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ON INJECTIVITY OF SEMIMODULES OVER ADDITIVELY IDEMPOTENT DIVISION SEMIRINGS AND CHAIN MV-SEMIRINGS

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Abstract. In this paper, we give a characterization of injective semimodules over additively idempotent semirings. Consequently, we provide a complete description of injective semimodules over the semifield of tropical integers, and give an explicit construction of the injective hulls of semimodules over chain division semirings. We also give a criterion for self-injective MV-semirings with an atomic Boolean center, and describe the structure of (finitely generated) injective semimodules over finite MV-semirings, as well as we show that every complete MV-semiring with an atomic Boolean center is an exact semiring which is defined by a Hahn-Banach-type separation property on semimodules arising in the tropical case from the phenomenon of tropical matrix duality. Moreover, we show that complete Boolean algebras are precisely the MV-semirings in which every principal ideal is injective.

Key words: MV-algebras; additively idempotent semirings and semifields; injective and projective semimodules; semilattices.

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1. Introduction

Semirings and semimodules, and their applications, arise in various branches of Mathematics, Computer Science, Physics, as well as in many other areas of

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modern science (see, for instance, [16]). In recent years, there has been a substantial amount of interest in additively idempotent semirings — among which the Boolean semifield, tropical semifields, and coordinate semirings of tropical varieties represent a set of well-known examples — originated in several extremely interesting contexts as Tropical Geometry [41] and [15], Tropical Algebra [24], $\mathbb{F}_1$-Geometry [9], [8] and [34], the Geometry of Blueprints [36], Cryptography [39], Cluster algebras [31], Dequantization [35], and in particular MV-algebras [2], [10] and [11], for example.

Let us briefly mention MV-algebras and connections between them and additively idempotent semirings. MV-algebras arose in the literature as the algebraic semantics of Łukasiewicz propositional logic, one of the longest-known many-valued logics. In the last decades the knowledge about MV-algebras benefited from the literature on lattice-ordered groups via the well-known and celebrated categorical equivalence between MV-algebras and lattice-ordered Abelian groups with a distinguished strong order unit [40].

A connection between MV-algebras and a special category of additively idempotent semirings (called *MV-semirings* or *Łukasiewicz semirings*) was first observed in [10] and eventually implemented in [2]. On the one hand, every MV-algebra has two *semiring reducts* isomorphic to each other by the involutive unary operation * of MV-algebras (see, e.g., [12, Proposition 4.8]); on the other hand, the category of MV-semirings defined in [4] is isomorphic to the one of MV-algebras. Such results led to interesting applications of MV-semirings and their semimodules to the theory of fuzzy weighted automata [42], and to an algebraic approach to fuzzy compression algorithms and reconstruction of digital images [11]. Another link between MV-algebras and semiring theory relies on the aforementioned categorical equivalence due to Mundici. It is also well-known that the category of lattice-ordered Abelian groups with a distinguished strong order unit is isomorphic to the one of additively idempotent semifields with a distinguished strong order unit. Consequently, the category of semimodules over a given MV-algebra A is a full subcategory of the one of semimodules over the positive cone of the additively idempotent semifield corresponding to A [13, Corollary 2.12].

It is well known that an effective way to understand the behavior of a ring R is to study the various ways in which R acts on its left and right modules. Thus, the theory of modules may be expected to be an essential chapter in the theory of rings. Two of the most important objects in the theory of modules are projective and injective modules. As algebraic objects, semirings are certainly the most natural generalization of such (at first glance different) algebraic systems as rings and bounded distributive lattices, and therefore, they form an extremely interesting, natural, and important, non-abelian/non-additive setting for furthering the structure theory of projective and injective semimodules. We shall mention that the structure theory of projective and injective semimodules has been considered by some authors (see, for example, [44], [45], [29], [20], [30], [12], [22], [1], [21], [26], [38], [23], and [25]).

Although, in general, describing the structure of projective and injective semimodules seems to be a quite difficult task, recently people have obtained a number of interesting results regarding the structures of projective and injective semimodules over special classes of semirings among which we mention, for example, the following ones. Il'in, Katsov and the third author [23] initiated a homological structure theory of semirings and investigated semirings all of whose cyclic semimodules are projective; Izhakian, Johnson and Kambites [26] characterized finitely generated projective semimodules over a tropical semifield in terms of rank functions of semimodules; and Macpherson [38] classified projective semimodules over additively idempotent semirings that are free on a monoid. Further, motivated by understanding direct sum decompositions of subsemimodules of free semimodules over the tropical semifield and related structures in tropical algebra, Izhakian, Knebusch and Rowen [25] developed a theory of the decomposition of socle for zerosumfree semimodules, and established the uniqueness of direct sum decompositions for some special finitely generated projective semimodules.

Horn and Kimura in [19], and independently Bruns and Lakser in [3], gave a characterization of injective semimodules, and the explicit description of injective hulls of injective semimodules, over the Boolean semifield $\mathbb{B} := \{0, 1\}$. Fofanova [14] provided a characterization of injective semimodules over Boolean algebras. Takahashi-Wang [44] proved a characterization of injective semimodules over chain division semirings. It should be mentioned that Fafonova and Takahashi-Wang did not give the description of injective hulls of semimodules over Boolean algebras and over chain division semirings in their papers cited above, respectively. Wang [45] showed that every semimodule over an additively idempotent semiring has an injective hull. Abuhlail, Il'in, Katsov and the third author [1] contributed to the theme of understanding a semiring through the category of its semimodules by considering semirings all whose simple and cyclic semimodules are injective; Il'in [21] gave a class of semimodules, and described the semirings for which every semimodule of this class has an injective envelope.

On the other hand, to our knowledge, in the literature, there is no characterization of an injective semimodule over an arbitrary additively idempotent semiring. In this article we give a criterion for injectivity of semimodules over additively idempotent semirings and an explicit description of injective hulls of semimodules over chain division semirings, and describe the structure of injective semimodules over MV-semirings with an atomic Boolean center. Our method is to characterize injective semimodules over additively idempotent semirings S in terms of the $\mathbb{B}$-dual of the additive reduct $(S, +, 0)$, i.e., the opposite to the lattice of ideals of the join-semilattice S for the natural order, which is different from Takahashi-Wang's method [44] which consists in investigating injectivity of a semimodule M over an additively idempotent semiring S via the S-semimodule $Hom_{\mathbb{B}}(S, P(M))$, where $P(M)$ is the monoid of all subsets of M together with the intersection operation.

The article is organized as follows. In Section 2, for the reader's convenience, we provide all necessary notions and facts on semirings and semimodules. In Section 3, we give a characterization for injective semimodules over additively idempotent semirings (Theorem 3.3). Consequently, we recover Takahashi-Wang's result cited above about a characterization of injective semimodules over chain division semirings (Corollary 3.8), and recover Il'in's result [20] that all left S-semimodules over a semiring S are injective if and only if S is a semisimple ring (Corollary 3.6). Also, based on Horn-Kimura's construction [19, Theorem 3.8], we give an explicit construction for the injective hulls of semimodules over chain division semirings (Theorem 3.11). Finally, we give a complete desription of injective semimodules over the semifield of tropical integers (Theorem 3.12).

In Section 4, based on Theorem 3.3, we characterize self-injective MV-semirings with an atomic Boolean center (Theorem 4.7), and give a description of (finitely generated) injective semimodules over finite MV-semirings (Theorems 4.10 and 4.11). An interesting consequence of Theorem 4.7 is to show that every complete MV-semiring with an atomic Boolean center is an exact semiring (Corollary 4.9). It should be mentioned that the concept of exact semirings was introduced by Wilding-Johnson-Kambites [46], and defined by a Hahn-Banach-type separation property on semimodules arising in the tropical case from the phenomenon of tropical matrix duality (see, e.g., [5], [6], [7] and [18]). Finally, we show that complete Boolean algebras are precisely the MV-semirings in which every principal ideal is injective (Proposition 4.13).

All notions and facts of categorical algebra, used here without any comments, can be found in [37]; for notions and facts from semiring theory we refer to [16].

2. Preliminaries

Recall [16] that a *semiring* is an algebra $(S, +, \cdot, 0, 1)$ such that the following conditions are satisfied:

(1) $(S, +, 0)$ is a commutative monoid with identity element 0;
(2) $(S, \cdot, 1)$ is a monoid with identity element 1;
(3) multiplication distributes over addition from either side;
(4) $0s = 0 = s0$ for all $s \in S$.

Given two semirings S and S', a map $\varphi : S \longrightarrow S'$ is a *homomorphism* if it satisfies $\varphi(0) = 0$, $\varphi(1) = 1$, $\varphi(x + y) = \varphi(x) + \varphi(y)$ and $\varphi(xy) = \varphi(x)\varphi(y)$ for all $x, y \in S$. As usual, the *direct product* of a family $(S_i)_{i\in I}$ of semirings, denoted by $\prod_{i\in I} S_i$, is the semiring obtained by endowing the set-theoretical cartesian product of the family with the semiring-operations defined pointwise.

A semiring S is *commutative* if $(S, \cdot, 1)$ is a commutative monoid; and S is *additively idempotent* if the monoid $(S, +, 0)$ is idempotent, i.e., $s + s = s$ for all $s \in S$. The semiring S is a *division semiring* if $(S \setminus \{0\}, \cdot, 1)$ is a group; and S is a *semifield* if it is a commutative division semiring. Two well-known important examples of additively idempotent semifields are the so-called the *Boolean semifield* $\mathbb{B} := (\{0,1\}, +, \cdot)$, and the *tropical semifield* $\mathbb{T} := (\mathbb{R} \cup \{-\infty\}, \max, +, -\infty, 0)$.

As usual, a *left S-semimodule* over a given semiring S is a commutative monoid $(M, +, 0_M)$ together with a scalar multiplication $(s, m) \mapsto ms$ from $S \times M$ to M which satisfies the identities $(s'sm) = s'(sm)$, $s(m+m') = sm+sm'$, $(s+s')m = sm + s'm$, $1m = m$, $s0_M = 0_M = 0m$ for all $s, s' \in S$ and $m, m' \in M$.

Right semimodules over S and homomorphisms between semimodules are defined in the standard manner. From now on, let $\mathcal{M}_S$ and ${}_S\mathcal{M}$ denote the categories of right and left S-semimodules, respectively, over a semiring S.

An element $\infty \in M$ of an S-semimodule M is called *infinite* if $\infty + m = \infty$ for all $m \in M$. A left S-semimodule M is called *additively idempotent* if $m + m = m$ for all $m \in M$. Notice that a semiring S is *additively idempotent* if ${}_SS \in |{}_S\mathcal{M}|$ is an additively idempotent semimodule, and every left semimodule over an additvely idempotent semiring is also additively idempotent. Moreover, we have the following simple fact:

Fact 2.1. *Let S be a semiring and M an additively idempotent left S-semimodule. Then, the monoid $(M, +, 0)$ is always a join-semilattice with the partial order relation $\leq$ (called the natural order) on M defined for any two elements $m, m' \in M$ by $m \leq m'$ if $m + m' = m'$; and $m \vee m' = m + m'$ for all $m, m' \in M$.*

Notice that an infinite element in an additively idempotent semimodule is exactly a maximum for the natural order. An additively idempotent semiring S is called *chain* if the partial ordering $\leq$ on S is total.

Let S be a semiring and I a nonempty set. The *direct product* of a family $(M_i)_{i\in I}$ of left S-semimodules, denoted by $\prod_{i\in I} M_i$, is the left S-semimodule obtained by endowing the set-theoretical cartesian product of the family with the semimodule-operations defined pointwise. In other words, $\prod_{i\in I} M_i$ is the set of all functions $f : I \longrightarrow \bigcup_{i\in I} M_i$ such that $f(i) \in A_i$ for all $i \in I$, with the addition and scalar operations defined by:

$$(f + g)(i) = f(i) + g(i) \text{ and } (sf)(i) = sf(i)$$

for all $i \in I$ and $s \in S$. In particular, for each left S-semimodule M and nonempty set X, the left S-semimodule M^X is the direct product of the family $(M_x)_{x\in X}$, where $M_x = M$ for all $x \in X$.

As usual (see, for example, [16, Ch. 17]), if S is a semiring, then in the category ${}_S\mathcal{M}$, a *free* (left) semimodule $\bigoplus_{i\in I} S_i, S_i \cong {}_SS$, $i \in I$, with a basis set I is an I-indexed direct sum of copies of copies of ${}_SS$. A semimodule ${}_SM \in |{}_S\mathcal{M}|$ is *finitely generated* (resp. *cyclic*) if it is a homomorphic image of a free semimodule with a finite basis set (resp. a homomorphic image of S_S).

A left S-semimodule P is *projective* if the following condition holds: if $\varphi : M \longrightarrow N$ is a surjective S-homomorphism of left S-semimodules and if $\alpha : P \longrightarrow$

N is an S-homomorphism then there exists an S-homomorphism $\beta : P \longrightarrow M$ satisfying $\varphi\beta = \alpha$.

A left S-semimodule N is a *retract* of a left S-semimodule M if there exist an S-homomorphism $\theta : M \longrightarrow N$ and an S-homomorphism $\psi : N \longrightarrow M$ such that $\theta\psi = id_N$. Recall (see, e.g., [16, Proposition 17.16]) that a projective left semimodule in ${}_S\mathcal{M}$ is exactly a retract of a free left semimodule.

A left S-semimodule E is *injective* if, given a left S-semimodule M and a subsemimodule N, any S-homomorphism from N to E can be extended to an S-homomorphism from M to E. A semiring S is called *left (right) self-injective* if the regular left (right) S-semimodule S is injective.

An injective S-homomorphism $\alpha : M \longrightarrow N$ of left S-semimodules is *essential* if, for any S-homomorphism $\beta : N \longrightarrow N'$, the map $\beta\alpha$ is an injective S-homomorphism only when β is an injective S-homomorphism.

Let M be a left S-semimodule. If there exists an injective left S-semimodule E and an essential injective S-homomorphism $\alpha : M \longrightarrow E$ then E is an *injective hull* of M.

3. Injectivity of semimodules over additively idempotent semirings

In this section, we give a necessary and sufficient condition for the injectivity of semimodules over additively idempotent semirings. Consequently, we give a complete description of injective semimodule over the semifield of tropical integers, as well as we give an explicit construction of the injective hulls of semimodules over chain division semirings. The semirings considered in this section are not necessarily commutative.

We begin by considering some simple facts on injectivity of semimodules.

Lemma 3.1. *For any semiring S, any retract of an injective left S-semimodule is injective.*

Proof. Assume that M is an injective left S-semimodule and N is a retract of M; that means, there exists a surjective S-homomorphism $\theta : M \longrightarrow N$ and an injective S-homomorphism $\psi : N \longrightarrow M$ such that $\theta\psi = id_N$.

Let K be a left S-semimodule, L a subsemimodule of K, and $\alpha : L \longrightarrow N$ an S-homomorphism. Then, since M is injective, there exists an S-homomorphism $\beta : K \longrightarrow M$ such that $\beta|_L = \psi\alpha$. It implies that $\theta\beta : K \longrightarrow N$ is an S-homomorphism such that $\theta\beta|_L = \alpha$, and hence, N is injective, finishing the proof. □

Let $\pi : T \longrightarrow S$ be a semiring homomorphism. S is canonically a left T-semimodule where the scalar multiplication is defined by $t \cdot s = \pi(t)s$ for all $t \in T$ and $s \in S$. Let M be a left T-semimodule. Then $Hom_T(S, M)$ is a left S-semimodule with respect to componentwise addition and scalar multiplication given by: $(s'\alpha)(s) = \alpha(ss')$ for all $\alpha \in Hom_T(S, M)$ and $s, s' \in S$.

Lemma 3.2. *For any additively idempotent semiring S, the left S-semimodule $Hom_{\mathbb{B}}(S,\mathbb{B})$ is injective.*

Proof. Let S be an additively idempotent semiring. We then have that $\mathbb{B}$ may be considered as a subsemiring of S; hence, every left S-semimodule is also a $\mathbb{B}$-semimodule. Let N be a subsemimodule of a left S-semimodule M and let $\alpha : N \longrightarrow Hom_{\mathbb{B}}(S,\mathbb{B})$ be an S-homomorphism. Notice that N is also a $\mathbb{B}$-subsemimodule of M. Define a map $\theta : N \longrightarrow \mathbb{B}$ by setting $\theta(n) = (\alpha(n))(1)$ for all $n \in N$. Then θ is a $\mathbb{B}$-homomorphism. Indeed, if $n, n' \in N$ then $\theta(n + n') = (\alpha(n + n'))(1) = (\alpha(n) + \alpha(n'))(1) = (\alpha(n))(1) + (\alpha(n'))(1) = \theta(n) + \theta(n')$.

By [45, Lemma 1] (see, also, [14, Corollary 2]), $\mathbb{B}$ is an injective $\mathbb{B}$-semimodule, and so there exists a $\mathbb{B}$-homomorphism $\varphi : M \longrightarrow \mathbb{B}$ such that $\varphi\theta = id_N$. Define a map $\beta : M \longrightarrow Hom_{\mathbb{B}}(S,\mathbb{B})$ by setting $(\beta(m))(s) = \varphi(sm)$ for all $m \in M$ and $s \in S$. We show that β is an S-homomorphism. Indeed, for all $m_1, m_2 \in M$ and $s_1, s_2 \in S$ we have $(\beta(s_1m_1 + s_2m_2))(s) = \varphi(s(s_1m_1 + s_2m_2)) = \varphi(s(s_1m_1) + s(s_2m_2)) = \varphi(s(s_1m_1)) + \varphi(s(s_2m_2)) = \varphi((ss_1)m_1)) + \varphi((ss_2)m_2)) = (\beta(m_1))(ss_1) + (\beta(m_2))(ss_2) = (s_1\beta(m_1)(s) + (s_2\beta(m_2)(s) = (s_1\beta(m_1)+s_2\beta(m_2))(s)$ for all $s \in S$. This implies that β is an S-homomorphism. Furthermore, β extends α since for each $n \in N$ and $s \in S$ we have $(\beta(n))(s) = \varphi(sn) = \theta(sn) = (\alpha(sn))(1) = (s\alpha(n))(1) = (\alpha(n))(s)$. Thus, $Hom_{\mathbb{B}}(S,\mathbb{B})$ is an injective left S-semimodule, finishing the proof. □

We are now able to give a necessary and sufficient condition for the injectivity of semimodules over additively idempotent semirings, which will play an important role in the paper. The argument of the following theorem is based on the proof of [28, Theorem 4.2].

Theorem 3.3. *Let S be an additively idempotent semiring and M a left S-semimodule. Then, M is injective if and only if there exists a set X such that M is a retract of the left S-semimodule $Hom_{\mathbb{B}}(S,\mathbb{B})^X$, where $\mathbb{B}$ is the Boolean semifield.*

Proof. ($\Longrightarrow$). Let M be an injective left S-semimodule. We then have that $Hom_{\mathbb{B}}(Hom_{\mathbb{B}}(M,\mathbb{B}),\mathbb{B})$ is a left S-semimodule, where the scalar multiplication defined by: $(s \cdot \beta)(\alpha) = \beta(\alpha \cdot s)$ for all $\beta \in Hom_{\mathbb{B}}(Hom_{\mathbb{B}}(M,\mathbb{B}),\mathbb{B})$, $\alpha \in Hom_{\mathbb{B}}(M,\mathbb{B})$ and $s \in S$. Note that $Hom_{\mathbb{B}}(M,\mathbb{B})$ is a right S-semimodule, where the scalar multiplication defined by $(\alpha \cdot s)(m) = \alpha(sm)$ for all $\alpha \in Hom_{\mathbb{B}}(M,\mathbb{B})$, $s \in S$ and $m \in M$. We claim that the map $\varphi : M \longrightarrow Hom_{\mathbb{B}}(Hom_{\mathbb{B}}(M,\mathbb{B}),\mathbb{B})$, defined by $\varphi(m)(f) = f(m)$ for all $m \in M$ and $f \in Hom_{\mathbb{B}}(M,\mathbb{B})$, is an S-homomorphism. Indeed, for all $s, s' \in S$ and $m, m' \in M$ we have $(\varphi(sm + s'm'))(f) = f(sm+s'm') = f(sm)+f(s'm') = (f\cdot s)(m)+(f\cdot s')(m') = (\varphi(m))(f\cdot s) + (\varphi(m'))(f \cdot s') = (s \cdot \varphi(m))(f) + (s' \cdot \varphi(m'))(f) = (s \cdot \varphi(m) + s' \cdot \varphi(m'))(f)$ for all $f \in Hom_{\mathbb{B}}(M,\mathbb{B})$, thus the claim is proved.

We next prove that φ is injective. Indeed, we first note that since S is an additively idempotent semiring, the semimodule M is additively idempotent. By

Fact 2.1, the monoid $(M, +, 0)$ is a join-semilattice with the partial ordering $\leq$ on M defined for any two elememts $m, m' \in M$ by $m \leq m'$ if $m + m' = m'$.

Let $m, m' \in M$ such that $m \neq m'$. We then must have that $m \nleq m'$ or $m' \nleq m$. Without loss of generality we may assume that $m \nleq m'$. We consider the $\mathbb{B}$-homomorphism $f : M \longrightarrow \mathbb{B}$, defined by: for all $x \in M$,

$$f(x) = \begin{cases} 0 & \text{if } x \leq m', \\ 1 & \text{otherwise} \end{cases}.$$

We then have that

$$\varphi(m)(f) = f(m) = 1 \neq 0 = f(m') = \varphi(m')(f),$$

that means, $\varphi(m) \neq \varphi(m')$. This implies that φ is injective.

For the right S-semimodule $Hom_{\mathbb{B}}(M, \mathbb{B})$, by [16, Proposition 17.11], there exists a surjective S-homomorphism $\theta : \bigoplus_{x \in X} S_x \longrightarrow Hom_{\mathbb{B}}(M, \mathbb{B})$ from a free right S-semimodule $\bigoplus_{x \in X} S_x$, $S_x \cong S$ as right S-semimodules for all $x \in X$, where X is any set of generators for the right S-semimodule $Hom_{\mathbb{B}}(M, \mathbb{B})$. This surjection induces an injective S-homomorphism

$$\theta^* : Hom_{\mathbb{B}}(Hom_{\mathbb{B}}(M, \mathbb{B}), \mathbb{B}) \longrightarrow Hom_{\mathbb{B}}(\bigoplus_{x \in X} S_x, \mathbb{B}),$$

defined by $\theta^*(\beta) = \beta\theta$, for all $\beta \in Hom_{\mathbb{B}}(Hom_{\mathbb{B}}(M, \mathbb{B}), \mathbb{B})$. Therefore, we obtain an injective S-homomorphism $\theta^*\varphi : M \longrightarrow Hom_{\mathbb{B}}(\bigoplus_{x \in X} S_x, \mathbb{B})$.

Consider for $x \in X$ the natural injection $\iota_x : S_x \longrightarrow \bigoplus_{x \in X} S_x$. We then have an S-isomorphism

$$Hom_{\mathbb{B}}(\bigoplus_{x \in X} S_x, \mathbb{B}) \longrightarrow \prod_{x \in X} Hom_{\mathbb{B}}(S_x, \mathbb{B}),$$

by the map $f \longmapsto (f\iota_x)_{x \in X}$, and so we have an injective S-homomorphism $\mu : M \longrightarrow Hom_{\mathbb{B}}(S, \mathbb{B})^X$. Since M is injective, there exists an S-homomorphism $\eta : Hom_{\mathbb{B}}(S, \mathbb{B})^X \longrightarrow M$ such that $\eta\mu = id_M$; that means, M is a retract of the left S-semimodule $Hom_{\mathbb{B}}(S, \mathbb{B})^X$.

$(\Longleftarrow)$. By Lemma 3.2, the left S-semimodule $Hom_{\mathbb{B}}(S, \mathbb{B})$ is injective, and so $Hom_{\mathbb{B}}(S, \mathbb{B})^X$ is also an injective left S-semimodule, by [16, Proposition 17.23 (1)]. Then, by Lemma 3.1, we immediately get that M is injective, finishing our proof. $\square$

The next goal of this section is to recover some results which have been established previously. To do so, we require the following useful notions.

Definition 3.4 (*cf.* [19, Definitions 2.6 and 2.7] and [14, Definition 1])**.** An additively idempotent left semimodule M over a semiring is called *complete* if the join-semilattice M is a complete lattice. A complete semimodule M is called *infinitely distributive* if for any family $(m_i)_{i \in I}$ of elements of M and any $m \in M$, we have $m + \bigwedge_{i \in I} m_i = \bigwedge_{i \in I}(m + m_i)$.

As an application of Theorem 3.3, we obtain a necessary condition for injectivity of a semimodule over an additively idempotent semiring, and in particular over an additively idempotent semiring which is a distributive lattice for the natural order, which extends Takahashi-Wang's result [44, Theorem 6 (5)].

Corollary 3.5. *Let S be an additively idempotent semiring and M an injective left S-semimodule. Then M is a complete semimodule. If in addition the join-semilattice S is a distributive lattice, then M is a complete and infinitely distributive semimodule.*

Proof. Since M is injective and by Theorem 3.3, there exists a set X such that M is a retract of a left S-semimodule $Hom_{\mathbb{B}}(S, \mathbb{B})^X$; that means, there exist an injective S-homomorphism

$$\alpha : M \longrightarrow Hom_{\mathbb{B}}(S, \mathbb{B})^X$$

and a surjective S-homomorphism

$$\beta : Hom_{\mathbb{B}}(S, \mathbb{B})^X \longrightarrow M$$

such that $\beta\alpha = id_M$.

Recall first that a submonoid A of the monoid $(S, +, 0)$ is called *subtractive* if s, $s + s' \in A$ imply $s' \in A$ for all $s, s' \in S$. In other words, a subtractive submonoid of the monoid $(S, +, 0)$ is a nonempty lower subset of the join-semilattice S (for the natural order) which is closed under the supremum. Then, the intersection of an arbitrary family of subtractive submonoids of the additive monoid S is clearly subtractive. We denote by $\mathcal{K}$ the set of all subtractive submonoids of the additive monoid S. We have that $(\mathcal{K}, \cap, S)$ is a complete $\mathbb{B}$-semimodule with the natural order is defined by for all $A, A' \in \mathcal{K}$, $A \le A'$ if $A \cap A' = A'$ (*i.e.*, $A' \subseteq A$); and for each family $(A_i)_{i\in I} \subseteq \mathcal{K}$, $\bigvee_{i\in I} A_i = \cap_{i\in I} A_i$ and $\bigwedge_{i\in I} A_i$ is the intersection of all subtractive submonoids of the additive monoid S containing all A_i.

We claim that if S is a distributive lattice for the natural order, then the $\mathbb{B}$-semimodule $(\mathcal{K}, \cap, S)$ is infinitely distributive. Indeed, we first show that $\bigwedge_{i\in I} A_i = \sum_{i\in I} A_i$ for any family $(A_i)_{i\in I} \subseteq \mathcal{K}$, where the symbol $\sum$ denotes sum of submonoids. To prove this fact we only need to check that $\sum_{i\in I} A_i$ is a subtractive submonoid of the additive monoid S. Let $s \in S$ and $s', a \in \sum_{i\in I} A_i$ such that $s + s' = a$, i.e., $s \le a$. We write a in the form $a = a_{i_1} + a_{i_2} + \cdots + a_{i_n}$, where $a_{i_k} \in A_{i_k}$. Since the join-semilattice S is a distributive lattice, $s = s \wedge a = s \wedge (\sum_{k=1}^n a_{i_k}) = \sum_{k=1}^n (s \wedge a_{i_k})$. For each $k = 1, .., n$, since A_{i_k} is a lower subset of the join-semilattice S and $a_{i_k} \in A_{i_k}$, $s \wedge a_{i_k} \in A_{i_k}$, so $s \in \sum_{i\in I} A_i$, that is, $\sum_{i\in I} A_i$ is subtractive.

Similarly, we have that $A \cap (\sum_{i\in I} A_i) \subseteq \sum_{i\in I}(A \cap A_i)$ for all $A, A_i \in \mathcal{K}$. The inverse inclusion is obvious, so $A \cap (\sum_{i\in I} A_i) = \sum_{i\in I}(A \cap A_i)$ for all $A, A_i \in \mathcal{K}$, that is, $\mathbb{B}$-semimodule $(\mathcal{K}, \cap, S)$ is infinitely distributive, proving the claim.

For each $f \in Hom_{\mathbb{B}}(S, \mathbb{B})$, $Ker(f)$ is clearly a subtractive submonoid of the additive monoid S. Inversely, for each subtractive submonoid A of the additive monoid S, define a $\mathbb{B}$-homomorphism $f_A : S \longrightarrow \mathbb{B}$ by setting for all $s \in S$,

$$f_A(s) = \begin{cases} 0 & \text{if } s \in A, \\ 1 & \text{otherwise} \end{cases}.$$

We obviously have that $Ker(f_A) = A$. We also note that for all $f, g \in Hom_{\mathbb{B}}(S, \mathbb{B})$ we have $Ker(f+g) = Ker(f) \cap Ker(g)$. From these remarks, one obtains that the assignment $f \longmapsto Ker(f)$, for all $f \in Hom_{\mathbb{B}}(S, \mathbb{B})$, determines an isomorphism of $\mathbb{B}$-semimodules

$$(Hom_{\mathbb{B}}(S, \mathbb{B}), +, 0) \longrightarrow (\mathcal{K}, \cap, S).$$

This implies that $Hom_{\mathbb{B}}(S, \mathbb{B})$ is a complete left S-semimodule, and it is a complete and infinitely distributive left S-semimodule when S is a distributive lattice for the natural order. Therefore, $Hom_{\mathbb{B}}(S, \mathbb{B})^X$ is also a complete left S-semimodule, and it is a complete and infinitely distributive left S-semimodule when S is a distributive lattice for the natural order. We then get similar facts for the semimodule M, by repeating verbatim the argument in the proof of [44, Lemma 2]; and just for the reader's convenience, we briefly sketch it here.

We first claim that M is a complete semimodule and

$$\bigvee_{i \in I} m_i = \beta(\bigvee_{i \in I} \alpha(m_i)), \text{ and } \bigwedge_{i \in I} m_i = \beta(\bigwedge_{i \in I} \alpha(m_i))$$

for all $m_i \in M$. Indeed, we firstly have that $m_i = \beta(\alpha(m_i)) \leq \beta(\bigvee_{i \in I} \alpha(m_i))$ for all $i \in I$. If $m' \in M$ and $m_i \leq m'$ for all $i \in I$, then we have that $\alpha(m_i) \leq \alpha(m')$ for all $i \in I$, so $\bigvee_{i \in I} \alpha(m_i) \leq \alpha(m')$. This implies that $\beta(\bigvee_{i \in I} \alpha(m_i)) \leq \beta(\alpha(m')) = m'$. Therefore, $\bigvee_{i \in I} m_i$ exists in M and is equal to $\beta(\bigvee_{i \in I} \alpha(m_i))$. By a dual argument, we get that $\bigwedge_{i \in I} m_i$ exists in M and is equal to $\beta(\bigwedge_{i \in I} \alpha(m_i))$. Consequently, M is a complete semimodule.

If in addition S is a distributive lattice for the natural order, then $Hom_{\mathbb{B}}(S, \mathbb{B})^X$ is an infinitely distributive semimodule, so we have

$$\begin{aligned} m + \bigwedge_{i \in I} m_i &= \beta(\alpha(m)) + \beta(\bigwedge_{i \in I} \alpha(m_i)) = \beta(\alpha(m) + \bigwedge_{i \in I} \alpha(m_i)) \\ &= \beta(\bigwedge_{i \in I}(\alpha(m) + \alpha(m_i))) = \beta(\bigwedge_{i \in I} \alpha(m + m_i)) \\ &= \bigwedge_{i \in I}(m + m_i) \end{aligned}$$

for all $m, m_i \in M$, so M is an infinitely distributive semimodule, finishing the proof.

□

In [20] Il'in proved that all left S-semimodules over a semiring S are injective if and only if S is a semisimple ring. The following corollary will recover Il'in's result, by using Corollary 3.5.

Corollary 3.6 ([20, Theorem 3.4])**.** *For any semiring S, the following conditions are equivalent:*

(1) *Every left S-semimodule is injective;*
(2) *S is a semisimple ring.*

Proof. (1)$\Longrightarrow$(2). Define a relation ρ on S by setting for all $s, s' \in S$, $s\,\rho\,s'$ if there exist elements $x, y \in S$ and nonnegative integers n and m satisfying $s + x = ns'$ and $s' + y = ms$. It is straightforward to check that ρ is a congruence relation on S such that $s\,\rho\,(s+s)$ for all $s \in S$. Moreover, if τ is a congruence relation on S such that $s\,\tau\,(s+s)$ for all $s \in S$, then $\rho \subseteq \tau$. Indeed, let $s, s' \in S$ with $s\,\rho\,s'$, that is, there exist elements $x, y \in S$ and nonnegative integers n and m satisfying $s + x = ns'$ and $s' + y = ms$. We denote by $[a]_\tau$ the equivalent class of an element $a \in S$ under τ. Since $[a]_\tau = [a+a]_\tau = [a]_\tau + [a]_\tau$ for all $a \in S$, $[s+x]_\tau = [ns']_\tau = [s']_\tau$ and $[s'+y]_\tau = [ms]_\tau = [s]_\tau$, so

$$[s + s' + x + y]_\tau = [s' + s' + y]_\tau = [s' + s']_\tau + [y]_\tau = [s' + y]_\tau = [s]_\tau$$

and

$$[s + s' + x + y]_\tau = [s + s + x]_\tau = [s + s]_\tau + [x]_\tau = [s + x]_\tau = [s']_\tau,$$

showing that $[s]_\tau = [s']_\tau$, that is, $s\,\tau\,s'$. Thus $\rho \subseteq \tau$.

From these observations, we have that $T := S/\rho$ is the maximal additively idempotent quotient of S. Assume that $T \neq 0$. Consider the free left T-semimodule $F := \bigoplus_{i\in\mathbb{Z}^+} T_i$, where $T_i = T$ for all $i \in \mathbb{Z}^+$. We claim that F is not an injective left T-semimodule. Indeed, suppose it is injective. By Corollary 3.5, F is a complete left T-semimodule. Let z be the maximum of F for the natural order. Write z in the form $z = (z_i)_{i\in\mathbb{Z}^+}$, where $z_i \in T_i$ such that $z_i = 0$ for all but finitely many i; that means, there exists a positive integer n such that $z_i = 0$ for all $i \geq n$. Let $x := (x_i)_{i\in\mathbb{Z}^+} \in F$, where $x_n = 1$ and $x_i = 0$ for all $i \neq n$. We then have that $z = x + z \neq z$, a contradiction. Therefore, F is not an injective left T-semimodule.

Let $\pi : S \longrightarrow T$ be the natural surjection. Then, F may be considered as a left S-semimodule by pullback along π, that is, by definition $s \cdot a = s\pi(a)$ for all $s \in S$ and $a \in F$. By [30, Lemma 5.2], F is not an injective left S-semimodule, which contradicts with hypothesis (1). Therefore, $T = 0$, that is, $1\,\rho\,0$. It implies that $1 + x = 0$ for some $x \in S$, so S is a ring. Then, by [32, Theorem 1.2.9], we immediately get that S is a semisimple ring.

(2)$\Longrightarrow$(1). It follows from [32, Theorem 1.2.9]. □

In [44, Theorem 6 (6)] Takahashi and Wang proved that a left semimodule over a chain division semiring is injective if and only if it is complete and infinitely distributive. We will recover this results in terms of another approach. Before doing so, we need the following simple lemma.

Lemma 3.7. *Let S be an additively idempotent division semiring and M a left S-semimodule. Then, for each $X \subseteq M$ and $s \in S \setminus \{0\}$, if $\bigwedge_{x\in X} x$ exists, then $\bigwedge_{x\in X} sx = s\bigwedge_{x\in X} x$.*

Proof. Since $\bigwedge_{x\in X} x \leq x$ for all $x \in X$, we have that $s\bigwedge_{x\in X} x \leq sx$ for all $x \in X$. Let $m \in M$ such that $m \leq sx$ for all $x \in X$. We then have that $s^{-1}m \leq x$ for all $x \in X$, and hence, $s^{-1}m \leq \bigwedge_{x\in X} x$, that means, $m \leq s \bigwedge_{x\in X} x$. This implies that $\bigwedge_{x\in X} sx = s \bigwedge_{x\in X} x$, finishing the proof. $\square$

Corollary 3.8 ([44, Theorem 6 (6)])**.** *Let S be a chain division semiring and M a left S-semimodule. Then, M is injective if and only if it is a complete and infinitely distributive semimodule.*

Proof. ($\Longrightarrow$). Since S is a chain division semiring, it is a distributive lattice for the natural order, so the statement immediately follows from Corollary 3.5.

($\Longleftarrow$). Since S is an additively idempotent semiring and by [28, Theorem 4.1] (see also the proof of Theorem 3.3), the semimodule M may be considered as a subsemimodule of some injective left S-semimodule I. For each $x \in I$, let $I_x := \{m \in M \mid x \leq m\}$. Consider the map $f : I \longrightarrow M$ defined by for any $x \in I$, $f(x) = \bigwedge_{m\in I_x} m$. We always assume that $\bigwedge_{m\in I_x} m = \bigvee_{m\in M} m$ if I_x is the empty set.

We claim that f is an S-homomorphism such that $f|_M = id_M$. Indeed, we first prove that $f(x + x') = f(x) + f(x')$ for all $x, x' \in I$. Indeed, we have that $I_{x+x'} \subseteq I_x$ and $I_{x+x'} \subseteq I_{x'}$, so $f(x) + f(x') \leq f(x + x')$. On the other hand, since M is an infinitely distributive semimodule, we have

$$f(x) + f(x') = \bigwedge_{m\in I_x} m + \bigwedge_{m'\in I_{x'}} m' = \bigwedge\{m + m' \mid m \in I_x, m' \in I_{x'}\}.$$

Furthermore, we always have

$$\bigwedge\{m + m' \mid m \in I_x, m' \in I_{x'}\} \geq \bigwedge_{m\in I_{x+x'}} m = f(x + x'),$$

so $f(x) + f(x') \geq f(x + x')$; that means, $f(x) + f(x') = f(x + x')$.

Let $x \in I$ and $s \in S \setminus \{0\}$. We show that $f(sx) = sf(x)$. Indeed, by Lemma 3.7, we have $sf(x) = s \bigwedge_{m\in I_x} m = \bigwedge_{m\in I_x} sm \geq \bigwedge_{m'\in I_{sx}} m' = f(sx)$. On the other hand, we have

$$\begin{aligned} f(sx) &= \bigwedge\{m' \in M \mid sx \leq m'\} = \bigwedge\{s(s^{-1}m') \in M \mid x \leq s^{-1}m'\} \\ &= s\bigwedge\{s^{-1}m' \in M \mid x \leq s^{-1}m'\} \qquad \text{(by Lemma 3.7)} \\ &\geq s\bigwedge\{m \in M \mid x \leq m\} = sf(x), \end{aligned}$$

that means, $sf(x) = f(sx)$. Therefore, f is an S-homomorphism. Furthermore, since $m \in I_m$ for all $m \in M$, $f(m) = \bigwedge_{m'\in I_m} m' = m$ for all $m \in M$, so $f|_M = id_M$, proving the claim. Since I is injective and by Lemma 3.1, we obtain that M is also injective, finishing the proof. $\square$

Horn and Kimura in [19], and independently Bruns and Lakser in [3], gave an explicit construction for the injective hull of any semilattice with zero (i.e., $\mathbb{B}$-semimodules). Based essentially on this construction, we will give an explicit construction for the injective hull of semimodules over chain division semirings. We firstly recall some important notions and notations.

Definition 3.9. (1) ([19, Definition 3.3]) A family $\{m_i \mid i \in I\}$ of elements of a join-semilattice M is called *distributive* if (1) $\bigwedge_{i\in I} m_i$ exists, and (2) for any $m \in M$, $\bigwedge_{i\in I}(m \vee m_i)$ exists and is equal to $m \vee \bigwedge_{i\in I} m_i$.

(2) ([19, Definition 3.6] and [14, Definition 2]) A subset J of a join-semilattice M is called a *d-ideal* if the following conditions are satisfied: (i) $x \in J$ and $x \leq y$ imply $y \in J$; and (ii) If $\{a_i \mid i \in I\} \subseteq M$ is distributive and $a_i \in J$ for all $i \in I$ then $\bigwedge_{i\in I} a_i \in J$.

The following lemma plays an important role in the construction of the injective hulls of semimodules over chain division semirings. (We note that parts (1) and (2) of Lemma 3.10 has appeared in [19, Lemma 3.7] for semilattices with zero.)

Lemma 3.10. *Let S be an additively idempotent division semiring, M a left S-semimodule, and let $\mathfrak{D}(M)$ be the set of all d-ideals of the join-semilattice M with the natural partial order. Then the following statements hold:*

(1) *$\mathfrak{D}(M)$ is closed under arbitrary intersections;*

(2) *If $< X >$ denotes the smallest d-ideal containing X, then for any nonempty set X of M satisfying condition (i) of Definition 3.9 (2), we have*

$$< X > = \{m \in M \mid m = \textstyle\bigwedge_{i\in I} m_i \text{ for some distributive family } \{m_i \mid i \in I\} \text{ contained in } X\};$$

(3) *$sJ := \{sx \mid x \in J\} \in \mathfrak{D}(M)$ for all $J \in \mathfrak{D}(M)$ and $0 \neq s \in S$;*

(4) *For all $J \in \mathfrak{D}(M)$ and $0 \neq s, s' \in S$, we have $(s + s')J \subseteq sJ \cap s'J$. If in addition S is chain, then $(s + s')J = sJ \cap s'J$;*

(5) *If S is a chain division semiring, the monoid $(\mathfrak{D}(M), \cap, M)$ is an injective left S-semimodule with the scalar multiplication $(s, J) \longmapsto s \cdot J$ from $S \times \mathfrak{D}(M)$ to $\mathfrak{D}(M)$ defined by*

$$s \cdot J = \begin{cases} sJ & \text{if } s \neq 0, \\ M & \text{otherwise} \end{cases} ;$$

(6) *For each $x \in M$ let $x^M := \{m \in M \mid x \leq m\}$. We have that $x^M \in \mathfrak{D}(M)$, $(x + y)^M = x^M \cap y^M$ and $sx^M = (sx)^M$ for all $x, y \in M$ and $0 \neq s \in S$;*

(7) *If S is a chain division semiring, then the map $f : M \longrightarrow \mathfrak{D}(M)$, defined by $f(x) = x^M$ for all $x \in M$, is an injective S-homomorphism.*

Proof. (1) Let $(J_k)_{k\in K}$ be a family of elements of $\mathfrak{D}(M)$. If $x \in \cap_{k\in K} J_k$ and $x \leq y$, then $x \in J_k$ for all $k \in K$, so $y \in J_k$ for all $k \in K$, that is, $y \in \cap_{k\in K} J_k$. If $\{m_i \mid i \in I\} \subseteq M$ is distributive and $m_i \in \cap_{k\in K} J_k$ for all $i \in I$, then for each $k \in K$ we have $m_i \in J_k$ for all $i \in K$, so $\bigwedge_{i\in I} m_i \in J_k$. This implies that $\bigwedge_{i\in I} m_i \in \cap_{k\in K} J_k$. These observations show that $\cap_{k\in K} J_k \in \mathfrak{D}(M)$.

(2) It is done similarly to [19, Lemma 3.7].

(3) Let $J \in \mathfrak{D}(M)$ and $0 \neq s \in S$. We claim that $sJ \in \mathfrak{D}(M)$. Indeed, let $x \in \mathfrak{D}(M)$ and $y \in M$ such that $sx \leq y$. We then have that $x \leq s^{-1}y$, and hence, $s^{-1}y \in J$, since J is a d-ideal of M. This implies that $y = sx'$ for some $x' \in J$, that means, $y \in sJ$. Let $\{sx_i \mid i \in I\} \subseteq M$ be distributive, where $x_i \in J$ for all

$i \in I$. By Lemma 3.7, we have that $s^{-1} \bigwedge_{i\in I} sx_i = \bigwedge_{i\in I} s^{-1}(sx_i) = \bigwedge_{i\in I} x_i$, i.e., $\bigwedge_{i\in I} sx_i = s \bigwedge_{i\in I}$. For each $m \in M$, since $\{sx_i \mid i \in I\}$ is distributive, we get that

$$s \bigwedge_{i\in I}(m + x_i) = \bigwedge_{i\in I}(sm + sx_i) = sm + \bigwedge_{i\in I} sx_i = s(m + \bigwedge_{i\in I} x_i),$$

that means, $\bigwedge_{i\in I}(m + x_i) = m + \bigwedge_{i\in I} x_i$. This implies that $\{x_i \mid i \in I\} \subseteq J$ is distributive, so $\bigwedge_{i\in I} x_i \in J$ and $\bigwedge_{i\in I} sx_i = s \bigwedge_{i\in I} x_i \in sJ$. Thus, sJ is a d-ideal of the join-semilattice M, proving the claim.

(4) Let $J \in \mathfrak{D}(M)$ and $0 \neq s, s' \in S$. We prove that $(s + s')J \subseteq sJ \cap s'J$. Indeed, for each $x \in J$, we always have that $sx \leq (s + s')x$ and $s'x \leq (s + s')x$. Since sJ and $s'J$ are d-ideals of the join-semilattice M, $(s+s')x \in sJ \cap s'J$, that means, $(s + s')J \subseteq sJ \cap s'J$.

If in addition S is chain, then without loss of generality we may assume that $s' \leq s$. We then have that $s + s' = s$ and $(s + s')J = sJ$. On the other hand, for each $x \in J$, we always have $s'x \leq sx$, so $sx \in s'J$, since $s'J$ is a d-ideal. This implies that $sJ \subseteq s'J$, that is, $sJ \cap s'J = sJ = (s + s')J$.

(5) We have that $(\mathfrak{D}(M), \cap, M)$ is a commutative monoid, by item (1). We next claim that $s \cdot (J \cap J') = s \cdot J \cap s \cdot J'$ for all $J, J' \in \mathfrak{D}(M)$ and $s \in S$. Indeed, the claim is obvious if either $s = 0$ or one of these two sets J and J' is empty. Otherwise, for each $m \in s \cdot J \cap s \cdot J'$, there exist $x \in J$ and $x' \in J'$ such that $sx = m = sx'$. Since $s \neq 0$, $x = s^{-1}(sx) = s^{-1}(sx') = x' \in J \cap J'$, so $m \in s \cdot (J \cap J')$, which shows that $s \cdot J \cap s \cdot J' \subseteq s \cdot (J \cap J')$. The inverse inclusion is obvious, thus the claim is proved. Using this observation, and items (3) and (4), we get that $(\mathfrak{D}(M), \cap, M)$ is a left S-semimodule with the above scalar multiplication.

We have that the natural order $\leq$ on the S-semimodule $\mathfrak{D}(M)$ is defined by for all $J, J' \in \mathfrak{D}(M)$, $J \leq J'$ if and only if $J \cap J' = J'$. Equivalently, $J \leq J'$ if and only if $J' \subseteq J$.

We prove that $\mathfrak{D}(M)$ is injective. Indeed, by items (1) and (2), $\mathfrak{D}(M)$ is a complete left S-semimodule, and

$$\bigvee_{i\in I} J_i = \cap_{i\in I} J_i \text{ and } \bigwedge_{i\in I} J_i =< \cup_{i\in I} J_i >$$

for any family $\{J_i \mid i \in I\}$ of d-ideals of the join-semillatice M. Suppose $J \in \mathfrak{D}(M)$ and $J_i \in \mathfrak{D}(M)$ for all $i \in I$. If $x \in J \cap \bigwedge_{i\in I} J_i$, then $x \in J$ and $x \in< \cup_{i\in I} J_i >$. By item (2), $x = \bigwedge_{k\in K} x_k$ for some distributive family $\{x_k \mid k \in K\}$ contained in $\cup_{i\in I} J_i$. Hence, $x_k \in J \cap (\cup_{i\in I} J_i) = \cup_{i\in I}(J \cap J_i)$, so $x \in< \cup_{i\in I}(J \cap J_i) >= \bigwedge_{i\in I}(J \cap J_i)$. This implies that $\bigwedge_{i\in I}(J \cap J_i) \leq J \cap \bigwedge_{i\in I} J_i$. The converse inclusion $J \cap \bigwedge_{i\in I} J_i \leq \bigwedge_{i\in I}(J \cap J_i)$ is obvious, so $\mathfrak{D}(M)$ is a complete and infinitely distributive left S-semimodule. Thus, the left S-semimodule $\mathfrak{D}(M)$ is injective, by Corollary 3.8.

(6) We obviously have that $x^M \in \mathfrak{D}(M)$ for all $x \in M$. Let $x, y \in M$. For each $m \in x^M \cap y^M$ we have that $x \leq m$ and $y \leq m$, so $x + y \leq m + m = m$, that

is, $m \in (x+y)^M$, which shows that $x^M \cap y^M \subseteq (x+y)^M$. The inverse inclusion is obvious, so $x^M \cap y^M = (x+y)^M$.

Let $x \in M$ and $0 \neq s \in S$. For each $m \in sx^M$ we have $m = sm'$ for some $m' \in M$ with $x \leq m'$, so $sx \leq sm' = m$, that is, $m \in (sx)^M$, which implies that $sx^M \subseteq (sx)^M$. Conversely, for each $m \in (sx)^M$ we have $sx \leq m$, so $x \leq s^{-1}m$ and $m = s(s^{-1}m) \in sx^M$. It shows that $(sx)^M \subseteq sx^M$, so $sx^M = (sx)^M$.

(7) We have that $f : M \longrightarrow \mathfrak{D}(M)$, defined by $f(x) = x^M$ for all $x \in M$, is an S-homomorphism, by item (6). Suppose $x, y \in M$ such that $f(x) = f(y)$, that means, $x^M = y^M$. We then have $x \in y^M$ and $y \in x^M$, so $y \leq x$ and $x \leq y$. It implies that $x = y$, thus f is injective. □

The following theorem gives an explicit construction for the injective hull of a semimodule over a chain division semiring. Its proof is based on the proof of [19, Theorem 3.8].

Theorem 3.11. *Let S be a chain division semiring and let M be a left S-semimodule. Then, the S-homomorphism $f : M \longrightarrow \mathfrak{D}(M)$, defined by $f(x) = x^M$ for all $x \in M$, is essential and $\mathfrak{D}(M)$ is the injective hull of M.*

Proof. We show that f is essential. We first have that f is an injective S-homomorphism, by Lemma 3.10 (7). Assume that $g : \mathfrak{D}(M) \longrightarrow N$ is an S-homomorphism such that gf is injective. We prove the following useful claims.

Claim 1. *If $m \in M$, $J \in \mathfrak{D}(M)$ and $gf(m) = g(J)$, then $m = \bigwedge_{x \in J} x$.*

Proof of the claim. We always have the formula

$$gf(m+x) = gf(m) + gf(x) = g(J \cap f(x))$$

for all $x \in M$. If $x \in J$, then $f(x) = x^M \subseteq J$, so $gf(m+x) = gf(x)$, by the above formula. Therefore, $m + x = x$, i.e, $m \leq x$ for all $x \in J$. Assume that $x \leq y$ for all $y \in J$. We then have that $f(x) = x^M \leq y^M = f(x)$ for all $y \in J$, i.e., $y^M \subseteq x^M$ for all $y \in J$. This implies that $J = \cup_{y \in J} y^M \subseteq x^M = f(x)$ and $gf(x+m) = g(J) = gf(m)$. Hence, $x + m = m$, i.e., $x \leq m$. Thus $m = \bigwedge_{y \in J} y$.

Claim 2. *If $m \in M$, $J \in \mathfrak{D}(M)$ and $gf(m) = g(J)$, then $J = f(m)$.*

Proof of the claim. For all $x \in M$, we always have $gf(m+x) = g(J \cap f(x))$. Therefore, by Claim 1, $m + x = \bigwedge(J \cap f(x))$, and in particular $m = \bigwedge_{y \in J} y$.

We prove that $J \cap f(x) = \{x+y \mid y \in J\}$. Indeed, for any $z \in J \cap f(x)$ we have $z \in J$ and $x \leq z$, so $z = x+z \in \{x+y \mid y \in J\}$, that is, $J \cap f(x) \subseteq \{x+y \mid y \in J\}$. Conversely, for any $y \in J$ we have $x \leq x+y$, so $x+y \in J \cap f(x)$, since J is a d-ideal. It implies that $\{x+y \mid y \in J\} \subseteq J \cap f(x)$.

From these observations we have $x + m = \bigwedge_{y \in J}(x+y)$, so J is distributive. Then, $m = \bigwedge_{y \in J} y \in J$, since J is a d-ideal of the join-semilattice M. This shows that $f(m) \geq \bigwedge_{y \in y} f(y)$. The inverse inclusion $f(m) \leq \bigwedge_{y \in J} f(y)$ is obvious. Therefore, $f(m) = \bigwedge_{y \in J} f(y)$. We then have $J = \cup_{y \in J} y^M = \cup_{y \in J} f(y) = \bigwedge_{y \in J} f(y) = f(m)$, hence proving the claim.

We are now ready to show that g is injective. Assume that $g(J_1) = g(J_2)$, where $J_1, J_2 \in \mathfrak{D}(M)$. If $x \in J_1$, then $f(x) = x^M \subseteq J_1$ and

$$gf(x) = g(f(x) \cap J_1) = gf(x) + g(J_1) = gf(x) + g(J_2) = g(f(x) \cap J_2).$$

By Claim 2, we have that $f(x) = f(x) \cap J_2$, so $x^M = f(x) \subseteq J_2$ for all $x \in J_1$. This implies that $J_1 = \cup_{x \in J_1} x^M \subseteq J_2$. Similarly, $J_2 \subseteq J_1$. Hence, $J_1 = J_2$, showing that g is injective. Thus, we get that f is essential.

By Lemma 3.10 (5), $\mathfrak{D}(M)$ is an injective left S-semimodule, so $\mathfrak{D}(M)$ is the injective hull of M, finishing the proof. $\square$

We end this section by giving a complete description of injective semimodules over the semifield of tropical integers. Let $\mathbb{Z}_{\max} := (\mathbb{Z} \cup \{-\infty\}, \max, +, -\infty, 0) \subseteq \mathbb{T}$ be the semifield of tropical integers. (Notice that $\mathbb{Z}_{\max}$ plays an important role in the *Arithmetic Site* which is introduced by Connes and Consani [8]. The reason for this is that the arithmetic site acquires its algebraic structure from its structure sheaf, and the structure sheaf of the arithmetic site is the semiring $\mathbb{Z}_{\max}$ on which the multiplicative monoid $\mathbb{N}^{\times}$ of non-zero positive integers acts by Frobenius endomorphisms.) We extend the semiring structure on $\mathbb{Z}_{\max}$ to a semiring structure on $\bar{\mathbb{Z}}_{\max} := \mathbb{Z}_{\max} \cup \{\infty\}$, where the extra element ∞ satisfies $s \vee \infty = \infty \vee s = \infty$ for all $s \in \mathbb{Z}_{\max}$, $(-\infty) + \infty = -\infty$ and $s + \infty = \infty$ for all $-\infty \neq s \in \mathbb{Z}_{\max}$. Clearly, $\bar{\mathbb{Z}}_{\max}$ is a $\mathbb{Z}_{\max}$-semimodule. The following result provides us with a complete description of injective $\mathbb{Z}_{\max}$-semimodules.

Theorem 3.12. *For any $\mathbb{Z}_{\max}$-semimodule M, the following statements are equivalent:*

1. *M is injective;*
2. *M is a retract of a $\mathbb{Z}_{\max}$-semimodule $(\bar{\mathbb{Z}}_{\max})^X$ for some set X;*
3. *M is a complete and infinitely distributive semimodule.*

Proof. (1)$\Longleftrightarrow$(3). It follows from Corollary 3.8.

(1)$\Longleftrightarrow$(2). By Theorem 3.3, the statement immediately follows from the claim that $\bar{\mathbb{Z}}_{\max} \cong Hom_{\mathbb{B}}(\mathbb{Z}_{\max}, \mathbb{B})$ as $\mathbb{Z}_{\max}$-semimodules.

We now prove the claim. Indeed, for each $x \in \mathbb{Z}_{\max}$, we define a map $f_x : \mathbb{Z}_{\max} \longrightarrow \mathbb{B}$ by setting

$$f_x(t) = \begin{cases} 0 & \text{if } t \leq x, \\ 1 & \text{otherwise} \end{cases}$$

for all $t \in \mathbb{Z}_{\max}$. Then f_x is clearly an $\mathbb{B}$-homomorphism.

For each $f \in Hom_{\mathbb{B}}(\mathbb{Z}_{\max}, \mathbb{B})$, we have that $Ker(f)$ is a lower subset of the lattice $\mathbb{Z}_{\max}$, so $Ker(f) = \mathbb{Z}_{\max}$ or $Ker(f) = \{y \in \mathbb{Z}_{\max} \mid y \leq x\}$ for some $x \in \mathbb{Z}_{\max}$, which shows that $f = 0$ or $f = f_x$. For the convenience, we denote by f_∞ the zero $\mathbb{Z}_{\max}$-homomorphism. We then have that $Hom_{\mathbb{B}}(\mathbb{Z}_{\max}, \mathbb{B}) = \{f_x \mid x \in \bar{\mathbb{Z}}_{\max}\}$. To avoid the confusion we denote the additive operation in the $\mathbb{Z}_{\max}$-semimodule $Hom_{\mathbb{B}}(\mathbb{Z}_{\max}, \mathbb{B})$ by the notation $\boxplus$.

Define a map $\theta : \bar{\mathbb{Z}}_{\max} \longrightarrow Hom_{\mathbb{B}}(\mathbb{Z}_{\max}, \mathbb{B})$ by setting $\theta(x) = f_{-x}$ for all $x \in \bar{\mathbb{Z}}_{\max}$. We claim that θ is a $\mathbb{Z}_{\max}$-homomorphism. Indeed, let $x, y \in \bar{\mathbb{Z}}_{\max}$, and assume that $x \leq y$. We then have that $f_{-x} \leq f_{-y}$ (since $-y \leq -x$) and

$$\theta(x \vee y) = \theta(y) = f_{-y} = f_{-x} \boxplus f_{-y} = \theta(x) \boxplus \theta(y).$$

Let $x \in \bar{\mathbb{Z}}_{\max}$ and $y \in \mathbb{Z}_{\max}$. We have that $\theta(y + x) = f_{-(x+y)}$ and

$$(y\theta(x))(t) = (yf_{-x})(t) = f_{-x}(y + t) = \begin{cases} 0 & \text{if } t \leq -(x + y), \\ 1 & \text{otherwise} \end{cases}$$

for all $t \in \mathbb{Z}_{\max}$, so $y\theta(x) = f_{-(x+y)} = \theta(y + x)$. Therefore, θ is a $\mathbb{Z}_{\max}$-homomorphism. Moreover, θ is clearly a surjective $\mathbb{Z}_{\max}$-homomorphism.

Assume that $x, y \in \bar{\mathbb{Z}}_{\max}$ such that $\theta(x) = \theta(y)$. We then have $f_{-x} = f_{-y}$, so $f_{-x}(-y) = 0 = f_{-y}(-x)$, that is, $-y \leq -x$ and $-x \leq -y$, hence $x = y$, proving that θ is injective, so it is an isomorphism, finishing the proof. $\square$

4. Injectivity of semimodules over chain MV-semirings

In this section, based on Section 3, we give a criterion for self-injectivity of an MV-semiring with an atomic Boolean center, and give a complete description of (finitely generated) injective semimodules over a finite MV-semiring. Consequently, we get that every complete MV-semiring with an atomic Boolean center is an exact semiring. Also, we show that complete Boolean algebras are precisely the MV-semirings in which every principal ideal is injective.

We begin by considering an important example of additively idempotent semirings, the so-called the *MV-semiring*, associated to an MV-algebra. MV-algebras arose in the literature as the algebraic semantics of Łukasiewicz propositional logic, one of the longest-known many-valued logics.

Recall [4] that an *MV-algebra* is an algebra $(A, \oplus, ^*, 0)$ with a binary operation $\oplus$, a unary operation * and a constant 0 such that $(A, \oplus, 0)$ is a commutative monoid with identity element 0, and, for all $x, y \in A$:

(1) $(x^*)^* = x$;
(2) $x \oplus 0^* = 0^*$;
(3) $(x^* \oplus y)^* \oplus y = (y^* \oplus x)^* \oplus x$.

As usual, let A and B be MV-algebras. A map $h : A \longrightarrow B$ is a *homomorphism* iff it satisfies the following conditions, for all $x, y \in A$:

(1) $h(0) = 0$,
(2) $h(x \oplus y) = h(x) \oplus h(y)$,
(3) $h(x^*) = h(x)^*$.

On each MV-algebra A we define the constant 1 and the operation $\odot$ as follows:

$$1 := 0^* \text{ and } x \odot y := (x^* \oplus y^*)^*.$$

For any MV-algebra A and $x, y \in A$, we write $x \leq y$ if there exists an element $z \in A$ such that $x \oplus z = y$. It is well-known that $\leq$ is a partial order on A, called the *natural order* of A. Moreover, the natural order determines a structure of bounded distributive lattice on A [4, Propositions 1.1.5 and 1.5.1], with 0 and 1 respectively bottom and top element, and $\vee$ and $\wedge$ defined by

$$\begin{aligned} x \vee y &= (x \odot y^*) \oplus y, \\ x \wedge y &= (x^* \vee y^*)^* = x \odot (x^* \oplus y). \end{aligned}$$

An MV-algebra A is called an *MV-chain* if the natural order of A is total; and the MV-algebra A is called *complete* if the natural order of A is complete.

A *strong order unit* in a lattice-ordered Abelian group G is an element $0 \leq u \in G$ such that for any $x \in G$, there exists a positive integer n such that $x \leq nu$.

In [40] Mundici constructed a categorical equivalence between the category $\mathcal{MV}$ of MV-algebras with MV-algebra homomorphisms and the category $\mathcal{L}_u$ of lattice-ordered Abelian groups with a distinguished strong order unit whose morphisms are lattice-ordered group homomorphisms which preserve the distinguished strong unit. The two functors of the equivalence are usually denoted by $\Gamma : \mathcal{L}_u \longrightarrow \mathcal{MV}$ and $\Xi : \mathcal{MV} \longrightarrow \mathcal{L}_u$; while the former is very easy to present and shall be recalled hereafter, the latter requires more work and the details of its construction are not really relevant to our discussion.

Let $G = (G, +, -, \leq, \vee, \wedge, 0)$ be a lattice-ordered Abelian group with a distinguished strong order unit u. Then the MV-algebra $\Gamma(G, u)$ is

$$([0,\, u] := \{x \in G \mid 0 \leq x \leq u\},\, \oplus,\, ^*,\, 0)$$

with $x \oplus y = (x + y) \wedge u$ and $x^* = u - x$ for all $x, y \in [0,\, u]$. In this case, the operation $\odot$ in $\Gamma(G, u)$ is defined by $x \odot y = u - (2u - x - y) \wedge u$ for all $x, y \in [0,\, u]$.

The following examples provide us with some fundamental examples of MV-algebras.

Example 4.1. (1) Let $\mathbb{R}$ be the additive groups of reals with the natural order. Then $\Gamma(\mathbb{R}, 1) = [0,\, 1]$ yields an MV-chain, often called the *standard MV-algebra*. In the standard MV-algebra the order relation (and therefore the lattice structure) is the usual one of real numbers; the operations $\oplus$, $*$ and $\odot$ are defined respectively by $x \oplus y = \min\{x + y, 1\}$, $x^* = 1 - x$ and $x \odot y = \max\{x + y - 1, 0\}$ for all $x, y \in [0,\, 1]$.

(2) Let $\mathbb{Z}$ be the additive groups of integers and $n \geq 2$ an integer. Consider the subgroup $\mathbb{Z}\frac{1}{n-1} = \{\frac{z}{n-1} \mid z \in \mathbb{Z}\}$ of the additive group of rationals with the natural order. Then

$$\mathbf{L}_n := \Gamma(\mathbb{Z}\frac{1}{n-1}, 1) = \{0, 1/(n-1), \cdots, (n-2)/(n-1)\}$$

yields an MV-chain with the operations defined as the restriction of the standard MV-algebra of these operations.

(3) For any Boolean algebra $(B, \vee, \wedge, ', 0, 1)$, the structure $(B, \vee, ', 0)$ is an MV-algebra.

In [10, Proposition 3.6] the first author and Gerla proved that for any MV-algebra A, $A^{\vee\odot} := (A, \vee, \odot, 0, 1)$ and $A^{\wedge\oplus} := (A, \wedge, \oplus, 1, 0)$ are additively idempotent commutative semirings isomorphic to each other by the involutive unary operation * of A. This remark allows us to limit our attention to one of these *two semiring reducts* of A; therefore, whenever not otherwise specified, we will refer only to $A^{\vee\odot}$, all the results holding also for $A^{\wedge\oplus}$ up to the application of *.

Definition 4.2. Let A be an MV-algebra. The semiring $A^{\vee\odot} := (A, \vee, \odot, 0, 1)$ is called the *MV-semiring associated with* A (for short, MV-semiring). The semiring $A^{\vee\odot}$ is called a *chain MV-semiring* if A is an MV-chain; and the semiring $A^{\vee\odot}$ is called a *complete chain MV-semiring* if A is a complete MV-chain.

Fact 4.3. *Let A and B be MV-algebras. Then, a map $h : A \longrightarrow B$ is an MV-algebra homomorphism if and only if $h : A^{\vee\odot} \longrightarrow B^{\vee\odot}$ is a semiring homomorphism satisfying $h(x^*) = h(x)^*$ for all $x \in A$.*

Proof. ($\Longrightarrow$). Assume that h is an MV-algebra homomorphism. Clearly, $h(1) = h(0^*) = h(0)^* = 0^* = 1$. Take any $x, y \in A$. We then have that $x \odot y = (x^* \oplus y^*)^*$ and

$$x \vee y = (x \odot y^*) \oplus y = (x^* \oplus y)^* \oplus y,$$

so

$$h(x \odot y) = h(x^* \oplus y^*)^* = (h(x)^* \oplus h(y)^*)^* = h(x) \odot h(y)$$

and

$$h(x \vee y) = h((x^* \oplus y)^* \oplus y) = (h(x) \odot h(y)^*) \oplus h(y) = h(x) \vee h(y).$$

This implies that h is a semiring homomorphism.

($\Longleftarrow$). Assume that h is a semiring homomorphism with $h(x^*) = h(x)^*$ for all $x \in A$. Take any $x, y \in A$. We then have that $x \oplus y = (x^* \odot y^*)^*$, so

$$h(x \oplus y) = h(x^* \odot y^*)^* = (h(x)^* \odot h(y)^*)^* = h(x) \oplus h(y).$$

This implies that h is an MV-algebra homomorphism, finishing the proof. □

As an application of Corollary 3.5, we obtain a necessary condition for injectivity of semimodules over MV-semirings.

Proposition 4.4. *Let A be an MV-algebra and M an injective $A^{\vee\odot}$-semimodule. Then M is a complete and infinitely distributive $A^{\vee\odot}$-semimodule.*

Proof. By [4, Propositions 1.1.5 and 1.5.1], the natural order determines a structure of distributive lattice on the semiring $A^{\vee\odot}$. Then, by Corollary 3.5, we immediately get that M is a complete and infinitely distributive $A^{\vee\odot}$-semimodule, finishing the proof. □

In [14, Theorem 4] Fofanova showed that a semimodule over a Boolean algebra is injective if and only if it is complete and infinitely distributive. It is also well-known (see, e.g., [40, Corollary 1.5.5]) that Boolean algebras are precisely the

MV-algebras satisfying the additional equation $x \oplus x = x$. In the light of these results and Proposition 4.4, it is natural to pose the following question.

Problem 1. Is every complete and infinitely distributive semimodule over an MV-semiring injective?

Recall ([40, Section 1.5]) that an element a of an MV-algebra A is called *idempotent* if $a \oplus a = a$. The set $\mathbf{B(A)}$ of all idempotent elements of an MV-algebra A is a Boolean algebra, usually called the *Boolean center* of the MV-algebra A. Following [40, Definition 6.7.1], an *atom* of an MV-algebra A is an atom of the lattice A for the natural order. We say that A is *atomic* if for each nonzero element $x \in A$ there exists an atom $a \in A$ with $a \leq x$.

In the following theorem we provide criteria for an MV-semiring with an atomic Boolean center is self-injective, which solves a part of Problem 1. To do this, we need the following useful lemmas.

Lemma 4.5. *(1) For each integer $n \geq 2$, $\boldsymbol{L}_n^{\vee\odot} \cong Hom_{\mathbb{B}}(\boldsymbol{L}_n^{\vee\odot}, \mathbb{B})$ as $\boldsymbol{L}_n^{\vee\odot}$-semimodules. Consequently, $\boldsymbol{L}_n^{\vee\odot}$ is a self-injective semiring.*

(2) $[0,\, 1]^{\vee\odot}$ is a self-injective semiring.

Proof. Let $G = (G, +, -, \leq, \vee, \wedge, 0)$ be a lattice-ordered Abelian group with a distinguished strong order unit u. To avoid the confusion we denote the additive operation in the $\Gamma(G,u)^{\vee\odot}$-semimodule $Hom_{\mathbb{B}}(\Gamma(G,u)^{\vee\odot}, \mathbb{B})$ by the notation $\boxplus$.

For each $x \in \Gamma(G,u)^{\vee\odot}$, we define $f_x \in Hom_{\mathbb{B}}(\Gamma(G,u)^{\vee\odot}, \mathbb{B})$ as follows:

$$f_x(t) = \begin{cases} 0 & \text{if } 0 \leq t \leq x, \\ 1 & \text{otherwise.} \end{cases}$$

We claim that the map $\varphi : \Gamma(G,u)^{\vee\odot} \longrightarrow Hom_{\mathbb{B}}(\Gamma(G,u)^{\vee\odot}, \mathbb{B})$, defined by $\varphi(x) = f_{(u-x)}$ for all $x \in \Gamma(G,u)^{\vee\odot}$, is an injective $\Gamma(G,u)^{\vee\odot}$-homomorphism. Indeed, let $x, y \in \Gamma(G,u)^{\vee\odot}$. We have that for all $t \in \Gamma(G,u)^{\vee\odot}$,

$$\begin{aligned} t \leq u - x \ \&\ t \leq u - y &\iff x \leq u - t \ \&\ y \leq u - t \\ &\iff x \vee y \leq u - t \iff t \leq u - (x \vee y), \end{aligned}$$

so $f_{(u-(x\vee y))} = f_{(u-x)} \boxplus f_{(u-y)}$, that means, $\varphi(x \vee y) = \varphi(x) \boxplus \varphi(y)$.

Let $x, y \in \Gamma(G,u)^{\vee\odot}$. We then have that $x \odot y = u - (2u - x - y) \wedge u$ and $\varphi(y \odot x) = f_{(2u-x-y)\wedge u}$. On the other hand, for any $t \in \Gamma(G,u)^{\vee\odot}$, we have that

$$(y\varphi(x))(t) = (yf_{(u-x)})(t) = f_{(u-x)}(y \odot t) = f_{(u-x)}(u - (2u - y - t) \wedge u)$$

and

$$\begin{aligned} u - (2u - y - t) \wedge u \leq u - x &\iff x \leq (2u - y - t) \wedge u \\ &\iff x \leq 2u - y - t \qquad (\textit{since } x \leq u) \\ &\iff t \leq 2u - x - y \\ &\iff t \leq (2u - x - y) \wedge u \quad (\textit{since } t \leq u), \end{aligned}$$

so $(y\varphi(x))(t) = 0$ if and only if $f_{(2u-x-y)\wedge u}(t) = 0$. This implies that $y\varphi(x) = f_{(2u-x-y)\wedge u} = \varphi(y \odot x)$. Therefore, φ is a $\Gamma(G,u)^{\vee\odot}$-homomorphism. Similar to the proof of Theorem 3.12, we get that φ is injective, proving the claim.

(1) Consider the case that $G = \mathbb{Z}\frac{1}{n-1}$ and $\mathbf{L}_n := \Gamma(\mathbb{Z}\frac{1}{n-1}, 1)$. Take any $f \in Hom_{\mathbb{B}}(\mathbf{L}_n^{\vee\odot}, \mathbb{B})$. Since $Ker(f)$ is a lower subset of the lattice $\mathbf{L}_n^{\vee\odot}$ for the natural order, $Ker(f) = \{x \in \mathbf{L}_n^{\vee\odot} \mid 0 \leq x \leq i/(n-1)\}$ for some $0 \leq i \leq n-1$, so $f = f_{i/(n-1)}$. We then have that $\varphi((n-i-1)/(n-1)) = f_{i/(n-1)} = f$, and so φ is surjective. Therefore, φ is an isomorphism $\mathbf{L}_n^{\vee\odot}$-semimodules, giving the statement (1).

(2) Consider the case that $G = \mathbb{R}$ and $\Gamma(\mathbb{R}, 1) = [0,\, 1]$. We claim that the map

$$\theta : Hom_{\mathbb{B}}([0,\, 1]^{\vee\odot}, \mathbb{B}) \longrightarrow [0,\, 1]^{\vee\odot}$$

defined by $\theta(f) = 1 - \bigvee_{t \in Ker(f)} t$ for all $f \in Hom_{\mathbb{B}}([0,\, 1]^{\vee\odot}, \mathbb{B})$, is a $[0,\, 1]^{\vee\odot}$-homomorphism. Indeed, let f and $g \in Hom_{\mathbb{B}}([0,\, 1]^{\vee\odot}, \mathbb{B})$. Then, for any $t \in [0,\, 1]^{\vee\odot}$, $(f \boxplus g)(t) = 0 \iff f(t) \vee g(t) = 0 \iff f(t) = 0 = g(t)$, and so $Ker(f \boxplus g) = Ker(f) \cap Ker(g)$.

Let $x := \bigvee_{t \in Ker(f)} t$ and $y := \bigvee_{t \in Ker(g)} t$. Since $[0,\, 1]^{\vee\odot}$ is an MV-chain, one of the two lower subsets $Ker(f)$ and $Ker(g)$ is included in the other, so $\bigvee\{t \in \Gamma(G)^{\vee\odot} \mid t \in Ker(f) \cap Ker(g)\} = x \wedge y$. Also, since $\Gamma(G)^{\vee\odot}$ is an MV-chain, $1 - (x \wedge y) = (1-x) \vee (1-y)$. This implies that

$$\theta(f \boxplus g) = 1 - (x \wedge y) = (1-x) \vee (1-y) = \theta(f) \vee \theta(y).$$

Let $y \in [0,\, 1]^{\vee\odot}$ and $f \in Hom_{\mathbb{B}}([0,\, 1]^{\vee\odot}, \mathbb{B})$. Then, for each $t \in [0,\, 1]^{\vee\odot}$, we have that

$$(yf)(t) = f(y \odot t) = f(1 - (2 - y - t) \wedge 1) = f((t + y - 1) \vee 0).$$

We show that

$$\bigvee_{t \in Ker(yf)} t = (1 + x - y) \wedge 1,$$

where $x := \bigvee_{t \in Ker(f)} t$. Indeed, for any $t \in [0,\, 1]^{\vee\odot}$ with $(yf)(t) = 0$, we have that $f((t+y-1) \vee 0) = 0$, so $(t+y-1) \vee 0 \leq x$. We also note that $(t+y-1) \vee 0 \leq x \implies t + y - 1 \leq x \implies t \leq 1 + x - y \implies t \leq (1 + x - y) \wedge 1$ (since $t \leq 1$). Therefore, $\bigvee_{t \in Ker(yf)} t \leq (1 + x - y) \wedge 1$.

Take any $a \in [0,\, 1]^{\vee\odot}$ with $a < (1 + x - y) \wedge 1$. We have that $a < 1 + x - y$, that is, $a + y - 1 < x$. If $x \in Ker(f)$ then $(a + y - 1) \vee 0 \leq x$, so $(yf)(a) = f((a + y - 1) \vee 0) = 0$, since $Ker(f)$ is a lower subset of the lattice $[0,\, 1]^{\vee\odot}$. Otherwise, we have that $0 < x$, and so $(a + y - 1) \vee 0 < x$. Then, since $x = \bigvee_{t \in Ker(f)} t$, there exists $t \in Ker(f)$ such that $(a+y-1) \vee 0 < t$, which shows that $(yf)(a) = f((a + y - 1) \vee 0) = 0$, since $Ker(f)$ is a lower subset of $[0,\, 1]^{\vee\odot}$. In any case we have that $a \in Ker(yf)$, and so $\bigvee_{t \in Ker(yf)} t = (1 + x - y) \wedge 1$, giving the fact.

From this observation, we get that

$$\theta(yf) = 1 - (1 + x - y) \wedge 1 = y \odot (1 - x) = y \odot \theta(f).$$

Therefore, θ is a $[0,\, 1]^{\vee\odot}$-homomorphism, proving the claim. Furthermore, for any $x \in [0,\, 1]^{\vee\odot}$, we have that $\theta\varphi(x) = \theta(f_{1-x}) = 1 - (1 - x) = x = id_{[0,\, 1]^{\vee\odot}}(x)$, that is, $\theta\varphi = id_{[0,\,1]^{\vee\odot}}$, and so $[0,\,1]^{\vee\odot}$ is a retract of the $[0,\,1]^{\vee\odot}$-semimodule $Hom_{\mathbb{B}}([0,\,1]^{\vee\odot}, \mathbb{B})$. Then, by Theorem 3.3, we get that $[0,\,1]^{\vee\odot}$ is a self-injective semiring, giving the statement (2), thus the proof is complete. $\square$

The following lemma is an analog of [33, Corollary 3.11B] for our semiring setting.

Lemma 4.6 (*cf.* [33, Corollary 3.11B])**.** *Let $S = \prod_{i\in I} S_i$ be a direct product of semirings S_i. Then S is left self-injective if and only if each S_i is left self-injective.*

Proof. We first note that every left S_i-semimodule may be viewed as a left S-semimodule via the natural projection $S \longrightarrow S_i$. This provides that for each $i \in I$ the left S_i-semimodule S_i is viewed as a left S-semimodule We then have that $S \cong \prod_{i\in I} S_i$ as left S-semimodules. By the dual of [16, Proposition 17.19], S is left self-injective if and only if each S_i is injective left S-semimodule.

We next claim that S_i is injective left S-semimodule if and only if S_i is left self-injective. Indeed, suppose S_i is injective left S-semimodule. Let $f : A \longrightarrow B$ be an injective S_i-homomorphism and $g : A \longrightarrow S_i$ an S_i-homomorphism. Then, by the above note, f and g may be viewed as S-homomorphisms. Since S_i is injective left S-semimodule, there exists an S-homomorphism $h : B \longrightarrow S_i$ such that $hf = g$, giving that S_i is left self-injective.

Conversely, suppose S_i is left self-injective. Let $f : A \longrightarrow B$ be an injective S-homomorphism and $g : A \longrightarrow S_i$ an S-homomorphism. Write $S = S_i \times S_i^c$, where $S_i^c = \prod_{j\in I, j\neq i} S_j$. We then have that $A = S_iA \oplus S_i^cA$ and $A = S_iB \oplus S_i^cB$. By f and g are S-homomorphisms, we have that $f(S_iA) = S_if(A) \subseteq S_iB$, $f(S_i^cA) = S_i^cf(A) \subseteq S_i^cB$ and $g(S_i^cA) = S_i^cg(A) \subseteq S_i^cS_i = 0$. Since S_i is left self-injective, there exists an S_i-homomorphism $h : S_i^cB \longrightarrow S_i$ such that $h \circ g|_{S_iB} = f|_{S_iA}$. We extend h to $h' : B \longrightarrow S_i$ by taking $h'|_{S_i^cB}$. We then get that $h'f = g$. This implies that S_i is an injective left S-semimodule, proving the claim.

From these observations we immediately get that S is left self-injective if and only if each S_i is left self-injective. $\square$

Theorem 4.7. *For any MV-algebra A with an atomic Boolean center, the following conditions are equivalent:*

1. *The semiring $A^{\vee\odot}$ is self-injective;*
2. *All finitely generated projective $A^{\vee\odot}$-semimodules are injective;*
3. *All cyclic projective $A^{\vee\odot}$-semimodules are injective;*
4. *A is a complete MV-algebra.*

Proof. (1)$\Longrightarrow$(2). Suppose $A^{\vee\odot}$ is a self-injective semiring and M is a finitely generated projective $A^{\vee\odot}$-semimodule. Then, by [16, Proposition 17.16], M is a retract of a free $A^{\vee\odot}$-semimodule $(A^{\vee\odot})^X$ with a finite set X. Since $A^{\vee\odot}$ is a self-injective semiring and by [16, Proposition 17.23 (1)], $(A^{\vee\odot})^X$ is an injective $A^{\vee\odot}$-semimodule, so M is also an injective $A^{\vee\odot}$-semimodule, by Lemma 3.1.

(2)$\Longrightarrow$(3). Since every cyclic semimodule is finitely generated, the statement is obvious.

(3)$\Longrightarrow$(4). Since $A^{\vee\odot}$ is a cyclic projective semimodule over itself, and by hypothesis (3), $A^{\vee\odot}$ is a self-injective semiring. From this and Proposition 4.4, we get that A is a complete lattice for the natural order, so A is a complete MV-algebra.

(4)$\Longrightarrow$(1). Since A is a complete MV-algebra with its Boolean center $\mathbf{B}(A)$ is atomic, A is a direct product of complete MV-chains, by [4, Theorem 6.8.1]. Also, by [4, Theorem 6.8.5], every complete MV-chain is either a finite MV-chain or isomorphic to the standard MV-algebra. Furthermore, by [40, Theorem 3.8], every nonzero finite MV-algebra is isomorphic to $\mathbf{L}_n$ for some integer $n \geq 2$. From these observations, Fact 4.3, and Lemmas 4.5 and 4.6, we immediately get the statement, finishing the proof. □

In [46] Wilding, Johnson and Kambites introduced exact semirings, defined in terms of a Hahn-Banach-type separation property on semimodules arising in the tropical case from the phenomenon of tropical matrix duality (see, e.g., [5], [6], [7] and [18]).

We write $M_{m\times n}(S)$ for the additive monoid of m row, n column matrices with entries in a semiring S, where $m, n \in \mathbb{N}$. Matrix multiplication behaves in the usual ways: where defined it is associative and distributes over matrix addition.

For each $A \in M_{m\times n}(S)$ has an associated *row space* $Row(A) = \{x \in M_{1\times n}(S) \mid x = uA \text{ for some } u \in M_{1\times m}(S)\}$ and an associated *column space* $Col(A) = \{y \in M_{m\times 1}(S) \mid y = Av \text{ for some } u \in M_{n\times 1}(S)\}$.

Definition 4.8 ([46, Definition 3.1]). A semiring S is *exact* if for every matrix $A \in M_{m\times n}(S)$, (i) for any matrix $x \in M_{1\times n}(S) \setminus Row(A)$ there exist t and $u \in M_{n\times 1}(S)$ satisfying $At = Au$, but $xt \neq xu$; and (ii) for any matrix $y \in M_{m\times 1}(S) \setminus Col(A)$ there exist v and $w \in M_{1\times m}(S)$ satisfying $vA = wA$, but $vy \neq wy$.

Recall (see, e.g., [27, p. 1895]) that a left S-semimodule M is *FP-injective* if every S-homomorphism $f : X \longrightarrow M$ from a finitely generated left subsemimodule X of a free S-semimodule F can be extended to F. A semiring S is *left (resp. right) FP-injective* if the regular left (resp. right) S-semimodule S is FP-injective. The semiring S is called *FP-injective* if S is both left and right FP-injective. In [27, Lemma 3.1], Johnson and the third author noted that a semiring S is exact if and only if it is FP-injective. In [43] Shitov proved the interesting result that a semifield S is exact if and only if S is either a field or an additively idempotent semifield. As immediate corollary of Theorem 4.7, we get the following result, which provides us with many examples of exact semirings.

Corollary 4.9. *Every complete MV-semiring with an atomic Boolean center is an exact semiring.*

Proof. Let S be a complete MV-semiring with an atomic Boolean center. By Theorem 4.7, S is a self-injective semiring, and so it is FP-injective. Then, by [27, Lemma 3.1], we get that S is an exact semiring, finishing the proof. □

The following theorem gives the structure of injective semimodules over finite MV-semirings.

Theorem 4.10. *Let A be a finite MV-algebra and M a $A^{\vee\odot}$-semimodule. Then M is injective if and only if there exists a set X such that M is a retract of the $A^{\vee\odot}$-semimodule $(A^{\vee\odot})^X$.*

Proof. By Theorem 3.3, an $A^{\vee\odot}$-semimodule M is injective if and only if there exists a set X such that M is a retract of the $A^{\vee\odot}$-semimodule $Hom_{\mathbb{B}}(A^{\vee\odot}, \mathbb{B})^X$. By [4, Proposition 3.6.5], $A \cong \mathbf{L}_{n_1} \times \cdots \times \mathbf{L}_{n_d}$ (as MV-algebras) for some integers $2 \leq n_1 \leq \cdots \leq n_d$, and so the semiring $A^{\vee\odot}$ is isomorphic to $\prod_{i=1}^d \mathbf{L}_{n_i}^{\vee\odot}$, by Fact 4.3. For each $1 \leq i \leq d$, $\mathbf{L}_{n_i}^{\vee\odot}$ may be viewed as an $A_{n_i}^{\vee\odot}$-semimodule via the natural projection $A_{n_i}^{\vee\odot} \longrightarrow \mathbf{L}_{n_i}^{\vee\odot}$. We then have that $A^{\vee\odot} \cong \prod_{i=1}^d \mathbf{L}_{n_i}^{\vee\odot}$ as $A^{\vee\odot}$-semimodules, and

$$Hom_{\mathbb{B}}(A^{\vee\odot}, \mathbb{B}) \cong Hom_{\mathbb{B}}(\textstyle\prod_{i=1}^d \mathbf{L}_{n_i}^{\vee\odot}, \mathbb{B}) \cong \prod_{i=1}^d Hom_{\mathbb{B}}(\mathbf{L}_{n_i}^{\vee\odot}, \mathbb{B})$$

as $A^{\vee\odot}$-semimodules. By Lemma 4.5 (1), we get that $Hom_{\mathbb{B}}(\mathbf{L}_{n_i}^{\vee\odot}, \mathbb{B}) \cong \mathbf{L}_{n_i}^{\vee\odot}$ as $A^{\vee\odot}$-semimodules, and so

$$Hom_{\mathbb{B}}(A^{\vee\odot}, \mathbb{B}) \cong \textstyle\prod_{i=1}^d Hom_{\mathbb{B}}(\mathbf{L}_{n_i}^{\vee\odot}, \mathbb{B}) \cong \prod_{i=1}^d \mathbf{L}_{n_i}^{\vee\odot} \cong A^{\vee\odot}$$

as $A^{\vee\odot}$-semimodules. This implies that $Hom_{\mathbb{B}}(A^{\vee\odot}, \mathbb{B})^X \cong (A^{\vee\odot})^X$ as $A^{\vee\odot}$-semimodules, so the statement is proved, finishing the proof. $\square$

The following theorem provides us with the structure of finitely generated injective semimodules over finite MV-semirings.

Theorem 4.11. *Let A be a finite MV-algebra and M a finitely generated $A^{\vee\odot}$-semimodule. Then the following statements are equivalent:*

(1) *M is injective;*
(2) *M is FP-injective;*
(3) *M is a retract of a $A^{\vee\odot}$-semimodule $(A^{\vee\odot})^X$ for some finite set X;*
(4) *M is projective.*

Proof. (1)$\Longrightarrow$(2). It is obvious.

(2)$\Longrightarrow$(3). As in the proof of Theorem 3.3, there always exists an injective $A^{\vee\odot}$-homomorphism $\mu : M \longrightarrow Hom_{\mathbb{B}}(A^{\vee\odot}, \mathbb{B})^X$, where X is any set of generators for the $A^{\vee\odot}$-semimodule $Hom_{\mathbb{B}}(A^{\vee\odot}, \mathbb{B})$. Since $A^{\vee\odot}$ is finite, the $A^{\vee\odot}$-semimodule $Hom_{\mathbb{B}}(A^{\vee\odot}, \mathbb{B})$ is finitely generated, so we can pick X which is a finite set. Similar to the proof of Theorem 4.10 we have that $Hom_{\mathbb{B}}(A^{\vee\odot}, \mathbb{B})^X \cong (A^{\vee\odot})^X$ as $A^{\vee\odot}$-semimodules. Therefore, we get an injective $A^{\vee\odot}$-homomorphism $\mu : M \longrightarrow (A^{\vee\odot})^X$. Since M is both finitely generated and FP-injective, there exists a surjective $A^{\vee\odot}$-homomorphism $\theta : (A^{\vee\odot})^X \longrightarrow M$ such that $\theta\mu = id_M$; that means, M is a retract of the $A^{\vee\odot}$-semimodule $(A^{\vee\odot})^X$.

(3)$\Longrightarrow$(4). Since X is finite, $A^{\vee\odot}$-semimodule $(A^{\vee\odot})^X$ is free, so the statement follows from [16, Proposition 17.16].

(4)$\Longrightarrow$(1). Since M is both finitely generated and projective, M is a retract of a free $A^{\vee\odot}$-semimodule F with a finite basis set X, by [16, Proposition 17.16]. Now applying Theorem 4.10, we get the statement, finishing the proof. $\square$

As shown in [23, Corollary 3.2], every semimodule can be represented, in a canonical way, as a colimit of its cyclic subsemimodules. This observation motivates the study of semirings over which any semimodule is a colimit of cyclic semimodules possessing some special properties (see, e.g. [1] and [23]). Thus, it is quite natural to present complete characterizations of MV-semirings in terms of the injectivity of cyclic semimodules. Notice that as a corollary of [1, Theorem 4.6], we obtain that all cyclic S-semimodules over an MV-semiring S are injective if and only if S is a finite Boolean algebra. The following result shows that complete Boolean algebras are precisely the MV-algebras in which every principal ideal is injective. Before doing so, we need the following notion and simple fact. A semiring S is called *von Neumann regular* if for any $x \in S$ there exists $y \in S$ such that $x = xyx$.

Lemma 4.12. *If S is a semiring in which every principal left ideal is injective then S is left self-injective and von Neumann regular.*

Proof. Let S be a semiring in which every principal left ideal is injective. Since S is a principal left ideal of S generated by 1, S is left self-injective, by the hypothesis. Take any $x \in S$. Then, by the hypothesis, Sx is an injective left S-semimodule, and so there exists an A-homomorphism $f : S \longrightarrow Sx$ such that $f|_{Sx} = id_{Sx}$. It implies that $x = f(x) = f(x.1) = xf(1)$. Since $f(1) \in Sx$, there exists $y \in S$ such that $f(1) = yx$, and so $x = xyx$. Thus, S is von Neumann regular, finishing the proof. $\square$

Proposition 4.13. *For every MV-algebra A, the following statements are equivalent:*

(1) Every principal ideal of $A^{\vee\odot}$ is injective;
(2) $A^{\vee\odot}$ is a self-injective and von Neumann regular semiring;
(3) A is a complete Boolean algebra.

Proof. (1)$\Longrightarrow$(2). It follows from Lemma 4.12.

(2)$\Longrightarrow$(3). Since $A^{\vee\odot}$ is a self-injective semiring and by Proposition 4.4, the lattice A is complete. Take any $a \in A$. Since $A^{\vee\odot}$ is a von Neumann regular semiring, $a = a \odot b \odot a$ for some $b \in A$. We then get

$$a \vee a \odot a = a \odot b \odot a \vee a \odot a = a \odot (b \vee 1) \odot a = a \odot a.$$

On the other hand,

$$a \vee a \odot a = a \odot (1 \vee a) = a \odot 1 = a.$$

Therefore, $a = a \odot a$ for all $a \in A$. Then, by [4, Theorem 1.5.3], $a = a \oplus a$ for all $a \in A$. This implies that A is a Boolean algebra, by [4, Corollary 1.5.5]. Thus A is a complete Boolean algebra.

(3)$\Longrightarrow$(1). Suppose A is a complete Boolean algebra. Then, by [4, Theorem 1.5.3], the semiring $A^{\vee\odot}$ is also a complete Boolean algebra. By [14, Corollary 2], $A^{\vee\odot}$ is a self-injective semiring. Take any $a \in A$. We have that $a \odot a = a$. Define two $A^{\vee\odot}$-homomorphisms $\alpha : A^{\vee\odot}a \longrightarrow A^{\vee\odot}$ and $\beta : A^{\vee\odot} \longrightarrow A^{\vee\odot}a$ by setting $\alpha(b \odot a) = b \odot a$ and $\beta(b) = b \odot a$ for all $b \in A$. It is obvious that $\beta\alpha = id_{A^{\vee\odot}a}$; that means, $A^{\vee\odot}a$ is a retract of the $A^{\vee\odot}$-semimodule $A^{\vee\odot}$. Since $A^{\vee\odot}$ is self-injective and by Lemma 3.1, $A^{\vee\odot}a$ is an injective $A^{\vee\odot}$-semimodule, and so statement (1) is proved, finishing the proof. $\square$

As was mentioned above, Boolean algebras are precisely the MV-algebras satisfying the additional equation $x \oplus x = x$; that means, Boolean algebras form a subvariety of the variety of MV-algebras which is generated by $\mathbf{L}_2$. In the light of this remark and Proposition 4.12, we end this article by posing the following problem.

Problem 2. Could one describe the subvarieties of the variety of MV-algebras generated by $\mathbf{L}_n$ in terms of the injectivity and projectivity of semimodules?

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Response to referee

We would like to thank the reviewer for his/her positive and insightful comments on the manuscript. Below we list the changes made in the present revision.

(1) We include in the introduction what is the key innovation in our work and what it gives beyond what Wang and Takahashi obtained.

(2) We remove the concept of the subdirect product of semirings in the Preliminaries.

(3) After Fact 2.1, we add a sentence "Notice that an infinite element in an additively idempotent semimodule is exactly a maximum for the natural order."

(4) We add a sentence "The semirings considered in this section are not necessarily commutative" in the top of Section 3.

(5) In the proof of Lemma 3.2, we prove that θ and β are homomorphisms of semimodules in more detail.

(6) In Corollary 3.5, we replace "In particular, if S is a chain semiring, then M is a complete and infinitely distributive semimodule." by "If in addition the join-semilattice S is a distributive lattice, then M is a complete and infinitely distributive semimodule". We would like to thank the reviewer for his/her suggestion "From this perspective, it is fairly obvious that your claim about infinite distributivity holds for any idempotent semiring, not only chain semirings." Unfortunately, we have not proved that the infinite distributivity holds for any idempotent semiring. I have only checked that this property holds for any additively idempotent semiring which is a distributive lattice for the natural order. (This may recover Proposition 4.4.)

(7) In the proof of Corollary 3.5, we add a note that a subtractive submonoid of the additive monoid $(S, +, 0)$ is a lower subset of the join-semilattice S (for the natural order) which is closed under the supremum, and the proof of the fact: for any additively idempotent semiring S which is a distributive lattice for the natural order, $\bigwedge_{i\in I} A_i = \sum_{i\in I} A_i$ for any family $(A_i)_{i\in I}$ of subtractive submonoids of the monoid $(S, +, 0)$.

(8) We re-write the proof of Corollary 3.8 via the reviewer's suggestion. In particular, we add a note that $T := S/\rho$ is the maximal additively idempotent quotient of S.

(9) In Lemma 3.7, we remove item (2).

(10) In the proof of Corollary 3.8, we remove "$\infty := \bigvee_{m\in M} m \qquad$ otherwise" in the definition of f.

(11) We move Theorem 3.9 to the end of Section 3, and we give the short proof of the theorem (please see Theorem 3.12).

(12) We remove the word “nonempty” in Definition 3.9 (2)–It is Definition 3.10 (2) of the old version.

(13) We move the construction of the homomorphism f which is part of the data of an injective hull into Lemma 3.10–It is Lemma 3.11 of the old version.

(14) We re-write Theorem 3.11 (It is Theorem 3.12 of the old version) and its proof via the review.

(15) Before Example 4.1, we add the concept of strong order unit and the categorical equivalence between the category MV-algebras and the category of lattice-ordered Abelian groups with a distinguished strong order unit which was constructed by Mundici.

(16) We remove Proposition 4.5, since it is not important in our article.

(17) After Problem 1, we add the concepts of the Boolean center of an MV-algebra, and atomic MV-algebras.

(18) We extend Theorem 4.6 to an MV-algebra with an atomic Boolean center (please see Theorem 4.7). To do so, we add Lemmas 4.5 and 4.6 into the Section 4.

(19) We make a numbered definition for the notion of exact semiring (see Definition 4.8). Corollary 4.7 of the old version is replaced by Corollary 4.9 “Every complete MV-semiring with an atomic Boolean center is an exact semiring”.

(20) We move Lemma 4.9 of the old version into Lemma 4.5 (1).

(21) We extend Theorems 4.10 and 4.11 to an arbitrary finite MV-algebra.

(22) We make a separate lemma for the argument (1)$\Longrightarrow$(2) of Proposition 4.12 of the old version working for any semiring (please see Lemma 4.12).

(23) Before Lemma 4.12, we add the concept of von Neumann regular semiring.

(24) We add references [5], [6], [7] and [33] into the References. The reference [33] is necessary to Lemma 4.6. Prof. Stéphane Gaubert recommended references [5], [6] and [7] which consist in investigating a Hahn-Banach-type separation property on semimodules, which is a motivation of the concept of exact semiring.