Optimal Schwarz Waveform Relaxation for fractional diffusion-wave equations

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Abstract

We introduce domain decomposition methods of Schwarz waveform relaxation (WR) type for fractional diffusion-wave equations. We show that the Dirichlet transmission conditions among the subdomains lead to slow convergence. So, we construct optimal transmission conditions at the artificial interfaces and we prove that optimal Schwarz WR methods on N subdomains converge in N iterations both on infinite spatial domains and on finite spatial domains. We also propose optimal transmission conditions when the original problem is spatially discretized and we prove the same result found in the continuous case.

Keywords: Schwarz Methods, Domain Decomposition, Fractional diffusion-wave equations, Waveform relaxation, Optimized transmission conditions

1. Introduction

In this paper we are interested in solving a class of integro-partial differential equations of the form

$$\frac{\partial u}{\partial t}(x,t) - \frac{\nu}{\Gamma(\beta)}\int_0^t (t-\tau)^{\beta-1}\frac{\partial^2 u}{\partial x^2}(x,\tau)d\tau = f(x,t), \quad \beta \in (-1,1) \tag{1.1}$$

for $x \in \Omega \subseteq \mathbb{R}$ and $t > 0$, taken together with Dirichlet, Neumann or transparent boundary conditions and with the initial condition $u(x,0) = u_0(x)$ for $x \in \Omega$. Such equations describe anomalous diffusion processes and wave propagations in viscoelastic materials and they have recently attracted increasing interest in the physical, chemical and engineering literature, see [19, 22, 36] and references therein. Numerical methods for the time discretization of (1.1) have been proposed by various authors, see [7, 19–21, 35, 37, 39, 40]. In this paper we introduce domain decomposition techniques, in order to solve the problem (1.1) in parallel.

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WR methods have been originally proposed for large systems of ordinary differential equations (see [3] and references therein), and then have been extended to other kind of evolution equations such as Volterra integral equations (refer for example to [4–6] and references therein). They are particularly convenient to solve large systems of equations, as they are designed in order to decouple the original large system in smaller subsystems: in this way, the iteration process can be implemented in a parallel computational environment, since each subsystem can be treated by a single processor/tread (see [13, 16] and references therein). This iteration process realizes what is commonly known as *parallelism across the system*, and so a massive parallelism. Other kinds of parallel methods perform instead a *parallelism across the method*, as for example [10, 23] in the context of multistage methods for ODEs and VIEs (see [9, 11, 12, 14, 15] and references therein).

Schwarz Waveform Relaxation methods have been mainly developed and analyzed for several kinds of PDEs (see [1, 2, 17, 18, 24–27, 31–34] and the related bibliography), and consist in decomposing the spatial domain into subdomains and solve iteratively time dependent problems on subdomains, exchanging information at the boundary. We will first analyze the convergence behaviour of the overlapping Schwarz waveform relaxation method, showing that Dirichlet boundary conditions at the artificial interfaces inhibit the information exchange between the subdomains and therefore slow down the convergence of the methods. Using the ideas introduced in [28–30], we will derive optimal transmission conditions for the convergence of the method. They lead to non-overlapping Schwarz WR methods which converge in a finite number of steps, identical to the number of subdomains.

By defining the operator

$$\mathcal{L}(u) = u_t - \nu \partial^{-\beta} u_{xx}, \tag{1.2}$$

where $\partial^{-\beta}$ denotes the fractional integral of order β, the problem (1.1) can be written as

$$\begin{cases} \mathcal{L}(u) = f & in \quad \Omega \times \mathbb{R}_+ \\ u = u_0 & x \in \Omega, \ t = 0. \end{cases} \tag{1.3}$$

We will consider the problem (1.3) both on the infinite spatial domain $\Omega = \mathbb{R}$ with the asymptotic condition $u(x,t) \to 0$ for $x \to \pm\infty$, and on finite spatial domains $\Omega = [a,b]$, $a, b \in \mathbb{R}$, with Dirichlet, Neumann or transparent boundary conditions.

The paper is organized as follows. In Section 2 we introduce the overlapping classical Schwarz method on infinite spatial domain, proving linear convergence rate on infinite time intervals and superlinear convergence rate on finite time intervals if $\beta \in (0,1)$. In Sections 3 and 4 we construct the optimal Schwarz WR methods on $N = 2$ and $N > 2$ subdomains respectively, by providing the transmission conditions which assure convergence in N iterations. In Section 5 we derive the optimal transmission conditions for the spatially discretized equation and we prove the same results of convergence. We present the conclusions in Section 6.

2. Overlapping classical Schwarz WR method on infinite spatial domain

We decompose the spatial domain $\Omega = \mathbb{R}$ into two overlapping subdomains $\Omega_1 = (-\infty, L]$ and $\Omega_2 = [0, \infty)$, $L > 0$. The overlapping classical Schwarz waveform relaxation consists then in solving iteratively subproblems on $\Omega_1 \times \mathbb{R}_+$ and $\Omega_2 \times \mathbb{R}_+$ with Dirichlet transmission conditions at the interface, i.e. using as boundary condition at the interfaces $x = 0$ and $x = L$ the values obtained from the previous iteration. Thus the method, for iteration index $k = 0, 1, 2, ...$, assumes the form

$$\begin{cases} \mathcal{L}(u_1^{k+1}) = f & in\ \Omega_1 \times \mathbb{R}_+ \\ u_1^{k+1}(L,t) = u_2^k(L,t) & t > 0 \\ u_1^{k+1}(x,0) = u_0(x) & x \in \Omega_1 \end{cases} \qquad \begin{cases} \mathcal{L}(u_2^{k+1}) = f & in\ \Omega_2 \times \mathbb{R}_+ \\ u_2^{k+1}(0,t) = u_1^k(0,t) & t > 0 \\ u_2^{k+1}(x,0) = u_0(x) & x \in \Omega_2 \end{cases} \tag{2.1}$$

where an initial guess $u_1^0(0,t)$ and $u_2^0(L,t)$, $t \in \mathbb{R}_+$, needs to be provided.

In order to analyze the convergence properties of the method (2.1), we observe that by linearity it is sufficient to analyze the method for homogeneous problems with zero initial conditions

$$\begin{cases} \mathcal{L}(u) = 0 & in \quad \Omega \times \mathbb{R}_+ \\ u = 0 & x \in \Omega,\ t = 0, \end{cases} \tag{2.2}$$

i.e.,

$$\begin{cases} \mathcal{L}(u_1^{k+1}) = 0 & in\ \Omega_1 \times \mathbb{R}_+ \\ u_1^{k+1}(L,t) = u_2^k(L,t) & t > 0 \\ u_1^{k+1}(x,0) = 0 & x \in \Omega_1 \end{cases} \qquad \begin{cases} \mathcal{L}(u_2^{k+1}) = 0 & in\ \Omega_2 \times \mathbb{R}_+ \\ u_2^{k+1}(0,t) = u_1^k(0,t) & t > 0 \\ u_2^{k+1}(x,0) = 0 & x \in \Omega_2 \end{cases} \tag{2.3}$$

and prove the convergence to zero.

We will apply in the following, in our proofs, the Laplace transform in time with parameter $s \in \mathbb{C}$, $\mathrm{Re}(s) > 0$ to the operator (1.2), thus obtaining

$$\widehat{\mathcal{L}}(\widehat{u}) := s\widehat{u}(x,s) - \nu s^{-\beta}\widehat{u}_{xx}(x,s) = (\lambda^2\widehat{u}(x,s) - \widehat{u}_{xx}(x,s))\frac{\nu}{s^\beta}, \tag{2.4}$$

where we have defined

$$\lambda = \lambda(s) := \frac{s^\gamma}{\sqrt{\nu}}, \tag{2.5}$$

$$\gamma = \frac{\beta+1}{2} \in (0,1). \tag{2.6}$$

Theorem 1. *The Schwarz method (2.1) converges with linear rate on unbounded time intervals.*

Proof. Applying the Laplace transform in time with parameter $s \in \mathbb{C}$, $\mathrm{Re}(s) > 0$, to the equation (2.3) we get the equation

$$\widehat{\mathcal{L}}\left(\widehat{u}_i^{k+1}\right) = 0, \quad i = 1, 2,$$

with $\widehat{\mathcal{L}}$ given by (2.4), whose characteristic equation is

$$\lambda^2 - y^2 = 0,$$

with λ given by (2.5), having solution $y = \pm\lambda$. Thus, by using the Dirichlet transmission conditions, we get that the transformed solutions are

$$\widehat{u}_1^{k+1}(x) = \widehat{u}_2^k(L)e^{\lambda(x-L)} \tag{2.7}$$

and

$$\widehat{u}_2^{k+1}(x) = \widehat{u}_1^k(0)e^{-\lambda x}, \tag{2.8}$$

where we have omitted the dependence on s for brevity of notation. By evaluating (2.7) for $x = 0$ and (2.8) at the previous iteration for $x = L$, we obtain by induction

$$\widehat{u}_1^{2k}(0) = r_{cla}^k \widehat{u}_1^0(0), \quad \widehat{u}_2^{2k}(L) = r_{cla}^k \widehat{u}_2^0(L), \tag{2.9}$$

where the convergence factor $r_{cla} = r_{cla}(s, L, \nu, \gamma)$ of the classical Schwarz method is given by

$$r_{cla}(s, L, \nu, \gamma) = e^{-2s^\gamma L/\sqrt{\nu}} < 1 \quad \forall \mathrm{Re}(s) > 0. \tag{2.10}$$

Thus the iterates converge to zero on the line $x = 0$ and $x = L$, respectively. Since with zero boundary conditions the solution vanishes identically, we have shown the convergence of the classical Schwarz method, with linear rate, for all frequencies with $\mathrm{Re}(s) > 0$. □

Theorem 1 shows that the convergence factor (2.10) depends on the problem parameters, ν and γ, on the size of the overlap L and on the frequency parameter s. We observe that, the method is also well defined without overlap, i.e. $L = 0$, but in this case it's not convergent, differently from the optimized methods proposed in next sections.

In the following we will use the notation

$$\|f\|_{p,T} = \left(\int_0^T |f(t)|^p \, dt\right)^{1/p}, \quad \|f\|_{\infty,T} = \sup_{0<t<T} |f(t)|.$$

Theorem 2. *For $\gamma \in (\frac{1}{2}, 1)$ the Schwarz method (2.1) has a superlinear asymptotic convergence rate on bounded time domains:*

$$\left\|u_i^{2k}(0,\cdot)\right\|_{2,T} \le e\sqrt{T}\rho_\gamma^k \operatorname{erfc}\left(\frac{kL}{\sqrt{\nu T}}\right) \left\|u_i^0(0,\cdot)\right\|_{2,T}, \quad i = 1, 2, \tag{2.11}$$

where

$$\rho_\gamma = e^{\frac{2LM_\gamma}{\sqrt{\nu}}}, \quad M_\gamma = \max_{\theta\in[-\pi/2,\pi/2]} \psi_\gamma(\theta), \quad \psi_\gamma(\theta) = \left(\frac{\cos(\theta/2)}{\sqrt{T\cos(\theta)}} - \frac{\cos(\gamma\theta)}{(T\cos(\theta))^\gamma}\right). \tag{2.12}$$

Proof. To obtain the convergence result for bounded time intervals, we rewrite the first of (2.9) in the form

$$\widehat{u}_1^{2k}(0,s) = \widehat{F}(s)\widehat{G}(s),$$

where

$$\widehat{F}(s) = e^{-cs^\gamma + c\sqrt{s}}, \quad \widehat{G}(s) = e^{-c\sqrt{s}}\widehat{u}_1^0(0,s), \tag{2.13}$$

with $c = 2kL/\sqrt{\nu}$. By using Parseval formula we obtain, for $\sigma = 1/T$,

$$\left\|u_1^{2k}(0,t)\right\|_{2,T} \leq e\left\|e^{-\sigma t}u_1^{2k}(0,t)\right\|_{2,T} \leq e \max_{\mathrm{Re}(s)=\sigma}\left|\widehat{F}(s)\right| \left\|e^{-\sigma t}G(t)\right\|_{2,T}. \tag{2.14}$$

As the inverse Laplace transform of $\widehat{K}(s) = e^{-c\sqrt{s}}$ is

$$K(t) = \frac{c}{2\sqrt{\pi}t^{3/2}}e^{-\frac{c^2}{4t}},$$

by inverting the second of (2.13), the function G is given by the convolution $G = K * u_1^0(0,\cdot)$. Hence, with $\sigma = 1/T$, we obtain

$$\left\|e^{-\sigma t}G(t)\right\|_{2,T} \leq \sqrt{T}\,\|G\|_{2,T} \leq \sqrt{T}\,\|K\|_{1,T}\left\|u_1^0(0,\cdot)\right\|_{2,T}, \tag{2.15}$$

so,

$$\left\|e^{-\sigma t}G(t)\right\|_{2,T} \leq \sqrt{T}\,\mathrm{erfc}\left(\frac{kL}{\sqrt{\nu T}}\right)\left\|u_1^0(0,\cdot)\right\|_{2,T}. \tag{2.16}$$

In order to calculate $\max\limits_{\mathrm{Re}(s)=\sigma}\left|\widehat{F}(s)\right|$ we observe that, for $\mathrm{Re}(s) = \sigma$, we have $s = \frac{\sigma}{\cos\theta}e^{\imath\theta}$, with $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, and

$$\left|\widehat{F}(s)\right| = e^{c\psi_\gamma(\theta)},$$

with $\psi_\gamma(\theta)$ defined in (2.12). For $\gamma \in (\frac{1}{2}, 1)$ we have

$$\lim_{\theta\to-\frac{\pi}{2}}\psi_\gamma(\theta) = \lim_{\theta\to\frac{\pi}{2}}\psi_\gamma(\theta) = -\infty,$$

and the function ψ_γ has maximum $M_\gamma = \max\limits_{\theta\in[-\pi/2,\pi/2]}\psi_\gamma(\theta)$. Then

$$\max_{\mathrm{Re}(s)=\sigma}\left|\widehat{F}(s)\right| = \rho_\gamma^k,$$

with $\rho_\gamma = e^{\frac{2LM_\gamma}{\sqrt{\nu}}}$, which, together with (2.14) and (2.16), leads to (2.11). □

3. The optimal Schwarz WR methods on two subdomains

Dirichlet boundary conditions at the interfaces are responsible for the slow rate of convergence. In this section we analyze the case of two subdomains and derive optimal trasmission conditions, which assure convergence in a finite number of iterations, thus obtaining the optimal Schwarz WR methods (see also [8]).

3.1. Infinite spatial domain

Let us consider the method (2.3) with different transmission conditions:

$$\begin{cases} \mathcal{L}(u_1^{k+1}) = 0 & in\ \Omega_1 \times \mathbb{R}_+ \\ u_{1,x}^{k+1} + \Lambda^+(u_1^{k+1}) = u_{2,x}^k + \Lambda^+(u_2^k) & x = L,\ t > 0 \\ u_1^{k+1} = 0 & x \in \Omega_1,\ t = 0 \end{cases} \tag{3.1}$$

$$\begin{cases} \mathcal{L}(u_2^{k+1}) = 0 & in\ \Omega_2 \times \mathbb{R}_+ \\ u_{2,x}^{k+1} + \Lambda^-(u_2^{k+1}) = u_{1,x}^k + \Lambda^-(u_1^k) & x = 0,\ t > 0 \\ u_2^{k+1} = 0 & x \in \Omega_2,\ t = 0 \end{cases} \tag{3.2}$$

where Λ^+ and Λ^- are linear operators acting on the boundary in time.

Theorem 3. *If the operators Λ^+ and Λ^- have corresponding symbols*

$$\lambda^+ = \lambda, \quad \lambda^- = -\lambda, \tag{3.3}$$

with λ given by (2.5), then the method (3.1)-(3.2) converges in two iterations independently of the initial guess, of the size of the overlap L and the problem parameters $\nu > 0, \gamma \in (0,1)$.

Proof. Applying the Laplace transform in time of equations (3.1)-(3.2), with parameter s, $\mathrm{Re}(s) > 0$, we find, for $k \geq 0$,

$$\widehat{\mathcal{L}}\left(\widehat{u}_i^{k+1}\right) = 0, \quad i = 1, 2,$$

with $\widehat{\mathcal{L}}$ given by (2.4), which, together with the conditions $\widehat{u}_1(x) \to 0$ for $x \to -\infty$, $\widehat{u}_2(x) \to 0$ for $x \to +\infty$, lead to

$$\widehat{u}_1^{k+1}(x) = \widehat{u}_1^{k+1}(0)e^{\lambda x}, \quad \widehat{u}_2^{k+1}(x) = \widehat{u}_2^{k+1}(0)e^{-\lambda x}, \tag{3.4}$$

with partial derivatives satisfying

$$\widehat{u}_{1,x}^{k+1}(x) = \lambda \widehat{u}_1^{k+1}(x), \quad \widehat{u}_2^{k+1}(x) = -\lambda \widehat{u}_2^{k+1}(x). \tag{3.5}$$

As λ^+ and λ^- are the symbols corresponding to Λ^+ and Λ^-, by considering the Laplace transform of the transmission conditions in (3.1)-(3.2) and from (3.5), we have

$$\left(\lambda^+ + \lambda\right) \widehat{u}_1^{k+2}(L) = \left(\lambda^+ - \lambda\right) \widehat{u}_2^{k+1}(L) \tag{3.6}$$

and

$$\left(\lambda^- - \lambda\right) \widehat{u}_2^{k+2}(0) = \left(\lambda^- + \lambda\right) \widehat{u}_1^{k+1}(0). \tag{3.7}$$

Then, by using (3.6)-(3.7) and (3.4) with $x = L$, we obtain, for $k \geq 0$,

$$\widehat{u}_1^{k+2}(L) = \frac{\lambda^+ - \lambda}{\lambda^+ + \lambda} e^{-\lambda L} \widehat{u}_2^{k+1}(0)$$

and

$$\widehat{u}_2^{k+2}(0) = \frac{\lambda^- + \lambda}{\lambda^- - \lambda} e^{-\lambda L} \widehat{u}_1^{k+1}(L).$$

Thus

$$\widehat{u}_1^{2k}(L) = r_{opt}^k \widehat{u}_1^0(L), \quad \widehat{u}_2^{2k}(0) = r_{opt}^k \widehat{u}_2^0(0),$$

where the new convergence factor r_{opt} is given by

$$r_{opt} = r_{opt}(s, L, \nu, \gamma) = \frac{\lambda^+ - \lambda}{\lambda^+ + \lambda} \frac{\lambda^- + \lambda}{\lambda^- - \lambda} e^{-2s^\gamma L/\sqrt{\nu}},$$

and differs from the one of the classical Schwarz method r_{cla} given in (2.10) only for the factor in front of the exponential. Then by choosing the symbols as in (3.3), the new convergence factor vanishes identically $r_{opt} \equiv 0$, and, since with zero boundary conditions the solution vanishes identically, the thesis immediately follows. □

We note from the previous theorem that the exponential factor in the convergence rate becomes irrelevant and thus the optimized Schwarz method is convergent also without overlap, i.e. $L = 0$, in contrast to the classical Schwarz method. In the next Section we will generalize the optimal convergence result to $N > 2$ subdomains and convergence in N iterations.

3.2. Finite spatial domain

Let us consider the homogeneous problem (2.2) on a finite domain $\Omega = [a, b]$, $a, b \in \mathbb{R}$, together with zero boundary conditions of the form

$$\begin{cases} \mathcal{B}^-(u) = 0, & x = a, \\ \mathcal{B}^+(u) = 0, & x = b. \end{cases} \tag{3.8}$$

Namely we will consider

- transparent Boundary conditions

$$\mathcal{B}^-(u) = u_x + \Lambda^-(u), \quad \mathcal{B}^+(u) = u_x + \Lambda^+(u), \tag{3.9}$$

  where operators Λ^+ and Λ^- have corresponding symbols (3.3);

- Dirichlet boundary conditions

$$\mathcal{B}^-(u) = \mathcal{B}^+(u) = u; \tag{3.10}$$

- Neumann boundary conditions

$$\mathcal{B}^-(u) = \mathcal{B}^+(u) = u_x. \tag{3.11}$$

The problem (1.3) with transparent boundary conditions (3.8)-(3.9) permits to reduce the computation of the solution of the problem from $\Omega = \mathbb{R}$ to a finite domain $\Omega = [a, b]$, for inhomogeneity f with support in $[a, b]$, see [38].

Let $a = L_1 < L_2 < L_3 = b$, and decompose the spatial domain $\Omega = [a, b]$ into two non-overlapping subdomains $\Omega_1 = [a, L_2]$ and $\Omega_2 = [L_2, b]$, with

$$\sigma_1 := L_2 - a \quad and \quad \sigma_2 := b - L_2. \tag{3.12}$$

Let us consider the Schwarz method, as before for the homogeneous problem (2.2) with zero initial condition and zero boundary conditions (3.8):

$$\begin{cases} \mathcal{L}(u_1^{k+1}) = 0 & in\ \Omega_1 \times \mathbb{R}_+ \\ \mathcal{B}^-(u_1^{k+1}) = 0 & x = a,\ t > 0 \\ u_{1,x}^{k+1} + \Lambda^+(u_1^{k+1}) = u_{2,x}^k + \Lambda^+(u_2^k) & x = L_2 \\ u_1^{k+1} = 0 & x \in \Omega_1,\ t = 0 \end{cases} \tag{3.13}$$

$$\begin{cases} \mathcal{L}(u_2^{k+1}) = 0 & in\ \Omega_2 \times \mathbb{R}_+ \\ u_{2,x}^{k+1} + \Lambda^-(u_2^{k+1}) = u_{1,x}^k + \Lambda^-(u_1^k) & x = L_2,\ t > 0 \\ \mathcal{B}^+(u_2^{k+1}) = 0 & x = b,\ t > 0 \\ u_2^{k+1} = 0 & x \in \Omega_2,\ t = 0 \end{cases} \tag{3.14}$$

Theorem 4. *If the operators Λ^+ and Λ^- have corresponding symbols given by (3.3)-(2.5), the method (3.13)-(3.14), applied to the problem (2.2) with transparent boundary conditions (3.8)-(3.9), converges in 2 iterations.*

Proof. Applying the Laplace transform in time of equations (3.13)-(3.14) and by using the transparent boundary conditions, it is immediate to verify that $\widehat{u}_i^{k+1}$, $i = 1, 2$, satisfy

$$\widehat{u}_1^{k+1}(x) = \widehat{u}_1^{k+1}(L_2) e^{\lambda(x-L_2)}, \quad \widehat{u}_2^{k+1}(x) = \widehat{u}_2^{k+1}(L_2) e^{-\lambda(x-L_2)},$$

with partial derivatives satisfying

$$\widehat{u}_{1,x}^{k+1}(x) = \lambda \widehat{u}_1^{k+1}(x), \quad \widehat{u}_{2,x}^{k+1}(x) = -\lambda \widehat{u}_2^{k+1}(x). \tag{3.15}$$

Then the proof proceeds as the proof of Theorem 3, i.e. from the Laplace transform of the transmission conditions in (3.13)-(3.14) and from (3.15), we have

$$\left(\lambda^+ + \lambda\right) \widehat{u}_1^{k+2}(L_2) = \left(\lambda^+ - \lambda\right) \widehat{u}_2^{k+1}(L_2) \tag{3.16}$$

and

$$\left(\lambda^- - \lambda\right) \widehat{u}_2^{k+2}(L_2) = \left(\lambda^- + \lambda\right) \widehat{u}_1^{k+1}(L_2). \tag{3.17}$$

Then, by using (3.16)-(3.17), we obtain, for $k \geq 0$,

$$\widehat{u}_1^{k+2}(L_2) = \frac{\lambda^+ - \lambda}{\lambda^+ + \lambda} \widehat{u}_2^{k+1}(L_2)$$

and

$$\widehat{u}_2^{k+2}(L_2) = \frac{\lambda^- + \lambda}{\lambda^- - \lambda} \widehat{u}_1^{k+1}(L_2),$$

thus obtaining

$$\widehat{u}_i^{2k}(L_2) = r_{opt}^k \widehat{u}_i^0(L_2), \quad i = 1, 2,$$

where the convergence factor r_{opt} is given by

$$r_{opt} = r_{opt}(s) = \frac{\lambda^+ - \lambda}{\lambda^+ + \lambda}\frac{\lambda^- + \lambda}{\lambda^- - \lambda}.$$

Then by choosing the symbols as in (3.3)-(2.5), the convergence factor vanishes identically $r_{opt} \equiv 0$, and, since with zero boundary conditions the solution vanishes identically, the thesis immediately follows. $\square$

The following theorem shows that the transmission conditions in the method (3.13)-(3.14) with the operators Λ^+ and Λ^- defined in Theorem (4) do not guarantee the convergence in 2 iterations when we solve the problem (2.2) together with Dirichlet or Neumann boundary conditions.

Theorem 5. *Let us consider the problem (2.2) with Dirichlet boundary conditions (3.8)-(3.10) or Neumann boundary conditions (3.8)-(3.11). Then the error of the method (3.13)-(3.14) with the operators Λ^+ and Λ^- having corresponding symbols (3.3)-(2.5), exponentially decays with the length $b-a$.*

Proof. Applying the Laplace transform in time to equations (3.13)-(3.14) and by using the Dirichlet boundary conditions

$$\widehat{u}_1^{k+1}(a) = 0, \quad \widehat{u}_2^{k+1}(b) = 0,$$

it is immediate to verify that $\widehat{u}_i^{k+1}$, $i = 1, 2$, satisfy

$$\begin{aligned} \widehat{u}_1^{k+1}(x) &= \frac{\widehat{u}_1^{k+1}(L_2)}{e^{\lambda\sigma_1} - e^{-\lambda\sigma_1}}\left(e^{\lambda(x-a)} - e^{-\lambda(x-a)}\right), \\ \widehat{u}_2^{k+1}(x) &= \frac{\widehat{u}_2^{k+1}(L_2)}{e^{-\lambda\sigma_2} - e^{\lambda\sigma_2}}\left(e^{\lambda(x-b)} - e^{-\lambda(x-b)}\right), \end{aligned} \tag{3.18}$$

with σ_1 and σ_2 given by (3.12) and from which we obtain

$$\begin{aligned} \widehat{u}_{1,x}^{k+1}(L_2) &= \lambda\frac{1+e^{-2\lambda\sigma_1}}{1-e^{-2\lambda\sigma_1}}\widehat{u}_1^{k+1}(L_2), \\ \widehat{u}_{2,x}^{k+1}(L_2) &= -\lambda\frac{1+e^{-2\lambda\sigma_2}}{1-e^{-2\lambda\sigma_2}}\widehat{u}_2^{k+1}(L_2). \end{aligned} \tag{3.19}$$

Then, by substituting (3.19) in the Laplace transform of the transmission conditions in (3.13)-(3.14), exploiting the fact that Λ^+ and Λ^- have corresponding symbols given by (3.3), we obtain, for $k \geq 0$

$$\begin{aligned} \widehat{u}_1^{k+2}(L_2) &= -e^{-2\lambda\sigma_2}\frac{1-e^{-2\lambda\sigma_1}}{1-e^{-2\lambda\sigma_2}}\widehat{u}_2^{k+1}(L_2), \\ \widehat{u}_2^{k+2}(L_2) &= -e^{-2\lambda\sigma_1}\frac{1-e^{-2\lambda\sigma_2}}{1-e^{-2\lambda\sigma_1}}\widehat{u}_1^{k+1}(L_2), \end{aligned}$$

from which it follows

$$\widehat{u}_i^{2k}(L_2) = r^k \widehat{u}_i^k(L_2), \quad i = 1, 2, \tag{3.20}$$

with

$$r(s) = e^{-2\lambda(b-a)}. \tag{3.21}$$

Analogously, by applying the Laplace transform in time to equations (3.13)-(3.14) and by using the Neumann boundary conditions

$$\widehat{u}_{1,x}^{k+1}(a) = 0, \quad \widehat{u}_{2,x}^{k+1}(b) = 0,$$

it is immediate to verify that $\widehat{u}_i^{k+1}$, $i = 1, 2$, satisfy

$$\widehat{u}_1^{k+1}(x) = \frac{\widehat{u}_1^{k+1}(L_2)}{e^{\lambda\sigma_1} + e^{-\lambda\sigma_1}} \left(e^{\lambda(x-a)} + e^{-\lambda(x-a)}\right),$$
$$\widehat{u}_2^{k+1}(x) = \frac{\widehat{u}_2^{k+1}(L_2)}{e^{-\lambda\sigma_2} + e^{\lambda\sigma_2}} \left(e^{\lambda(x-b)} - e^{-\lambda(x-b)}\right),$$

which differs from (3.18) only for the sign "+" instead of the sign "−" between the two exponentials. The same argument as for Dirichlet boundary conditions leads again to (3.20)-(3.21). □

Theorems 6 and 7 show how to modify the transmission conditions in case of Dirichlet or Neumann boundary conditions in order to guarantee the convergence in 2 iterations.

Theorem 6. *If the operators Λ^+ and Λ^- have corresponding symbols given by*

$$\lambda^+ = \overline{\lambda}_2, \quad \lambda^- = -\overline{\lambda}_1, \tag{3.22}$$

where

$$\overline{\lambda}_i = \lambda \frac{1 + e^{-2\lambda\sigma_i}}{1 - e^{-2\lambda\sigma_i}}, \quad i = 1, 2,$$

with λ given by (2.5) and $\sigma_i, i = 1, 2$ given by (3.12), then the method (3.13)-(3.14), applied to the problem (2.2)-(3.8) with Dirichlet boundary conditions (3.8),(3.10) converges in 2 iterations.

Proof. As in the proof of Theorem 5 we get that the derivatives of $\widehat{u}_i^{k+1}$, $i = 1, 2$, satisfy (3.19). By substituting (3.19) in the Laplace transform of the transmission conditions in (3.13)-(3.14), and by (3.22), we get

$$\widehat{u}_1^{k+2}(L_2) = \frac{\lambda^+ - \overline{\lambda}_2}{\lambda^+ + \overline{\lambda}_1} \widehat{u}_2^{k+1}(L_2),$$
$$\widehat{u}_2^{k+2}(L_2) = \frac{\lambda^- + \overline{\lambda}_1}{\lambda^- - \overline{\lambda}_2} \widehat{u}_1^{k+1}(L_2),$$

thus obtaining

$$\widehat{u}_i^{2k}(L_2) = r_{opt}^k \widehat{u}_i^0(L_2), \quad i = 1, 2,$$

where the convergence factor r_{opt} is given by

$$r_{opt} = r_{opt}(s) = \frac{\lambda^+ - \overline{\lambda}_2}{\lambda^+ + \overline{\lambda}_1} \frac{\lambda^- + \overline{\lambda}_1}{\lambda^- - \overline{\lambda}_2},$$

and the thesis immediately follows. □

Theorem 7. *If the operators Λ^+ and Λ^- have corresponding symbols given by*

$$\lambda^+ = \widetilde{\lambda}_2, \quad \lambda^- = -\widetilde{\lambda}_1,$$

$$\widetilde{\lambda}_i = \lambda \frac{1 - e^{-2\lambda\sigma_i}}{1 + e^{-2\lambda\sigma_i}},$$

with λ given by (2.5), then the method (3.13)-(3.14), applied to the problem (2.2)-(3.8) with Neumann boundary conditions (3.8),(3.11) converges in 2 *iterations.*

Proof. The proof is analogous to that of Theorem 6. □

In particular if we consider the limit for $\sigma_i \to \infty$, we have $\overline{\lambda}_i$, $\widetilde{\lambda}_i \to \lambda$ and the optimal conditions coincide with that on infinite domain.

4. The optimal Schwarz WR methods on N subdomains

In this section we will generalize the optimal convergence results of the previous Section to $N > 2$ subdomains and convergence in N iterations. Let us split the spatial domain $\Omega = [a, b] \subseteq \mathbb{R}$, with $a < b$ and $a \in \mathbb{R} \cup \{-\infty\}$, $b \in \mathbb{R} \cup \{+\infty\}$, on N non-overlapping subdomains $\Omega_i = [L_i, L_{i+1}]$, $i = 1, ..., N$, where $a = L_1 < L_2 < ... < L_N < L_{N+1} = b$.

As before, let us consider the Schwarz WR method for the homogeneous problem with zero initial condition (2.2) and zero boundary conditions (3.8), where, for a finite domain, the boundary conditions $\mathcal{B}^-$ and $\mathcal{B}^+$ are given by (3.9), (3.10) or (3.11), while for an infinite domain, with $a = -\infty$, $b = +\infty$, the asymptotic condition $u(x, t) \to 0$ for $x \to \pm\infty$ is obtained by using boundary conditions $\mathcal{B}^-$ and $\mathcal{B}^+$ given by (3.10).

By considering transmission conditions which depend on the domain Ω_i, the Schwarz WR method assumes the form

$$\begin{cases} \mathcal{L}(u_i^{k+1}) = 0 & in\ \Omega_i \times \mathbb{R}_+ \\ \mathcal{S}_i^-(u_i^{k+1}) = \mathcal{S}_i^-(u_{i-1}^k) & x = L_i,\ t > 0 \\ \mathcal{S}_i^+(u_i^{k+1}) = \mathcal{S}_i^+(u_{i+1}^k) & x = L_{i+1},\ t > 0 \\ u_i^{k+1} = 0 & x \in \Omega_i,\ t = 0 \end{cases}, \quad i = 1, ..., N, \tag{4.1}$$

where we set $u_0^k = u_{N+1}^k \equiv 0$ for $k \geq 0$ and we define the transmission operators $\mathcal{S}_i^\pm$ as

$$\mathcal{S}_i^-(u) = \begin{cases} \mathcal{B}^-(u), & i = 1 \\ u_x + \Lambda_i^-(u), & i = 2, ..., N \end{cases}, \tag{4.2}$$

$$\mathcal{S}_i^+(u) = \begin{cases} u_x + \Lambda_i^+(u), & i = 1, ..., N-1 \\ \mathcal{B}^+(u), & i = N, \end{cases} \tag{4.3}$$

where Λ_i^+ and Λ_i^- are linear operators acting on the boundary in time.

In order to find the optimal transmission conditions we apply the Laplace transform in time to (4.1), obtaining

$$\begin{cases} \widehat{\mathcal{L}}(\widehat{u}_i^{k+1}) = 0 & in\ \Omega_i \\ \widehat{\mathcal{S}}_i^-\left(\widehat{u}_i^{k+1}\right) = \widehat{\mathcal{S}}_i^-\left(\widehat{u}_{i-1}^k\right) & x = L_i \\ \widehat{\mathcal{S}}_i^+\left(\widehat{u}_i^{k+1}\right) = \widehat{\mathcal{S}}_i^+\left(\widehat{u}_{i+1}^k\right) & x = L_{i+1}. \end{cases} \tag{4.4}$$

We denote the interface values which subdomain Ω_i obtains from its neighbors Ω_{i-1} and Ω_{i+1} as

$$g_i^{k+1,-} := \widehat{\mathcal{S}}_i^-(\widehat{u}_{i-1}^k)\Big|_{x=L_i} \tag{4.5}$$

and

$$g_i^{k+1,+} := \widehat{\mathcal{S}}_i^+(\widehat{u}_{i+1}^k)\Big|_{x=L_{i+1}}, \tag{4.6}$$

for $k \geq 0$. Then the method (4.4) takes the form

$$\begin{cases} \widehat{\mathcal{L}}(\widehat{u}_i^{k+1}) = 0 & in\ \Omega_i \\ \widehat{\mathcal{S}}_i^-\left(\widehat{u}_i^{k+1}\right) = g_i^{k+1,-} & x = L_i \\ \widehat{\mathcal{S}}_i^+\left(\widehat{u}_i^{k+1}\right) = g_i^{k+1,+} & x = L_{i+1}. \end{cases} \tag{4.7}$$

In order to prove convergence in N iterations we will determine the operators Λ_i^+ and Λ_i^- which assure that

$$g_i^{k+1,-} = 0 \Longrightarrow g_{i+1}^{k+2,-} = 0, \quad i = 2, ..., N, \tag{4.8}$$

$$g_i^{k+1,+} = 0 \Longrightarrow g_{i-1}^{k+2,+} = 0, \quad i = 1, ..., N-1, \tag{4.9}$$

and

$$g_2^{k+2,-} = 0, \quad g_{N-1}^{k+2,+} = 0, \forall k \geq 0, \tag{4.10}$$

which will imply, at iteration N, $g_i^{N,-} = 0$, $g_i^{N,+} = 0$, $i = 1, ..., N$. Then from (4.7) with $k = N$ we get $\widehat{u}_i^N \equiv 0$, $i = 1, ..., N$, as with zero boundary conditions the solution identically vanishes, and then convergence in N iterations.

4.1. Infinite spatial domain

Let us consider the case $a = -\infty$, $b = +\infty$. The following theorem estabilishes the convergence in a finite number of iterations, equal to the number of subdomains, by using the same transmission conditions used in the case of 2 domains.

Theorem 8. *The method (4.1) converges in N iterations, if the operators Λ_i^+ and Λ_i^- are given by $\Lambda_i^+ = \Lambda^+$, $\Lambda_i^- = \Lambda^-$, $i = 1, ..., N$, with Λ^+ and Λ^- having corresponding symbols given by (3.3).*

Proof. In order to prove the conditions (4.8)-(4.10), we observe that, from (4.5)-(4.6), and the hypothesis on Λ_i^+ and Λ_i^-, we have

$$g_i^{k+1,-} = \widehat{u}_{i-1,x}^k(L_i) - \lambda \widehat{u}_{i-1}^k(L_i), \qquad for \quad i \geq 2,$$

and

$$g_i^{k+1,+} = \widehat{u}_{i+1,x}^k(L_{i+1}) + \lambda \widehat{u}_{i+1}^k(L_{i+1}), \qquad for \quad i \leq N-1.$$

Let us suppose $g_i^{k+1,-} = 0$ for $i \geq 2$. Then, from (4.7) and the hypothesis on the operator Λ_i^-, $\widehat{u}_i^{k+1}(x)$ is solution of $\widehat{\mathcal{L}}(\widehat{u}_i^{k+1}) = 0$ with $\widehat{u}_{i,x}^{k+1}(L_i) - \lambda \widehat{u}_i^{k+1}(L_i) = 0$, from which it follows

$$\widehat{u}_i^{k+1}(x) = c_i e^{\lambda x} + d_i e^{-\lambda x} \tag{4.11}$$

and

$$\widehat{u}_{i,x}^{k+1}(L_i) = \lambda \widehat{u}_i^{k+1}(L_i). \tag{4.12}$$

The condition (4.12) leads to $d_i = 0$ in (4.11) and then

$$\widehat{u}_{i,x}^{k+1}(x) = \lambda \widehat{u}_i^{k+1}(x) \quad \forall x \in \Omega_i. \tag{4.13}$$

By evaluating (4.13) for $x = L_{i+1}$ we obtain $g_{i+1}^{k+2,-} = 0$, and so (4.8) is proved. An analogous argument leads to (4.9).

The equations $\widehat{\mathcal{L}}(\widehat{u}_i^{k+1}) = 0$ for $i = 1$ and $i = N$, together with the conditions $\widehat{u}_1^{k+1} \to 0$ for $x \to -\infty$ and $\widehat{u}_N^{k+1} \to 0$ for $x \to +\infty$, lead to (4.10), which completes the proof. □

The previous result can be explained by observing that zero boundary conditions require N iterations in order to be transmitted along the N subdomains by means of the optimal transmission conditions.

4.2. Finite spatial domain

Let us consider the case $a, b \in \mathbb{R}$ and let us denote by $\sigma_i = L_{i+1} - L_i$, $i = 1, ..., N$, the length of the intervals Ω_i. The following theorems provide the conditions for having convergence in N iterations.

Theorem 9. *If we consider the problem (2.2) with transparent boundary conditions (3.8)-(3.9), the method (4.1) converges in N iterations, if the operators Λ_i^+ and Λ_i^- are defined as in Theorem 8.*

Proof. The proof is analogous to that of the Theorem 8 in the infinite case. □

Theorem 10. *If we consider the problem (2.2) with Dirichlet boundary conditions (3.8)-(3.10), the method (4.1) converges in N iterations, if the operators*

Λ_i^+ and Λ_i^- have corresponding symbols given by

$$\overline{\lambda}_i^- = -\lambda \frac{1 + e^{-2\lambda \sum_{j=1}^{i-1} \sigma_j}}{1 - e^{-2\lambda \sum_{j=1}^{i-1} \sigma_j}}, \quad i = 2, ..., N,$$

$$\overline{\lambda}_i^+ = \lambda \frac{1 + e^{-2\lambda \sum_{j=i+1}^{N} \sigma_j}}{1 - e^{-2\lambda \sum_{j=i+1}^{N} \sigma_j}}, \quad i = 1, ..., N-1,$$

with λ given by (2.5).

Proof. As in the proof of Theorem 8, we will prove the conditions (4.8)-(4.10). From (4.5)-(4.6), we have

$$g_i^{k+1,-} = \widehat{u}_{i-1,x}^k(L_i) + \overline{\lambda}_i^- \widehat{u}_{i-1}^k(L_i), \qquad for \quad i \geq 2,$$

and

$$g_i^{k+1,+} = \widehat{u}_{i+1,x}^k(L_{i+1}) + \overline{\lambda}_i^+ \widehat{u}_{i+1}^k(L_{i+1}), \qquad for \quad i \leq N-1.$$

Let us suppose $g_i^{k+1,-} = 0$ for $i \geq 2$. Then, from (4.7) $\widehat{u}_i^{k+1}(x)$ is solution of $\widehat{\mathcal{L}}(\widehat{u}_i^{k+1}) = 0$ with $\widehat{u}_{i,x}^{k+1}(L_i) + \overline{\lambda}_i^- \widehat{u}_i^{k+1}(L_i) = 0$, from which it follows

$$\widehat{u}_i^{k+1}(x) = \frac{\widehat{u}_i^{k+1}(L_{i+1}) e^{-\lambda \sigma_i}}{1 - e^{-2\lambda \sum_{j=1}^{i} \sigma_j}} \left(e^{\lambda(x - L_i)} - e^{-2\lambda \sum_{j=1}^{i-1} \sigma_j} e^{-\lambda(x - L_i)} \right), \quad i = 2, ..., N,$$

and then

$$\widehat{u}_{i,x}^{k+1}(L_{i+1}) = -\overline{\lambda}_{i+1}^- \widehat{u}_i^{k+1}(L_{i+1}),$$

i.e. $g_{i+1}^{k+2,-} = 0$, and so (4.8) is proved. An analogous argument leads to (4.9).

The equations $\widehat{\mathcal{L}}(\widehat{u}_i^{k+1}) = 0$ for $i = 1$ and $i = N$, together with the boundary conditions $\widehat{u}_1^{k+1} = 0$ for $x = L_1$ and $\widehat{u}_N^{k+1} = 0$ for $x = L_{N+1}$, lead to

$$\widehat{u}_1^{k+1}(x) = \frac{\widehat{u}_1^{k+1}(L_2) e^{-\lambda \sigma_1}}{1 - e^{-2\lambda \sigma_1}} \left(e^{\lambda(x - L_1)} - e^{-\lambda(x - L_1)} \right)$$

and

$$\widehat{u}_N^{k+1}(x) = \frac{\widehat{u}_N^{k+1}(L_N) e^{-\lambda \sigma_N}}{1 - e^{-2\lambda \sigma_N}} \left(e^{\lambda(x - L_{N+1})} - e^{-\lambda(x - L_{N+1})} \right),$$

which then lead to (4.10), which completes the proof. □

Theorem 11. *If we consider the problem (2.2) with Neumann boundary conditions (3.8)-(3.11), the method (4.1) converges in N iterations, if the operators*

Λ_i^+ *and* Λ_i^- *have corresponding symbols given by*

$$\widetilde{\lambda}_i^- = -\lambda \frac{1 - e^{-2\lambda \sum_{j=1}^{i-1} \sigma_j}}{1 + e^{-2\lambda \sum_{j=1}^{i-1} \sigma_j}}, \quad i = 2, ..., N,$$
$$\widetilde{\lambda}_i^+ = \lambda \frac{1 - e^{-2\lambda \sum_{j=i+1}^{N} \sigma_j}}{1 + e^{-2\lambda \sum_{j=i+1}^{N} \sigma_j}}, \quad i = 1, ..., N-1,$$

with λ *given by (2.5).*

Proof. The proof is analogous to that of Theorem 10. □

Note that the optimal conditions $\overline{\lambda}_i^-$, $\overline{\lambda}_i^+$, $\widetilde{\lambda}_i^-$ and $\widetilde{\lambda}_i^+$ tend to the optimal conditions on infinite intervals when the dimension σ_1 of the first interval and σ_N of the last interval tend to infinite, respectively.

5. Optimal conditions for spatially discretized equation

In this section we construct optimal transmission conditions, which assure the convergence in a finite number of iterations, when the original equation is spatially discretized. We will consider an infinity spatial domain $\Omega = \mathbb{R}$ which will be reduced to a finite spatial domain $[a, b]$ with $a < b, a, b \in \mathbb{R}$, by a suitable choice of transparent boundary conditions at $x = a$ and $x = b$. By considering the mesh

$$\tilde{\Omega} = \{x_j := a + j\Delta x, \quad j = 0, ..., M\}, \tag{5.1}$$

with $x_M = b$ and by replacing in the $\mathcal{L}$ operator (1.2) $\frac{\partial^2}{\partial x^2}$ with

$$\delta_{xx} u(x_j, t) := \frac{u(x_{j+1}, t) - 2u(x_j, t) + u(x_{j-1}, t)}{\Delta x^2}, \quad j = 1, ..., M-1, \tag{5.2}$$

we define the operator

$$\mathcal{L}_{\Delta x}(u) := u_t - \nu \partial^{-\beta} \delta_{xx} u. \tag{5.3}$$

We consider the following discretization of the homogeneous problem (2.2):

$$\begin{cases} \mathcal{L}_{\Delta x}(u) = 0, & in \quad \tilde{\Omega} \times \mathbb{R}_+, \\ u = 0 & x_j \in \tilde{\Omega}, \ t = 0, \end{cases} \tag{5.4}$$

with zero transparent boundary conditions of the form

$$\begin{cases} \mathcal{B}_{\Delta x}^-(u) = \Lambda^-(\delta_x u) + u = 0, & x = a, \\ \mathcal{B}_{\Delta x}^+(u) = \Lambda^+(\delta_x u) + u = 0, & x = b, \end{cases} \tag{5.5}$$

where δ_x represents a discretization of $\frac{\partial}{\partial x}$ given by

$$\delta_x u(x_j,t) := \begin{cases} \frac{u(x_{j+1},t)-u(x_j,t)}{\Delta x}, & j=0,...,M-1, \\ \frac{u(x_j,t)-u(x_{j-1},t)}{\Delta x}, & j=M, \end{cases} \tag{5.6}$$

and the operators Λ^- and Λ^+ have corresponding symbols

$$\lambda^- = -\frac{\Delta x}{\left(\frac{\lambda\Delta x}{\sqrt{2}}\right)^2 + \sqrt{\left(1+\left(\frac{\lambda\Delta x}{\sqrt{2}}\right)^2\right)^2 - 1}}, \lambda^+ = -\lambda^- \tag{5.7}$$

with λ given by (2.5). The expression (5.7) for λ^- and λ^+ follows by imposing the conditions

$$u(x,t) \to 0, \quad x \to \pm\infty \tag{5.8}$$

in order to reduce the computation of the solution from $\Omega = \mathbb{R}$ to $\Omega = [a,b]$. The computations on the function u which permitts to obtain (5.7) from conditions (5.8) are analogous to the computations done on functions u_1 and u_2 in Remark 1 of next subsection.

5.1. Two subdomains

Let us consider the Schwarz WR method for the homogeneous problem (5.4) with zero initial boundary conditions (5.5). Let $a = L_1 < L_2 < L_3 = b$, and decompose the spatial domain Ω=[a,b] into two non-overlapping subdomains $\Omega_1 = [a, L_2]$ and $\Omega_2 = [L_2, b]$. By considering suitable transmission conditions at the interface, the Schwarz method assumes the following form

$$\begin{cases} \mathcal{L}_{\Delta x}(u_1^{k+1}) = 0, & in\ \Omega_1 \times \mathbb{R}_+, \\ \Lambda_1^-(\delta_x u_1^{k+1}) + u_1^{k+1} = 0, & x = a, t > 0, \\ \Lambda_1^+(\delta_x u_1^{k+1}) + u_1^{k+1} = \Lambda_1^+(\delta_x u_2^k) + u_2^k, & x = L_2, t > 0, \\ u_1^{k+1} = 0, & x \in \Omega_1, t = 0, \end{cases} \tag{5.9}$$

$$\begin{cases} \mathcal{L}_{\Delta x}(u_2^{k+1}) = 0, & in\ \Omega_2 \times \mathbb{R}_+, \\ \Lambda_2^-(\delta_x u_2^{k+1}) + u_2^{k+1} = \Lambda_2^-(\delta_x u_1^k) + u_1^k, & x = L_2, t > 0, \\ \Lambda_2^+(\delta_x u_2^{k+1}) + u_2^{k+1} = 0, & x = b, t > 0, \\ u_2^{k+1} = 0, & x \in \Omega_2, t = 0, \end{cases} \tag{5.10}$$

where the operators Λ_1^- and Λ_2^+ have corresponding symbols respectively λ^- and λ^+ given by (5.7), while the operators Λ_1^+ and Λ_2^- have corresponding symbols

$$\lambda_1^+ = -\frac{\Delta x}{\left(\frac{\lambda\Delta x}{\sqrt{2}}\right)^2 - \sqrt{\left(1+\left(\frac{\lambda\Delta x}{\sqrt{2}}\right)^2\right)^2 - 1}}, \lambda_2^- = -\lambda_1^+ \tag{5.11}$$

with λ given by (2.5).

Remark 1. *We observe as the zero transparent boundary conditions*

$$\Lambda_1^-(\delta_x u_1^{k+1}) + u_1^{k+1} = 0, \qquad x = a, \tag{5.12}$$

$$\Lambda_2^+(\delta_x u_2^{k+1}) + u_2^{k+1} = 0, \qquad x = b, \tag{5.13}$$

in the method (5.9)-(5.10) replace the asymptotic conditions (5.8) when we reduce the computation of the solution from $\Omega = \mathbb{R}$ to $\Omega = [a, b]$.
As a matter of fact, by applying the Laplace transform in time to equations (5.9) and (5.10) we obtain the difference equations:

$$\widehat{u}_i(x_{j+1}, s) - [2 + (\Delta x \lambda)^2]\widehat{u}_i(x_j, s) + \widehat{u}_i(x_{j-1}, s) = 0, \quad i = 1, 2 \tag{5.14}$$

with general solution of form:

$$\widehat{u}_i(x_j, s) = C\lambda_1^j + D\lambda_2^j, \qquad C, D \in \mathbb{R},$$

where x_j is given by (5.1), λ_1 and λ_2 are roots of the characteristic polynomial associated with the equations (5.14), i.e.:

$$\lambda_{1,2} = \left(1 + \left(\frac{\Delta x \lambda}{\sqrt{2}}\right)^2\right) \pm \sqrt{\left(1 + \left(\frac{\Delta x \lambda}{\sqrt{2}}\right)^2\right)^2 - 1}. \tag{5.15}$$

By considering the conditions (5.8), we impose that

$$\lim_{j \to -\infty} \widehat{u}_1^{(k+1)}(x_j, s) = 0, \quad \lim_{j \to +\infty} \widehat{u}_2^{(k+1)}(x_j, s) = 0.$$

Since $|\lambda_1(s)| < 1$ and $|\lambda_2(s)| > 1$, we find

$$\widehat{u}_1^{(k+1)}(x_j, s) = C\,[\lambda_1(s)]^j\,, \quad \widehat{u}_2^{(k+1)}(x_j, s) = D\,[\lambda_2(s)]^j\,. \tag{5.16}$$

By computing

$$\delta_x \widehat{u}_1^{(k+1)}(x_0, s) = \frac{\widehat{u}_1^{(k+1)}(x_1, s) - \widehat{u}_1^{(k+1)}(x_0, s)}{\Delta x} = \frac{C\,[\lambda_1(s)] - C\,[\lambda_1(s)]^0}{\Delta x}$$
$$= -\frac{C\,[1 - \lambda_1(s)]}{\Delta x},$$

we have

$$C = -\frac{\delta_x \widehat{u}_1^{(k+1)}(x_0, s)}{1 - \lambda_1(s)} \Delta x, \tag{5.17}$$

and, by replacing the (5.17) in (5.16) we obtain

$$\lambda_1^- \delta_x \widehat{u}_1^{(k+1)}(a, s) + \widehat{u}_1^{(k+1)}(a, s) = 0, \tag{5.18}$$

with $\lambda_1^- = \lambda^-$ given by (5.7). A similar argument proves

$$\lambda_2^+ \delta_x \widehat{u}_2^{(k+1)}(b, s) + \widehat{u}_2^{(k+1)}(b, s) = 0, \tag{5.19}$$

with $\lambda_2^+ = \lambda^+$ defined in (5.7). Applying the inverse Laplace transform to (5.18) and (5.19) we have (5.12) and (5.13).

Theorem 12. *If the operators Λ_1^-,Λ_1^+,Λ_2^- and Λ_2^+ have corresponding symbols given by (5.7)-(5.11), the method (5.9) and (5.10) applied to the problem (5.4) with transparent boundary conditions (5.12)-(5.13), converges in 2 iterations.*

Proof. In order to prove convergence in 2 iterations we will derive the operators Λ_2^- and Λ_1^+ in (5.9)-(5.10) which assure the following implications:

$$\Lambda_1^-\delta_x u_1^{k+1}(a,s)+u_1^{k+1}(a,s)=0 \Rightarrow \Lambda_2^-\delta_x u_1^{k+1}(L_2,s)+u_1^{k+1}(L_2,s)=0, \quad (5.20)$$

$$\Lambda_2^+\delta_x u_2^{k+1}(b,s)+u_2^{k+1}(b,s)=0 \Rightarrow \Lambda_1^+\delta_x u_2^{k+1}(L_2,s)+u_2^{k+1}(L_2,s)=0. \quad (5.21)$$

Then from (5.9)-(5.10) we will find that the solution identically vanishes in 2 iterations, i.e. $u_i^{k+2}\equiv 0, i=1,2$ since it has to satisfy a system with all zero boundary conditions.
Using the expression of $\widehat{u}_1^{(k+1)}(x_j,s)$ in (5.16) we compute $\delta_x\widehat{u}_1^{(k+1)}(L_2,s)$ and we find the constant

$$C=\frac{\Delta x\delta_x\widehat{u}_1^{(k+1)}(L_2,s)}{1-\lambda_2}, \quad (5.22)$$

with λ_2 defined in (5.15). By replacing (5.22) in (5.16) we obtain

$$\lambda_2^-\delta_x\widehat{u}_1^{(k+1)}(L_2,s)+\widehat{u}_1^{(k+1)}(L_2,s)=0, \quad (5.23)$$

with $\lambda_2^-=-\lambda_1^+$ given by (5.11). Inverse Laplace transform of (5.23) leads to (5.20). An analogous argument proves (5.21). □

5.2. N subdomains

In this subsection we will generalize the optimal convergence result of the previous subsection to $N>2$ subdomains and convergence in N iterations. We consider the more general case in which the stepsize Δx can be different in every subdomain. We split the spatial finite domain $\Omega=[a,b]\subseteq\mathbb{R}, a,b\in\mathbb{R}$ in N non-overlapping subdomains $\Omega_i=[L_i,L_{i+1}]$, i=1,...N, where $a=L_1<L_2<\ldots<L_N<L_{N+1}=b$ and we discretize the problem (2.2) in each subdomain, by considering the meshes

$$\tilde{\Omega}_i=\{x_j^{(i)}=L_i+j\Delta_i x, \quad j=0,...,N_i\}, \quad (5.24)$$

with $x_{N_i}^{(i)}=L_{i+1}$ and by replacing $\frac{\partial^2}{\partial x^2}$ with (5.2). So, in each subdomain $\tilde{\Omega}_i$, for i=1,...,N, the Schwarz WR method assumes the form:

$$\begin{cases} \mathcal{L}_{\Delta_i x}(u_i^{k+1})=0, & in\ \tilde{\Omega}_i\times\mathbb{R}_+ \\ \mathcal{S}_i^-(u_i^{k+1})=\mathcal{S}_i^-(u_{i-1}^k), & x=L_i, t>0 \\ \mathcal{S}_i^+(u_i^{k+1})=\mathcal{S}_i^+(u_{i+1}^k), & x=L_{i+1}, t>0 \\ u_i^{k+1}=0, & x\in\ \tilde{\Omega}_i, t=0 \end{cases} \quad (5.25)$$

where we set $u_0^k = u_{N+1}^k \equiv 0$, for $k \geq 0$, $\mathcal{L}_{\Delta_i x}$ given by (5.3) with $\Delta x = \Delta_i x$, and the transmission operators $\mathcal{S}_i^{\pm}$ are defined by:

$$\mathcal{S}_i^-(u_i^{k+1}) = \begin{cases} \mathcal{B}^-_{\Delta_i x}(u_i^{k+1}), & i = 1, \\ \Lambda_i^-(\delta_x u_i^{k+1}) + u_i^{k+1}, & i = 2, ..., N \end{cases} \tag{5.26}$$

$$\mathcal{S}_i^-(u_i^{k+1}) = \begin{cases} \Lambda_i^+(\delta_x u_i^{k+1}) + u_i^{k+1}, & i = 1, ..., N-1, \\ \mathcal{B}^+_{\Delta_i x}(u_i^{k+1}), & i = N \end{cases} \tag{5.27}$$

where $\Lambda_i^{\pm}$ are linear operators acting on the boundary in time, and $\mathcal{B}^{\pm}_{\Delta_i x}$ are given by (5.5) with $\Delta x = \Delta_i x$. By considering the roots of characteristic polynomial associated with the equations $\widehat{\mathcal{L}}_{\Delta_i x}(\widehat{u}_i^{k+1}) = 0 \quad i = 1, ..., N$, i.e.

$$\lambda_{1,i} = \left(1 + \left(\frac{\lambda \Delta_i x}{\sqrt{2}}\right)^2\right) + \sqrt{\left(1 + \left(\frac{\lambda \Delta_i x}{\sqrt{2}}\right)^2\right)^2 - 1},$$

$$\lambda_{2,i} = \left(1 + \left(\frac{\lambda \Delta_i x}{\sqrt{2}}\right)^2\right) - \sqrt{\left(1 + \left(\frac{\lambda \Delta_i x}{\sqrt{2}}\right)^2\right)^2 - 1},$$

with λ given by (2.5), we can proceed as in Section 4 and prove the following result:

Theorem 13. *If we consider the problem (5.4) with transparent boundary conditions (5.5) and if the operators $\Lambda_i^{\pm}$, for i=1,...,N, have corresponding symbols respectively given by:*

$$\lambda_i^- = \begin{cases} -\frac{\Delta_i x}{\lambda_{1,i} - 1}, & i = 1, \\ -\frac{\Delta_{i-1} x}{1 - \lambda_{2,i-1}}, & i = 2, \\ -\frac{\Delta_{i-1} x}{1 - \frac{\lambda_{1,i-1}^{N_{i-1}-1} - \alpha_i^- \lambda_{2,i-1}^{N_{i-1}-1}}{\lambda_{1,i-1}^{N_{i-1}} - \alpha_i^- \lambda_{2,i-1}^{N_{i-1}}}}, & i = 3, ..., N, \end{cases} \tag{5.28}$$

with

$$\alpha_i^- = \frac{\lambda_{i-1}^-(\lambda_{1,i-1} - 1) + \Delta_{i-1} x}{\lambda_{i-1}^-(\lambda_{2,i-1} - 1) + \Delta_{i-1} x}, \tag{5.29}$$

and

$$\lambda_i^+ = \begin{cases} -\frac{\Delta_j x}{1 - \lambda_{1,j}}, & j = N, \\ -\frac{\Delta_{j+1} x}{\lambda_{2,j+1} - 1}, & j = N-1, \\ -\frac{\Delta_{j+1} x}{\frac{\lambda_{1,j+1} - \alpha_j^+ \lambda_{2,j+1}}{1 - \alpha_j^+} - 1} & j = N-2, ..., 1, \end{cases} \tag{5.30}$$

with

$$\alpha_j^+ = \frac{\lambda_{j+1}^+(\lambda_{1,j+1}^{N_{j+1}} - \lambda_{1,j+1}^{N_{j+1}-1}) + \lambda_{1,j+1}^{N_{j+1}} \Delta_{j+1} x}{\lambda_{j+1}^+(\lambda_{2,j+1}^{N_{j+1}} - \lambda_{2,j+1}^{N_{j+1}-1}) + \lambda_{2,j+1}^{N_{j+1}} \Delta_{j+1} x}, \tag{5.31}$$

then the method (5.25) converges in N iterations.

Proof. In order to prove the conditions (4.8)-(4.10) we use (4.5)-(4.6) and the hypothesis (5.28,5.29,5.30,5.31). So, we have

$$g_i^{k+1,-} = \lambda_i^- \delta_x \widehat{u}_{i-1}^k(L_i) + \widehat{u}_{i-1}^k(L_i), \quad for \quad i \geq 2, \tag{5.32}$$

and

$$g_i^{k+1,+} = \lambda_i^+ \delta_x \widehat{u}_{i+1}^k(L_{i+1}) + \widehat{u}_{i+1}^k(L_{i+1}), \quad for \quad i \leq N-1. \tag{5.33}$$

By supposing $g_i^{k+1,-} = 0$ for $i \geq 2$, we find

$$\widehat{u}_i^{k+1}(x_j^{(i)}) = C\lambda_{1,i}^j + D\lambda_{2,i}^j \tag{5.34}$$

and

$$\delta_x \widehat{u}_i^{k+1}(L_i) = \frac{\widehat{u}_i^{k+1}(x_1^{(i)}) - \widehat{u}_i^{k+1}(x_0^{(i)})}{\Delta_i x} = \frac{C\lambda_{1,i} + D\lambda_{2,i} - C - D}{\Delta_i x}$$

So, we have

$$D = -\frac{\lambda_i^-(\lambda_{1,i} - 1) + \Delta_i x}{\lambda_i^-(\lambda_{2,i} - 1) + \Delta_i x} C \tag{5.35}$$

and (5.34) becomes

$$\widehat{u}_i^{k+1}(x_j^{(i)}) = C(\lambda_{1,i}^j - \alpha_i^- \lambda_{2,i}^j) \tag{5.36}$$

with α_i^- given by (5.29). From (5.28,5.29,5.30,5.31) and (5.36) we obtain $g_{i+1}^{k+2,-} = 0$, and so (4.8) is proved. An analogous argument leads to (4.9). The equations $\widehat{\mathcal{L}}(\widehat{u}_i^{k+1}) = 0$ for $i = 1$ and $i = N$, together with the conditions $\widehat{u}_1^{k+1} \to 0$ for $x \to -\infty$ and $\widehat{u}_N^{k+1} \to 0$ for $x \to +\infty$, lead to (4.10), which completes the proof. □

The previous result can be applied when the stepsize Δx is the same on each subdomain. In this case, the operators Λ_i^- and Λ_i^+ assume a more simple form. For example, in the case of two subdomains, they have the symbols given by (5.7) and (5.11).

In order to solve the system (4.1) with $u_i^{k+1}(x_j^{(i)}, 0) = u_0(x_j^{(i)})$, by denoting with $\eta_i^\pm$ the inverse Laplace transform of $\lambda_i^\pm$, i.e.

$$\hat{\eta}_i^\pm = \lambda_i^\pm,$$

the system (4.1) can be written as follows:

$$\left\{\begin{array}{l} \frac{\partial u_i^{k+1}}{\partial t}(x,t) - \frac{\nu}{\Gamma(\beta)} \int_0^t (t-\tau)^{\beta-1} \delta_{xx} u_i^{k+1}(x,\tau) d\tau = 0, \\ \delta_x u_i^{k+1}(L_i,t) + \int_0^t \eta_i^-(t-\tau) u_i^{k+1}(L_i,\tau) d\tau = \\ \qquad\qquad \delta_x u_{i-1}^k(L_i,t) + \int_0^t \eta_i^-(t-\tau) u_{i-1}^k(L_i,\tau) d\tau, \\ \delta_x u_i^{k+1}(L_{i+1},t) + \int_0^t \eta_i^+(t-\tau) u_i^{k+1}(L_{i+1},\tau) d\tau = \\ \qquad\qquad \delta_x u_{i+1}^k(L_{i+1},t) + \int_0^t \eta_i^+(t-\tau) u_{i+1}^k(L_{i+1},\tau) d\tau, \\ u_i^{k+1}(x,0) = u_0(x), \end{array}\right. , \tag{5.37}$$

for $i = 2, ..., N-1$ and where $x \in \tilde{\Omega}_i, x \neq L_i, x \neq L_{i+1}$ and $t > 0$. If we consider for example the case of a finite spatial domain with transparent boundary conditions, in which $\lambda_i^{\pm} = \pm\lambda = \pm\frac{s^\gamma}{\sqrt{\nu}}$, then we have $\eta_i^{\pm}(t) = \pm\frac{t^{-\gamma-1}}{\sqrt{\nu}}$ with $-\gamma - 1 \in (-2, -1)$ and then we can have problems in the convergence of the integrals appearing in the boundary conditions of the method (5.37). For this reason, by integrating with respect to time, we rewrite the method (5.37) as

$$\left\{\begin{array}{l} u_i^{k+1}(x,t) - \frac{\nu}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\delta_{xx}u_i^{k+1}(x,\tau)d\tau = 0 \\ \int_0^t \xi_i^-(t-\tau)\delta_x u_i^{k+1}(L_i,\tau)d\tau + u_i^{k+1}(L_i,t) = \\ \qquad\qquad \int_0^t \xi_i^-(t-\tau)\delta_x u_{i-1}^{k}(L_i,\tau)d\tau + u_{i-1}^{k}(L_i,t) \\ \int_0^t \xi_i^+(t-\tau)\delta_x u_i^{k+1}(L_{i+1},\tau)d\tau + u_i^{k+1}(L_{i+1},t) = \\ \qquad\qquad \int_0^t \xi_i^+(t-\tau)\delta_x u_{i+1}^{k}(L_{i+1},\tau)d\tau + u_{i+1}^{k}(L_{i+1},t) \end{array}\right. , \tag{5.38}$$

where $x \in \tilde{\Omega}_i, x \neq L_i, x \neq L_{i+1}$ and $t > 0$ and where now

$$\hat{\xi}_i^{\pm} = \frac{1}{\lambda_i^{\pm}},$$

and, in the particular case $\lambda_i^{\pm} = \pm\lambda = \pm\frac{s^\gamma}{\sqrt{\nu}}$, we have $\xi_i^{\pm}(t) = \pm\frac{t^{\gamma-1}}{\sqrt{\nu}}$ with $\gamma - 1 \in (-1, 0)$.

Then, by defining

$$\mu_\alpha(t) = \frac{\nu}{\Gamma(\alpha)}t^{\alpha-1}$$

we obtain the system of VIEs

$$U_i^{k+1}(t) = \int_0^t A_i(t-\tau)U_i^{k+1}(\tau)d\tau + G_i^k(t), \quad i = 2, ..., N-1, \tag{5.39}$$

where

$$U_i^{k+1}(t) = \left[\begin{array}{c} u_i^{k+1}(L_i,t) \\ ... \\ u_i^{k+1}(x_j^{(i)},t) \\ ... \\ u_i^{k+1}(L_{i+1},t) \end{array}\right],$$

for j=1,...,$N_i - 1$,

$$G_i^k(t) = \left[\begin{array}{c} u_{i-1}^k(L_i,t) + \int_0^t \xi_i^-(t-\tau)\frac{u_{i-1}^k(L_i,\tau) - u_{i-1}^k(x_{N_i-1},\tau)}{\Delta_{i-1}x}d\tau \\ \\ 0 \qquad\qquad j = 1, ..., N_i - 1 \\ \\ u_{i+1}^k(L_{i+1},t) + \int_0^t \xi_i^+(t-\tau)\frac{u_{i+1}^k(x_1,\tau) - u_{i+1}^k(L_{i+1},\tau)}{\Delta_{i+1}x}d\tau \end{array}\right],$$

and

$$A_i(t) = \begin{bmatrix} \frac{\xi_i^-(t)}{\Delta_i x} & -\frac{\xi_i^-(t)}{\Delta_i x} & 0 & ... & ... & 0 \\ \frac{\mu_\alpha(t)}{\Delta_i x^2} & \frac{-2\mu_\alpha(t)}{\Delta_i x^2} & \frac{\mu_\alpha(t)}{\Delta_i x^2} & 0 & ... & 0 \\ 0 & ... & ... & ... & ... & ... \\ ... & ... & ... & ... & ... & 0 \\ 0 & ... & 0 & \frac{\mu_\alpha(t)}{\Delta_i x^2} & \frac{-2\mu_\alpha(t)}{\Delta_i x^2} & \frac{\mu_\alpha(t)}{\Delta_i x^2} \\ 0 & ... & ... & 0 & \frac{\xi_i^+(t)}{\Delta_i x} & -\frac{\xi_i^+(t)}{\Delta_i x} \end{bmatrix}.$$

If we consider a finite domain with non reflecting boundary conditions, then we have to solve the system of VIEs (5.39) for $i = 1, ..., N$. If we consider Dirichlet or Neumann boundary conditions then the system of VIEs for $i = 1$ and $i = N$ has the form (5.39) but we have to change accordingly the matrices $A_1(t)$, $A_N(t)$, and the vectors $G_1^k(t)$, $G_N^k(t)$. The system of VIEs (5.39) can be solved by using known methods for systems of weakly singular Volterra integral equations (see for example [4–6, 16]).

6. Concluding remarks

We have introduced domain decomposition methods for fractional diffusion-wave equations both on infinite spatial domains and on finite spatial domains. The proofs of the slow convergence of the classical Schwarz waveform relaxation method have been given on unbounded time intervals and on bounded time domains. We have constructed the optimal transmission conditions at first on 2 subdomains and then on $N > 2$ subdomains, showing the convergence of the methods in a finite number of iterations. In the case of finite spatial domain we have considered the original problem together with transparent, Dirichlet or Neumann boundary conditions, providing optimal transmission conditions for each of these cases. We also proved analogous results of convergence when the original problem is spatially discretized.

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