NORMS SUPPORTING THE LEBESGUE DIFFERENTIATION THEOREM

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Abstract. A version of the Lebesgue differentiation theorem is offered, where the L^p norm is replaced with any rearrangement-invariant norm. Necessary and sufficient conditions for a norm of this kind to support the Lebesgue differentiation theorem are established. In particular, Lorentz, Orlicz and other customary norms for which Lebesgue's theorem holds are characterized.

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1. Introduction and main results

A standard formulation of the classical Lebesgue differentiation theorem asserts that, if $u \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, $n \geq 1$, then

$$\lim_{r\to 0^+} \frac{1}{\mathcal{L}^n(B_r(x))}\int_{B_r(x)} u(y)\,d\mathcal{L}^n(y) \qquad \text{exists and is finite for a.e. } x\in\mathbb{R}^n, \tag{1.1}$$

where $\mathcal{L}^n$ denotes the Lebesgue measure in $\mathbb{R}^n$, and $B_r(x)$ the ball, centered at x, with radius r. Here, and in what follows, "a.e." means "almost every" with respect to Lebesgue measure. In addition to (1.1), one has that

$$\lim_{r\to 0^+} \|u-u(x)\|^{\oslash}_{L^1(B_r(x))} = 0 \qquad \text{for a.e. } x\in\mathbb{R}^n, \tag{1.2}$$

where $\|\cdot\|^{\oslash}_{L^1(B_r(x))}$ stands for the averaged norm in $L^1(B_r(x))$ with respect to the normalized Lebesgue measure $\frac{1}{\mathcal{L}^n(B_r(x))}\mathcal{L}^n$. Namely,

$$\|u\|^{\oslash}_{L^1(B_r(x))} = \frac{1}{\mathcal{L}^n(B_r(x))}\int_{B_r(x)} |u(y)|\,d\mathcal{L}^n(y)$$

for $u \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$.

A slight extension of this property ensures that an analogous conclusion holds if the L^1-norm in (1.2) is replaced with any L^p-norm, with $p\in[1,\infty)$. Indeed, if $u \in L^p_{\mathrm{loc}}(\mathbb{R}^n)$, then

$$\lim_{r\to 0^+} \|u-u(x)\|^{\oslash}_{L^p(B_r(x))} = 0 \qquad \text{for a.e. } x\in\mathbb{R}^n, \tag{1.3}$$

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the averaged norm $\|\cdot\|^{\oslash}_{L^p(B_r(x))}$ being defined accordingly. By contrast, property (1.3) fails when $p=\infty$.

The question thus arises of a characterization of those norms, defined on the space $L^0(\mathbb{R}^n)$ of measurable functions on $\mathbb{R}^n$, for which a version of the Lebesgue differentiation theorem continues to hold.

In the present paper we address this issue in the class of all rearrangement-invariant norms, i.e. norms which only depend on the "size" of functions, or, more precisely, on the measure of their level sets. Let us mention that the Lebesgue point property of rearrangement-invariant norms was also investigated in [5, 6, 20, 22]. The analysis of those papers is however limited to the case of functions of one variable, and the results have a somewhat technical nature – see the comments after Theorem 1.1 below.
A precise definition of rearrangement-invariant norm, as well as other notions employed hereafter, can be found in Section 2 below, where the necessary background material is collected. Let us just recall here that, if $\|\cdot\|_{X(\mathbb{R}^n)}$ is a rearrangement-invariant norm, then

$$\|u\|_{X(\mathbb{R}^n)} = \|v\|_{X(\mathbb{R}^n)} \qquad \text{whenever } u^* = v^*, \tag{1.4}$$

where u^* and v^* denote the decreasing rearrangements of the functions $u, v \in L^0(\mathbb{R}^n)$. Moreover, by Luxemburg theorem [4, Theorem 4.10, Chapter 2], given any norm of this kind, there exists another rearrangement-invariant function norm $\|\cdot\|_{\overline{X}(0,\infty)}$ on $L^0(0,\infty)$, called the representation norm of $\|\cdot\|_{X(\mathbb{R}^n)}$, such that

$$\|u\|_{X(\mathbb{R}^n)} = \|u^*\|_{\overline{X}(0,\infty)} \tag{1.5}$$

for every $u \in L^0(\mathbb{R}^n)$. By $X(\mathbb{R}^n)$ we denote the Banach function space, in the sense of Luxemburg, of all functions $u \in L^0(\mathbb{R}^n)$ such that $\|u\|_{X(\mathbb{R}^n)} < \infty$. Classical instances of rearrangement-invariant function norms are Lebesgue, Lorentz, Orlicz, and Marcinkiewicz norms.

In analogy with (1.3), a rearrangement-invariant norm $\|\cdot\|_{X(\mathbb{R}^n)}$ will be said to satisfy the Lebesgue point property if, for every $u \in X_{\rm loc}(\mathbb{R}^n)$,

$$\lim_{r\to 0^+} \|u - u(x)\|^{\oslash}_{X(B_r(x))} = 0 \qquad \text{for a.e. } x \in \mathbb{R}^n. \tag{1.6}$$

Here, $\|\cdot\|^{\oslash}_{X(B_r(x))}$ denotes the norm on $X(B_r(x))$ with respect to the normalized Lebesgue measure $\frac{1}{\mathcal{L}^n(B_r(x))}\mathcal{L}^n$ – see (2.21), Section 2.
Let us mention that an alternative notion of Lebesgue point with respect to a general norm is available in the literature, where the expression $\|u-u(x)\|^{\oslash}_{X(B_r(x))}$ is replaced with $\frac{\|u-u(x)\|_{X(B_r(x))}}{\|1\|_{X(B_r(x))}}$ in (1.6). Although the two notions agree in some customary cases – e.g. for Lebesgue and Lorentz norms [7, Section 3] – they may differ in general. In particular, the averaged norm appearing in (1.6) has a correct scaling in r, as also shown by the analysis of approximate differentiability properties of Sobolev type functions in [7]. Moreover, there exist norms in quite classical Orlicz spaces, such as some Zygmund spaces, which satisfy the Lebesgue point property in the sense of (1.6) (Proposition 1.3 below), but not in the other one [10].

We shall exhibit necessary and sufficient conditions for $\|\cdot\|_{X(\mathbb{R}^n)}$ to enjoy the Lebesgue point property. To begin with, a necessary condition for $\|\cdot\|_{X(\mathbb{R}^n)}$ to satisfy the Lebesgue point property is to be locally absolutely continuous (Proposition 3.1, Section 3). This means that, for each function $u \in X_{\rm loc}(\mathbb{R}^n)$, one has $\lim_{j\to\infty} \|u\chi_{K_j}\|_{X(\mathbb{R}^n)} = 0$ for every non-increasing sequence $\{K_j\}$ of bounded measurable sets in $\mathbb{R}^n$ such that $\cap_{j\in\mathbb{N}} K_j = \emptyset$.
The local absolute continuity of $\|\cdot\|_{X(\mathbb{R}^n)}$ is in turn equivalent to the local separability of

$X(\mathbb{R}^n)$, namely to the separability of each subspace of $X(\mathbb{R}^n)$ consisting of all functions which are supported in any given bounded measurable subset of $\mathbb{R}^n$.

As will be clear from applications of our results to special instances, this necessary condition is not yet sufficient. A full characterization of the norms $\|\cdot\|_{X(\mathbb{R}^n)}$ supporting the Lebesgue point property requires some additional assumption. Criteria of diverse nature will be provided. The first one is easily checked in most customary instances, and involves a new functional $\mathcal{G}_X$, associated with the representation norm $\|\cdot\|_{\overline{X}(0,\infty)}$, and defined as

$$\mathcal{G}_X(f) = \|f^{-1}\|_{\overline{X}(0,\infty)} \tag{1.7}$$

for every non-increasing function $f\colon [0,\infty) \to [0,\infty]$. Here, $f^{-1}\colon [0,\infty) \to [0,\infty]$ denotes the (generalized) right-continuous inverse of f. The relevant assumption in connection with the Lebesgue point property of $\|\cdot\|_{X(\mathbb{R}^n)}$ amounts to requiring that $\mathcal{G}_X$ be "almost concave". By this expression, we mean that the functional $\mathcal{G}_X$, restricted to the convex set $\mathcal{C}$ of all non-increasing functions from $[0,\infty)$ into $[0,1]$, fulfils the inequality in the definition of concavity possibly up to a multiplicative positive constant c. Namely,

$$c\sum_{i=1}^{k} \lambda_i \mathcal{G}_X(f_i) \le \mathcal{G}_X\left(\sum_{i=1}^{k} \lambda_i f_i\right) \tag{1.8}$$

for any numbers $\lambda_i \in (0,1)$, $i = 1,\dots,k$, $k \in \mathbb{N}$, such that $\sum_{i=1}^{k} \lambda_i = 1$, and any functions $f_i \in \mathcal{C}$, $i = 1,\dots,k$. Clearly, the functional $\mathcal{G}_X$ is concave on $\mathcal{C}$, in the usual sense, if inequality (1.8) holds with $c = 1$.

In order to give an idea of how the functional $\mathcal{G}_X$ looks like in classical instances, consider the case when $\|\cdot\|_{X(\mathbb{R}^n)} = \|\cdot\|_{L^p(\mathbb{R}^n)}$. One has that

$$\mathcal{G}_{L^p}(f) = \begin{cases} \left(p\int_0^\infty s^{p-1} f(s)\, d\mathcal{L}^1(s)\right)^{\frac{1}{p}} & \text{if } p \in [1,\infty), \\ \mathcal{L}^1(\{s \in [0,\infty)\colon f(s) > 0\}) & \text{if } p = \infty\,, \end{cases}$$

for every non-increasing function $f\colon [0,\infty) \to [0,\infty]$. Note that the functional $\mathcal{G}_{L^p}$ is concave for every $p \in [1,\infty]$. However, $\|\cdot\|_{L^p(\mathbb{R}^n)}$ is locally absolutely continuous only for $p < \infty$.

Alternative characterizations of the rearrangement-invariant norms satisfying the Lebesgue point property call into play a maximal function operator associated with the norms in question. The relevant operator, denoted by $\mathcal{M}_X$, is defined, at each $u \in X_{\text{loc}}(\mathbb{R}^n)$, as

$$\mathcal{M}_X u(x) = \sup_{B \ni x} \|u\|_{X(B)}^{\oslash} \qquad \text{for } x \in \mathbb{R}^n, \tag{1.9}$$

where B stands for any ball in $\mathbb{R}^n$. The operator $\mathcal{M}_X$ has been exploited, for instance, in [16, 17, 18] in the proof of certain weighted inequalities for some classical operators of harmonic analysis. Further investigations can be found in [15].
In the case when $X(\mathbb{R}^n) = L^1(\mathbb{R}^n)$, the operator $\mathcal{M}_X$ coincides with the classical Hardy-Littlewood maximal operator $\mathcal{M}$. It is well known that $\mathcal{M}$ is of weak type from $L^1(\mathbb{R}^n)$ into $L^1(\mathbb{R}^n)$. Moreover, since

$$\|u^*\|_{L^1(0,s)}^{\oslash} = \frac{1}{s}\int_0^s u^*(t)\, d\mathcal{L}^1(t) \qquad \text{for } s \in (0,\infty),$$

for every $u \in L^1_{\text{loc}}(\mathbb{R}^n)$, the celebrated Riesz-Wiener inequality takes the form

$$(\mathcal{M}u)^*(s) \le C\|u^*\|_{L^1(0,s)}^{\oslash} \qquad \text{for } s \in (0,\infty),$$

for some constant $C = C(n)$ [4, Theorem 3.8, Chapter 3].

The validity of the Lebesgue point property for a rearrangement-invariant norm $\|\cdot\|_{X(\mathbb{R}^n)}$ turns out to be intimately connected to a suitable version of these two results for the maximal operator $\mathcal{M}_X$ defined by (1.9). Its statement makes use of a notion of weak-type operators between local rearrangement-invariant spaces. We say that $\mathcal{M}_X$ is of weak type from $X_{\rm loc}(\mathbb{R}^n)$ into $L^1_{\rm loc}(\mathbb{R}^n)$ if for every bounded measurable set $K \subseteq \mathbb{R}^n$, there exists a constant $C = C(K)$ such that

$$\mathcal{L}^n(\{x \in K \colon \mathcal{M}_X u(x) > t\}) \leq \frac{C}{t}\|u\|_{X(\mathbb{R}^n)} \qquad \text{for } t \in (0,\infty), \tag{1.10}$$

for every function $u \in X_{\rm loc}(\mathbb{R}^n)$ whose support is contained in K.

We are now in a position to state our main result.

Theorem 1.1. *Let $\|\cdot\|_{X(\mathbb{R}^n)}$ be a rearrangement-invariant norm. Then the following statements are equivalent:*

(i) $\|\cdot\|_{X(\mathbb{R}^n)}$ *satisfies the Lebesgue point property;*

(ii) $\|\cdot\|_{X(\mathbb{R}^n)}$ *is locally absolutely continuous and the functional $\mathcal{G}_X$ is almost concave;*

(iii) $\|\cdot\|_{X(\mathbb{R}^n)}$ *is locally absolutely continuous, and the Riesz-Wiener type inequality*

$$(\mathcal{M}_X u)^*(s) \leq C\|u^*\|^{\oslash}_{\overline{X}(0,s)} \qquad \textit{for } s \in (0,\infty), \tag{1.11}$$

holds for some constant C, and for every $u \in X_{\rm loc}(\mathbb{R}^n)$;

(iv) $\|\cdot\|_{X(\mathbb{R}^n)}$ *is locally absolutely continuous, and the operator $\mathcal{M}_X$ is of weak type from $X_{\rm loc}(\mathbb{R}^n)$ into $L^1_{\rm loc}(\mathbb{R}^n)$;*

(v) *for every function $u \in X(\mathbb{R}^n)$, supported in a set of finite measure,*

$$\mathcal{L}^n(\{x \in \mathbb{R}^n \colon \mathcal{M}_X u(x) > 1\}) < \infty;$$

(vi) *for every function $u \in X(\mathbb{R}^n)$, supported in a set of finite measure,*

$$\lim_{s\to\infty} (\mathcal{M}_X u)^*(s) = 0.$$

Some comments on Theorem 1.1 are in order.

The local absolute continuity of a rearrangement-invariant norm $\|\cdot\|_{X(\mathbb{R}^n)}$ and the almost concavity of the functional $\mathcal{G}_X$ are independent properties.
For instance, as noticed above, the norm $\|\cdot\|_{L^\infty(\mathbb{R}^n)}$ is not locally absolutely continuous, although the functional $\mathcal{G}_{L^\infty}$ is concave. On the other hand, whenever $q < \infty$, the Lorentz norm $\|\cdot\|_{L^{(p,q)}(\mathbb{R}^n)}$ is locally absolutely continuous, but $\mathcal{G}_{L^{(p,q)}}$ is almost concave if and only if $q \leq p$. The Luxemburg norm $\|\cdot\|_{L^A(\mathbb{R}^n)}$ in the Orlicz space $L^A(\mathbb{R}^n)$ is almost concave for every N-function A, but is locally absolutely continuous if and only if A satisfies the Δ_2-condition near infinity.
These properties are established in Section 6 below, where the validity of the Lebesgue point property for various classes of norms is discussed.

The local absolute continuity of the norm $\|\cdot\|_{X(\mathbb{R}^n)}$ is also independent of the other assumptions in (iii) or (iv). For instance, both these assumptions are fulfilled by the rearrangement-invariant norm $\|\cdot\|_{L^\infty(\mathbb{R}^n)}$, which, however, is not locally absolutely continuous, and, in fact, does not satisfy the Lebesgue point property.

Unlike (ii), (iii) and (iv), conditions (v) and (vi) in Theorem 1.1 do not make explicit reference to the local absolute continuity of the norm in question.

As mentioned above, results in the case when $n = 1$ are available in the literature. For instance, it is proved in [20] that a rearrangement-invariant norm $\|\cdot\|_{X(0,1)}$ enjoys the Lebesgue point property if and only if it is absolutely continuous (or, equivalently, the space $X(0,1)$ is separable) and

$$\inf \left\| \sum_{i=1}^{k} f_i \left(\frac{t-\tau_{i-1}}{\tau_i - \tau_{i-1}} \right) \chi_{[\tau_{i-1},\tau_i]}(t) \right\|_{X(0,1)} > 0, \tag{1.12}$$

where the infimum is extended over all $k \in \mathbb{N}$, all functions $f_i \in X(0,1)$ such that $\|f_i\|_{X(0,1)} = 1$, and all $\tau_i \in [0,1]$ such that $0 = \tau_0 < \tau_1 < \cdots < \tau_k = 1$, $i = 1, 2, \dots, k$.

Riesz-Wiener type inequalities for special classes of rearrangement-invariant norms have been investigated in the literature – see e.g. [3, 2, 12, 13]. In particular, in [2] inequality (1.11) is shown to hold when $\|\cdot\|_{X(\mathbb{R}^n)}$ is an Orlicz norm $\|\cdot\|_{L^A(\mathbb{R}^n)}$ associated with any Young function A. The case of Lorentz norms $\|\cdot\|_{L^{(p,q)}(\mathbb{R}^n)}$ is treated in [3], where it is proved that (1.11) holds if, and only if, $1 \le q \le p$. In fact, a different definition of maximal operator is considered in [3], where $\|u\|^{\oslash}_{X(B_r)}$ is replaced with $\frac{\|u\|_{X(B_r(x))}}{\|1\|_{X(B_r)}}$. However, these notions are equivalent when $\|\cdot\|_{X(\mathbb{R}^n)}$ is a Lorentz norm, as is easily seen from [7, Equation (3.7)].
A simple sufficient condition for the validity of the Riesz-Wiener type inequality for very general maximal operators is proposed in [13]. In our framework, where maximal operators built upon rearrangement-invariant norms are taken into account, such condition turns out to be also necessary, as will be shown in Proposition 4.2. The approach introduced in [13] leads to alternative proofs of the Riesz-Wiener type inequality for Orlicz and Lorentz norms, and was also used in [15] to prove the validity of (1.11) for further families of rearrangement-invariant norms, including, in particular, all Lorentz endpoint norms $\|\cdot\|_{\Lambda_\varphi(\mathbb{R}^n)}$. A kind of rearrangement inequality for the maximal operator built upon these Lorentz norms already appears in [12].

Results on weak type boundedness of the maximal operator $\mathcal{M}_X$ are available in the literature as well [1, 8, 14, 16, 25]. For instance, in [25] it is pointed out that the operator $\mathcal{M}_{L^{(p,q)}}$ is of weak type from $L^{(p,q)}(\mathbb{R}^n)$ into $L^p(\mathbb{R}^n)$, if $1 < p < \infty$ and $1 \le q \le p$, and hence, in particular, it is of weak type from $L^{(p,q)}_{\mathrm{loc}}(\mathbb{R}^n)$ into $L^1_{\mathrm{loc}}(\mathbb{R}^n)$.

Theorem 1.1 enables us to characterize the validity of the Lebesgue point property in customary classes of rearrangement-invariant norms. The following proposition deals with the Lorentz norms $\|\cdot\|_{L^{(p,q)}(\mathbb{R}^n)}$, defined if either $1 < p < \infty$ and $1 \le q \le \infty$, or $p = q = \infty$.

Proposition 1.2. *The Lorentz norm $\|\cdot\|_{L^{(p,q)}(\mathbb{R}^n)}$ satisfies the Lebesgue point property if, and only if, $1 \le q \le p < \infty$.*

Since $L^{(p,p)}(\mathbb{R}^n) = L^p(\mathbb{R}^n)$, with equivalent norms, for $p > 1$, Proposition 1.2 recovers, in particular, the standard result, mentioned above, that the norm $\|\cdot\|_{L^p(\mathbb{R}^n)}$ enjoys the Lebesgue point property if $1 < p < \infty$. This fact is also reproduced, including the case when $p = 1$, by the following proposition, which concerns Orlicz norms $\|\cdot\|_{L^A(\mathbb{R}^n)}$ built upon a Young function A.

Proposition 1.3. *The Orlicz norm $\|\cdot\|_{L^A(\mathbb{R}^n)}$ satisfies the Lebesgue point property if, and only if, the Young function A satisfies the Δ_2-condition near infinity.*

The last two results concern the so-called Lorentz and Marcinkiewicz endpoint norms $\|\cdot\|_{\Lambda_\varphi(\mathbb{R}^n)}$ and $\|\cdot\|_{M_\varphi(\mathbb{R}^n)}$, respectively, associated with a (non identically vanishing) concave function $\varphi\colon [0,\infty) \to [0,\infty)$.

Proposition 1.4. *The Lorentz norm $\|\cdot\|_{\Lambda_\varphi(\mathbb{R}^n)}$ satisfies the Lebesgue point property if, and only if, $\lim_{s\to 0^+} \varphi(s) = 0$.*

Proposition 1.5. *The Marcinkiewicz norm $\|\cdot\|_{M_\varphi(\mathbb{R}^n)}$ satisfies the Lebesgue point property if, and only if, $\lim_{s\to 0^+}\frac{s}{\varphi(s)}>0$, namely, if and only if, $(M_\varphi)_{\rm loc}(\mathbb{R}^n)=L^1_{\rm loc}(\mathbb{R}^n)$.*

2. Background

In this section we recall some definitions and basic properties of decreasing rearrangements and rearrangement-invariant function norms. For more details and proofs, we refer to [4, 19].

Let E be a Lebesgue-measurable subset of $\mathbb{R}^n$, $n\geq 1$. The Riesz space of measurable functions from E into $[-\infty,\infty]$ is denoted by $L^0(E)$. We also set $L^0_+(E)=\{u\in L^0(E)\colon u\geq 0 \text{ a.e. in } E\}$, and $L^0_0(E)=\{u\in L^0(E)\colon u \text{ is finite a.e. in } E\}$. The *distribution function* $u_*\colon[0,\infty)\to[0,\infty]$ and the *decreasing rearrangement* $u^*\colon[0,\infty)\to[0,\infty]$ of a function $u\in L^0(E)$ are defined by

$$u_*(t)=\mathcal{L}^n(\{y\in E\colon |u(y)|>t\}) \qquad \text{for } t\in[0,\infty), \tag{2.1}$$

and by

$$u^*(s)=\inf\{t\geq 0\colon u_*(t)\leq s\} \qquad \text{for } s\in[0,\infty), \tag{2.2}$$

respectively.

The Hardy-Littlewood inequality tells us that

$$\int_E |u(y)v(y)|\,d\mathcal{L}^n(y)\leq \int_0^\infty u^*(s)v^*(s)\,d\mathcal{L}^1(s) \tag{2.3}$$

for every $u,v\in L^0(E)$. The function $u^{**}\colon(0,\infty)\to[0,\infty]$, given by

$$u^{**}(s)\;=\;\frac{1}{s}\int_0^s u^*(t)d\mathcal{L}^1(t) \qquad \text{for } s\in\;(0,\infty)\,, \tag{2.4}$$

is non-increasing and satisfies $u^*\leq u^{**}$. Moreover,

$$(u+v)^{**}\;\leq\;u^{**}+v^{**} \tag{2.5}$$

for every $u,v\in L^0_+(E)$.

A *rearrangement-invariant norm* is a functional $\|\cdot\|_{X(E)}\colon L^0(E)\to[0,\infty]$ such that:

$$\begin{aligned}
&\|u+v\|_{X(E)}\;\leq\;\|u\|_{X(E)}+\|v\|_{X(E)} \quad \text{for all } u,v\in L^0_+(E);\\
&\|\lambda u\|_{X(E)}\;=\;|\lambda|\,\|u\|_{X(E)} \quad \text{for all } \lambda\in\mathbb{R},\ u\in L^0(E);\\
&\|u\|_{X(E)}\;>\;0 \quad \text{if } u \text{ does not vanish a.e. in } E;
\end{aligned} \tag{2.6}$$

(2.7) $\|u\|_{X(E)}\;\leq\;\|v\|_{X(E)}$ whenever $0\leq u\;\leq\;v$ a.e. in E;

(2.8) $\sup_k\|u_k\|_{X(E)}\;=\;\|u\|_{X(E)}$ if $\{u_k\}\subset L^0_+(E)$ with $u_k\nearrow u$ a.e. in E;

(2.9) $\|\chi_G\|_{X(E)}\;<\;\infty$ for every measurable set $G\subseteq E$, such that $\mathcal{L}^n(G)<\infty$;

(2.10) for every measurable set $G\subseteq E$, with $\mathcal{L}^n(G)<\infty$, there exists a constant $C(G)$ such that $\|u\|_{L^1(G)}\leq C(G)\,\|u\chi_G\|_{X(E)}$ for all $u\in L^0(E)$;

(2.11) $\|u\|_{X(E)}\;=\;\|v\|_{X(E)}$ for all $u,v\in L^0(E)$ such that $u^*=v^*$.

The functional $\|\cdot\|_{X(E)}$ is a norm in the standard sense when restricted to the set

$$X(E)=\{u\in L^0(E)\colon \|u\|_{X(E)}<\infty\}. \tag{2.12}$$

The latter is a Banach space endowed with such norm, and is called a rearrangement-invariant Banach function space, briefly, a *rearrangement-invariant space*.
Given a measurable subset E' of E and a function $u \in L^0(E')$, define the function $\widehat{u} \in L^0(E)$ as

$$\widehat{u}(x) = \begin{cases} u(x) & \text{if } x \in E', \\ 0 & \text{if } x \in E \setminus E'. \end{cases}$$

Then the functional $\|\cdot\|_{X(E')}$ given by

$$\|u\|_{X(E')} = \|\widehat{u}\|_{X(E)}$$

for $u \in L^0(E')$ is a rearrangement-invariant norm.
If $\mathcal{L}^n(E) < \infty$, then

$$L^\infty(E) \to X(E) \to L^1(E), \tag{2.13}$$

where $\to$ stands for a continuous embedding.
The local r.i. space $X_{\mathrm{loc}}(E)$ is defined as

$$X_{\mathrm{loc}}(E) = \{u \in L^0(E):\ u\chi_K \in X(E) \text{ for every bounded measurable set } K \subset E\}.$$

The *fundamental function* of $X(E)$ is defined by

$$\varphi_{X(E)}(s) = \|\chi_G\|_{X(E)} \qquad \text{for } s \in [0, \mathcal{L}^n(E)), \tag{2.14}$$

where G is any measurable subset of E such that $\mathcal{L}^n(G) = s$. It is non-decreasing on $[0, \mathcal{L}^n(E))$, $\varphi_{X(E)}(0) = 0$ and $\varphi_{X(E)}(s)/s$ is non-increasing for $s \in (0, \mathcal{L}^n(E))$.
Hardy's Lemma tells us that, given $u, v \in L^0(E)$ and any rearrangement-invariant norm $\|\cdot\|_{X(E)}$,

$$\text{if } u^{**} \le v^{**}, \text{ then } \|u\|_{X(E)} \le \|v\|_{X(E)}. \tag{2.15}$$

The *associate rearrangement-invariant norm* of $\|\cdot\|_{X(E)}$ is the rearrangement-invariant norm $\|\cdot\|_{X'(E)}$ defined by

$$\|v\|_{X'(E)} = \sup\Big\{ \int_E |u(y)v(y)|\, d\mathcal{L}^n(y):\ u \in L^0(E),\ \|u\|_{X(E)} \le 1\Big\}. \tag{2.16}$$

The corresponding rearrangement-invariant space $X'(E)$ is called the *associate space* of $X(E)$. The Hölder type inequality

$$\int_E |u(y)v(y)|\, d\mathcal{L}^n(y) \le \|u\|_{X(E)} \|v\|_{X'(E)} \tag{2.17}$$

holds for every $u \in X(E)$ and $v \in X'(E)$. One has that $X(E) = X''(E)$.
The rearrangement-invariant norm, defined as

$$\|f\|_{\overline{X}(0,\mathcal{L}^n(E))} = \sup_{\|u\|_{X'(E)} \le 1} \int_0^\infty f^*(s) u^*(s)\, d\mathcal{L}^1(s)$$

for $f \in L^0(0, \mathcal{L}^n(E))$, is a *representation norm* for $\|\cdot\|_{X(E)}$. It has the property that

$$\|u\|_{X(E)} = \|u^*\|_{\overline{X}(0,\mathcal{L}^n(E))} \tag{2.18}$$

for every $u \in X(E)$. For customary rearrangement-invariant norms, an expression for $\|\cdot\|_{\overline{X}(0,\mathcal{L}^n(E))}$ is immediately derived from that of $\|\cdot\|_{X(E)}$.

The *dilation operator* $D_\delta\colon \overline{X}(0,\mathcal{L}^n(E)) \to \overline{X}(0,\mathcal{L}^n(E))$ is defined for $\delta>0$ and $f\in \overline{X}(0,\mathcal{L}^n(E))$ as

$$(D_\delta f)(s) = \begin{cases} f(s\delta) & \text{if } s\delta\in(0,\mathcal{L}^n(E)),\\ 0 & \text{otherwise,}\end{cases} \tag{2.19}$$

and is bounded [4, Chapter 3, Proposition 5.11].

We shall make use of the subspace $\overline{X}_1(0,\infty)$ of $\overline{X}(0,\infty)$ defined as

$$\overline{X}_1(0,\infty) = \{f\in\overline{X}(0,\infty)\colon\ f(s)=0 \text{ for a.e. } s>1\}. \tag{2.20}$$

Now, assume that E is a measurable positive cone in $\mathbb{R}^n$ with vertex at 0, namely, a measurable set which is closed under multiplication by positive scalars. In what follows, we shall focus the nontrivial case when $\mathcal{L}^n(E)$ does not vanish, and hence $\mathcal{L}^n(E)=\infty$. Let $\|\cdot\|_{X(E)}$ be a rearrangement-invariant norm, and let G be a measurable subset of E such that $0<\mathcal{L}^n(G)<\infty$. We define the functional $\|\cdot\|^{\oslash}_{X(G)}$ as

$$\|u\|^{\oslash}_{X(G)} = \big\|(u\chi_G)(\sqrt[n]{\mathcal{L}^n(G)}\ \cdot)\big\|_{X(E)} \tag{2.21}$$

for $u\in L^0(E)$. We call it the *averaged norm of* $\|\cdot\|_{X(E)}$ *on* G, since

$$\|u\|^{\oslash}_{X(G)} = \|u\chi_G\|_{X\big(G,\frac{\mathcal{L}^n}{\mathcal{L}^n(G)}\big)} \tag{2.22}$$

for $u\in L^0(E)$, where $\|\cdot\|_{X\big(G,\frac{\mathcal{L}^n}{\mathcal{L}^n(G)}\big)}$ denotes the rearrangement-invariant norm, defined as $\|\cdot\|_{X(G)}$, save that the Lebesgue measure $\mathcal{L}^n$ is replaced with the normalized Lebesgue measure $\frac{\mathcal{L}^n}{\mathcal{L}^n(G)}$. Notice that

$$\|u\|^{\oslash}_{X(G)} = \|(u\chi_G)^*(\mathcal{L}^n(G)\,\cdot)\|_{\overline{X}(0,\infty)} \tag{2.23}$$

for $u\in L^0(E)$. Moreover,

$$\|1\|^{\oslash}_{X(G)} \quad \text{is independent of } G. \tag{2.24}$$

The Hölder type inequality for averaged norms takes the form:

$$\frac{1}{\mathcal{L}^n(G)}\int_G |u(y)v(y)|\,d\mathcal{L}^n(y) \le \|u\|^{\oslash}_{X(G)}\ \|v\|^{\oslash}_{X'(G)} \tag{2.25}$$

for $u,v\in L^0(E)$.

We conclude this section by recalling the definition of some customary, and less standard, instances of rearrangement-invariant function norms of use in our applications. In what follows, we set $p'=\frac{p}{p-1}$ for $p\in(1,\infty)$, with the usual modifications when $p=1$ and $p=\infty$. We also adopt the convention that $1/\infty=0$.
Prototypal examples of rearrangement-invariant function norms are the classical Lebesgue norms. Indeed, $\|u\|_{L^p(\mathbb{R}^n)}=\|u^*\|_{L^p(0,\infty)}$, if $p\in[1,\infty)$, and $\|u\|_{L^\infty(\mathbb{R}^n)}=u^*(0)$.

Let $p,q\in[1,\infty]$. Assume that either $1<p<\infty$ and $1\le q\le\infty$, or $p=q=\infty$. The functional defined as

$$\|u\|_{L^{(p,q)}(\mathbb{R}^n)} = \|s^{\frac1p-\frac1q}\ u^{**}(s)\|_{L^q(0,\infty)} \tag{2.26}$$

for $u\in L^0(\mathbb{R}^n)$ is a rearrangement-invariant norm. The corresponding rearrangement-invariant space $L^{(p,q)}(\mathbb{R}^n)$ is called a *Lorentz space*. Note that $\|\cdot\|_{L^{(p,q)}(0,\infty)}$ is the representation norm of $\|\cdot\|_{L^{(p,q)}(\mathbb{R}^n)}$, and $L^{(p,p)}(\mathbb{R}^n)=L^p(\mathbb{R}^n)$ when $p>1$. Moreover, $L^{(p,q)}(\mathbb{R}^n)\to L^{(p,r)}(\mathbb{R}^n)$ if

$1 < p < \infty$ and $1 \le q \le r \le \infty$.
The set of functions $u \in L^0(\mathbb{R}^n)$, for which the functional

$$\|u\|_{L^{p,q}(\mathbb{R}^n)} = \|s^{\frac{1}{p}-\frac{1}{q}}\, u^*(s)\|_{L^q(0,\infty)} \tag{2.27}$$

is finite, will be denoted by $L^{p,q}(\mathbb{R}^n)$. This functional is a rearrangement-invariant norm if and only if $1 \le q \le p$ or $p = q = \infty$.
Note that, however, the quantities $\|u\|_{L^{p,q}(\mathbb{R}^n)}$ and $\|u\|_{L^{(p,q)}(\mathbb{R}^n)}$ are equivalent, up to multiplicative constants, for every $p > 1$. Indeed,

$$\|u\|_{L^{p,q}(\mathbb{R}^n)} \le \|u\|_{L^{(p,q)}(\mathbb{R}^n)} \le p'\|u\|_{L^{p,q}(\mathbb{R}^n)}, \tag{2.28}$$

see e.g. [4, Chapter 4, Lemma 4.5, p. 219].

Let A be a Young function, namely a left-continuous convex function from $[0,\infty)$ into $[0,\infty]$, which is neither identically equal to 0, nor to ∞. The *Luxemburg rearrangement-invariant norm* associated with A is defined as

$$\|u\|_{L^A(\mathbb{R}^n)} = \inf\left\{\lambda > 0\colon \int_{\mathbb{R}^n} A\left(\frac{|u(x)|}{\lambda}\right) d\mathcal{L}^n(x) \le 1\right\} \tag{2.29}$$

for $u \in L^0(\mathbb{R}^n)$. Its representation norm is $\|u\|_{L^A(0,\infty)}$. The space $L^A(\mathbb{R}^n)$ is called an *Orlicz space*. In particular, $L^A(\mathbb{R}^n) = L^p(\mathbb{R}^n)$ if $A(t) = t^p$ for $p \in [1,\infty)$, and $L^A(\mathbb{R}^n) = L^\infty(\mathbb{R}^n)$ if $A(t) = \infty\chi_{(1,\infty)}(t)$.
Recall that A is said to satisfy the *Δ_2–condition near infinity* if it is finite valued and there exist constants $C > 0$ and $t_0 \ge 0$ such that

$$A(2t) \le CA(t) \qquad \text{for } t \in [t_0,\infty)\,. \tag{2.30}$$

If A satisfies the Δ_2–condition near infinity, and $u \in L^A(\mathbb{R}^n)$ has support of finite measure, then

$$\int_{\mathbb{R}^n} A(c|u(x)|)d\mathcal{L}^n(x) < \infty$$

for every positive number c.
A subclass of Young functions which is often considered in the literature is that of the so-called N-functions. A Young function A is said to be an N-function if it is finite-valued, and

$$\lim_{t\to 0^+}\frac{A(t)}{t} = 0, \qquad \lim_{t\to\infty}\frac{A(t)}{t} = \infty\,.$$

Let $\varphi\colon [0,\infty) \to [0,\infty)$ be a concave function which does not vanish identically. Hence, in particular, φ is non-decreasing, and $\varphi(t) > 0$ for $t \in (0,\infty)$. The *Marcinkiewicz* and *Lorentz endpoint norm* associated with φ are defined as

$$\|u\|_{M_\varphi(\mathbb{R}^n)} = \sup_{s\in(0,\infty)} u^{**}(s)\varphi(s), \tag{2.31}$$

$$\|u\|_{\Lambda_\varphi(\mathbb{R}^n)} = \int_0^\infty u^*(s)d\varphi(s), \tag{2.32}$$

for $u \in L^0(\mathbb{R}^n)$, respectively. The representation norms are $\|\cdot\|_{M_\varphi(0,\infty)}$ and $\|\cdot\|_{\Lambda_\varphi(0,\infty)}$, respectively. The spaces $M_\varphi(\mathbb{R}^n)$ and $\Lambda_\varphi(\mathbb{R}^n)$ are called *Marcinkiewicz endpoint space* and *Lorentz endpoint space* associated with φ. The fundamental functions of $M_\varphi(\mathbb{R}^n)$ and $\Lambda_\varphi(\mathbb{R}^n)$ coincide with φ. In fact, $M_\varphi(\mathbb{R}^n)$ and $\Lambda_\varphi(\mathbb{R}^n)$ are respectively the largest and the smallest rearrangement-invariant space whose fundamental function is φ, and this accounts for the expression "endpoint" which is usually attached to their names. Note the alternative expression

$$\|u\|_{\Lambda_\varphi(\mathbb{R}^n)} = u^*(0)\varphi(0^+) + \int_0^\infty u^*(s)\varphi'(s)d\mathcal{L}^1(s), \tag{2.33}$$

for $u \in L^0(\mathbb{R}^n)$, where $\varphi(0^+) = \lim_{s\to 0^+} \varphi(s)$.

3. A necessary condition: local absolute continuity

In the present section we are mainly concerned with a proof of the following necessary conditions for a rearrangement-invariant norm to satisfy the Lebesgue point property.

Proposition 3.1. *If $\|\cdot\|_{X(\mathbb{R}^n)}$ is a rearrangement-invariant norm satisfying the Lebesgue point property, then:*

(i) *$\|\cdot\|_{X(\mathbb{R}^n)}$ is locally absolutely continuous;*
(ii) *$X(\mathbb{R}^n)$ is locally separable.*

The proof of Proposition 3.1 is split in two steps, which are the content of the next two lemmas.

Lemma 3.2. *Let $\|\cdot\|_{X(\mathbb{R}^n)}$ be a rearrangement-invariant norm which satisfies the Lebesgue point property. Then:*

(H) *Given any function $f \in \overline{X}_1(0,\infty)$, any sequence $\{I_k\}$ of pairwise disjoint intervals in $(0,1)$, and any sequence $\{a_k\}$ of positive numbers such that $a_k \geq \mathcal{L}^1(I_k)$ and*

$$\|(f\chi_{I_k})^*\|^{\oslash}_{\overline{X}(0,a_k)} > 1, \tag{3.1}$$

one has that

$$\sum_{k=1}^\infty a_k < \infty.$$

Remark 3.3. Condition (H) in Lemma 3.2 is not only necessary, but also sufficient for a rearrangement-invariant norm to satisfy the Lebesgue point property. This follows from Proposition 4.3, in next section, and Theorem 1.1.

Lemma 3.4. *If a rearrangement-invariant norm $\|\cdot\|_{X(\mathbb{R}^n)}$ fulfills condition (H) of Lemma 3.2, then $\|\cdot\|_{X(\mathbb{R}^n)}$ is locally absolutely continuous.*

Proof of Proposition 3.1. Owing to
[4, Chapter 1, Corollary 5.6], assertions (i) and (ii) are equivalent. Assertion (i) follows from Lemmas 3.2 and 3.4. □

The remaining part of this section is devoted to the proofs of Lemmas 3.2 and 3.4. The proof of the former in turn exploits the following property, which will also be of later use.

Lemma 3.5. *Let $\|\cdot\|_{X(\mathbb{R}^n)}$ be a rearrangement-invariant norm. Given any function $f \in \overline{X}(0,\infty)$, the function $F:(0,\infty)\to[0,\infty)$, defined as*

$$F(r) = r\,\|f^*\|^{\oslash}_{\overline{X}(0,r)} \qquad \text{for } r \in (0,\infty), \tag{3.2}$$

is non-decreasing on $(0,\infty)$, and the function $\frac{F(r)}{r}$ is non-increasing on $(0,\infty)$. In particular, the function F is continuous on $(0,\infty)$.

Proof. Let $0 < r_1 < r_2$. An application of (2.16) tells us that

$$\begin{aligned} F(r_1) =&\ r_1 \|(f^* \chi_{(0,r_1)})(r_1 \cdot)\|_{\overline{X}(0,\infty)} = r_1 \sup_{\|g\|_{\overline{X}'(0,\infty)} \leq 1} \int_0^1 g^*(s) f^*(r_1 s)\, d\mathcal{L}^1(s) \\ =& \sup_{\|g\|_{\overline{X}'(0,\infty)} \leq 1} \int_0^{r_1} g^*\left(\tfrac{t}{r_1}\right) f^*(t)\, d\mathcal{L}^1(t) \leq \sup_{\|g\|_{\overline{X}'(0,\infty)} \leq 1} \int_0^{r_2} g^*\left(\tfrac{t}{r_1}\right) f^*(t)\, d\mathcal{L}^1(t) \\ \leq& \sup_{\|g\|_{\overline{X}'(0,\infty)} \leq 1} \int_0^{r_2} g^*\left(\tfrac{t}{r_2}\right) f^*(t)\, d\mathcal{L}^1(t) = r_2 \sup_{\|g\|_{\overline{X}'(0,\infty)} \leq 1} \int_0^1 g^*(s) f^*(r_2 s)\, d\mathcal{L}^1(s) \\ =&\ r_2 \|(f^* \chi_{(0,r_2)})(r_2 \cdot)\|_{\overline{X}(0,\infty)} = F(r_2). \end{aligned}$$

Namely, F is non-decreasing on $(0, \infty)$. The fact that the function $\frac{F(r)}{r}$ is non-increasing on $(0, \infty)$ is a consequence of property (2.7) and of the inequality

$$f^*(r_1 \cdot)\chi_{(0,r_1)}(r_1 \cdot) \geq f^*(r_2 \cdot)\chi_{(0,r_2)}(r_2 \cdot)$$

if $0 < r_1 < r_2$. Hence, in particular, the function F is continuous on $(0, \infty)$ (see e.g. [11, Chapter 2, p. 49]). □

Proof of Lemma 3.2. Assume that $\|\cdot\|_{X(\mathbb{R}^n)}$ satisfies the Lebesgue point property. Suppose, by contradiction, that condition (H) fails, namely, there exist a function $f \in \overline{X}_1(0,\infty)$, a sequence $\{I_k\}$ of pairwise disjoint intervals in $(0, 1)$ and a sequence $\{a_k\}$ of positive numbers, with $a_k \geq \mathcal{L}^1(I_k)$, fulfilling (3.1) and such that $\sum_{k=1}^{\infty} a_k = \infty$.

We may assume, without loss of generality, that

$$\lim_{k\to\infty} a_k = 0. \tag{3.3}$$

Indeed, if (3.3) fails, then the sequence $\{a_k\}$ can be replaced with another sequence, enjoying the same properties, and also (3.3). To verify this assertion, note that, if (3.3) does not hold, then there exist $\varepsilon > 0$ and a subsequence $\{a_{k_j}\}$ of $\{a_k\}$ such that $a_{k_j} \geq \varepsilon$ for all j. Consider the sequence $\{b_j\}$, defined as

$$b_j = \max\left\{\tfrac{\varepsilon}{j},\ \mathcal{L}^1(I_{k_j})\right\} \quad \text{for } j \in \mathbb{N}. \tag{3.4}$$

Equation (3.4) immediately tells us that $b_j \geq \mathcal{L}^1(I_{k_j})$, and $\sum_{j=1}^{\infty} b_j = \infty$. Moreover, Lemma 3.5 and the inequality $b_j \leq a_{k_j}$ for $j \in \mathbb{N}$ ensure that (3.1) holds with a_k and I_k replaced by b_j and I_{k_j}, respectively. Finally, $\lim\limits_{j\to\infty} b_j = 0$, since $\sum_{j=1}^{\infty} \mathcal{L}^1(I_{k_j}) \leq 1$, and hence $\lim\limits_{j\to\infty} \mathcal{L}^1(I_{k_j}) = 0$.

Moreover, by skipping, if necessary, a finite number of terms in the relevant sequences, we may also assume that

$$\sum_{k=1}^{\infty} \mathcal{L}^1(I_k) < 1\,. \tag{3.5}$$

Now, set $a_0 = 0$, and $J_k = (\sum_{j=0}^{k-1} a_j, \sum_{j=0}^{k} a_j)$ for each $k \in \mathbb{N}$. We define the function $g\colon (0,\infty) \to [0,\infty)$ as

$$g(s) = \sum_{k=1}^{\infty} (f\chi_{I_k})^* \Big(s - \sum_{j=0}^{k-1} a_j\Big) \chi_{J_k}(s) \qquad \text{for } s \in (0,\infty),$$

and the function $u\colon \mathbb{R}^n \to [0,\infty)$ as

$$u(y) = \begin{cases} \sup\limits_{k\in\mathbb{N}} \ g(y_1+k-1) & \text{for } y=(y_1,\dots,y_n)\in(0,1)^n,\\ 0 & \text{for } y\in\mathbb{R}^n\setminus(0,1)^n. \end{cases}$$

The function u belongs to $X(\mathbb{R}^n)$. To verify this fact, note that

$$\mathcal{L}^n(\{y\in\mathbb{R}^n\colon u(y)>t\}) \le \mathcal{L}^1(\{s\in(0,\infty)\colon |f\chi_{\cup_{k\in\mathbb{N}}I_k}|(s)>t\}) \tag{3.6}$$

for every $t\ge 0$. Indeed, thanks to the equimeasurability of g and $f\chi_{\cup_{k\in\mathbb{N}}I_k}$,

$$\begin{aligned}\mathcal{L}^n(\{y\in\mathbb{R}^n\colon u(y)>t\}) &= \mathcal{L}^1(\{s\in(0,1)\colon \sup_{k\in\mathbb{N}} g(s+k-1)>t\})\\ &= \mathcal{L}^1(\cup_{k\in\mathbb{N}}\{s\in(0,1)\colon g(s+k-1)>t\}) \le \sum_{k=1}^\infty \mathcal{L}^1(\{s\in(0,1)\colon g(s+k-1)>t\})\\ &= \mathcal{L}^1(\{s\in(0,\infty)\colon g(s)>t\}) = \mathcal{L}^1(\{s\in(0,\infty)\colon |f\chi_{\cup_{k\in\mathbb{N}}I_k}|(s)>t\}).\end{aligned}$$

From (3.6) it follows that

$$\|u\|_{X(\mathbb{R}^n)} \le \|f\chi_{\cup_{k\in\mathbb{N}}I_k}\|_{\overline{X}(0,\infty)} \le \|f\|_{\overline{X}(0,\infty)} < \infty\,,$$

whence $u\in X(\mathbb{R}^n)$. Next, one has that

$$\limsup_{r\to0^+} \|u\|^{\oslash}_{X(B_r(x))} > 0 \qquad \text{for a.e. } x\in(0,1)^n. \tag{3.7}$$

To prove (3.7), set

$$\Lambda_k = \{l\in\mathbb{N}\colon J_l\subseteq[k-1,k]\} \qquad \text{for } k\in\mathbb{N}.$$

Since $\sum_{l=0}^\infty a_l=\infty$ and $\lim\limits_{l\to\infty}|J_l| = \lim\limits_{l\to\infty} a_l = 0$, we have that

$$\lim_{k\to\infty} \bigcup_{l\in\Lambda_k} (\overline{J_l}-k+1) = (0,1), \tag{3.8}$$

where $\overline{J_l}$ denotes the closure of the open interval J_l. Equation (3.8) has to be interpreted in the following set-theoretic sense: fixed any $x\in(0,1)^n$, there exist k_0 and an increasing sequence $\{l_k\}_{k=k_0}^\infty$ in $\mathbb{N}$ such that $l_k\in\Lambda_k$ and $x_1\in(\overline{J_{l_k}}-k+1)$ for all $k\in\mathbb{N}$ greater than k_0. Such a k_0 can be chosen so that $B_{\sqrt{n}a_{l_k}}(x)\subseteq(0,1)^n$ for all $k\ge k_0$, since $\lim\limits_{k\to\infty} a_{l_k}=0$.

On the other hand, for every $k\ge k_0$, one also has

$$B_{\sqrt{n}a_{l_k}}(x) \ \supseteq\ \prod_{i=1}^n [x_i-a_{l_k}, x_i+a_{l_k}] \ \supseteq\ (J_{l_k}-k+1)\ \times\ \prod_{i=2}^n [x_i-a_{l_k}, x_i+a_{l_k}].$$

Consequently, for every $y\in\mathbb{R}^n$,

$$\begin{aligned} u\chi_{B_{\sqrt{n}a_{l_k}}(x)}(y) \ &\ge\ g(y_1+k-1)\ \chi_{(J_{l_k}-k+1)}(y_1) \prod_{i=2}^n \chi_{[x_i-a_{l_k},x_i+a_{l_k}]}(y_i)\\ &=\ (f\chi_{I_{l_k}})^*\Big(y_1+k-1-\sum_{j=0}^{k-1}a_j\Big)\ \chi_{(J_{l_k}-k+1)}(y_1)\prod_{i=2}^n \chi_{[x_i-a_{l_k},x_i+a_{l_k}]}(y_i).\end{aligned}$$

Hence,

$$(u\chi_{B_{\sqrt{n}a_{l_k}}(x)})^*(s)\ \ge\ (f\chi_{I_{l_k}})^*\left((2a_{l_k})^{1-n}s\right) \qquad \text{for } s\in(0,\infty). \tag{3.9}$$

Therefore, thanks to the boundedness on rearrangement-invariant spaces of the dilation operator, defined as in (2.19), one gets

$$\begin{aligned}\|u\|^{\oslash}_{X(B_{\sqrt{n}a_{l_k}}(x))} &= \|(u\chi_{B_{\sqrt{n}a_{l_k}}})^*(\mathcal{L}^n(B_{\sqrt{n}a_{l_k}}(x))\,\cdot)\|_{\overline{X}(0,\infty)}\\ &\geq C\,\|(f\chi_{I_{l_k}})^*(a_{l_k}\,\cdot)\|_{\overline{X}(0,\infty)} = C\|(f\chi_{I_{l_k}})^*\|^{\oslash}_{\overline{X}(0,a_{l_k})} > 1\end{aligned}$$

for some positive constant $C = C(n)$. Hence inequality (3.7) follows, since $\lim_{k\to\infty} a_{l_k} = 0$.

To conclude, consider the set $M = \{y \in (0,1)^n \colon u(y) = 0\}$. This set M has positive measure. Indeed, (3.6) with $t = 0$ and (3.5) imply

$$\begin{aligned}\mathcal{L}^n(M) = 1 - \mathcal{L}^n(\{y \in (0,1)^n \colon u(y) > 0\}) &\geq 1 - \mathcal{L}^1(\{s \in (0,\infty) \colon |f\chi_{\cup_{k\in\mathbb{N}} I_k}|(s) > 0\})\\ &\geq 1 - \sum_{k=1}^{\infty} \mathcal{L}^1(I_k) > 0.\end{aligned}$$

Then, estimate (3.7) tells us that

$$\limsup_{r\to 0^+} \|u - u(x)\|^{\oslash}_{X(B_r(x))} = \limsup_{r\to 0^+} \|u\|^{\oslash}_{X(B_r(x))} > 0 \qquad \text{for a.e. } x \in M.$$

This contradicts the Lebesgue point property for $\|\cdot\|_{X(\mathbb{R}^n)}$. Thus $\|\cdot\|_{X(\mathbb{R}^n)}$ fulfils condition (H). □

Proof of Lemma 3.4. Let $\|\cdot\|_{X(\mathbb{R}^n)}$ be a rearrangement-invariant norm satisfying condition (H). We first prove that, if (H) is in force, then

$$\lim_{t\to 0^+} \|g^*\chi_{(0,t)}\|_{\overline{X}(0,\infty)} = 0 \tag{3.10}$$

for every $g \in \overline{X}_1(0,\infty)$.

Arguing by contradiction, assume the existence of some $g \in \overline{X}_1(0,\infty)$ for which (3.10) fails. From property (2.7) of rearrangement-invariant norms, this means that some $\varepsilon > 0$ exists such that $\|g^*\chi_{(0,t)}\|_{\overline{X}(0,\infty)} \geq \varepsilon$ for every $t \in (0,1)$. Thanks to (2.6), we may assume, without loss of generality, that $\varepsilon = 2$.

Then, by induction, construct a decreasing sequence $\{b_k\}$, with $0 < b_k \leq 1$, such that

$$\|g^*\chi_{(b_{k+1},b_k)}\|_{\overline{X}(0,\infty)} > 1 \tag{3.11}$$

for every $k \in \mathbb{N}$. To this purpose, set $b_1 = 1$, and assume that b_k is given for some $k \in \mathbb{N}$. Then define

$$h_l = g^*\chi_{\left(\frac{b_k}{l},b_k\right)} \qquad \text{for } l \in \mathbb{N}, \text{ with } l \geq 2.$$

Since $0 \leq h_l \nearrow g^*\chi_{(0,b_k)}$, property (2.8) tells us that $\|h_l\|_{\overline{X}(0,\infty)} \nearrow \|g^*\chi_{(0,b_k)}\|_{\overline{X}(0,\infty)}$. Inasmuch as $\|g^*\chi_{(0,b_k)}\|_{\overline{X}(0,\infty)} \geq 2$, then there exists an l_0, with $l_0 \geq 2$, such that $\|h_{l_0}\|_{\overline{X}(0,\infty)} > 1$. Defining $b_{k+1} = \frac{b_k}{l_0}$ entails that $0 < b_{k+1} < b_k$ and $\|g^*\chi_{(b_{k+1},b_k)}\|_{\overline{X}(0,\infty)} = \|h_{l_0}\|_{\overline{X}(0,\infty)} > 1$, as desired.

Observe that choosing $f = g^*\chi_{(0,1)}$, and $a_k = 1$, $I_k = (b_{k+1}, b_k)$ for each $k \in \mathbb{N}$ provides a contradiction to assumption (H). Indeed, inequality (3.1), which agrees with (3.11) in this case, holds for every k, whereas $\sum_{k=1}^{\infty} a_k = \infty$. Consequently, (3.10) does hold.

Now, take any $u \in X_{\mathrm{loc}}(\mathbb{R}^n)$ and any non-increasing sequence $\{K_j\}$ of measurable bounded sets in $\mathbb{R}^n$ such that $\cap_{j\in\mathbb{N}} K_j = \emptyset$. Clearly, $u\chi_{K_1} \in X(\mathbb{R}^n)$ and $\lim_{j\to\infty} \mathcal{L}^n(K_j) = 0$. We may assume that $\mathcal{L}^n(K_1) < 1$, whence $(u\chi_{K_j})^* \in \overline{X}_1(0,\infty)$ for each $j \in \mathbb{N}$.

From (3.10) it follows that

$$\lim_{j\to\infty} \|u\chi_{K_j}\|_{X(\mathbb{R}^n)} = \lim_{j\to\infty} \|(u\chi_{K_j})^*\|_{\overline{X}(0,\infty)} \le \lim_{j\to\infty} \|(u\chi_{K_1})^*\chi_{(0,\mathcal{L}^n(K_j))}\|_{\overline{X}(0,\infty)} = 0,$$

namely the local absolute continuity of $\|\cdot\|_{X(\mathbb{R}^n)}$. □

4. The functional $\mathcal{G}_X$ and the operator $\mathcal{M}_X$

This section is devoted to a closer analysis of the functional $\mathcal{G}_X$ and the operator $\mathcal{M}_X$ associated with a rearrangement-invariant norm $\|\cdot\|_{X(\mathbb{R}^n)}$.

We begin with alternate characterizations of the almost concavity of the functional $\mathcal{G}_X$. In what follows, we shall make use of the fact that

$$\|h\|_{\overline{X}(0,\infty)} = \|h^*\|_{\overline{X}(0,\infty)} = \|(h_*)_*\|_{\overline{X}(0,\infty)} = \mathcal{G}_X(h_*) \tag{4.1}$$

for every $h \in L^0(0,\infty)$.
Moreover, by a *partition* of the interval $(0,1)$ we shall mean a finite collection $\{I_k\colon k=1,\dots,m\}$, where $I_k=(\tau_{k-1},\tau_k)$ with $0=\tau_0<\tau_1<\dots<\tau_m=1$.

Proposition 4.1. *Let $\|\cdot\|_{X(\mathbb{R}^n)}$ be a rearrangement-invariant norm. Then the following conditions are equivalent:*

(i) *the functional $\mathcal{G}_X$ is almost concave;*
(ii) *there exists a constant C such that*

$$\sum_{k=1}^m \mathcal{L}^1(I_k)\,\|f\|^{\oslash}_{\overline{X}(I_k)} \le C\|f\|_{\overline{X}(0,\infty)} \tag{4.2}$$

for every $f\in \overline{X}_1(0,\infty)$, and for every partition $\{I_k\colon k=1,\dots,m\}$ of $(0,1)$;
(iii) *there exists a constant C such that*

$$\sum_{k=1}^m \mathcal{L}^n(B_k)\,\|u\|^{\oslash}_{X(B_k)} \le C\ \mathcal{L}^n\big(\cup_{k=1}^m B_k\big)\ \|u\|^{\oslash}_{X(\cup_{k=1}^m B_k)} \tag{4.3}$$

for every $u\in X_{\rm loc}(\mathbb{R}^n)$, and for every finite collection $\{B_k\colon k=1,\dots,m\}$ of pairwise disjoint balls in $\mathbb{R}^n$.

Proof. (i) ⇒ (ii) Assume that $\mathcal{G}_X$ is almost concave. Fix any function $f\in\overline{X}_1(0,\infty)$, and any partition $\{I_k\colon k=1,\dots,m\}$ of $(0,1)$. It is easily verified that

$$\Big((f\chi_{I_k})^*(\mathcal{L}^1(I_k)\cdot)\Big)_* = \frac{(f\chi_{I_k})_*}{\mathcal{L}^1(I_k)} \quad\text{and}\quad \big(f\chi_{\cup_{k=1}^m I_k}\big)_* = \sum_{k=1}^m (f\chi_{I_k})_*\,.$$

Hence, by (4.1) and the almost concavity of $\mathcal{G}_X$, there exists a constant C such that

$$\begin{aligned}\sum_{k=1}^m \mathcal{L}^1(I_k)\,\|f\|^{\oslash}_{\overline{X}(I_k)} &= \sum_{k=1}^m \mathcal{L}^1(I_k)\,\|(f\chi_{I_k})^*(\mathcal{L}^1(I_k)\cdot)\|_{\overline{X}(0,\infty)} = \sum_{k=1}^m \mathcal{L}^1(I_k)\,\mathcal{G}_X\Big(\frac{(f\chi_{I_k})_*}{\mathcal{L}^1(I_k)}\Big)\\ &\le C\,\mathcal{G}_X\Big(\sum_{k=1}^m (f\chi_{I_k})_*\Big) = C\,\mathcal{G}_X\Big((f\chi_{\cup_{k=1}^m I_k})_*\Big) \le C\,\mathcal{G}_X(f_*) = C\,\|f\|_{\overline{X}(0,\infty)}.\end{aligned}$$

This yields inequality (4.2).
(ii) ⇒ (i) Take any finite collections $\{g_k\colon k=1,\dots,m\}$ in $\mathcal{C}$ and $\{\lambda_k\colon k=1,\dots,m\}$ in $(0,1)$,

respectively, with $\sum_{k=1}^m \lambda_k = 1$. For each $k = 1, \dots, m$, write $f_k = (g_k)_*$, $a_k = \sum_{i=1}^k \lambda_i$, and $I_k = (a_{k-1}, a_k)$ with $a_0 = 0$. Then define

$$f(t) = \sum_{k=1}^m f_k(\tfrac{t-a_{k-1}}{\lambda_k})\chi_{I_k}(t) \qquad \text{for } t \in (0,\infty).$$

Observe that $f_* = \sum_{k=1}^m \lambda_k (f_k)_* = \sum_{k=1}^m \lambda_k g_k$. Owing to (4.1) and (4.2), one thus obtains

$$\begin{aligned}\mathcal{G}_X\Big(\sum_{k=1}^m \lambda_k g_k\Big) &= \mathcal{G}_X(f_*) = \|f\|_{\overline{X}(0,\infty)} \geq \frac{1}{C}\sum_{k=1}^m \lambda_k \|f\|^{\oslash}_{\overline{X}(I_k)} = \frac{1}{C}\sum_{k=1}^m \lambda_k \|f_k^*(\tfrac{1}{\lambda_k}\,\cdot)\|^{\oslash}_{\overline{X}(0,\lambda_k)} \\ &= \frac{1}{C}\sum_{k=1}^m \lambda_k \|f_k^*\|_{\overline{X}(0,\infty)} = \frac{1}{C}\sum_{k=1}^m \lambda_k \mathcal{G}_X\big((f_k)_*\big) = \frac{1}{C}\sum_{k=1}^m \lambda_k \mathcal{G}_X(g_k),\end{aligned}$$

whence the almost concavity of $\mathcal{G}_X$ follows.
(ii) $\Rightarrow$ (iii) Fix any function $u \in X_{\rm loc}(\mathbb{R}^n)$, and any finite collection $\{B_k\colon k = 1, \dots, m\}$ of pairwise disjoint balls in $\mathbb{R}^n$. Set $a_k = \mathcal{L}^n(B_k)$, for $k = 1, \dots, m$, and $a_0 = 0$. Define

$$I_k = \left(\frac{\sum_{j=0}^{k-1} a_j}{\sum_{j=1}^m a_j}, \frac{\sum_{j=0}^{k} a_j}{\sum_{j=1}^m a_j}\right) \qquad \text{for } k = 1, \dots, m.$$

Thanks to rearrangement invariance of $\overline{X}(0,\infty)$, assumption (ii) ensures that

$$\begin{aligned}\|u\|^{\oslash}_{X(\cup_{k=1}^m B_k)} &= \Big\|(u\chi_{\cup_{k=1}^m B_k})^*\big(\sum_{k=1}^m a_k\ \cdot\big)\Big\|_{\overline{X}(0,\infty)} = \Big\|\sum_{k=1}^m (u\chi_{B_k})^*\big(\sum_{j=1}^m a_j\ \cdot - \sum_{j=0}^{k-1} a_j\big)\ \chi_{I_k}\Big\|_{\overline{X}(0,\infty)} \\ &\geq \frac{1}{C}\ \sum_{k=1}^m \frac{a_k}{\sum_{j=1}^m a_j}\Big\|(u\chi_{B_k})^*\big(\sum_{k=1}^m a_k\ \cdot - \sum_{j=0}^{k-1} a_j\big)\ \chi_{I_k}\Big\|^{\oslash}_{\overline{X}(I_k)} \\ &= \frac{1}{C\sum_{j=1}^m a_j}\ \sum_{k=1}^m a_k\, \|(u\chi_{B_k})^*(a_k\,\cdot)\|_{\overline{X}(0,\infty)} = \frac{1}{C\,\mathcal{L}^n(\cup_{k=1}^m B_k)}\sum_{k=1}^m \mathcal{L}^n(B_k)\|u\|^{\oslash}_{X(B_k)}.\end{aligned}$$

Inequality (4.3) is thus established.
(iii) $\Rightarrow$ (ii) Assume that $f \in \overline{X}_1(0,\infty)$, and that $\{I_k\colon k = 1, \dots, m\}$ is a partition of $(0,1)$. Let $\{B_k\colon k = 1, \dots, m\}$ be a family of pairwise disjoint balls in $\mathbb{R}^n$ such that $\mathcal{L}^n(B_k) = \mathcal{L}^1(I_k)$, and let u be a measurable function on $\mathbb{R}^n$ vanishing outside of $\cup_{k=1}^m B_k$ and fulfilling $(u\chi_{B_k})^* = (f\chi_{I_k})^*$, for $k = 1, 2, \dots, m$. Assumption (iii) then tells us that

$$\sum_{k=1}^m \mathcal{L}^1(I_k)\|f\|^{\oslash}_{\overline{X}(I_k)} \leq C\,\mathcal{L}^1(\cup_{k=1}^m I_k)\|f\|^{\oslash}_{\overline{X}(\cup_{k=1}^m I_k)} = C\,\|f\|^{\oslash}_{\overline{X}(0,1)} = C\,\|f\|_{\overline{X}(0,\infty)},$$

namely (4.2). □

We next focus on the maximal operator $\mathcal{M}_X$. Criteria for the validity of the Riesz-Wiener type inequality (1.11) are the content of the following result.

Proposition 4.2. *Let $\|\cdot\|_{X(\mathbb{R}^n)}$ be a rearrangement-invariant norm. Then the following conditions are equivalent:*

(i) *the Riesz-Wiener type inequality* (1.11) *holds for some constant C, and for every $u \in X_{\rm loc}(\mathbb{R}^n)$;*

(ii) *there exists a constant* C_1 *such that*

$$\min_{k=1,\dots,m} \|u\|^{\oslash}_{X(B_k)} \leq C_1 \|u\|^{\oslash}_{X(\cup_{k=1}^m B_k)} \tag{4.4}$$

for every $u \in X_{\rm loc}(\mathbb{R}^n)$*, and for every finite collection* $\{B_k \colon k = 1, \dots, m\}$ *of pairwise disjoint balls in* $\mathbb{R}^n$*;*

(iii) *there exists a constant* C_2 *such that*

$$\min_{k=1,\dots,m} \|f\|^{\oslash}_{\overline{X}(I_k)} \leq C_2 \|f\|_{\overline{X}(0,\infty)}$$

for every $f \in \overline{X}_1(0,\infty)$*, and for every partition* $\{I_k \colon k = 1, \dots, m\}$ *of* $(0,1)$*.*

Proof. (i) $\Rightarrow$ (ii) Let $u \in X_{\rm loc}(\mathbb{R}^n)$, and let $\{B_k \colon k = 1, \dots, m\}$ be a collection of pairwise disjoint balls in $\mathbb{R}^n$. When $\min_{k=1,\dots,m} \|u\|^{\oslash}_{X(B_k)} = 0$, then (4.4) trivially holds. Assume that $\min_{k=1,\dots,m} \|u\|^{\oslash}_{X(B_k)} > 0$. Fix any $s \in \big(0, \mathcal{L}^n(\cup_{k=1}^m B_k)\big)$, and any $t \in \big(0, \min_{k=1,\dots,m} \|u\|^{\oslash}_{X(B_k)}\big)$. If $x \in B_j$ for some $j = 1, \dots, m$, then

$$\mathcal{M}_X(u\chi_{\cup_{k=1}^m B_k})(x) \geq \|u\chi_{\cup_{k=1}^m B_k}\|^{\oslash}_{X(B_j)} \geq \min_{k=1,\dots,m} \|u\|^{\oslash}_{X(B_k)} > t.$$

Thus,

$$\mathcal{L}^n(\{x \in \mathbb{R}^n \colon \mathcal{M}_X(u\chi_{\cup_{k=1}^m B_k})(x) > t\}) \geq \mathcal{L}^n(\cup_{k=1}^m B_k) > s,$$

and, consequently,

$$\big(\mathcal{M}_X(u\chi_{\cup_{k=1}^m B_k})\big)^*(s) \geq t.$$

Since the last inequality holds for every $t < \min_{k=1,\dots,m} \|u\|^{\oslash}_{X(B_k)}$, one infers that

$$\big(\mathcal{M}_X(u\chi_{\cup_{k=1}^m B_k})\big)^*(s) \ \geq \ \min_{k=1,\dots,m} \|u\|^{\oslash}_{X(B_k)}. \tag{4.5}$$

On the other hand, an application of assumption (i) with u replaced by $u\chi_{\cup_{k=1}^m B_k}$ tells us that

$$\big(\mathcal{M}_X(u\chi_{\cup_{k=1}^m B_k})\big)^*(s) \leq C \, \|(u\chi_{\cup_{k=1}^m B_k})^*\|^{\oslash}_{\overline{X}(0,s)} \qquad \text{for } s \in \big(0, \mathcal{L}^n(\cup_{k=1}^m B_k)\big). \tag{4.6}$$

Coupling (4.5) with (4.6) implies that

$$\min_{k=1,\dots,m} \|u\|^{\oslash}_{X(B_k)} \leq C \, \|(u\chi_{\cup_{k=1}^m B_k})^*\|^{\oslash}_{\overline{X}(0,s)} \qquad \text{for } s \in \big(0, \mathcal{L}^n(\cup_{k=1}^m B_k)\big).$$

Thus, owing to the continuity of the function $s \mapsto \|(u\chi_{\cup_{k=1}^m B_k})^*\|^{\oslash}_{\overline{X}(0,s)}$, which is guaranteed by Lemma 3.5, we deduce that

$$\min_{k=1,\dots,m} \|u\|^{\oslash}_{X(B_k)} \leq C \|(u\chi_{\cup_{k=1}^m B_k})^*\|^{\oslash}_{\overline{X}(0,\mathcal{L}^n(\cup_{k=1}^m B_k))} = C \|u\|^{\oslash}_{X(\cup_{k=1}^m B_k)},$$

namely (4.4).

(ii) $\Rightarrow$ (i) By [15, Proposition 3.2], condition (ii) implies the existence of a constant C' such that

$$(\mathcal{M}_X u)^*(s) \leq C' \|u^*\big(3^{-n}s \ \cdot\big)\chi_{(0,1)}(\,\cdot\,)\|_{\overline{X}(0,\infty)} \qquad \text{for } s \in (0,\infty), \tag{4.7}$$

for every $u \in X_{\rm loc}(\mathbb{R}^n)$. By the boundedness of the dilation operator on rearrangement-invariant spaces, there exists a constant C'', independent of u, such that

$$\begin{aligned} \|u^*\big(3^{-n}s \ \cdot\big)\chi_{(0,1)}(\,\cdot\,)\|_{\overline{X}(0,\infty)} &\leq C'' \|u^*(s\,\cdot)\chi_{(0,1)}(3^n\,\cdot)\|_{\overline{X}(0,\infty)} \\ &\leq C'' \|u^*(s\,\cdot)\chi_{(0,1)}(\,\cdot\,)\|_{\overline{X}(0,\infty)} = C'' \|u^*\|^{\oslash}_{\overline{X}(0,s)} \qquad \text{for } s \in (0,\infty). \end{aligned} \tag{4.8}$$

Inequality (1.11) follows from (4.7) and (4.8).
(ii) $\Leftrightarrow$ (iii) The proof is completely analogous to that of the equivalence between conditions (ii) and (iii) in Proposition 4.1. We omit the details for brevity. $\square$

Condition (H) introduced in Lemma 3.2 can be characterized in terms of the maximal operator $\mathcal{M}_X$ as follows.

Proposition 4.3. *Let $\|\cdot\|_{X(\mathbb{R}^n)}$ be a rearrangement-invariant norm. Then the following assertions are equivalent:*

(i) $\|\cdot\|_{X(\mathbb{R}^n)}$ *fulfils condition* (H) *in Lemma 3.2;*
(ii) *For every function $u \in X(\mathbb{R}^n)$, supported in a set of finite measure,*

$$\mathcal{L}^n(\{x \in \mathbb{R}^n \colon \mathcal{M}_X u(x) > 1\}) \; < \infty.$$

Proof. (i) $\Rightarrow$ (ii) Let $u \in X(\mathbb{R}^n)$ be supported in a set of finite measure. Set $E = \{x \in \mathbb{R}^n : \mathcal{M}_X u(x) > 1\}$. According to (1.9), for any $y \in E$, there exists a ball B_y in $\mathbb{R}^n$ such that $y \in B_y$ and $\|u\|^{\oslash}_{X(B_y)} > 1$. Define

$$E_1 = \Big\{ y \in E \colon \mathcal{L}^n(B_y) > \max\{1, \mathcal{L}^n(\{|u| > 0\})\} \Big\}. \tag{4.9}$$

We claim that, if $y \in E_1$, then

$$(u\chi_{B_y})^*\big(\mathcal{L}^n(B_y)s\big) \le (u\chi_{B_y})^*(s)\chi_{\big(0, \frac{\mathcal{L}^n(\{|u|>0\})}{\mathcal{L}^n(B_y)}\big)}(s) \qquad \text{for } s \in (0,\infty). \tag{4.10}$$

Indeed, since $\mathcal{L}^n(B_y) \ge 1$, by the monotonicity of the decreasing rearrangement

$$(u\chi_{B_y})^*(s) \ge (u\chi_{B_y})^*\big(\mathcal{L}^n(B_y)s\big) \qquad \text{for } s \in (0,\infty).$$

Thus, inequality (4.10) certainly holds if $s \in \big(0, \frac{\mathcal{L}^n(\{|u|>0\})}{\mathcal{L}^n(B_y)}\big]$. On the other hand, $\mathcal{L}^n(B_y) \ge \mathcal{L}^n(\{|u| > 0\})$, and since $(u\chi_{B_y})^*(s) = 0$ for $s \ge \mathcal{L}^n(\{|u| > 0\})$, we have that $(u\chi_{B_y})^*\big(\mathcal{L}^n(B_y)s\big) = 0$ if $s \in \big(\frac{\mathcal{L}^n(\{|u|>0\})}{\mathcal{L}^n(B_y)}, \infty\big)$. Thereby, inequality (4.10) also holds for these values of s.

Owing to (4.10),

$$\begin{aligned} 1 < \|u\|^{\oslash}_{X(B_y)} &= \|(u\chi_{B_y})^*(\mathcal{L}^n(B_y)\ \cdot)\|_{\overline{X}(0,\infty)} \\ &\le \Big\|(u\chi_{B_y})^*\chi_{\big(0, \frac{\mathcal{L}^n(\{|u|>0\})}{\mathcal{L}^n(B_y)}\big)}\Big\|_{\overline{X}(0,\infty)} \le \Big\|u^*\chi_{\big(0, \frac{\mathcal{L}^n(\{|u|>0\})}{\mathcal{L}^n(B_y)}\big)}\Big\|_{\overline{X}(0,\infty)}. \end{aligned} \tag{4.11}$$

Since (i) is in force, equation (3.10) holds with $g = u^*\chi_{(0,1)} \in \overline{X}_1(0,\infty)$, namely,

$$\lim_{t \to 0^+} \|u^*\chi_{(0,t)}\|_{\overline{X}(0,\infty)} = 0.$$

This implies the existence of some $t_0 \in (0,1)$ such that $\|u^*\chi_{(0,t)}\|_{\overline{X}(0,\infty)} < 1$ for every $t \in (0, t_0)$. Thus, (4.11) entails that

$$\frac{\mathcal{L}^n(\{|u| > 0\})}{\mathcal{L}^n(B_y)} \ge t_0$$

for every $y \in E_1$. Hence, by (4.9),

$$\sup_{y \in E} \mathcal{L}^n(B_y) \le \max\Big\{1, \frac{\mathcal{L}^n(\{|u| > 0\})}{t_0}\Big\}. \tag{4.12}$$

An application of Vitali's covering lemma, in the form of [24, Chapter 1, Lemma 1.6], ensures that there exists a countable set $\mathcal{I} \subseteq E$ such that the family $\{B_y \colon y \in \mathcal{I}\}$ consists of pairwise disjoint balls, such that $E \subseteq \cup_{y \in \mathcal{I}} 5B_y$. Here, $5B_y$ denotes the ball, with the same center as B_y, whose radius is 5 times the radius of B_y. If $\mathcal{I}$ is finite, then trivially $\mathcal{L}^n(E) \le 5^n \sum_{y \in \mathcal{I}} \mathcal{L}^n(B_y) < \infty$.

Assume that, instead, $\mathcal{I}$ is infinite, and let $\{y_k\}$ be the sequence of its elements. For each $k \in \mathbb{N}$, set, for simplicity, $B_k = B_{y_k}$, and

$$\alpha_k = \mathcal{L}^n(\{y \in B_k \colon u(y) \neq 0\}), \qquad I_k = \left(\frac{\sum_{i=0}^{k-1} \alpha_i}{\alpha}, \frac{\sum_{i=1}^{k} \alpha_i}{\alpha} \right), \qquad a_k = \frac{\mathcal{L}^n(B_k)}{\alpha}, \tag{4.13}$$

where $\alpha = \mathcal{L}^n(\{y \in \mathbb{R}^n \colon u(y) \neq 0\})$ and $\alpha_0 = 0$. Note that $\{I_k\}$ is a sequence of pairwise disjoint intervals in $(0,1)$, and $a_k \geq \mathcal{L}^1(I_k)$ for each $k \in \mathbb{N}$. The function $f \colon (0,\infty) \to [0,\infty)$, defined by

$$f(s) = \sum_{k=1}^{\infty} \big(u\chi_{\{x \in B_k \colon u(x) \neq 0\}}\big)^* \Big(\alpha s - \sum_{i=1}^{k-1} \alpha_i\Big) \chi_{I_k}(s) \quad \text{for } s \in (0,\infty),$$

belongs to $\overline{X}_1(0,\infty)$, and

$$\begin{aligned}\|(f\chi_{I_k})^*\|^{\oslash}_{\overline{X}(0,a_k)} &= \|(u\chi_{\{x \in B_k \colon u(x) \neq 0\}})^*(\alpha\,\cdot)\|^{\oslash}_{\overline{X}(0,a_k)} \\ &= \|(u\chi_{B_k})^*(\mathcal{L}^n(B_k)\,\cdot)\|_{\overline{X}(0,\infty)} = \|u\|^{\oslash}_{X(B_k)} > 1.\end{aligned}$$

By (i), one thus obtains that $\sum_{k=1}^{\infty} \mathcal{L}^n(B_k) = \alpha \sum_{k=1}^{\infty} a_k < \infty$. Hence $\mathcal{L}^n(E) \leq 5^n \sum_{k=1}^{\infty} \mathcal{L}^n(B_k) < \infty$, also in this case.

(ii) $\Rightarrow$ (i) Let $f \in \overline{X}_1(0,\infty)$, let $\{I_k\}$ be any sequence of pairwise disjoint intervals in $(0,1)$, and let $\{a_k\}$ be a sequence of real numbers, such that $a_k \geq \mathcal{L}^1(I_k)$, fulfilling (3.1).

Consider any sequence $\{B_k\}$ of pairwise disjoint balls in $\mathbb{R}^n$, such that $\mathcal{L}^n(B_k) = a_k$ for $k \in \mathbb{N}$. For each $k \in \mathbb{N}$, choose a function $g_k \colon \mathbb{R}^n \to [0,\infty)$, supported in B_k, and such that g_k is equimeasurable with $f\chi_{I_k}$. Then, define $u = \sum_{k=1}^{\infty} g_k$. Note that $u \in X(\mathbb{R}^n)$, since $u^* = (f\chi_{\bigcup_{k=1}^{\infty} I_k})^* \leq f^*$. Furthermore, u is supported in a set of finite measure. Thus, assumption (ii) implies that

$$\mathcal{L}^n(\{x \in \mathbb{R}^n \colon \mathcal{M}_X u(x) > 1\}) < \infty. \tag{4.14}$$

If $x \in B_k$ for some $k \in \mathbb{N}$, then

$$\mathcal{M}_X u(x) \geq \|u\|^{\oslash}_{X(B_k)} = \|g_k\|^{\oslash}_{X(B_k)} = \|g_k^*\|^{\oslash}_{\overline{X}(0,\mathcal{L}^n(B_k))} = \|(f\chi_{I_k})^*\|^{\oslash}_{\overline{X}(0,a_k)} > 1. \tag{4.15}$$

Consequently,

$$\cup_{k=1}^{\infty} B_k \subseteq \{x \in \mathbb{R}^n \colon \mathcal{M}_X u(x) > 1\}$$

and

$$\sum_{k=1}^{\infty} a_k = \mathcal{L}^n(\cup_{k=1}^{\infty} B_k) \leq \mathcal{L}^n(\{x \in \mathbb{R}^n \colon \mathcal{M}_X u(x) > 1\}) < \infty.$$

Condition (i) is thus fulfilled. □

5. Proof of Theorem 1.1

The present section is devoted to a proof of Theorem 1.1. We preliminarily state and prove a further lemma.

Lemma 5.1. *Let $\|\cdot\|_{X(\mathbb{R}^n)}$ be a rearrangement-invariant norm such that*

$$\lim_{s \to 0^+} \varphi_{X(\mathbb{R}^n)}(s) = 0. \tag{5.1}$$

If $u \colon \mathbb{R}^n \to \mathbb{R}$ is any simple function, then

$$\lim_{r \to 0^+} \|u - u(x)\|^{\oslash}_{X(B_r(x))} = 0 \qquad \textit{for a.e. } x \in \mathbb{R}^n.$$

Proof. Let E be any measurable subset of $\mathbb{R}^n$. By the Lebesgue density theorem,

$$\begin{cases} \lim_{r\to 0^+} \frac{\mathcal{L}^n(B_r(x)\setminus E)}{\mathcal{L}^n(B_r(x))} &= 0 \qquad \text{for a.e. } x \in E \\ \lim_{r\to 0^+} \frac{\mathcal{L}^n(B_r(x)\cap E)}{\mathcal{L}^n(B_r(x))} &= 0 \qquad \text{for a.e. } x \in \mathbb{R}^n \setminus E. \end{cases} \tag{5.2}$$

Since

$$\lim_{r\to 0^+} \|\chi_E - \chi_E(x)\|^{\oslash}_{X(B_r(x))} = \begin{cases} \lim\limits_{r\to 0^+} \left\|\chi_{\left(0,\frac{\mathcal{L}^n(B_r(x)\setminus E)}{\mathcal{L}^n(B_r(x))}\right)}\right\|_{\overline{X}(0,\infty)} & \text{for a.e. } x \in E, \\ \lim\limits_{r\to 0^+} \left\|\chi_{\left(0,\frac{\mathcal{L}^n(B_r(x)\cap E)}{\mathcal{L}^n(B_r(x))}\right)}\right\|_{\overline{X}(0,\infty)} & \text{for a.e. } x \in \mathbb{R}^n \setminus E, \end{cases} \tag{5.3}$$

it follows from (5.1) that

$$\lim_{r\to 0^+} \|\chi_E - \chi_E(x)\|^{\oslash}_{X(B_r(x))} = 0 \quad \text{for a.e. } x \in \mathbb{R}^n. \tag{5.4}$$

Hence, if u is any simple function having the form $u = \sum_{i=1}^k a_i \chi_{E_i}$, where $E_1, \dots, E_k$ are pairwise disjoint measurable subsets of $\mathbb{R}^n$, and $a_1, \dots, a_k \in \mathbb{R}$, then

$$\lim_{r\to 0^+} \|u - u(x)\|^{\oslash}_{X(B_r(x))} \le \lim_{r\to 0^+} \sum_{i=1}^k |a_i| \, \|\chi_{E_i} - \chi_{E_i}(x)\|^{\oslash}_{X(B_r(x))} = 0 \tag{5.5}$$

for a.e. $x \in \mathbb{R}^n$. □

Proof of Theorem 1.1. (v) ⇔ (vi) Let us observe that, thanks to the positive homogeneity of the maximal operator $\mathcal{M}_X$, one has that (v) holds if, and only if, $\mathcal{L}^n(\{x \in \mathbb{R}^n : \mathcal{M}_X u(x) > t\}) < \infty$ for every $u \in X(\mathbb{R}^n)$ supported in a set of finite measure and for every $t \in (0, \infty)$. The latter condition is equivalent to (vi).

(i) ⇒ (v) It suffices to apply Lemma 3.2 and Proposition 4.3.

(v) ⇒ (ii) By Proposition 4.3 and Lemma 3.4, assumption (v) implies the local absolute continuity for $\|\cdot\|_{X(\mathbb{R}^n)}$.

Next, assume, by contradiction, that the functional $\mathcal{G}_X$ is not almost concave. Owing to Proposition 4.1, this amounts to assuming that, for every $k \in \mathbb{N}$, there exist a function $f_k \in \overline{X}_1(0,\infty)$ and a partition $\{J_{k,l} : l = 1, \dots, m_k\}$ of $(0,1)$ such that

$$\sum_{l=1}^{m_k} \mathcal{L}^1(J_{k,l}) \, \|f_k\|^{\oslash}_{\overline{X}(J_{k,l})} \ > \ 4^k \|f_k\|_{\overline{X}(0,\infty)}. \tag{5.6}$$

Define the function $f : (0,\infty) \to \mathbb{R}$ as

$$f(t) = \sum_{k=1}^{\infty} \frac{\chi_{(2^{-k},2^{-k+1})}(t) \, f_k(2^k t - 1)}{2^k \left\|\chi_{(2^{-k},2^{-k+1})} \, f_k(2^k \cdot -1)\right\|_{\overline{X}(0,\infty)}} \qquad \text{for } t \in (0,\infty). \tag{5.7}$$

Since $f \in L^0(0,\infty)$, $f = 0$ on $(1,\infty)$ and $\|f\|_{\overline{X}(0,\infty)} \le \sum_{k=1}^{\infty} 2^{-k}$, we have that $f \in \overline{X}_1(0,\infty)$. Let us denote by Λ the set $\{(k,l) \in \mathbb{N}^2 : l \le m_k\}$, ordered according to the lexicographic order, and define the sequence $\{I_{k,l}\}$ as

$$I_{k,l} = \frac{1}{2^k} J_{k,l} + \frac{1}{2^k} \qquad \text{for } (k,l) \in \Lambda. \tag{5.8}$$

Each element $I_{k,l}$ is an open subinterval of $(0,1)$. Moreover, the intervals $I_{k,l}$ and $I_{h,j}$ are disjoint if $(k,l) \neq (h,j)$. Actually, if $k \neq h$, then

$$I_{k,l} \cap I_{h,j} \ \subseteq \ (2^{-k}, 2^{1-k}) \cap (2^{-h}, 2^{1-h}) \ = \ \emptyset;$$

if, instead, $k=h$ but $l\neq j$, then the same conclusion immediately follows from the fact that the intervals $J_{k,l}$ and $J_{k,j}$ are disjoint. Owing to (5.6) and (5.7),

$$
\begin{aligned}
\sum_{(k,l)\in\Lambda}\mathcal{L}^1(I_{k,l})\,\|f\|^{\oslash}_{\overline{X}(I_{k,l})} &= \sum_{(k,l)\in\Lambda}\mathcal{L}^1(I_{k,l})\,\|(f\chi_{I_{k,l}})^*(\mathcal{L}^1(I_{k,l})\,\cdot)\|_{\overline{X}(0,\infty)}\\
&= \sum_{(k,l)\in\Lambda}\frac{\mathcal{L}^1(J_{k,l})}{2^k}\,\frac{\|(f_k\chi_{J_{k,l}})^*(\mathcal{L}^1(J_{k,l})\,\cdot)\|_{\overline{X}(0,\infty)}}{2^k\|\chi_{(2^{-k},2^{1-k})}f_k(2^k\,\cdot-1)\|_{\overline{X}(0,\infty)}}\\
&= \sum_{k=1}^{\infty}\frac{1}{4^k\|f_k^*(2^k\,\cdot)\|_{\overline{X}(0,\infty)}}\sum_{l=1}^{m_k}\mathcal{L}^1(J_{k,l})\,\|f_k\|^{\oslash}_{\overline{X}(J_{k,l})}\\
&\geq \sum_{k=1}^{\infty}\frac{4^k\|f_k\|_{\overline{X}(0,\infty)}}{4^k\|f_k^*(2^k\,\cdot)\|_{\overline{X}(0,\infty)}} \geq \sum_{k=1}^{\infty}\frac{4^k\|f_k\|_{\overline{X}(0,\infty)}}{4^k\|f_k^*\|_{\overline{X}(0,\infty)}} = \sum_{k=1}^{\infty}1=\infty.
\end{aligned}
\tag{5.9}
$$

Set $M=\{(k,l)\in\Lambda\colon \|f\|^{\oslash}_{\overline{X}(I_{k,l})}\leq 2\}$, and observe that

$$
\sum_{(k,l)\in M}\mathcal{L}^1(I_{k,l})\,\|f\|^{\oslash}_{\overline{X}(I_{k,l})}\leq 2\sum_{(k,l)\in M}\mathcal{L}^1(I_{k,l})\leq 2. \tag{5.10}
$$

From (5.9) and (5.10) we thus infer that

$$
\sum_{(k,l)\in\Lambda\setminus M}\mathcal{L}^1(I_{k,l})\,\|f\|^{\oslash}_{\overline{X}(I_{k,l})}=\infty.
$$

On the other hand, assumption (v) implies property (3.10). This property, applied with $g=f\chi_{I_{k,l}}$, in turn ensures that, for every $(k,l)\in\Lambda$,

$$
\lim_{t\to+\infty}\|(f\chi_{I_{k,l}})^*\|^{\oslash}_{\overline{X}(0,t)}=\lim_{t\to+\infty}\|(f\chi_{I_{k,l}})^*(t\,\cdot)\|_{\overline{X}(0,\infty)}\leq\lim_{t\to+\infty}\|(f\chi_{I_{k,l}})^*\,\chi_{(0,\frac{1}{t})}\|_{\overline{X}(0,\infty)}=0.
$$

Note that the inequality holds since the function $(f\chi_{I_{k,l}})^*$ belongs to $\overline{X}_1(0,\infty)$, and is non-increasing, and hence $(f\chi_{I_{k,l}})^*(ts)\leq(f\chi_{I_{k,l}})^*(s)\,\chi_{(0,\frac{1}{t})}(s)$ for $s\in(0,\infty)$. Thus, owing to Lemma 3.5, if $(k,l)\in\Lambda\setminus M$, there exists a number $a_{k,l}\geq\mathcal{L}^1(I_{k,l})$ such that

$$
\|(f\chi_{I_{k,l}})^*\|^{\oslash}_{\overline{X}(0,a_{k,l})}=2\,. \tag{5.11}
$$

Furthermore, by (5.11),

$$
\begin{aligned}
\sum_{(k,l)\in\Lambda\setminus M}a_{k,l} &= \frac{1}{2}\sum_{(k,l)\in\Lambda\setminus M}a_{k,l}\,\|(f\chi_{I_{k,l}})^*\|^{\oslash}_{\overline{X}(0,a_{k,l})}\\
&\geq \frac{1}{2}\sum_{(k,l)\in\Lambda\setminus M}\mathcal{L}^1(I_{k,l})\,\|(f\chi_{I_{k,l}})^*\|^{\oslash}_{\overline{X}(0,\mathcal{L}^1(I_{k,l}))}\\
&= \frac{1}{2}\sum_{(k,l)\in\Lambda\setminus M}\mathcal{L}^1(I_{k,l})\,\|f\|^{\oslash}_{\overline{X}(I_{k,l})}=\infty.
\end{aligned}
\tag{5.12}
$$

Thanks to (5.12), the function $f\in\overline{X}_1(0,\infty)$, defined by (5.7), the sequence $\{I_{k,l}\}$, defined by (5.8), and the sequence $\{a_{k,l}\}$ contradict condition (H) in Lemma 3.2, and, thus, assumption (v).

(ii) ⇒ (iii) It follows from Propositions 4.1 and 4.2, since condition (ii) of Proposition 4.1 trivially implies condition (iii) of Proposition 4.2.

(iii) $\Rightarrow$ (iv) We have to show that assumption (iii) implies that for every bounded measurable set $K \subseteq \mathbb{R}^n$, there exists a constant $C = C(K)$ such that

$$\mathcal{L}^n(\{x \in K \colon \mathcal{M}_X u(x) > t\}) \le \frac{C}{t}\|u\|_{X(\mathbb{R}^n)} \qquad \text{for } t \in (0,\infty), \tag{5.13}$$

for every function $u \in X_{\rm loc}(\mathbb{R}^n)$ whose support is contained in K.
Let K be a bounded subset of $\mathbb{R}^n$. Fix any function $u \in X_{\rm loc}(\mathbb{R}^n)$ whose support is contained in K. Clearly, $u = u\chi_K$. From (iii), we infer that

$$\begin{aligned}\sup_{t>0}\ t\,\mathcal{L}^n(\{x \in K \colon \mathcal{M}_X u(x) > t\}) &= \sup_{t>0}\ t\,\mathcal{L}^n(\{x \in \mathbb{R}^n \colon \chi_K \mathcal{M}_X u(x) > t\}) \\ &= \sup_{s>0}\ s\big(\chi_K\,\mathcal{M}_X u\big)^*(s) \le \sup_{s\in(0,\mathcal{L}^n(K))} s\big(\mathcal{M}_X u\big)^*(s) \le C \sup_{s\in(0,\mathcal{L}^n(K))} s\|u^*\|^{\oslash}_{\overline{X}(0,s)} \\ &\le C\,\mathcal{L}^n(K)\,\|u^*\|^{\oslash}_{\overline{X}(0,\mathcal{L}^n(K))} \le C'\|u^*\|_{\overline{X}(0,\infty)} = C'\|u\|_{X(\mathbb{R}^n)},\end{aligned}$$

for some constants C and C', where the last but one inequality follows from Lemma 3.5, and the last one from the boundedness of the dilation operator on rearrangement-invariant spaces. Property (iv) is thus established.

(iv) $\Rightarrow$ (i) Since $\mathbb{R}^n$ is the countable union of balls, in order to prove (i) it suffices to show that, given any $u \in X_{\rm loc}(\mathbb{R}^n)$ and any ball B in $\mathbb{R}^n$,

$$\lim_{r\to0^+} \|u - u(x)\|^{\oslash}_{X(B_r(x))} = 0 \qquad \text{for a.e. } x \in B. \tag{5.14}$$

Equation (5.14) will in turn follow if we show that, for every $t > 0$, the set

$$A_t = \{x \in B \colon \limsup_{r\to0^+} \|u - u(x)\|^{\oslash}_{X(B_r(x))} > 2t\} \tag{5.15}$$

has measure zero. To prove this, we begin by observing that, since $\|\cdot\|_{X(\mathbb{R}^n)}$ is locally absolutely continuous, [4, Chapter 1, Theorem 3.11] ensures that for any $\varepsilon > 0$ there exists a simple function v_ε supported on B such that $u\chi_B = v_\varepsilon + w_\varepsilon$ and $\|w_\varepsilon\|_{X(B)} < \varepsilon$. Clearly, w_ε is supported on B as well. Moreover, [4, Chapter 2, Theorem 5.5, Part (b)] and Lemma 5.1 imply that

$$\lim_{r\to0^+} \|v_\varepsilon - v_\varepsilon(x)\|^{\oslash}_{X(B_r(x))} = 0 \qquad \text{for a.e. } x \in B.$$

Fix any $\varepsilon > 0$. Then

$$\begin{aligned}\limsup_{r\to0^+} \|u - u(x)\|^{\oslash}_{X(B_r(x))} &\le \limsup_{r\to0^+} \|v_\varepsilon - v_\varepsilon(x)\|^{\oslash}_{X(B_r(x))} + \limsup_{r\to0^+} \|w_\varepsilon - w_\varepsilon(x)\|^{\oslash}_{X(B_r(x))} \\ &= \limsup_{r\to0^+} \|w_\varepsilon - w_\varepsilon(x)\|^{\oslash}_{X(B_r(x))} \le \mathcal{M}_X w_\varepsilon(x) + |w_\varepsilon(x)|\,\|\chi_{(0,1)}\|_{\overline{X}(0,\infty)}.\end{aligned}$$

Therefore,

$$A_t \subseteq \{x \in B \colon \mathcal{M}_X w_\varepsilon(x) > t\} \cup \{y \in B \colon |w_\varepsilon(y)|\,\|\chi_{(0,1)}\|_{\overline{X}(0,\infty)} > t\} \qquad \text{for } t \in (0,\infty). \tag{5.16}$$

Owing to (iv),

$$\mathcal{L}^n(\{x \in B \colon \mathcal{M}_X w_\varepsilon(x) > t\}) \le \frac{C}{t}\,\|w_\varepsilon\|_{X(B)} \qquad \text{for } t \in (0,\infty).$$

On the other hand,

$$\mathcal{L}^n(\{y \in B \colon |w_\varepsilon(y)|\|\chi_{(0,1)}\|_{\overline{X}(0,\infty)} > t\}) \le \frac{1}{t}\,\|\chi_{(0,1)}\|_{\overline{X}(0,\infty)}\,\|w_\varepsilon\|_{L^1(B)} \le \frac{C_0}{t}\|\chi_{(0,1)}\|_{\overline{X}(0,\infty)}\|w_\varepsilon\|_{X(B)}$$

for every $t \in (0,\infty)$, where C_0 is the norm of the embedding $X(B) \to L^1(B)$. Inasmuch as $\|w_\varepsilon\|_{X(B)} < \varepsilon$, the last two inequalities, combined with (5.16) and with the subadditivity of the outer Lebesgue measure, imply that the outer Lebesgue measure of A_t does not exceed

$$\frac{\varepsilon}{t}\big(C + C_0\|\chi_{(0,1)}\|_{\overline{X}(0,\infty)}\big)$$

for every $t \in (0,\infty)$. Hence, $\mathcal{L}^n(A_t) = 0$, thanks to the arbitrariness of $\varepsilon > 0$. □

6. Proofs of Propositions 1.2–1.5

In this last section, we show how our general criteria can be specialized to characterize those rearrangement-invariant norms, from customary families, which satisfy the Lebesgue point property, as stated in Propositions 1.2–1.5. In fact, these propositions admit diverse proofs, based on the different criteria provided by Theorem 1.1. For instance, Propositions 1.2–1.4 can be derived via Theorem 1.1, combined with results on the local absolute continuity of the norms in question and on Riesz-Wiener type inequalities contained in [2] (Orlicz norms), [3] (norms in the Lorentz spaces $L^{(p,q)}(\mathbb{R}^n)$), and [15] (norms in the Lorentz endpoint spaces $\Lambda_\varphi(\mathbb{R}^n)$). Let us also mention that, at least in the one-dimensional case, results from these propositions overlap with those of [5, 6, 22].

Hereafter, we provide alternative, more self-contained proofs of Propositions 1.2–1.5, relying upon our general criteria. Let us begin with Proposition 1.2, whose proof requires two preliminarily lemmas. For techincal reasons, we extend, with the same definition, the notions of Lebesgue point property, of $\mathcal{G}_X$, $\|\cdot\|^{\oslash}_X$, $\mathcal{M}_X$, and the condition (H) of Lemma 3.2 to the case when $\|\cdot\|_{X(\mathbb{R}^n)} = \|\cdot\|_{L^{p,q}(\mathbb{R}^n)}$. These notions are meaningful also for those values of p and q for which this functional is not a norm.

Lemma 6.1. *Assume that either $1 < p < \infty$ and $1 \le q \le \infty$, or $p = q = 1$, or $p = q = \infty$. Then*

$$\mathcal{G}_{L^{p,q}}(f) = \begin{cases} \Big(p\int_0^\infty s^{q-1}(f(s))^{\frac{q}{p}}\, d\mathcal{L}^1(s)\Big)^{\frac{1}{q}} & \text{if } 1<p<\infty \text{ and } 1\le q<\infty, \text{or } p=q=1;\\ \sup\limits_{s\in(0,\infty)} s(f(s))^{\frac{1}{p}} & \text{if } 1<p<\infty \text{ and } q=\infty;\\ \mathcal{L}^1(\{s\in(0,\infty)\colon f(s)>0\}) & \text{if } p=q=\infty\,, \end{cases} \tag{6.1}$$

for every non-increasing function $f\colon [0,\infty)\to[0,\infty]$. Hence, the functional $\mathcal{G}_{L^{p,q}}$ is concave if $1 \le q \le p$.

Proof. Equation (6.1) follows from a well-known expression of the functional $\|\cdot\|_{L^{p,q}(\mathbb{R}^n)}$ in terms of the distribution function (see, e.g., [9, Proposition 1.4.9]), from equality (4.1) and from the fact that every non-increasing function $f\colon [0,\infty)\to[0,\infty]$ agrees a.e. with the function $f = (f_*)_*$.

The fact that $\mathcal{G}_{L^{p,q}}$ is concave if $1 \le q \le p$ is an easy consequence of the representation formulas (6.1). In particular, the fact that the function $[0,\infty) \ni t \mapsto t^\alpha$ is concave if $0 < \alpha \le 1$ plays a role here. □

Lemma 6.2. *Assume that either $1 < p < \infty$ and $1 \le q \le \infty$, or $p = q = 1$, or $p = q = \infty$. Then there exists a function $u \in L^{p,q}(\mathbb{R}^n)$, having support of finite measure, such that*

$$\mathcal{L}^n(\{x\in\mathbb{R}^n\colon \mathcal{M}_{L^{p,q}}u(x) > 1\}) = \infty. \tag{6.2}$$

Proof. We shall prove that the functional $\|\cdot\|_{L^{p,q}(\mathbb{R}^n)}$ does not satisfy condition (H) from Lemma 3.2, if $1 \le p < q < \infty$. The conclusion will then follow via Proposition 4.3.

To this purpose, define $f\colon (0,\infty)\to[0,\infty)$ as

$$f(s)=c\sum_{k=1}^{\infty}\left(\frac{3^k}{k}\right)^{\frac{1}{p}}\chi_{\left(\frac{1}{2\cdot 3^k},\frac{1}{2\cdot 3^{k-1}}\right)}(s)\qquad\text{for } s\in(0,\infty),\tag{6.3}$$

where $c>\left(\frac{q}{p}\right)^{\frac{1}{q}}$. Observe that $f=f^*\chi_{(0,1)}$ a.e., since f is a nonnegative decreasing function in $(0,\infty)$ with support in $(0,1)$. Moreover, $f\in L^{p,q}(0,\infty)$, since

$$\|f\|^q_{L^{p,q}(0,\infty)}=c^q\sum_{k=1}^{\infty}\int_{\frac{1}{2\cdot 3^k}}^{\frac{1}{2\cdot 3^{k-1}}}\left(\frac{3^k}{k}\right)^{\frac{q}{p}}s^{\frac{q}{p}-1}\,d\mathcal{L}^1(s)\le c^q\sum_{k=1}^{\infty}\left(\frac{3^k}{k}\right)^{\frac{q}{p}}\int_0^{\frac{1}{2\cdot 3^{k-1}}}s^{\frac{q}{p}-1}\,d\mathcal{L}^1(s)<\infty\,,$$

thanks to the assumption that $q>p$. For each $k\in\mathbb{N}$, set $I_k=\left(\frac{1}{2\cdot 3^k},\frac{1}{2\cdot 3^{k-1}}\right)$ and $a_k=\frac{1}{k}$. Then

$$\|(f\chi_{I_k})^*\|^{\oslash}_{L^{p,q}(0,a_k)}=c\left(\int_0^{\infty}\left(\frac{3^k}{k}\right)^{\frac{q}{p}}\chi_{(0,\frac{1}{3^k})}\left(\frac{s}{k}\right)s^{\frac{q}{p}-1}\,d\mathcal{L}^1(s)\right)^{\frac{1}{q}}=c\left(\frac{p}{q}\right)^{\frac{1}{q}}>1,$$

and hence condition (H) of Lemma 3.2 fails for the functional $\|\cdot\|_{L^{p,q}(\mathbb{R}^n)}$. □

We are now in a position to accomplish the proof of Proposition 1.2.

Proof of Proposition 1.2. Assume first that $1\le q\le p<\infty$. Then $\|\cdot\|_{L^{p,q}(\mathbb{R}^n)}$ is a rearrangement-invariant norm. By Lemma 6.1, the functional $\mathcal{G}_{L^{p,q}}$ is concave. Moreover, the norm $\|\cdot\|_{L^{p,q}(\mathbb{R}^n)}$ is locally absolutely continuous since $q\le p$, and therefore $q<\infty$ – see e.g. [19, Theorem 8.5.1]. Thereby, an application of Theorem 1.1 tells us that the norm $\|\cdot\|_{L^{p,q}(\mathbb{R}^n)}$ has the Lebesgue point property. Now assume that $p<q<\infty$. Then, coupling Theorem 1.1 with Lemma 6.2 implies that the functional $\|\cdot\|_{L^{p,q}(\mathbb{R}^n)}$ does not have the Lebesgue point property. By (2.28), neither does the norm $\|\cdot\|_{L^{(p,q)}(\mathbb{R}^n)}$.
In the remaining case when $q=\infty$, the functional $\|\cdot\|_{L^{p,q}(\mathbb{R}^n)}$ is not locally absolutely continuous, and so neither is the norm $\|\cdot\|_{L^{(p,q)}(\mathbb{R}^n)}$. Hence, by Theorem 1.1, it does not have the Lebesgue point property. □

One proof of Proposition 1.3, dealing with Orlicz norms, will follow from Theorem 1.1 via the next lemma.

Lemma 6.3. *Let A be a Young function satisfying the Δ_2-condition near infinity. Then*

$$\mathcal{L}^n(\{x\in\mathbb{R}^n\colon \mathcal{M}_{L^A}u(x)>1\})<\infty$$

for every $u\in L^A(\mathbb{R}^n)$, supported in a set of finite measure.

Proof. Owing to Proposition 4.3, it suffices to show that condition (H) from Lemma 3.2 is fulfilled by the Luxemburg norm.
Consider any function $f\in L^A_1(0,\infty)$, any sequence $\{I_k\}$ of pairwise disjoint intervals in $(0,1)$, and any sequence $\{a_k\}$ of positive real numbers such that

$$a_k\ge\mathcal{L}^1(I_k)\quad\text{and}\quad \|(f\chi_{I_k})^*\|^{\oslash}_{L^A(0,a_k)}>1\quad\text{for } k\in\mathbb{N}.$$

Since

$$a_k<\int_0^{a_k}A((f\chi_{I_k})^*(s))\,d\mathcal{L}^1(s)=\int_{I_k}A(|f(s)|)\,d\mathcal{L}^1(s)$$

for every $k\in\mathbb{N}$, one has that

$$\sum_{k=1}^{\infty}a_k<\sum_{k=1}^{\infty}\int_{I_k}A(|f(s)|)\,d\mathcal{L}^1(s)\le\int_0^1A(|f(s)|)\,d\mathcal{L}^1(s)<\infty.$$

Notice that the last inequality holds owing to the assumption that A satisfies the Δ_2-condition near infinity, and f has support of finite measure. Altogether, condition (H) is satisfied by the norm $\|\cdot\|_{L^A(\mathbb{R}^n)}$. □

Proof of Proposition 1.3. If A satisfies the Δ_2-condition near infinity, then the norm $\|\cdot\|_{L^A(\mathbb{R}^n)}$ fulfills the Lebesgue point property, by Lemma 6.3 and Theorem 1.1. Conversely, assume that the norm $\|\cdot\|_{L^A(\mathbb{R}^n)}$ fulfills the Lebesgue point property. Then it has to be locally absolutely continuous, by Theorem 1.1. Owing to [21, Theorem 14 and Corollary 5, Section 3.4], this implies that A satisfies the Δ_2-condition near infinity. □

In the next proposition, we point out the property, of independent interest, that the functional $\mathcal{G}_{L^A}$ is almost concave for *any* N-function A. Such a property, combined with the fact that the norm $\|\cdot\|_{L^A(\mathbb{R}^n)}$ is locally absolutely continuous if and only if A satisfies the Δ_2-condition near infinity, leads to an alternative proof of Proposition 1.3, at least when A is an N-function, via Theorem 1.1.

Proposition 6.4. *The functional $\mathcal{G}_{L^A}$ is almost concave for every N-function A.*

Proof. The norm $\|\cdot\|_{L^A(\mathbb{R}^n)}$ is equivalent, up to multiplicative constants, to the norm $\|\cdot\|_{L_A(\mathbb{R}^n)}$ defined as

$$\|u\|_{L_A(\mathbb{R}^n)} = \inf\left\{\frac{1}{k}\left(1+\int_{\mathbb{R}^n} A(k|u(x)|)\,dx\right) : k>0\right\}$$

for $u \in L^0(\mathbb{R}^n)$ – see [21, Section 3.3, Proposition 4 and Theorem 13]. One has that

$$\frac{1}{k}\left(1+\int_{\mathbb{R}^n} A(k|u(x)|)\,dx\right) = \frac{1}{k} + \int_0^\infty A'(kt)u_*(t)\,dt,$$

where A' denotes the left-continuous derivative of A. Altogether, we have that

$$\mathcal{G}_{L_A}(f) = \inf\left\{\frac{1}{k} + \int_0^\infty A'(kt)f(t)\,dt\right\}$$

for every non-increasing function $f\colon [0,\infty) \to [0,\infty)$. The functional $\mathcal{G}_{L_A}$ is concave, since it is the infimum of a family of affine functionals, and hence the functional $\mathcal{G}_{L^A}$ is almost concave. □

Let us next focus on the case of Lorentz endpoint norms, which is the object of Proposition 1.4.

Lemma 6.5. *Assume that $\varphi\colon [0,\infty) \to [0,\infty)$ is a (non identically vanishing) concave function. Then*

$$\mathcal{G}_{\Lambda_\varphi}(f) = \int_0^{\mathcal{L}^1(\{f>0\})} \varphi(f(t))\,d\mathcal{L}^1(t) \tag{6.4}$$

for every non-increasing function $f\colon [0,\infty) \to [0,\infty]$. In particular, the functional $\mathcal{G}_{\Lambda_\varphi}$ is concave.

Proof. Take any non-increasing function $f\colon [0,\infty) \to [0,\infty]$. Set $h = f_*$, whence $f = f^* = (f_*)_* = h_*$ a.e., and $h^*(0) = f_*(0) = \mathcal{L}^1(\{f>0\})$. From equations (2.33) and (4.1), one has, via

Fubini's theorem,

$$\begin{aligned}\mathcal{G}_{\Lambda_\varphi}(f) = \mathcal{G}_{\Lambda_\varphi}(h_*) = \|h\|_{\Lambda_\varphi(0,\infty)} &= h^*(0)\varphi(0^+) + \int_0^\infty h^*(s)\varphi'(s)\, d\mathcal{L}^1(s)\\ &= h^*(0)\varphi(0^+) + \int_0^\infty \int_0^{h^*(s)} d\mathcal{L}^1(t)\, \varphi'(s)\, d\mathcal{L}^1(s)\\ &= h^*(0)\varphi(0^+) + \int_0^{h^*(0)} \int_0^{h_*(t)} \varphi'(s)\, d\mathcal{L}^1(s)\, d\mathcal{L}^1(t)\\ &= h^*(0)\varphi(0^+) + \int_0^{h^*(0)} [\varphi(h_*(t)) - \varphi(0^+)]\, d\mathcal{L}^1(t)\\ &= \int_0^{h^*(0)} \varphi(h_*(t))\, d\mathcal{L}^1(t) = \int_0^{\mathcal{L}^1(\{f>0\})} \varphi(f(t))\, d\mathcal{L}^1(t).\end{aligned}$$

Hence, formula (6.4) follows.
In order to verify the concavity of $\mathcal{G}_{\Lambda_\varphi}$, fix any pair of non-increasing functions $f, g\colon [0,\infty) \to [0,\infty]$ and $\lambda \in (0,1)$. Observe that

$$\{t \in [0,\infty)\colon \lambda f(t) + (1-\lambda)g(t) > 0\} = \{t \in [0,\infty)\colon f(t) > 0\} \cup \{t \in [0,\infty)\colon g(t) > 0\}.$$

The monotonicity of f and g ensures that the two sets on the right-hand side of the last equation are intervals whose left endpoint is 0. Consequently,

$$\mathcal{L}^1(\{\lambda f + (1-\lambda)g > 0\}) = \max\{\mathcal{L}^1(\{f > 0\}), \mathcal{L}^1(\{g > 0\})\}. \tag{6.5}$$

On making use of equations (6.4) and (6.5), and of the concavity of φ, one infers that the functional $\mathcal{G}_{\Lambda_\varphi}$ is concave as well. □

Proof of Proposition 1.4. By Lemma 6.5, the functional $\mathcal{G}_{\Lambda_\varphi}$ is concave for every non identically vanishing concave function $\varphi\colon [0,\infty) \to [0,\infty)$. On the other hand, it is easily verified, via equation (2.33), that the norm $\|\cdot\|_{\Lambda_\varphi(\mathbb{R}^n)}$ is locally absolutely continuous if, and only if, $\varphi(0^+) = 0$. The conclusion thus follows from Theorem 1.1. □

We conclude with a proof of Proposition 1.5.

Proof of Proposition 1.5. Assume first that $\lim_{s\to 0^+} \frac{s}{\varphi(s)} = 0$. Then we claim that the norm $\|\cdot\|_{M_\varphi(\mathbb{R}^n)}$ is not locally absolutely continuous, and hence, by 1.1, it does not have the Lebesgue point property. To verify this claim, observe that the function $(0,\infty) \ni s \mapsto \frac{s}{\varphi(s)}$ is quasiconcave in the sense of [4, Definition 5.6, Chapter 2], and hence, by [4, Chapter 2, Proposition 5.10], there exists a concave function $\psi\colon (0,\infty) \to [0,\infty)$ such that $\frac{1}{2}\psi(s) \le \frac{s}{\varphi(s)} \le \psi(s)$ for $s \in (0,\infty)$. Let ψ' denote the right-continuous derivative of ψ, and define $u(x) = \psi'(\omega_n|x|^n)$ for $x \in \mathbb{R}^n$, where ω_n is the volume of the unit ball in $\mathbb{R}^n$. Then $u^* = \psi'$ in $(0,\infty)$, so that

$$1 \le u^{**}(s)\varphi(s) \le 2 \quad \text{for } s \in (0,\infty).$$

The second inequality in the last equation ensures that $u \in M_\varphi(\mathbb{R}^n)$, whereas the first one tells us that u does not have a locally absolutely continuous norm in $M_\varphi(\mathbb{R}^n)$.
Conversely, assume that $\lim_{s\to 0^+} \frac{s}{\varphi(s)} > 0$, then $(M_\varphi)_{\mathrm{loc}}(\mathbb{R}^n) = L^1_{\mathrm{loc}}(\mathbb{R}^n)$, with equivalent norms on any given subset of $\mathbb{R}^n$ with finite measure (see e.g. [23, Theorem 5.3]). Hence, the norm $\|\cdot\|_{M_\varphi(\mathbb{R}^n)}$ has the Lebesgue point property, since $\|\cdot\|_{L^1(\mathbb{R}^n)}$ has it. □

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