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TWO NEW CRITERIA FOR SOLVABILITY OF FINITE GROUPS

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Abstract. We prove the following two new criteria for the solvability of finite groups. Theorem 1. Let G be a finite group of order n containing a subgroup A of prime power index p^s. Suppose that A contains a normal cyclic subgroup B satisfying the following condition: A/B is a cyclic group of order 2^r for some non-negative integer r. Then G is a solvable group. Theorem 3. Let G be a finite group of order n and suppose that $\psi(G) \geq \frac{1}{6.68}\psi(C_n)$, where $\psi(G)$ denotes the sum of the orders of all elements of G and C_n denotes the cyclic group of order n. Then G is a solvable group.

I. Introduction

The aim of this paper is to prove the following two criteria for solvability of finite groups. The first criterion is the subject of the following theorem.

Theorem 1. *Let G be a finite group of order n containing a subgroup A of prime power index p^s. Suppose that A contains a normal cyclic subgroup B satisfying the following condition: A/B is a cyclic group of order 2^r for some non-negative integer r. Then G is a solvable group.*

Before stating the second criterion, we need some definitions and some remarks. If G is a finite group, then $\psi(G)$ denotes the sum of the orders of all elements of G. More generally, if X is a subset of G, then $\psi(X)$ denotes the sum of the orders of all elements of X. Moreover,

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Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX the cyclic group of order n is denoted by C_n. For example, $\psi(C_4) = 1+2+4+4 = 11$ and $\psi(C_2 \times C_2) = 1+2+2+2 = 7$. In the paper [1] H. Amiri, S.M. Jafarian Amiri and I.M. Isaacs proved that, if G is a finite group and $|G| = n$, then $\psi(G) \leq \psi(C_n)$, and $\psi(G) = \psi(C_n)$ if and only if $G \simeq C_n$. Thus the sum of element orders of C_n is bigger than that of any other group of order n. In a previous paper we proved the following result (see [5], Theorem 1 and Corollary 4).

Theorem 2. *Let G be a finite non-cyclic group of order n. Then*

$$\psi(G) \leq \frac{7}{11}\psi(C_n).$$

Moreover, if n is odd, then

$$\psi(G) < \frac{1}{2}\psi(C_n).$$

Thus the sum of element orders of C_n is by far bigger than that of any other group of order n. A result in the even case is contained in the paper [6].

Our second solvability criterion for a group G of order n refers to the ratio $\psi(G)/\psi(C_n)$. We proved:

Theorem 3. *Let G be a finite group of order n and suppose that*

$$\psi(G) \geq \frac{1}{6.68}\psi(C_n).$$

Then G is a solvable group.

In particular, this theorem implies the following result.

Theorem 4. *If G is a non-solvable group of order n, then*

$$\psi(G) < \frac{1}{6.68}\psi(C_n).$$

In particular, this holds for all non-abelian simple groups.

We continue now with a series of remarks related to the above mentioned results. In these remarks G denotes a finite group.

Remark 1 For the proof of Theorem 1, we used the folowing two results.

The first result is the following theorem of H. Wielandt and O.H. Kegel (see [9] and [13]).

[WK]-theorem. *If $G = AB$, where A and B are nilpotent subgroups of G, then G is solvable.*

The second result is the Szep's conjecture, which was proved by Elsa Fisman and Zvi Arad (see [11], [12] and [2]).

[FA]-theorem. *If $G = AB$, where A and B are subgroups of G with non-trivial centers, then G is not a non-abelian simple group.*

The proof of the [WK]-theorem does not rely upon the classification of finite simple groups, but the proof of the [FA]-theorem does rely on it. Therefore our proof of Theorem 1 relies upon the classification of finite simple groups.

The assumptions of the [FA] theorem do not imply the solvability of G. For example, the non-solvable group $G = SL(2,5) \times C_2$ is a product of two subgroups with non-trivial centers.

Remark 2 For the proof of Theorem 3 we used the following six results.

The first result is the following Corollary B in the paper [1] of H. Amiri, S.M. Jafarian Amiri and I.M. Isaacs.

[AAI]-theorem. *If R is a normal cyclic Sylow subgroup of G, then*

$$\psi(G) \leq \psi(R)\psi(G/R),$$

with equality if and only if R is central in G.

The second result is the following theorem of I.N. Herstein (see [4]).

[IH]-theorem. *If G contains an abelian maximal subgroup, then G is solvable.*

The third result is the following Proposition 2.5 in our previous paper [5].

[HLM]-theorem. *Let p be the maximal prime divisor of $|G|$ and suppose that $[G : \langle x \rangle] < 2p$ for some $x \in G$. Then either the Sylow p-subgroup of G is cyclic and normal in G or G is a solvable group.*

The fourth result is the following theorem of Marshall Hall, Jr (see Theorem 3.1 in [3]).

[MH]-theorem. *Let p be a prime and let $n = 1 + rp$, with r being an integer satisfying $1 < r < (p+3)/2$. Then no group has n Sylow p-subgroups, unless either $n = q^t$ for some prime q, or $r = (p-3)/2$ and $p > 3$ is a Fermat prime.*

The fifth result is the following theorem of Andrea Lucchini (see Theorem 2.20 in [8]).

[AL]-theorem. *Let A be a cyclic proper subgroup of G and let $K = core_G(A)$. Then $[A : K] < [G : A]$, and in particular, if $|A| \geq [G : A]$, then $K > 1$.*

The sixth result is the following theorem of B. Huppert and N. Ito (see [7] and Theorem 13.10.1 in [10]).

[HI]-theorem. *If $G = PB$, where P is a nilpotent subgroup of G and B is a subgroup of G containing a cyclic subgroup H of index $[B : H] \leq 2$. then G is solvable.*

In our proof of Theorem 3 we used the following corollary of the [HI]-theorem:

[HI1]-theorem. *Suppose that G contains a subgroup B of prime power index and B contains a cyclic subgroup H of index $[B : H] \leq 2$. Then G is a solvable group.*

Our Theorem 1 is clearly a generalization of the [HI1]-theorem. In our proof of Theorem 3 we prefered to refer to [HI1]-theorem, rather than to our Theorem 1, since our proof of Theorem 1 uses the [FA]-theorem, which relies on the classification of simple groups, while the proof of the [HI]-theorem does not rely on the classification of simple groups.

Remark 3 In this remark we shall discuss the question how close are our results mentioned above to the best possible ones.

The upper bound $\frac{7}{11}$ in Theorem 2 is best possible. For example, as shown above, $\psi(C_2 \times C_2) = 7$ and $\psi(C_4) = 11$. Therefore

$$\psi(C_2 \times C_2) = \frac{7}{11}\psi(C_4).$$

Moreover, it is easy to see that if $n = 4k$ for some odd integer k, then the group $G = C_{2k} \times C_2$ satisfies the above equality (see [5], Proposition 2).

The criterion for solvability in the [HI1]-theorem is quite delicate. The smallest non-abelian simple group A_5 contains a dihedral subgroup A of order 10, but $[A_5 : A] = 6$, not a prime power. On the other hand, the simple group PSL(2,7) contains a subgroup A of index 8 and A contains a normal cyclic subgroup B of order 7, but $[A : B] = 3$, not 2.

Remark 4 Here we shall discuss the solvability criterion in Theorem 3 and we shall state some related conjectures.

Notice that $\psi(A_5) = 211$ and $\psi(C_{60}) = 1617$. Therefore

$$\psi(A_5) = \frac{211}{1617}\psi(C_{60}) > \frac{1}{7.67}\psi(C_{60}).$$

So our lower bound $\frac{1}{6.68}$ in Theorem 3 is not very far from the best possible one.

As a matter of fact, we believe that the following conjecture holds:

Conjecture 5. *If G is a group of order n and*

$$\psi(G) > \frac{211}{1617}\psi(C_n),$$

then G is solvable.

If true, this lower bound is certainly best possible.

We also state a stronger version of Conjecture 5.

Conjecture 6. *If G is a non-solvable group of order n, then*

$$\psi(G) \leq \frac{211}{1617}\psi(C_n),$$

with equality if and only if $G = A_5$. In particular, this inequality holds for all non-abelian simple groups.

As mentioned above, the simple group A_5 satisfies the equality:

$$\psi(G)/\psi(C_n) = \frac{211}{1617} \sim \frac{1}{7.66},$$

but for the other five smallest finite non-abelian simple groups this ratio is lower than $\frac{1}{18}$.

Finally we mention our Proposition 2.6, which was used in the proof of Theorem 3.

Proposition 2.6. *If H is a normal subgroup of G, then*

$$\psi(G) \leq \psi(G/H)|H|^2.$$

We conjecture that the following "companion" inequality is also true:

Conjecture 7. *If H is a subgroup of G, then*

$$\psi(G) \leq \psi(H)(|G|/|H|)^2.$$

This was our last remark, which concludes the Introduction section. Our other sections are: Preliminary results related to $\psi(G)$, Proof of Theorem 1 and Proof of Theorem 3.

II. Preliminary results related to $\psi(G)$

Notation 2.1. *Let $\{q_1, q_2, q_3, \dots\}$ be the set of* **all** *primes in an increasing order: $2 = q_1 < q_2 < q_3 < \dots$ Let also $q_0 = 1$.*

If r is a positive integer, we define the function $f(r)$ as follows:

$$f(r) = \prod_{i=1}^{r} \frac{q_i}{q_i + 1}.$$

We also define $f(0) = 1$. Clearly $f(r+1) = f(r) \cdot \frac{q_{r+1}}{q_{r+1}+1}$. Since $q_1 = 2$, $q_2 = 3$, $q_3 = 5$, $q_4 = 7$ and $q_5 = 11$, we have:

$$\text{(A)} \qquad f(0) = 1, \quad f(1) = \frac{2}{3}, \quad f(2) = \frac{1}{2}, \quad f(3) = \frac{5}{12}, \quad f(4) = \frac{35}{96}, \quad f(5) = \frac{385}{1152}.$$

Notation 2.2. *If r is a positive integer, we define the function $h(r)$ as follows: $h(1) = f(0)q_1 = 2$ and for $r > 1$*

$$h(r) = \left(\prod_{i=1}^{r-1} \frac{q_i}{q_i + 1} \right) q_r = f(r-1)q_r.$$

Thus:

$$\text{(B)} \qquad h(1) = 2, \quad h(2) = 2, \quad h(3) = \frac{5}{2}, \quad h(4) = \frac{35}{12}, \quad \textit{and} \quad h(5) = \frac{385}{96}.$$

When convenient, we shall use the notation $\psi(n)$ for $\psi(C_n)$.

Lemma 2.3. *Let n be a positive integer and suppose that $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$, where p_i are primes, $p_1 < p_2 < \cdots < p_r = p$ and α_i are positive integers. Then the following inequalities hold:*

(1) $\frac{p_i}{p_i+1} \geq \frac{q_i}{q_i+1}$ *for all* i,
(2) $\prod_{i=1}^{r} \frac{p_i}{p_i+1} \geq f(r)$,
(3) $\psi(n) > (\prod_{i=1}^{r} \frac{p_i}{p_i+1})n^2 \geq f(r)n^2$,
(4) $\psi(n) > f(r-1)\frac{p}{p+1}n^2$,
(5) $\psi(n) > h(r)\frac{n^2}{p+1}$, *and*
(6) *if* $1 \leq s < r$, *then*

$$\psi(n) > h(s)\frac{p_r}{q_{r-1}+1} \cdot \frac{n^2}{p_r+1} \geq h(s) \cdot \frac{n^2}{p_r+1}.$$

Proof. As shown in [5],

$$\psi(C_{p_i^{\alpha_i}}) = \frac{p_i^{2\alpha_i+1}+1}{p_i+1} \quad \text{and} \quad \psi(C_n) = \prod_{i=1}^{r} \psi(C_{p_i^{\alpha_i}}).$$

(1) Since $p_i \geq q_i$ for all i, it follows that $\frac{p_i}{p_i+1} \geq \frac{q_i}{q_i+1}$ for all i.
(2) Follows from (1).
(3)
$$\psi(n) = \prod_{i=1}^{r} \frac{p_i^{2\alpha_i+1}+1}{p_i+1} > \prod_{i=1}^{r} \frac{p_i^{2\alpha_i+1}}{p_i+1} = \left(\prod_{i=1}^{r} \frac{p_i}{p_i+1}\right)n^2 \geq f(r)n^2.$$
(4) $\psi(n) > (\prod_{i=1}^{r} \frac{p_i}{p_i+1})n^2 \geq f(r-1)\frac{p}{p+1}n^2$.
(5) Follows from (4), since $h(r) = f(r-1)q_r \leq f(r-1)p$.
(6) By (4)

$$\psi(n) > \left(\prod_{i=1}^{r-1} \frac{q_i}{q_i+1}\right)\frac{p_r}{p_r+1} \cdot n^2 = \left(\prod_{i=1}^{s-1} \frac{q_i}{q_i+1}\right)q_s\left(\prod_{i=s}^{r-2} \frac{q_{i+1}}{q_i+1}\right)\frac{p_r}{q_{r-1}+1} \cdot \frac{n^2}{p_r+1}$$

where $\prod_{i=s}^{r-2} \frac{q_{i+1}}{q_i+1} = 1$ if $s = r-1$. Thus

$$\psi(n) > h(s)\frac{p_r}{q_{r-1}+1} \cdot \frac{n^2}{p_r+1} \geq h(s)\frac{n^2}{p_r+1}.$$

□

If n is as in Lemma 2.3, we denote n by $n(r)$ and $\psi(n)$ by $\psi(n(r))$. Then Lemma 2.3(4) implies that $\psi(n(r)) > f(r-1)\frac{p}{p+1}n^2$ and denoting $\frac{p}{p+1}n^2 = L$, we have:

$$\psi(n(1)) > L, \quad \psi(n(2)) > \frac{2}{3}L, \quad \psi(n(3)) > \frac{1}{2}L, \quad \psi(n(4)) > \frac{5}{12}L, \quad \text{and} \quad \psi(n(5)) > \frac{35}{96}L.$$

We continue with two useful lemmas.

Lemma 2.4. *If $1 \le r \le 4$, $n = n(r)$ and $p_r = p > 7$, then*

$$\psi(n(r)) \ge \frac{385}{96} \cdot \frac{n^2}{p+1}.$$

Proof. Notice that $p \ge 11$ and by Lemma 2.3(4) $\psi(n(r)) > f(r-1)p\frac{n^2}{p+1}$. Hence it suffices to show that $f(r-1)p \ge \frac{385}{96}$ for $1 \le r \le 4$. Since $f(r)$ is a decreasing function, it suffices to show it for $r = 4$. If $r = 4$, then $f(r-1)p = \frac{5}{12}p \ge \frac{55}{12} > \frac{385}{96}$, as required. $\square$

Lemma 2.5. *If $n = n(r)$ and $r \ge 5$, then $\psi(n(r)) \ge \frac{385}{96} \cdot \frac{n^2}{p+1}$.*

Proof. By Lemma 2.3(5),(6) and (B) we have $\psi(n(r)) \ge h(5)\frac{n^2}{p+1} = \frac{385}{96} \cdot \frac{n^2}{p+1}$. $\square$

The last result in this section is the following important proposition.

Proposition 2.6. *Let H be a normal subgroup of the finite group G. Then*

$$\psi(G) \le \psi(G/H)|H|^2.$$

Proof. Write $|G/H| = s$, $G/H = \{x_1H, x_2H, \dots, x_sH\}$ and for every $i \in \{1, \dots, s\}$ denote the order of x_iH in G/H by t_i. Then

$$\psi(G/H) = t_1 + t_2 + \cdots + t_s.$$

Since $G = x_1H \dot{\cup} x_2H \dot{\cup} \cdots \dot{\cup} x_sH$, it follows that

$$\psi(G) = \psi(x_1H) + \psi(x_2H) + \cdots + \psi(x_sH).$$

Now we claim that

$$\psi(x_iH) \le t_i|H|^2 \quad \text{for every } i \in \{1, \dots, s\}.$$

Indeed, if $h \in H$, then $(x_ih)^{t_i} \in (x_iH)^{t_i} = H$. Thus $(x_ih)^{t_i} \in H$ and therefore $(x_ih)^{t_i|H|} = 1$. Hence $o(x_ih) \le t_i|H|$, implying that $\psi(x_iH) \le t_i|H|^2$, as claimed. Therefore

$$\psi(G) = \psi(x_1H) + \psi(x_2H) + \cdots + \psi(x_sH) \le (t_1 + t_2 + \cdots + t_s)|H|^2 = \psi(G/H)|H|^2,$$

as required. $\square$

III. Proof of Theorem 1

Proof of Theorem 1. We assume that $B \trianglelefteq A < G$, with B a cyclic subgroup of A, A/B a cyclic group of order 2^r for some non-negative integer r and $[G:A] = p^r$. It is easy to see that if $C \leq A$ and $D \trianglelefteq A$, then both C and A/D contain a normal cyclic subgroup of index 2^t, for some t being an integer satisfying $0 \leq t \leq r$.

Let P be a Sylow p-subgroup of G. Then $G = PA$. Our proof is by induction on the order of G. We may assume that $p^r > 1$ and $|B| > 1$, since otherwise either $G = A$ and hence it is solvable or A is cyclic and G is solvable by the [WK]-theorem.

First we claim that G is non-simple. If $|A/B| = 1$, then $G = PB$, and G is solvable by the [WK]-theorem. So we may assume in the proof of our claim that $[A:B] = 2^r$ for some positive integer r.

If B is of even order, then the involution in B is central in A. Thus both P and A have non-trivial centers, which implies by the [FA]-theorem that G is non-simple, as claimed. So we may also assume in the proof of our claim that B is of odd order.

Suppose that $p = 2$. Then $G = PB$ and G is solvable by the [WK]-theorem. On the other hand, since B is of odd order, $p > 2$ implies that a Sylow 2-subgroup of G is cyclic and hence G is solvable. Thus in all cases G is non-simple, as claimed.

So let N be a minimal normal subgroup of G. Then $G/N = (PN/N)(AN/N)$ and $AN/N \cong A/A \cap N$. Thus AN/N satisfies our assumptions concerning A and hence G/N satisfies the assumptions of our theorem. By the inductive hypothesis G/N is solvable.

Since NA is a subgroup of G containing A and $G = PA$, it follows that $NA = (NA \cap P)A$. Thus NA satisfies the assumptions of our theorem. If $NA < G$, then by the inductive hypothesis NA is solvable. Thus N is solvable, which implies the solvability of G, as required. Finally, if $NA = G$, then $G = PA$ implies that $|N||A|/|N \cap A| = |P||A|/|P \cap A|$. Thus $|N|/|A \cap N| = |P|/|A \cap P|$ and $N = (A \cap N)S$, where S is a Sylow p-subgroup of N. Hence N satisfies the assumptions of our theorem and by the inductive hypothesis N is solvable. So G is solvable, as required. The proof of the theorem is complete. $\square$

IV. Proof of Theorem 3

Proof of Theorem 3. Suppose that G be a group of order n and

$$\psi(G) \geq \frac{1}{6.68}\psi(C_n).$$

Our aim is to prove that G is solvable.

Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$, where p_i are primes, $p_1 < p_2 < \cdots < p_r = p$ and α_i are positive integers. Our proof is by induction on r. If $r \leq 2$ then G is solvable, as required. So suppose that $r \geq 3$ and the theorem holds for groups of order which is a product of less than r distinct prime powers.

Suppose that G contains a normal cyclic Sylow subgroup R. Then by the [AAI]-theorem and our assumptions we have

$$\psi(R)\psi(G/R) \geq \psi(G) \geq \frac{1}{6.68}\psi(|R|)\psi(|G/R|).$$

Since $\psi(R) = \psi(|R|)$, it follows that $\psi(G/R) \geq \frac{1}{6.68}\psi(|G/R|)$ and by our inductive hypothesis G/R is a solvable group. But then also G is solvable, as required. So we may assume that G has no normal cyclic Sylow subgroups. This assumption will be called "assumption A".

Suppose, next, that H is an abelian subgroup of G with a prime index $[G : H]$. Then H is an abelian maximal subgroup of G and hence by the [IH]-theorem G is solvable, as required. So we may also assume that if H is an abelian subgroup of G, then the index $[G : H]$ is not a prime number. This assumption will be called "assumption B".

If either $p > 7$ or $r \geq 5$, then by Lemmas 2.4 and 2.5 we have $\psi(n) \geq \frac{385}{96} \cdot \frac{n^2}{p+1}$. Hence

$$\psi(G) \geq \frac{1}{6.68}\psi(n) \geq \frac{1}{6.68} \cdot \frac{385}{96} \cdot \frac{n^2}{p+1} = \frac{385}{641.28} \cdot \frac{n^2}{p+1},$$

which implies that there exists $x \in G$ such that $|x| > \frac{385}{641.28} \cdot \frac{n}{p+1}$. Hence $[G : \langle x \rangle] < \frac{641.28}{385}(p+1)$. Since $r \geq 3$, it follows that $p \geq 5$ and $p + 1 \leq \frac{6}{5}p$. Hence $[G : \langle x \rangle] < \frac{3847.68}{1925}p < 2p$, and in view of "assumption A", the [HLM]-theorem implies that G is solvable, as required.

Therefore we may assume that $3 \leq r \leq 4$ and $p \leq 7$. This implies that we need to deal only with the following three cases: (i) $r = 4$ and $p = 7$, (ii) $r = 3$ and $p = 7$, and (iii) $r = 3$ and $p = 5$.

Suppose, first, that case (i) holds: $r = 4$ and $p = 7$. Then $p_4 = 7$ and by Lemma 2.3(3) we have

$$\psi(G) \geq \frac{1}{6.68}\psi(n) > \frac{1}{6.68}f(4)n^2 = \frac{1}{6.68} \cdot \frac{35}{96}n^2 = \frac{35}{641.28}n^2,$$

which implies that there exists $x \in G$ such that $|x| > \frac{35}{641.28}n$. Hence $[G : \langle x \rangle] < \frac{641.28}{35} < 19$.

If $7 \mid [G : \langle x \rangle]$, then by "assumption B" we have $[G : \langle x \rangle] = 14$. Let $Q \leq \langle x \rangle$ be a cyclic Sylow 5-subgroup of G and let $N = N_G(Q)$. Then $N \geq \langle x \rangle$ and

$$14 = [G : N][N : \langle x \rangle] = (1 + 5k)[N : \langle x \rangle].$$

Since $k > 0$ by "assumption A", we have reached a contradiction.

If $7 \nmid [G : \langle x \rangle]$, let $P \leq \langle x \rangle$ be a cyclic Sylow 7-subgroup of G and let $N = N_G(P)$. Then $N \geq \langle x \rangle$ and

$$19 > [G : \langle x \rangle] = [G : N][N : \langle x \rangle] = (1 + 7k)[N : \langle x \rangle].$$

By "assumption A" $k = 0$ is impossible and by the [MH]-theorem also $k = 2$ is impossible. Hence $k = 1$, which implies that $[G : \langle x \rangle] = 8[N : \langle x \rangle]$ and $[N : \langle x \rangle] \leq 2$. Since $[G : N] = 8$, it follows by the [HI1]-theorem that G is solvable, as required. The proof in case (i) is complete.

Suppose, next, that case (ii) holds: $r = 3$ and $p = 7$. By Lemma 2.3(3) we have

$$\psi(G) \geq \frac{1}{6.68}\psi(n) > \frac{1}{6.68}f(3)n^2 = \frac{1}{6.68} \cdot \frac{5}{12}n^2 = \frac{5}{80.16}n^2,$$

which implies that there exists $x \in G$ such that $|x| > \frac{5}{80.16}n$. Hence $[G : \langle x \rangle] < \frac{80.16}{5} < 17$.

If $7 \mid [G : \langle x \rangle]$, then it follows from $[G : \langle x \rangle] < 17$ by "assumption B" that $[G : \langle x \rangle] = 14$. If $5 \mid n$, let $Q \leq \langle x \rangle$ be a cyclic Sylow 5-subgroup of G and let $N = N_G(Q)$. Then $N \geq \langle x \rangle$ and

$$14 = [G : N][N : \langle x \rangle] = (1 + 5k)[N : \langle x \rangle],$$

with $k > 0$ by "assumption A", a contradiction. If $5 \nmid n$, then $3 \mid n$ and let $R \leq \langle x \rangle$ be a cyclic Sylow 3-subgroup of G. Then $M = N_G(R) \geq \langle x \rangle$ and

$$14 = [G : M][M : \langle x \rangle] = (1 + 3k)[M : \langle x \rangle].$$

It follows by "assumption A" that $k = 2$, $[G : M] = 7$ and $[M : \langle x \rangle] = 2$, which implies by the [HI1]-theorem that G is solvable, as required.

If $7 \nmid [G : \langle x \rangle]$, let $P \leq \langle x \rangle$ be a cyclic Sylow 7-subgroup of G. Then $N = N_G(P) \geq \langle x \rangle$ and

$$17 > [G : \langle x \rangle] = [G : N][N : \langle x \rangle] = (1 + 7k)[N : \langle x \rangle].$$

By "assumption A" $k > 0$ and by the [MH]-theorem $k = 2$ is impossible. So we must have $k = 1$. Thus $[G : N] = 8$ and $[N : \langle x \rangle] \leq 2$, which implies by the [HI1]-theorem that G is solvable, as required. The proof in case (ii) is complete.

Suppose, finally, that case (iii) holds: $r = 3$ and $p = 5$. By Lemma 2.3(3) we have

$$\psi(G) \geq \frac{1}{6.68}\psi(n) > \frac{1}{6.68}f(3)n^2 = \frac{1}{6.68} \cdot \frac{5}{12}n^2 = \frac{5}{80.16}n^2,$$

which implies that there exists $x \in G$ such that $|x| > \frac{5}{80.16}n$. Hence $[G : \langle x \rangle] < \frac{80.16}{5} < 17$.

If $5 \mid [G : \langle x \rangle]$, then it follows from $[G : \langle x \rangle] < 17$ by "assumption B" that either $[G : \langle x \rangle] = 10$ or $[G : \langle x \rangle] = 15$. If $[G : \langle x \rangle] = 10$, let $U \leq \langle x \rangle$ be a cyclic Sylow 3-subgroup of G and let $N = N_G(U)$. Then $N \geq \langle x \rangle$ and

$$10 = [G : N][N : \langle x \rangle] = (1 + 3k)[N : \langle x \rangle].$$

By "assumption A" it follows that $k = 3$ and $N = \langle x \rangle$. Hence N is an abelian maximal subgroup of G and G is solvable by the [IH]-theorem, as required. If $[G : \langle x \rangle] = 15$, then a Sylow 2-subgroup of G is cyclic and hence G is solvable, as required.

If $5 \nmid [G : \langle x \rangle]$, let $P \leq \langle x \rangle$ be a cyclic Sylow 5-subgroup of G. Then $M = N_G(P) \geq \langle x \rangle$ and by "assumption A" we have

$$17 > [G : \langle x \rangle] = [G : M][M : \langle x \rangle] = (1 + 5k)[M : \langle x \rangle],$$

with either $k = 1$ or $k = 2$ or $k = 3$.

If either $k = 2$ or $k = 3$, then $M = \langle x \rangle$ and either $[G : M] = 11$ or $[G : M] = 16$, respectively. Hence G is solvable by the [HI1]-theorem, as required.

If $k = 1$, then $[G : M] = 6$, $[M : \langle x \rangle] \leq 2$ and M is a maximal subgroup of G. If $M = \langle x \rangle$, then M is an abelian maximal subgroup of G and G is solvable by the [IH]-theorem, a contradiction, since $[G : M]$ is not a prime power.

So suppose that $[G : M] = 6$, $[M : \langle x\rangle] = 2$ and G is a non-solvable $(2, 3, 5)$-group satisfying $\psi(G) > \frac{5}{80.16}n^2$. Our aim is to reach a contradiction.

Denote $\langle x\rangle = A$. Since $[M : A] = 2$ and $[G : A] = 12$, it follows by the [AL]-theorem that if $H = core_G(A)$, then $[A : H] < [G : A] = 12$. If $5 \nmid [A : H]$, then H contains a cyclic Sylow 5-subgroup Q of G, and since H is cyclic, Q is normal in G, contradicting "assumption A". Hence $[A : H] = 5v < 12$, which implies that either $[A : H] = 5$ or $[A : H] = 10$. Thus either $|G/H| = 60$ or $|G : H| = 120$.

If $|G/H| = 60$, then G/H is a non-solvable group of order 60 and hence $G/H \simeq A_5$. Since $\psi(A_5) = 211$, Proposition 2.6 implies that

$$\psi(G) \leq \psi(G/H)|H|^2 = 211(n/60)^2 = \frac{211}{3600}n^2.$$

But $\psi(G) > \frac{5}{80.16}n^2$ and $\frac{5}{80.16} > \frac{211}{3600}$, so we have reached a contradiction.

If $|G/H| = 120$ and G/H is non-solvable, then using a list of such groups and their ψ-values, we have $\psi(G/H) \leq 663$. Thus Proposition 2.6 implies that

$$\psi(G) \leq \psi(G/H)|H|^2 \leq 663(n/120)^2 = \frac{663}{4 \times 3600}n^2 < \frac{211}{3600}n^2,$$

and arguing as before we reach a contradiction. This final contradiction completes the proof of the theorem. $\square$

References

1. H. Amiri, S.M. Jafarian Amiri and I.M. Isaacs, *Sums of element orders in finite groups*, Comm. Algebra **37** (2009), 2978-2980.
2. E. Fisman and Z. Arad, *A proof of Szep's conjecture on nonsimplicity of certain finite groups*, J. Algebra **108(2)** (1987), 340-354.
3. M. Hall, Jr., *On the number of Sylow subgroups in a finite group*, J. Algebra **7** (1967), 363-371.
4. I.N. Herstein, *A remark on finite groups*, Proc. Amer. Math. Soc. **9** (1958), 255-257.
5. M. Herzog, P. Longobardi and M. Maj, *An exact upper bound for sums of element orders in non-cyclic finite groups*, J. Pure Appl. Algebra **222(7)** (2018), 1628-1642.
6. M. Herzog, P. Longobardi and M. Maj, *Sums of element orders in groups of order* $2m$ *with* m *odd,*, submitted.
7. B. Huppert and N. Itô, *Über die Auflosbarkeit faktorisierbarer Gruppen II*, Math. Zeit. **61** (1954), 94-99.
8. I.M. Isaacs, *Finite group theory*, Amer. Math. Soc., Providence, Rhode Island, 2008.
9. O.H. Kegel, *Produkte nilpotenten Gruppen*, Arch. Math. **12** (1961), 90-93.
10. W.R. Scott, *Group theory*, Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1964.
11. J. Szep, *Zur Theorie der faktorisierbaren Gruppen*, Acta Sci. Math. (Szeged) **16** (1955), 54-57.
12. J. Szep, *Sui gruppi fattorizzabili non semplici*, Rend. Mat. Appl. **22** (1963), 245-252.
13. H. Wielandt, *Über Produkte nilpotenten Gruppen*, Illinois J. Math. **2** (1958), 611-618.