \mathcal{D}(\Omega)$ and $\xi \in \mathcal{C}^1_{per}(Y^*)$. By density we obtain

$$\int_{\Omega\times Y^*} A(y,u_0)(\nabla u_0 + \nabla_y \widehat{u})(\nabla\varphi + \nabla_y\psi)dxdy + |Y|c_\gamma \int_\Omega h(u_0)\varphi dx \\ = |Y^*| \int_\Omega f\zeta(u_0)\varphi\chi_{\{u_0>0\}}dx + |\partial T|\mathcal{M}_{\partial T}(g)\int_\Omega \varphi dx, \tag{6.20}$$

for every $\varphi \in H_0^1(\Omega) \cap L^\infty(\Omega)$ and $\psi \in L^2(\Omega; H^1_{per}(Y^*))$.
Let us now notice that formula (2.17) is a known consequence of the expression of $\widehat{u}$ given in Proposition 2.6. Moreover, in view of (2.17), a standard computation shows that u_0 is a solution of the following problem:

$$\begin{cases} -div(A^0(u_0)\nabla u_0) + c_\gamma h(u_0) = \theta f\zeta(u_0)\chi_{\{u_0>0\}} + \dfrac{|\partial T|}{|Y|}\mathcal{M}_{\partial T}(g) & \text{in } \Omega, \\ u_0 = 0 & \text{on } \partial\Omega. \end{cases} \tag{6.21}$$

Also, u_0 results the unique solution of this problem, since [25, Theorem 6.2] holds under assumptions H_2)- H_6) and the conditions satisfied by A^0 given in (2.10). This implies the uniqueness of $\widehat{u}$ under the condition $\mathcal{M}_{Y^*}(\widehat{u}) = 0$, in view of Proposition 2.6. The uniqueness of u_0 provides the validity of convergence (2.11) for the whole sequence.
It remains to show that $u_0 > 0$ almost everywhere in Ω. To do that, we prove that $v \doteq -u_0 < 0$ a.e. in Ω. Observe that, since (2.13) holds, $v \le 0$ in Ω and $v = 0$ on $\partial\Omega$, so that $\sup_\Omega v = 0$. Now, if we suppose there exists a ball B well contained in Ω such that $sup_B v = 0 = \sup_\Omega v$, then, from the strong maximum principle, by using the boundary condition on $\partial\Omega$, we derive $v \equiv 0$ in Ω, that is $u_0 \equiv 0$ in Ω.
Since u_0 is a solution of problem (6.21), whose variational formulation is

$$\int_\Omega A^0(u_0)\nabla u_0 \nabla\varphi dx + c_\gamma \int_\Omega h(u_0)\varphi dx = \theta \int_\Omega f\zeta(u_0)\chi_{\{u_0>0\}}\varphi dx + \frac{|\partial T|}{|Y|}\mathcal{M}_{\partial T}(g)\int_\Omega \varphi dx,$$

$\forall \varphi \in H_0^1(\Omega) \cap L^\infty(\Omega)$. The hypothesis $h(0) = 0$, the positivity of θ, f, ζ and g imply that each integral on the right-hand side above is zero, for every $\varphi \in \mathcal{D}(\Omega)$. This means that $f \equiv 0$ on Ω and $g \equiv 0$ on ∂T, which contradicts assumption H_4). Consequently $u_0 > 0$ almost everywhere in Ω and $\chi_{\{u_0>0\}} \equiv 1$. Then u_0 satisfies the limit equation (2.16). □

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