ON OPTIMAL CONTROLS IN COEFFICIENTS FOR ILL-POSED NON-LINEAR ELLIPTIC DIRICHLET BOUNDARY VALUE PROBLEMS

Olha P. Kupenko

National Mining University
Department of System Analysis and Control
Yavornitskii av., 19, 49005 Dnipro, Ukraine
and
Institute for Applied System Analysis
National Academy of Sciences and Ministry of Education and Science of Ukraine
Peremogy av., 37/35, IASA, 03056 Kyiv, Ukraine

Rosanna Manzo

Università degli Studi di Salerno
Dipartimento di Ingegneria dell'Informazione
Ingegneria Elettrica e Matematica Applicata
Via Giovanni Paolo II, 132, 84084 Fisciano (SA), Italy

Abstract. We consider an optimal control problem associated to Dirichlet boundary value problem for non-linear elliptic equation on a bounded domain Ω. We take the coefficient $u(x) \in L^\infty(\Omega) \cap BV(\Omega)$ in the main part of the non-linear differential operator as a control and in the linear part of differential operator we consider coefficients to be unbounded skew-symmetric matrix $A_{skew} \in L^q(\Omega; \mathbb{S}^N_{skew})$. We show that, in spite of unboundedness of the non-linear differential operator, the considered Dirichlet problem admits at least one weak solution and the corresponding OCP is well-possed and solvable. At the same time, optimal solutions to such problem can inherit a singular character of the matrices A^{skew}. We indicate two types of optimal solutions to the above problem and show that one of them can be attained by optimal solutions of regularized problems for coercive elliptic equations with bounded coefficients, using the two-parametric regularization of the initial OCP.

1. **Introduction.** Optimal control in coefficients for partial differential equations is a classical subject initiated by Lurie [25], Lions [23, 24], and Pironneau [29]. Since the range of such optimal control problems is very wide, including as well optimal shape design problems, some problems originating in mechanics and others, this topic has been widely studied by many authors. However, most of these results and methods rely on linear PDEs with bounded coefficients in the main part of elliptic operators, while only a few articles deal with with unbounded and degenerate controls in coefficients, see [2, 8, 12, 20, 21]. The constrained optimal control problem in the coefficients of the principle part of elliptic operator was first

---

2010 *Mathematics Subject Classification.* Primary: 49J20, 35J92; Secondary: 49J45, 49K20.

Key words and phrases. Generalized p-Laplace equations, control in coefficients, variational convergence.

This is a pre-copy-editing, author-produced PDF of an article accepted for publication in DISCRETE AND CONTINUOUS DYNAMICAL SYSTEMS. SERIES B following peer review. The definitive publisher-authenticated version Olha P. Kupenko and Rosanna Manzo (2018). On Optimal Controls in Coefficients for Ill-Posed Non-Linear Elliptic Dirichlet Bounday Value Problems. DISCRETE AND CONTINUOUS DYNAMICAL SYSTEMS. SERIES B, 23(4), 1363-1393, ISSN: 1531-3492, doi: 10.3934/dcdsb.2018155 is available online at https://www.aimsciences.org/article/doi/10.3934/dcdsb.2018155 .

discussed in detail by Casas [3] in case of the classical Laplace equation, where the scalar coefficient u in the $\operatorname{div}(u\nabla\cdot)$ formulation was taken as control. The problem of existence and uniqueness of the underlying boundary value problem and the corresponding optimal control problem was treated and an optimality system has been derived and analyzed. Analogous results for the case of of the weighted p-Laplacian $\operatorname{div}(u|\nabla y|^{p-2}\nabla y)$ were recently obtained by Casas, Kogut, and Leugering in [5]. However, the principle questions related with the study of optimal control problems in coefficients for the general case of quasilinear elliptic operator $\operatorname{div}(a(u,\nabla\cdot)$ remain open.

In this article we treat the case of the perturbed p-Laplacian $\operatorname{div}(u|\nabla y|^{p-2}\nabla y + A_{skew}\nabla y)$ with unbounded matrix of coefficients A_{skew}. Namely, we study the following optimal control problem (OCP) for a nonlinear elliptic equation containing generalized p-Laplacian with unbounded coefficients in the linear part of the elliptic operator:

$$\text{Minimize } \left\{ I(u,y) = \int_\Omega |y-y_d|^2\,dx + \int_\Omega |\nabla y|^p u\,dx \right\}, \tag{1}$$

subject to constrains

$$-\operatorname{div}\left(u|\nabla y|^{p-2}\nabla y\right) - \operatorname{div}(A_{skew}\nabla y) = -\operatorname{div} f \quad \text{in } \Omega, \tag{2}$$

$$u \in \mathfrak{A}_{ad} \subset L^\infty(\Omega)\cap BV(\Omega),\ y \in W_0^{1,p}(\Omega), \tag{3}$$

where $p\ge 2$, $\mathfrak{A}_{ad}$ is a class of admissible controls, $y_d\in L^2(\Omega)$ and $f\in L^2(\Omega;\mathbb{R}^N)$ are given distributions, $A_{skew}\in L^q(\Omega;\mathbb{R}^{N\times N})$ is a given skew-symmetric matrix, $q=p/(p-1)$.

It is worth to emphasize that the first term in (1) is of the tracking-type, whereas the second term in (1) refers to the energy of a system, described by the given boundary value problem (for the details, we refer to the book [19]). So, the physical motivation of the cost functional is as follows: the state of the system is being driven as close to a target function y_d as possible, while the energy term should be minimal as well.

The characteristic feature of such optimal control problem is the unboundedness of skew-symmetric matrix $A_{skew}\in L^q(\Omega;\mathbb{R}^{N\times N})$. As it was indicated in [12, 13, 14, 17, 37], this circumstance can lead to the existence of elements $y\in W_0^p(\Omega)$ such that $y\notin L^\infty(\Omega)$,

$$\int_\Omega \left(\nabla\varphi_n, A_{skew}\nabla\varphi_n\right)_{\mathbb{R}^N} dx = 0\ \forall\, n\in\mathbb{N},\ \text{and}\ \lim_{n\to\infty}\int_\Omega \left(\nabla\varphi_n, A_{skew}\nabla y\right)_{\mathbb{R}^N} dx \ne 0$$

where $C_0^\infty(\Omega)\ni\varphi_n\to y$ strongly in $W_0^p(\Omega)$. As a result, the existence, uniqueness, and variational properties of the weak solution to the Dirichlet problem (2)–(3) usually are drastically different from the corresponding properties of solutions to the elliptic equations with coercive L^∞-matrices in coefficients (we refer to [6, 30, 31, 32, 34] for the details and other results in this field). However, it is worth to emphasize that under some special restrictions on the skew-symmetric matrix A_{skew} that do not eliminate its local unboundedness property, the corresponding Dirichlet boundary value problem (2)–(3) can admit a unique solution for each $u\in\mathfrak{A}_{ad}$ (see, for instance, [7, 26, 28]).

The aim of this work is to study the existence of optimal controls to the problem (1)–(3) under rather general assumptions on matrix $A_{skew}\in L^q(\Omega;\mathbb{R}^{N\times N})$ and propose the scheme of their approximation. Using the direct method in the Calculus of variations, we show in Section 3 that the original OCP admits at least one solution $(u^0,y^0)\in L^\infty(\Omega)\times W_0^{1,p}(\Omega)$ such that y^0 is a weak solution in the sense of Minty to the corresponding Dirichlet problem (2)–(3). However, in this case it is unknown whether the optimal state y^0 is a weak solution to BVP (2)–(3) in the sense of distributions. Besides, an important aspect in the study of any optimization problem is deriving of the corresponding optimality conditions. In the case of OCP (1)–(3), one faces up a number of difficulties on this way concerned with degeneracies the weighted p-Laplacian $-\mathrm{div}\left(u|\nabla y|^{p-2}\nabla y + A_{skew}\nabla y\right)$ as ∇y tends to zero and also if u approaches zero. Moreover, when the term $u|\nabla y|^{p-2}I + A_{skew}$ is regarded as the coefficient of the Laplace operator, we also have the case of unbounded coefficients. In order to avoid degeneracy and singularity in this case, we introduce in Section 4 the regularization of the differential operator in the left-hand side of equation (2) (see [5] for comparison) that leads to a sequence of monotone and bounded operators

$$\mathcal{A}_{\varepsilon,k,u}y := -\mathrm{div}(u(\varepsilon + \mathcal{F}_k(|\nabla y|^2))^{\frac{p-2}{2}}\nabla y) - \mathrm{div}(A^k_{skew}\nabla y),$$

where A^k_{skew} is an L^∞-approximations of the unbounded skew-symmetric matrix A_{skew}. As a result, we deal with a two-parameter family of optimal control problems in the coefficients for elliptic Dirichlet boundary value problems with coercive and strictly monotone operators. We show that such OCPs with regularized p-Laplacian are well-posed and have nonempty set of solutions in the classical space $L^\infty(\Omega) \times H^1_0(\Omega)$ for each $k \in \mathbb{N}$ and $\varepsilon > 0$. The main question we study in Section 5 is about the asymptotic behaviour of optimal solutions to the approximate OCPs as $k \to \infty$ and $\varepsilon \to 0$. We show that any sequence of optimal pairs $\left\{\left(u^0_{\varepsilon_n,k_n}, y^0_{\varepsilon_n,k_n}\right)\right\}_{n\in\mathbb{N}}$ to the approximate OCPs is compact with respect to the product of weak-$*$ topology of $BV(\Omega)$ and the weak topology of $H^1_0(\Omega)$. Moreover, all its cluster pairs belong to $(L^\infty(\Omega) \cap BV(\Omega)) \times W^{1,p}_0(\Omega)$. As a main result of Section 5, we propose the sufficient conditions under which some optimal solutions of the original optimal control problem (1)–(3) can be attained through optimal solutions to regularized problems. In the last section (Section 6) we focus on the deriving of the first order optimality conditions to regularized OCPs.

2. **Notation and preliminaries.** Throughout the paper Ω is a bounded open subset of $\mathbb{R}^N$, $N \geq 2$, for which Poincaré's inequality holds. For real numbers $2 \leq p < +\infty$ and $1 < q < +\infty$ such that $1/p + 1/q = 1$, let $W^{1,p}_0(\Omega)$ be the closure of $C^\infty_0(\Omega)$ in the Sobolev space $W^{1,p}(\Omega)$ and $W^{1,p}_0(\Omega)$ is endowed with the norm $\|y\|_{W^{1,p}_0(\Omega)} = \left(\int_\Omega |\nabla y|^p_{\mathbb{R}^N}\right)^{1/p}$. Let $W^{-1,q}(\Omega)$ be the dual space of $W^{1,p}_0(\Omega)$.

Let $\mathbb{M}^N$ be the set of all $N \times N$ real matrices. We denote by $\mathbb{S}^N_{skew}$ the set of all skew-symmetric $N \times N$-matrices $C = [c_{ij}]^N_{i,j=1}$ and by $\mathbb{S}^N_{sym}$ the set of all $N \times N$ symmetric matrices. Thus, if $C \in \mathbb{S}^N_{skew}$ then $c_{ij} = -c_{ji}$ and, hence, $c_{ii} = 0$. By matrix norm in $\mathbb{M}^N$ (and for functions with values in $\mathbb{S}^N_{skew}$ as well) we mean a sub-multiplicative norm $\|A\| := \sup\left\{|A\xi|_{\mathbb{R}^N} \ : \ \xi \in \mathbb{R}^N \text{ with } |\xi|_{\mathbb{R}^N} = 1\right\}$. It is worth to note that, in the case of Euclidean norm $|\cdot|_{\mathbb{R}^N}$, the norm $\|A\|$ can be computed as the spectral norm $\|A\| = \sqrt{\lambda_{max}(A^tA)}$, where $\lambda_{max}(A^tA)$ is the largest eigenvalue of the positive-semidefinite matrix A^tA. Let $L^q\left(\Omega; \mathbb{S}^N_{skew}\right)$ be the normed space of measurable integrable with power q functions whose values are skew-symmetric matrices.

Let χ_E be the characteristic function of a set $E \subset \mathbb{R}^N$ and let $|E|$ be its N-dimensional Lebesgue measure. For any vector field $\mathbf{v} \in L^q(\Omega; \mathbb{R}^N)$, its divergence is an element of the space $W^{-1,q}(\Omega)$ defined by the formula

$$\langle \mathrm{div}\,\mathbf{v}, \varphi\rangle_{W^{-1,q}(\Omega);W^{1,p}_0(\Omega)} = -\int_\Omega (\mathbf{v}, \nabla\varphi)_{\mathbb{R}^N}\,dx, \quad \forall\,\varphi \in W^{1,p}_0(\Omega). \tag{4}$$

Here $\langle\cdot,\cdot\rangle_{W^{-1,q}(\Omega);W^{1,p}_0(\Omega)}$ denotes the duality pairing between spaces $W^{-1,q}(\Omega)$ and $W^{1,p}_0(\Omega)$, and $(\cdot,\cdot)_{\mathbb{R}^N}$ denotes the scalar product of two vectors in $\mathbb{R}^N$.

Functions with bounded variations. Let $f:\Omega\to\mathbb{R}$ be a function of $L^1(\Omega)$. Define

$$TV(f):=\int_\Omega |Df| \\ = \sup\Big\{\int_\Omega f\,(\nabla,\varphi)_{\mathbb{R}^N}\,dx:\ \varphi\in C_0^1(\Omega;\mathbb{R}^N),\ |\varphi(x)|\le 1 \text{ for } x\in\Omega\Big\},$$

where $(\nabla,\varphi)_{\mathbb{R}^N}=\sum_{i=1}^N\frac{\partial\varphi_i}{\partial x_i}$.

According to the Radon-Nikodym theorem, if $TV(f)<+\infty$ then the distribution Df is a measure and there exist a vector-valued function $\nabla f\in L^1(\Omega;\mathbb{R}^N)$ and a measure D_sf, singular with respect to the N-dimensional Lebesgue measure $\mathcal{L}^N\lfloor\Omega$ restricted to Ω, such that $Df=\nabla f\mathcal{L}^N\lfloor\Omega+D_sf$.

Definition 2.1. A function $f\in L^1(\Omega)$ is said to have a bounded variation in Ω if $TV(f)<+\infty$. By $BV(\Omega)$ we denote the space of all functions in $L^1(\Omega)$ with bounded variation, i.e. $BV(\Omega)=\left\{f\in L^1(\Omega)\ :\ TV(f)<+\infty\right\}$.

Under the norm $\|f\|_{BV(\Omega)}=\|f\|_{L^1(\Omega)}+TV(f)$, $BV(\Omega)$ is a Banach space. For our further analysis, we need the following properties of BV-functions (see [10]):

Proposition 2.1. (i) *Let $\{f_k\}_{k=1}^\infty$ be a sequence in $BV(\Omega)$ strongly converging to some f in $L^1(\Omega)$ and satisfying condition $\sup_{k\in\mathbb{N}}TV(f_k)<+\infty$. Then*

$$f\in BV(\Omega)\quad\text{and}\quad TV(f)\le\liminf_{k\to\infty}TV(f_k);$$

(ii) *for every $f\in BV(\Omega)\cap L^r(\Omega)$, $r\in[1,+\infty)$, there exists a sequence $\{f_k\}_{k=1}^\infty\subset C^\infty(\Omega)$ such that $\lim_{k\to\infty}\int_\Omega|f-f_k|^r\,dx=0$ and $\lim_{k\to\infty}TV(f_k)=TV(f)$;*

(iii) *for every bounded sequence $\{f_k\}_{k=1}^\infty\subset BV(\Omega)$ there exists a subsequence, still denoted by f_k, and a function $f\in BV(\Omega)$ such that $f_k\to f$ in $L^1(\Omega)$.*

Admissible Controls and Generalized p-Laplacian. Let α, β, γ, and m be given positive constants such that $0<\alpha\le\beta<+\infty$ and $\alpha|\Omega|\le m\le\beta|\Omega|$. We define the class of admissible controls $\mathfrak{A}_{ad}$ as follows

$$\mathfrak{A}_{ad}=\Big\{u\in BV(\Omega)\cap L^\infty(\Omega)\ \Big| \\ TV(u)\le\gamma,\ \|u\|_{L^1(\Omega)}=m,\ \alpha\le u(x)\le\beta \text{ a.e. in }\ \Omega\Big\}. \tag{5}$$

Remark 2.1. Typically the controls in the coefficients of the principal part of non-linear elliptic operator reflect the physical properties of materials or systems such as conductivity, elasticity, and etc. From this point of view it is reasonable to suppose that such coefficients are essentially bounded and strictly positive functions. So, $L^\infty(\Omega)$ arguably looks as the natural functional space for the controls in this case. However, the subtle point here is the choice of the appropriate topology with respect to which each minimizing sequence is convergent. Moreover, since the solvability of the considered optimal control problem strictly depends on properties of the mapping $L^\infty(\Omega)\ni u\mapsto y(u)\in W_0^{1,p}(\Omega)$ (this mapping must be at least continuous), it follows from our further analysis that it would be reasonable to consider the mapping $u\mapsto y(u)$ with respect to the strong topology of $L^p(\Omega)$ for the controls and the weak topology of $W_0^{1,p}(\Omega)$ for the corresponding states. This is the main reason why we choose the control set to be bounded in the space of BV-functions. In this case it can be shown (see Proposition 3.2 below) that the set of admissible controls $\mathfrak{A}_{ad}$ given by the rule (5) is a nonempty convex and compact subset of $L^p(\Omega)$ with an empty topological interior. Moreover, as it can be seen further, in this case the the corresponding optimal control problem admits a nonempty set of solutions.

The optimal control problem we consider in this paper is to minimize the cost functional (1), where $y_d \in L^2(\Omega)$ and $y \in W_0^{1,p}(\Omega)$ is a weak solution to the boundary value problem (2) by choosing an appropriate function $u \in \mathfrak{A}_{ad}$ as control. Here by $\Delta_p : \mathfrak{A}_{ad} \times W_0^{1,p}(\Omega) \to W^{-1,q}(\Omega)$ we denote the generalized p-Laplacian which can be defined by the rule

$$\Delta_p(u,y) = -\mathrm{div}\left(u(x)|\nabla y|^{p-2}\nabla y\right), \text{ where } |\nabla y|^{p-2} := |\nabla y|^{p-2}_{\mathbb{R}^N} = \left(\sum_{i=1}^N \left|\frac{\partial y}{\partial x_j}\right|^2\right)^{\frac{p-2}{2}},$$

or via the paring

$$\langle \Delta_p(u,y), v\rangle_{W^{-1,\,q}(\Omega);W_0^{1,p}(\Omega)} = \int_\Omega u(x)|\nabla y|^{p-2}\left(\nabla y, \nabla v\right)_{\mathbb{R}^N}\,dx,\ \forall\, v \in W_0^{1,p}(\Omega).$$

It is easy to see that for every admissible control $u \in \mathfrak{A}_{ad}$, the operator $\Delta_p(u,\cdot)$ turns out to be strictly monotone, coercive, and semi-continuous, where the above mentioned properties for the operator $\mathcal{A} : X \to X^*$ acting in a Banach space X have the following meaning (see [11, 23, 33]):

$$\langle \mathcal{A}y - \mathcal{A}v, y-v\rangle_{X^*;X} \ge 0,\ \forall\, y, v \in X;\ \langle \mathcal{A}y - \mathcal{A}v, y-v\rangle_{X^*;X} = 0 \implies y = v; \quad (6)$$

$$\frac{\langle \mathcal{A}y, y\rangle_{X^*;X}}{\|y\|_X} \to +\infty \text{ provided } \|y\|_{X\to\infty}; \quad (7)$$

$$\mathbb{R} \ni t \mapsto \langle \mathcal{A}(y+tv), w\rangle_{X^*;X} \text{ is continuous } \forall\, y, v, w \in X, \quad (8)$$

respectively. In what follows, we associate with $A_{skew} \in L^q(\Omega; \mathbb{S}^N_{skew})$ the bilinear skew-symmetric form

$$\Phi(y,v) = \int_\Omega \left(\nabla v, A_{skew}\nabla y\right)_{\mathbb{R}^N} dx, \quad \forall\, y, v \in C_0^\infty(\Omega). \quad (9)$$

It is easy to see, that the form $\Phi(y,v)$ is unbounded on $W_0^{1,p}(\Omega)$, since there is no reason to assume that $\left(\nabla v, A_{skew}\nabla y\right)_{\mathbb{R}^N} \in L^1(\Omega)$ for all $y, v \in W_0^{1,p}(\Omega)$, in general. However, if we temporary assume that $A_{skew} \in L^\infty(\Omega; \mathbb{S}^N_{skew})$, then the bilinear form $\Phi(\cdot,\cdot)$ becomes bounded for each $y, v \in W_0^{1,p}(\Omega)$.

In order to deal with the case $A_{skew} \notin L^\infty(\Omega; \mathbb{S}^N_{skew})$, we notice that if $A_{skew} \in L^q\left(\Omega; \mathbb{S}^N_{skew}\right)$ then the integral $\int_\Omega \left(\nabla\varphi, A_{skew}\nabla y\right)_{\mathbb{R}^N} dx$ is well defined for every $y \in W_0^{1,p}(\Omega)$ and $\varphi \in C_0^\infty(\Omega)$. Indeed,

$$\begin{aligned}|\Phi(y,\varphi)| &:= \left|\int_\Omega \left(\nabla\varphi, A_{skew}\nabla y\right)_{\mathbb{R}^N} dx\right| \le \|\varphi\|_{C^1(\overline{\Omega})}\left(\int_\Omega |A_{skew}\nabla y|_{\mathbb{R}^N}\,dx\right)\\ &\le \|\varphi\|_{C^1(\overline{\Omega})}\left(\int_\Omega \left\|A_{skew}\right\|^q dx\right)^{1/q}\left(\int_\Omega |\nabla y|^p_{\mathbb{R}^N}\,dx\right)^{1/p}\\ &\le \|\varphi\|_{C^1(\overline{\Omega})}\|A_{skew}\|_{L^q(\Omega;\mathbb{S}^N_{skew})}\|y\|_{W_0^{1,p}(\Omega)} < +\infty.\end{aligned}$$

In what follows, we set

$$[y,\varphi]_{A_{skew}} = \int_\Omega \left(\nabla\varphi, A_{skew}\nabla y\right)_{\mathbb{R}^N} dx\ \forall\, y \in W_0^{1,p}(\Omega),\ \forall\,\varphi \in C_0^\infty(\Omega). \quad (10)$$

Let $I_k : \mathbb{U}_k \times \mathbb{Y}_k \to \overline{\mathbb{R}}$ be a cost functional, $\mathbb{Y}_k$ be a space of states, and $\mathbb{U}_k$ be a space of controls. Let $\min\left\{I_k(u,y) \ :\ (u,y) \in \Xi_k\right\}$ be a parameterized OCP, where

$$\Xi_k \subset \left\{(u_k, y_k) \in \mathbb{U}_k \times \mathbb{Y}_k \ :\ u_k \in U_k,\ I_k(u_k,y_k) < +\infty\right\}$$

is a set of all feasible pairs linked by some state equation. In what follows we make a difference between the notations $\inf\limits_{(u,y)\in\,\Xi_k} I_k(u,y)$ and $\left\langle \inf\limits_{(u,y)\in\,\Xi_k} I_k(u,y)\right\rangle$. The first one means the infimum $m = \inf\left\{I_k(u,y) \ :\ (u,y) \in \Xi_k\right\}$ of I_k over the set Ξ_k, whereas by the second one, we mean the constrained minimization problem as an object being defined by the triplet $(I_k, \Xi_k, \mathbb{U}_k \times \mathbb{Y}_k)$. Hereinafter we always associate to such OCP the corresponding constrained minimization problem:

$$(\mathrm{CMP_k}) \ : \qquad \left\langle \inf_{(u,y)\in\,\Xi_k} I_k(u,y) \right\rangle. \tag{11}$$

Since each of constrained minimization problems (11) lives in variable spaces $\mathbb{U}_k \times \mathbb{Y}_k$, we assume that there exists a Banach space $\mathbb{U} \times \mathbb{Y}$ with respect to which a convergence in the scale of spaces $\{\mathbb{U}_k \times \mathbb{Y}_k\}_{k\in\mathbb{N}}$ is well defined (for the details, we refer to [19, 36]). In the sequel, we use the following notation for this convergence $(u_k, y_k) \xrightarrow{\tau} (u,y)$ in $\mathbb{U}_k \times \mathbb{Y}_k$. Moreover, we assume that every bounded sequence in variable space $\mathbb{U}_k \times \mathbb{Y}_k$ is sequentially compact with respect to the τ-convergence.

In order to study the asymptotic behaviour of a family of $(\mathrm{CMP_k})$, the passage to the limit in (11) as the parameter k tends to $+\infty$ has to be realized. The expression "passing to the limit" means that we have to find a kind of "limit cost functional" I and "limit set of constraints" Ξ with a clearly defined structure such that the limit object $\left\langle \inf_{(u,y)\in\Xi} I(u,y)\right\rangle$ could be interpreted as some OCP.

Following the scheme of the direct variational convergence [19], we adopt the following definition for the convergence of minimization problems in variable spaces.

Definition 2.2. A problem $\left\langle \inf_{(u,y)\in\Xi} I(u,y)\right\rangle$ is the variational τ-limit of (11) as $k\to\infty$, if and only if the following conditions are satisfied:

(d) If sequences $\{k_n\}_{n\in\mathbb{N}}$ and $\{(u_n,y_n)\}_{n\in\mathbb{N}}$ are such that $k_n\to\infty$ as $n\to\infty$, $(u_n,y_n)\in\Xi_{k_n}$ $\forall\, n\in\mathbb{N}$, and $(u_n,y_n)\xrightarrow{\tau}(u,y)$ in $\mathbb{U}_{k_n}\times\mathbb{Y}_{k_n}$, then

$$(u,y)\in\Xi; \quad I(u,y)\le \liminf_{n\to\infty} I_{k_n}(u_n,y_n); \tag{12}$$

(dd) For every $(u,y)\in\Xi\subset\mathbb{U}\times\mathbb{Y}$, there are an integer $k^0>0$ and a sequence $\{(u_k,y_k)\}_{k\in\mathbb{N}}$ (called a Γ-realizing sequence) such that

$$(u_k,y_k)\in\Xi_k,\ \forall\, k\ge k^0, \quad (u_k,y_k)\xrightarrow{\tau}(\widehat{u},\widehat{y}) \ \text{ in } \ \mathbb{U}_k\times\mathbb{Y}_k, \tag{13}$$

$$I(u,y)\ge \limsup_{k\to\infty} I_k(u_k,y_k). \tag{14}$$

Then the following result takes place [19].

Theorem 2.3. *Assume that the constrained minimization problem*

$$\left\langle \inf_{(u,y)\in\Xi_0} I_0(u,y)\right\rangle \tag{15}$$

is the variational τ-limit of (11) in the sense of Definition 2.2 and this problem has a nonempty set of solutions

$$\Xi_0^{opt} := \left\{ (u^0,y^0)\in\Xi_0 \ : \ I_0(u^0,y^0) = \inf_{(u,y)\in\Xi_0} I_0(u,y)\right\}.$$

For every $k\in\mathbb{N}$, let $(u_k^0,y_k^0)\in\Xi_k$ be a minimizer of I_k on the corresponding set Ξ_k. If the sequence $\{(u_k^0,y_k^0)\}_{k\in\mathbb{N}}$ is relatively compact with respect to the τ-convergence in variable spaces $\mathbb{U}_k\times\mathbb{Y}_k$, then there exists a pair $(u^0,y^0)\in\Xi_0^{opt}$ such that (up to a subsequence)

$$(u_k^0,y_k^0)\xrightarrow{\tau}(u^0,y^0) \quad \text{in} \ \ \mathbb{U}_k\times\mathbb{Y}_k, \tag{16}$$

$$\inf_{(u,y)\in\,\Xi_0} I_0(u,y) = I_0\left(u^0,y^0\right) = \lim_{k\to\infty} I_k(u_k^0,y_k^0) = \lim_{k\to\infty}\ \inf_{(u_k,y_k)\in\,\Xi_k} I_k(u_k,y_k). \tag{17}$$

3. **Setting of the optimal control problem.** Let $f:\Omega\to\mathbb{R}^N$ be a vector-valued function such that $f\in L^2(\Omega;\mathbb{R}^N)$. The optimal control problem (1)–(3) we consider in this paper is to minimize the discrepancy (tracking error) between a given distribution $y_d\in L^2(\Omega)$ and a solution y of the Dirichlet boundary value problem (2)–(3) by choosing an appropriate control function $u(x)\in\mathfrak{A}_{ad}$.

Definition 3.1. Let $u \in \mathfrak{A}_{ad}$, $A_{skew} \in L^q(\Omega; \mathbb{S}^N_{skew})$, and $f \in L^2(\Omega; \mathbb{R}^N)$ be given distributions. We say that a function $y = y(u, A_{skew}, f)$ is a weak solution in the Minty sense (weak Minty solution) to boundary value problem (2),(3) if $y \in W_0^{1,p}(\Omega)$ and the inequality

$$\int_\Omega u(x)|\nabla\varphi|^{p-2}(\nabla\varphi, \nabla\varphi - \nabla y)_{\mathbb{R}^N}\,dx + \int_\Omega (A_{skew}\nabla\varphi, \nabla\varphi - \nabla y)_{\mathbb{R}^N}\,dx \geq \int_\Omega (f, \nabla\varphi - \nabla y)_{\mathbb{R}^N}\,dx \quad (18)$$

holds for any $\varphi \in C_0^\infty(\Omega)$.

We note that by the initial assumptions and Hölder's inequality, this definition makes a sense because $(A_{skew}\nabla y) \in L^1(\Omega; \mathbb{R}^N)$ for each $y \in W_0^{1,p}(\Omega)$.

It is worth to notice that the original boundary value problem (2)–(3) is ill-possed, in general. Moreover, since the skew-symmetric form (9) can be unbounded on $W_0^{1,p}(\Omega)$, the existence of a weak solution in the Minty sense to (2),(3) for fixed $u \in \mathfrak{A}_{ad}$, $A_{skew} \in L^q(\Omega; \mathbb{S}^N_{skew})$, and $f \in L^2(\Omega; \mathbb{R}^N)$ seems to be an open question.

Further, we restrict our analysis to the following set of feasible solutions for the original optimal control problem. Namely, we indicate the set

$$\Xi = \left\{ (u, y) \;\middle|\; u \in \mathfrak{A}_{ad},\ y \in W_0^{1,p}(\Omega),\ (u, y) \text{ are related by (18)} \right\}. \quad (19)$$

Definition 3.2. We say that an element $y \in W_0^{1,p}(\Omega)$ belongs to the set $D \subset W_0^{1,p}(\Omega)$ if

$$\left| \int_\Omega \big(\nabla\varphi, A_{skew}(x)\nabla y\big)_{\mathbb{R}^N}\,dx \right| \leq c(y) \left(\int_\Omega |\nabla\varphi|^p\,dx \right)^{1/p}, \ \forall\, \varphi \in C_0^\infty(\Omega) \quad (20)$$

with some constant $c(y)$ depending on y.

As a result, if $y \in D$ then the mapping $\varphi \mapsto [y, \varphi]_{A_{skew}}$ can be defined for all $\varphi \in W_0^{1,p}(\Omega)$ using (10) and the standard rule

$$[y, \varphi]_{A_{skew}} = \lim_{\varepsilon \to 0}\, [y, \varphi_\varepsilon]_{A_{skew}}, \quad (21)$$

where $\{\varphi_\varepsilon\}_{\varepsilon>0} \subset C_0^\infty(\Omega)$ and $\varphi_\varepsilon \to \varphi$ strongly in $W_0^{1,p}(\Omega)$. In particular, if $y \in D$, then we can define the value $[y, y]_{A_{skew}}$ and this one is finite for every $y \in D$, although the "integrand" $\big(\nabla y, A_{skew}\nabla y\big)_{\mathbb{R}^N}$ needs not be integrable on Ω, in general. We adopt the following hypothesis.

Hypothesis A. *For each $u \in \mathfrak{A}_{ad}$ the following implication holds:*

If $(u, y) \in \Xi$, then $y \in D$.

Remark 3.1. Assume $y \in W_0^{1,p}(\Omega)$ is a weak solution to problem (2)–(3) in the sense of distributions, i.e. in this case the integral identity

$$\int_\Omega u|\nabla y|^{p-2}(\nabla y, \nabla\varphi)_{\mathbb{R}^N}dx + \int_\Omega (A_{skew}(x)\nabla y, \nabla\varphi)_{\mathbb{R}^N}\,dx = \int_\Omega (f, \nabla\varphi)_{\mathbb{R}^N}\,dx \quad (22)$$

holds for all $\varphi \in C_0^\infty(\Omega)$. Then, obviously, $y \in D$. Indeed,

$$\left| \int_\Omega \big(\nabla\varphi, A_{skew}(x)\nabla y\big)_{\mathbb{R}^N}\,dx \right| \leq \left| \int_\Omega (f, \nabla\varphi)_{\mathbb{R}^N}\,dx - \int_\Omega u|\nabla y|^{p-2}(\nabla y, \nabla\varphi)_{\mathbb{R}^N}dx \right| \leq \big(\|f\|_{L^q(\Omega;\mathbb{R}^N)} + \beta\|y\|^{p-1}_{W_0^{1,p}(\Omega)}\big)\|\varphi\|_{W_0^{1,p}(\Omega)}, \ \forall\, \varphi \in C_0^\infty(\Omega).$$

However, this property is not clear in the case of Minty weak solutions.

As it follows from the definition of the bilinear form $[y,\varphi]_{A_{skew}}$, in general, the value $[y,y]_{A_{skew}}$ is not equal to zero and does not preserve a constant sign for all $y \in D$. In particular, in the case $p = q = 2$ there can be found an element $y_0 \in D$ and a matrix $A_{skew} \in L^2(\Omega; \mathbb{S}^N_{skew})$ such that $\big(\nabla y(x), A_{skew}(x)\nabla y(x)\big)_{\mathbb{R}^N}$ is identical to zero for a.e. $x \in \Omega$ whereas $[y_d, y_d]_{A_{skew}} < 0$ (for the corresponding examples, we refer to [37, 12, 18]). This fact does not allow us to derive a reasonable a priory estimate in $\|\cdot\|_{W_0^{1,p}(\Omega)}$-norm for Minty weak solutions. Moreover, if $A_{skew} \in L^q(\Omega; \mathbb{S}^N_{skew})$, then the differential operator in the left-hand side of equation (2) is neither monotone nor coercive, so two definitions of weak solutions, given by (18) and (22) are not equivalent [27]. Besides, the mapping $u \mapsto y(u,f)$ can be multivalued (see [13] for the details).

Here we make one more hypothesis, which is mainly motivated by the previous reasonings.

Hypothesis B. *The set of feasible solutions Ξ is nonempty.*

Remark 3.2. It should be mentioned here that Hypotheses A and B are satisfied provided the matrix A_{skew} possesses some special BMO-properties (for the details we refer to [7, 28]).

For our further analysis, we make use of some auxiliary results. We begin with the following property.

Proposition 3.1. *If $u_k \to u$ in $L^1(\Omega)$ and $\{u_k\}_{k\in\mathbb{N}} \subset \mathfrak{A}_{ad}$, then $u \in \mathfrak{A}_{ad}$, $u_k \to u$ in $L^r(\Omega)$ for any $r \in [1,+\infty)$, and $u_k \overset{*}{\to} u$ in $L^\infty(\Omega)$.*

Proof. Since the estimate

$$\|u_k - u\|^r_{L^r(\Omega)} \le \operatorname*{vrai\,sup}_{x\in\Omega} |u_k(x) - u(x)\|^{r-1}\|u_k - u\|_{L^1(\Omega)} \le (\beta-\alpha)^{r-1}\|u_k - u\|_{L^1(\Omega)}$$

holds true for any $r \in [1,+\infty)$, it follows that $u_k \to u$ in $L^r(\Omega)$.

To deduce the weak-$*$ convergence property $u_k \overset{*}{\to} u$ in $L^\infty(\Omega)$, it is enough to note that the strong convergence $u_k \to u$ in $L^1(\Omega)$ implies, up to a subsequence, the convergence $u_k(x) \to u(x)$ almost everywhere in Ω. Hence, by Lebesgue Theorem, we have

$$\int_\Omega (u_k - u)\varphi\,dx \to 0, \quad \forall\,\varphi \in L^1(\Omega),$$

that is $u_k \overset{*}{\to} u$ in $L^\infty(\Omega)$. Since this conclusion is true for any weakly-$*$ convergent subsequence of $\{u_k\}_{k\in\mathbb{N}}$, it follows that u is the weak-$*$ limit for the whole sequence $\{u_k\}_{k\in\mathbb{N}}$.

As for the inclusion $u \in \mathfrak{A}_{ad}$, this fact immediately follows from definition of the set $\mathfrak{A}_{ad}$, the pointwise convergence $u_k \to u$ in Ω, and Proposition 2.1(i). □

Proposition 3.2. *$\mathfrak{A}_{ad}$ is a sequentially compact subset of $L^r(\Omega)$ for any $r \in [1,+\infty)$, and it is a sequentially weakly-$*$ compact subset of $L^\infty(\Omega)$.*

Proof. Let $\{u_k\}_{k\in\mathbb{N}}$ be any sequence of $\mathfrak{A}_{ad}$. Then $\{u_k\}_{k\in\mathbb{N}}$ is bounded in $BV(\Omega)\cap L^\infty(\Omega)$. As a result, the statement immediately follows from Propositions 3.1 and 2.1(iii). □

Proposition 3.3. *Assume Hypothesis B holds true. Then the set Ξ is sequentially closed in the following sense: for any sequence $\{(u_k, y_k) \in \Xi\}_{k\in\mathbb{N}}$ such that*

$$u_k \overset{*}{\rightharpoonup} u_0 \text{ in } BV(\Omega), \quad y_k \rightharpoonup y_0 \text{ in } W_0^{1,p}(\Omega), \tag{23}$$

we have $(u_0, y_0) \in \Xi$, where y_0 is a Minty weak solution to the Dirichlet boundary value problem (2)–(3) with $u = u_0$.

Proof. Since $u_k \in \mathfrak{A}_{ad}$ for every $k \in \mathbb{N}$ then in view of Proposition 3.1, we immediately get $u_0 \in \mathfrak{A}_{ad}$. Moreover, the Minty inequality

$$\int_\Omega u_k |\nabla\varphi|^{p-2} (\nabla\varphi, \nabla\varphi - \nabla y_k)_{\mathbb{R}^N}\, dx + \int_\Omega (A_{skew}\nabla\varphi, \nabla\varphi - \nabla y_k)_{\mathbb{R}^N}\, dx \ge \int_\Omega (f, \nabla\varphi - \nabla y_k)_{\mathbb{R}^N}\, dx \quad (24)$$

holds true for all $\varphi \in C_0^\infty(\Omega)$ and $k \in \mathbb{N}$. Since $u_k|\nabla\varphi|^{p-2}\nabla\varphi \to u_0|\nabla\varphi|^{p-2}\nabla\varphi$ strongly in $L^q(\Omega; \mathbb{R}^N)$ (see Proposition 3.1) and $\nabla y_k \rightharpoonup \nabla y$ in $L^p(\Omega; \mathbb{R}^N)$, we can pass to the limit in (24) as $k \to \infty$. As a result, we have

$$\int_\Omega u_0 |\nabla\varphi|^{p-2}_{\mathbb{R}^N} (\nabla\varphi, \nabla\varphi - \nabla y)_{\mathbb{R}^N}\, dx + \int_\Omega (A_{skew}\nabla\varphi, \nabla\varphi - \nabla y)_{\mathbb{R}^N}\, dx \ge \int_\Omega (f, \nabla\varphi - \nabla y)_{\mathbb{R}^N}\, dx.$$

Therefore $(u_0, y) \in \Xi$. □

We are now in a position to establish the main result of this section.

Theorem 3.3. *Assume that Hypothesis B is valid. Then, for given distributions $A_{skew} \in L^q(\Omega; \mathbb{S}^N_{skew})$, $f \in L^2(\Omega; \mathbb{R}^N)$ and $y_d \in L^2(\Omega)$, the optimal control problem* (1)–(3) *admits at least one solution.*

Proof. Since $\Xi \neq \emptyset$ and the cost functional is bounded from below on Ξ, it follows that there exists a minimizing sequence $\{(u_k, y_k)\}_{k\in\mathbb{N}} \subset \Xi$ such that $I(u_k, y_k) \xrightarrow[k\to\infty]{} I_{\min} \equiv \inf_{(u,y)\in\Xi} I(u, y) \ge 0$. Hence, $\sup_{k\in\mathbb{N}} I(u_k, y_k) \le C$, where the constant C is independent of k. We have

$$\sup_{k\in\mathbb{N}} \|y_k\|^p_{W_0^{1,p}(\Omega)} = \int_\Omega |\nabla y_k|^p\, dx \le \alpha^{-1} \int_\Omega \left(u_k|\nabla y_k|^p + y^2\right) dx \le 2\alpha^{-1} \left(\sup_{k\in\mathbb{N}} I(u_k, y_k) + \|y_d\|^2_{L^2(\Omega)}\right) \le 2\alpha^{-1} \left(C + \|y_d\|^2_{L^2(\Omega)}\right). \quad (25)$$

Therefore, passing to a subsequence if necessary and taking into account Proposition 3.2, we may assume the existence of a pair $(u_0, y_0) \in \mathfrak{A}_{ad} \times W_0^{1,p}(\Omega)$ such that

$$u_k \overset{*}{\rightharpoonup} u_0 \;\text{ in }\; BV(\Omega),\; u_k \to u_0 \;\text{ strongly in } L^1(\Omega) \quad (26)$$

$$\nabla y_k \rightharpoonup \nabla y_0 \;\text{ in }\; L^p(\Omega; \mathbb{R}^N), \quad (27)$$

$$I(u_0, y_0) < +\infty. \quad (28)$$

In remains to show that (u_0, y_0) is an optimal pair. Since $u_k \to u$ strongly in $L^r(\Omega)$ for every $1 \le r < +\infty$, $\{u_k\}_{k\in\mathbb{N}}$ is bounded in $L^\infty(\Omega)$ and $u_k(x) \ge \alpha$ for almost all $x \in \Omega$, using conditions (26)–(28), it is easy to check that $\nabla y_k u_k^{1/p} \rightharpoonup \nabla y_0 u_0^{1/p}$ in $L^p(\Omega; \mathbb{R}^N)$. Taking into account the property of lower semicontinuity of the norms $\|\cdot\|_{L^2(\Omega)}$ and $\|\cdot\|_{L^p(\Omega;\mathbb{R}^N)}$ with respect to the the weak topologies of $L^2(\Omega)$ and $L^p(\Omega; \mathbb{R}^N)$, respectively, we get

$$\liminf_{k\to\infty} \int_\Omega |\nabla y_k - \nabla y_d|^2\, dx + \int_\Omega |\nabla y_k|^p u_k\, dx$$
$$= \liminf_{k\to\infty} \left[\|y_k - y_d\|^2_{L^2(\Omega)} + \|u_k^{1/p}\nabla y_k\|^p_{L^p(\Omega;\mathbb{R}^N)}\right]$$
$$\ge \|y_0 - y_d\|^2_{L^2(\Omega)} + \|u_0^{1/p}\nabla y_0\|^p_{W_0^{1,p}(\Omega)} = \int_\Omega |\nabla y_0 - \nabla y_d|^2\, dx + \int_\Omega |\nabla y_0|^p u_0\, dx.$$

Thus,

$$I(u_0, y_0) \ge \inf_{(u,y)\in\Xi} I(u, y) = \lim_{k\to\infty} I(u_k, y_k) \ge \liminf_{k\to\infty} I(u_k, y_k) \ge I(u_0, y_0),$$

and, hence, the pair (u_0, y_0) is optimal for problem (1)–(3). The proof is complete. □

4. **Regularization of OCP** (1)–(3)**.** In this section we introduce the two-parameter regularization of the considered optimization problem for the case when the term $[\nabla y]^2$ may grow large or tend to zero. Moreover, in the approximating OCP we consider an L^∞-regularization of the unbounded matrix $A_{skew} \in L^q(\Omega; \mathbb{S}^N_{skew})$. As a result, we show that in suitable topologies optimal solutions of regularized problems tend to some optimal solutions of the initial problem.

As was pointed out in [33], the p-Laplacian $\Delta_p(u, y)$ provides an example of a quasi-linear operator in divergence form with a so-called degenerate nonlinearity for $p > 2$. In this context we have non-differentiability of the state y with respect to the control u. As follows from Theorem 3.3, this fact is not an obstacle to prove existence of considered optimal controls in the coefficients, but it causes certain difficulties when deriving the optimality conditions for the considered problem. To overcome this difficulty, we introduce a special family of approximating control problems (see, for comparison, the approach of Casas and Fernandez [4] for quasi-linear elliptic equations with a distributed control in the right hand side).

Let $A^k_{skew} \in L^\infty(\Omega; \mathbb{S}^N_{skew})$ be the sequence such that $A^k_{skew} \to A_{skew}$ strongly in $L^q(\Omega; \mathbb{S}^N_{skew})$ as $k \to \infty$. Existence of such sequence is a well-known fact of functional analysis as well as methods of its construction. Indeed, let for every $k \in \mathbb{N}$, $T_k : \mathbb{R} \to \mathbb{R}$ be the truncation function defined by

$$T_k(s) = \max\left\{\min\left\{s, k\right\}, -k\right\}. \tag{29}$$

Then, for an arbitrary $g \in L^q(\Omega)$, we have (see, for example [16])

$$T_k(g) \in L^\infty(\Omega)\ \forall\, k \in \mathbb{N} \quad \text{and} \quad T_k(g) \to g \text{ strongly in } \ L^q(\Omega). \tag{30}$$

So, for a given matrix $A_{skew} \in L^q(\Omega; \mathbb{S}^N_{skew})$, $A_{skew} = [a_{ij}]^N_{i,j=1}$, we can set up $A^k_{skew} = [T_k(a_{ij})]^N_{i,j=1}$.

In what follows we associate with the initial optimal control problem (1)–(3) the following sequence of optimization problems (see, for comparison, [4])

$$\text{Minimize } \left\{ I_{\varepsilon,k}(u, y) = \int_\Omega |y - y_d|^2\, dx + \int_\Omega u \left(\varepsilon + \mathcal{F}_k\big(|\nabla y|^2\big)\right)^{\frac{p-2}{2}} |\nabla y|^2\, dx \right\}, \tag{31}$$

subject to constrains

$$-\text{div}\left(u \left(\varepsilon + \mathcal{F}_k\big(|\nabla y|^2\big)\right)^{\frac{p-2}{2}} \nabla y \right) - \text{div}(A^k_{skew} \nabla y) = -\text{div} f, \tag{32}$$

$$u \in \mathfrak{A}_{ad} \subset L^\infty(\Omega) \cap BV(\Omega),\ y \in H^1_0(\Omega), \tag{33}$$

Here, $\mathfrak{A}_{ad}$ is defined in (5), $k \in \mathbb{N}$, ε is a small parameter, which varies within a strictly decreasing sequence of positive numbers converging to 0, and $\mathcal{F}_k : \mathbb{R}_+ \to \mathbb{R}_+$ is a non-decreasing $C^1(\mathbb{R}_+)$-function such that

$$\begin{gathered} \mathcal{F}_k(t) = t, \ \text{ if } \ t \in \left[0, k^2\right], \quad \mathcal{F}_k(t) = k^2 + 1, \ \text{ if } \ t > k^2 + 1, \quad \text{and} \\ t \le \mathcal{F}_k(t) \le t + \delta, \ \text{ if } \ k^2 \le t < k^2 + 1 \quad \text{for some } \ \delta \in (0, 1), \\ \mathcal{F}'_k(t) \le \delta^*, \ \text{ if } \ k^2 \le t < k^2 + 1 \quad \text{for some } \ \delta^* > 1, \end{gathered} \tag{34}$$

and the constants δ and δ^* are independent of $k \in \mathbb{N}$. In particular, if

$$\mathcal{F}_k(t) = \begin{cases} t, & \text{if } \ 0 \le t \le k^2, \\ (k^2 - t)^3 + (k^2 - t)^2 + t, & \text{if } \ k^2 \le t \le k^2 + 1, \\ k^2 + 1, & \text{if } \ t \ge k^2 + 1. \end{cases}$$

then $\delta = 4/27$ and $\delta^* = 4/3$ satisfy(34).

It is clear that the effect of such perturbations of $\Delta_p(u,y)$ is in its regularization around critical points and points where $|\nabla y(x)|$ becomes unbounded. In particular, if $y \in W_0^{1,p}(\Omega)$ and $\Omega_k(y) := \left\{x \in \Omega \ : \ |\nabla y(x)| > \sqrt{k^2+1}\right\}$, then the following chain of inequalities

$$
\begin{aligned}
|\Omega_k(y)| &:= \int_{\Omega_k(y)} 1\,dx \le \frac{1}{\sqrt{k^2+1}} \int_{\Omega_k(y)} |\nabla y(x)|\,dx \\
&\le \frac{1}{\sqrt{k^2+1}} |\Omega_k(y)|^{\frac{1}{q}} \left(\int_\Omega |\nabla y|^p\,dx \right)^{\frac{1}{p}} = \frac{\|y\|_{W_0^{1,p}(\Omega)}}{\sqrt{k^2+1}} |\Omega_k(y)|^{\frac{p-1}{p}}
\end{aligned}
$$

shows that the Lebesgue measure of the set $\Omega_k(y)$ satisfies the estimate

$$
|\Omega_k(y)| \le \left(\frac{1}{\sqrt{k^2+1}} \right)^p \|y\|^p_{W_0^{1,p}(\Omega)} \le \|y\|^p_{W_0^{1,p}(\Omega)} k^{-p}, \quad \forall\, y \in W_0^{1,p}(\Omega), \tag{35}
$$

i.e. the approximation $\mathcal{F}_k(|\nabla y|^2)$ is essential on sets with small Lebesgue measure. The main goal of this section is to show that for each $\varepsilon > 0$ and $k \in \mathbb{N}$, the perturbed optimal control problem (31)–(33) is well posed and its solutions can be considered as a reasonable approximation of optimal pairs to the original problem (1)–(3). To begin with, we establish a few auxiliary results concerning monotonicity and growth conditions for the regularized operator $\mathcal{A}_{\varepsilon,k,u} : H_0^1(\Omega) \to H^{-1}(\Omega)$

$$
\mathcal{A}_{\varepsilon,k,u}\, y = -\mathrm{div}\left(u \left(\varepsilon + \mathcal{F}_k\big(|\nabla y|^2\big)\right)^{\frac{p-2}{2}} \nabla y \right) - \mathrm{div}(A^k_{skew} \nabla y) \tag{36}
$$

where $u \in \mathfrak{A}_{ad}$ is a fixed control.

For our further analysis, we make use of the following the notation

$$
\|\varphi\|_{\varepsilon,k,u} = \left(\int_\Omega u \left(\varepsilon + \mathcal{F}_k(|\nabla \varphi|^2) \right)^{\frac{p-2}{2}} |\nabla \varphi|^2\,dx \right)^{1/p} \quad \forall \varphi \in H_0^1(\Omega).
$$

Remark 4.1. For an arbitrary element $y^* \in H_0^1(\Omega)$ let us consider the level set $\Omega_k(y^*) := \left\{x \in \Omega \ : \ |\nabla y^*(x)| > \sqrt{k^2+1}\right\}$. Then

$$
\begin{aligned}
|\Omega_k(y^*)| &:= \int_{\Omega_k(y^*)} 1\,dx \le \frac{1}{\sqrt{k^2+1}} \int_{\Omega_k(y^*)} |\nabla y^*(x)|\,dx \\
&\le \frac{1}{k} |\Omega_k(y^*)|^{\frac{1}{2}} \left(\int_{\Omega_k(y^*)} |\nabla y^*|^2\,dx \right)^{\frac{1}{2}} \\
&= \frac{1}{k} \left(\frac{1}{\varepsilon + k^2 + 1} \right)^{\frac{p-2}{4}} \left(\int_{\Omega_k(y^*)} \left(\varepsilon + \mathcal{F}_k(|\nabla y^*|^2) \right)^{\frac{p-2}{2}} |\nabla y^*|^2\,dx \right)^{\frac{1}{2}} |\Omega_k(y^*)|^{\frac{1}{2}} \\
&\le \frac{1}{k}\, |\Omega_k(y^*)|^{\frac{1}{2}}\, \alpha^{-\frac{1}{2}} \|y^*\|^{\frac{p}{2}}_{\varepsilon,k,u}.
\end{aligned}
$$

Hence, the Lebesgue measure of the set $\Omega_k(y^*)$ satisfies the estimate

$$
|\Omega_k(y^*)| \le \frac{\alpha^{-1}}{k^2} \|y^*\|^p_{\varepsilon,k,u}, \quad \forall\, y^* \in H_0^1(\Omega). \tag{37}
$$

Now, we establish the following results.

Proposition 4.1. *For every $u \in \mathfrak{A}_{ad}$, $k \in \mathbb{N}$ and $\varepsilon > 0$, the operator $\mathcal{A}_{\varepsilon,k,u}$: $H_0^1(\Omega) \to H^{-1}(\Omega)$, defined in* (36) *is bounded, strictly monotone, coercive and semi-continuous in the sense of relations* (6)–(8) *and*

$$
\|\mathcal{A}_{\varepsilon,k,u}\| \le \left(\varepsilon + k^2 + 1 \right)^{\frac{p-2}{2}} \beta + |\Omega| k N.
$$

Proof. Boundedness. Following the definition of $\mathcal{F}_k$, A^k_{skew}, and the boundedness of u, we obtain

$$\|\mathcal{A}_{\varepsilon,k,u}\| = \sup_{\|y\|_{H^1_0(\Omega)}\le 1} \|\mathcal{A}_{\varepsilon,k,u}\, y\|_{H^{-1}(\Omega)} = \sup_{\substack{\|y\|_{H^1_0(\Omega)}\le 1\\ \|v\|_{H^1_0(\Omega)}\le 1}} \langle \mathcal{A}_{\varepsilon,k,u}\, y, v\rangle_{H^{-1}(\Omega);H^1_0(\Omega)}$$

$$= \sup_{\substack{\|y\|_{H^1_0(\Omega)}\le 1\\ \|v\|_{H^1_0(\Omega)}\le 1}} \int_\Omega \left[u\big(\varepsilon + \mathcal{F}_k(|\nabla y|^2)\big)^{\frac{p-2}{2}} (\nabla y, \nabla v)_{\mathbb{R}^N} + (A^k_{skew}\nabla y, \nabla v)_{\mathbb{R}^N}\right] dx$$
$$\le \left(\beta(\varepsilon + k^2 + 1)^{\frac{p-2}{2}} + \|A^k_{skew}\|_{L^\infty(\Omega;\mathbb{S}^N_{skew})}\right) \le \left(\beta(\varepsilon + k^2 + 1)^{\frac{p-2}{2}} + |\Omega| k N\right).$$

Strict monotonicity. We make use of the following algebraic inequality, which is proved in [22, Proposition 4.4]:

$$\left(\big(\varepsilon + \mathcal{F}_k(|a|^2)\big)^{\frac{p-2}{2}} a - \big(\varepsilon + \mathcal{F}_k(|b|^2)\big)^{\frac{p-2}{2}} b, a-b\right)_{\mathbb{R}^N} \ge \varepsilon^{\frac{p-2}{2}} |a-b|^2,\ a, b \in \mathbb{R}^N.$$

With this, having put $a := \nabla y$, $b := \nabla v$ we obtain

$$\big\langle \mathcal{A}_{\varepsilon,k,u}\, y - \mathcal{A}_{\varepsilon,k,u}\, v, y - v\big\rangle_{H^{-1}(\Omega);H^1_0(\Omega)} \ge \varepsilon^{\frac{p-2}{2}} \int_\Omega u|\nabla y - \nabla v|^2 dx$$
$$+ \int_\Omega \big(A^k_{skew}(\nabla y - \nabla v), \nabla y - \nabla v\big)_{\mathbb{R}^N} dx \ge \alpha\varepsilon^{\frac{p-2}{2}} \|y - v\|^2_{H^1_0(\Omega)} \ge 0,$$

as far as $A^k_{skew} \in L^\infty(\Omega; \mathbb{S}^N_{skew})$ and $(A^k_{skew}\xi, \xi)_{\mathbb{R}^N} = -(\xi, A^k_{skew}\xi)_{\mathbb{R}^N} = 0$ for all $\xi \in \mathbb{R}^N$. Since the relation $\big\langle \mathcal{A}_{\varepsilon,k,u}\, y - \mathcal{A}_{\varepsilon,k,u}\, v, y - v\big\rangle_{H^{-1}(\Omega);H^1_0(\Omega)} = 0$ implies $y = v$, it follows that the strict monotonicity property (6) holds true for each $u \in \mathfrak{A}_{ad}$, $k \in \mathbb{N}$, and $\varepsilon > 0$.

Coercivity. The coercivity property obviously follows from the estimate

$$\big\langle \mathcal{A}_{\varepsilon,k,u}\, y, y\big\rangle_{H^{-1}(\Omega);H^1_0(\Omega)} \ge \alpha\varepsilon^{\frac{p-2}{2}} \|y\|^2_{H^1_0(\Omega)}. \tag{38}$$

Semi-continuity. In order to get the equality

$$\lim_{t\to 0} \langle \mathcal{A}_{\varepsilon,k,u}\,(y + tv), w\rangle_{H^{-1}(\Omega);H^1_0(\Omega)} = \langle \mathcal{A}_{\varepsilon,k,u}\, y, w\rangle_{H^{-1}(\Omega);H^1_0(\Omega)},$$

it is enough to observe that

$$u(\varepsilon + \mathcal{F}_k(|(\nabla y + t\nabla v)|^2))^{\frac{p-2}{2}} (\nabla y + t\nabla v) \to u(\varepsilon + \mathcal{F}_k(|\nabla y|^2))^{\frac{p-2}{2}} \nabla y,$$
$$\big(A^k_{skew}(\nabla y + t\nabla v), \nabla w\big)_{\mathbb{R}^N} \to \big(A^k_{skew}\nabla y, \nabla w\big)_{\mathbb{R}^N}$$

as $t \to 0$ almost everywhere in Ω, and make use of Lebesgue's dominated convergence theorem. □

The existence of a unique solution to the boundary value problem (32)–(33) follows from the following abstract theorem on monotone operators (see, for instance, [23] or [35, §II.2]).

Theorem 4.1. *Let V be a reflexive separable Banach space. Let V^* be the dual space, and let $A : V \to V^*$ be a bounded, semicontinuous, coercive and strictly monotone operator. Then the equation $Ay = f$ has a unique solution for each $f \in V^*$. Moreover, $Ay = f$ if and only if $\langle A\varphi, \varphi - y\rangle \ge \langle f, \varphi - y\rangle$ for all $\varphi \in V^*$.*

Indeed, applying the above theorem to the equation $\mathcal{A}_{\varepsilon,k,u} y = \mathrm{div} f$ with $\mathrm{div} f \in H^{-1}(\Omega)$, we arrive at the following assertion.

Theorem 4.2. *For each* $\varepsilon > 0$, $k \in \mathbb{N}$, $u \in \mathfrak{A}_{ad}$, $A_{skew} \in L^q(\Omega;\mathbb{S}^N_{skew})$, *and* $f \in L^2(\Omega;\mathbb{R}^N)$, *the boundary value problem* (32)–(33) *admits a unique weak solution* $y_{\varepsilon,k} \in H^1_0(\Omega)$ *in the sense of distributions, i.e.*

$$\int_\Omega u(\varepsilon + \mathcal{F}_k(|\nabla y_{\varepsilon,k}|^2))^{\frac{p-2}{2}} (\nabla y_{\varepsilon,k}, \nabla\varphi)_{\mathbb{R}^N}\; dx + \int_\Omega \left(A^k_{skew}\nabla y_{\varepsilon,k}, \nabla\varphi\right)_{\mathbb{R}^N} dx = \int_\Omega (f, \nabla\varphi)_{\mathbb{R}^N} dx, \quad \forall\,\varphi \in C^\infty_0(\Omega). \tag{39}$$

Moreover, this solution satisfies the Minty inequality

$$\int_\Omega u(x)(\varepsilon + \mathcal{F}_k(|\nabla\varphi|^2))^{\frac{p-2}{2}} (\nabla\varphi, \nabla\varphi - \nabla y_{\varepsilon,k})_{\mathbb{R}^N}\; dx + \int_\Omega \left(A^k_{skew}\nabla\varphi, \nabla\varphi - \nabla y_{\varepsilon,k}\right)_{\mathbb{R}^N} dx \geq \int_\Omega (f, \nabla\varphi - \nabla y_{\varepsilon,k})_{\mathbb{R}^N} dx, \quad \forall\,\varphi \in C^\infty_0(\Omega). \tag{40}$$

and the energy equality

$$\int_\Omega u(\varepsilon + \mathcal{F}_k(|\nabla y_{\varepsilon,k}|^2))^{\frac{p-2}{2}} |\nabla y_{\varepsilon,k}|^2\, dx = \int_\Omega (f, \nabla y_{\varepsilon,k})_{\mathbb{R}^N} dx. \tag{41}$$

Thus, as follows from Theorem 4.2, for every $\varepsilon > 0$ and $k \in \mathbb{N}$, the set of feasible pairs to problem (31)–(33)

$$\Xi_{\varepsilon,k} = \left\{(u, y)\ \middle|\ u \in \mathfrak{A}_{ad},\ y \in H^1_0(\Omega),\ (u, y) \text{ are related by equality (39)}\right\} \tag{42}$$

is always nonempty.

Now we are in a position to prove the existence result for regularized OCPs (31)–(33).

Theorem 4.3. *For every positive value* $\varepsilon > 0$ *and integer* $k \in \mathbb{N}$, *the optimal control problem* (31)–(33) *admits at least one solution.*

Proof. Indeed, we have already established that the set $\Xi_{\varepsilon,k}$ is not empty. Let $\{(u_i, y_i)\}_{i\in\mathbb{N}} \subset \Xi_{\varepsilon,k}$ be a minimizing sequence , i.e.

$$\lim_{i\to\infty} I_{\varepsilon,k}(u_i, y_i) = \inf_{(u,y)\in\Xi_{\varepsilon,k}} I_{\varepsilon,k}(u, y), \quad \sup_{i\in\mathbb{N}} I_{\varepsilon,k}(u_i, y_i) \leq C.$$

Following the proof of Theorem 3.3 it is easy to see that there exists $u_0 \in \mathfrak{A}_{ad}$ such that $u_i \overset{*}{\rightharpoonup} u_0$ in $BV(\Omega)$ and $u_i \to u_0$ strongly in $L^1(\Omega)$. Also we have

$$\sup_{i\in\mathbb{N}} \|y_i\|^2_{H^1_0(\Omega)} = \sup_{i\in\mathbb{N}} \int_\Omega |\nabla y_i|^2\, dx \leq \alpha^{-1}\varepsilon^{\frac{2-p}{2}} \sup_{i\in\mathbb{N}} \int_\Omega u_i\left(\varepsilon + \mathcal{F}_k(|\nabla y_i|^2)\right)^{\frac{p-2}{2}} |\nabla y_i|^2\, dx \leq \alpha^{-1}\varepsilon^{\frac{2-p}{2}} \sup_{i\in\mathbb{N}} I_{\varepsilon,k}(u_i, y_i) \leq C$$

and, therefore, $\exists\, y_0 \in H^1_0(\Omega)$ such that, within a subsequence, $y_i \rightharpoonup y_0$ in $H^1_0(\Omega)$ as $i \to \infty$. Similarly to the proof of Proposition 3.3, we pass to the limit in the Minty relation (see (40))

$$\int_\Omega u_i(x)(\varepsilon + \mathcal{F}_k(|\nabla\varphi|^2))^{\frac{p-2}{2}} (\nabla\varphi, \nabla\varphi - \nabla y_i)_{\mathbb{R}^N}\; dx + \int_\Omega \left(A^k_{skew}\nabla\varphi, \nabla\varphi - \nabla y_i\right)_{\mathbb{R}^N} dx \geq \int_\Omega (f, \nabla\varphi - \nabla y_i)_{\mathbb{R}^N} dx, \quad \forall\,\varphi \in C^\infty_0(\Omega),$$

as $i \to \infty$ and get the assertion $(u_0, y_0) \in \Xi_{\varepsilon,k}$.

As follows from the direct method of the Calculus of Variations, the lower semi-continuous property of the cost functional with respect to suitable topologies in ‘control-state’ space plays a crucial role in the proof of solvability of optimization problems. However, in our case we make use of a weaker property, namely, we establish the fulfilment of the inequality $I_{\varepsilon,k}(u_0, y_0) \leq \liminf_{i\to\infty} I_{\varepsilon,k}(u_i, y_i)$ only for elements $(u_i, y_i) \in \Xi_{\varepsilon,k}$ of minimizing sequence. To this end, we proceed as in the proof of Theorem 3.3. We have to show that the following relation

$$\int_\Omega (\varepsilon + \mathcal{F}_k(|\nabla y_0|^2))^{\frac{p-2}{2}} |\nabla y_0|^2 u_0\, dx \le \liminf_{i\to\infty} \int_\Omega (\varepsilon + \mathcal{F}_k(|\nabla y_i|^2))^{\frac{p-2}{2}} |\nabla y_i|^2 u_i\, dx$$

is valid. In fact, this is an equality and it can be proved as follows. Since pairs (u_i, y_i) and (u_0, y_0) satisfy (41), we infer from (41) that

$$\lim_{i\to\infty} \int_\Omega (\varepsilon + \mathcal{F}_k(|\nabla y_i|^2))^{\frac{p-2}{2}} |\nabla y_i|^2 u_i\, dx = \lim_{i\to\infty} \int_\Omega (f, \nabla y_i)_{\mathbb{R}^N}\, dx = \\ = \int_\Omega (f, \nabla y_0)_{\mathbb{R}^N}\, dx = \int_\Omega (\varepsilon + \mathcal{F}_k(|\nabla y_0|^2))^{\frac{p-2}{2}} |\nabla y_0|^2 u_0\, dx.$$

□

Remark 4.2. It is worth to note that the cost functional (31) does not possess the lower semicontinuity property on $BV(\Omega) \times H_0^1(\Omega)$ with respect to the weak-$*$ convergence in $BV(\Omega)$ and the weak convergence in $H_0^1(\Omega)$, in general. As follows from Theorem 4.3, this property is valid only on the set of feasible pairs $\Xi_{\varepsilon,k} \subset BV(\Omega) \times H_0^1(\Omega)$.

For our further analysis, we need to obtain some appropriate a priory estimates for the weak solutions to the problem (32)–(33). With that in mind, we make use of the following auxiliary results (see for the comparison Proposition 4.2 in [5]).

Proposition 4.2. *Let $u \in \mathfrak{A}_{ad}$, $k \in \mathbb{N}$, and $\varepsilon > 0$ be given. Then, for arbitrary $g \in L^2(\Omega; \mathbb{R}^N)$ and $y \in H_0^1(\Omega)$, we have*

$$\left| \int_\Omega (g, \nabla y)_{\mathbb{R}^N} dx \right| \le \|g\|_{L^2(\Omega;\mathbb{R}^N)} \left[\alpha^{-\frac{1}{p}} |\Omega|^{\frac{p-2}{2p}} \|y\|_{\varepsilon,k,u} + \alpha^{-\frac{1}{2}} \|y\|^{\frac{p}{2}}_{\varepsilon,k,u} \right]. \tag{43}$$

Proof. Let us fix an arbitrary element y of $H_0^1(\Omega)$. We associate with this element the set $\Omega_k(y)$, where

$$\Omega_k(y) := \left\{ x \in \Omega \;:\; |\nabla y(x)| > \sqrt{k^2+1} \right\}. \tag{44}$$

Then, by Cauchy-Bunyakovsky inequality,

$$\int_\Omega (g, \nabla y)_{\mathbb{R}^N} dx \le \|g\|_{L^2(\Omega;\mathbb{R}^N)} \|\nabla y\|_{L^2(\Omega;\mathbb{R}^N)} \\ \le \|g\|_{L^2(\Omega;\mathbb{R}^N)} \left(\|\nabla y\|_{L^2(\Omega\setminus\Omega_k(y);\mathbb{R}^N)} + \|\nabla y\|_{L^2(\Omega_k(y);\mathbb{R}^N)} \right). \tag{45}$$

Using the fact that

$$\|\nabla y\|_{L^2(\Omega\setminus\Omega_k(y);\mathbb{R}^N)} \le |\Omega|^{\frac{p-2}{2p}} \|\nabla y\|_{L^p(\Omega\setminus\Omega_k(y);\mathbb{R}^N)} \\ \le |\Omega|^{\frac{p-2}{2p}} \left(\int_{\Omega\setminus\Omega_k(y)} (\varepsilon + |\nabla y|^2)^{\frac{p-2}{2}} |\nabla y|^2\, dx \right)^{\frac{1}{p}}$$

and

$$\mathcal{F}_k(|\nabla y|^2) = |\nabla y|^2 \;\text{ a.e. in } \Omega \setminus \Omega_k(y), \text{ and} \\ k^2 \le \mathcal{F}_k(|\nabla y|^2) \le k^2 + 1 \;\text{ a.e. in } \Omega_k(y), \quad \forall\, k \in \mathbb{N},$$

we obtain

$$\|\nabla y\|_{L^2(\Omega\setminus\Omega_k(y);\mathbb{R}^N)} \le |\Omega|^{\frac{p-2}{2p}}\left(\int_{\Omega\setminus\Omega_k(y)}(\varepsilon+\mathcal{F}_k(|\nabla y|^2))^{\frac{p-2}{2}}|\nabla y|^2\,dx\right)^{\frac{1}{p}} \le |\Omega|^{\frac{p-2}{2p}}\alpha^{-\frac{1}{p}}\|y\|_{\varepsilon,k,u}, \tag{46}$$

$$\|\nabla y\|_{L^2(\Omega_k(y);\mathbb{R}^N)} \le \left(\int_{\Omega_k(y)}(\varepsilon+\mathcal{F}_k(|\nabla y|^2))^{\frac{p-2}{2}}|\nabla y|^2\,dx\right)^{\frac{1}{2}} \le \alpha^{-\frac{1}{2}}\|y\|^{\frac{p}{2}}_{\varepsilon,k,u}. \tag{47}$$

As a result, inequality (43) immediately follows from (45)–(47). The proof is complete. □

Definition 4.4. Let $\{u_{\varepsilon,k}\}_{\substack{\varepsilon>0\\k\in\mathbb{N}}}\subset\mathfrak{A}_{ad}$ be an arbitrary sequence of admissible controls. We say that a two-parametric sequence $\{y_{\varepsilon,k}\}_{\substack{\varepsilon>0\\k\in\mathbb{N}}}\subset H^1_0(\Omega)$ is bounded with respect to the $\|\cdot\|_{\varepsilon,k,u_{\varepsilon,k}}$-quasi-seminorm if $\sup_{\substack{\varepsilon>0\\k\in\mathbb{N}}}\|y_{\varepsilon,k}\|_{\varepsilon,k,u_{\varepsilon,k}}<+\infty$.

To conclude this section, let us show that for every $u\in\mathfrak{A}_{ad}$ and $f\in L^2(\Omega;\mathbb{R}^N)$, the sequence of weak solutions to problem (32)–(33) $\left\{y_{\varepsilon,k}=y_{\varepsilon,k}(u,A^k_{skew},f)\right\}_{\substack{\varepsilon>0\\k\in\mathbb{N}}}$ is bounded with respect to the $\|\cdot\|_{\varepsilon,k,u}$-quasi-seminorm in the sense of Definition 4.4.

Indeed, the integral identity (39) together with estimate (43) (for $g=f$) immediately lead us to the relation

$$\|y_{\varepsilon,k}\|^p_{\varepsilon,k,u}:=\int_\Omega\left(\varepsilon+\mathcal{F}_k(|\nabla y_{\varepsilon,k}|^2)\right)^{\frac{p-2}{2}}|\nabla y_{\varepsilon,k}|^2u\,dx=\int_\Omega(f,\nabla y_{\varepsilon,k})_{\mathbb{R}^N}\,dx \le \|f\|_{L^2(\Omega;\mathbb{R}^N)}\left[\alpha^{-\frac{1}{p}}|\Omega|^{\frac{p-2}{2p}}\|y_{\varepsilon,k}\|_{\varepsilon,k,u}+\alpha^{-\frac{1}{2}}\|y_{\varepsilon,k}\|^{\frac{p}{2}}_{\varepsilon,k,u}\right]. \tag{48}$$

As a result, it follows from (48) that

$$\|y_{\varepsilon,k}\|_{\varepsilon,k,u}\le\max\left\{C_f^{\frac{2}{p}},C_f^{\frac{1}{p-1}}\right\},\quad\forall\,\varepsilon>0,\ \forall\,k\in\mathbb{N},\ \forall\,u\in\mathfrak{A}_{ad}, \tag{49}$$

where $C_f:=C\|f\|_{L^2(\Omega;\mathbb{R}^N)}=\left(\alpha^{-\frac{1}{p}}|\Omega|^{\frac{p-2}{2p}}+\alpha^{-\frac{1}{2}}\right)\|f\|_{L^2(\Omega;\mathbb{R}^N)}$. Moreover, taking $g=\nabla y=\nabla y_{\varepsilon,k}$ in (43) and using (49), we also have $\forall\,\varepsilon>0,\ \forall\,k\in\mathbb{N}$,

$$\|y_{\varepsilon,k}\|_{H^1_0(\Omega)}\le\max\left\{C^2\|f\|_{L^2(\Omega;\mathbb{R}^N)},C^{\frac{p}{p-1}}\|f\|^{\frac{1}{p-1}}_{L^2(\Omega;\mathbb{R}^N)}\right\},\quad\forall\,u\in\mathfrak{A}_{ad}. \tag{50}$$

5. **Asymptotic analysis of the approximate OCP** (31)–(33)**.** Our main intention in this section is to show that some optimal solutions to the original OCP (1)–(3) can be attained (in certain sense) by optimal solutions to the approximate problems (31)–(33). With that in mind, we make use of the concept of variational convergence of constrained minimization problems (see Definition 2.2) and study the asymptotic behaviour of a family of OCPs (31)–(33) as $\varepsilon\to0$ and $k\to\infty$. We begin with some auxiliary results concerning the weak compactness in $H^1_0(\Omega)$ of $\|\cdot\|_{\varepsilon,k,u}$-bounded sequences.

Lemma 5.1 (see [5])**.** *Let $\{u_{\varepsilon,k}\}_{\substack{\varepsilon>0\\k\in\mathbb{N}}}\subset\mathfrak{A}_{ad}$ be an arbitrary sequence of admissible controls with associated states $\{y_{\varepsilon,k}\}_{\substack{\varepsilon>0\\k\in\mathbb{N}}}\subset H^1_0(\Omega)$, $y_{\varepsilon,k}=y_{\varepsilon,k}(u_{\varepsilon,k})$. Then each cluster point y of the sequence $\{y_{\varepsilon,k}\}_{\substack{\varepsilon>0\\k\in\mathbb{N}}}$ with respect to the weak convergence in $H^1_0(\Omega)$, satisfies: $y\in W^{1,p}_0(\Omega)$.*

Definition 5.1. We say that a sequence of pairs $\left\{(u_k,y_k)\in BV(\Omega)\times H^1_0(\Omega)\right\}_{k\in\mathbb{N}}$ τ-converges to a pair $(u,y)\in BV(\Omega)\times H^1_0(\Omega))$ if

$$u_k\overset{*}{\rightharpoonup}u\text{ in }BV(\Omega);\quad y_k\rightharpoonup y\ \text{ in }H^1_0(\Omega).$$

Theorem 5.2. *Let* $\{\varepsilon_n\}_{n\in\mathbb{N}}$, $\{k_n\}_{n\in\mathbb{N}}$, $\{u_n\}_{n\in\mathbb{N}}\subset\mathfrak{A}_{ad}$, *and* $\{y_n\}_{n\in\mathbb{N}}\subset H_0^1(\Omega)$ *be sequences such that*

$$\varepsilon_n\to 0,\quad k_n\to\infty,\ (u_n,y_n)\in\Xi_{\varepsilon_n,k_n}. \tag{51}$$

Then, the sequence $\{(u_n,y_n)\}_{i\in\mathbb{N}}$ *is* τ*-compact and for each its* τ*-cluster pair* (u,y) *there exists a subsequence* $\{(u_i,y_i)\}_{i\in\mathbb{N}}\subset\{(u_n,y_n)\}_{n\in\mathbb{N}}$ *such that the following assertions hold true*

$$(u,y)\in\Xi, \tag{52}$$

$$\chi_{\Omega\setminus\Omega_{k_i}(y_i)}\nabla y_i\rightharpoonup\nabla y\ \text{in}\ \ L^p(\Omega;\mathbb{R}^N), \tag{53}$$

$$\liminf_{i\to\infty}\int_\Omega\left(\varepsilon_i+\mathcal{F}_{k_i}(|\nabla y_i|^2)\right)^{\frac{p-2}{2}}|\nabla y_i|^2u_i\,dx\ge\int_\Omega|\nabla y|^pu\,dx, \tag{54}$$

and the pair (u,y) *satisfies the energy inequality*

$$\int_\Omega|\nabla y|^pu\,dx\le\int_\Omega(f,\nabla y)_{\mathbb{R}^N}\,dx. \tag{55}$$

Where $\Omega_{k_i}(y_i)$ is defined by (44).

Proof. In view of Propositions 2.1 and 3.2, the sequence $\{u_n\}_{n\mathbb{N}}\subset\mathfrak{A}_{ad}$ is compact with respect to the weak-$*$ convergence in $BV(\Omega)$ and the strong convergence in $L^1(\Omega)$. Hence, as far as $\sup_{n\in\mathbb{N}}\|y_n\|_{H_0^1(\Omega)}\le const$ (see (50)), using Lemma 5.1, we can claim that there exists a subsequence $\{(u_i,y_i)\}_{i\in\mathbb{N}}\subset\{(u_n,y_n)\}_{n\in\mathbb{N}}$ and a pair $(u,y)\in\mathfrak{A}_{ad}\times W_0^{1,p}(\Omega)$ such that $(u_i,y_i)\xrightarrow{\tau}(u,y)$ in $BV(\Omega)\times H_0^1(\Omega)$.

The rest of proof is divided into three steps.

Step 1. On this step we prove that $\chi_{\Omega\setminus\Omega_k(y_i)}\nabla y_i\rightharpoonup\nabla y$ in $L^p(\Omega;\mathbb{R}^N)$. Following the definition of the sets $\Omega_{k_i}(y_i)$ and using (49), we obtain

$$\begin{aligned}\int_\Omega|\chi_{\Omega\setminus\Omega_{k_i}(y_i)}\nabla y_i|^p\,dx&=\int_{\Omega\setminus\Omega_{k_i}(y_i)}|\nabla y_i|^p\,dx\\&\le\alpha^{-1}\int_{\Omega\setminus\Omega_{k_i}(y_i)}\left(\varepsilon_i+\mathcal{F}_{k_i}(|\nabla y_i|^2)\right)^{\frac{p-2}{2}}|\nabla y_i|^2u_i\,dx,\\&\le\alpha^{-1}\|y_i\|^p_{\varepsilon_i,k_i,u_i}\le C<+\infty,\quad\forall\,i\in\mathbb{N}.\end{aligned}$$

Hence, taking a new subsequence if necessary, we infer the existence of a vector-valued function $g\in L^p(\Omega;\mathbb{R}^N)$ such that $\chi_{\Omega\setminus\Omega_{k_i}(y_i)}\nabla y_i\rightharpoonup g$ in $L^p(\Omega;\mathbb{R}^N)$ as $i\to\infty$. Since $u_i\to u$ in $L^q(\Omega)$, we conclude that

$$\lim_{i\to\infty}\int_{\Omega\setminus\Omega_{k_i}(y_i)}(\nabla y_i,\nabla\varphi)\,u_i\,dx=\int_\Omega(g,\nabla\varphi)\,u\,dx,\quad\forall\varphi\in C_0^\infty(\Omega). \tag{56}$$

On the other hand, in view of the weak convergence $\nabla y_i\rightharpoonup\nabla y$ in $L^2(\Omega;\mathbb{R}^N)$,

$$\begin{aligned}\int_\Omega(\nabla y,\nabla\varphi)\,u\,dx&=\lim_{i\to\infty}\int_\Omega(\nabla y_i,\nabla\varphi)\,u_i\,dx\\&=\lim_{i\to\infty}\int_{\Omega\setminus\Omega_{k_i}(y_i)}(\nabla y_i,\nabla\varphi)\,u_i\,dx+\lim_{i\to\infty}\int_{\Omega_{k_i}(y_i)}(\nabla y_i,\nabla\varphi)\,u_i\,dx.\end{aligned} \tag{57}$$

Since

$$\left|\int_{\Omega_{k_i}(y_i)} (\nabla y_i, \nabla \varphi)\, u_i\, dx\right| \le \beta \|\varphi\|_{C^1(\overline{\Omega})} |\Omega_{k_i}(y_i)|^{1/2} \left(\int_{\Omega_{k_i}(y_i)} |\nabla y_i|^2\, dx\right)^{1/2}$$

$$= \beta \|\varphi\|_{C^1(\overline{\Omega})} |\Omega_{k_i}(y_i)|^{1/2} \sup_{i\in\mathbb{N}} \|y_i\|_{H_0^1(\Omega)}$$

$$\stackrel{\text{by (37)}}{\le} \beta \|\varphi\|_{C^1(\overline{\Omega})} \sup_{i\in\mathbb{N}} \|y_i\|_{H_0^1(\Omega)} \left(\frac{\alpha^{-1}}{k_i^2} \|y_i\|^p_{\varepsilon_i,k_i,u_i}\right)^{1/2} \stackrel{\text{by (49)}}{\le} \frac{\widetilde{C}}{k_i} \to 0 \ \text{ as } \ i\to\infty,$$

it follows from (56) and (57) that

$$\int_\Omega (g, \nabla\varphi)\, u\, dx = \int_\Omega (\nabla y, \nabla \varphi)\, u\, dx, \quad \forall \varphi \in C_0^\infty(\Omega).$$

Hence, $g = \nabla y$ almost everywhere in Ω and $\chi_{\Omega\setminus\Omega_k(y_i)} \nabla y_i \rightharpoonup \nabla y$ in $L^p(\Omega;\mathbb{R}^N)$.

Step 2. On this step we prove assertion (54). Since $u_i \to u$ in $L^r(\Omega)$ for every $1 \le r < +\infty$, $\{u_i\}_{i\in\mathbb{N}}$ is bounded in $L^\infty(\Omega)$, and $u_i(x) \ge \alpha$ for a. a. $x\in\Omega$, due to the result of the previous step, it is easy to check that $\chi_{\Omega\setminus\Omega_{k_i}(y_i)} \nabla y_i u_i^{1/p} \rightharpoonup \nabla y u^{1/p}$ in $L^p(\Omega;\mathbb{R}^N)$. Using this convergence and lower semi-continuity of norm in $L^p(\Omega;\mathbb{R}^N)$ we get

$$\liminf_{i\to\infty} \int_\Omega \left(\varepsilon_i + \mathcal{F}_{k_i}(|\nabla y_i|^2)\right)^{\frac{p-2}{2}} |\nabla y_i|^2 u_i\, dx$$

$$\ge \liminf_{i\to\infty} \int_{\Omega\setminus\Omega_{k_i}(y_i)} \left(\varepsilon_i + \mathcal{F}_{k_i}(|\nabla y_i|^2)\right)^{\frac{p-2}{2}} |\nabla y_i|^2 u_i\, dx$$

$$\stackrel{\text{by (34)}}{\ge} \liminf_{i\to\infty} \int_{\Omega\setminus\Omega_{k_i}(y_i)} \left(\varepsilon_i + |\nabla y_i|^2\right)^{\frac{p-2}{2}} |\nabla y_i|^2 u_i\, dx$$

$$\ge \liminf_{i\to\infty} \int_\Omega |\chi_{\Omega\setminus\Omega_{k_i}(y_i)} \nabla y_i|^p u_i\, dx \ge \int_\Omega |\nabla y|^p u\, dx. \tag{58}$$

Step 3. Here we show that the pair (u, y) is admissible to the initial OCP (1)–(3). Let us prove that y is a solution of (2)-(3) corresponding to the control function $u \in \mathfrak{A}_{ad}$. Let us fix an arbitrary test function $\varphi \in C_0^\infty(\Omega)$ and pass to the limit in the Minty inequality

$$\int_\Omega u_i(x)(\varepsilon_i + \mathcal{F}_{k_i}(|\nabla\varphi|^2))^{\frac{p-2}{2}} (\nabla\varphi, \nabla\varphi - \nabla y_i)_{\mathbb{R}^N}\ dx$$
$$+ \int_\Omega (A^{k_i}_{skew} \nabla\varphi, \nabla\varphi - \nabla y_i)_{\mathbb{R}^N}\, dx \ge \int_\Omega (f, \nabla\varphi - \nabla y_i)_{\mathbb{R}^N}\, dx, \tag{59}$$

as $i\to\infty$. Taking into account that

$$(\varepsilon_i + \mathcal{F}_{k_i}(|\nabla\varphi|^2))^{\frac{p-2}{2}} \nabla\varphi \to |\nabla\varphi|^{p-2}\nabla\varphi \ \text{ strongly in } L^q(\Omega;\mathbb{R}^N), \tag{60}$$

$$(A^{k_i}_{skew}\nabla\varphi, \nabla\varphi)_{\mathbb{R}^N} \to (A_{skew}\nabla\varphi, \nabla\varphi)_{\mathbb{R}^N} \text{ strongly in } L^1(\Omega;\mathbb{R}^N). \tag{61}$$

In view of the weak convergence $\nabla y_i \rightharpoonup \nabla y$ in $L^2(\Omega;\mathbb{R}^N)$ and the strong convergence $u_i \to u$ in $L^r(\Omega)$, for all $r<\infty$, the boundedness of $\{u_i\}_{i\in\mathbb{N}}$ in $L^\infty(\Omega)$ and the fact that $A^{k_i}_{skew} \to A_{skew}$ strongly in $L^q(\Omega; \mathbb{S}^N_{skew})$, we obtain

$$\lim_{i\to\infty} \int_\Omega (\varepsilon_i + \mathcal{F}_{k_i}(|\nabla\varphi|^2))^{\frac{p-2}{2}} (\nabla\varphi, \nabla\varphi)_{\mathbb{R}^N} u_i\, dx = \int_\Omega |\nabla\varphi|^{p-2} (\nabla\varphi, \nabla\varphi)_{\mathbb{R}^N} u\, dx,$$
$$\lim_{i\to\infty} \int_\Omega (\varepsilon_i + \mathcal{F}_{k_i}(|\nabla\varphi|^2))^{\frac{p-2}{2}} (\nabla\varphi, \nabla y_i)_{\mathbb{R}^N} u_i\, dx = \int_\Omega |\nabla\varphi|^{p-2} (\nabla\varphi, \nabla y)_{\mathbb{R}^N} u\, dx,$$
$$\lim_{i\to\infty} \int_\Omega (A^{k_i}_{skew}\nabla\varphi, \nabla y_i)_{\mathbb{R}^N} dx = \int_\Omega (A_{skew}\nabla\varphi, \nabla y)_{\mathbb{R}^N} dx. \tag{62}$$

Let us explain two last relations in details. Indeed, we see that

$$\begin{aligned}
&\int_\Omega (\varepsilon_i + \mathcal{F}_{k_i}(|\nabla\varphi|^2))^{\frac{p-2}{2}} (\nabla\varphi, \nabla y_i)_{\mathbb{R}^N} u_i\, dx - \int_\Omega |\nabla\varphi|^{p-2} (\nabla\varphi, \nabla y)_{\mathbb{R}^N} u\, dx \\
&\quad\le \left| \int_\Omega (\varepsilon_i + \mathcal{F}_{k_i}(|\nabla\varphi|^2))^{\frac{p-2}{2}} \left(\nabla\varphi, \chi_{\Omega\setminus\Omega_{k_i}(y_i)}\nabla y_i\right)_{\mathbb{R}^N} u_i\, dx \right.\\
&\qquad\left. - \int_\Omega |\nabla\varphi|^{p-2} (\nabla\varphi, \nabla y)_{\mathbb{R}^N} u\, dx \right| \\
&\qquad + \int_{\Omega_{k_i}(y_i)} \left| (\varepsilon_i + \mathcal{F}_{k_i}(|\nabla\varphi|^2))^{\frac{p-2}{2}} (\nabla\varphi, \nabla y_i)_{\mathbb{R}^N} u_i \right| dx = I_1 + I_2.
\end{aligned}$$

In view of (60) and the weak convergence $\chi_{\Omega\setminus\Omega_{k_i}(y_i)}\nabla y_i \to \nabla y$ in $L^p(\Omega; \mathbb{R}^N)$, we immediately get $I_1 \to 0$. As for I_2, we have

$$\begin{aligned}
&\int_{\Omega_{k_i}(y_i)} \left| (\varepsilon_i + \mathcal{F}_{k_i}(|\nabla\varphi|^2))^{\frac{p-2}{2}} (\nabla\varphi, \nabla y_i)_{\mathbb{R}^N} u_i \right| dx \\
&\le \beta 2^{\frac{p-2}{2}} \int_{\Omega_{k_i}(y_i)} \left(\varepsilon^{\frac{p-2}{2}} + |\nabla\varphi|^{p-2} \right) |(\nabla\varphi, \nabla y_i)_{\mathbb{R}^N}|\, dx \le C_1 \|y_i\|_{H^1_0(\Omega_{k_i}(y_i))} < \infty.
\end{aligned}$$

Hence, this integrand is obviously equi-integrable. Equi-integrability property implies that $I_2 \to 0$ as far as $|\Omega_{k_i}(y_i)| \to 0$. To get (62), the similar argumentation can be used together with the estimate

$$\begin{aligned}
\int_{\Omega_{k_i}(y_i)} (A^{k_i}_{skew}\nabla\varphi, \nabla y_i)_{\mathbb{R}^N} dx &\le \left(\int_{\Omega_{k_i}(y_i)} |A^{k_i}_{skew}\nabla\varphi|^2 dx \right)^{1/2} \|y_i\|_{H^1_0(\Omega_{k_i}(y_i))} \\
&\overset{\text{by (29)}}{\le} \sqrt{|\Omega_{k_i}(y_i)|} \|\varphi\|_{C^1(\overline{\Omega})} k_i N \|y_i\|_{H^1_0(\Omega_{k_i}(y_i))} \\
&\overset{\text{by (37) and (49)}}{\le} C_2 \|y_i\|_{H^1_0(\Omega_{k_i}(y_i))} \to 0 \text{ as } i \to \infty.
\end{aligned}$$

Thus, passing to the limit in relation (59) as $i \to \infty$, we arrive at the inequality (18) for every $\varphi \in C^\infty_0(\Omega)$.

Step 4. On this step we prove energy inequality (55). For each $i \in \mathbb{N}$, we have the energy equalities

$$\int_\Omega u_i (\varepsilon_i + \mathcal{F}_{k_i}(|\nabla y_i|^2))^{\frac{p-2}{2}} |\nabla y_i|^2\, dx = \int_\Omega (f, \nabla y_i)_{\mathbb{R}^N}\, dx. \tag{63}$$

Taking into account the weak convergence $y_i \rightharpoonup y$ in $H^1_0(\Omega)$ and (54), after passing to the limit in (63), we immediately arrive at the desired assertion. □

As follows from the statement of Theorem 5.2, Hypothesis B can be eliminated from Proposition 3.3 and Theorem 3.3.

Corollary 1. *If $\mathfrak{A}_{ad} \neq \emptyset$, then the set of feasible solutions Ξ to OCP* (1)–(3) *is nonempty for every $f \in L^2(\Omega; \mathbb{R}^N)$ and $y_d \in L^2(\Omega)$.*

Remark 5.1. As follows from Theorem 5.2, feasible solutions to the regularized OCPs (31)–(33) always lead in the limit to some admissible solution (u, y) of the original OCP (1)–(3). It is reasonable to call such pair attainable. However, up to now the structure of the entire set of all attainable pairs remains unclear. For instance, it is unknown whether this set is convex and closed in Ξ. It is also unknown whether all optimal solutions to OCP (1)–(3) can be attainable in such way.

Taking these observations into account, we make use of the following notion.

Definition 5.3. We say that a pair $(\widehat{u},\widehat{y}) \in BV(\Omega)\times W_0^{1,p}(\Omega)$ is a variational solution to OCP (1)–(3) if

$$I(\widehat{u},\widehat{y}) = \inf_{(u,y)\in\Xi} I(u,y), \quad (\widehat{u},\widehat{y}\,) \in \Xi, \tag{64}$$

and $(\widehat{u},\widehat{y}\,)$ is related by energy equality

$$\int_\Omega |\nabla\widehat{y}|^p\,\widehat{u}\,dx = \int_\Omega (f,\nabla\widehat{y})_{\mathbb{R}^N}\;dx. \tag{65}$$

As a consequence of Theorem 5.2 and properties of the variational limits of constrained minimization problems (see Theorem 2.3), we have the following result.

Proposition 5.1. *Assume that initial OCP* (1)–(3) *is a variational* τ*-limit of the sequence* $\left\langle \inf_{(u,y)\in\,\Xi_{\varepsilon,k}} I(u,y)\right\rangle_{\substack{\varepsilon>0\\ k\in\mathbb{N}}}$ *of constrained minimization problems* (31)–(33) *(see Definition 2.2). Let* $\left\{(u^0_{\varepsilon,k},y^0_{\varepsilon,k})\in\Xi_{\varepsilon,k}\right\}_{\substack{\varepsilon>0\\ k\in\mathbb{N}}}$ *be a sequence of optimal solutions to the corresponding regularized OCPs. Then this sequence is relatively compact with respect to the* τ*-convergence and each its* τ*-cluster pair* $(\widehat{u},\widehat{y}\,)\in BV(\Omega)\times W_0^{1,p}(\Omega)$ *is a variational solution to OCP* (1)–(3) *in the sense of Definition 5.3. Moreover, up to a subsequence, we have*

$$y^0_{\varepsilon,k}\to\widehat{y} \;\; in \;\; H^1_0(\Omega) \;\; and \quad as\ \varepsilon\to 0,\ k\to\infty, \tag{66}$$

$$\lim_{\substack{\varepsilon\to 0\\ k\to\infty}} \int_\Omega \left(\varepsilon+\mathcal{F}_k(|\nabla y^0_{\varepsilon,k}|^2)\right)^{\frac{p-2}{2}} |\nabla y^0_{\varepsilon,k}|^2\, u^0_{\varepsilon,k}\,dx = \int_\Omega |\nabla\widehat{y}|^p\,\widehat{u}\,dx, \tag{67}$$

Proof. To begin with, we note that the τ-compactness property of the sequence $\left\{(u^0_{\varepsilon,k},y^0_{\varepsilon,k})\in\Xi_{\varepsilon,k}\right\}_{\substack{\varepsilon>0\\ k\in\mathbb{N}}}$ is a direct consequence of Theorem 5.2. In order to prove the strong convergence (66), we make use of the main properties of the variational convergence. Following Theorems 2.3, 3.3, and 5.2 (see also Corollary 1), we can claim that OCP (1)–(3) is solvable and there exists an optimal pair $(\widetilde{u},\widetilde{y}\,)\in\Xi$ to this problem such that

$$\begin{aligned}\inf_{(u,y)\in\,\Xi} I(u,y) &= I\left(\widetilde{u},\widetilde{y}\,\right) = \|\widetilde{y}-y_d\|^2_{L^2(\Omega)} + \int_\Omega \widetilde{u}|\nabla\widetilde{y}|^p\,dx\\ &= \lim_{\substack{\varepsilon\to 0\\ k\to\infty}} \inf_{(u_{\varepsilon,k},y_{\varepsilon,k})\in\,\Xi_{\varepsilon,k}} I(u_k,y_k) = \lim_{\substack{\varepsilon\to 0\\ k\to\infty}} I(u^0_{\varepsilon,k},y^0_{\varepsilon,k})\\ &= \lim_{\substack{\varepsilon\to 0\\ k\to\infty}} \left[\left\|y^0_{\varepsilon,k}-y_d\right\|^2_{L^2(\Omega)} + \int_\Omega u^0_{\varepsilon,k}\left(\varepsilon+\mathcal{F}_k(|\nabla y^0_{\varepsilon,k}|^2)\right)^{\frac{p-2}{2}}|\nabla y^0_{\varepsilon,k}|^2\,dx\right].\end{aligned} \tag{68}$$

However, because of the lower semicontinuity of $\|\cdot\|_{L^2(\Omega)}$ and $\|\cdot\|_{L^p(\Omega;\mathbb{R}^N)}$ with respect to the weak convergence, the convergence $(u^0_{\varepsilon,k},y^0_{\varepsilon,k})\xrightarrow{\tau}(\widehat{u},\widehat{y}\,)$ implies that $\liminf_{\substack{\varepsilon\to 0\\ k\to\infty}} \int_\Omega u^0_{\varepsilon,k}\left(\varepsilon+\mathcal{F}_k(|\nabla y^0_{\varepsilon,k}|^2)\right)^{\frac{p-2}{2}}|\nabla y^0_{\varepsilon,k}|^2\,dx \ge \int_\Omega\widehat{u}|\nabla\widehat{y}|^p\,dx$ (see (58) and Step 2 of the proof of Theorem 5.2), and, hence,

$$\begin{aligned}\inf_{(u,y)\in\,\Xi} I(u,y) \overset{\text{by (68)}}{=} &\lim_{\substack{\varepsilon\to 0\\ k\to\infty}} \Big[\int_\Omega u^0_{\varepsilon,k}\left(\varepsilon+\mathcal{F}_k(|\nabla y^0_{\varepsilon,k}|^2)\right)^{\frac{p-2}{2}}|\nabla y^0_{\varepsilon,k}|^2\,dx\\ &+\left\|y^0_{\varepsilon,k}-y_d\right\|^2_{L^2(\Omega)}\Big] \ge \|\widehat{y}-y_d\|^2_{L^2(\Omega)} + \int_\Omega\widehat{u}|\nabla\widehat{y}|^p\,dx.\end{aligned} \tag{69}$$

Since the pair $(\widehat{u}, \widehat{y}\,)$ is feasible for the problem (1)–(3) (see Theorem 5.2), it follows that $(\widehat{u}, \widehat{y}\,)$ is an optimal pair. Therefore, the last relation in (69) becomes equality. The weak convergence $y^0_{\varepsilon,k} \rightharpoonup \widehat{y}$ in $H^1_0(\Omega)$ implies that $y_{\varepsilon,k} \to \widehat{y}$ strongly in $L^2(\Omega)$ and, therefore, $\lim_{\substack{\varepsilon\to 0\\ k\to\infty}} \|y^0_{\varepsilon,k} - y_d\|^2_{L^2(\Omega)} = \|\widehat{y} - y_d\|^2_{L^2(\Omega)}$. Whence, we immediately get (65).

To prove the energy equality (65) it is enough to pass to the limit in relation

$$\int_\Omega u^0_{\varepsilon,k}(\varepsilon + \mathcal{F}_k(|\nabla y^0_{\varepsilon,k}|^2))^{\frac{p-2}{2}} |\nabla y^0_{\varepsilon,k}|^2\,dx = \int_\Omega (f, \nabla y^0_{\varepsilon,k})_{\mathbb{R}^N}\,dx,$$

which holds for each ε and k. As a result, using (67), we finally have

$$\begin{aligned} 0 = \lim_{\substack{\varepsilon>0\\ k\in\mathbb{N}}} [y^0_{\varepsilon,k}, y^0_{\varepsilon,k}]_{A^k_{skew}} &= -\lim_{\substack{\varepsilon>0\\ k\in\mathbb{N}}} \int_\Omega u^0_{\varepsilon,k}(\varepsilon + \mathcal{F}_k(|\nabla y^0_{\varepsilon,k}|^2))^{\frac{p-2}{2}} |\nabla y^0_{\varepsilon,k}|^2\,dx \\ &+ \lim_{\substack{\varepsilon>0\\ k\in\mathbb{N}}} \int_\Omega \left(f, \nabla y^0_k\right)_{\mathbb{R}^N}\,dx = -\int_\Omega |\nabla \widehat{y}|^p\,\widehat{u}\,dx + \int_\Omega (f, \nabla \widehat{y})_{\mathbb{R}^N}\,dx. \end{aligned}$$

In fact, from (67) it follows that

$$\begin{aligned} &\int_\Omega \widehat{u}|\nabla\widehat{y}|^p\,dx = \lim_{\substack{\varepsilon>0\\ k\in\mathbb{N}}} \int_\Omega \left(\varepsilon + \mathcal{F}_k(|\nabla y^0_{\varepsilon,k}|^2)\right)^{\frac{p-2}{2}} |\nabla y^0_{\varepsilon,k}|^2 u^0_{\varepsilon,k}\,dx \\ &\geq \limsup_{\substack{\varepsilon>0\\ k\in\mathbb{N}}} \int_{\Omega\setminus\Omega_k(y^0_{\varepsilon,k})} \left(\varepsilon + \mathcal{F}_k(|\nabla y^0_{\varepsilon,k}|^2)\right)^{\frac{p-2}{2}} |\nabla y^0_{\varepsilon,k}|^2 u^0_{\varepsilon,k}\,dx \\ &\overset{\text{by (34)}}{\geq} \limsup_{\substack{\varepsilon>0\\ k\in\mathbb{N}}} \int_{\Omega\setminus\Omega_k(y^0_{\varepsilon,k})} \left(\varepsilon + |\nabla y^0_{\varepsilon,k}|^2\right)^{\frac{p-2}{2}} |\nabla y^0_{\varepsilon,k}|^2 u^0_{\varepsilon,k}\,dx \\ &\geq \limsup_{\substack{\varepsilon>0\\ k\in\mathbb{N}}} \int_\Omega |\chi_{\Omega\setminus\Omega_k(y^0_{\varepsilon,k})} \nabla y^0_{\varepsilon,k}|^p u^0_{\varepsilon,k}\,dx \geq \liminf_{\substack{\varepsilon>0\\ k\in\mathbb{N}}} \int_\Omega |\chi_{\Omega\setminus\Omega_k(y^0_{\varepsilon,k})} \nabla y^0_{\varepsilon,k}|^p u^0_{\varepsilon,k}\,dx. \end{aligned} \tag{70}$$

Using the weak convergence $\chi_{\Omega\setminus\Omega_k(y^0_{\varepsilon,k})} \nabla y^0_{\varepsilon,k} {u^0_{\varepsilon,k}}^{1/p} \rightharpoonup \nabla\widehat{y}\,\widehat{u}^{1/p}$ in $L^p(\Omega;\mathbb{R}^N)$ (see the proof of Theorem 5.2, Step 2 for similar arguments), and (70), we get

$$\begin{aligned} &\int_\Omega \widehat{u}|\nabla\widehat{y}|^p\,dx \geq \limsup_{\substack{\varepsilon>0\\ k\in\mathbb{N}}} \int_\Omega |\chi_{\Omega\setminus\Omega_k(y^0_{\varepsilon,k})} \nabla y^0_{\varepsilon,k}|^p u^0_{\varepsilon,k}\,dx \\ &\geq \liminf_{\substack{\varepsilon>0\\ k\in\mathbb{N}}} \int_\Omega |\chi_{\Omega\setminus\Omega_k(y^0_{\varepsilon,k})} \nabla y^0_{\varepsilon,k}|^p u^0_{\varepsilon,k}\,dx \geq \|\nabla\widehat{y}\widehat{u}^{1/p}\|^p_{L^p(\Omega;\mathbb{R}^N)} = \int_\Omega \widehat{u}|\nabla\widehat{y}|^p\,dx. \end{aligned}$$

The weak convergence $\chi_{\Omega\setminus\Omega_k(y^0_{\varepsilon,k})} \nabla y^0_{\varepsilon,k} {u^0_{\varepsilon,k}}^{1/p} \rightharpoonup \nabla\widehat{y}\,\widehat{u}^{1/p}$ and the convergence of their norms $\|\chi_{\Omega\setminus\Omega_k(y^0_{\varepsilon,k})} \nabla y^0_{\varepsilon,k} {u^0_{\varepsilon,k}}^{1/p}\|_{L^p(\Omega;\mathbb{R}^N)} \to \|\nabla\widehat{y}\,\widehat{u}^{1/p}\|_{L^p(\Omega;\mathbb{R}^N)}$ imply the strong convergence in $L^p(\Omega;\mathbb{R}^N)$. Now, it is a simple exercise to check the strong convergence $\chi_{\Omega\setminus\Omega_k(y^0_{\varepsilon,k})} \nabla y^0_{\varepsilon,k} \to \nabla\widehat{y}$ in $L^p(\Omega;\mathbb{R}^N)$. Further, from here and (70) we obtain

$$\lim_{\substack{\varepsilon>0\\ k\in\mathbb{N}}} \int_{\Omega_k(y^0_{\varepsilon,k})} \left(\varepsilon + \mathcal{F}_k(|\nabla y^0_{\varepsilon,k}|^2)\right)^{\frac{p-2}{2}} |\nabla y^0_{\varepsilon,k}|^2 u^0_{\varepsilon,k}\,dx = 0 \tag{71}$$

Applying (71), it is easy to deduce from this that

$$\begin{aligned} &\lim_{\substack{\varepsilon\to 0\\ k\to\infty}} \int_{\Omega_k(y^0_{\varepsilon,k})} |\nabla y^0_{\varepsilon,k}|^2\,dx \\ &\qquad\leq \frac{1}{\alpha} \lim_{\substack{\varepsilon\to 0\\ k\to\infty}} \int_{\Omega_k(y^0_{\varepsilon,k})} (\varepsilon + \mathcal{F}_k(|\nabla y^0_{\varepsilon,k}|^2))^{\frac{p-2}{2}} |\nabla y^0_{\varepsilon,k}|^2 u^0_{\varepsilon,k}\,dx = 0. \end{aligned}$$

Combining this estimate and the strong convergence $\chi_{\Omega\setminus\Omega_k(y^0_{\varepsilon,k})}\nabla y^0_{\varepsilon,k} \to \nabla\widehat{y}$ in $L^p(\Omega;\mathbb{R}^N)$, we conclude that

$$\nabla y^0_{\varepsilon,k} = \chi_{\Omega_k(y^0_{\varepsilon,k})}\nabla y^0_{\varepsilon,k} + \chi_{\Omega\setminus\Omega_k(y^0_{\varepsilon,k})}\nabla y^0_{\varepsilon,k} \to \nabla\widehat{y} \text{ strongly in } L^2(\Omega;\mathbb{R}^N).$$

□

Remark 5.2. As follows from Proposition 5.1 and Theorem 5.2, even if the OCP (1)–(3) has a unique solution (u^0, y^0), it does not ensure that this pair is the variational solution to the above problem. In other words, the existence of Γ-realizing sequence for the pair $(u^0, y^0) \in \Xi$ (see Definition 2.2) is not established.

We are now in a position to discuss the existence of variational solutions to the OCP (1)–(3).

Theorem 5.4. *Assume that*

- *Hypothesis A holds true;*
- *for given matrix $A_{skew} \in L^q(\Omega;\mathbb{S}^N_{skew})$, we have*

$$[y,y]_{A_{skew}} = 0 \quad \forall\, y \in D. \tag{72}$$

Then OCP (1)–(3) has variational solutions for every $f \in L^2(\Omega;\mathbb{R}^N)$ and $y_d \in L^2(\Omega)$.

Proof. To begin with, we note that in this case every weak solution in the Minty sense to the boundary value problem (2)–(3) satisfies energy equality (65). Indeed, since, by Hypothesis A, every weak solution $y = y(u) \in W_0^{1,p}(\Omega)$ such that $(u,y) \in \Xi$, belongs to the set D, the energy inequality can be obtained, using the argumentation similar to the so-called Minty trick (see Pastukhova [27]). Starting from Minty inequality

$$\int_\Omega u|\nabla\varphi|^{p-2}(\nabla\varphi, \nabla\varphi - \nabla y)_{\mathbb{R}^N}\,dx \\ + \int_\Omega (A_{skew}\nabla\varphi, \nabla\varphi - \nabla y)_{\mathbb{R}^N}\,dx \ge \int_\Omega (f, \nabla\varphi - \nabla y)_{\mathbb{R}^N}\,dx, \tag{73}$$

as a test functions in (73) we can take $\varphi = y \pm tv$, where $t \in \mathbb{R}^N$, $v \in D$. Taking into account, that $(A_{skew}\nabla\varphi, \nabla\varphi)_{\mathbb{R}^N} = 0$ and

$$(A_{skew}\nabla\varphi, \nabla\varphi - \nabla y)_{\mathbb{R}^N} = -(A_{skew}\nabla\varphi, \nabla y)_{\mathbb{R}^N} = (\nabla\varphi, A_{skew}\nabla y)_{\mathbb{R}^N},$$

and using the semi-continuity of $\Delta_p(u,\cdot)$, after passing to the limit as $t \to 0$ we have

$$\pm\int_\Omega u|\nabla y|^{p-2}(\nabla y, \nabla v)_{\mathbb{R}^N}\,dx + \int_\Omega (A_{skew}\nabla y, \nabla y)_{\mathbb{R}^N}\,dx \\ \ge \pm\int_\Omega (f, \nabla v)_{\mathbb{R}^N}\,dx,\ \forall\, v \in D,$$

which yields in view of (72) the validity of the integral identity

$$\int_\Omega u|\nabla y|^{p-2}(\nabla y, \nabla v)_{\mathbb{R}^N}\,dx = \int_\Omega (f, \nabla v)_{\mathbb{R}^N}\,dx,\ \forall\, v \in D, \tag{74}$$

and, taking $v = y$ we get (65).

Further, our aim is to show that OCP (1)–(3) can be interpreted as the variational limit of the sequence of constrained minimization problems (31)–(33) for $\varepsilon > 0$ and $k \in \mathbb{N}$. To do so, we have to verify the fulfilment of all conditions of Definition 2.2. Let $\{(u_{\varepsilon,k}, y_{\varepsilon,k}) \subset \Xi_{\varepsilon,k}\}_{\substack{\varepsilon>0\\ k\in\mathbb{N}}}$ be a sequence in $BV(\Omega) \times H_0^1(\Omega)$ with the following properties:

(a) $(u_n, y_n) \in \Xi_{\varepsilon_n,k_n}$ for every $n \in \mathbb{N}$, where $\{\varepsilon_n, k_n\}_{n\in\mathbb{N}}$ is a subsequence of $\{\varepsilon, k\}$ such that $\varepsilon_n \to 0$, $k_n \to \infty$ as $n \to \infty$;

(aa) $(u_n, y_n) \xrightarrow{\tau} (u, y)$ in the sense of Definition 5.1.

Then proceeding as in the proof of Theorem 5.2, it can be shown that the limit pair (u, y) is feasible to the original OCP (1)–(3). Hence, this problem is solvable by Theorem 3.3. Moreover, similar to (54), we have

$$\liminf_{n\to\infty} \int_\Omega u_n \left(\varepsilon_n + \mathcal{F}_{k_n}(|\nabla y_n|^2)\right)^{\frac{p-2}{2}} |\nabla y_n|^2\, dx \geq \int_\Omega u|\nabla y|^p\, dx,$$

Hence, property (d) immediately follows from the relation

$$\begin{aligned}
&\liminf_{n\to\infty} I(u_n, y_n)\\
&= \liminf_{n\to\infty} \left[\|y_n - y_d\|^2_{L^2(\Omega)} + \int_\Omega u_n \left(\varepsilon_n + \mathcal{F}_{k_n}(|\nabla y_n|^2)\right)^{\frac{p-2}{2}} |\nabla y_n|^2\, dx\right]\\
&\geq \|y - y_d\|^2_{L^2(\Omega)} + \int_\Omega u|\nabla y|^p\, dx = I(u, y),
\end{aligned}$$

which holds true for any sequence $\left\{(u_n, y_n) \in \mathfrak{A}_{ad} \times H^1_0(\Omega)\right\}_{n\in\mathbb{N}}$ with properties (a)–(aa).

We focus now on the verification of condition (dd) of Definition 2.2. Let $(u^\sharp, y^\sharp)$ be an arbitrary admissible pair to the original problem. We set $\{u_{\varepsilon,k} \equiv u^\sharp\}_{\substack{\varepsilon>0\\ k\in\mathbb{N}}}$ and $\left\{y_{\varepsilon,k} = y(u_{\varepsilon,k}, f, A^k_{skew})\right\}_{\substack{\varepsilon>0\\ k\in\mathbb{N}}}$ we choose as corresponding solutions to the regularized boundary value problems (32)–(33). Then having applied the arguments of the proof of Theorem 5.2, it can be shown that the sequence $\{y_{\varepsilon,k}\}_{\substack{\varepsilon>0\\ k\in\mathbb{N}}}$ is uniformly bounded in $H^1_0(\Omega)$ and there exists an element $\widehat{y} \in W^{1,p}_0(\Omega)$ such that $\widehat{y} \in D$, $(u^\sharp, \widehat{y}) \in \Xi$, and, up to a subsequence,

$$y_{\varepsilon,k} \to \widehat{y} \ \text{ in } \ L^2(\Omega), \quad \nabla y_k \rightharpoonup \nabla\widehat{y} \ \text{ in } \ L^2(\Omega;\mathbb{R}^N). \tag{75}$$

Our aim is to show that $\widehat{y} = y^\sharp$ and the following identity

$$I(u^\sharp, y^\sharp) = \limsup_{\substack{\varepsilon\to 0\\ k\to\infty}} I(u^\sharp_k, y_k) \tag{76}$$

holds true.

Indeed, since $(u^\sharp, y^\sharp) \in \Xi$ and $(u^\sharp, \widehat{y}) \in \Xi$, in view of preconditions of the theorem, it follows that this pairs satisfy the integral identity (74)

$$\begin{aligned}
\int_\Omega u^\sharp |\nabla y^\sharp|^{p-2} (\nabla y^\sharp, \nabla v)_{\mathbb{R}^N}\, dx &= \int_\Omega (f, \nabla v)_{\mathbb{R}^N}\, dx,\ \forall\, v \in D,\\
\int_\Omega u^\sharp |\nabla\widehat{y}|^{p-2} (\nabla\widehat{y}, \nabla v)_{\mathbb{R}^N}\, dx &= \int_\Omega (f, \nabla v)_{\mathbb{R}^N}\, dx,\ \forall\, v \in D.
\end{aligned}$$

After subtraction and setting $v = y^\sharp - \widehat{y}$ (since the form $[y,\varphi]_{A_{skew}}$ is bilinear, it is clear that the set D is a linear manifold), we get

$$\int_\Omega u^\sharp (|\nabla y^\sharp|^{p-2}\nabla y^\sharp - |\nabla\widehat{y}|^{p-2}\nabla\widehat{y}, y^\sharp - \widehat{y})_{\mathbb{R}^N}\, dx = 0. \tag{77}$$

The strict monotonicity of $\Delta_p(u^\sharp, \cdot)$ and (77) immediately imply $y^\sharp = \widehat{y}$. Moreover, in view of preconditions of the theorem, for the limit pair $(u^\sharp, \widehat{y})$ instead of the energy inequality (55) (see Theorem 5.2), we have the energy equality (73). Hence, passing to a subsequence, if necessary, we have

$$\lim_{\substack{\varepsilon\to 0\\ k\to\infty}} \int_\Omega \left(\varepsilon + \mathcal{F}_k(|\nabla y_{\varepsilon,k}|^2)\right)^{\frac{p-2}{2}} |\nabla y_{\varepsilon,k}|^2 u^\sharp\, dx = \int_\Omega |\nabla\widehat{y}|^p u^\sharp\, dx,$$

i.e. desired condition (dd) obviously holds:

$$\begin{aligned}\limsup_{\substack{\varepsilon\to 0\\ k\to\infty}} I(u_{\varepsilon,k}, y_{\varepsilon,k}) &\ge \lim_{\substack{\varepsilon\to 0\\ k\to\infty}} I(u^\sharp, y_{\varepsilon,k})\\ &= \lim_{\substack{\varepsilon\to 0\\ k\to\infty}} \left[\|y_{\varepsilon,k}-y_d\|^2_{L^2(\Omega)} + \int_\Omega u^\sharp \left(\varepsilon + \mathcal{F}_k(|\nabla y_{\varepsilon,k}|^2)\right)^{\frac{p-2}{2}} |\nabla y_{\varepsilon,k}|^2\,dx\right]\\ &= \|\widehat{y}-y_d\|^2_{L^2(\Omega)} + \int_\Omega u^\sharp |\nabla y^\sharp|^p\,dx = I(u^\sharp, y^\sharp).\end{aligned}$$

This concludes the proof. □

Our next observation shows that variational solutions do not exhaust the entire set of all possible solutions to the original OCP (1)–(3). With that in mind, we adopt the following concept.

Definition 5.5. We say that a pair $(u_0, y_0) \in \Xi$ is a non-variational solution to OCP (1)–(3) if

$$I(u_0, y_0) = \inf_{(u,y)\in\Xi} I(u,y), \quad (u_0,y_0)\in\Xi, \quad \text{and} \tag{78}$$

$$\int_\Omega u_0 |\nabla y_0|^p\,dx \ne \int_\Omega (f, \nabla y_0)_{\mathbb{R}^N}\,dx. \tag{79}$$

Lemma 5.2. *Assume that there exists an element $u_0 \in \mathfrak{A}_{ad}$, $A_{skew} \in L^q(\Omega; \mathbb{S}^N_{skew})$ and an element $v \in D \subset W^{1,p}_0(\Omega)$ with property $[v,v]_{A_{skew}} > 0$ and such that $|\nabla v|^{p-2}\nabla v + A_{skew}\nabla v \in L^2(\Omega;\mathbb{R}^N)$. Then there are distributions $f \in L^2(\Omega;\mathbb{R}^N)$ and $y_d \in W^{1,p}_0(\Omega) \subset L^2(\Omega)$ such that the optimal control problem*

$$\textit{Minimize } I(u,y) = \|y-y_d\|^2_{L^2(\Omega)} + \int_\Omega u|\nabla y - \nabla y_d|^p\,dx \tag{80}$$

$$\textit{subject to the constraints } (2)\text{–}(3) \tag{81}$$

has a non-variational solution in the sense of Definition 5.5.

Proof. Further we put

$$y_d = v, \quad f = u_0|\nabla v|^{p-2}\nabla v + A_{skew}\nabla v. \tag{82}$$

Since $v \in D$, it follows that $y_d \in W^{1,p}_0(\Omega)$, and in view of preconditions of the lemma, $f \in L^2(\Omega;\mathbb{R}^N)$. Then, taking into account monotonicity of $\Delta_p(u_0,\cdot)$ and the fact, that $(A_{skew}\nabla\varphi, \nabla v)_{\mathbb{R}^N} = -(\nabla\varphi, A_{skew}\nabla v)_{\mathbb{R}^N}$ we have

$$\begin{aligned}&\int_\Omega u_0(|\nabla\varphi|^{p-2}\nabla\varphi, \nabla\varphi-\nabla v)_{\mathbb{R}^N}\,dx + \int_\Omega (A_{skew}\nabla\varphi, \nabla\varphi - \nabla v)_{\mathbb{R}^N}\\ &- \int_\Omega (f, \nabla\varphi-\nabla v)_{\mathbb{R}^N} = \int_\Omega u_0(|\nabla\varphi|^{p-2}\nabla\varphi - |\nabla v|^{p-2}\nabla v, \nabla\varphi-\nabla v)_{\mathbb{R}^N}\,dx\\ &+ \int_\Omega (A_{skew}\nabla\varphi - \nabla v, \nabla\varphi-\nabla v)_{\mathbb{R}^N} \ge \int_\Omega (A_{skew}\nabla v, \nabla v) > 0, \ \forall\,\varphi \in C^\infty_0(\Omega).\end{aligned}$$

Hence, the distribution y_d is a weak Minty solution to boundary value problem (2)–(3). Moreover, using the fact that $I(u_0, y_d) = 0$, we can conclude: (u_0, y_d) is the unique optimal pair to the above OCP. It remains to observe that in view of condition $[y_d, y_d]_{A_{skew}} := [v,v]_{A_{skew}} > 0$, the strict energy inequality

$$\int_\Omega u_0|\nabla y_d|^p\,dx < \int_\Omega (f, \nabla y_d)_{\mathbb{R}^N}\,dx$$

takes place and, hence, (u_0, y_d) is a non-variational solution to the above problem. The proof is complete. □

6. **Optimality conditions for perturbed OCPs (31)–(33).** In view of Proposition 5.1 and Theorem 5.4 under fulfillment of Hypothesis A and condition (72), some optimal solutions to initial OCP (1)–(3) can be attained through optimal solutions to optimization problems (31)–(33). Mainly motivated by these reasoning and for the sake of numerical computations, we focus on establishment of optimality conditions to OCPs (31)–(33). Before deriving the optimality conditions for OCP (31)–(33), we recall the well-known Ioffe & Tikhomirov theorem (see [15]) justifying the Lagrange principle for well-posed boundary value problems.

Theorem 6.1. *Let Y, U and V be Banach spaces. Let $I : U \times Y \to \mathbb{R}\cup\{+\infty\}$ be a cost functional, $F : U \times Y \to V$ a mapping, and U_∂ a nonempty subset of U. Let $(u^0, y^0) \in U \times Y$ be a solution to the following extremal problem*

$$I(u, y) \to \inf, \qquad F(u, y) = 0, \quad u \in U_\partial. \tag{83}$$

Assume that for every $u \in U_\partial$, the mappings $y \mapsto I(u, y)$ and $y \mapsto F(u, y)$ are continuously differentiable for $y \in \mathcal{O}(y^0)$, where $\mathcal{O}(y^0)$ is some neighborhood of the point y^0. Let $\operatorname{Im} F'_y(u^0, y^0)$ be a closed subset of V and has a finite codimension in V. Assume that for each $y \in \mathcal{O}(y^0)$ the function $u \mapsto I(u, y)$ is convex, and the mapping $u \mapsto F(u, y)$ is continuous from U to V and affine, i.e.,

$$F(\gamma u_1 + (1-\gamma)u_2, y) = \gamma F(u_1, y) + (1-\gamma)F(u_2, y), \quad \forall\, u_1, u_2 \in U,\ \gamma \in \mathbb{R}. \tag{84}$$

Then there exists a pair $(\lambda, \mu) \in (\mathbb{R}_+ \times V^*)\setminus\{0\}$ such that for the Lagrange function to the problem (83), that is defined by the equality

$$\Lambda(u, y, \lambda, \mu) = \lambda I(u, y) + \langle \mu, F(u, y)\rangle_{V^*;V}\,, \tag{85}$$

we have

$$\left\langle \Lambda'_y(u^0, y^0, \lambda, \mu), h\right\rangle_{Y^*,Y} = 0, \quad \forall\, h \in Y, \tag{86}$$

$$\Lambda(u, y^0, \lambda, \mu) - \Lambda(u^0, y^0, \lambda, \mu) \ge 0, \quad \forall\, u \in U_\partial. \tag{87}$$

Moreover, if $\operatorname{Im} F'_y(u^0, y^0) = V$, then we can assume that $\lambda = 1$ in (86)–(87).

In order to derive the optimality conditions to OCP (31)–(33), we begin with the following result.

Lemma 6.1. *Let $u \in \mathfrak{A}_{ad}$ be given control, and let y be any element of $H^1_0(\Omega)$. Then the mapping $\mathcal{A}_{\varepsilon,k,u} : H^1_0(\Omega) \to H^{-1}(\Omega)$ given by the rule*

$$y \mapsto \mathcal{A}_{\varepsilon,k,u}(y) = -\operatorname{div}\left(u\left(\varepsilon + \mathcal{F}_k\left(|\nabla y|^2\right)\right)^{\frac{p-2}{2}}\nabla y\right) - \operatorname{div}\left(A^k_{skew}\nabla y\right)$$

is Fréchet differentiable at y for all $\varepsilon > 0$ and $k \in \mathbb{N}$, its Fréchet derivative

$$\left(\mathcal{A}_{\varepsilon,k,u}\right)'_F \in \mathcal{L}\left(H^1_0(\Omega), H^{-1}(\Omega)\right)$$

is well defined on $H^1_0(\Omega)$ and takes the following implicit form:

$$\begin{aligned}\left(\mathcal{A}_{\varepsilon,k,u}(y)\right)'_F[h] &= -\operatorname{div}\left(u(\varepsilon + \mathcal{F}_k\left(|\nabla y|^2\right))^{\frac{p-2}{2}}\nabla h\right)\\ -(p-2)\operatorname{div}&\left(u(\varepsilon + \mathcal{F}_k\left(|\nabla y|^2\right))^{\frac{p-4}{2}}\mathcal{F}'_k\left(|\nabla y|^2\right)(\nabla y, \nabla h)_{\mathbb{R}^N}\nabla y\right) - \operatorname{div}\left(A^k_{skew}\nabla h\right).\end{aligned} \tag{88}$$

Proof. To begin with, we note that for every $u \in \mathfrak{A}_{ad}$, $k \in \mathbb{N}$, and $\varepsilon > 0$, the mapping $y \mapsto \mathcal{A}_{\varepsilon,k,u}(y)$ is bounded as an operator acting from $H^1_0(\Omega)$ in $H^{-1}(\Omega)$ (see Proposition 4.1). Let $y, h \in H^1_0(\Omega)$ be arbitrary distributions. Let us consider the vector-valued function

$$g(t) := \left(\varepsilon + \mathcal{F}_k\left(|\nabla y + t\nabla h|^2\right)\right)^{\frac{p-2}{2}}(\nabla y + t\nabla h) + A^k_{skew}(\nabla y + t\nabla h) \tag{89}$$

Since this function is continuously differentiable and

$$g'(t) = A^k_{skew}\nabla h + \left(\varepsilon + \mathcal{F}_k\left(|\nabla y + t\nabla h|^2\right)\right)^{\frac{p-2}{2}}\nabla h + (p-2)\times$$
$$\times\left(\varepsilon + \mathcal{F}_k\left(|\nabla y + t\nabla h|^2\right)\right)^{\frac{p-4}{2}}\mathcal{F}'_k\left(|\nabla y + t\nabla h|^2\right)(\nabla y + t\nabla h, \nabla h)_{\mathbb{R}^N}(\nabla y + t\nabla h)\,.$$

It follows that the mapping $y \mapsto \mathcal{A}_{\varepsilon,k,u}(y)$ has a Gâteaux derivative

$$\left(\mathcal{A}_{\varepsilon,k,u}(y)\right)'_G[h] = -\mathrm{div}\left(u(\varepsilon + \mathcal{F}_k\left(|\nabla y|^2\right))^{\frac{p-2}{2}}\nabla h\right)$$
$$-(p-2)\mathrm{div}\left(u(\varepsilon + \mathcal{F}_k\left(|\nabla y|^2\right))^{\frac{p-4}{2}}\mathcal{F}'_k\left(|\nabla y|^2\right)(\nabla y, \nabla h)_{\mathbb{R}^N}\nabla y\right) - \mathrm{div}\left(A^k_{skew}\nabla h\right), \tag{90}$$

and this derivative satisfies the following relation

$$\lim_{\lambda\to+0}\left\|\frac{\mathcal{A}_{\varepsilon,k,u}(y+\lambda h) - \mathcal{A}_{\varepsilon,k,u}(y)}{\lambda} - \left(\mathcal{A}_{\varepsilon,k,u}(y)\right)'_G[h]\right\|_{H^{-1}(\Omega)} = 0.$$

As follows from (90), the Gâteaux derivative $\left(\mathcal{A}_{\varepsilon,k,u}(y)\right)'_G$ is well defined in any neighborhood of a given point $y \in H^1_0(\Omega)$ and is continuous at this point (for the technical details we refer to [1]). As a result, the Gâteaux derivative implies the existence of the Fréchet derivative and these derivatives coincide [9]. □

As an obvious consequence of this result and the fact that Fréchet differentiability of operator $y \mapsto \mathcal{A}_{\varepsilon,k,u}(y)$ implies the existence of the gradient for the functional $\varphi : H^1_0(\Omega) \to \mathbb{R}$, where

$$\varphi(y) = \langle\mathcal{A}_{\varepsilon,k,u}(y), \mu\rangle_{H^{-1}(\Omega);H^1_0(\Omega)}$$
$$= \int_\Omega u(\varepsilon + \mathcal{F}_k\left(|\nabla y|^2\right))^{\frac{p-2}{2}}(\nabla y, \nabla\mu)_{\mathbb{R}^N}\,dx + \int_\Omega (A^k_{skew}\nabla y, \nabla\mu)_{\mathbb{R}^N}\,dx,$$

we have

$$\left\langle\varphi'_y(y), h\right\rangle_{H^{-1}(\Omega);H^1_0(\Omega)} = \left\langle\left(\mathcal{A}_{\varepsilon,k,u}(y)\right)'_F[h], \mu\right\rangle_{H^{-1}(\Omega);H^1_0(\Omega)},$$

for all $\mu \in H^1_0(\Omega)$, where $\left(\mathcal{A}_{\varepsilon,k,u}(y)\right)'_F[h]$ is given by (90).

Similarly, for $I_{\varepsilon,k}(u,\cdot) : H^1_0(\Omega) \to \mathbb{R}$, we have

$$\left\langle I'_{\varepsilon,k_y}(u,y), h\right\rangle_{H^{-1}(\Omega);H^1_0(\Omega)}$$
$$= 2\int_\Omega (y - y_d)h\,dx + 2\int_\Omega u\left(\varepsilon + \mathcal{F}_k\left(|\nabla y|^2\right)\right)^{\frac{p-2}{2}}(\nabla y, \nabla h)_{\mathbb{R}^N}\,dx$$
$$+ (p-2)\int_\Omega u\left(\varepsilon + \mathcal{F}_k\left(|\nabla y|^2\right)\right)^{\frac{p-4}{2}}\mathcal{F}'_k\left(|\nabla y|^2\right)|\nabla y|^2\left(\nabla y, \nabla h\right)_{\mathbb{R}^N}dx.$$

As a result, we arrive at the following obvious result.

Corollary 2. *Let $u \in \mathfrak{A}_{ad}$, $f \in L^2(\Omega;\mathbb{R}^N)$, and $y_d \in L^2(\Omega)$ be given distributions. Then, for arbitrary $\varepsilon > 0$ and $k \in \mathbb{N}$, the mapping*

$$y \mapsto \Lambda(u,y,\mu) := I_{\varepsilon,k}(u,y) + \int_\Omega u(\varepsilon + \mathcal{F}_k\left(|\nabla y|^2\right))^{\frac{p-2}{2}}(\nabla y, \nabla\mu)_{\mathbb{R}^N}\,dx$$
$$+ \int_\Omega (A^k_{skew}\nabla y, \nabla\mu)_{\mathbb{R}^N}\,dx - \int_\Omega (f, \nabla\mu)_{\mathbb{R}^N}\,dx$$

is Fréchet differentiable at any $y \in H_0^1(\Omega)$ and its gradient $\Lambda'_y(u,y,\mu) \in H^{-1}(\Omega)$ exists and takes the form:

$$\begin{aligned}
&\left\langle \Lambda'_y(u,y,\mu), h\right\rangle_{H^{-1}(\Omega);H_0^1(\Omega)} \\
&= 2\int_\Omega \big(y-y_d\big)h\,dx + 2\int_\Omega u\left(\varepsilon + \mathcal{F}_k\left(|\nabla y|^2\right)\right)^{\frac{p-2}{2}}(\nabla y, \nabla h)_{\mathbb{R}^N}\,dx \\
&+ (p-2)\int_\Omega u\left(\varepsilon + \mathcal{F}_k\left(|\nabla y|^2\right)\right)^{\frac{p-4}{2}}\mathcal{F}'_k\left(|\nabla y|^2\right)|\nabla y|^2\big(\nabla y, \nabla h\big)_{\mathbb{R}^N}\,dx \\
&+ \int_\Omega u(\varepsilon + \mathcal{F}_k\left(|\nabla y|^2\right))^{\frac{p-2}{2}}(\nabla \mu, \nabla h)_{\mathbb{R}^N}\,dx - \int_\Omega (A^k_{skew}\nabla\mu, \nabla h)_{\mathbb{R}^N}dx \\
&+ (p-2)\int_\Omega u(\varepsilon + \mathcal{F}_k\left(|\nabla y|^2\right))^{\frac{p-4}{2}}\mathcal{F}'_k\left(|\nabla y|^2\right)(\nabla y, \nabla\mu)_{\mathbb{R}^N}\big(\nabla y, \nabla h\big)_{\mathbb{R}^N}\,dx. \quad (91)
\end{aligned}$$

Remark 6.1. In view of the equality $(\nabla y, \nabla \mu)_{\mathbb{R}^N}\nabla y = [\nabla y \otimes \nabla y]\,\nabla\mu$, the third and the last term in (91) can be rewritten as follows

$$(p-2)\int_\Omega u(\varepsilon + \mathcal{F}_k\left(|\nabla y|^2\right))^{\frac{p-4}{2}}\mathcal{F}'_k\left(|\nabla y|^2\right)\Big([\nabla y\otimes\nabla y]\,\nabla y, \nabla h\Big)_{\mathbb{R}^N}dx,$$

$$(p-2)\int_\Omega u(\varepsilon + \mathcal{F}_k\left(|\nabla y|^2\right))^{\frac{p-4}{2}}\mathcal{F}'_k\left(|\nabla y|^2\right)\Big([\nabla y\otimes\nabla y]\,\nabla \mu, \nabla h\Big)_{\mathbb{R}^N}dx.$$

Hence, using the notion of identity matrix $I_N := \operatorname{diag}(1,\dots,1) \in \mathcal{L}(\mathbb{R}^N;\mathbb{R}^N)$, we have

$$\begin{aligned}
&\left\langle \Lambda'_y(u,y,\mu), h\right\rangle_{H^{-1}(\Omega);H_0^1(\Omega)} = 2\int_\Omega \big(y-y_d\big)h\,dx \\
&+ \int_\Omega \rho_{\varepsilon,k}(u,y)\,(\mathcal{B}_{\varepsilon,k}(y)\nabla y, \nabla h)_{\mathbb{R}^N}\,dx + \int_\Omega \rho_{\varepsilon,k}(u,y)\,(\mathcal{C}_{\varepsilon,k}(y)\nabla \mu, \nabla h)_{\mathbb{R}^N}\,dx \quad (92)
\end{aligned}$$

where the matrices $\mathcal{B}_{\varepsilon,k}(y), \mathcal{C}_{\varepsilon,k}(y) \in \mathcal{L}(\mathbb{R}^N,\mathbb{R}^N)$ and the scalar function $\rho_{\varepsilon,k}(u,y)$ are defined as follows

$$\mathcal{B}_{\varepsilon,k}(y) = \left(2I_N + (p-2)\frac{\mathcal{F}'_k\left(|\nabla y|^2\right)}{\varepsilon + \mathcal{F}_k\left(|\nabla y|^2\right)}\left[\nabla y\otimes\nabla y\right]\right), \quad (93)$$

$$\mathcal{C}_{\varepsilon,k}(y) = \left(I_N + (p-2)\frac{\mathcal{F}'_k\left(|\nabla y|^2\right)}{\varepsilon + \mathcal{F}_k\left(|\nabla y|^2\right)}\left[\nabla y\otimes\nabla y\right] - A^k_{skew}\right), \quad (94)$$

$$\rho_{\varepsilon,k}(u,y) = u(\varepsilon + \mathcal{F}_k\left(|\nabla y|^2\right))^{\frac{p-2}{2}}. \quad (95)$$

Therefore,

$$\begin{aligned}
\left\langle \Lambda'_y(u,y,\lambda,\mu), h\right\rangle_{H^{-1}(\Omega);H_0^1(\Omega)} &= 2\lambda\,(y-y_d, h)_{L^2(\Omega)} \\
&+ \left\langle -\operatorname{div}\left(\rho_{\varepsilon,k}(u,y)\mathcal{B}_{\varepsilon,k}(y)\nabla y\right), h\right\rangle_{H^{-1}(\Omega);H_0^1(\Omega)} \\
&+ \left\langle -\operatorname{div}\left(\rho_{\varepsilon,k}(u,y)\mathcal{C}_{\varepsilon,k}(y)\nabla \mu\right), h\right\rangle_{H^{-1}(\Omega);H_0^1(\Omega)}. \quad (96)
\end{aligned}$$

We note that the matrices $\mathcal{B}_{\varepsilon,k}(y), \mathcal{C}_{\varepsilon,k}(y)$ and the scalar function $\rho_{\varepsilon,k}(u,y)$, given by (94)–(95), possess the following properties (see also (34)):

$$\mathcal{B}_{\varepsilon,k}(u,y) \in L^\infty(\Omega;\mathbb{S}^N_{sym}),\ \mathcal{C}_{\varepsilon,k}(u,y) \in L^\infty(\Omega;\mathbb{R}^N); \quad (97)$$

$$2|\eta|^2 \le (\eta, \mathcal{B}_{\varepsilon,k}(y)\eta)_{\mathbb{R}^N} \le p\,\delta^*|\eta|^2 \ \text{ a.e. in } \ \Omega,\ \forall\,\eta \in \mathbb{R}^N; \quad (98)$$

$$|\eta|^2 \le (\eta, \mathcal{C}_{\varepsilon,k}(y)\eta)_{\mathbb{R}^N} \le (p-1)\,\delta^*|\eta|^2 \ \text{ a.e. in } \ \Omega,\ \forall\,\eta \in \mathbb{R}^N; \quad (99)$$

$$\rho_{\varepsilon,k}(u,y) \in L^\infty(\Omega), \ \text{ and } \ \rho_{\varepsilon,k}(y) \ge \varepsilon^{\frac{p-2}{2}}\alpha > 0. \quad (100)$$

Indeed, properties (97), (98) and (99) follow from (94) and definition of the $C^1(\mathbb{R}_+)$-function $\mathcal{F}_k : \mathbb{R}_+ \to \mathbb{R}_+$ (here, $\mathcal{F}'_k\left(|\nabla y|^2\right) = 0$ a.e. on the set $\Omega_k(y)$. As for the relation (100), it follows from

$$\varepsilon^{\frac{p-2}{2}}\alpha \le \rho_{\varepsilon,k}(u,y) \le \beta(\varepsilon+k^2+1)^{\frac{p-2}{2}}, \quad \forall\, u\in\mathfrak{A}_{ad},\ \forall\, y\in H_0^1(\Omega). \tag{101}$$

To prove the property (98) (and (99) by analogy), it is enough to take into account the definition of the class of admissible controls $\mathfrak{A}_{ad}$ and the following chain of estimates

$$2|\eta|^2 \le 2\,(\eta, I\eta)_{\mathbb{R}^N} \le 2\,(\eta, I\eta)_{\mathbb{R}^N} + (p-2)\mathcal{F}_k'\left(|\nabla y|^2\right)\left(\frac{\nabla y}{\sqrt{\varepsilon+\mathcal{F}_k\left(|\nabla y|^2\right)}},\eta\right)^2_{\mathbb{R}^N}$$

$$= (\eta, \mathcal{B}_{\varepsilon,k}(y)\eta)_{\mathbb{R}^N} \le 2|\eta|^2 + (p-2)\mathcal{F}_k'\left(|\nabla y|^2\right)\left|\frac{\nabla y}{\sqrt{\varepsilon+\mathcal{F}_k\left(|\nabla y|^2\right)}}\right|^2|\eta|^2 \le p\,\delta^*|\eta|^2.$$

As a result, for every $v\in H_0^1(\Omega)$, we have

$$\|v\|^2_{H_0^1(\Omega)} := \int_\Omega |\nabla v|^2\,dx \le \alpha^{-1}\varepsilon^{\frac{2-p}{2}}\int_\Omega u\big(\varepsilon+\mathcal{F}_k\left(|\nabla y|^2\right)\big)^{\frac{p-2}{2}}|\nabla v|^2\,dx$$

$$\le \alpha^{-1}\varepsilon^{\frac{2-p}{2}}\int_\Omega \rho_{\varepsilon,k}(u,y)\left(\nabla v, \mathcal{C}_{\varepsilon,k}(y)\nabla v\right)_{\mathbb{R}^N}\,dx$$

by (99), (101), and Friedrich's inequality

$$\le \alpha^{-1}\varepsilon^{\frac{2-p}{2}}\beta(\varepsilon+k^2+1)^{\frac{p-2}{2}}(p-1)\delta^*\int_\Omega|\nabla v|^2\,dx = C(k,\varepsilon)\|v\|^2_{H_0^1(\Omega)}. \tag{102}$$

We are now in a position to establish the main result of this section.

Theorem 6.2. *For given $\varepsilon>0$, $k\in\mathbb{N}$, $f\in L^2(\Omega;\mathbb{R}^N)$, and $y_d\in L^2(\Omega)$, let $(u^0_{\varepsilon,k}, y^0_{\varepsilon,k})\in\Xi_{\varepsilon,k}$ be an optimal pair to the perturbed problem (31)–(33). Then there exists a distribution $\mu_{\varepsilon,k}\in H_0^1(\Omega)$ such that the triplet $(u^0_{\varepsilon,k}, y^0_{\varepsilon,k}, \mu_{\varepsilon,k})$ satisfies the following Euler-Lagrange system to the problem (31)–(33)*

$$\left.\begin{aligned} -\mathrm{div}\left(\rho_{\varepsilon,k}(u^0_{\varepsilon,k},y^0_{\varepsilon,k})\nabla y^0_{\varepsilon,k}\right) - \mathrm{div}\left(A^k_{skew}y^0_{\varepsilon,k}\right) &= -\mathrm{div} f \quad in\ \ \Omega,\\ y^0_{\varepsilon,k} &= 0 \quad on\ \ \partial\Omega, \end{aligned}\right\} \tag{103}$$

$$\left.\begin{aligned} &-\mathrm{div}\Big(\rho_{\varepsilon,k}(u^0_{\varepsilon,k},y^0_{\varepsilon,k})\mathcal{C}_{\varepsilon,k}(y^0_{\varepsilon,k})\nabla\mu_{\varepsilon,k}\Big) + \mathrm{div}\left(A^k_{skew}\nabla\mu_{\varepsilon,k}\right)\\ &\quad = 2\left(y_d - y^0_{\varepsilon,k}\right) - \mathrm{div}\Big(\rho_{\varepsilon,k}(u^0_{\varepsilon,k},y^0_{\varepsilon,k})\mathcal{B}_{\varepsilon,k}(y^0_{\varepsilon,k})\nabla y^0_{\varepsilon,k}\Big) \quad in\ \ \Omega,\\ &\qquad\qquad \mu_{\varepsilon,k} = 0 \quad on\ \ \partial\Omega, \end{aligned}\right\} \tag{104}$$

$$\left.\begin{aligned} &\int_\Omega \rho_{\varepsilon,k}(\widehat{u}-u^0_{\varepsilon,k},y^0_{\varepsilon,k})\left(\nabla y^0_{\varepsilon,k},\nabla\mu_{\varepsilon,k}\right)_{\mathbb{R}^N}\,dx\\ &\qquad + \int_\Omega \rho_{\varepsilon,k}(\widehat{u}-u^0_{\varepsilon,k},y^0_{\varepsilon,k})|\nabla y^0_{\varepsilon,k}|^2\,dx \ge 0, \quad \forall\,\widehat{u}\in\mathfrak{A}_{ad}. \end{aligned}\right\} \tag{105}$$

Proof. To deduce relations (103)–(105), we show that all assumptions of Theorem 6.1 hold true. For given $\varepsilon>0$ and $k\in\mathbb{N}$, we set:

$$Y = H_0^1(\Omega),\ V = H^{-1}(\Omega),\ U = BV(\Omega),\ U_\partial = \mathfrak{A}_{ad}, \tag{106}$$

$$\text{and}\ \ F(u,y) = \mathcal{A}_{\varepsilon,k,u}(y) - \mathrm{div} f. \tag{107}$$

As properties (97)–(100) indicate, the bilinear form $a_{\varepsilon,k}: H_0^1(\Omega)\times H_0^1(\Omega)\to\mathbb{R}$, where

$$a_{\varepsilon,k}[z,v] := \int_\Omega \rho_{\varepsilon,k}(u^0_{\varepsilon,k},y^0_{\varepsilon,k})\left(\nabla z, \mathcal{C}_{\varepsilon,k}(y^0_{\varepsilon,k})\nabla v\right)_{\mathbb{R}^N}\,dx - \int_\Omega (A^k_{skew}\nabla z,\nabla v)_{\mathbb{R}^N}\,dx,$$

satisfies all conditions of the Lax-Milgram theorem. Hence, the boundary value problem

$$-\text{div}\left(\rho_{\varepsilon,k}(u^0_{\varepsilon,k},y^0_{\varepsilon,k})\mathcal{C}_{\varepsilon,k}(y^0_{\varepsilon,k})\nabla\mu\right)+\text{div}\left(A^k_{skew}\nabla\mu\right)=g \text{ in } \Omega, \quad \mu=0 \text{ on } \partial\Omega$$

admits a unique solution $\mu \in H^1_0(\Omega)$ for every $g \in H^{-1}_0(\Omega)$. Following Lemma 6.1 and representation (93)–(95), we conclude that $\operatorname{Im} F'_y(u^0_{\varepsilon,k},y^0_{\varepsilon,k}) = V$. Moreover, the convexity of the function $u \mapsto I(u,y)$ and continuity of the mapping $u \mapsto F(u,y) : U \to V$ together with its affine property (84) are obvious and immediately follow from the choice rules (106)–(107). It remains to note that the continuous differentiability of the mappings $y \mapsto I(u,y)$ and $y \mapsto F(u,y)$ for all $y \in \mathcal{O}(y^0)$ is guaranteed by Lemma 6.1 and its Corollary 2. Thus, all assumptions of Theorem 6.1 are satisfied.

Following these observations and taking into account that, for each $\varepsilon > 0$ and $k \in \mathbb{N}$, the mapping $y \mapsto \mathcal{A}_{\varepsilon,k,u}(y)$ acting from $H^1_0(\Omega)$ to $H^{-1}(\Omega)$ is bounded (see Proposition 4.1), we can define the Lagrangian to regularized OCP (31)–(33) by the rule (85) with $\lambda = 1$, i.e.

$$\begin{aligned}\Lambda(u,y,\mu) := &\int_\Omega u(\varepsilon+\mathcal{F}_k\left(|\nabla y|^2\right))^{\frac{p-2}{2}}\,(\nabla y,\nabla\mu)_{\mathbb{R}^N}\,dx\\ &+\int_\Omega u(\varepsilon+\mathcal{F}_k\left(|\nabla y|^2\right))^{\frac{p-2}{2}}|\nabla y|^2\,dx+\int_\Omega (y-y_d)^2\,dx\\ &+\int_\Omega \left(A^k_{skew}\nabla y,\nabla\mu\right)_{\mathbb{R}^N}dx-\int_\Omega (f,\nabla\mu)_{\mathbb{R}^N}\,dx, \end{aligned}\tag{108}$$

where $\mu \in H^1_0(\Omega)\setminus\{0\}$ is a Lagrange multiplier.

Then, by Theorem 6.1, there exists a distribution $\mu \in H^1_0(\Omega) = V^*$ such that the Lagrange function Λ satisfies relations (86) and (87). Moreover, in view of representation (96), equality (86) takes the form (104). To conclude the proof, it remains to note that inequality (87) coincides with (105). □

REFERENCES

[1] A. B. Al'shin, M. O. Korpusov and A. G. Sveshnikov, *Blow-up in Nonlinear Sobolev Type Equations*, De Gruyter, Berlin, 2011.

[2] G. Buttazzo and P. I. Kogut, Weak optimal controls in coefficients for linear elliptic problems, *Revista Matematica Complutense*, **24** (2011), 83–94.

[3] E. Casas, Optimal control in the coefficients of elliptic equations with state constraints, *Appl. Math. Optim.*, **26** (1992), 21–37.

[4] E. Casas and L. A. Fernandez, Optimal control of quasilinear elliptic equations with non differentiable coefficients at the origin, *Rev. Matematica Univ. Compl. Madrid*, **4** (1991), 227–250.

[5] E. Casas, P. I. Kogut and G. Leugering, Approximation of optimal control problems in the coefficient for the p-Laplace equation. I. Convergence result, *SIAM Journal on Control and Optimization* , **54** (2016), 1406–1422.

[6] M. Chicco and M. Venturino, Dirichlet problem for a divergence form elliptic equation with unbounded coefficients in an unbounded domain, *Annali di Matematica Pura ed Applicata*, **178** (2000), 325–338.

[7] R. Coifman, P.-L. Lions, Y. Meyer and S. Semmes, Compensated compactness and Hardy spaces, *J. Math. Pures Appl.*, **72** (1993), 247–286.

[8] C. D'Apice, U. De Maio and O. P. Kogut, Optimal control problems in coefficients for degenerate equations of monotone type: shape stability and attainability problems, *SIAM Journal on Control and Optimization*, **50** (2012), 1174–1199.

[9] P. Drabek and Y. Milota, *Methods of Nonlinear Analysis. Applications to Differential Equations*, Birkhäuser, Berlin, 2007.

[10] L. C. Evans and R. F. Gariepy, *Measure Theory and Fine Properties of Functions*, CRC Press, Boca Raton, 1992.

[11] H. Gajewski, K. Gröger and K. Zacharias, *Nichtlineare Operatorgleichungen und Operatordifferentialgleichungen*, Academie-Varlar, Berlin, 1974.

[12] S. A. Gorbonos, On approximation of solutions to one class of the optimal control problems for parabolic equation with unbounded coefficients, *Journal of Automation and Information Sciences*, **46** (2014), 12–28.

[13] T. Horsin and P. I. Kogut, Optimal L^2-control problem in coefficients for a linear elliptic equation. I. Existence result, *Mathematical Control and Related Fields*, **5** (2015), 73–96.
[14] T. Horsin and P. I. Kogut, On unbounded optimal controls in coefficients for ill-posed elliptic Dirichlet boundary value problems, *Asymptotic Analysis*, **98** (2016), 155–188.
[15] A. Ioffe and V. Tikhomirov, *Extremal Problems*, North-Holland, Amsterdam, 1979.
[16] D. Kinderlehrer and G. Stampacchia, *An Introduction to Variational Inequalities and Their Applications*, Academic Press, New York, 1980.
[17] P. I. Kogut, On Approximation of an optimal boundary control problem for linear elliptic equation with unbounded coefficients, *Discrete and Continuous Dynamical Systems - Series A*, **34** (2014), 2105–2133.
[18] P. I. Kogut, On some properties of unbounded bilinear forms associated with skew-symmetric $L^2(Omega)$-matrices, *Bulletin of Dnipro National University, Series: Modelling*, **5** (2013), 84–97.
[19] P. I. Kogut and G. Leugering, *Optimal Control Problems for Partial Differential Equations on Reticulated Domains: Approximation and Asymptotic Analysis*, Birkhäuser, Boston, 2011.
[20] O. P. Kupenko, Optimal control problems in coefficients for degenerate variational inequalities of monotone type. I. Existence of optimal solutions, *J. Computational and Appl. Mathematics*, **106** (2011), 88–104.
[21] O. P. Kupenko, Optimal control problems in coefficients for degenerate variational inequalities of monotone type. II. Attainability problem, *J. Computational and Appl. Mathematics*, **107** (2012), 15–34.
[22] O. P. Kupenko and R. Manzo, Approximation of an optimal control problem in coefficient for variational inequality with anisotropic p-Laplacian, *Nonlinear Differential Equations and Applications*, **23** (2016), Art. 35, 18 pp.
[23] J.-L. Lions, *Some Methods of Solving Non-Linear Boundary Value Problems*, Dunod-Gauthier-Villars, Paris, 1969.
[24] J.-L. Lions, *Optimal Control of Systems Governed by Partial Differential Equations*, Springer-Verlag, New York-Berlin, 1971.
[25] K. A. Lurie, Optimum control of conductivity of a fluid moving in a channel in a magnetic field, *J. Appl. Math. Mech.*, **28** (1964), 316–327.
[26] V. G. Mazya and I. E. Verbitsky, Form boundedness of the general second-order differential operator, *Comm. Pure Appl. Math.*, **59** (2006), 1286–1329.
[27] S. E. Pastukhova, Degenerate equations of monotone type: Lavrentiev phenomenon and attainability problems, *Sbornik: Mathematics*, **198** (2007), 1465–1494.
[28] T. Phan, Regularity gradient estimates for weak solutions of singular quasi-linear parabolic equations, *J. Differential Equations*, **263** (2017), 8329–8361.
[29] O. Pironneau, *Optimal Shape Design for Elliptic Systems*, Springer-Verlag, Berlin, 1984.
[30] F. Punzo and A. Tesei, Uniqueness of solutions to degenerate elliptic problems with unbounded coefficients, *Ann. I.H. Poincaré*, **26** (2009), 2001–2024.
[31] T. Radice, Regularity result for nondivergence elliptic equations with unbounded coefficients, *Diff. Integral Equa.*, **23** (2010), 989–1000.
[32] T. Radice and G. Zecca, Existence and uniqueness for nonlinear elliptic equations with unbounded coefficients, *Ricerche Mat.*, **63** (2014), 355–367.
[33] T. Roubíček, *Nonlinear Partial Differential Equations with Applications*, Birkhäuser, Basel, 2013.
[34] M. V. Safonov, Non-divergence elliptic equations of second order with unbounded drift, *Nonlinear Partial Diff. Equa. and Related Topics*, **229** (2010), 211–232.
[35] R. E. Showalter, *Monotone Operators in Banach Space and Nonlinear Partial Differential Equations*, American Mathematical Society, 1997.
[36] V. V. Zhikov, Weighted Sobolev spaces, *Sbornik: Mathematics*, **189** (1998), 27–58.
[37] V. V. Zhikov, Remarks on the uniqueness of a solutionof the Dirichlet problem for second-order elliptic equations with lower-order terms, *Functional Analysis and Its Applications*, **38** (2004), 173–183.

E-mail address: kogut_olga@bk.ru
E-mail address: rmanzo@unisa.it