



# On infinite groups in which all abelian subgroups are locally cyclic

Costantino Delizia<sup>1</sup> · Chiara Nicotera<sup>1</sup>

Received: 1 May 2021 / Accepted: 22 June 2021 / Published online: 6 July 2021  
© The Author(s) 2021

## Abstract

The structure of locally soluble periodic groups in which every abelian subgroup is locally cyclic was described over 20 years ago. We complete the aforementioned characterization by dealing with the non-periodic case. We also describe the structure of locally finite groups in which all abelian subgroups are locally cyclic.

**Keywords** Locally cyclic group · Locally finite group · Abelian subgroups

**Mathematics Subject Classification** 20E34 · 20F50 · 20E25

## 1 Introduction and results

In this paper a group is termed *anticommutative* if two of its elements commute only when they generate a cyclic subgroup. The class of anticommutative groups is obviously subgroup closed. On the other hand it is not quotient closed, since all free groups are anticommutative. Notice that a group which has two normal subgroups with trivial intersection both having an element of the same prime order cannot be anticommutative, since these elements generate a non-cyclic abelian subgroup.

It is easy to see that an abelian group is anticommutative if and only if it is locally cyclic. Hence anticommutative groups are precisely those groups in which every abelian subgroup is locally cyclic. Clearly, a locally cyclic group is either periodic or torsion-free. A torsion-free group is locally cyclic if and only if it is a subgroup of the additive group of rational numbers (see, for instance, [8, Exercise 4.2.6]). These

---

Communicated by John S. Wilson.

---

✉ Costantino Delizia  
cdelizia@unisa.it

Chiara Nicotera  
cnicotera@unisa.it

<sup>1</sup> Department of Mathematics, University of Salerno, Via Giovanni Paolo II, 132, Fisciano, SA, Italy

groups have been completely classified by Baer (see, for instance, [3]). Non-trivial subgroups of the additive group of rational numbers are called rational groups. On the other hand, a periodic group is locally cyclic if and only if it is a subgroup of the additive group  $\mathbb{Q}/\mathbb{Z}$ . Such groups decompose into a direct product of Sylow subgroups, which are cyclic or quasicyclic.

A finite group is anticommutative if and only if each of its abelian subgroups is cyclic. This is equivalent to require that its Sylow subgroups are either cyclic or a generalized quaternion group

$$Q_{2^n} = \langle a, x \mid a^{2^{n-1}} = 1, x^2 = a^{2^{n-2}}, x^{-1}ax = a^{-1} \rangle$$

of order  $2^n$  with  $n > 2$ . As usual, we set  $Q_2 = Q_4 = Q_8$ . It is an immediate consequence of the Brauer–Suzuki theorem (see [5, Chap.12, Theorem 1.1]) that a finite anticommutative group cannot be simple. Finite anticommutative groups occur, for instance, as the finite multiplicative subgroups of any division ring. Their structure has been fully described by Zassenhaus [11] and Suzuki [10].

It is much harder to deal with infinite anticommutative groups. First of all, there exist infinite simple anticommutative groups. Easy examples are the Tarski  $p$ -groups, which are infinite groups whose proper non-trivial subgroups have prime order  $p$ . In the sequel, we will refer to the group

$$Q_{2^\infty} = \bigcup_{n \in \mathbb{N}} Q_{2^n}$$

as the infinite quaternion group. This group has a quasicyclic subgroup  $C_{2^\infty}$  of index 2, so it is metabelian. Moreover  $Q_{2^\infty}$  is locally nilpotent but not nilpotent, and every element of  $Q_{2^\infty} \setminus C_{2^\infty}$  has order 4. Let  $C_t$  denote the cyclic group of order  $t \in \mathbb{N}$ . It is well known that a 2-group has a unique element of order 2 if and only if it is isomorphic to  $C_{2^n}$  or  $Q_{2^n}$  for some  $n \in \mathbb{N} \cup \{\infty\}$  (see, for instance, [1, Theorem 8.1]). It easily follows that the infinite quaternion group is anticommutative.

In [6, Lemma 3] locally nilpotent anticommutative groups have been completely classified. It turns out that an infinite nilpotent (respectively, locally nilpotent) group is anticommutative if and only if it is either locally cyclic or isomorphic to the direct product of a group isomorphic to  $Q_{2^n}$  for some  $n \in \mathbb{N}$  (respectively,  $n \in \mathbb{N} \cup \{\infty\}$ ) and a periodic locally cyclic group without elements of order 2. In particular, every infinite locally nilpotent anticommutative group is metabelian. Furthermore, every torsion-free locally nilpotent anticommutative group is abelian.

The structure of periodic locally soluble anticommutative groups was described in [6, Theorems 1 and 2]. The authors also considered the non-periodic case, but then they added the condition that the group has a non-trivial normal periodic subgroup. As they noticed, this condition, together with anticommutativity, implies periodicity. Then essentially only the periodic case is covered in [6]. In Sect. 2 we complete the above description by dealing with non-periodic locally soluble anticommutative groups. Our main results are the following.

**Theorem A.** *A torsion-free locally soluble group is anticommutative if and only if it is isomorphic to a semidirect product  $A \rtimes \langle x \rangle$ , where  $A$  is a rational group and  $x$  is an automorphism of  $A$  having infinite order.*

**Theorem B.** *A mixed locally soluble group is anticommutative if and only if it is isomorphic to a semidirect product  $A \rtimes \langle x \rangle$ , where  $A$  is a rational group,  $x$  has order 2 and  $a^x = a^{-1}$  for all  $a \in A$ .*

Locally soluble anticommutative groups are soluble. This, together with a result due to Šunkov [9], implies that every locally finite anticommutative group is soluble-by-finite (that is, a finite extension of a soluble group). On the other hand, since the direct product of a quasicyclic  $p$ -group by a finite non-soluble anticommutative group of order coprime to  $p$  is anticommutative (see [6, Lemma 1]), there exist locally finite anticommutative groups which are neither soluble nor finite. In Sect. 3 we completely describe the structure of locally finite anticommutative groups. Our result is the following.

**Theorem C.** *A locally finite non-soluble group  $G$  is anticommutative if and only if one of the following holds:*

- (i)  $G = R \times S$ , where  $S \simeq \text{SL}(2, p)$  for some prime  $p > 3$ ,  $R = A \rtimes B$  for some periodic locally cyclic groups  $A$  and  $B$  with  $\pi(A) \cap \pi(B) = \emptyset$ , and  $\pi(R) \cap \pi(\text{SL}(2, p)) = \emptyset$ ;
- (ii)  $G = K \langle x \rangle$ , where  $K = (A \rtimes B) \times S$  is as in (i),  $|x| = 4$ ,  $x^2 \in Z(S)$ ,  $b^x = b$  for every  $b \in B$ ,  $x$  acts on  $A$  as a power automorphism of order at most 2, and on  $S$  as a non-inner automorphism of order 2.

The notation used throughout the paper is standard, and can be found for instance in [8]. In particular,  $Z(G)$  denotes the center of the group  $G$ . As usual,  $\pi(G)$  denotes the set of all prime numbers involved in the factorization of the orders of elements of any periodic group  $G$ .

## 2 Locally soluble anticommutative groups

In this section we describe the structure of locally soluble anticommutative groups. As already mentioned, the periodic case was treated by Kuzennyi and Maznichenko.

**Theorem 2.1** [6, Theorems 1 and 2] *A locally soluble periodic group  $G$  is anticommutative if and only if one of the following holds:*

- (1)  $G$  is locally supersoluble, and either
  - (1.1)  $G = A \rtimes B$ , where  $A$  and  $B$  are periodic locally cyclic groups with  $\pi(A) \cap \pi(B) = \emptyset$ , or
  - (1.2)  $G = A \rtimes (B \times Q)$ , where  $A$  and  $B$  are periodic locally cyclic groups without elements of order 2,  $\pi(A) \cap \pi(B) = \emptyset$ ,  $Q \simeq Q_{2^n}$  for some  $n \in \mathbb{N} \cup \{\infty\}$ , and  $[A, Q'] = 1$ ;

or

(2)  $G$  is not locally supersoluble, and either

- (2.1)  $G = A \rtimes (B \times (Q \rtimes T))$ , where  $A$  and  $B$  are periodic locally cyclic groups without elements of order 2 and 3,  $\pi(A) \cap \pi(B) = \emptyset$ ,  $Q \simeq Q_8$  is a Sylow 2-subgroup of  $G$ ,  $T$  is a cyclic Sylow 3-subgroup of  $G$ ,  $[A, B \times T] = A$ ,  $[A, Q] = 1$ , and  $[Q, T] = Q$ , or
- (2.2)  $G = A \rtimes (B \times ((Q \rtimes T)\langle x \rangle))$ , where  $A$  and  $B$  are periodic locally cyclic groups without elements of order 2 and 3,  $\pi(A) \cap \pi(B) = \emptyset$ ,  $Q \simeq Q_8$ ,  $T$  is a cyclic Sylow 3-subgroup of  $G$ ,  $|x| = 4$ ,  $[A, B \times \langle x \rangle] = A$ ,  $[A, Q \rtimes T] = 1$ ,  $[Q, T] = Q$ ,  $[T, \langle x \rangle] = T$ , and  $Q\langle x \rangle \simeq Q_{16}$ .

In particular, a periodic locally soluble anticommutative group is soluble with derived length at most 4. We start by showing that every non-periodic locally soluble anticommutative group is metabelian.

**Lemma 2.2** *A non-periodic soluble anticommutative group  $G$  has a torsion-free maximal normal abelian subgroup  $A$ , and  $G/A$  is abelian.*

**Proof** Let  $G$  be a non-periodic soluble anticommutative group. We claim that  $G$  has an infinite normal abelian subgroup. Let  $F$  denote the Fitting subgroup of  $G$ . Clearly  $F$  is infinite, otherwise  $G/F$  would be also finite as it is isomorphic to a subgroup of  $\text{Aut}(F)$ , giving the contradiction that  $G$  is finite. Since  $F$  is locally nilpotent and anticommutative, it is metabelian (see Sect. 1). Hence we may assume that the commutator subgroup  $F'$  of  $F$  is finite. This implies that  $F/Z_2(F)$  is finite (see, for instance, [8, 14.5.3]). Thus  $Z_2(F)$  is infinite, and we may assume that it is not abelian. Then, by [6, Lemma 3],  $Z_2(F)$  is isomorphic to a direct product  $D \times C$ , where  $D$  is a generalized quaternion group of order  $2^n$  with  $n \geq 3$  and  $C$  is a locally cyclic group without elements of order 2. Therefore  $C$  is the required subgroup.

Let  $A$  be an infinite maximal normal abelian subgroup of  $G$ . If  $A$  is periodic then it is a direct product of cyclic and quasicyclic  $p$ -groups for pairwise different primes  $p$ . It follows that there exists a finite subgroup  $L$  of  $A$  which is normal in  $G$ , and hence  $G/C_G(L)$  is finite. Let  $g$  be an element of infinite order of  $G$ . Thus  $g^n \in C_G(L)$  for some positive integer  $n$ . Then  $\langle g^n, L \rangle$  is abelian but not locally cyclic, a contradiction since  $G$  is anticommutative. Therefore  $A$  is torsion-free.

Obviously  $A \leq C_G(A)$ . If  $C_G(A) \neq A$  then there exists a non-trivial normal abelian subgroup  $B/A$  of  $C_G(A)/A$ . Since  $A \leq Z(B)$ , this implies that  $B$  is nilpotent. Moreover  $B$  is torsion-free, otherwise there exists an element  $b \in Z(B)$  of finite order, giving the contradiction that  $\langle b, A \rangle$  is abelian and non-cyclic. Therefore  $B$  is abelian, a contradiction by the choice of  $A$ . Hence  $C_G(A) = A$ . It follows that  $G/A$  is isomorphic to a subgroup of  $\text{Aut}(A)$ , and therefore it is abelian (see, for instance, [4, Exercise 4, p. 254]).  $\square$

**Corollary 2.3** *A locally soluble anticommutative group is either periodic or metabelian.*

**Proof** Let  $G$  be a non-periodic locally soluble anticommutative group, and let  $g \in G$  be an element of infinite order. Let  $x_1, x_2, x_3, x_4$  be arbitrary elements of  $G$ . Then  $\langle x_1, x_2, x_3, x_4, g \rangle$  is metabelian by Lemma 2.2. Therefore  $G$  is metabelian.  $\square$

Let  $A$  be an abelian group. Following [7], we denote by  $r_0(A)$  the torsion-free rank of  $A$ , that is the cardinality of any maximal linearly independent subset of elements of infinite order of  $A$ . Moreover, if  $p$  is a prime, we denote by  $r_p(A)$  the  $p$ -rank of  $A$ , that is the cardinality of any maximal linearly independent subset of elements of order  $p$  of  $A$ . Let  $\pi(A)$  denote the set of all primes  $p$  for which there exist elements of order  $p$  in  $A$ . Then we denote by

$$r(A) = r_0(A) + \sum_{p \in \pi(A)} r_p(A)$$

the total rank of  $A$ , and by

$$\widehat{r}(A) = r_0(A) + \max_{p \in \pi(A)} \{r_p(A)\}$$

the reduced rank of  $A$ .

A group  $G$  is said to have finite Prüfer rank  $r$  if every finitely generated subgroup of  $G$  can be generated by  $r$  elements, and  $r$  is the least integer with such property. An abelian group  $A$  has finite Prüfer rank if and only if  $\widehat{r}(A)$  is finite (and then  $\widehat{r}(A)$  equals the Prüfer rank of  $A$ ). It is also easy to see that a soluble group has finite Prüfer rank if and only if has a series of finite length in which each factor is abelian of finite reduced rank. Obviously, anticommutative groups are precisely those groups in which every abelian subgroup has Prüfer rank 1.

A soluble group is said to have finite abelian total rank if it has a series of finite length in which each factor is abelian of finite total rank. Of course, such groups have finite Prüfer rank.

Now we can complete the description of the structure of locally soluble anticommutative groups.

**Theorem A.** *A torsion-free locally soluble group is anticommutative if and only if it is isomorphic to a semidirect product  $A \rtimes \langle x \rangle$ , where  $A$  is a rational group and  $x$  is an automorphism of  $A$  having infinite order.*

**Proof** Let  $G$  be a torsion-free locally soluble anticommutative group. By Corollary 2.3,  $G$  is metabelian. By Lemma 2.2, there exists a normal subgroup  $A$  of  $G$  which is a rational group, and  $G/A$  is abelian.

Let  $g \in G$ , and assume that  $g^n \in A$  for some positive integer  $n$ . Since  $G$  is anticommutative, for every element  $a \in A$  there exists an element  $b \in G$  such that  $\langle a, g^n \rangle = \langle b \rangle$ . So  $a = b^h$  and  $g^n = b^k$  for some integers  $h$  and  $k$ . It follows that  $a^k = (g^n)^h$ , and thus the group  $A/\langle g^n \rangle$  is periodic. Since  $A\langle g \rangle/A \simeq \langle g \rangle/A \cap \langle g \rangle$  is finite (of order dividing  $n$ ), we conclude that  $A\langle g \rangle/\langle g^n \rangle$  is periodic. As  $G$  is soluble,  $A\langle g \rangle/\langle g^n \rangle$  is locally finite. Moreover  $\langle g^n \rangle \leq Z(A\langle g \rangle)$ , hence  $A\langle g \rangle/Z(A\langle g \rangle)$  is locally finite. Thus the commutator subgroup  $(A\langle g \rangle)'$  is locally finite (see [8, Exercise 10.1.3]), and therefore it is trivial since  $G$  which is torsion-free. Hence  $A\langle g \rangle$  is abelian. Since  $G/A$  is abelian,  $A\langle g \rangle$  is normal in  $G$ , and so  $g \in A$  by the maximality of  $A$ . Therefore  $G/A$  is torsion-free.

By [8, Exercise 15.2.7] the group  $G$  has finite abelian total rank. Thus there exists a nilpotent subgroup  $X$  of  $G$  such that  $G/AX$  is finite (see [7, Theorem 10.4.3]). Then

$X$  is infinite, so it is abelian and hence locally cyclic. Hence  $AX/A \simeq X/A \cap X$  is locally cyclic. Let  $g$  and  $h$  be arbitrary elements of  $G$ . Then there exists a positive integer  $n$  such that  $g^n, h^n \in AX$ . Write  $g^n = ay$  and  $h^n = bz$ , where  $a, b \in A$  and  $y, z \in X$ . Since  $AX/A$  is locally cyclic, the subgroup  $\langle g^n, h^n \rangle A = \langle y, z \rangle A$  is cyclic. As  $G/A$  is torsion-free abelian, it follows that  $\langle g, h \rangle A \simeq \langle g, h \rangle^n A = \langle g^n, h^n \rangle A$  is cyclic. This means that  $G/A$  is locally cyclic.

The group  $G/A$  is isomorphic to a subgroup of  $\text{Aut}(A)$ . By [4, Example 5, p. 250], the latter is the direct product of a cyclic group of order 2 by a free abelian group. Since  $G/A$  is torsion-free and locally cyclic, it is infinite cyclic. Hence  $G/A = \langle Ax \rangle = A\langle x \rangle/A \simeq \langle x \rangle/\langle x \rangle \cap A$ . It follows that  $\langle x \rangle \cap A$  is trivial. By the maximality of  $A$ , this implies that  $x^n$  cannot act trivially on  $A$  for any integer  $n \neq 0$ . Therefore  $x$  is an automorphism of  $A$  having infinite order, and  $G = A \rtimes \langle x \rangle$ , as required.

Conversely, let  $G = A \rtimes \langle x \rangle$ , where  $A$  is a rational group and  $x$  is an automorphism of  $A$  having infinite order. Since  $x$  is induced by an automorphism of the additive group of rational numbers, it acts on  $A$  via multiplication by a suitable rational number. Hence every power of  $x$  acts fixed point freely on  $A$ . This implies that an element of  $G$  which commutes with a non-trivial element of  $A$  must belong to  $A$ . Now let consider any abelian subgroup  $B$  of  $G$ . If  $A \cap B = \{1\}$  then  $B \simeq AB/A$  is cyclic. Otherwise, let  $a$  be any non-trivial element of  $A \cap B$ . Since every element of  $B$  commutes with  $a$ , we get  $B \leq A$ . Therefore  $B$  is locally cyclic, and  $G$  is anticommutative.  $\square$

**Theorem B.** *A mixed locally soluble group is anticommutative if and only if it is isomorphic to a semidirect product  $A \rtimes \langle x \rangle$ , where  $A$  is a rational group,  $x$  has order 2 and  $a^x = a^{-1}$  for all  $a \in A$ .*

**Proof** Let  $G$  be a mixed locally soluble anticommutative group. By Corollary 2.3,  $G$  is metabelian. By Lemma 2.2, there exists a normal subgroup  $A$  of  $G$  which is a rational group, and  $G/A$  is abelian. The group  $G/A$  is isomorphic to a subgroup of  $\text{Aut}(A)$ . Hence, as above,  $G/A = T/A \times C/A$  where  $T/A$  has order 2 and  $C/A$  is free abelian. We claim that  $C/A$  is trivial.

Assume, for a contradiction, that  $C/A$  is not trivial. Then Theorem A yields  $C = A \rtimes \langle x \rangle$  for some element  $x$  of infinite order. Let  $g \in G$  be any element of finite order. Since  $G/C$  has order 2 we get  $g^2 \in C$ , hence  $g^2 = 1$ . Therefore  $G = C \rtimes \langle y \rangle$  for some element  $y$  of order 2. For all elements  $a \in A$  we get  $(aa^y)^y = a^y a = aa^y$ , hence  $\langle aa^y, y \rangle$  is abelian. Since  $G$  is anticommutative,  $A$  is torsion-free and  $y$  has finite order, this implies that  $aa^y = 1$ , and thus  $a^y = a^{-1}$  for all  $a \in A$ . Now let  $B = A^2$ . Clearly  $B$  is normal in  $G$  and  $A/B$  is an elementary abelian 2-group of rank 1, so it has order 2. Hence  $A/B \leq Z(G/B)$ . Then  $C/B = A/B \times \langle x \rangle B/B$ . It follows that  $\langle x^2 \rangle B/B = \langle x^2 B \rangle = (C/B)^2$  is normal in  $G/B$ , and so  $\langle x^2 \rangle B$  is normal in  $G$ . Since  $\langle x \rangle \cap A$  is trivial, the group  $\langle x^2 \rangle B/B$  is infinite cyclic. Thus either  $(x^2)^y B = x^2 B$  or  $(x^2)^y B = x^{-2} B$ . First suppose  $(x^2)^y B = x^2 B$ . Then  $(x^2)^y = x^2 b$  for a suitable  $b \in B$ , and of course there exists  $a \in A$  such that  $b = a^2$ . It follows that  $(x^2 a)^y = (x^2)^y a^y = x^2 a^2 a^{-1} = x^2 a$ , so  $\langle x^2 a, y \rangle$  is abelian and hence cyclic. As  $C$  is torsion-free, this implies that  $x^2 a = 1$  and so  $x^2 \in A$ , a contradiction. Now suppose  $(x^2)^y B = x^{-2} B$ . Then  $x^{-4} = x^{-2} (x^2)^y b$  for a suitable  $b \in B$ . It follows that  $x^{-4} = [x^2, y] b \in G'A = A$ , again a contradiction. Therefore  $C/A$  is trivial, as claimed.

Then  $G/A$  has order 2. Let  $x$  be an element of finite order of  $G$ . Then  $\langle x \rangle \cap A$  is trivial. For every  $a \in A$ , from  $(aa^x)^x = a^x a = aa^x$  we obtain that  $\langle aa^x, x \rangle$  is abelian, and so it is cyclic. But  $A$  is torsion-free, hence  $aa^x = 1$  and  $a^x = a^{-1}$ . Thus  $x^2 = 1$  and  $G$  has the required structure.

Conversely, let  $G = A \rtimes \langle x \rangle$ , where  $A$  is a rational group,  $x$  has order 2 and  $a^x = a^{-1}$  for all  $a \in A$ . Then an abelian subgroup of  $G$  is either cyclic or contained in  $A$ , and hence it is locally cyclic. Hence  $G$  is anticommutative.  $\square$

### 3 Locally finite anticommutative groups

A finite group is called a  $\mathfrak{3}$ -group if its Sylow subgroups are cyclic. These groups are obviously anticommutative. Their structure is well known (see, for instance, [8, 10.1.10]). In particular, every  $\mathfrak{3}$ -group is metacyclic.

As noticed in Sect. 1, a finite anticommutative group cannot be simple. The structure of finite anticommutative groups was described by Zassenhaus in the soluble case, and by Suzuki in the non-soluble case.

**Theorem 3.1** ([10,11], see also [2]) *A finite group  $G$  is anticommutative if and only if one of the following holds:*

(1)  $G$  is soluble, and either

(1.1)  $G$  is a  $\mathfrak{3}$ -group, or

(1.2)  $G = LQ$ , where  $L$  is a  $\mathfrak{3}$ -group of odd order, and  $Q \simeq Q_{2^n}$  for some  $n \in \mathbb{N}$ ;

or

(2)  $G$  is not soluble, and either

(2.1)  $G = L \times S$ , where  $L$  is a  $\mathfrak{3}$ -group,  $S \simeq \text{SL}(2, p)$  for some prime  $p > 3$ , and  $(|L|, |S|) = 1$ , or

(2.2)  $G = ((\langle \sigma \rangle \rtimes \langle \rho \rangle) \times S) \langle x \rangle$ , where  $S \simeq \text{SL}(2, p)$  for some prime  $p > 3$ ,  $|\sigma| = m$ ,  $|\rho| = n$ ,  $|x| = 4$ ,  $\sigma^\rho = \sigma^r$  with  $(m, n) = 1$ ,  $(mn, p(p^2 - 1)) = 1$  and  $r^n \equiv 1 \pmod{m}$ ,  $\sigma^x = \sigma^t$  with  $t^2 \equiv 1 \pmod{m}$ ,  $\rho^x = \rho$ ,  $x^2 \in Z(S)$ , and  $x$  acts on  $S$  as a non-inner automorphism of order 2.

Notice that a group  $G$  as in (2.2) of Theorem 3.1 has a subgroup  $H = L \times S$  of index 2, where  $L = \langle \sigma \rangle \rtimes \langle \rho \rangle$  is a  $\mathfrak{3}$ -group,  $S \simeq \text{SL}(2, p)$  for some prime  $p > 3$ , and  $(|L|, |S|) = 1$ . Thus the structure of  $H$  is like that of the group  $G$  in (2.1) of Theorem 3.1.

A locally finite group is anticommutative if and only if its Sylow subgroups are either locally cyclic (and hence cyclic or quasicyclic) or isomorphic to  $Q_{2^n}$  for some  $n \in \mathbb{N} \cup \{\infty\}$  (see [6, Lemma 1]). In what follows we shall say that a locally finite group is a  $\overline{\mathfrak{3}}$ -group if its Sylow subgroups are locally cyclic. Clearly, locally finite  $\overline{\mathfrak{3}}$ -groups are anticommutative. Furthermore, it has been proved by Sesekin and Starostin (see [6, Lemma 2]) that a locally finite group is a  $\overline{\mathfrak{3}}$ -group if and only if it is isomorphic to a semidirect product  $A \rtimes B$ , where  $A$  and  $B$  are periodic locally cyclic groups with  $\pi(A) \cap \pi(B) = \emptyset$ .



The structure of locally finite anticommutative groups which are soluble was described in Theorem 2.1. We complete the above description by dealing with the non-soluble case.

**Theorem C.** *A locally finite non-soluble group  $G$  is anticommutative if and only if one of the following holds:*

- (i)  $G = R \times S$ , where  $S \simeq \text{SL}(2, p)$  for some prime  $p > 3$ ,  $R = A \rtimes B$  for some periodic locally cyclic groups  $A$  and  $B$  with  $\pi(A) \cap \pi(B) = \emptyset$ , and  $\pi(R) \cap \pi(\text{SL}(2, p)) = \emptyset$ ;
- (ii)  $G = K \langle x \rangle$ , where  $K = (A \rtimes B) \times S$  is as in (i),  $|x| = 4$ ,  $x^2 \in Z(S)$ ,  $b^x = b$  for every  $b \in B$ ,  $x$  acts on  $A$  as a power automorphism of order at most 2, and on  $S$  as a non-inner automorphism of order 2.

**Proof** Let  $G$  be a locally finite non-soluble anticommutative group. A result due to Šunkov [9] ensures that there exists a normal soluble subgroup  $R$  of  $G$  such that  $G/R$  is finite. Choose a transversal  $T$  of  $R$  in  $G$ , and set  $X = \langle T \rangle$ . As  $T$  is finite,  $X$  is finite as well. Then  $G = RX$ , hence  $X$  is not soluble. First we show that we can assume that all elements of  $R$  are of odd order.

In fact, if  $Y$  is any finite non-soluble subgroup of  $G$ , by (2) of Theorem 3.1 there exists a normal subgroup  $S$  of  $Y$  such that  $S \simeq \text{SL}(2, p)$  for some prime  $p > 3$ , and  $y^2 \in S$  for every 2-element  $y$  of  $Y$ . Now let  $g \in R$  be a 2-element, and write  $F = \langle X, g \rangle$ . Since  $F$  is a finite non-soluble subgroup of  $G$ , by above there exists a normal subgroup  $S$  of  $F$  such that  $S \simeq \text{SL}(2, p)$  for some prime  $p > 3$ , and  $g^2 \in S$ . Hence  $g^2 \in R \cap S$ , and the latter is a soluble normal subgroup of  $S$ . Then  $g^2 \in Z(S)$  (see, for instance, [8, 3.2.13]). As  $Z(S)$  has order 2, it follows that  $|g| \leq 4$ . Thus  $R$  cannot have 2-elements of order greater than 4. Therefore  $R$  has the structure which is described in (1.1), (1.2) or (2.1) of Theorem 2.1. It easily follows that there exists a subgroup  $V$  of  $R$  such that  $R = V \rtimes L$ , with  $L$  finite and  $2 \notin \pi(V)$ . At expense of replacing  $R$  by  $V$  and  $X$  by  $\langle X, L \rangle$ , we can assume that all elements of  $R$  have odd order.

Therefore  $G = RX$ , where  $X$  is a finite non-soluble subgroup of  $G$ ,  $R$  is a normal subgroup of  $G$ , and every element of  $R$  is of odd order. Hence the structure of  $X$  is that described in (2.1) or in (2.2) of Theorem 3.1. We will analyze these two cases separately.

First suppose that the structure of  $X$  is that in (2.1) of Theorem 3.1. Hence  $X = L \times S$ , where  $L$  is a 3-group,  $S \simeq \text{SL}(2, p)$  for some prime  $p > 3$ , and  $(|L|, |S|) = 1$ . Then  $RL$  is a normal soluble subgroup of  $G$ , and all its elements are of odd order. After replacing  $R$  by  $RL$ , we can assume  $G = RS$ , where  $R$  is soluble and  $S \simeq \text{SL}(2, p)$  for some prime  $p > 3$ . We claim that  $G = R \times S$ .

Clearly  $R \cap S$  is soluble and normal in  $S$ , so  $R \cap S \leq Z(S)$ . Since the elements of  $R$  have odd order, it follows that  $R \cap S = \{1\}$ . Notice that if there exists a subgroup  $W$  of  $G$  containing  $S$ , with  $W \simeq \text{SL}(2, q)$  for some prime  $q > p$ , then from  $W = (R \cap W)S$  it follows that  $R \cap W$  is non-trivial, again a contradiction since  $R \cap W < Z(W)$  which has order 2. Now, given an element  $g \in R$ , consider the finite subgroup  $N = \langle g, S \rangle$  of  $G$ . By (2) of Theorem 3.1 there exists a normal subgroup  $I$  of  $N$  such that  $I \simeq \text{SL}(2, s)$  for some prime  $s > 3$  and  $N/I$  is soluble. As  $I \cap S$  is normal in  $S$ , we obtain either



$I \cap S \leq Z(S)$  or  $I \cap S = S$ . If  $I \cap S \leq Z(S)$  then  $IS/I$  has a quotient which is isomorphic to  $S/Z(S)$ , and the latter is not soluble. This is a contradiction since  $N/I$  is soluble. Therefore  $I \cap S = S$  and  $S \leq I$ . By above this implies  $S = I$ . Thus  $g$  normalizes  $S$ . This holds for every  $g \in R$ , hence  $S$  is normalized by  $R$  and  $RS = R \times S$ , as claimed.

Finally  $R$  is a  $\bar{3}$ -group, so it is isomorphic to a semidirect product  $A \rtimes B$ , where  $A$  and  $B$  are locally cyclic groups with  $\pi(A) \cap \pi(B) = \emptyset$ . Therefore  $G$  has the structure described in (i).

Now suppose that the structure of  $X$  is that in (2.2) of Theorem 3.1, hence  $X$  has a subgroup  $H = L \times S$  of index 2, where  $L$  is a  $\bar{3}$ -group,  $S \simeq \text{SL}(2, p)$  for some prime  $p > 3$ , and  $(|L|, |S|) = 1$ . Moreover  $X = H\langle x \rangle$ , where  $|x| = 4, x^2 \in Z(S)$ , and there exists a non-inner automorphism  $\theta$  of  $S$  of order 2 such that  $s^x = \theta(s)$  for all  $s \in S$ . Write  $K_1 = RH$ , so  $G = K_1\langle x \rangle$ . Arguing as before we get  $K_1 = (A_1 \rtimes B_1) \times S$ , where  $A_1$  and  $B_1$  are locally cyclic groups,  $\pi(A_1) \cap \pi(B_1) = \emptyset$ , and  $\pi(R) \cap \pi(\text{SL}(2, p)) = \emptyset$ . Finally consider the group  $D = (A_1 \rtimes B_1)\langle x \rangle$ . Then  $D$  is a  $\bar{3}$ -group having a Sylow 2-subgroup  $\langle x \rangle$  which is cyclic of order 4, and  $x^2 \in Z(D)$ . By [8, 14.3.4], all Sylow 2-subgroups of  $D$  are cyclic of order 4. By above,  $D = A_2 \rtimes B_2$ , where  $A_2$  and  $B_2$  are locally cyclic groups, and  $\pi(A_2) \cap \pi(B_2) = \emptyset$ . Clearly, either  $2 \in \pi(A_2)$  or  $2 \in \pi(B_2)$ .

If  $2 \in \pi(A_2)$ , then  $\langle x \rangle$  is normal in  $D$ . Hence  $D/C_D(x)$  is isomorphic to a subgroup of  $\text{Aut}(\langle x \rangle)$ , and thus it has order at most 2. This means that  $b^2 \in C_D(x)$  for all  $b \in B_2$ , and so  $b \in C_D(x)$  since the elements of  $B_2$  have odd order. Therefore  $x \in Z(D)$ , and  $G = K_1\langle x \rangle = ((A_1 \rtimes B_1) \times S)\langle x \rangle$  has the structure described in (ii) with  $A = A_1, B = B_1$  and  $K = K_1$ .

If  $2 \in \pi(B_2)$ , then  $x$  fixes every element of  $B_2$ . Moreover  $x$  fixes all subgroups of every Sylow subgroup of  $A_2$ , and thus it fixes all subgroups of  $A_2$ . Therefore  $x$  acts as a power automorphism of order at most 2 on  $A_2$ . Thus again  $G$  has the structure described in (ii), where  $A = A_2$  and  $B$  is the direct product of all Sylow  $p$ -subgroups of  $B_2$  with  $p > 2$ .

Conversely, if (i) holds then clearly  $G$  is anticommutative. Now suppose that (ii) holds, and let  $\theta$  be the non-inner automorphism of order 2 which describes the action of  $x$  on  $S$ . We shall prove that  $G$  is anticommutative by proving that every Sylow subgroup of  $G$  is either locally cyclic or isomorphic to a generalized quaternion group. Let  $p$  be an odd prime. Then every Sylow  $p$ -subgroup of  $G$  is contained in  $K$ , and therefore it is locally cyclic because  $K$  is a  $\bar{3}$ -group. Let now  $W$  be a Sylow 2-subgroup of  $G$  containing  $x$ . Then  $K \cap W$  is a Sylow 2-subgroup of  $K$  and hence of  $S$ , so it is a generalized quaternion group. Moreover  $W = \langle x \rangle K \cap W = \langle x \rangle (K \cap W)$ , and  $x^2 \in K \cap W$ . We claim that  $x^2$  is the unique element of  $W$  having order 2. Indeed, let  $w = xk$  with  $k \in K \cap W$  and  $1 = w^2 = xkxk = x^2k^xk$ . Then  $k^xk = x^2 \in Z(S)$ , hence  $k\theta(k) \in Z(S)$ . It follows that  $k^xk = 1$  (see [10, Lemma 9]). Thus  $x^2 = 1$ , a contradiction. So  $W$  has only one element of order 2, as claimed. Therefore it is isomorphic to  $Q_{2^n}$  for some  $n \in \mathbb{N}$ . □

**Open Access** This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article's Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article's Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit <http://creativecommons.org/licenses/by/4.0/>.

## References

1. Banakh, T., Gavrylkiv, V.: Algebra in the superextensions of twinic groups. *Dissert. Math.* **243**, 74 (2010)
2. Bianchi, M., Gillio Berta Mauri, A.: Gruppi a sottogruppi abeliani localmente ciclici. *Istit. Lombardo Accad. Sci. Lett. Rend. A* **115**, 81–93 (1982)
3. Fuchs, L.: *Abelian Groups*. Pergamon Press, Oxford (1960)
4. Fuchs, L.: *Infinite Abelian Groups*, vol. II. Academic Press, New York-London (1973)
5. Gorenstein, D.: *Finite Groups*. Chelsea Publishing Company, New York (2007)
6. Kuzennyi, M.F., Maznichenko, S.V.: Structure of certain classes of groups with locally cyclic abelian subgroups. *Ukrain. Math. J.* **51**, 1824–1838 (1999)
7. Lennox, J.C., Robinson, D.J.S.: *The Theory of Infinite Soluble Groups*. Clarendon Press, Oxford (2004)
8. Robinson, D.J.S.: *A Course in the Theory of Groups*. Springer, New York (1996)
9. Šunkov, V.P.: On locally finite groups of finite rank. *Algebra Logic* **10**, 127–142 (1971)
10. Suzuki, M.: On finite groups with cyclic Sylow subgroups for all odd primes. *Am. J. Math.* **77**, 657–691 (1955)
11. Zassenhaus, H.: Über endliche Fastkörper. *Abh. Math. Sem. Univ. Hamburg* **11**, 187–220 (1935)

**Publisher's Note** Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.