Bayesian estimation and prediction for the transformed gamma degradation process

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Abstract

Very recently, a new degradation process model, named the transformed gamma (TG) process, has been proposed to describe Markovian degradation processes whose increments over disjoint intervals are not independent, so that the degradation growth over a future time interval can depend both on the current age and the current state (degradation level) of the unit. This paper introduces a Bayesian estimation approach for such a process, based on prior information on physical characteristics of the observed degradation process. Several different prior distributions are then proposed, reflecting different degrees of knowledge of the analyst on the observed phenomenon. A Monte Carlo Markov Chain technique is adopted to estimate the TG parameters and some functions thereof, such as the residual reliability of a unit, as well as to predict future degradation growth and residual lifetime. Finally, the proposed approach is applied to a real dataset consisting of wear measures of the liners of the 8-cylinder engine which equips a cargo ship.

Keywords: degradation process; transformed gamma process; Bayesian estimation; degradation growth prediction; MCMC.

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1. Introduction

A large body of literature has addressed the problem of developing stochastic process models able to provide an effective description of real degradation phenomena. Very recently, some stochastic process models have been proposed to describe degradation phenomena where the degradation increment over a future time interval depends on the current state of the unit (and, possibly, on its current age), so that the degradation increments over disjoint time intervals are dependent random variables (Giorgio *et al.*[1-3], Guida and Pulcini[4]).

Within these (Markovian) state-dependent degradation process models, the transformed gamma (TG) process (Giorgio *et al.*[3]) seems to be very attractive due to its mathematical tractability. In particular, unlike the other Markovian state-dependent processes proposed in the literature, the conditional distribution of the degradation growth under the TG process is in closed form. In addition, since the TG process can be viewed as a non-linear transformation of the gamma process (Abdel-Hameed[5]), it constitutes a natural choice for modelling degradation phenomena when degradation growth takes place gradually over time in a sequence of tiny increments. Thus, the TG process seems to be suitable to describe degradation phenomena caused by continuous use, such as wear, chemical corrosion, fatigue, and so on and has proved to be statistically tractable.

Estimation procedures of the TG process parameters based on the maximum likelihood method have been discussed in Giorgio *et al.*[3] and, more recently, in Giorgio *et al.*[6]. On the contrary, Bayesian inference under the TG process has not yet been considered, although engineers often possess prior knowledge on the observed degradation process that can be profitably used to improve the estimation procedures. In this paper, in order to fill in this gap, a Bayesian procedure is proposed that allows for technological information on the observed degradation phenomenon to be directly incorporated into the inferential procedure. In particular, the prior information is formulated in terms of: *a*) the (possible) correlation between the degradation growth in a future time interval and the current degradation level, and *b*) the behavior of the mean degradation curve.

Posterior inference on the process parameters and on several functions thereof, such as the residual reliability and the mean residual life, was carried out by Monte Carlo Markov Chain (MCMC) techniques. Prediction of the degradation increment over a future time interval is also provided. The proposed procedure is then applied to a real dataset given in Giorgio *et al.*[7], consisting of wear measures in the liners of the eight-cylinder engine, which equips a cargo ship of the Grimaldi Lines.

Finally, it must be emphasized that the estimation and prediction procedures provided in this paper can be usefully exploited in defining the optimal condition-based maintenance policy of a degrading unit, by using, for example, the approach proposed in Giorgio *et al.*[7] which is instead based on the maximum likelihood estimators.

2. The transformed gamma process

Let $\eta(t)$ be a non-negative, monotone increasing function of time t, hereinafter called "age function", with $\eta(0) = 0$, and let $g(w)$ be a non-negative, monotone increasing and differentiable function of the degradation level w, hereinafter called "state function", with $g(0) = 0$. An increasing degradation process $\{W(t); t \geq 0\}$ is said to be a TG process with age function $\eta(t)$ and state function $g(w)$ if it possesses the following properties:

1. the degradation increments over disjoint time intervals are (possibly) dependent random variables,
2. the degradation increment $\Delta W(t, t + \Delta t) \equiv W(t + \Delta t) - W(t)$ over the time interval $(t, t + \Delta t)$ depends on the process history up to t through the current time t and the current state (degradation level) $w_t = W(t)$, only,
3. the (conditional) probability density function (pdf) of $\Delta W(t, t + \Delta t)$, given $W(t) = w_t$, can be expressed as:

$$f_{\Delta W(t,t+\Delta t)}(\delta|w_t) = g'(w_t + \delta) \frac{[g(w_t, w_t + \delta)]^{\eta(t,t+\Delta t)-1}}{\Gamma[\eta(t, t + \Delta t)]} \exp[-g(w_t, w_t + \delta)] \,, \qquad \delta > 0 \tag{1}$$

   where $g'(w_t + \delta)$ is the derivative of the state function $g(w)$ evaluated at $w_t + \delta$, $g(w_t, w_t + \delta) = g(w_t + \delta) - g(w_t)$, $\eta(t, t + \Delta t) = \eta(t + \Delta t) - \eta(t)$, and $\Gamma(\cdot)$ is the complete gamma function.

On the one hand, if $\eta(t)$ is linear with t, the (conditional) pdf of $\Delta W(t, t + \Delta t)$ depends on the interval width Δt and not on the current age t, so that the TG process is said to be age-independent. On the other hand, if $g(w)$ is linear with w, the distribution of $\Delta W(t, t + \Delta t)$ does not depend on the current degradation level w_t, and then the TG process reduces to a (state-independent) gamma process.

From (1), the pdf and the cumulative distribution function (Cdf) of the degradation level $W(t)$ at the time t of a new (unused) unit are given, respectively, by

$$f_{W(t)}(w) = g'(w) \frac{[g(w)]^{\eta(t)-1}}{\Gamma[\eta(t)]} \exp[-g(w)] \,, \tag{2}$$

$$F_{W(t)}(w) = \frac{\mathrm{IG}[g(w);\eta(t)]}{\Gamma[\eta(t)]}\ , \tag{3}$$

where $\mathrm{IG}[y;s] = \int_0^y z^{s-1}\exp(-z)\mathrm{d}z$ is the (lower) incomplete gamma function.

Several functional forms for the age and state functions can be chosen. Following Giorgio *et al.*[7], in this paper a power-law function is used both for $\eta(t)$ and for $g(w)$:

$$\eta(t) = (t/a)^b \quad \text{and} \quad g(w) = (w/\alpha)^\beta, \tag{4}$$

where a, b, α, and β are positive parameters. Under such a formulation, the TG process becomes age-independent when $b = 1$, and is state-independent when $\beta = 1$. When $g(w) = (w/\alpha)^\beta$, the mean and variance of the degradation level $W(t)$ are in closed form, and given by:

$$E\{W(t)\} = \alpha\frac{\Gamma[\eta(t)+1/\beta]}{\Gamma[\eta(t)]} = \alpha\frac{\Gamma[(t/a)^b+1/\beta]}{\Gamma[(t/a)^b]} \tag{5}$$

$$V\{W(t)\} = \alpha^2\frac{\Gamma[\eta(t)+2/\beta]}{\Gamma[\eta(t)]} - E^2\{W(t)\} = \alpha^2\left(\frac{\Gamma[(t/a)^b+2/\beta]}{\Gamma[(t/a)^b]} - \frac{\Gamma^2[(t/a)^b+1/\beta]}{\Gamma^2[(t/a)^b]}\right) \tag{6}$$

Unfortunately, even when $g(w) = (w/\alpha)^\beta$, the conditional moments of the degradation increment $\Delta W(t, t+\Delta t)$, given the current level $W(t) = w_t$, are not in closed form, and involve numerical integration. In particular, the conditional mean is given by:

$$E\{\Delta W(t,t+\Delta t)|w_t\} = \int_0^\infty \delta\frac{\beta(w_t+\delta)^{\beta-1}}{\alpha^{\beta\eta(t,t+\Delta t)}}\frac{[(w_t+\delta)^\beta - w_t^\beta]^{\eta(t,t+\Delta t)-1}}{\Gamma[\eta(t,t+\Delta t)]}\exp\left\{-\left(\frac{w_t+\delta}{\alpha}\right)^\beta + \left(\frac{w_t}{\alpha}\right)^\beta\right\}\mathrm{d}\delta \tag{7}$$

In the context of an increasing degradation processes, a unit is assumed to fail when its degradation level exceeds a threshold limit $w_{\max}$. Then, the lifetime T of a unit can be defined as the operating time to the first, and sole, passage beyond the limit $w_{\max}$. The residual lifetime X of a unit still functioning at time t, that is $X = T - t$, is defined as the further operating time it takes the unit to exceed the level $w_{\max}$, when starting from the current degradation level $W(t) = w_t$.

Thus, by using (1), the (conditional) residual reliability $R_t(x|w_t)$, that is the probability that, given the current degradation level $W(t) = w_t$, the level $W(t+x)$ reached at the future time $t+x$ does not exceed $w_{\max}$, is given by:

$$R_t(x|w_t) = \Pr\{W(t+x) \le w_{\max}|w_t\} = \Pr\{\Delta W(t,t+x) \le w_{\max} - w_t|w_t\}$$
$$= \frac{\mathrm{IG}[(w_{\max}/\alpha)^\beta - (w_t/\alpha)^\beta;\ [(t+x)/a]^b - (t/a)^b]}{\Gamma\{[(t+x)/a]^b - (t/a)^b\}}\ . \tag{8}$$

From (3) or from (8), the reliability function of a new (unused) unit is derived:

$$R(t) = \Pr\{W(t) \le w_{\max}\} = \frac{\mathrm{IG}\left[\left(\frac{w_{\max}}{\alpha}\right)^{\beta};\left(\frac{t}{a}\right)^{b}\right]}{\Gamma\left[\left(\frac{t}{a}\right)^{b}\right]}. \tag{9}$$

In addition, from (8), since $\Pr\{\Delta W(t, t+x) \le w_{\max} - w_t | w_t\} = 1 - \Pr\{X < x | w_t\}$, the conditional pdf of the residual lifetime X given the current degradation level $W(t) = w_t$, is:

$$f_X(x|w_t) = -\frac{\mathrm{d}}{\mathrm{d}t}\frac{\mathrm{IG}\left[\left(\frac{w_{\max}}{\alpha}\right)^{\beta} - \left(\frac{w_t}{\alpha}\right)^{\beta};\left[\frac{t+x}{a}\right]^{b} - \left(\frac{t}{a}\right)^{b}\right]}{\Gamma\left\{\left[\frac{t+x}{a}\right]^{b} - \left(\frac{t}{a}\right)^{b}\right\}}. \tag{10}$$

Likewise, from (9), the pdf of the lifetime T is given by:

$$f_T(t) = -\frac{\mathrm{d}}{\mathrm{d}t}\frac{\mathrm{IG}\left[\left(\frac{w_{\max}}{\alpha}\right)^{\beta};\left(\frac{t}{a}\right)^{b}\right]}{\Gamma\left[\left(\frac{t}{a}\right)^{b}\right]}. \tag{11}$$

Both the above pdfs involve the numerical derivative of the (lower) incomplete gamma function. However, by using arguments in Giorgio *et al.*[3], the pdfs in (10) and (11) can be given in a more mathematically convenient form that does not involve derivatives:

$$\begin{aligned} f_X(x|w_t) &= \frac{b}{a}\left(\frac{t+x}{a}\right)^{b-1}\frac{1}{\Gamma\{[(t+x)/a]^b - (t/a)^b\}} \\ &\times\left\{\mathrm{IG}\left[\left(\frac{w_{\max}}{\alpha}\right)^{\beta} - \left(\frac{w_t}{\alpha}\right)^{\beta};\left(\frac{t+x}{a}\right)^{b} - \left(\frac{t}{a}\right)^{b}\right]\left(\psi\left[\left(\frac{t+x}{a}\right)^{b} - \left(\frac{t}{a}\right)^{b}\right] - \ln\left[\left(\frac{w_{\max}}{\alpha}\right)^{\beta} - \left(\frac{w_t}{\alpha}\right)^{\beta}\right]\right)\right. \\ &\left. + \sum_{k=0}^{\infty}\frac{(-1)^k[(w_{\max}/\alpha)^{\beta} - (w_t/\alpha)^{\beta}]^{(t/a)^b + k}}{\{[(t+x)/a]^b - (t/a)^b + k\}^2\, k!}\right\} \end{aligned} \tag{12}$$

$$\begin{aligned} f_T(t) &= \frac{b}{a}\left(\frac{t}{a}\right)^{b-1}\frac{1}{\Gamma[(t/a)^b]} \\ &\times\left\{\mathrm{IG}\left[\left(\frac{w_{max}}{\alpha}\right)^{\beta};\left(\frac{t}{a}\right)^{b}\right]\left(\psi\left[\left(\frac{t}{a}\right)^{b}\right] - \ln\left[\left(\frac{w_{\max}}{\alpha}\right)^{\beta}\right]\right) + \sum_{k=0}^{\infty}\frac{(-1)^k(w_{\max}/\alpha)^{\beta[(t/a)^b + k]}}{[(t/a)^b + k]^2\, k!}\right\}, \end{aligned} \tag{13}$$

where $\psi(z) = \mathrm{d}\ln[\Gamma(z)]/\mathrm{d}z$ denotes the digamma function. The mean lifetime $E\{T\}$ and the mean residual lifetime $E\{X|w_t\}$ can be obtained by integrating the reliability functions (9) and (8), respectively, over the interval $(0, +\infty)$.

3. The likelihood function

Let us suppose that m units operate under identical conditions, and that each unit is inspected n_i times at possibly not equal ages $t_{i,k}$ $(k = 1, \dots, n_i)$. Let $w_{i,k} = W(t_{i,k})$ denote the degradation level of the unit i measured at the k-th inspection time $t_{i,k}$. Then, under the assumption that the degradation process is TG with $\eta(t) = (t/a)^b$ and $g(w) = (w/\alpha)^\beta$, the likelihood function relative to the observed data $\boldsymbol{w} = (w_{1,1}, \dots, w_{1,n_1}, \dots, w_{m,1}, \dots, w_{m,n_m})$ is given by:

$$L(\boldsymbol{\theta}|\boldsymbol{w}) = \prod_{i=1}^{m}\prod_{k=1}^{n_i} f_{\Delta W(t_{i,k-1},t_{i,k})}(w_{i,k} - w_{i,k-1} |\, w_{i,k-1})$$

$$= \left(\frac{\beta}{\alpha}\right)^N \prod_{i=1}^{m}\left(\prod_{k=1}^{n_i}\left(\frac{w_{i,k}}{\alpha}\right)^{\beta-1} \frac{[(w_{i,k}/\alpha)^\beta - (w_{i,k-1}/\alpha)^\beta]^{(t_{i,k}/a)^b-(t_{i,k-1}/a)^b-1}}{\Gamma[(t_{i,k}/a)^b - (t_{i,k-1}/a)^b]}\right) \exp\left[-\left(\frac{w_{i,n_i}}{\alpha}\right)^\beta\right], \tag{14}$$

where $N = \sum_{i=1}^{m} n_i$ is the total number of observations, $t_{i,0} = 0$ and $w_{i,0} = 0$ for all i, and $\boldsymbol{\theta} = (a, b, \alpha, \beta)$ denotes the vector of the TG parameters.

Maximum likelihood estimates of the TG parameters can be obtained by numerical maximization of the log-likelihood function (14), with respect to $\boldsymbol{\theta}$. Approximate confidence intervals for the model parameters can be obtained by using asymptotic results, see Giorgio *et al.*[3].

4. The Bayesian inferential procedure

A Bayesian inferential approach is here proposed that allows technological information on the observed degradation phenomenon to be incorporated in the inferential procedure. Both vague and informative priors are here considered. In particular, the proposed informative priors have been chosen to achieve a good trade-off between simplicity and flexibility. Indeed, although they are indexed by no more than two parameters, they are quite flexible. These distributions are used to model the prior technological information the analysist possesses or might conceivably possess on the shape parameters β and b of the state and age functions, respectively, which characterize the behavior of the TG process.

To this aim, from (7) we note that the behavior of the state function $g(w)$ affects the conditional mean $E\{\Delta W(t, t + \Delta t)|w_t\}$; in particular, if $g(w) = (w/\alpha)^\beta$ is concave, as occurs when $\beta < 1$, then $E\{\Delta W(t, t + \Delta t)|w_t\}$ is a monotone increasing function of the current degradation level $W(t) = w_t$, whereas if $g(w)$ is convex, as occurs when $\beta > 1$, then $E\{\Delta W(t, t + \Delta t)|w_t\}$ decreases monotonically with the current degradation level w_t. This implies (see, e.g., Lehmann[8]) that the degradation increment $\Delta W(t, t+\Delta t)$ during the future time interval $(t, t+\Delta t)$ is positively (negatively) correlated to $W(t)$, given the current age t and Δt, when the shape parameter β is less than 1 (greater than 1). It means, for example, that when $\beta < 1$, the larger the degradation $W(t)$ at age t, the more rapidly it will grow in the future. On the contrary, if $\beta > 1$, the larger $W(t)$, the more slowly it will grow.

For illustrative purpose, Figure 1 shows the coefficient of correlation $\rho(W(t), \Delta W(t, t+\Delta t))$ of $W(t)$ and $\Delta W(t, t+\Delta t)$ evaluated at selected values of β, in particular $\beta = 0.3$, 0.5, 0.7, 0.9, 1.2, 1.5, 2.0, 2.5, 3.0, and 4.0, for arbitrarily chosen values of a, α, t, and Δt which do not affect the behavior of the coefficient of correlation. We have that, regardless of the value of the parameters a, b, and α, the coefficient of correlation is always positive for $\beta < 1$, and always negative for $\beta > 1$. In addition, given a, b, and α, the further β from 1, the greater, in absolute value, the coefficient of correlation. As mentioned before, if $g(w) \propto w$, the process is state-independent and, consequently, $\Delta W(t, t+\Delta t)$ and $W(t)$ are uncorrelated.

Thus, prior information on the dependence of the degradation increment $\Delta W(t, t+\Delta t)$ on the current level $W(t)$ can be easily converted into a prior information on the shape parameter β of the state function $g(w)$ in (4). In particular, the analyst can make one of the following assumptions depending on the available prior information:

1. No information is available on the dependence between $\Delta W(t, t+\Delta t)$ and $W(t)$, and then the (improper) vague prior $g(\beta) \propto 1/\beta$, $\beta > 0$, is adopted.

2. The correlation between $\Delta W(t, t+\Delta t)$ and $W(t)$ is known to be positive:
   a. if this is the only prior information available on $\rho(W(t), \Delta W(t, t+\Delta t))$, then the (improper) vague prior $g(\beta) \propto 1/\beta$, $0 < \beta < 1$ is used;
   b. if the analyst is also able to formulate a prior mean $E\{\beta\}$ and a prior variance $V\{\beta\}$ on β ($0 < \beta < 1$), then the following Beta prior is used:

$$g(\beta) = \frac{\beta^{p-1}(1-\beta)^{q-1}}{B(p,q)}\ , \quad 0 \le \beta \le 1;\ \ p, q > 0\ , \tag{15}$$

   where parameters p and q can be obtained by $p = E^2\{\beta\}(1 - E\{\beta\})/V\{\beta\} - E\{\beta\}$ and $q = p/E\{\beta\} - p$.

3. The correlation between $\Delta W(t, t+\Delta t)$ and $W(t)$ is known to be negative:
   a. if the only prior information available is that $\beta > 1$, then the following 3-parameter gamma distribution, with unit location parameter, is used:

$$g(\beta) = \frac{q^p(\beta-1)^{p-1}}{\Gamma(p)} \exp[-q(\beta-1)], \quad \beta > 1;\; p, q > 0 \tag{16}$$

with parameters $q = (E\{\beta\} - 1)/V\{\beta\}$ and $p = q(E\{\beta\} - 1)$ calculated in correspondence of very large $E\{\beta\}$ and $V\{\beta\}$ values, so that the resulting pdf (16) is "flat" over the region supported by the likelihood;

b. if the analyst is also able to formulate a prior mean $E\{\beta\}$ and a prior variance $V\{\beta\}$ on $\beta > 1$, the prior (16) is still adopted, whose parameters q and p are calculated in correspondence of these prior moments.

4. The correlation between $\Delta W(t, t + \Delta t)$ and $W(t)$ is known to be null or weak: the following gamma prior with mean equal to 1 is used:

$$g(\beta) = \frac{p^p \beta^{p-1}}{\Gamma(p)} \exp(-p\beta)\,, \;\; \beta > 0\,;\;\; p > 0\,, \tag{17}$$

where the parameter p is determined as $p = 1/V\{\beta\}$ by formulating a prior variance $V\{\beta\}$.

Figure 2 depicts the behavior of the mean degradation $E\{W(t)\}$ given in (5) for selected values of the shape parameters b and β, and arbitrarily chosen values of the scale parameters a and α which do not affect the behavior of the mean degradation curve. From these plots, we notice that $E\{W(t)\}$ increases almost linearly with the observation time when $b = \beta$, is concave when $b < \beta$, while is convex when $b > \beta$. In addition, the smaller (larger) the ratio b/β, the more concave (convex) $E\{W(t)\}$ is. Of course, if $b = \beta = 1$, the degradation mean $E\{W(t)\}$ increases exactly linearly, because the TG process with $b = \beta = 1$ is a homogeneous gamma process.

The same conclusions can be derived by approximating the expression of the mean degradation in (5). In particular, by using the asymptotic first order approximation $\Gamma(z + c)/\Gamma(z + d) \cong z^{c-d}$ (Abramowitz and Stegun[9]), we obtain:

$$E\{W(t)\} = \alpha \frac{\Gamma[(t/a)^b + 1/\beta]}{\Gamma[(t/a)^b]} \cong \alpha\, (t/a)^{b/\beta} \quad . \tag{18}$$

From (18), the $E\{W(t)\}$ curve (approximately) increases linearly with the observation time t when $b/\beta = 1$, is concave when $b/\beta < 1$, and is convex when $b/\beta > 1$. Thus, if the observation period is sufficiently large, the behavior of the mean degradation depends (approximately) only on the ratio b/β. It is worth noting that, even when $b = \beta$, although the mean increases almost linearly with the operating time, the TG process does not tend to a (homogeneous) gamma process. Indeed, Figure 3 shows that the variance-to-mean ratio of the TG process with $b = \beta \neq 1$ is not constant with t, as under the gamma process, but varies with t, being monotonically decreasing when $b = \beta < 1$, and monotonically increasing when $b = \beta > 1$.

As a consequence of the above considerations, prior information on the behavior of the $E\{W(t)\}$ curve can be easily converted into (conditional) prior information on b, given β. In particular, depending on the available information, the analyst can make one of the following assumptions:

1. No information is available on the behavior of the mean function, and hence the (improper) vague prior $g(b) \propto 1/b$, $b > 0$, is adopted.
2. Prior information on the ratio b/β is available; it is expressed in terms of the ratio γ between the conditional mean of $b|\beta$ and β, that is $E\{b|\beta\}/\beta = \gamma$, and in terms of the coefficient of variation of $b|\beta$, that is $\sigma(b|\beta)/E\{b|\beta\} = \rho$. The (conditional) gamma prior is used:
$$g(b|\beta) = \frac{{s_\beta}^r b^{r-1}}{\Gamma(r)} \exp[-s_\beta b] \ , \ \ b > 0\,;\ s_\beta, r > 0 \tag{19}$$
with parameters $s_\beta = 1/(\rho^2 \gamma \beta)$ and $r = 1/\rho^2$;
3. It is known that $E\{W(t)\}$ is convex, so that $b > \beta$;
    a. If this is the only information available on b, then the (conditional) 3-parameter gamma prior on $b|\beta$:
    $$g(b|\beta) = \frac{{s_\beta}^r (b-\beta)^{r-1}}{\Gamma(r)} \exp\left[-s_\beta(b-\beta)\right],\ \ b > \beta;\ \ r, s_\beta > 0, \tag{20}$$
    is adopted, with location parameter β, and hyper-parameters values resulting in a very large prior mean and variance, so that the resulting pdf (20) is "flat" over the region supported by the likelihood;
    b. If the analyst is also able to provide prior values of the ratio $E\{b|\beta\}/\beta = \gamma > 1$ and of the coefficient of variation $\sigma(b|\beta)/E\{b|\beta\} = \rho$ (for example, by exploiting information on the mean degradation curve relative to previous data sets of similar units), then pdf (20) is used, where the hyper-parameters are equal to $r = [(\gamma-1)/(\gamma\rho)]^2$ and $s_\beta = (\sqrt{r}/\rho - r)/\beta$.
4. It is known that $E\{W(t)\}$ is concave, and hence that $b < \beta$:
    a. If this is the only information available on b, then the conditional (improper) vague distribution on $b|\beta$:
    $$g(b|\beta) = 1/b\ ,\ \ 0 \le b \le \beta\,, \tag{21}$$
    is used;

b. If the analyst is also able to provide a prior value of the ratio $E\{b|\beta\}/\beta = \gamma < 1$ and a prior value of the coefficient of variation $\sigma(b|\beta)/E\{b|\beta\} = \rho$, then a (conditional) Beta prior on $b|\beta$ is adopted:

$$g(b|\beta) = \frac{b^{r-1}(\beta - b)^{s-1}}{B(r,s)\beta^{r+s-1}} \quad , \quad 0 < b < \beta \; ; \; r, s > 0, \tag{22}$$

where $r = (1-\gamma)/\rho^2 - \gamma$ and $s = r/\gamma - r$.

Finally, we assume that no prior information is available on the scale parameters a and α, so that the Uniform vague prior pdfs over the intervals (0, a_U) and (0, α_U), that is $g(a) = 1/a_U$ and $g(\alpha) = 1/\alpha_U$, respectively, are adopted, where a_U and α_U are sufficiently large values. Thus, the joint posterior pdf of the TG parameters is given by:

$$\pi(a, b, \alpha, \beta|\boldsymbol{w}) \propto L(\boldsymbol{w}|\boldsymbol{\theta}) g(b, \beta)\,, \tag{23}$$

from which the posterior pdf of any process parameter or function thereof can be derived.

From (23), the posterior predictive distribution of the degradation increment $\Delta W_i = W(t_{i,n_i} + \Delta t) - W(t_{i,n_i})$ of unit i during the future time interval $(t_{i,n_i}, t_{i,n_i} + \Delta t)$ given $W(t_{i,n_i}) = w_{i,n_i}$, can be formulated as:

$$f_{\Delta W_i}(\delta|\boldsymbol{w}) = \int_\beta \int_\alpha \int_b \int_a \pi(a, b, \alpha, \beta|\boldsymbol{w}) f_{\Delta W_i}(\delta|w_{i,n_i}) \mathrm{d}a \, \mathrm{d}b \, \mathrm{d}\alpha \, \mathrm{d}\beta \,, \tag{24}$$

where from (1) and (4):

$$f_{\Delta W_i}(\delta|w_{i,n_i}) = \frac{\beta}{\alpha}\left(\frac{w_{i,n_i} + \delta}{\alpha}\right)^{\beta-1} \frac{\{[(w_{i,n_i} + \delta)/\alpha]^\beta - (w_{i,n_i}/\alpha)^\beta\}^{\eta(t_{i,n_i}, t_{i,n_i} + \Delta t) - 1}}{\Gamma[\eta(t_{i,n_i}, t_{i,n_i} + \Delta t)]}$$
$$\times \exp\left\{-\left(\frac{w_{i,n_i} + \delta}{\alpha}\right)^\beta + \left(\frac{w_{i,n_i}}{\alpha}\right)^\beta\right\} \,. \tag{25}$$

Similarly, by using the (approximate) pdf (13) of the lifetime T, we obtain the posterior predictive distribution $f_T(t|\boldsymbol{w})$ of the lifetime T of a new unit:

$$f_T(t|\boldsymbol{w}) = \int_\beta \int_\alpha \int_b \int_a \pi(a, b, \alpha, \beta|\boldsymbol{w}) \, \frac{b}{a}\left(\frac{t}{a}\right)^{b-1} \frac{1}{\Gamma[(t/a)^b]}$$
$$\times \left\{\mathrm{IG}\left[\left(\frac{w_{\max}}{\alpha}\right)^\beta ; \left(\frac{t}{a}\right)^b\right]\left(\psi\left[\left(\frac{t}{a}\right)^b\right] - \ln\left[\left(\frac{w_{\max}}{\alpha}\right)^\beta\right]\right) + \sum_{k=0}^{\infty} \frac{(-1)^k (\mathrm{w}_{\max}/\alpha)^{\beta[(t/a)^b + k]}}{[(t/a)^b + k]^2 \, k!}\right\} \mathrm{d}a \, \mathrm{d}b \, \mathrm{d}\alpha \, \mathrm{d}\beta \tag{26}$$

From the posterior pdf, the posterior mean, which is the most commonly adopted point estimator within the Bayes framework, can be obtained. However, it is well known that the posterior mean is also the Bayes estimator under the squared error loss (SEL) function. Such a loss function is symmetric, and hence its use is generally inappropriate when a point estimate of the reliability function or of the mean lifetime is required, because in these cases overestimation is often much more serious than underestimation. For this reason, an asymmetric loss function, namely the General Entropy loss (GEL) initially proposed in Calabria and Pulcini[10-11]

$$L_{GEL}(\tilde{h}(\boldsymbol{\theta}), h(\boldsymbol{\theta})) \propto \left(\tilde{h}(\boldsymbol{\theta})/\, h(\boldsymbol{\theta})\right)^d - d\ln(\tilde{h}(\boldsymbol{\theta})/\, h(\boldsymbol{\theta})) - 1 \ , \tag{27}$$

is used, where the loss parameter $d > 0$ implies that an overestimation causes more severe consequences than underestimation, and vice versa. The Bayes estimate of $h(\boldsymbol{\theta})$ under the loss (27) is in closed form:

$$h^{GEL}(\boldsymbol{\theta}|\boldsymbol{w}) = \left(E_{h(\boldsymbol{\theta})}\{[h(\boldsymbol{\theta})]^{-d}|\boldsymbol{w}\}\right)^{-\frac{1}{d}} \ , \tag{28}$$

provided that the posterior mean $E_{h(\boldsymbol{\theta})}\{[h(\boldsymbol{\theta})]^{-d}|\boldsymbol{w}\}$ exists and is finite.

The value of the loss parameter d can be set once a suitable value of the ratio r between the expected loss caused by overestimating $h(\boldsymbol{\theta})$ of δ times and the expected loss for an underestimation of δ times, that is

$$r = \frac{L_{GEL}\left(h(\boldsymbol{\theta})\cdot\delta, h(\boldsymbol{\theta})\right)}{L_{GEL}\left(\frac{h(\boldsymbol{\theta})}{\delta}, h(\boldsymbol{\theta})\right)} \ ,$$

is formulated. In particular, d can be obtained solving the following equation:

$$\frac{\delta^d - d\ln(\delta) - 1}{\delta^{-d} + d\ln(\delta) - 1} = r \ . \tag{29}$$

From (28) it is easy to see that $h^{GEL}(\boldsymbol{\theta}|\boldsymbol{w})$ coincides with the posterior mean $E\{h(\boldsymbol{\theta})|\boldsymbol{w}\}$ when $d = -1$. Likewise, when $d = 1$, the Bayes estimate (28) coincides with the Bayes estimate under the weighted squared-error loss function $[\tilde{h}(\boldsymbol{\theta}) - h(\boldsymbol{\theta})]^2/h(\boldsymbol{\theta})$, used for example in Varde[12].

5. The Monte Carlo Markov Chain procedure

The Bayesian inferential procedure presented in Section 3 could be implemented, in line of principle, by adopting numerical multivariate integration that, however, is often unfeasible or highly time consuming in the practice. Thus, in this paper we adopt an MCMC technique for posterior sampling in order to reduce the computational burden and thus the execution time of the computer code. The software package OpenBUGS (Lunn *et al.*[13]), that implements different families of MCMC algorithms, such as Gibbs, Metropolis and slice sampling, is used and the adaptive Metropolis algorithm (Haario *et al.*[14]) is adopted. The main reason of such a choice is that the adaptive Metropolis algorithm typically provides good convergence characteristics in OpenBUGS also in the presence of probability distributions not included in the software tool, such as the distribution of the degradation increment $\Delta W(t, t+\Delta t)$, the 3-parameters gamma prior distribution (16) and the conditional improper prior distribution (21).

We first draw a four-dimensional vector sample of size M, that is $\boldsymbol{\theta}_j = \left(a_j, b_j, \alpha_j, \beta_j\right), j = 1, \ldots, M$, from the posterior pdf $\pi(a, b, \alpha, \beta|\boldsymbol{w})$ in (23), generated after a sufficiently large burn-in period performed to make the influence of the starting point of the numerical procedure negligible. Convergence to the stationary (target) distribution $\pi(a, b, \alpha, \beta|\boldsymbol{w})$ of the Markov Chain is also monitored and assessed.

From the vector sample $\boldsymbol{\theta}_j$, the posterior mean and the $(1-\gamma)$ highest posterior density (HPD) interval of each parameter can be estimated: the former is given by the mean of the corresponding elements of the posterior sample, for instance

$$E\{\alpha|\boldsymbol{w}\} = \int_a \int_b \int_\beta \int_\alpha \alpha\, \pi(a, b, \alpha, \beta|\boldsymbol{w}) \mathrm{d}\alpha\, \mathrm{d}\beta\, \mathrm{d}b\, \mathrm{d}a \;\cong \sum_{j=1}^{M} \alpha_j / M\,,$$

while the latter is obtained by ordering the posterior sample and selecting the shortest interval containing the fraction $(1-\gamma)$ of the sample.

The posterior sample of any function $h(\boldsymbol{\theta})$ of the TG parameters, such as the residual reliability $R_t(\tau|w_t)$ in (8) or the mean degradation $E\{W(t)\}$ in (5), is simply given by $h_j = h(\boldsymbol{\theta}_j)$, $i = 1, \ldots, M$, from which the posterior pdf, the mean and the HPD interval of such a quantity are easily obtained (Robert and Casella[15]).

The posterior sample of the (conditional) degradation increment $\Delta W(t, t+\Delta t)|w_t$, given the current degradation level w_t, is obtained by using the conditional pdf (1). In particular, by applying the method of composition (see, e.g., Tanner[16]), the conditional increment $\delta_j|w_t,\ j = 1, \ldots, M$, given w_t, is obtained by firstly generating a sample of size M, that is $z_j|w_t,\ j = 1, \ldots, M$, from a gamma distribution with unit scale parameter and shape parameter $\eta_j(t, t+\Delta t) = [(t+\Delta t)/a_j]^{b_j} - (t/a_j)^{b_j}$, and then by transforming each element $z_j|w_t$ of the pseudo-random sample by

$$\delta_j|w_t = \alpha_j \left[z_j|w_t + \left(\frac{w_t}{\alpha_j}\right)^{\beta_j} \right]^{\frac{1}{\beta_j}} - w_t\,. \tag{30}$$

A posterior sample of the lifetime T, that is t_j, $i = 1, \dots, M$, is obtained by first generating a sample of size M, that is u_j, $j = 1, \dots, M$, from a Uniform standard distribution, and then by searching the value of t_j such that

$$\frac{\mathrm{IG}[(w_{\max}/\alpha_j)^{\beta_j}; (t_j/a_j)^{b_j}]}{\Gamma[(t_j/a_j)^{b_j}]} - u_j = 0 . \tag{31}$$

Likewise, a posterior sample of the (conditional) residual lifetime $X|w_t$, that is x_j, $i = 1, \dots, M$, is obtained by the uniformly distributed random sample u_j, $j = 1, \dots, M$, by searching the value of x_j such that

$$\frac{\mathrm{IG}[(w_{\max}/\alpha)^{\beta} - (w_t/\alpha)^{\beta};\ [(t + x_j)/a]^b - (t/a)^b]}{\Gamma\{[(t + x_j)/a]^b - (t/a)^b\}} - u_j = 0 . \tag{32}$$

From the posterior samples obtained from (30)-(32), the posterior pdf, the posterior mean, and the $(1-\gamma)$ HPD interval of the conditional increment $\Delta W(t, t + \Delta t)|w_t$, of the lifetime T, and of the conditional residual lifetime $\mathrm{X}|w_t$, respectively, are easily derived.

Finally, from (28), the Bayes estimate under the GEL function of the generic function $h(\boldsymbol{\theta})$ is given by:

$$h^{GEL}(\boldsymbol{\theta}|\boldsymbol{w}) = \left(\frac{\sum_{j=1}^{M} \left[h_j(\boldsymbol{\theta})\right]^{-d}}{M}\right)^{-\frac{1}{d}} . \tag{33}$$

6. Numerical application

Let us now consider the wear measures, given in Table 1, of the liners of the 8-cylinder engine which equips a cargo ship of the Grimaldi Lines. A total of 23 inspections were carried out during a total operating time of 185,000 hours. Due to the caliper sensitivity, all of the wear measures are rounded up to the nearest multiple of 0.05 mm. In Figure 4, the observed paths are depicted, where the measured points are linearly connected for graphical display.

This wear dataset was analyzed within the TG process in Giorgio *et al.*[7] in order to illustrate a condition-based maintenance policy for deteriorating units. Maximum likelihood estimates were then obtained. The liners were assumed to fail when the accumulated wear exceeds the threshold value $w_{\max} = 4$ mm.

In order to perform the Bayesian estimation and prediction procedure, we have to formulate the prior information on the TG parameters. We then assume that the analyst, on the basis of previously observed similar wear processes, knows that: *a*) the wear increment $\Delta W(t, t + \Delta t)$ during a future time interval is strongly negatively correlated to the current level $W(t)$, so that β should be larger than 1, and *b*) mean degradation curve is quite concave, so that $b \ll \beta$.

On the basis of previous data sets relative to similar wear processes, the analyst is also able to provide prior values of the mean and variance of β, that is $E\{\beta\} = 3.0$ and $V\{\beta\} = 1.0$, and prior values of the ratio $\gamma = E\{b|\beta\}/\beta = 0.7$ and of the coefficient of variation $\rho = \sigma(b|\beta)/E\{b|\beta\} = 0.2$. In particular, the prior values of γ and ρ have been derived from the behavior of the mean degradation curve relative to data sets of similar wear processes. Thus, the analyst chooses:

- the 3-parameter gamma pdf (16) with parameters $q = (E\{\beta\} - 1)/V\{\beta\} = 2$ and $p = q(E\{\beta\} - 1) = 4$ as prior on β, and
- the Beta prior (22) with parameters $r = (1-\gamma)/\rho^2 - \gamma = 6.8$ and $s = r/\gamma - r = 2.9143$ as conditional prior on $b|\beta$.

The parameters a_U and α_U of the Uniform vague prior pdfs on a and α are set equal to 10^5 and 10^2, respectively, so as to ensure that these priors are "flat" over the region supported by the likelihood. Thus, the joint prior pdf on the TG parameters results in:

$$g(a, b, \alpha, \beta)$$
$$\propto \begin{cases} \dfrac{b^{r-1}(\beta - b)^{s-1}(\beta - 1)^{p-1}}{\beta^{r+s-1}} \exp[-q(\beta - 1)], & \text{for } a < 10^5, b < \beta, \alpha < 10^2, \beta > 1 \\ 0, & \text{elsewhere} \end{cases} \tag{34}$$

with $q = 2$, $p = 4$, $r = 6.8$, and $s = 2.9143$.

The inferential procedure used to estimate the TG parameters from the joint posterior pdf (23) is based on the adaptive Metropolis algorithm implemented in the OpenBUGS software. Given that the distributions of degradation increment and the prior (16), as well as the improper prior (21) considered in the last part of this section, are not included in the software library, we implemented some "zero tricks" procedures in the OpenBUGS code for the likelihood (14) and the nonstandard prior distributions. For further information on zero tricks, refer to Lunn *et al.*[17] and Méndez-González *et al.*[18].

The OpenBUGS code is launched by an R code (R Core Team[19]), that exploits R2OpenBUGS (Sturtz *et al.*[20]) and CODA (Plummer *et al.*[21]) packages to run OpenBUGS, to import the generated samples and to check MCMC convergence.

Convergence to the stationary posterior distribution is assessed by visual inspection of some plots (trace plots, running mean plots) and by running some diagnostic tests. In particular, the Gelman-Rubin convergence tool (with three chains), the Geweke test, the Raftery-Lewis diagnostics and the

Heidelberger-Welch stationary and half-width tests are performed and the autocorrelation function of all the parameters are computed (Robert and Casella[15]).

In order to collect posterior samples of (a, b, α, β) composed by $M = 10^5$ four-dimensional vector elements, we use a burn-in period of $2{\cdot}10^5$ iterations and a thinning interval equal to 300, that guarantee convergence of the MCMC algorithm to the target distribution, a proper mixing, and a negligible correlation between consecutive points of the Markov chain, for all the prior models of interest. The execution time of the OpenBUGS routine depends on the adopted priors, and is between 60 and 300 minutes (for each chain) on a notebook based on an Intel® Core™ i7 CPU@2.60GHz, showing the feasibility of the proposed Bayesian MCMC procedure.

The Bayesian inferential procedure on the posterior sample is implemented by an R code, as described in Section 5. The posterior means of the process parameters are $E\{a|\boldsymbol{w}\} = 6794$ hours, $E\{b|\boldsymbol{w}\} = 1.870$, $E\{\alpha|\boldsymbol{w}\} = 0.9562$ mm and $E\{\beta|\boldsymbol{w}\} = 2.662$, while the corresponding 0.90 HPD intervals are (3307 h, 10110 h), (1.318, 2.401), (0.5433 mm, 1.370 mm), and (1.739, 3.541), respectively. For a comparative purpose, the ML estimates given in Giorgio *et al.*[7] are $\hat{a} = 5107$ hours, $\hat{b} = 1.701$, $\hat{\alpha} = 0.750$ mm, and $\hat{\beta} = 2.31$, whereas the approximate 0.90 confidence intervals, based on the log-normal approximation for the distribution of the ML estimators of the (positive) parameter, are, respectively (1985 h, 13138 h), (1.077, 2.686), (0.322 mm, 1.746 mm), and (1.305, 4.099).

It should be noted that the HPD interval of b does not include the value 1, and that the lower limit is greater than 1. Thus, the wear increment of the cylinder liners is positively correlated to the current age. Indeed, from (7), we have that, given the current degradation level w_t, the conditional mean $E\{\Delta W(t, t + \Delta t)|w_t\}$ increases (decreases) with the current time t when the shape parameter b of the age function is larger than 1 (smaller than 1). This implies (see, e.g., Lehmann[8]) that, given $W(t) = w_t$, the wear increment $\Delta W(t, t + \Delta t)$, in the future interval of width Δt is positively (negatively) correlated to the age t at which the liner reaches the (given) wear level w_t when $b > 1$ ($b < 1$). We also note that all the HPD intervals are quite narrower than the corresponding confidence intervals, thus showing how the Bayes procedure based on informative priors provides more accurate estimates.

In Figure 5, the joint prior $g(\beta, b)$ and the marginal joint posterior pdf $\pi(\beta, b|\boldsymbol{w})$ are compared to depict the effect of the observed data on the knowledge on the shape parameters b and β. The plot of the posterior pdf also shows that the shape parameters β and b are, a posteriori, strongly positively correlated.

In Figure 6 the posterior mean and the 0.90 HPD interval of the mean and variance of the wear level $W(t)$ are depicted, and compared to the empirical estimates. Note that, since the inspection times can vary from unit to unit, and hence the wear measures generally refer to different operating time of the liners, the empirical estimate of the variance is obtained by using an interpolation procedure at selected equispaced times as suggested in Giorgio *et al.*[1]. From Figure 6(a) we have that the posterior mean of the mean wear $E\{W(t)\}$ is very close to the empirical estimate, and that the 0.90 HPD interval is very narrow and includes all the empirical estimates. In addition, from Figure 6(b), we note that the empirical estimate of the variance does not monotonically increase with the age t, as it should happen if the degradation process were state-independent. Thus, only a degradation model which is not purely age-dependent can adequately describe the observed wear process. The posterior mean of the wear variance reproduces the non-monotone behavior of the empirical estimates, even if increases initially more quickly that the empirical estimates and decreases quite more slowly. However, it should be considered that the empirical estimates of the variance are based on a few number of "points", and that the "points" are not observed but obtained through linear interpolation of the observations.

Figures 7 and 8 give the posterior mean and the Bayes estimator under the GEL function, with $d = 2.87$, of the reliability function (9) of a new liner and of the residual reliability (8) of the liners #1, 5, and 6, with $w_{\max} = 4$ mm. The value $d = 2.87$ has been obtained from (34), having assumed that an overestimation of 10% ($\delta = 1.1$) produces an expected loss which is 1.2 times larger than the expected loss caused by an underestimation of 10%. Note that, because the GEL parameter $d > 0$, the Bayes estimate under the GEL function is smaller than the posterior mean, and decreases with the age t very sharply. The very sharply decline in reliability under the GEL function, depicted in Figures 7 and 8, is due to the adopted value of d that strongly penalizes the reliability overestimation; a smaller positive value of d would produce less rapid declines.

We have also predicted the lifetime of a new liner and the residual lifetime of the liners #1-8. In Table 2, the posterior mean and the 0.90 HPD interval are given.

In Table 3, the posterior mean is compared to the Bayes estimator under the GEL function, with $d = 2.16$. Such a d value has been obtained by assuming that an underestimation of 20% ($\delta = 1.2$) of the mean lifetime produces an expected loss $r = 1.3$ times greater than that caused by an underestimation of 20%. We have that the GEL estimates are equal to 95%-98% of the posterior means.

In Figure 9 the posterior predictive distribution (24) of the wear increment $\Delta W(t_{i,n_i}, t_{i,n_i} + \Delta t)$ during the future time interval of width $\Delta t = 20{,}000$ hours, relative to the liners #3, 6, and 8, are depicted. The dependence of the growth of the wear process on the current wear level and current age is there highlighted. In particular, the wear increment relative to the liner #3 is much larger than the increment of liner #8 because, although their current age is the same, the current wear level $w_{3,2} = 1.35$ mm of liner #3 is smaller than the current wear level $w_{8,4} = 2.10$ mm of liner #8 and the wear increment is negatively correlated to the current level (the shape parameter β is larger than 1). Likewise, the wear increment relative to the liner #6 is much larger than the increment of liner #8 because, although their current wear level is about the same, the current age $t_{6,3} = 24{,}710$ hours of liner #6 is larger than the current age $t_{8,4} = 16{,}300$ hours of liner #8 and this wear increment is positively correlated to the current age because the shape parameter b is larger than 1.

Finally, in order to make a check on the effect of the prior information on b and β on the posterior results, we have performed the Bayes estimation assuming that the analyst possesses only little information on the wear process, in particular that:

- there is a negative correlation between $\Delta W(t, t + \Delta t)$ and $W(t)$, without being able to anticipate any plausible value for β, and hence the 3-parameter gamma pdf (16) with prior mean $E\{\beta\} = 100 + 1 = 101$ and prior variance $V\{\beta\} = 10^4$ (so that $p = 1$ and $q = 0.01$) is used, and
- the mean wear curve $E\{W(t)\}$ is surely concave, without being able to anticipate any plausible value for the ratio b/β, and hence the conditional (improper) vague prior (21) on $b|\beta$, in $0 < b < \beta$, is used.

Thus, using the uniform prior pdfs $g(a) = 1/a_U$ and $g(\alpha) = 1/\alpha_U$ over the intervals (0, a_U) and (0, α_U), respectively, where $a_U = 10^5$ and $\alpha_U = 10^2$, the resulting joint prior pdf is:

$$g(a, b, \alpha, \beta) \propto \begin{cases} \dfrac{\exp[-q(\beta - 1)]}{b}, & \text{for } a < 10^5, b < \beta, \alpha < 10^2, \beta > 1 \\ 0 \quad , & \text{elsewhere} \end{cases} \tag{35}$$

with $q = 0.01$. Under the weak joint prior (35), the posterior means are $E\{a|\boldsymbol{w}\} = 6585$ hours, $E\{b|\boldsymbol{w}\} = 1.815$, $E\{\alpha|\boldsymbol{w}\} = 0.9407$ mm and $E\{\beta|\boldsymbol{w}\} = 2.641$, and the corresponding 0.90 HPD intervals are (2266 h, 10650 h), (1.124, 2.452), (0.3886 mm, 1.474 mm), and (1.399, 3.831), respectively. Such posterior means are close to the posterior means under the strong informative prior pdf (34), whereas these HPD intervals are much wider that the intervals under the pdf (34), due to the little informative nature of the weak prior pdf (35).

7. Conclusions

In this paper, a Bayesian estimation procedure for the parameters of the transformed gamma (TG) degradation process has been proposed, when physical/technological prior information on the correlation between the future degradation increment and the current state and on the behavior of the mean degradation curve is available. The use of different types of prior distributions, reflecting different degrees of information on the degradation process under study, has been proposed and motivated. Computations have been performed using a Monte Carlo Markov Chain technique.

The posterior distribution of the parameters of the TG process, as well as of other quantities of interest such as the residual reliability, has been derived. From these posterior distributions, the posterior mean and the 0.90 highest posterior density credibility interval have been obtained. The Bayes estimates of the reliability function and unit lifetime under the (asymmetric) General Entropy loss function have been derived, too. Prediction of the liner lifetime and of the degradation increment over a future time interval, in the case of both new and used liners, has been also specifically addressed. The applicative example, referring to the wear process of the liners of a cargo ship engine, demonstrates the feasibility of the suggested Bayesian procedure.

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Table 1. Wear $w_{i,k}$ [mm] accumulated by liner i up to the inspection time $t_{i,k}$ [hours].

i	$t_{i,1}$	$w_{i,1}$	$t_{i,2}$	$w_{i,2}$	$t_{i,3}$	$w_{i,3}$	$t_{i,4}$	$w_{i,4}$
1	11,300	0.90	14,680	1.30	31,270	2.85		
2	11,300	1.50	21,970	2.00				
3	12,300	1.00	16,300	1.35				
4	14,810	1.90	18,700	2.25	28,000	2.75		
5	10,000	1.20	30,450	2.75	37,310	3.05		
6	6,860	0.50	17,200	1.45	24,710	2.15		
7	2,040	0.40	12,580	2.00	16,620	2.35		
8	7,540	0.50	8,840	1.10	9,770	1.15	16,300	2.10

Table 2. Prediction of the lifetime [in hours] of new and used liners.

	Posterior mean	0.90 HPD interval
New liner	51,327	(40,652 , 61,703)
Liner #1	20,003	(12,110 , 27,828)
Liner #2	30,749	(21,107 , 40,156)
Liner #3	36,841	(26,793 , 47,003)
Liner #4	22,210	(13,470 , 30,170)
Liner #5	16,057	(9,101 , 22,954)
Liner #6	28,484	(19,088 , 37,628)
Liner #7	31,215	(20,970 , 40,938)
Liner #8	33,207	(23,139 , 43,089)

Table 3. Posterior mean and Bayes estimator under the GEL function of the life [in hours] of new and used liners.

	Posterior mean	GEL estimate
New liner	51,327	50,715
Liner #1	20,003	19,103
Liner #2	30,749	29,835
Liner #3	36,841	36,027
Liner #4	22,210	21,266
Liner #5	16,057	15,221
Liner #6	28,484	27,557
Liner #7	31,215	30,206
Liner #8	33,207	32,292

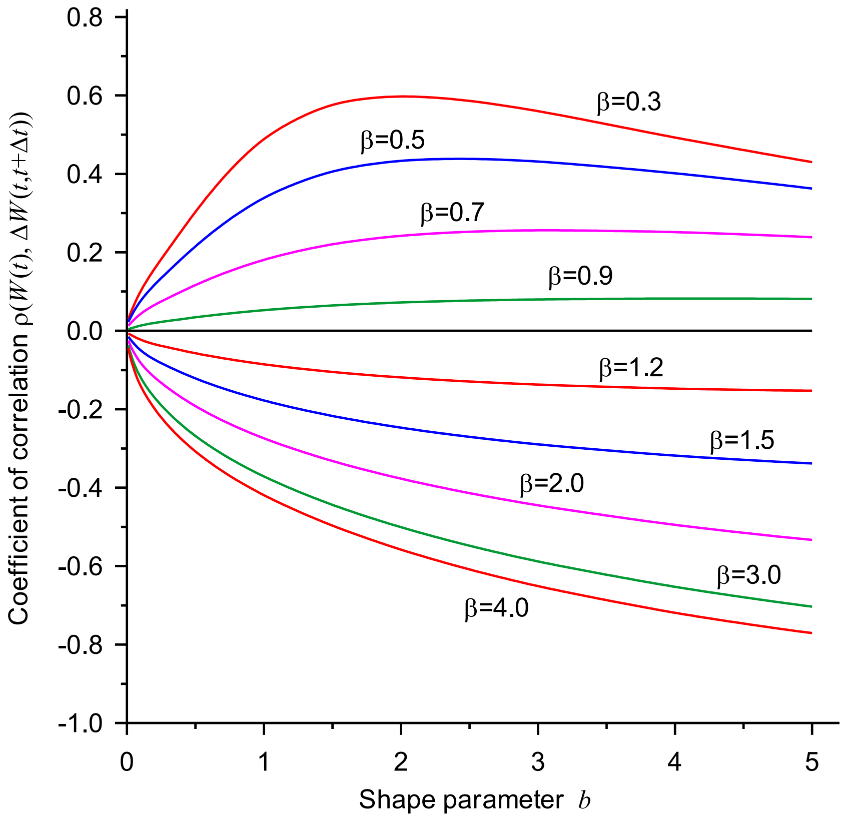

Figure 1. Plots of the coefficient of correlation between $W(t)$ and $\Delta W(t, t+\Delta t)$, for arbitrary values of a, α, t, and Δt.

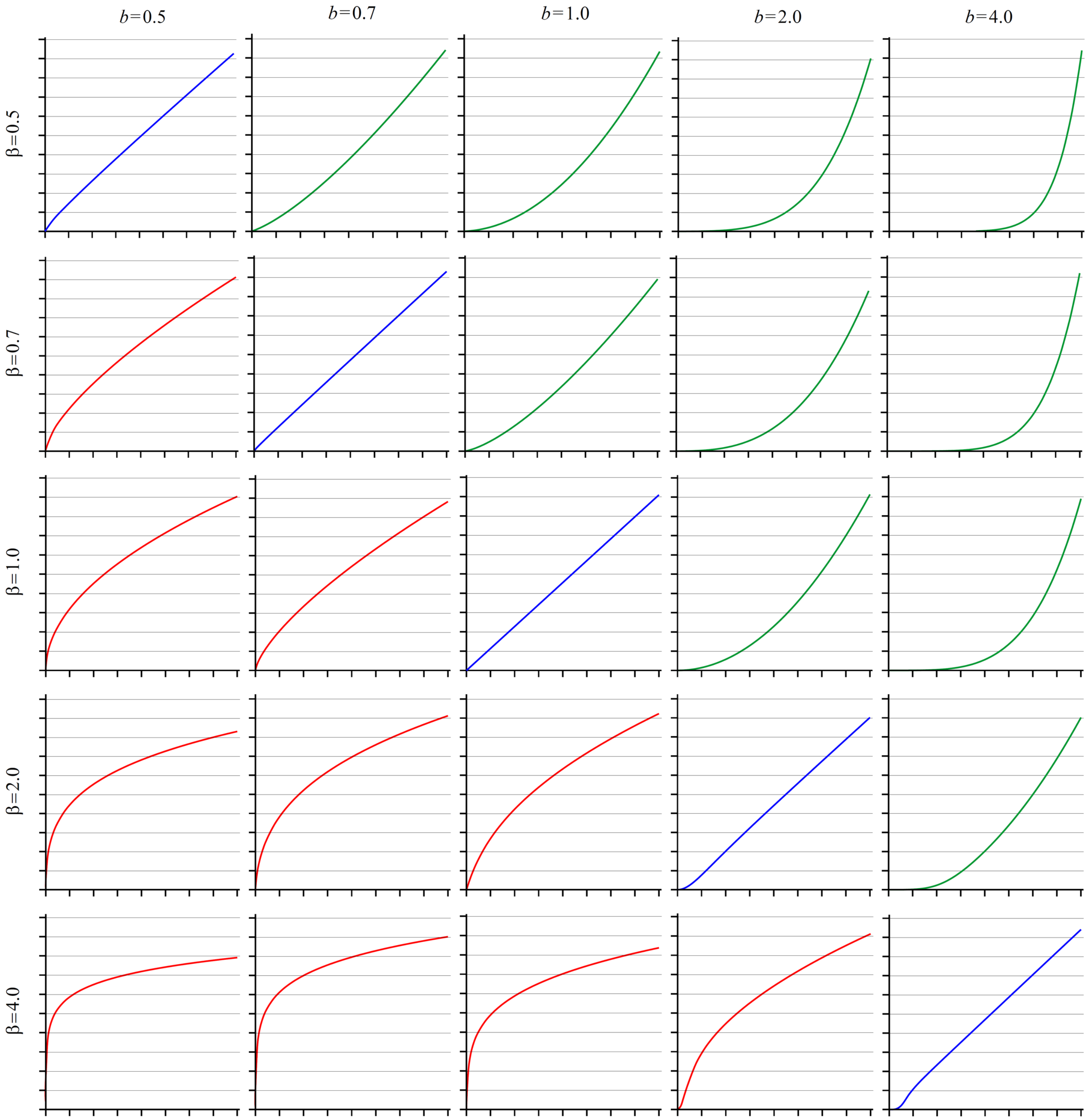

Figure 2. Behavior of the mean degradation $E\{W(t)\}$ for different values of the parameters b and β (the blue, green, and red curves refer to, respectively, $b=\beta$, $b>\beta$, and $b<\beta$).

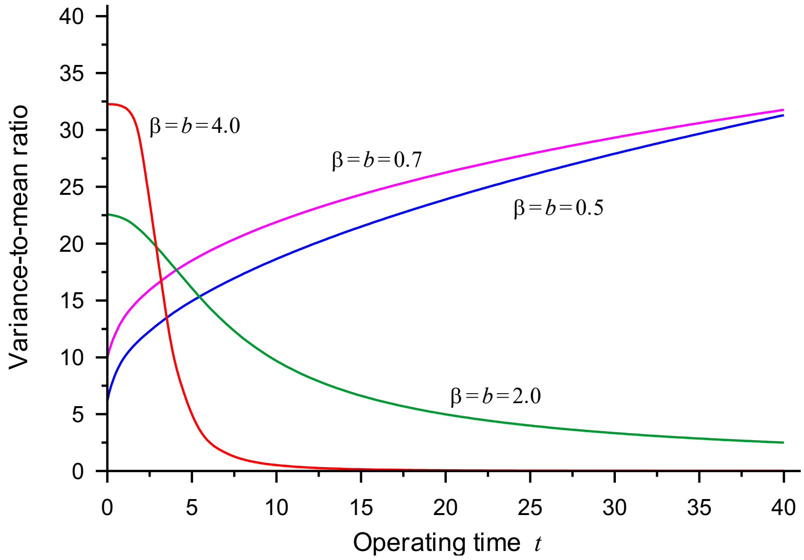

Figure 3. The variance-to-mean ratio of the TG process for different values of $b = \beta$.

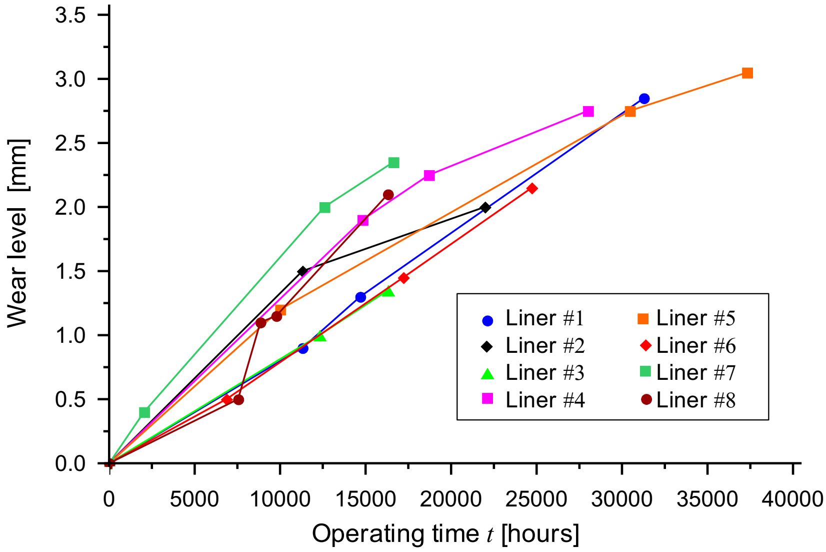

Figure 4. Observed paths of the liner wear (the measured points are linearly connected for graphical opportunity).

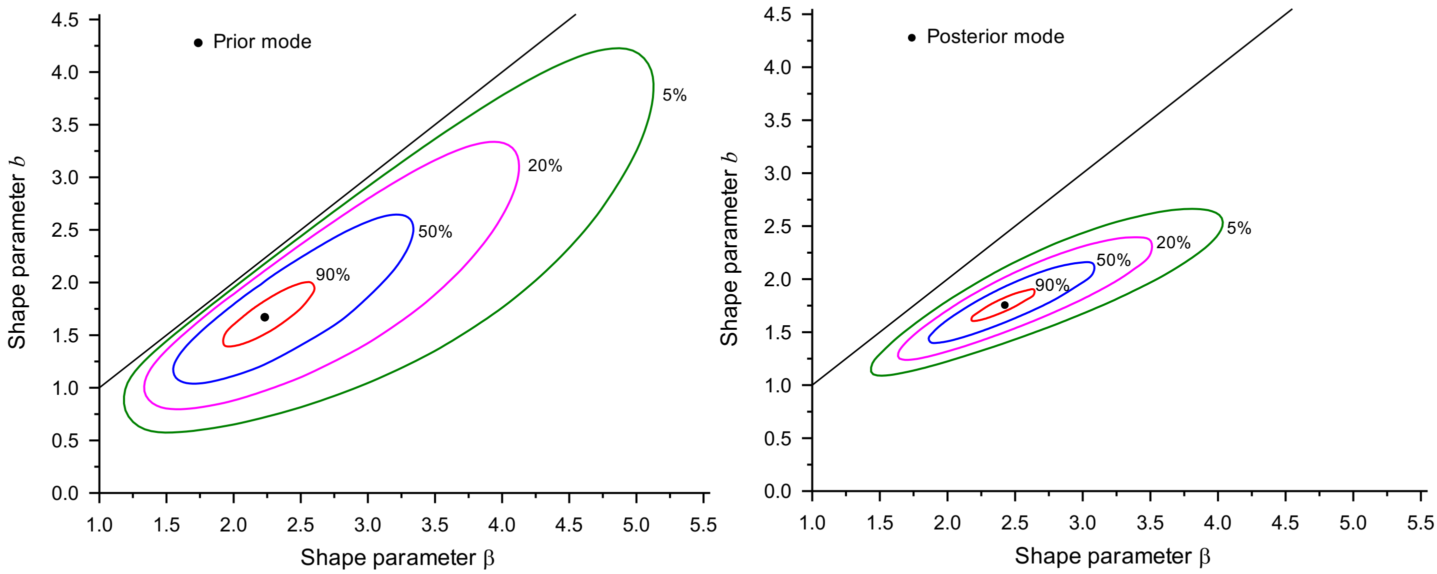

Figure 5. The joint prior $g(\beta, b)$, on the left, and the marginal joint posterior pdf $\pi(\beta, b|\boldsymbol{w})$, on the right (the level lines in the maps are for 5%, 20%, 50%, and 90% of the maximum).

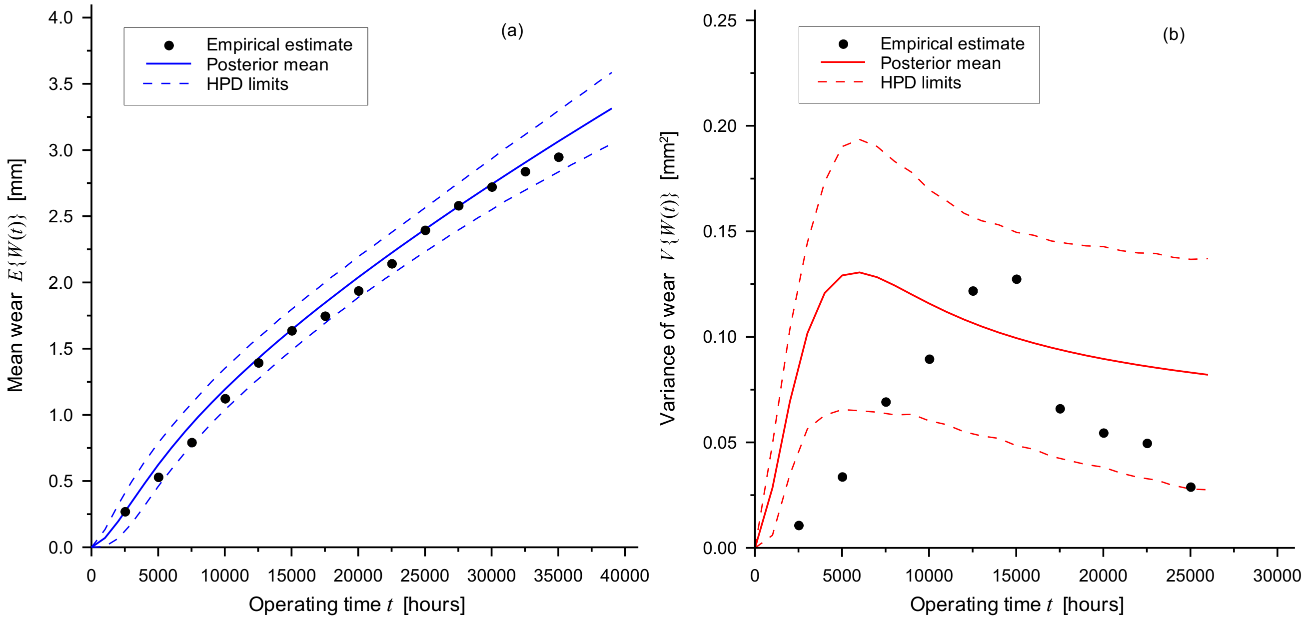

Figure 6. Empirical and Bayesian estimates (posterior mean and 0.90 HPD intervals) of the mean (a) and variance (b) of the wear process.

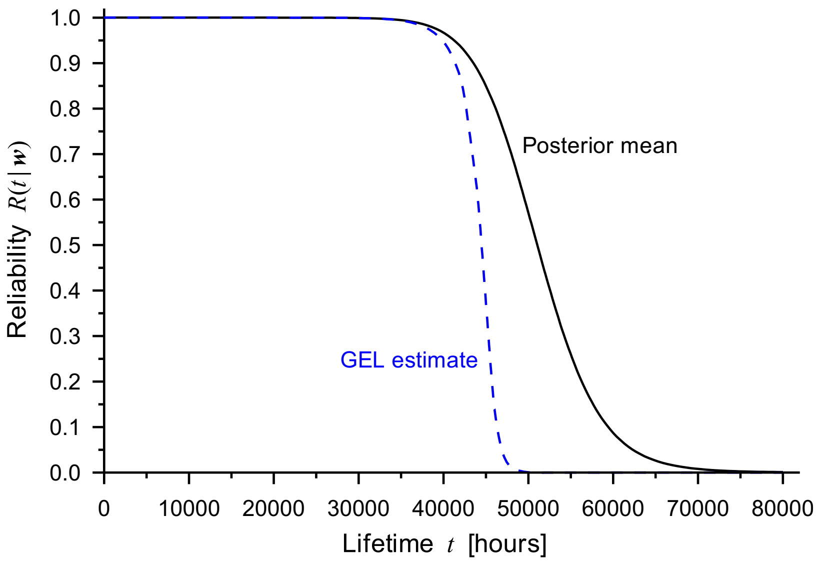

Figure 7. Posterior mean (solid line) and Bayes estimator under the GEL function (dashed line) of the reliability of a new liner.

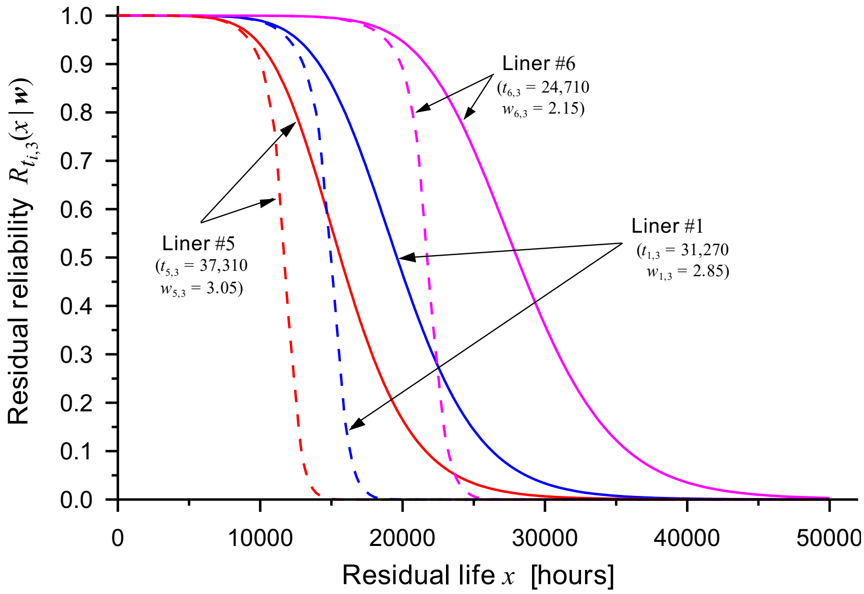

Figure 8. Posterior mean (solid line) and Bayes estimator under the GEL function (dashed line) of the residual reliability of liners #1, 5, and 6.

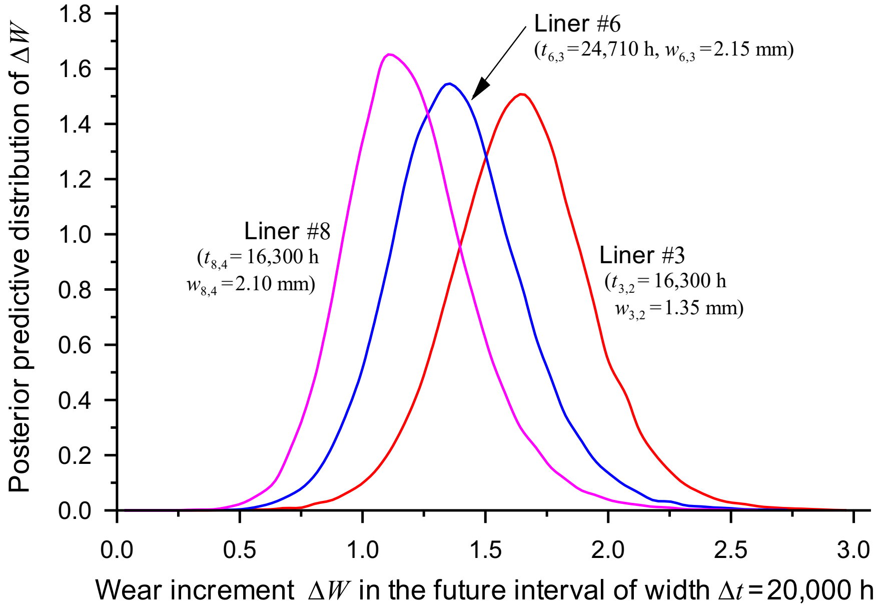

Figure 9. Posterior predictive distribution of the wear increment during the future interval of 20,000 hours process of liners #3, 6, and 8.