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Construction and implementation of two-step continuous methods for Volterra Integral Equations

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Abstract

It is the purpose of this paper to construct an error estimation for highly stable two-step continuous methods derived in [6], in order to use it in a variable stepsize implementation. New families of two step almost collocation methods are constructed, by using a collocation technique which permits to increase the uniform order of one step collocation methods, without increasing the computational cost and by maintaining good stability properties, thus avoiding the order reduction phenomenon. Numerical experiments confirm the effectiveness of the proposed methods.

Keywords:
collocation, Volterra integral equations, stability, error estimation

1. Introduction

This paper concerns the construction and implementation of both efficient and highly stable numerical methods for Volterra Integral Equations (VIEs) of the form

$$y(t) = g(t) + \int_0^t k(t,\tau,y(\tau))d\tau, \quad t \in [0,T], \tag{1.1}$$

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where the *forcing function* $g : \mathbb{R} \to \mathbb{R}^d$ and the *kernel* $k : \mathbb{R}^2 \times \mathbb{R}^d \to \mathbb{R}^d$ are assumed to be sufficiently smooth. A special interest in literature has been reserved to collocation methods [1, 2], which are based on the idea of approximating the exact solution with a suitable function belonging to a finite dimensional space, usually a piecewise algebraic polynomial, which exactly satisfies the equation on a certain subset of the integration interval, called the set of collocation points. It is well known that the best stability properties are reached by implicit numerical methods, with the disadvantage that they lead to nonlinear systems of equations to be solved at each time step. With the aim of reducing the computational cost associated to the solution of the above nonlinear systems, Two-Step Almost Collocation (TSAC) methods have been introduced in [6] and further analyzed in [7], as a modification of multistep collocation methods [8, 9, 10]: the resulting methods possess higher order of convergence with respect to one step collocation methods without any additional computational cost and preserve strong stability properties. The idea of TSAC methods has been for the first time introduced in [11] and further analyzed in [12, 13, 14, 15], in the context of Ordinary Differential Equations. They are constructed by using a collocation technique and then by relaxing some of the collocation and interpolation conditions in order to obtain desirable stability properties. In this paper we aim to derive an error estimation for TSAC methods in order to provide a variable stepsize implementation which can be competitive with the most used and widespread variable stepsize code for VIEs, namely the COLVI2 code of [19], based instead on one step collocation methods. New families of TSAC methods are constructed together with their error estimation.

The paper is structured as follows. In Section 2 we describe the class of TSAC methods, in Section 3 an error estimation of such methods is derived, Section 4 regards the effective construction of new families of TSAC methods together with the associated error estimation, while in Section 5 some numerical experiments are carried out. Finally concluding remarks are reported in Section 6.

2. Two step almost collocation methods

In this section we describe the class of TSAC methods introduced in [6]. Let us consider a discretization of the interval $[0, T]$ by means of a uniform mesh

$$\{t_n := nh, n = 0, ..., N,\ h \geq 0, Nh = T\}.$$

Then the equation (1.1) can be rewritten, by relating it to this mesh, as

$$y(t) = F^{[n]}(t, y(\cdot)) + \Phi^{[n+1]}(t, y(\cdot)), \quad t \in [t_n, t_{n+1}], \tag{2.1}$$

where

$$F^{[n]}(t, y(\cdot)) := g(t) + \int_0^{t_n} k(t, \tau, y(\tau))d\tau, \quad \Phi^{[n+1]}(t, y(\cdot)) := \int_{t_n}^{t} k(t, \tau, y(\tau))d\tau \tag{2.2}$$

are the *lag term* and the *increment term* respectively.

Let us consider m collocation parameters $c_1, \ldots, c_m$, which identify the points $t_{nj} = t_n + c_j h$ associated to the interval $[t_n, t_{n+1}]$. TSAC methods for VIEs look for a dense approximation $P(t)$ to the solution $y(t)$ of (1.1) on $[0, T]$, such that its restriction to each interval $(t_n, t_{n+1}]$ is a polynomial:

$$P(t)|_{(t_n, t_{n+1}]} = P_n(t).$$

The polynomial $P_n(t)$ is computed by employing the information about the equation on two consecutive steps:

$$P_n(t_n + sh) = \varphi_0(s)y_{n-1} + \varphi_1(s)y_n + \sum_{j=1}^{m} \chi_j(s)Y_j^{[n]} + \sum_{j=1}^{m} \psi_j(s)Y_j^{[n+1]}, \tag{2.3}$$

where $Y_j^{[n]} = P(t_{n-1,j})$ and, as in [6] it was assumed that the polynomial P_n satisfies the collocation conditions in c_j, $Y_j^{[n+1]} = P_n(t_{nj})$.

The basis functions $\varphi_0(s)$, $\varphi_1(s)$, $\chi_j(s)$ and $\psi_j(s)$, $j = 1, 2, \ldots, m$, are polynomials of degree p, determined from the continuous order conditions, according to the following theorem [6]:

Theorem 2.1. *Assume that the kernel $k(t, \eta, y)$ and the function $g(t)$ in (1.1) are sufficiently smooth. Then the method (2.3)-(2.6) has uniform order p, i.e.,*

$$\eta(t_n + sh) = O(h^{p+1}), \quad h \to 0,$$

for $s \in [0, 1]$, if the polynomials $\varphi_0(s)$, $\varphi_1(s)$, $\chi_j(s)$ and $\psi_j(s)$, $j = 1, 2, ..., m$ satisfy the system of equations

$$\begin{cases} 1 - \varphi_0(s) - \varphi_1(s) - \displaystyle\sum_{j=1}^{m} \chi_j(s) - \sum_{j=1}^{m} \psi_j(s) = 0, \\ s^k - (-1)^k \varphi_0(s) - \displaystyle\sum_{j=1}^{m} (c_j - 1)^k \chi_j(s) - \sum_{j=1}^{m} c_j^k \psi_j(s) = 0, \end{cases} \tag{2.4}$$

$s \in [0,1],\ k = 1,2,...,p,$ *where*

$$\begin{aligned} \eta(t_n + sh) &= y(t_n + sh) - \varphi_0(s)y(t_n - h) - \varphi_1(s)y(t_n) \\ &- \sum_{j=1}^{m} \Big(\chi_j(s)y(t_n + (c_j - 1)h) + \psi_j(s)y(t_n + c_j h) \Big). \end{aligned} \tag{2.5}$$

is the local truncation error.

The solvability of system (2.4) is guaranteed in the hypothesis (3.14).

Then the method assumes the form:

$$\begin{cases} Y_j^{[n+1]} = F_j^{[n]} + \Phi_j^{[n+1]}, \\ y_{n+1} = \varphi_0(1)y_{n-1} + \varphi_1(1)y_n + \sum_{j=1}^{m} \chi_j(1)Y_j^{[n]} + \sum_{j=1}^{m} \psi_j(1)Y_j^{[n+1]} \end{cases}, \tag{2.6}$$

where $F_h^{[n]}(t)$ and $\Phi_j^{[n+1]}$ are suitable sufficiently high order quadrature formulae for the discretization of $F^{[n]}(t, P(\cdot))$ and $\Phi^{[n+1]}(t_{nj}, P(\cdot))$ respectively, assuming the form

$$F_h^{[n]}(t) = g(t) + h \sum_{\nu=1}^{n} \sum_{l=0}^{m+1} b_l k\left(t, t_{\nu-1,l}, Y_l^{[\nu]}\right), \tag{2.7}$$

and

$$\Phi_j^{[n+1]} = h \sum_{l=0}^{m} w_{jl} k\left(t_{nj}, t_{nl}, Y_l^{[n+1]}\right), \tag{2.8}$$

and $F_j^{[n]} = F_h^{[n]}(t_{nj})$. In the quadrature formulae (2.7)–(2.8) we mean $t_{\nu-1,0} = t_{\nu-1}$, $t_{\nu-1,m+1} = t_\nu$, $Y_0^{[\nu]} = P_n(t_{\nu-1})$, $Y_{m+1}^{[\nu]} = P_n(t_\nu)$ and $t_{n0} = t_n$.We observe as the computation of the stages $Y_j^{[n+1]}$ requires to solve a nonlinear system of dimension md. Quadrature formulas of the form (2.7)–(2.8) of order q are obtained by choosing b_j and w_{ij} as solutions of the following linear systems

$$\begin{gathered} \sum_{j=0}^{m+1} b_j f(c_j) = \int_0^1 f(s)ds, \\ f(x) = 1, x, ..x^{q-1}, \end{gathered} \tag{2.9}$$

$$\begin{gathered} \sum_{j=0}^{m} w_{ij} f(c_j) = \int_0^{c_i} f(s)ds, \\ f(x) = 1, x, ..x^{q-2}, \end{gathered} \tag{2.10}$$

where we assume $c_0 = 0$ and $c_{m+1} = 1$.

As regards the global error, the method has uniform order of convergence $p^* = \min\{l+1, q, p+1\}$, where l and q are the order of the starting procedure (for the computation of the starting values y_1 and $Y_i^{[1]}$, $i = 1, 2, ..., m$) and the order of the quadrature formulas (2.7)-(2.8) respectively (see Theorem 2.5 in [6]). Then we use as starting procedure a one step collocation method having uniform order of convergence $l = p$.

Two-step collocation methods are obtained by solving the system of order conditions up to the maximum uniform attainable order $p = 2m+1$, and, in this way, all the basis functions are determined as the unique solution of such system. However, as observed in [6], it is not convenient to impose all the order conditions because it is not possible to achieve high stability properties (e.g. $A-$stability) without getting rid of some of them. Therefore, *almost* collocation methods have been introduced by relaxing a specified number r of order conditions, i.e. by a priori opportunely fixing r basis functions, and determining the remaining ones as the unique solution of the system of order conditions up to $p = 2m + 1 - r$. Within the class of TSAC methods, $A-$stable methods have been constructed in [6] by fixing one (case $r = 1$) or both (case $r = 2$) of the polynomials $\varphi_0(s)$ and $\varphi_1(s)$ as

$$\begin{aligned} \varphi_0(s) &= \prod_{k=1}^{m}(s - c_k)(q_0 + q_1 s + \ldots + q_{p-m} s^{p-m}), \\ \varphi_1(s) &= \prod_{k=1}^{m}(s - c_k)(p_0 + p_1 s + \ldots + p_{p-m} s^{p-m}), \end{aligned} \tag{2.11}$$

where α_j and β_j, $j = 0, 1, \ldots, p-m$, are free parameters, which have to be determined in order to obtain desired stability properties.

3. Error estimation

As in [18, 14] we look for an error estimation which can be computed in terms of the stages and the numerical approximation of the solution in the mesh points. We observe that, from Theorem 2.1 (see also formula (2.3) in [6]), the local error for an order p TSAC method assumes the form

$$\eta(t_n + sh) = C_p(s) h^{p+1} y^{(p+1)}(t_n) + O(h^{p+2})$$

where

$$C_p(s) = \frac{s^{p+1}}{(p+1)!} - \frac{(-1)^{p+1}}{(p+1)!}\varphi_0(s) - \sum_{j=1}^{m} \frac{(c_j-1)^{p+1}}{(p+1)!}\chi_j(s) - \sum_{j=1}^{m} \frac{c_j^{p+1}}{(p+1)!}\psi_j(s). \tag{3.12}$$

Then the local discretization error at the point t_{n+1} of the m-stage TSAC method is

$$\eta(t_{n+1}) = C_p(1)h^{p+1}y^{(p+1)}(t_n) + O(h^{p+2}).$$

As in [19] we define the local solution $\tilde{y}(t)$ as the solution obtained from a single step of the approximating method, not taking into account the errors inherited from previous steps:

$$\tilde{y}(t) = F_h^{[n]}(t) + \Phi^{[n+1]}(t, \tilde{y}(\cdot)), \quad t \in [t_n, t_{n+1}], \tag{3.13}$$

where $F_h^{[n]}(t)$ and $\Phi^{[n+1]}(t, \tilde{y}(\cdot))$ are defined in (2.7) and (2.2), respectively.

Let us assume that the kernel k satisfies the Lipschitz condition

$$\|k(t,s,y) - k(t.s.z)\| \leq L\,\|y - z\|$$

for all $(t,s) \in S = \{(t,s) : 0 \leq s \leq t \leq T\}$, with Lipschitz constant $L \geq 0$ being independent of y and z. Then, by subtracting (3.13) from (2.1), we obtain

$$\|y(t) - \tilde{y}(t)\| \leq \left\| F^{[n]}(t, y(\cdot)) - F_h^{[n]}(t) \right\| + L \int_{t_n}^{t} \|y(\tau) - \tilde{y}(\tau)\|\, d\tau.$$

Then, if the lag term quadrature formula (2.7) has order q, we obtain, using Gronwall's inequalities [2]

$$\|y(t) - \tilde{y}(t)\| \leq Ch^q e^{L(t-t_n)}.$$

Hence

$$\|y(t) - \tilde{y}(t)\| = O(h^q), t \in [t_n, t_{n+1}].$$

Assuming that the kernel k is sufficiently smooth, we have an analogous conclusion for the derivatives of $y(t)$ and $\tilde{y}(t)$. Then we can conclude that the local discretization error can be written as

$$\eta(t_n + sh) = C_p(s)h^{p+1}\tilde{y}^{(p+1)}(t_n) + O(h^{p+2}).$$

As in [14] we now look for estimates of $h^{p+1}\tilde{y}^{(p+1)}(t_n)$. We have the following theorem.

Theorem 3.1. *Assume that the solution to (3.13) is sufficiently smooth. Assume moreover that the collocation parameters satisfy*

$$\begin{array}{l} c_i \neq c_j,\ i \neq j, \\ c_i \neq c_j - 1,\ i \neq j, \\ c_i \neq -1, 0, 1,\ \forall i, \end{array} \tag{3.14}$$

Then

$$h^{p+1}\tilde{y}^{(p+1)}(t_n) \simeq \alpha_0 \tilde{y}(t_{n-1}) + \alpha_1 \tilde{y}(t_n) + \sum_{j=1}^{m} \left(\beta_j \tilde{y}(t_{n-1,j}) + \gamma_j \tilde{y}(t_{nj})\right)$$

where the constants α_0, α_1, β_j and γ_j, $j = 1, 2, ..., m$ satisfy the system of equations

$$\begin{cases} \alpha_0 + \alpha_1 + \sum\limits_{j=1}^{m} (\beta_j + \gamma_j) = 0, \\ (-1)^k \alpha_0 + \sum\limits_{j=1}^{m} \left(\beta_j (c_j - 1)^k + \gamma_j c_j^k\right) = 0, k = 1, ..., p \\ (-1)^{p+1} \alpha_0 + \sum\limits_{j=1}^{m} (\beta_j (c_j - 1)^{p+1} + \gamma_j {c_j}^{p+1}) = (p+1)! \end{cases} \tag{3.15}$$

Proof: By expanding $\tilde{y}(t_{n-1})$, $\tilde{y}(t_{n-1,j})$ and $\tilde{y}(t_{nj})$ into Taylor series around the point t_n and comparing the terms of order $O(h^k)$ for $k = 0, 1, ..., p+1$, we obtain system (3.15). By defining $\xi_1 = -1$, $\xi_2 = 0$, $\xi_{2+k} = c_k - 1$ and $\xi_{m+2+k} = c_k$ for $k = 1, ..., m$, then the coefficient matrix A of the linear system (3.15) is a Vandermonde type matrix with respect to nodes ξ_j, as $a_{ij} = \xi_j^{i-1}$ for $i = 1, ..., p+2$ and $j = 1, ..., 2m+2$. Then the matrix A is full rank, as $\xi_i \neq \xi_j$ for $i \neq j$, as a consequence of hypothesis (3.14). This ensures solvability of system (3.15). □

As $\tilde{y}(t)$ is the local solution, we can write an estimation to $\eta(t_{n+1})$ of the form

$$C_p(1) \left(\alpha_0 y_{n-1} + \alpha_1 y_n + \sum_{j=1}^{m} \beta_j Y_j^{[n]} + \sum_{j=1}^{m} \gamma_j Y_j^{[n+1]} \right), \tag{3.16}$$

to which we have to add the error due to the use of the quadrature formula in the increment term, which we denote by E_{nj}, $j = 1, ..., m$:

$$E_{nj} = \int_{t_n}^{t_{nj}} k(t_{nj}, \tau, \tilde{y}(\tau)) d\tau - \Phi_j^{[n+1]}$$

As in [19], an estimation of E_{nj} can be obtained by approximating the integral by a higher-order method using the same stepsize, i.e.

$$\int_{t_n}^{t_{nj}} k(t_{nj}, \tau, \tilde{y}(\tau))d\tau \approx h \sum_{l=0}^{\hat{m}} \tilde{w}_{jl} k\left(t_{nj}, t_n + \hat{c}_l h, \hat{Y}_l^{[n+1]}\right), \tag{3.17}$$

with $\hat{Y}_l^{[n+1]}$ computed as solution of the nonlinear system

$$\hat{Y}_j^{[n+1]} = F_h^{[n]}(t_n + \hat{c}_j h) + h \sum_{l=0}^{\hat{m}} \hat{w}_{jl} k\left(t_n + \hat{c}_j h, t_n + \hat{c}_l h, \hat{Y}_l^{[n+1]}\right). \tag{3.18}$$

We can choose $\hat{m} = m+1$, and, for fixed values $\hat{c}_l$, $l = 0, ..., m+1$, the matrices $\tilde{W}$ and $\hat{W}$ are determined by imposing that the corresponding quadrature formulae in (3.17)- (3.18) are of interpolatory type. The order of quadrature rules (3.17)- (3.18) is chosen to be $q+1$ if q is the order of quadrature rules (2.7)-(2.8).

So the local error estimation in the point t_{n+1} is given by

$$est(t_{n+1}) = C_p(1)\left(\alpha_0 y_{n-1} + \alpha_1 y_n + \sum_{j=1}^{m} \beta_j Y_j^{[n]} + \sum_{j=1}^{m} \gamma_j Y_j^{[n+1]}\right) + \|E_n\|, \tag{3.19}$$

where $E_n = (E_{n1}, E_{n2}, ..., E_{nm})^T$.

Having a local error estimation, the stepsize strategy in a variable stepsize implementation consists in advancing the solution from t_n to t_{n+1} with a trial stepsize h_n, then accept or reject the result and repeating the process with a modification of the previous stepsize. The trial solution is accepted if

$$\|est(t_{n+1})\| \leq \frac{tol \cdot h_n}{t_{n+1}}$$

and the stepsize is modified by

$$h_{new} = h_n \left(\frac{tol \cdot h_n}{t_{n+1} \cdot \|est(t_{n+1})\|}\right)^{\frac{1}{q}},$$

where q is the order of the quadrature formula to approximate the integral. Of course in a variable stepsize implementation the quadrature formulae (2.7) and (2.8) assume the form

$$F_h^{[n]}(t) = g(t) + \sum_{\nu=1}^{n} h_{\nu-1} \sum_{l=0}^{m+1} b_l k\left(t, t_{\nu-1,l}, Y_l^{[\nu]}\right), \tag{3.20}$$

and

$$\Phi_j^{[n+1]} = h_n \sum_{l=0}^{m} w_{jl} k\left(t_{nj}, t_{nl}, Y_l^{[n+1]}\right), \tag{3.21}$$

respectively, where $t_{nj} = t_n + c_j h$.

Moreover when we advance from t_n to t_{n+1} with stepsize h_n we have also to compute the missing approximations $\bar{y}_{n-1} \simeq y(t_n - h_n)$, and $\bar{Y}_j^{[n]} \simeq y(t_n - h_n + c_j h_n)$, $j = 1, ..., m$. This missing approximations are computed by evaluating the polynomial P. In particular the approximation $y(\bar{t})$ of the solution in a point $\bar{t} < t_n$ is calculated by first detecting the minimum integer k such that $\bar{t}$ belongs to the interval $[t_k, t_{k+1}]$ of lenght h_k and then the needed approximation is determined by evaluating the continuous approximant $P(t_k + sh_k)$ at the scaled variable $\bar{s} = \frac{\bar{t}-t_k}{h_k}$.

4. Derivation of the error estimation for specific methods

4.1. Methods with one stage

Let us consider the methods of order $p = 2m = 2$ derived in [6]. The associated error constant is $C_2(1) = -((-1+c_1)(2+c_1(-1+q_0+q_1)+c_1^2(q_0+q_1)))/6$, while the coefficients in the error estimation (3.19) are

$$\alpha_0 = -6/(c_1 + c_1^2),\ \alpha_1 = 6/((-1+c_1)c_1),\ \beta_1 = 6/(c_1 - c_1^2),\ \gamma_1 = 6/(c_1 + c_1^2).$$

4.2. Methods with two stages

Let us consider the methods of order $p = 4$, $m = 2$. The stability analysis of TSAC methods has been carried out in Section 4 of [6]. Classes of $A-$stable methods were then derived by considering

$$\varphi_0(s) = s(s - c_1)(s - c_2)(q_0 + q_1 s),$$

where c_1, c_2, q_0, q_1 are free parameters. The weights in (2.7)-(2.8) were computed in [6] by solving the systems (2.9)-(2.10) for $q = 4$:

$$\mathbf{b} = \left[\ \frac{-1+2c_1+2c_2-6c_1c_2}{12c_1c_2}, \frac{1-2c_2}{12c_1(c_1-1)(c_1-c_2)}, \frac{2c_1-1}{12c_2(c_2-1)(c_2-c_1)}, \frac{-3+4c_1+4c_2-6c_1c_2}{12(c_1-1)(c_2-1)}\ \right]^T$$

$$\mathbf{W} = \begin{bmatrix} -\frac{c_1^2-3c_1c_2}{6c_2} & \frac{c_1(2c_1-3c_2)}{6(c_1-c_2)} & \frac{c_1^3}{6c_2(c_1-c_2)} \\ -\frac{c_2^2-3c_1c_2}{6c_1} & -\frac{c_1^3}{6c_1(c_1-c_2)} & -\frac{c_2(2c_2-3c_1)}{6(c_1-c_2)} \end{bmatrix}.$$

The choice $q_1 = -q_0$ leads to the derivation of A-stable methods, but both components of the abscissa vector were far outside the interval $[0,1]$. However in variable stepsize implementation is preferable to have collocation parameters inside the interval $[0,1]$ in order to avoid extrapolation, expecially when the stepsize increases. For this reason in [6] lower order methods were derived. We now improve this result, by letting q_0 and q_1 assume different values. We performed a computer search based on the Schur criterion (see [6]) looking for $A-$stable methods in the parameter space (q_0, q_1, c_1, c_2) and we obtained, with the choice $q_0 = 15/10$, $q_1 = -1$, the families of A-stable methods reported in Figure 1.

The error constant associated to these methods is $C_4(1) = \frac{1}{240}(c_1-1)(c_2-1)\left(c_1^2c_2(c_2+1) + c_1\left(c_2^2+3c_2-4\right) - 4c_2 + 8\right)$, while the coefficients in the error estimation (3.19) are

$$\begin{aligned}
\alpha_0 &= -120/(c_1(1+c_1)c_2(1+c_2)),\\
\alpha_1 &= 120/((-1+c_1)c_1(-1+c_2)c_2),\\
\beta_1 &= -120/((-1+c_1)c_1(-1+c_1-c_2)(c_1-c_2)),\\
\beta_2 &= -120/((c_1-c_2)(1+c_1-c_2)(-1+c_2)c_2),\\
\gamma_1 &= 120/(c_1(1+c_1)(c_1-c_2)(1+c_1-c_2)),\\
\gamma_2 &= 120/(c_2(1+c_2)(-c_1+c_1^2+c_2-2c_1c_2+c_2^2)).
\end{aligned}$$

5. Numerical examples

We present first of all some fixed stepsize numerical results which confirm that, differently from one step collocation methods, the TSAC methods do not suffer form the order reduction in the integration of stiff systems, as we expect from the uniform order of convergence stated in Theorem 2.1. Superconvergent one-step collocation methods [2, 1], instead, have higher order of convergence at the mesh points and lower uniform order elsewhere in the time interval. For this reason, in stiff problems, it can happen that the effective order is the lower one also at the mesh points (order reduction).

In order to illustrate this phenomenon, we show the results obtained on both a non stiff and a stiff equation:

- the non stiff Volterra integral equation

$$y(t) = 2 - \cos(t) - \int_0^t \sin(ty(\tau) - \tau)d\tau, \quad t \in [0,3], \tag{5.1}$$

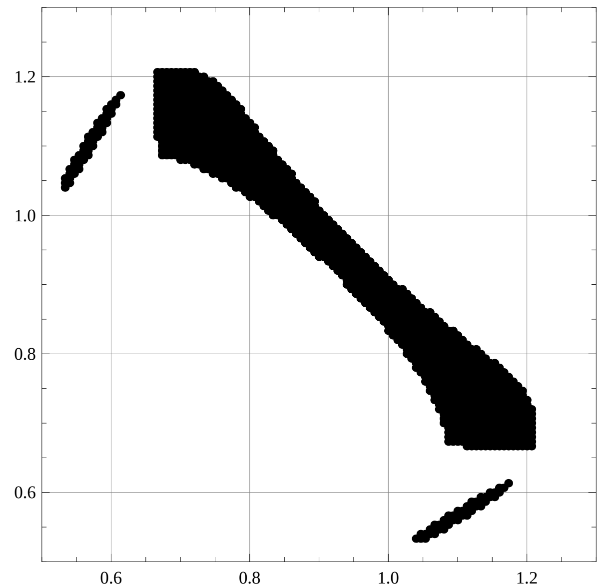


Figure 1: Region of A-stability in the (c_1, c_2) plane for the TSAC method with $m = 2$, $p = 4$, $q = 4$, $q_0 = 15/10$, $q_1 = -1$.

with exact solution $y(t) \equiv 1$;

- the stiff Volterra integral equation,

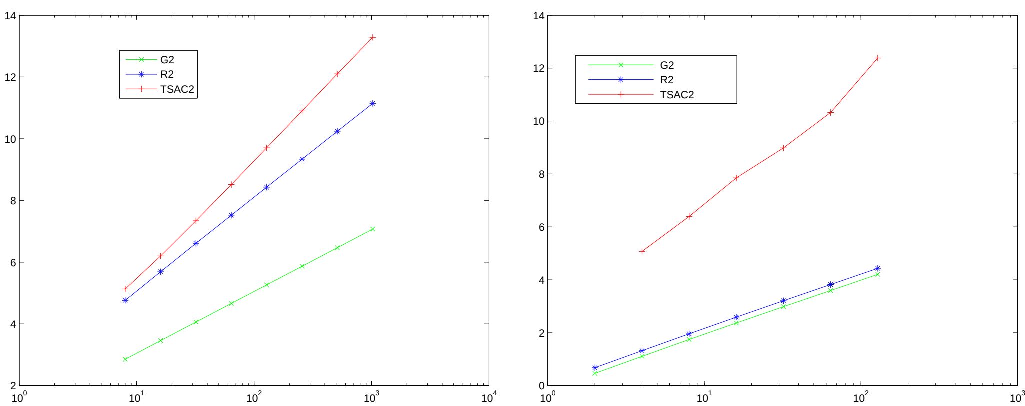
$$y(t) = \int_0^t (\lambda (y(\tau) - \sin(\tau)) + \cos(\tau)) \, d\tau, \quad t \in [0, \frac{3}{4}\pi], \tag{5.2}$$

  with $\lambda = -10^4$ and exact solution $y(t) = sin(t)$. This is a stiff problem because it is equivalent to the Prothero-Robinson problem for ODEs.

We compare TSAC methods with superconvergent one step collocation methods of [1, 2], where m denotes the number of collocation points and p denotes the order of the method:

$$\begin{array}{ll} G2: & \text{1 point Gauss collocation } + c_2 = 1,\ m = 2,\ p = 2 \\ R2: & \text{2 points Radau collocation, } m = 2,\ p = 3 \\ TSAC2: & \text{2 points TSAC method, } m = 2,\ p = 4 \end{array} \tag{5.3}$$

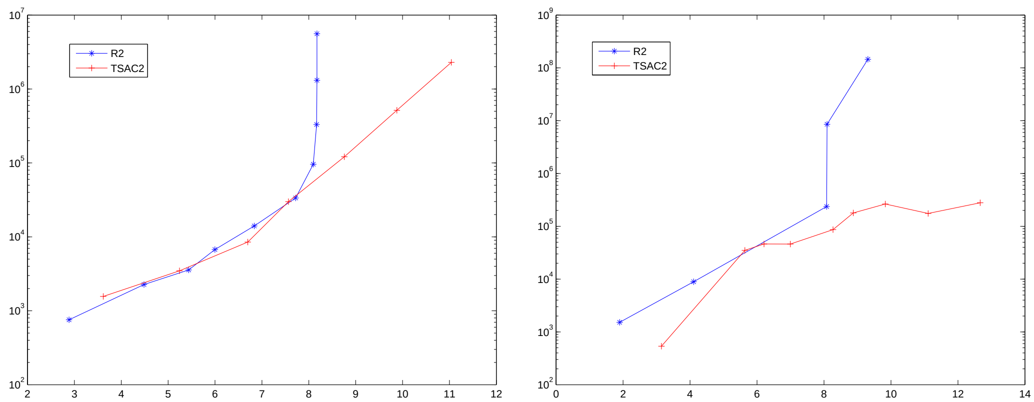
In particular the method TSAC2 is the two-stage TSAC method described in Section 4.2 with $m = 2$, $p = q = 4$, $q_0 = \frac{15}{10}$, $q_1 = -1$ and with collocation parameters $c_1 = 0.9$ and $c_2 = 0.95$, so that the point (c_1, c_2) belongs to the A-stability region of Figure 1. The starting procedure consists in a one step collocation method having uniform order of convergence $p = 4$ with $m = 4$ gaussian collocation parameters. The nonlinear systems in (2.6) and (3.18) have been solved with the fsolve function of Matlab. The accuracy is defined by the number of correct significant digits cd at the end point (the maximal absolute end point error is written as 10^{-cd}). For each test we plot in Figure 1 the number of cd versus the number of mesh points N. We observe as for non stiff Problem (5.1) the effective order of the all methods is coherent with (5.3), while for stiff Problem (5.2) the one step methods show order reduction as the effective order reduces to $p = 2$.



(a) Problem (5.1) (b) Problem (5.2)

Figure 1: Number of correct significant digits with respect to the number of mesh points obtained with methods (5.3).

As regards the variable stepsize implementation, which we carried out as described in Section 3, with the error estimation provided by Theorem 3.1, we aim to show the improvement, in terms of computational cost, obtained by TSAC methods with respect to COLVI2 variable stepsize implementation of one step collocation methods discussed in [19]. With this purpose we report in Figure 2 the work precision diagrams, which show as TSAC methods gain a higher number of correct significant digits with less kernel evaluations. Here the the minimum value for the stepsize is set to $h_{min} = 0.005$ for both methods. In particular we observe as two step collocation method exhibits a significative improvement with respect to one step collocation method, expecially in the case of the stiff problem, as it does not suffer from order reduction.



(a) Problem (5.1) (b) Problem (5.2)

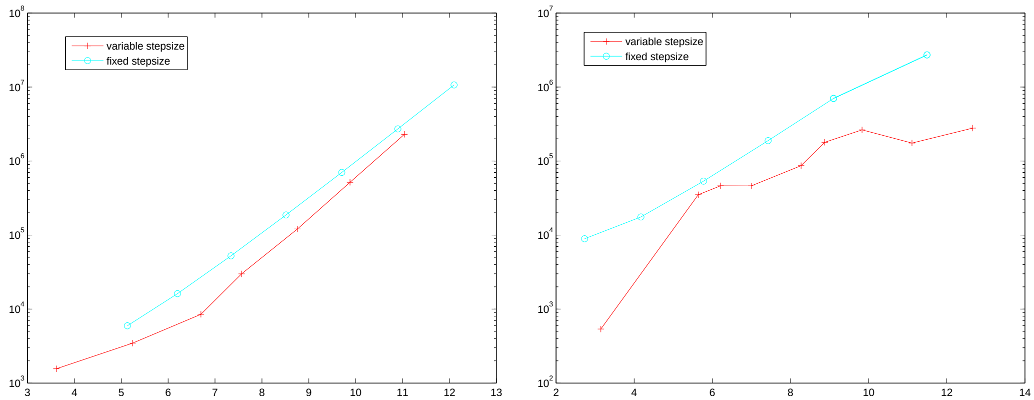
Figure 2: Number of kernel evaluations with respect to correct significant digits obtained with methods (5.3).

We compare in Figure 3 the performances of fixed and variable stepsize implementation of TSAC2 method on both non stiff problem (5.1) and stiff problem (5.2). We observe as the variable stepsize implementation gains higher number of correct significant digits with less kernel evaluations, expecially on the stiff problem.

We finally report in Figure 4 the error estimation of Theorem 3.1 compared with the global error for problem (5.2) with $toll = 10^{-8}$, showing the effectiveness of the chosen error estimation.

6. Concluding remarks

We derived an error estimation for two-step almost collocation methods for the numerical solution of Volterra integral equations. These are A-stable methods which have an uniform order of convergence and do not suffer from order reduction phenomenon. Numerical experiments show the improvement obtained by TSAC methods with respect to one step collocation methods both in fixed and in variable stepsize implementation. Future work will address the construction of higher order methods in order to provide a competitive software for the numerical solution of Voterra integral equations.



(a) Problem (5.1) (b) Problem (5.2)

Figure 3: Number of kernel evaluations with respect to correct significant digits obtained with TSAC2 method.

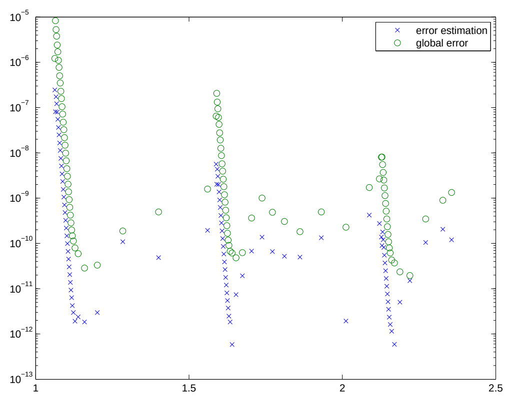


Figure 4: Comparison between the error estimation and the global error in the integration interval.

7. Acknowledgment

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References

[1] H. Brunner, Collocation Methods for Volterra Integral and Related Functional Equations, Cambridge University Press, 2004.

[2] H. Brunner, P. J. van der Houwen, The Numerical Solution of Volterra Equations, CWI Monographs, 3, North-Holland, Amsterdam, 1986.

[3] G. Capobianco, D. Conte, I. D. Prete, E. Russo, Fast runge–kutta methods for nonlinear convolution systems of volterra integral equations, BIT 47 (2007) 259–275.

[4] D. Conte, I. D. Prete, Fast collocation methods for volterra integral equations of convolution type, J. Comput. Appl. Math. 196 (2006) 652–663.

[5] A. Schädle, M. López-Fernández, C. Lubich, Fast and oblivious convolution quadrature, J. Sci. Comput. 28 (2006) 421–438.

[6] D. Conte, Z. Jackiewicz, B. Paternoster, Two-step almost collocation methods for volterra integral equations, Appl. Math. Comput. 204 (2008) 839–853.

[7] D. Conte, R. D'Ambrosio, B. Paternoster, wo–step diagonally–implicit collocation based methods for volterra integral equations, Appl. Numer. Math. 62 (2012) 1312–1324.

[8] A. Cardone, D. Conte, Multistep collocation methods for volterra integro-differential equations, Appl. Math. Comput. 221 (2013) 770–782.

[9] D. Conte, B. Paternoster, Multistep collocation methods for volterra integral equations, Appl. Numer. Math. 59 (2009) 1721–1736.

[10] B. P. D. Conte, A family of multistep collocation methods for volterra integral equations, AIP Conf. Proc. 936 (2007) 128–131.

[11] R. D'Ambrosio, M. Ferro, Z. Jackiewicz, B. Paternoster, Two-step almost collocation methods for ordinary differential equations, Numer. Algor. 53 (2010) 195–217.

[12] R. D'Ambrosio, M. Ferro, B. Paternoster, Collocation-based two-step runge-kutta methods for ordinary differential equations, Lecture Notes in Comput. Sci. 5073 (2008) 736–751.

[13] R. D'Ambrosio, Z. Jackiewicz, Continuous two-step runge-kutta methods for ordinary differential equations, Numer. Algor. 54 (2010) 169–193.

[14] R. D'Ambrosio, Z. Jackiewicz, Construction and implementation of highly stable two-step continuous methods for stiff differential systems, Math. Comput. Simul. 81 (2011) 1707–1728.

[15] R. D'Ambrosio, B. Paternoster, Two-step modified collocation methods with structured coefficients matrix for ordinary differential equations, Appl. Numer. Math. 62 (2012) 1325–1334.

[16] Z. Jackiewicz, General linear methods for ordinary differential equations, Wiley-Interscience, John Wiley & Sons, 2009.

[17] Z. Jackiewicz, S. Tracogna, A general class of two-step runge-kutta methods for ordinary differential equations, SIAM J. Numer. Anal. 32 (1995) 1390–1427.

[18] M. Bras, A. Cardone, R. D'Ambrosio, Implementation of explicit nordsieck methods with inherent quadratic stability, Math. Model. Anal. 18 (2013) 289–307.

[19] J. G. Blom, H. Brunner, The numerical solution of nonlinear volterra integral equations of the second kind by collocation and iterated collocation methods, SIAM J. Sci. Stat. Comput. 8 (1987) 806–830.