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Scalable Wake-up of Multi-Channel Single-Hop Radio Networks

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Abstract

We consider waking up a single-hop radio network with multiple channels. There are n stations connected to b channels without collision detection. Some k stations may become active spontaneously at arbitrary times, where k is unknown, and the goal is for all the stations to hear a successful transmission as soon as possible after the first spontaneous activation. We present a deterministic algorithm for the general problem that wakes up the network in $\mathcal{O}(k \log^{1/b} k \log n)$ time. We prove a lower bound that any deterministic algorithm requires $\Omega((k/b)\log(n/k))$ time. We give a deterministic algorithm for the special case when $b > d \log\log n$, for some constant $d > 1$, which wakes up the network in $\mathcal{O}((k/b)\log n \log(b \log n))$ time. This algorithm misses time optimality by at most a factor of $\log n \log b$. We give a randomized algorithm that wakes up the network within $\mathcal{O}(k^{1/b}\ln(1/\epsilon))$ rounds with the probability of at least $1-\epsilon$, for any unknown $0 < \epsilon < 1$. We also consider a model of jamming, in which each channel in any round may be jammed to prevent a successful transmission, which happens with some known parameter probability p, independently across all channels and rounds. For this model, we give a deterministic algorithm that wakes up the network in $\mathcal{O}(\log^{-1}(1/p) k \log n \log^{1/b} k)$ time with the probability of at least $1 - 1/\mathrm{poly}(n)$.

Keywords: multiple access channel, radio network, multi-channel, wake-up, synchronization, deterministic algorithms, randomized algorithms, distributed algorithms.

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1 Introduction

We consider wireless networks organized as a group of stations connected to a number of channels. Each channel provides the functionality of a single-hop radio network. A station can use any of these channels to communicate directly with any other station.

This topology is called multi-channel in the literature. The assumption usually made is that a station can connect to at most one channel at a time for either transmitting or listening. We depart from this restriction and consider the apparently stronger model in which a station can use all the available channels simultaneously and independently from each other, some for transmitting and others for listening. On the other hand, channels do not provide collision detection, which makes the model weaker than a multi-channel with carrier sensing capabilities.

The algorithmic problem that we consider is to wake up the network. Initially, all the stations are dormant but connected and listening to all the channels. Some stations become active spontaneously and want the whole network to be activated and synchronized. The first successful transmission on any channel suffices to accomplish this goal.

We use the following parameters to characterize a multi-channel network. We denote by n the number of stations and b is the number of shared channels. At most k stations become active spontaneously at arbitrary times and join an execution with the goal to wake up the network. When the first message is heard on some channel then the network is considered woken up and synchronized. Stations know n and b, but k is an unknown parameter used only to characterize the scalability of a given wake up algorithm.

Our results. We present a deterministic algorithm which wakes up the network in $\mathcal{O}(k \log^{1/b} k \log n)$ rounds. We give a deterministic algorithm for the special case of sufficiently many channels, which wakes up the network in $\mathcal{O}((k/b) \log n \log(b \log n))$ rounds when $b > \log(128 b \log n)$. We prove a lower bound of $\Omega((k/b) \log(n/k))$ rounds, which are required by any deterministic algorithm. In view of this lower bound, the algorithm of time performance $\mathcal{O}((k/b) \log n \log(b \log n))$ misses time optimality by at most a factor of $\log n \log b$. We give a randomized algorithm that wakes up the network within $\mathcal{O}(k^{1/b} \ln(1/\epsilon))$ rounds with the probability of at least $1 - \epsilon$, for any unknown $0 < \epsilon < 1$. We also consider a model of jamming, in which each channel in any round may be jammed to prevent a successful transmission, which happens with some known parameter probability p, independently across all channels and rounds. For this model, we give a deterministic algorithm that wakes up the network in $\mathcal{O}(\log^{-1}(1/p) k \log n \log^{1/b} k)$ time with the probability of at least $1 - 1/\mathrm{poly}(n)$.

For a multiple access channel, Jurdziński and Stachowiak [37] gave two randomized algorithms, one working in $\mathcal{O}(\log^2 n)$ time steps with high probability with respect to n, and another working in $\mathcal{O}(k)$ time steps with high probability with respect to k. Our randomized algorithm for multi-channel networks has performance sublinear in k for even just two channels.

Our deterministic general algorithm to wake up the network runs in $\mathcal{O}(k \log^{1/b} k \log n)$ time. When $b = \Omega(\log \log n)$ then wake-up is performed in $\mathcal{O}(k \log n)$ time. This is similar to the time bound $\mathcal{O}(k + k \log(n/k))$ given by Komlós and Greenberg [38] to resolve conflict for access to the channel among any k stations that start an execution in the same round.

A preliminary version of this paper appeared as [8].

Previous and related work. Shi et al. [42] considered the model of a multi-channel in which a node can simultaneously obtain different messages on different channels, while each channel is a single-hop radio network. They studied the information-exchange problem, in which some ℓ nodes start with a rumor each and the goal is to disseminate all rumors across all stations. They gave and algorithm of time performance $\mathcal{O}(\log \ell \log \log \ell)$ with n channels available. Most of the previous work on algorithms for multi-channel single-hop radio networks used the model defined as a collection of multiple-access channels such that a node has to choose a channel per round to participate in communication in this particular channel either as a listener or transmitter. Variants to this model with adversarial disruptions of channels were also considered. To the best out our knowledge, [42] was the only previous paper that used the strong model in which nodes can use all the available channels simultaneously and independently.

Next we review work done for the multi-channel model in which a station can use at most one channel for communication at a time. Dolev et al. [28] studied a parametrized variant of gossip for multi-channel radio networks. They gave oblivious deterministic algorithms for an adversarial setting in which a malicious adversary can disrupt one channel per round. Daum et al. [20] considered leader election and Dolev et al. [27] gave algorithms to synchronize a network, both papers about an adversarial setting in which the adversary can disrupt a number of channels in each round, this number treated as a parameter for performance bounds.

Information exchange has been investigated extensively for multi-channel wireless networks. The problem is about some ℓ nodes initialized with a rumor each and the goal is either to disseminate the rumors across the whole network or, when the communication environment is prone to failures, to have each node learn as many rumors as possible. Gilbert et al. [33] gave a randomized algorithm for the scenario when an adversary can disrupt a number of channels per round, this number being an additional parameter in performance bounds. Holzer et al. [36] and [35] gave deterministic and randomized algorithms to accomplish the information-exchange task in time $\mathcal{O}(\ell)$, for ℓ rumors and for suitable numbers of channels that make this achievable. This time bound $\mathcal{O}(\ell)$ is optimal when multiple rumors cannot be combined into compound messages. Wang et al. [44] considered information-exchange in a model when collision detection is available and rumors can be combined into compound messages. They gave an algorithm of time performance $\mathcal{O}(\ell/b + n\log^2 n)$, for ℓ rumors and b channels.

A multi-channel single-hop network is a generalization of a multiple-access channel, which consists of just one channel. For recent work on algorithms for multiple-access channels see [3, 4, 5, 6, 7, 12, 13, 18, 39].

The problem of waking up a radio network was first investigated by Gąsieniec et al. [32] in the case of multiple access channels, see [24, 25, 26, 37] for more on a related work. A broadcast from a synchronized start in a radio network was considered in [9, 15, 16, 17, 22, 23, 40]. The general problem of waking up a multi-hop radio network was studied in [10, 11, 14].

A lower bound for a multiple access channel was given by Greenberg and Winograd [34]. Lower bounds for multi-hop radio networks we proved by Alon et al. [1], Clementi et al. [16], Farach-Colton et al. [30] and Kushilevitz and Mansour [41].

Ad-hoc multi-hop multi-channel networks were studied by Alonso et al. [2], Daum et al. [19] and [21], Dolev et al. [29], and So and Vaidya [43].

2 Technical preliminaries

The model of a multi-channel single-hop radio network is defined as follows. There are n nodes attached to a spectrum of b frequencies. We use the term "station" and "node" interchangeably. The set of all stations is denoted by V. Each frequency determines a multiple access channel. All these b channels operate concurrently and independently from each other. All stations listen to all channels all the time and obtain the same feedback from each channel. A station can transmit on any set of channels at any time. A station obtains the respective feedback from each channel separately and concurrently.

When a station successfully receives a message transmitted on some channel then we say that the station *hears* the message. When no station transmits on a channel then the channel is *silent*. When more than one stations transmit on one channel such that their transmissions overlap then we say that a *collision* occurs on this channel during the time of overlap.

The semantics of channels. When a station transmits on a channel and no collision occurs during the transmission on this channel then each station hears the transmitted message. When a station transmits a message and a collision occurs during the transmission on the channel of transmission then no station hears this transmitted message. Channels operate independently, in particular, there could be a collision on one channel and at the same time a message may be heard on some other channel. There is no collision detection, which means that when a station listens to a channel then it receives the same feedback when the channel is silent and when a collision occurs on this channel.

Transmissions on all channels are synchronized. This means that an execution of an algorithm is partitioned into rounds of equal length so that each transmission occurs in some round. Each station has its private clock which is ticking at the rate of rounds. Rounds begin and end at the same time on all channels. When we refer to a round number then this refers to the indiction of some station's private clock and this station is understood. Messages are scaled to duration of rounds so that transmitting a message takes a whole round. Two transmissions overlap in time precisely when they are performed in the same round.

Spontaneous activations and waking up the network. Initially, all stations are *passive*, in that they do not execute any communication algorithm, and in particular do not transmit any messages on any channel. Passive stations listen to all channels all the time, in that when a message is heard on a channel then all passive stations hear it. At a point in time, some stations become *activated* spontaneously and afterwards they are *active*. Passive stations may keep getting activated spontaneously after the round of the first activations. A specific scenario of timings of certain stations being activated is called an *activation pattern*.

An activated station resets its private clock to zero at the round of activation. When a station becomes active, it starts from the first round of its private clock to execute an algorithm with the goal to wake up the whole network. This goal of *waking up* the network is accomplished in the first round when some active station transmits on some channel as the only station transmitting in this round on this particular channel. This moment is understood as all passive stations receiving a signal to wake up and proceed with executing a predetermined communication algorithm. The moment of wake-up can be used to synchronize local clocks so that they begin to reflect the coordinated time.

Performance of a wake-up algorithm is measured as the number of rounds measured from the first spontaneous activation to the round of the first message heard on the network. We use an additional parameter k, which is a natural number such that $1 \le k \le n$ and denotes an upper bound on the number of stations that may get activated spontaneously in an execution. Performance bounds of wake-up algorithms employ the following three variables: n, b, and k. A parameter of a system or executions is *known* when it can be used in codes of algorithms. The numbers n and b are assumed to be known while the parameter k is unknown.

Definitions. Next we summarize definitions used throughout the paper.

Definition 1 (Global time.) *The term* time step *refers to the time as measured by an external observer. We call this time* global. *The first round of spontaneous activation of some station becomes the first time step of this global time. The time step in which a station u becomes activated spontaneously is denoted by σ_u. The set of stations that are active by time step t is denoted by $W(t)$.*

We consider oblivious algorithms that have schedules of transmission precomputed for each station. Each such a schedule is represented as a sequence of 0s and 1s. The schedules are organized as rows of a binary matrix for the sake of visualization and discussion.

Definition 2 (Transmission arrays.) *Let ℓ be positive integer treated as a parameter. An array $\mathcal{T}$ of entries of the form $T(u,\beta,j)$, where $u \in V$ is a station, β such that $1 \le \beta \le b$ is a channel, and integer j is such that $0 \le j \le \ell$, is a* transmission array *when each entry is either a 0 or a 1. The parameter $\ell = \ell(\mathcal{T})$ is called the* length *of array $\mathcal{T}$. Entries of a transmission array $\mathcal{T}$ are called* transmission bits *of $\mathcal{T}$. The number j is the* position *of a transmission bit $T(u,\beta,j)$.*

Every station $u \in V$ is provided with a copy of all entries $T(u,*,*)$ of some transmission array $\mathcal{T}$ as a way to instantiate the code of a wake-up algorithm.

Definition 3 (Schedules.) *For a transmission array $\mathcal{T}$, a station u and channel β, the sequence of entries $T(u,\beta,j)$, for $j = 1,\ldots,\ell$, is called a (u,β)-*schedule *and is denoted $\mathcal{T}(u,\beta,*)$. A (u,β)-schedule $\mathcal{T}(u,\beta,*)$ defines the following* schedule of transmissions *for station u: it transmits on channel β in the jth round exactly when $T(u,\beta,j) = 1$.*

When a station u became active then it executes the following oblivious algorithm:

Algorithm Wake-Up$(u,\mathcal{T})$:

Execute the schedule of transmissions determined by $\mathcal{T}(u,\beta,*)$ on each channel β.

Observe that if a station u is active in time step t then u perceives this time step t as round $t-\sigma_u$. The concept of a transmission that wakes up the network is defined as follows.

Definition 4 (Isolation.) *A station v is β-*isolated *at time step t when $v \in W(t)$ and when both $T(v,\beta,t-\sigma_v) = 1$ and $T(u,\beta,t-\sigma_u) = 0$, for every $u \in W(t) \setminus \{v\}$. A station v is* isolated at time step t *when v is β-isolated at time step t for some channel $1 \le \beta \le b$.*

For a given transmission array, by an *isolated position* we understand a pair (t,β) of time step t and channel β such that there is a β-isolated station at time step t. Note that an isolated position (t,β) means that successful wake-up has occurred by time t.

We impose structure on a transmission array by partitioning it into sections of increasing length. We use the fact that a mapping $i \mapsto 2^i \cdot i^{1/b}$ is strictly increasing, which can be verified directly.

Definition 5 (Stages.) *Let c be a positive integer and let $\varphi(0) = 0$ and $\varphi(i) = c2^i \cdot i^{1/b} \lg n$, for positive integers i. The ith* section *of a (u,β)-schedule $T(u,\beta,*)$, for $1 \le i \le \lg n$, consists of all subsequences $T(u,\beta,\varphi(i)), T(u,\beta,\varphi(i)+1),\ldots,T(u,\beta,\varphi(i+1)-1)$ of consecutive transmission bits. A station executing the ith section of its schedules is said to be* in stage i. *The stations that are in stage i at a time step j are denoted by $W_i(j)$.*

The constant c in Definition 3 is be determined later as needed. The identity $\bigcup_{i=1}^{\lg n} W_i(j) = W(j)$ holds for every time step j, because an active station is in some stage. The length of the ith section for any (u,β)-schedule is $\varphi(i+1) - \varphi(i)$, which is at least as large as $\varphi(i)$.

Those time steps at which sufficiently many stations are in a stage, say, ω, and no station is involved in a stage with index larger than ω play a special role in the analysis:

Definition 6 (Balanced time steps.) *For a stage ω, where $1 \le \omega \le \log k$, a time step j is ω-*balanced *when the following hold:* (a) $2^\omega \le |W_\omega(j)| \le 2^{\omega+2}$ *and* (b) $|W_i(j)| = 0$, *for all stages i such that $i > \omega$.*

When we refer to time intervals then this means intervals of time steps of the global time. We identify time intervals with sufficiently large sequences of consecutive time steps that contain only balanced time steps:

Definition 7 (Balanced time intervals.) *Let ω be a stage, where $1 \le \omega \le \log k$. A time interval $[t_1, t_2]$ of size $\varphi(\omega - 1)$, is said to be ω-*balanced*, if every time step $j \in [t_1, t_2]$ is ω-balanced. An interval is called* balanced *when there exists a stage ω, for $1 \le \omega \le \log k$, such that it is ω-balanced.*

For a time step j, we define $\Psi(j)$ as follows:

$$\Psi(j) = \sum_{\omega=1}^{\log k} \frac{|W_\omega(j)|}{2^i}.$$

We specialize balanced time intervals even more by considering their useful properties:

Definition 8 (Light time intervals.) *Let ω be a stage, where $1 \le \omega \le \log k$. An ω-balanced time interval $[t_1, t_2]$ is called ω-*light *when (1) the inequality $\left|\bigcup_{i=1}^{\omega} W_i(j)\right| \le 2^{\omega+4}$ holds for every time step $j \in [t_1, t_2]$, and (2) interval $[t_1, t_2]$ contains at least $\varphi(\omega - 2)$ time steps j such that*

$$1 \le \Psi(j) \le 128 \cdot \omega \ . \tag{1}$$

An interval is called light *when there exists a stage ω, for $1 \le \omega \le \log n$, such that it is ω-light.*

We will use transmission arrays in which entries are independent random variables.

Definition 9 (Regular randomized transmission arrays.) *A* randomized transmission array $\mathcal{T}$ *has the structure of a transmission array. Transmission bits $T(u, \beta, j)$ are not fixed but instead are independent Bernoulli random variables. Let u be a station and β denote a channel. For $1 \le i \le \lg n$, the entries of the ith section of the (u, β)-schedule are stipulated to have the following probability distribution, for $j = \varphi(i), \dots, \varphi(i + 1) - 1$:*

$$\Pr(T(u, \beta, j) = 1) = 2^{-i} \cdot i^{-\beta/b} \ .$$

We say that the number of channels b is *n-large*, or simply *large*, or they there are *n-many channels*, when the inequality $b > \log(128b \log n)$ holds. We set $\varphi(i) = c \cdot (2^i/b) \lg n \log(128b \log n)$ for such b, where c is a sufficiently large constant to be specified later. Recall the notation $\Psi(j) = \sum_{i=1}^{\log k} \frac{|W_i(j)|}{2^i}$that, for a time step j. For n-many channels, we use a modified version of a light time interval (cf., Definition 8), where condition (2) is replaced by the following one:

$$1 \le \Psi(j) \le 128 \cdot \log n. \tag{2}$$

For a channel β, we use the notation $\beta^* = \beta \bmod \log(128b \log n)$.

Definition 10 (Modified randomized transmission arrays.) *A* modified randomized transmission array $\mathcal{T}$ *has the structure of a transmission array. Transmission bits $T(u, \beta, j)$ are not fixed but instead are independent Bernoulli random variables. Let u be a station and β denote a channel. For $1 \le i \le \lg n$, the entries of the ith section of the (u, β)-schedule are stipulated to have the following probability distribution, for $j = \varphi(i), \dots, \varphi(i + 1) - 1$:*

$$\Pr(T(u, \beta, j) = 1) = b \cdot 2^{-i-\beta^*} \ .$$

A randomized transmission array, whether regular or modified, is used to represent a randomized wake-up algorithm. To decide if a station u transmits on channel β in the jth round, this station first carries out a Bernoulli trial with the probability of success as stipulated in the definition of the respective randomized array, and transmits when the experiment results in success. Regular arrays are used in the general case and modified arrays when there are n-many channels.

Definition 11 (Waking arrays.) *A transmission array $\mathcal{T}$ is said to be* waking *when for every k such that $1 \le k \le n$ and a light interval $[t_1, t_2]$ such that $|W(t)| \le k$, whenever $t_1 \le t \le t_2$, there exist both a time step $j \in [t_1, t_2]$ and a station $w \in W(j)$ such that w is isolated at time step j.*

The length of a waking array is the worst-case time bound on performance of the wake-up algorithm determined by this transmission array.

We denote by $\mathcal{F}_k^n$ the family of sets with exactly k elements out of n possible elements, interpreted as k-sets of stations taken from all n stations. For $\lambda \le \kappa \le n$, a family $\mathcal{F} \subseteq \mathcal{F}_\kappa^n$ is said to be (n, κ, λ)-*intersection free* if $|F_1 \cap F_2| \ne \lambda$ for every F_1 and F_2 in $\mathcal{F}_\kappa^n$. The following fact is an upper bound on the size of intersection-free families.

Fact 1 ([31]) *For any (n, κ, λ)-intersection free family $\mathcal{F}$ the following inequality holds true:*

$$|\mathcal{F}| \le \binom{n}{\lambda} \cdot \frac{\binom{2\kappa-\lambda-1}{\kappa}}{\binom{2\kappa-\lambda-1}{\lambda}} \ ,$$

assuming that $2\lambda + 1 \ge \kappa$ and $\kappa - \lambda$ is a prime power. □

3 A lower bound for deterministic algorithms

We prove a lower bound on time performance of any deterministic wake-up algorithm. We assume that all stations start simultaneously and have access to a global clock. This means that the lower bound is valid in a much stronger setting than the one for which we design efficient algorithms.

We define a *query* to be a set of ordered pairs (x, β) for $x \in V$ and $1 \le \beta \le b$. An interpretation of a pair $(x, \beta) \in Q$, for a query Q, is that station x is to transmit on channel β at the time step assigned for the query. In this section, an algorithm $\mathcal{A}$ is represented as a sequence of queries $\mathcal{A} = \{Q_1, \ldots, Q_t\}$. The index i of a query Q_i in such a sequence $\mathcal{A}$ is interpreted as the time step assigned for the query. We use the notation $Q_{i,\beta} = \{x \in V : (x, \beta) \in Q_i\}$, for a query Q_i. This represents the subset of all stations that at time step i transmit on channel β.

We use the Iverson's bracket $[\mathcal{P}]$, where $\mathcal{P}$ is a statement that is either true or false, defined as follows: $[\mathcal{P}] = 1$ if $\mathcal{P}$ is true and $[\mathcal{P}] = 0$ if $\mathcal{P}$ is false. We use the notation $\lg x$ for $\log_2 x$.

Lemma 1 *Let $\mathcal{A} = \{Q_1, Q_2, \ldots, Q_t\}$ be a sequence of queries representing an algorithm. There exists a sub-family $\mathcal{S} \subseteq \mathcal{F}_k^n$ with at least $|\mathcal{F}_k^n|/2^{bt}$ elements such that any two sets $A, B \in \mathcal{S}$ satisfy $[A \cap Q_{i,\beta} \ne \emptyset] = [B \cap Q_{i,\beta} \ne \emptyset]$ for all i and β such that $1 \le \beta \le b$ and $1 \le i \le t$.*

Proof: The proof is by induction on t. The base of induction relies on the identity $\mathcal{S}(0) = \mathcal{F}_k^n$. For the inductive step, assume that the claim holds for i such that $0 \le i < t$. Let $\mathcal{S}(i+1)$ be a largest sub-family of $\mathcal{S}(i)$ with the property that for all sets A and B in $\mathcal{S}(i+1)$, the following equality holds for every $1 \le \beta \le b$:

$$[A \cap Q_{i+1,\beta} \ne \emptyset] = [B \cap Q_{i+1,\beta} \ne \emptyset] \ .$$

The inequity $|\mathcal{S}(i+1)| \ge |\mathcal{S}(i)|/2^b$ holds by the pigeonhole principle. □

Lemma 2 *Let $\mathcal{A} = \{Q_1, Q_2, \ldots, Q_t\}$ be an algorithm, where $t \leq \frac{k}{2b} \lg \frac{n}{k} - \frac{k+1}{b}$. There exist two sets $A, B \subseteq \mathcal{F}_k^n$ such that the following are satisfied:*

(a) $|A \cap B| = k/2$,

(b) $[A \cap Q_{i,\beta} \neq \emptyset] = [B \cap Q_{i,\beta} \neq \emptyset]$, *for every* $1 \leq \beta \leq b$ *and* $1 \leq i \leq t$.

Proof: By Lemma 1, there exists a sub-family $\mathcal{S} \subseteq \mathcal{F}_k^n$ of at least

$$|\mathcal{S}| \geq |\mathcal{F}_k^n|/2^{bt} = \binom{n}{k}/2^{bt} \tag{3}$$

elements in $\mathcal{F}_k^n$ such that $[A \cap Q_{i,\beta} \neq \emptyset] = [B \cap Q_{i,\beta} \neq \emptyset]$, for every $A, B \in \mathcal{S}$, $1 \leq \beta \leq b$ and $1 \leq i \leq t$. Therefore, any two sets A and B in $\mathcal{S} \subseteq \mathcal{F}_k^n$, satisfy condition (b).

It remains to show that there are at least two sets in $\mathcal{S}$ satisfying also condition (a), that is, intersecting in a set of $k/2$ elements. We use Fact 1 for $\kappa = k$ and $\lambda = k/2$ to obtain that any sub-family of $\mathcal{F}_k^n$ containing sets that have pairwise intersections of size different from $k/2$ has at most these many elements:

$$\binom{n}{k/2} \cdot \binom{(3/2)k - 1}{k} \Big/ \binom{(3/2)k - 1}{k/2} = \binom{n}{k/2} \cdot \frac{1}{2} \; .$$

It follows that it is sufficient to show that the following inequality holds:

$$|\mathcal{S}| > \binom{n}{k/2} \cdot \frac{1}{2} \; .$$

We show it, starting from (3), in the following manner:

$$\begin{aligned}
|\mathcal{S}| &\geq \binom{n}{k}/2^{bt} \\
&\geq 2^{k \lg(n/k) - bt} \\
&\geq 2^{k \lg(n/k) - (k/2)\lg(n/k) + k + 1} \quad \text{(substituting the assumed bound for } t\text{)} \\
&= 2^{(k/2)\lg(n/k) + k + 1} \\
&\geq 2^{(k/2)\lg(2ne/k)} - 1 \\
&= \left(\frac{2ne}{k}\right)^{k/2} \cdot \frac{1}{2} \\
&> \binom{n}{k/2} \cdot \frac{1}{2} \; ,
\end{aligned}$$

where in the last step in the derivation we used the inequality $\binom{n}{k} < \left(\frac{ne}{k}\right)^k$. Therefore, there exist two sets in $\mathcal{S}$ with an intersection with $k/2$ elements, which completes the proof of (a). $\square$

Now we proceed to prove the lower bound which is formulates as follows.

Theorem 1 *Any deterministic algorithm that solves the wake-up problem on a multi-channel network with b channels requires $\Omega(\frac{k}{b} \log \frac{n}{k})$ time steps.*

Proof: We show that for any family $\mathcal{A} = \{Q_1, Q_2, \ldots, Q_t\}$ of queries, where t satisfies the following inequality:

$$t \leq \frac{k}{2b} \lg \frac{n}{k} - \frac{k+1}{b} \; , \tag{4}$$

there exists a k-set X such that $X \cap Q_i = \emptyset$, for all $i = 1, 2, \ldots, t$. To this end, let $\mathcal{A} = \{Q_1, Q_2, \ldots, Q_t\}$ be an algorithm such that (4) holds. Let A and B be two sets $A, B \subseteq \mathcal{F}_k^n$, with the properties as stated in Lemma 2. Set $A' = A \setminus B$ and $B' = B \setminus A$. Observe that if A and B have properties (a) and (b) of Lemma 2 then the following holds for A' and B':

(a*) $|A'| = |B'| = k/2$,

(b*) $A' \cap B' = \emptyset$,

(c*) $[A' \cap Q_{i,\beta} \neq \emptyset] = [B' \cap Q_{i,\beta} \neq \emptyset]$, for every $1 \leq \beta \leq b$ and $1 \leq i \leq t$.

We set $X = A' \cup B'$ to obtain that (a*) and (b*) imply $|X| = k$. Moreover, from (c*) it follows that $X \cap Q_{i,\beta} = \emptyset$, for all $1 \leq \beta \leq b$ and $1 \leq i \leq t$. This implies that $X \cap Q_i = \emptyset$, for all $i = 1, 2, \ldots, t$. Consider an execution in which the stations in X are simultaneously activated spontaneously as the only stations activated spontaneously. Then during the first t time steps after activations, no station in X transmits on any channel. We conclude that if an algorithm $\mathcal{A}$ always wakes up the network then (4) cannot be the case. □

4 A general deterministic algorithm

The purpose of this section is to show the following fact:

Theorem 2 *There exists a deterministic waking array of length $\mathcal{O}(n \log n \log^{1/b} k)$ guaranteeing wake-up in $\mathcal{O}(k \log n \log^{1/b} k)$ time for any number $k \leq n$ of activated stations.*

This fact is proved by the probabilistic method. The proof proceeds through a sequence of Lemmas. Let X be the set of stations that are activated first. Let σ_X be the time step at which they wake up. There is no active station woken up before time σ_X.

Let $\gamma_0 = 0$ and for $i = 1, 2, \ldots, \lg n$, define γ_i as the sum of the lengths of the first i sections. We have $\gamma_i = \sum_{h=1}^{i} (\varphi(h+1) - \varphi(h)) = \varphi(i+1)$.

Lemma 3 *For $i = 1, 2, \ldots, \lg n$, all stations in X are in section i of their transmission schedules between time $\sigma_X + \gamma_{i-1}$ and time $\sigma_X + \gamma_i - 1$.*

Proof: Any station $x \in X$, woken up at time σ_X, for $1 \leq i \leq \lg n$, reaches section i at time $\sigma_X + \gamma_{i-1}$ and continues to transmit according to transmission bits in section i until time $\sigma_X + \gamma_i - 1$. □

Lemma 4 *Fix a time step j' and an integer ω, with $1 \leq \omega \leq \log n$. For any integer $h \geq 1$, there exists a time step $j'' \geq j'$ such that the following holds for $j = j'', \ldots, j'' + \varphi(\omega + h)$:*

$$\bigcup_{i=1}^{\omega} W_i(j') \subseteq W_{\omega+h+1}(j) \ .$$

Proof: Let us fix $h \geq 1$. Recall that the sum of the lengths of the first i sections is $\gamma_i = \varphi(i+1)$. Any station $x \in W_1(j')$ is in section $\omega + h + 1$ by time $j' + \varphi(\omega + h + 1)$. Analogously, a station $y \in W_\omega(j')$ cannot leave section $\omega + h + 1$ before time step $j' + \varphi(\omega + h + 2) - \varphi(\omega + 1)$. Therefore, all stations in $W_i(j')$, for $1 \leq i \leq \omega$, are in section $\omega + h + 1$ between time step $j'' = j' + \varphi(\omega + h + 1)$

and time $\tau = j' + \varphi(\omega + h + 2) - \varphi(\omega + 1)$. It remains to count the number of time steps between time j'' and τ. We have therefore that

$$\begin{aligned} \tau - j'' &= \varphi(\omega + h + 2) - \varphi(\omega + h + 1) - \varphi(\omega + 1) \\ &\geq \varphi(\omega + h + 1) - \varphi(\omega + 1) \\ &\geq \varphi(\omega + h), \end{aligned}$$

for every $h \geq 1$. $\square$

Lemma 5 *Fix a time step j' and an integer ω, with $1 \leq \omega \leq \log n$. Assume the following two inequalities:*

$$\Big|\bigcup_{i=1}^{\omega-1} W_i(j')\Big| \geq 3 \cdot |W_\omega(j')| \quad \text{and} \quad |W_\omega(j')| \geq 2^\omega \ .$$

Then there exists an interval $[t_1, t_2]$ of size $\varphi(\omega + 1)$ with $t_1 \geq j'$ such that $|W_{\omega+2}(j)| \geq 2^{\omega+2}$.

Proof: We have the following estimate:

$$\begin{aligned} \Big|\bigcup_{i=1}^{\omega} W_i(j')\Big| &= \Big|\bigcup_{i=1}^{\omega-1} W_i(j')\Big| + |W_\omega(j')| \\ &\geq 3 \cdot |W_\omega(j')| + |W_\omega(j')| \\ &= 2^{\omega+2}. \end{aligned} \tag{5}$$

By Lemma 4, there exists a round $j'' \geq j'$ such that for $j = j'', \ldots, j'' + \varphi(\omega + 1)$,

$$\bigcup_{i=1}^{\omega} W_i(j') \subseteq W_{\omega+2}(j).$$

Therefore, for $j = j'', \ldots, j'' + \varphi(\omega + 1)$,

$$\Big|W_{\omega+2}(j)\Big| \geq \Big|\bigcup_{i=1}^{\omega} W_i(j')\Big| \geq 2^{\omega+2}.$$

The last step follows from (5). We conclude the proof by setting $t_1 = j''$ and $t_2 = j'' + \varphi(\omega+1)$. $\square$

Lemma 6 *Let $[t_1, t_2]$ be an interval of size $\varphi(\omega - 1)$, for some $1 \leq \omega \leq \log k$, such that for every round $j \in [t_1, t_2]$, the following conditions hold:*

(a') $|W_\omega(j)| \geq 2^\omega$;

(b') *for $i > \omega$, $|W_i(j)| = 0$.*

There exists an ω'-balanced interval for some $\omega \leq \omega' \leq \log k$.

Proof: If $|W_\omega(j)| \leq 2^{\omega+2}$ for every $j \in [t_1, t_2]$, condition (a) of Definition 6 holds and there is nothing to prove. Therefore, assume that there exists $j' \in [t_1, t_2]$ such that $|W_\omega(j')| > 2^{\omega+2}$. Observe that since at most k stations can be activated, we must have $\omega < \log k - 2$. Let $h \geq 1$ be an integer such that $2^{\omega+h+1} < |W_\omega(j')| \le 2^{\omega+h+3}$. By Lemma 4, there exists a round $j'' \ge j'$ such that, for $j = j'', \ldots, j'' + \varphi(\omega + h)$, the following holds:

$$\bigcup_{i=1}^{\omega} W_i(j') \subseteq W_{\omega+h+1}(j).$$

Therefore $|W_{\omega+h+1}(j)| \ge 2^{\omega+h+1}$ for $j = j'', \ldots, j'' + \varphi(\omega + h)$. Let $t'_1 = j''$, $t'_2 = j'' + \varphi(\omega + h)$ and $\omega' = \omega + h + 1$. We have found an interval $[t'_1, t'_2]$ of size $\varphi(\omega' - 1)$ such that for every round $j \in [t'_1, t'_2]$, the following conditions hold:

1. $|W_{\omega'}(j)| \ge 2^{\omega'}$;
2. for $i > \omega'$, $|W_i(j)| = 0$.

If $|W_{\omega'}(j)| \le 2^{\omega'+2}$ for every $j \in [t'_1, t'_2]$ then interval $[t'_1, t'_2]$ is ω'-balanced and we are done; otherwise we repeat the same reasoning to find a new interval. Since the number of stations that can be woken up is bounded by k, there must exist an interval $[\tau_1, \tau_2]$ of size $\varphi(\iota - 1)$, for some $1 \le \iota \le \log k$, such that $|W_\iota(j)| \le 2^{\iota+2}$ for every $j \in [\tau_1, \tau_2]$. □

Lemma 7 *There exists an ω-balanced interval $[t_1, t_2]$, for some $1 \le \omega \le \log k$, such that*

$$\left|\bigcup_{i=1}^{\omega-1} W_i(j)\right| < 3 \cdot |W_\omega(j)| \tag{6}$$

for every $j \in [t_1, t_2]$.

Proof: Pick $1 \le \iota \le \log k$ such that $2^\iota \le |X| \le 2^{\iota+1}$. By Lemma 3, all stations in X are in section ι, for $j \in [\sigma_X + \gamma_{\iota-1}, \sigma_X + \gamma_\iota - 1]$. Therefore the inequality $|W_\iota(j)| \ge |X| \ge 2^\iota$ holds for every $j \in [\sigma_X + \gamma_{\iota-1}, \sigma_X + \gamma_\iota - 1]$. Since there is no active station woken up before t_X we also have that $|W_i(j)| = 0$ for $i > \iota$ and every $j \in [\sigma_X + \gamma_{\iota-1}, \sigma_X + \gamma_\iota - 1]$. By Lemma 6, there is a ι^*-balanced interval $[\tau_1, \tau_2]$ for some $\iota^* \ge \iota$.

Let us assume that (6) does not hold, otherwise we are done. Let $j' \in [\tau_1, \tau_2]$ be such that $\left|\bigcup_{i=1}^{\iota^*-1} W_i(j')\right| \ge 3 \cdot |W_{\iota^*}(j')|$. By Lemma 5, there exists an interval $[t_1, t_2]$ of size $\varphi(\iota^* + 1)$ with $t_1 \ge j'$ such that $|W_{\iota^*+2}(j)| \ge 2^{\iota^*+2}$. Letting $\omega = \iota^* + 2$, we have found an interval of size $\varphi(\omega - 1)$ such that $|W_\omega(j)| \ge 2^\omega$. By Lemma 6, there is an ω'-balanced interval for some $\omega' \ge \omega$. This process can be iterated until a balanced interval that satisfies condition (6) is identified. □

Lemma 8 *There exists an ω-light interval $[t_1, t_2]$, for some $1 \le \omega \le \log k$.*

Proof: Let $[t_1, t_2]$ be an ω-balanced interval for some $1 \le \omega \le \log k$, whose existence is guaranteed by Lemma 7. We can assume that every $j \in [t_1, t_2]$ satisfies condition (6), by this very Lemma. Moreover, since the interval is ω-balanced, we also have that $|W_\omega(j)| \le 2^{\omega+2}$ for every $j \in [t_1, t_2]$, by condition (a) of Definition 6. This yields

$$\begin{aligned} \left|\bigcup_{i=1}^{\omega} W_i(j)\right| &= \left|\bigcup_{i=1}^{\omega-1} W_i(j)\right| + |W_\omega(j)| \\ &< 3|W_\omega(j)| + |W_\omega(j)| \\ &\le 4 \cdot 2^{\omega+2} = 2^{\omega+4}, \end{aligned} \tag{7}$$

for every $j \in [t_1, t_2]$. Thus condition (1) of Definition 8 is proved.

Next, we demonstrate condition (2). By condition (a) of Definition 6 we have that $|W_\omega(j)| \geq 2^\omega$ for every $j \in [t_1, t_2]$. Therefore the following inequalities hold for every $j \in [t_1, t_2]$:

$$\Psi(j) \geq \frac{|W_\omega(j)|}{2^\omega} \geq 1 \ .$$

It remains to prove that the upper bound of (1) holds for at least $\varphi(\omega - 2)$ time steps. Suppose, with the goal to arrive at a contradiction, that the number of time steps j in $[t_1, t_2]$ that satisfies the rightmost inequality of condition (1), is less than $\varphi(\omega - 2)$. Let $B \subseteq [t_1, t_2]$ be the set of balanced time steps $j \in [t_1, t_2]$ such that condition (1) is not satisfied. By the assumption, the following holds:

$$|B| > |[t_1, t_2]| - \varphi(\omega - 2) \geq \frac{\varphi(\omega - 2)}{2}. \tag{8}$$

For any $j \in [t_1, t_2]$, let

$$U(j) = \bigcup_{i=1}^{\lg n} W_i(j) = \bigcup_{i=1}^{\omega} W_i(j) \ ,$$

where the last equality follows by condition (b) of Definition 6. We have that $|W_i(j)| = 0$ for $\omega < i \leq \lg n$ and for every $j \in [t_1, t_2]$ because $[t_1, t_2]$ is ω-balanced. Hence all stations in $W(j)$ lie on sections $i \leq \omega$ for every $j \in [t_1, t_2]$. By the specification of sections, a station is in section i, for $1 \leq i \leq \omega$, during $\varphi(i + 1) - \varphi(i) \geq \varphi(i)$ time steps. Therefore, for every $1 \leq i \leq \omega$

$$\varphi(i) \max_{t_1 \leq j \leq t_2} |U(j)| \geq \sum_{j=t_1}^{t_2} |W_i(j)| \geq \sum_{j \in B} |W_i(j)|.$$

We continue with the following estimates:

$$\begin{aligned}
\sum_{i=1}^{\omega} \max_{t_1 \leq j \leq t_2} |U(j)| &\geq \sum_{i=1}^{\omega} \sum_{j \in B} \frac{|W_i(j)|}{\varphi(i)} \\
&= \sum_{j \in B} \sum_{i=1}^{\log k} \frac{|W_i(j)|}{\varphi(i)} \\
&= \frac{1}{c \log n} \sum_{j \in B} \sum_{i=1}^{\omega} \frac{|W_i(j)|}{2^i \cdot i^{1/b}} \\
&\geq \frac{1}{c \log n \log^{1/b} k} \sum_{j \in B} \sum_{i=1}^{\omega} \frac{|W_i(j)|}{2^i} \\
&> \frac{1}{c \log n \log^{1/b} k} \sum_{j \in B} 128 \cdot \omega \quad \text{by the assumption} \\
&= \frac{|B| \cdot 128 \cdot \omega}{c \log n \log^{1/b} k}.
\end{aligned}$$

Therefore the following inequality holds:

$$\max_{t_1 \leq j \leq t_2} |U(j)| > \frac{|B| \cdot 128}{c \log n \log^{1/b} k}.$$

By (8) we obtain that

$$\max_{t_1 \le j \le t_2} |U(j)| > \frac{2^{\omega-3} c \log n \log^{1/b} k \cdot 128}{c \log n \log^{1/b} k} = 2^{\omega+4}.$$

This implies that there exists $j' \in [t_1, t_2]$ such that

$$\left| \bigcup_{i=1}^{\omega} W_i(j') \right| > 2^{\omega+4},$$

which contradicts (7). $\square$

Lemma 9 *Let β be a channel, for $1 \le \beta \le b$. Let every station be executing the randomized algorithm as represented by the regular randomized transmission array. Let $[t_1, t_2]$ be a light interval. The probability that there exists a station $w \in W(t)$ β-isolated at an arbitrary time step j such that $j \le t$ and $t_1 \le j \le t_2$, is at least*

$$\frac{\Psi(j)}{\log^{\beta/b} k} \cdot 4^{-\frac{\Psi(j)}{\log^{\beta/b} k}}.$$

Proof: Let $E_1(\beta, i, j)$ be the event "there exists $w \in W_i(j)$ such that $T(\beta, w, j) = 1$", and let $E_2(\beta, i, j)$ be the event "$T(u, \beta, j) = 0$ for all l with $l \ne i$ and for every $u \in W_l(j)$." Let us say that $W(t)$ is β-isolated at time step $j \le t$ if and only if there exists a station $w \in W(t)$ that is β-isolated at time step j. Clearly, $W(t)$ is β-isolated at time j if and only if the following event occurs:

$$\bigcup_{i=1}^{\log n} \left(E_1(\beta, i, j) \cap E_2(\beta, i, j) \right) .$$

We use the following estimate on probability:

$$\begin{aligned}
\Pr(E_1(\beta, i, j)) &\ge \frac{|W_i(j)|}{2^i i^{\beta/b}} \left(1 - \frac{1}{2^i i^{\beta/b}}\right)^{|W_i(j)|-1} \\
&\ge \frac{|W_i(j)|}{2^i i^{\beta/b}} \left(1 - \frac{1}{2^i i^{\beta/b}}\right)^{|W_i(j)|}
\end{aligned}$$

and the following identity:

$$\Pr(E_2(\beta, i, j)) = \prod_{l=1, l \ne i}^{\log n} \left(1 - \frac{1}{2^l l^{\beta/b}}\right)^{|W_l(j)|} .$$

Events $E_1(\beta, i, j)$ and $E_2(\beta, i, j)$ are independent, so it follows that

$$\begin{aligned}
\Pr(E_1(\beta, i, j) \cap E_2(\beta, i, j)) &\ge \frac{|W_i(j)|}{2^i i^{\beta/b}} \prod_{l=1}^{\log n} \left(1 - \frac{1}{2^l l^{\beta/b}}\right)^{|W_l(j)|} \\
&= \frac{|W_i(j)|}{2^i i^{\beta/b}} \prod_{l=1}^{\log n} \left(1 - \frac{1}{2^l l^{\beta/b}}\right)^{2^l l^{\beta/b} \frac{|W_l(j)|}{2^l l^{\beta/b}}} \\
&\ge \frac{|W_i(j)|}{2^i i^{\beta/b}} \cdot 4^{-\sum_{l=1}^{\log n} \frac{|W_l(j)|}{2^l l^{\beta/b}}} .
\end{aligned}$$

The events $E_1(\beta, i, j) \cap E_2(\beta, i, j)$ are mutually exclusive, for any fixed j and $1 \le i \le \lg n$. Additionally, $W_i(j) = \emptyset$ for all $i > \log k$, as $[t_1, t_2]$ is a light interval. Combining all this gives

$$\begin{aligned}
\Pr(E_1(\beta,i,j) \cap E_2(\beta,i,j)) &\ge \sum_{l=1}^{\log n} \frac{|W_l(j)|}{2^l l^{\beta/b}} \cdot 4^{-\sum_{l=1}^{\log n} \frac{|W_l(j)|}{2^l l^{\beta/b}}} \\
&= \sum_{l=1}^{\log k} \frac{|W_l(j)|}{2^l l^{\beta/b}} \cdot 4^{-\sum_{l=1}^{\log k} \frac{|W_l(j)|}{2^l l^{\beta/b}}} .
\end{aligned}$$

Observe that the function $x \cdot 4^{-x}$ is monotonically decreasing in x. We apply this for $x = \sum_{l=1}^{\log k} \frac{|W_l(j)|}{2^l l^{\beta/b}}$. Observe moreover the following inequality

$$\sum_{l=1}^{\log k} \frac{|W_l(j)|}{2^l l^{\beta/b}} < \frac{1}{\log^{\beta/b} k} \sum_{l=1}^{\log k} \frac{|W_l(j)|}{2^l} .$$

Combining these facts together justifies the following estimates

$$\begin{aligned}
\Pr(E_1(\beta,i,j) \cap E_2(\beta,i,j)) &\ge \frac{1}{\log^{\beta/b} k} \sum_{l=1}^{\log k} \frac{|W_l(j)|}{2^l} \cdot 4^{-\frac{1}{\log^{\beta/b} k} \sum_{l=1}^{\log k} \frac{|W_l(j)|}{2^l}} \\
&\ge \frac{\Psi(j)}{\log^{\beta/b} k} \cdot 4^{\frac{\Psi(j)}{\log^{\beta/b} k}} ,
\end{aligned}$$

which completes the proof. $\square$

Lemma 10 *Let every station be executing the randomized algorithm as represented by the regular randomized transmission array. There exists an ω-light interval $[t_1, t_2]$, for some $1 \le \omega \le \log k$, that contains at least $\varphi(\omega - 2)$ time steps $j \in [t_1, t_2]$ such that the probability that there exists a station $w \in W(j)$ isolated at time j is at least*

$$\frac{1}{4^{128}} \omega^{1/b} .$$

Proof: By Lemma 8, there exists an ω-light interval $[t_1, t_2]$, for some $1 \le \omega \le \log k$. There are at least $\varphi(\omega - 2)$ time steps $j \in [t_1, t_2]$ with $1 \le \Psi(j) \le 128 \cdot \omega$. Let T be the set of such time steps j. We define sets T_i as follows, for $1 \le i \le b$:

$$T_1 = \{j \in T |\ 1 \le \Psi(j) \le 128 \cdot \omega^{1/b}\}$$

and, for $q = 2, \ldots, b$,

$$T_q = \{j \in T |\ 128 \cdot \omega^{(q-1)/b} < \Psi(j) \le 128 \cdot \omega^{q/b}\} .$$

It suffices to show that for every time step $t \in T$ there exists a channel β, $1 \le \beta \le b$, such that the probability of β-isolating a station $w \in W(t)$ at time t is at least

$$\frac{1}{4^{128} \omega^{1/b}} .$$

Let us consider a time step $t \in T$, so that $t \in T_q$ for some $1 \le q \le b$. By Lemma 9, we have that if $t \in T_1$ then the probability that a station is 1-isolated at time step t is at least

$$\frac{\Psi(j)}{\omega^{\beta/b}} \cdot 4^{-\frac{\Psi(j)}{\omega^{\beta/b}}} > \frac{1}{\omega^{\beta/b}} \cdot 4^{-128 \frac{\omega^{\beta/b}}{\omega^{\beta/b}}} = \frac{1}{4^{128} \omega^{\beta/b}} .$$

If $t \in T_\beta$, for $2 \le \beta \le b$, then the probability that a station is β-isolated at time step t is at least

$$\frac{\Psi(j)}{\omega^{\beta/b}} \cdot 4^{-\frac{\Psi(j)}{\omega^{\beta/b}}} > \frac{128}{\omega^{\beta/b}} \cdot 4^{-\frac{\Psi(j)}{\omega^{\beta/b}}} = \frac{128}{4^{128}\omega^{\beta/b}} ,$$

which completes the proof. □

Lemma 11 *Let s be the time at which the first station wakes up and let $[t_1, t_2]$ be an ω-light interval, for some $\omega \le \log k$. Then $t_2 \le s + \varphi(\omega + 1)$.*

Proof: Interval $[t_1, t_2]$ is ω-balanced, by Definition 8. We have $W_j(t_2)$ is empty for every $j > \omega$, by Definitions 7 and 6(b). This means that no station, including those activated first in time s, are in section bigger than ω. Each station is activated after at most $\varphi(\omega + 1)$ time steps because the sum of the lengths of the first i sections is $\gamma_i = \varphi(i + 1)$. □

Lemma 12 *Let c in the definition of φ be bigger than some sufficiently large constant. There exists an waking array of length $2cn \log n \log^{1/b} k$ such that, for any transmission array, there is an integer $0 \le \omega \le \log k$ with the following properties:*

(1) *There are at least $c \cdot 2^{\omega-259} \log n$ isolated positions by time $c \cdot 2^{\omega+1} \log n \log^{1/b} k$.*

(2) *At least $c \cdot 2^{\omega-259} \log n$ isolated positions occur at time steps with at least 2^ω but no more than $2^{\omega+4}$ activated stations.*

Proof: Consider a regular randomized transmission array, as defined in Definition 9. Assume also a sufficiently large $c > 0$ in the definition of $\varphi(i) = c \cdot 2^i \log n \cdot i^{1/b}$, for any $1 \le i \le \log n$. Consider an activation pattern, with the first activation at point 0. By Lemma 10, there is $\omega \le \log k$ and an ω-light interval $[t_1, t_2]$ such that there are at least $\varphi(\omega-2)$ time steps $j \in [t_1, t_2]$ with the probability of existing a station $w \in W(j)$ isolated at time j being at least $1/(4^{128}\omega^{1/b})$. We choose the smallest such ω and associate the corresponding ω-light interval $[t_1, t_2]$ with the activation pattern. Note that we can partition all activation patterns into disjoint classes based on the intervals associated with them. The expected number of isolated positions in the ω-light interval $[t_1, t_2]$ is at least

$$\varphi(\omega - 2) \cdot \frac{1}{4^{128}\omega^{1/b}} \ge \varphi(\omega - 2) \cdot \frac{1}{4^{128}(\omega - 2)^{1/b}} \cdot \frac{(\omega - 2)^{1/b}}{\omega^{1/b}} \ge c \cdot 2^{\omega-258} \log n ,$$

where we take $\omega \ge 3$. By the Chernoff bound, the probability that the number of isolated positions is smaller than $c \cdot 2^{\omega-259} \log n$ is at most $\exp(-c \cdot 2^{\omega-261} \log n)$.

We want to apply the probabilistic method argument to the class of activation patterns associated with the ω-light time interval $[t_1, t_2]$. To this end, we need an estimate from above of the number of all such activation patterns. By Lemma 11, the end t_2 of this time interval is not bigger than

$$\varphi(\omega + 1) \le c \cdot 2^{\omega+1} \log n \log^{1/b} k .$$

There are no more than $2^{\omega+4}$ stations activated by time step t_2, because $[t_1, t_2]$ is ω-light. The number of different activation patterns in the class associated with the ω-light interval $[t_1, t_2]$ is at most $\binom{n}{2^{\omega+4}}(t_2)^{2^{\omega+4}}$. This quantity can be estimated from above as

$$\begin{aligned} \left(\frac{ne}{2^{\omega+4}}\right)^{2^{\omega+4}} \left(c \cdot 2^{\omega+1} \log n \log^{1/b} k\right)^{2^{\omega+4}} &= \exp\left(2^{\omega+4} \cdot \ln((ce/8) \cdot n \log n \log^{1/b} k)\right) \\ &\le \exp\left(3 \ln c \cdot 2^{\omega+4} \cdot \log n\right) . \end{aligned}$$

This bound is smaller than $\exp(c \cdot 2^{\omega-261} \log n - 4 \log(2cn \log n \log^{1/b} k))$ for a sufficiently large constant c. We combine the following two bounds:

- this upper bound $\exp(c \cdot 2^{\omega-261} \log n - 4\log(2cn \log n \log^{1/b} k))$ on the number all activation patterns in the class associated with the ω-light time interval $[t_1, t_2]$, with
- the upper bound $\exp(-c \cdot 2^{\omega-261} \log n)$ on the probability that for any fixed such activation pattern the number of isolated positions is smaller than $c \cdot 2^{\omega-259} \log n$.

We conclude that the probability of the event that there is an activation pattern associated with the ω-light time interval $[t_1, t_2]$ with less than $c \cdot 2^{\omega-259} \log n$ isolated positions, is smaller than

$$\exp(c \cdot 2^{\omega-261} \log n - 4\log(2cn \log n \log^{1/b} k)) \cdot \exp(-c \cdot 2^{\omega-261} \log n) = \exp(-4\log(2cn \log n \log^{1/b} k)) \ .$$

Finally, observe that there are at most $2cn \log n \log^{1/b} k$ candidates for time step t_1 and also for t_2, by Lemma 11 and the bound $\omega \le \log n$. Hence, applying the union bound to the above events over all such feasible intervals, we obtain that the probability of the event that there is $\omega \le \log n$ and an activation pattern associated with some ω-light time interval $[t_1, t_2]$ with less than $c \cdot 2^{\omega-259} \log n$ isolated positions is smaller than

$$\exp(-4\log(2cn \log n \log^{1/b} k)) \cdot (2cn \log n \log^{1/b} k)^2 < 1/n^2 \le 1 \ .$$

By the probabilistic-method argument, there is an instantiation of the random array, which is a deterministic array, for which the complementary event holds. Note that more than the fraction $1 - 1/n^2$ of random arrays defined in the beginning of the proof satisfy the complementary event. Hence, this array satisfies Claim (1) with respect to any activation pattern. Claim (2) follows by noticing that these occurrences of isolated positions take place in the corresponding ω-light interval. The interval, by definition, has no more than $2^{\omega+4}$ stations activated by its end, and at least 2^{ω} activated stations in the beginning. This is because ω-light interval is by definition an ω-balanced interval, according to Definitions 6, 7 and 8. $\square$

Proof of Theorem 2: There is an isolated position for every activation pattern by time $\mathcal{O}(k \log n \log^{1/b} k)$. This follows from point (1) of Lemma 12. To see this, notice that otherwise the ω-light interval, which is also ω-balanced, would have at least $2^{\omega} > k$ stations activated, by Definitions 6 and 7, contradicting the assumption of the theorem.

Channels with random jamming. Assume that at each time step and on every channel there is a jamming error with probability $0 \le p < 1$, independently over time steps and channels. The case $p = 0$ is covered by Theorem 2.

Theorem 3 *For a given error probability $0 < p < 1$, there exists a waking array of length $\mathcal{O}(\log^{-1}(1/p) n \log n \log^{1/b} k)$ providing wake-up in expected time $\mathcal{O}(\log^{-1}(1/p) k \log n \log^{1/b} k)$, for any number $k \le n$ of spontaneously activated stations.*

Proof: Let us set $c = c' \cdot \log^{-1}(1/p)$ for sufficiently large constant c', and consider any activation pattern. By Lemma 12, at least $c \cdot 2^{\omega-259} \log n$ isolated positions occur by time $c \cdot 2^{\omega+1} \log^{1+1/b} n$ and by that time no more than $2^{\omega+4}$ stations are activated. Each such an isolated position can be jammed independently with probability p. Therefore, the probability that all these positions are jammed, and thus no successful transmission occurs by time

$$c \cdot 2^{\omega+1} \log n \log^{1/b} k = \mathcal{O}(\log^{-1}(1/p) k \log n \log^{1/b} k) \ ,$$

is at least

$$p^{c \cdot 2^{\omega-259} \log n} = \exp\left(c' \cdot \log^{-1}(1/p) \cdot 2^{\omega-259} \log n \cdot \ln p\right) \ ,$$

This is smaller than $1/\text{poly}(n)$ for sufficiently large constant c'. Here we use the fact that $\frac{\ln p}{\log(1/p)}$ is a negative constant for $p \in (0,1)$. When estimating the time of a successful wake-up we relied on the fact that 2^ω, which is the lower bound on the number of activated stations by Lemma 12(2), must be smaller than k, by the assumption. $\square$

5 A deterministic algorithm for sufficiently many channels

The main result of this section is as follows:

Theorem 4 *There exists a waking array of $\mathcal{O}((n/b)\log n \log(b \log n))$ length, for $b > \log(128 b \log n)$, which guarantees a wake-up in time $\mathcal{O}((k/b)\log n \log(b\log n))$ for any number $k \le n$ of spontaneously activated stations.*

The proof of this fact is by way of showing the existence of a waking array, as defined in Definition 11, for a section length defined as $\varphi(i) = c\cdot(2^i/b)\lg n \log(128b\log n)$. Note that Lemmas 3 to 7 as well as Lemma 11 hold for the current setting of function φ, as their proofs do not refer to the value of this function. The following lemma is an analogous to Lemma 8, which was proved for $\varphi(i) = c\cdot 2^i \cdot i^{1/b}\log n$, while now we prove the same statement for $\varphi(i) = c\cdot(2^i/b)\lg n\log(128b\log n)$.

Lemma 13 *There exists an ω-light interval $[t_1, t_2]$, for some $1 \le \omega \le \log n$.*

Proof: Let $[t_1,t_2]$ be an ω-balanced interval, as guaranteed to exist by Lemma 7. By that very Lemma, we can assume that every $j \in [t_1,t_2]$ satisfies condition (6). Moreover, since the interval is ω-balanced, we also have that $|W_\omega(j)| \le 2^{\omega+2}$ for every $j \in [t_1,t_2]$, by condition (a) of Definition 6). We conclude with the following upper bound, for every $j \in [t_1,t_2]$:

$$\begin{aligned}\left|\bigcup_{i=1}^{\omega} W_i(j)\right| &= \left|\bigcup_{i=1}^{\omega-1} W_i(j)\right| + |W_\omega(j)| \\ &< 3|W_\omega(j)| + |W_\omega(j)| \\ &\le 4\cdot 2^{\omega+2} = 2^{\omega+4} \ . \end{aligned} \tag{9}$$

This proves condition (1) of Definition 8.

We proceed to prove condition (2) next. By condition (a) of Definition 6, we know that $|W_\omega(j)| \ge 2^\omega$ for every $j \in [t_1,t_2]$. Therefore, the following bounds hold for every $j \in [t_1,t_2]$:

$$\Psi(j) \ge \frac{|W_\omega(j)|}{2^\omega} \ge 1 \ .$$

What remains is to prove the upper bound of (2). Assume, to arrive at a contradiction, that the number of time steps j in $[t_1,t_2]$ that satisfies the rightmost inequality of condition (2), is less than $\varphi(\omega-2)$. Let $B \subseteq [t_1,t_2]$ be the set of balanced time steps $j \in [t_1,t_2]$ such that condition (2) is not satisfied. By the assumption, the following is the case:

$$|B| > |[t_1,t_2]| - \varphi(\omega-2) = \frac{\varphi(\omega-2)}{2}. \tag{10}$$

For any $j \in [t_1,t_2]$, let

$$U(j) = \bigcup_{i=1}^{\lg n} W_i(j) = \bigcup_{i=1}^{\omega} W_i(j) \ ,$$

where the last equality follows by condition (b) of Definition 6. By the specification of the array, any station belongs section i during $\varphi(i+1)-\varphi(i) \geq \varphi(i)$ time steps, for $1 \leq i \leq \lg n$. Therefore, for every $1 \leq i \leq \lg n$ we have the following bounds:

$$\varphi(i) \max_{t_1 \leq j \leq t_2} |U(j)| \geq \sum_{j=t_1}^{t_2} |W_i(j)| \geq \sum_{j \in B} |W_i(j)| \ .$$

This in turn allows to obtain the following bound:

$$\begin{aligned}
\sum_{i=1}^{\lg n} \max_{t_1 \leq j \leq t_2} |U(j)| &\geq \sum_{i=1}^{\lg n} \sum_{j \in B} \frac{|W_i(j)|}{\varphi(i)} \\
&= \sum_{j \in B} \sum_{i=1}^{\lg n} \frac{|W_i(j)|}{\varphi(i)} \\
&= \frac{1}{c(\lg n \log(128 b \log n))/b} \sum_{j \in B} \sum_{i=1}^{\lg n} \frac{|W_i(j)|}{2^i} \\
&> \frac{1}{c(\lg n \log(128 b \log n))/b} \sum_{j \in B} 128 \cdot \log n \\
&= \frac{128 b |B|}{c \log(128 b \log n)} \ .
\end{aligned}$$

Therefore we have the bound

$$\max_{t_1 \leq j \leq t_2} |U(j)| > \frac{128 b |B|}{c \lg n \log(128 b \log n)} \ .$$

By applying (10), we obtain

$$\max_{t_1 \leq j \leq t_2} |U(j)| > \frac{128 b \cdot c(2^{\omega-3}/b) \lg n \log(128 b \log n)}{c \lg n \log(128 b \log n)} = 2^{\omega+4} \ .$$

This implies that there exists $j' \in [t_1, t_2]$ such that

$$\left| \bigcup_{i=1}^{\omega} W_i(j') \right| > 2^{\omega+4} \ ,$$

which contradicts (9). □

Lemma 14 *Let β be a channel, for $1 \leq \beta \leq b$. Let every station be executing the randomized algorithm as represented by the modified randomized transmission array. The probability that there exists a station $w \in W(t)$ that is β-isolated at any time step $j \leq t$ is at least*

$$\Psi(j) \cdot b \cdot 2^{-\beta^*} \cdot 4^{-\Psi(j) \cdot b \cdot 2^{-\beta^*}} \ .$$

Proof: Let $E_1(\beta, i, j)$ be the event "there exists $w \in W_i(j)$ such that $T(\beta, w, j) = 1$", and let $E_2(\beta, i, j)$ be the event "$T(u, \beta, j) = 0$ for all l with $l \neq i$ and for every $u \in W_l(j)$." Let us say that

$W(t)$ is β-isolated at time step $j \le t$ if and only if there exists a station $w \in W(t)$ that is β-isolated at time step j. Clearly, $W(t)$ is β-isolated at time j if and only if the following event occurs:

$$\bigcup_{i=1}^{\log n} \left(E_1(\beta, i, j) \cap E_2(\beta, i, j) \right) .$$

We use the following inequality:

$$\begin{aligned} \Pr(E_1(\beta, i, j)) &\ge |W_i(j)| \cdot b \cdot 2^{-i-\beta^*} \left(1 - b \cdot 2^{-i-\beta^*}\right)^{|W_i(j)|-1} \\ &\ge |W_i(j)| \cdot b \cdot 2^{-i-\beta^*} \left(1 - b \cdot 2^{-i-\beta^*}\right)^{|W_i(j)|} \end{aligned}$$

and the following equality:

$$\Pr(E_2(\beta, i, j)) = \prod_{l=1, l \ne i}^{\log n} \left(1 - b \cdot 2^{-l-\beta^*}\right)^{|W_l(j)|} .$$

Events $E_1(\beta, i, j)$ and $E_2(\beta, i, j)$ are independent. It follows that

$$\begin{aligned} \Pr(E_1(\beta, i, j) \cap E_2(\beta, i, j)) &\ge |W_i(j)| \cdot b \cdot 2^{-i-\beta^*} \prod_{l=1}^{\log n} \left(1 - b \cdot 2^{-l-\beta^*}\right)^{|W_l(j)|} \\ &= |W_i(j)| \cdot b \cdot 2^{-i-\beta^*} \prod_{l=1}^{\log n} \left(1 - b \cdot 2^{-l-\beta^*}\right)^{(2^{l+\beta^*}/b) \cdot b |W_l(j)| 2^{-l-\beta^*}} \\ &\ge |W_i(j)| \cdot b \cdot 2^{-i-\beta^*} \cdot 4^{-\sum_{l=1}^{\log n} (b |W_l(j)| 2^{-l-\beta^*})} \\ &= \frac{|W_i(j)|}{2^i} \cdot (b \cdot 2^{-\beta^*}) \cdot 4^{-\Psi(j) \cdot (b \cdot 2^{-\beta^*})} . \end{aligned}$$

To conclude, observe that the events $E_1(\beta, i, j) \cap E_2(\beta, i, j)$ are mutually exclusive, for any j and $1 \le i \le \lg n$. $\square$

Lemma 15 *Let every station be executing the randomized algorithm as represented by the modified randomized transmission array. There exists an ω-light interval $[t_1, t_2]$, for some $1 \le \omega \le \log n$, that contains at least $\varphi(\omega - 2)$ time steps $j \in [t_1, t_2]$ such that in each of these steps the number of channels β, with the probability of a β-isolated station being at least $1/8$, is at least $\left\lfloor \frac{b}{\log(128 b \log n)} \right\rfloor$.*

Proof: By Lemma 13, there are at least $\varphi(\omega - 2)$ time steps $j \in [t_1, t_2]$ such that the inequalities $1 \le \Psi(j) \le 128 \cdot \lg n$ hold. Let T be the set of such time steps. Let us define a partition of T into sets $T_q = \{j \in T | \; 2^q < b \cdot \Psi(j) \le 2^{q+1}\}$, for $0 \le q < \log(128 b \log n)$. It suffices to show that for every time step $t \in T$ there exists $\left\lfloor \frac{b}{\log(128 b \log n)} \right\rfloor$ channels β, for $1 \le \beta \le b$, such that the probability of β-isolating a station $w \in W(t)$ at time t is at least $1/8$.

Let us take any time step $t \in T$, so that $t \in T_q$ for some $0 \le q < \log(128 b \log n)$. By Lemma 14, if $t \in T_q$ then for each β such that $\beta^* = q$, the probability that a station is β-isolated at time step t is at least

$$\Psi(j) \cdot b \cdot 2^{-\beta^*} \cdot 4^{-\Psi(j) \cdot b \cdot 2^{-\beta^*}} \ge \left(2^{q+1} \cdot 2^{-q}\right) \cdot 4^{-2^{q+1} \cdot 2^{-q}} = 1/8 ,$$

where we use the fact that the function $x \cdot 4^{-x}$ is monotonically decreasing in x. To conclude, notice that there are at least $\left\lfloor \frac{b}{\log(128 b \log n)} \right\rfloor$ channels β satisfying $\beta^* = q$, for any given $0 \le q < \log(128 b \log n)$. $\square$

Lemma 16 *Let c in the definition of φ be bigger than some sufficiently large constant. There exists a waking array of length $2c(n/b)\log n \log(128b\log n)$ such that for any activation pattern, there is an integer $0 \le \omega \le \log n$ with the following properties:*

(1) *There are at least $c \cdot 2^{\omega-6}\log n$ isolated positions by time step $c \cdot (2^{\omega+1}/b)\log n \log(128b\log n)$*

(2) *These positions occur at time step with at least 2^{ω} but no more than $2^{\omega+4}$ activated stations.*

Proof: Consider a modified randomized transmission array. Let us assume that $c > 0$ in the specification

$$\varphi(i) = c \cdot (2^i/b)\log n \log(128b\log n)$$

is sufficiently large, for any $1 \le i \le \log n$. Observe that the length of the schedules is not bigger than

$$\varphi(i+1) \le 2c \cdot (n/b)\log n \log(128b\log n) \ .$$

Consider an activation pattern with the first activation at time step 0. By Lemma 15, there is $\omega \le \log n$ and ω-light interval $[t_1, t_2]$ such that there are at least $\varphi(\omega - 2)$ time steps $j \in [t_1, t_2]$ such that each of them has at least $\left\lfloor \frac{b}{\log(128b\log n)} \right\rfloor$ channels β with the probability of β-isolation of a station $w \in W(j)$ being at least $1/8$. We choose the smallest such an ω and associate the corresponding ω-light interval $[t_1, t_2]$ with the activation pattern. We can clearly partition all activation patterns into disjoint classes based on the intervals associated with them.

Observe that the expected number of isolated positions in the ω-light interval $[t_1, t_2]$ is at least

$$\varphi(\omega-2) \cdot \left\lfloor \frac{b}{\log(128b\log n)} \right\rfloor \cdot \frac{1}{8} = c \cdot 2^{\omega-5}\log n \ .$$

By the Chernoff bound, the probability that the number of isolated positions is smaller than $c \cdot 2^{\omega-6}\log n$ is at most $\exp(-c \cdot 2^{\omega-8}\log n)$.

In order to apply a probabilistic argument to the class of activation patterns associated with the ω-light time interval $[t_1, t_2]$, it remains to estimate from above the number of all such activation patterns. By Lemma 11, the end-point t_2 of this time interval is not bigger than

$$\sum_{i=1}^{\omega} \varphi(i) \le c \cdot (2^{\omega+1}/b)\log n \log(128b\log n) \ .$$

Next observe that since $[t_1, t_2]$ is ω-light, there are no more than $2^{\omega+4}$ stations activated by time step t_2. Hence, the number of different activation patterns in the class associated with the ω-light interval $[t_1, t_2]$ is at most

$$\begin{aligned}
\binom{n}{2^{\omega+4}} (t_2)^{2^{\omega+4}} &\le \left(\frac{ne}{2^{\omega+4}}\right)^{2^{\omega+4}} \left(c \cdot (2^{\omega+1}/b)\log n \log(128b\log n)\right)^{2^{\omega+4}} \\
&= \exp\left(2^{\omega+4} \cdot \ln((ce/8)\cdot(n/b))\log n \log(128b\log n))\right) \\
&\le \exp\left(3\ln c \cdot 2^{\omega+4} \cdot \log n\right) \ ,
\end{aligned}$$

which is smaller than $\exp(c \cdot 2^{\omega-8}\log n - 4\log(2cn\log n))$ for sufficiently large constant c. Next we combine the following two bounds:

- the upper bound $\exp(c \cdot 2^{\omega-8}\log n - 4\log(2cn\log n))$ on the number all activation patterns in the class associated with the ω-light time interval $[t_1, t_2]$, with

- the upper bound $\exp(-c \cdot 2^{\omega-8} \log n)$ on the probability that for any fixed such an activation pattern the number of isolated positions is smaller than $c \cdot 2^{\omega-6} \log n$.

This allows to conclude that the probability of the event that there is an activation pattern associated with the ω-light time interval $[t_1, t_2]$ with less than $c \cdot 2^{\omega-6} \log n$ isolated positions is smaller than

$$\exp(c \cdot 2^{\omega-8} \log n - 4 \log(2cn \log n)) \cdot \exp(-c \cdot 2^{\omega-8} \log n) = \exp(-4 \log(2cn \log n)) \ .$$

There are at most $2cn \log n$ candidates for time step t_1 and also for t_2, by Lemma 11 applied to $\varphi(\omega) = c(2^\omega/b) \log n \log(128b \log n)$ and the bound $\omega \leq \log n$. We apply the union bound to these events over all such feasible intervals. This gives that the probability that there is $\omega \leq \log n$ and an activation pattern associated with some ω-light time interval $[t_1, t_2]$ with less than $c \cdot 2^{\omega-6} \log n$ isolated positions is smaller than

$$\exp(-4 \log(2cn \log n)) \cdot (2cn \log n)^2 < 1/n^2 \leq 1 \ .$$

Thus, by the probabilistic-method argument, there is an instantiation of the random array, which is a deterministic array, for which the complementary event holds. Hence, this array satisfies claim (1) of the lemma with respect to any activation pattern. Claim (2) follows when one observes that these occurrences of isolated positions take place in the corresponding ω-light interval, which by definition has no more than $2^{\omega+4}$ stations activated by its end, and at least 2^ω activated stations in its beginning. This is because ω-light interval is by definition an ω-balanced interval, according to Definitions 6, 7 and 8. $\square$

Proof of Theorem 4: There is an isolated position by $\mathcal{O}((k/b) \log n \log(128b \log n))$ time for every activation pattern. This follows from point (1) of Lemma 16. Indeed, otherwise the ω-light interval, which is also ω-balanced, would have at least $2^\omega > k$ stations activated, by Definitions 6 and 7, contrary to the assumptions.

Channels with random jamming. Assume that at each time step and on every channel there is a jamming error with probability $0 \leq p < 1$, independently over time steps and channels. The case $p = 0$ is subsumed by Theorem 2.

Theorem 5 *There exists a waking array of $\mathcal{O}(\log^{-1}(1/p)(n/b) \log n \log(b \log n))$ length, for a probability p such that $0 < p < 1$, which guarantees wake-up in $\mathcal{O}(\log^{-1}(1/p)(k/b) \log n \log(b \log n))$ time with the probability of at least $1 - 1/\mathrm{poly}(n)$.*

Proof: Let us set $c = c' \cdot \log^{-1}(1/p)$, for sufficiently large constant c', and consider any activation pattern. By Lemma 16, $c \cdot 2^{\omega-6} \log n$ isolated positions occur by time $c \cdot (2^{\omega+1}/b) \log n \log(128b \log n)$ and by that time no more than $2^{\omega+4}$ stations are activated. Each such isolated position can be jammed independently with probability p. Therefore, the probability that all these positions are jammed, and thus no successful transmission occurs by time $c \cdot (2^{\omega+1}/b) \log n \log(128b \log n) = \mathcal{O}(\log^{-1}(1/p)(k/b) \log n \log(b \log n))$, is at least

$$p^{c \cdot 2^{\omega-6} \log n} = \exp\left(c' \cdot \log^{-1}(1/p) \cdot (2^{\omega-6}/b) \log n \cdot \ln p\right) \ ,$$

which is smaller than $1/\mathrm{poly}(n)$ for sufficiently large constant c'. Here we use the fact that $\frac{\ln p}{\log(1/p)}$ is a negative constant for $p \in (0, 1)$. When bounding time of a successful wake-up to occur, we rely on the fact that 2^ω, which is the lower bound on the number of activated stations by Lemma 16(2), must be smaller than k by the assumption. $\square$

6 A randomized algorithm

In this Section, we present a randomized wake-up algorithm, which is complementary to deterministic algorithms considered so far. The code for a station u is as follows:

Algorithm CHANNEL-SCREENING(u):

For $\beta = 1, 2, \ldots, b$ transmit a message on channel β with probability $k^{-\beta/b}$.

Lemma 17 *Let t be a time step and let $1 \leq \beta \leq b$ be such that bounds $k^{(\beta-1)/b} \leq |W(t)| \leq k^{\beta/b}$ hold. Algorithm* CHANNEL-SCREENING *guarantees that the probability of hearing a message at time step t on channel β is at least $1/2ek^{1/b}$.*

Proof: Let $E(\beta, t)$ be the event of a successful transmission on channel β at time t. The probability that a station $w \in W(t)$ transmits at time t on channel β while all the others remain silent is

$$\begin{aligned} \Pr(E(\beta,t)) &\geq \frac{|W(t)|}{k^{\beta/b}}\left(1-\frac{1}{k^{\beta/b}}\right)^{|W(t)|-1} \\ &\geq \frac{k^{(\beta-1)/b}}{k^{\beta/b}}\left(1-\frac{1}{k^{\beta/b}}\right)^{k^{\beta/b}} , \end{aligned}$$

where the last inequality follows from the hypothesis that $k^{(\beta-1)/b} \leq |W(t)| \leq k^{\beta/b}$. Hence

$$\Pr(E(\beta,t)) \geq \frac{1}{2ek^{1/b}} ,$$

which completes the proof. □

An estimate the number of rounds needed to make the probability of failure smaller than a threshold ϵ is as follows:

Theorem 6 *Algorithm* CHANNEL-SCREENING *on b channels succeeds in waking up the network, with at most k active stations out of n in $\mathcal{O}(k^{1/b}\ln(1/\epsilon))$ time with the probability of at least $1-\epsilon$.*

Proof: Let us consider a set of contiguous time steps T. For $1 \leq \beta \leq b$, let

$$T_\beta = \{t \in T |\ k^{(\beta-1)/b} \leq |W(t)| \leq k^{\beta/b}\} .$$

Let $\bar{E}(t)$ be the event of an unsuccessful time step t, in which no station transmits as the only transmitted on any channel, and let $\bar{E}(\beta,t)$ be the event of an unsuccessful time step t on channel β, with $1 \leq \beta \leq b$. By Lemma 17, the probability of having a sequence of $\lambda = |T|$ unsuccessful time steps can be estimated as follows:

$$\begin{aligned} \Pr\Big(\bigcap_{t\in T}\bar{E}(t)\Big) &\leq \Pr\Big(\bigcap_{t\in T_1}\bar{E}(1,t)\Big)\cdot\Pr\Big(\bigcap_{t\in T_2}\bar{E}(2,t)\Big)\cdots\Pr\Big(\bigcap_{t\in T_b}\bar{E}(b,t)\Big) \\ &\leq \left(1-\frac{1}{2ek^{1/b}}\right)^{|T_1|}\cdot\left(1-\frac{1}{2ek^{1/b}}\right)^{|T_2|}\cdots\left(1-\frac{1}{2ek^{1/b}}\right)^{|T_b|} \\ &\leq \left(1-\frac{1}{2ek^{1/b}}\right)^{\lambda} \\ &\leq \epsilon, \end{aligned}$$

for $\lambda \geq 2ek^{1/b}\ln(1/\epsilon)$. □

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