Constructing transient birth-death processes by means of suitable transformations

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Abstract

For a birth-death process $N(t)$ with a reflecting state at 0 we propose a method able to construct a new birth-death process $M(t)$ defined on the same state-space. The birth and death rates of $M(t)$ depend on the rates of $N(t)$ and on the probability law of the process $N(t)$ evaluated at an exponentially distributed random time. Under a suitable assumption we obtain the conditional probabilities, the mean of the process, and the Laplace transforms of the downward first-passage-time densities of $M(t)$. We also discuss the connection between the proposed method and the notion of ν-similarity, as well as a relation between the distribution of $M(t)$ and the steady-state probabilities of $N(t)$ subject to catastrophes governed by a Poisson process. We investigate new processes constructed from (i) a birth-death process with constant rates, and (ii) a linear immigration-death process. Various numerical computations are performed to illustrate the obtained results.

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1. Introduction

Birth-death processes and related stochastic models are relevant in several fields, such as population dynamics, evolutionary genomics, ecology, queueing theory and inventory, among others (see, for instance, Ricciardi [1], Renshaw [2], Crawford and Suchard [3], Di Crescenzo *et al.* [4], Dharmaraja *et al.* [5]). Moreover, birth-death processes have recently been considered prominently in spatial evolutionary games for the analysis of cooperation and evolution in binary birth-death dynamics and for the expansion of cooperation by means of self-organized growth (see Szolnoki *et al.* [6]).

Many applications often demand for the determination of the probability laws of such processes, which is not an easy task. In fact various tecniques have been devoloped in the past decades aiming to obtain the transition probabilities of birth-death processes. Some methods are based on transforms, such as generating functions and Laplace transforms, spectral decompositions (see van Doorn [7], [8]) or continued fractions. Attention has been given also to the use of suitable transformations (see Lenin and Parthasarathy [9] and Lenin *et al.* [10]) and direct methods (see Parthasarathy [11]).

Along this line, in the present paper we propose a method able to determine closed-form transition probabilities of certain time-homogeneous birth-death processes with 0 reflecting state. This method is based on a transformation between two birth-death processes such that the birth and death rates of the new process depends on those of the former process and on the survival probability of a compound random variable, say Z. Specifically, Z describes the former birth-death process evaluated at an exponentially distributed random time T. We point out that the proposed method leads to transient birth-death processes, which are appropriate for describing population subject to rapid growth (such as unbounded bacterial growth). For these processes we determine various quantities of interest via computationally effective procedures, such as the conditional probabilities, the mean and Laplace transforms of some first-passage-time densities.

We point out that a further object of the paper is to illustrate a suitable connection between the obtained processes and the former processes subject to total catastrophes. We remark that some results on stochastic processes subject to catastrophes have been shown recently in Cairns and Pollett [12], van Doorn and Zeifman [13], Pollett *et al.* [14], Di Crescenzo *et al.* [15], Dimou and Economou [16], Zeifman *et al.* [17].

It is worth noting that the potential applicability of the proposed theory can be extended also to the physics of social systems, as reviewed in Castellano *et al.* [18], as well as to statistical mechanics of evolutionary and coevolutionary games, as reviewed recently in Perc and Grigolini [19].

We recall that the problem of determining the probability distribution of Markov chains is often unfeasible and thus one is forced to resort to suitable approximations, such as those based on convergences of truncated processes (see, e.g. Tweedie [20]). Moreover, the criteria grounded on truncated birth-death processes are often constrained by certain conditions, such as monotonicity or boundedness of transition rates (see Zeifman *et al.* [21], [22]). The procedure proposed in this paper allows to obtain exact distributions rather then approximate ones. Furthermore, the approximation based on truncated birth-death processes in our case is not necessarily successful, since the transition rates can be unbounded (see Section 6.2).

This is the plan of the paper. In Section 2 we present the method, based on the transformation between birth-death processes, both having state-space $\mathbb{N}_0$, with 0 a reflecting state. Starting from a birth-death process $N(t)$, we define a new birth-death process $M(t)$, whose birth and death rates depend on those of $N(t)$ and on the probability law of the process $N(t)$ evaluated at an exponentially distributed random time T with mean $\xi^{-1} > 0$. We obtain the conditional probabilities and the mean of $M(t)$ in closed form, and investigate the special case when $\xi \downarrow 0$. A remark on the case dealing with general birth processes is also given.

Section 3 is centered on the connection between the proposed method and the notion of ν-similarity, which is shown to hold for a special family of immigration-birth-death processes.

In Section 4 we show a connection between the distribution of $M(t)$ and the steady-state probability of $N(t)$ subject to catastrophes occurring according to a Poisson process with rate ξ.

Section 5 deals with Laplace transforms and first-passage time. We determine the Laplace transforms of the conditional probabilities of $M(t)$. Such functions are used to obtain the Laplace transform of the downward first-passage-time densities. Some results on the asymptotic behavior of the rates of $M(t)$ are also shown.

In Section 6 we analyze some special cases. We apply the proposed method to (i) a birth-death process with constant rates, and (ii) a linear immigration-death process. Various numerical computations are performed by means of MATHEMATICA® to illustrate the obtained results and to elucidate the role of the parameters.

2. Main results

Let $\{N(t), t \geq 0\}$ be a continuous-time birth-death process with state space $\mathbb{N}_0 = \{0, 1, \ldots\}$, 0 being a reflecting state. Assume that the birth rates $\{\lambda_n,\ n \in \mathbb{N}_0\}$ and the death rates $\{\mu_n,\ n \in \mathbb{N}\}$ are strictly positive, so that the process $N(t)$ is irreducible. As usual, we denote by

$$\pi_0 = 1, \qquad \pi_n = \frac{\lambda_0 \lambda_1 \cdots \lambda_{n-1}}{\mu_1 \mu_2 \cdots \mu_n}, \qquad n \in \mathbb{N} \tag{1}$$

the potential coefficients of $N(t)$ (cf. Keilson [23]). We remark that

$$\lambda_n\,\pi_n = \mu_{n+1}\,\pi_{n+1} \qquad \text{for all } n \in \mathbb{N}_0. \tag{2}$$

Let us introduce the following notation (cf. Callaert and Keilson [24], Kijima [25]):

$$\begin{aligned} A &= \sum_{n=0}^{+\infty} \frac{1}{\lambda_n \pi_n}, & B &= \sum_{n=0}^{+\infty} \pi_n, \\ C &= \sum_{n=0}^{+\infty} \frac{1}{\lambda_n \pi_n} \sum_{i=0}^{n} \pi_i, & D &= \sum_{n=0}^{+\infty} \frac{1}{\lambda_n \pi_n} \sum_{i=n+1}^{\infty} \pi_i. \end{aligned} \tag{3}$$

We recall that the process $N(t)$ is recurrent if $A = +\infty$ and is transient if A is finite. If $N(t)$ is recurrent, then it is positive recurrent if B is finite and null recurrent if B diverges. Moreover, the boundary at infinity of $N(t)$ is regular if C and D are finite, exit if $C < +\infty$ and $D = +\infty$, entrance if $C = +\infty$ and $D < +\infty$ and natural if $C = D = +\infty$. If the series C diverges, then the birth-death process $N(t)$ is nonexplosive. This classification is summarized in Anderson [26], pag. 262. Other general aspects are treated in Feller [27].

Hereafter, we assume that the boundary at infinity of $N(t)$ is natural, so that the process is nonexplosive and all the states are transient or recurrent. We denote the transition probabilities of $N(t)$ by

$$p_{j,n}(t) = P\{N(t) = n \,|\, N(0) = j\} \qquad (j, n \in \mathbb{N}_0). \tag{4}$$

They satisfy the forward Kolmogorov equations:

$$\begin{aligned} \frac{dp_{j,0}(t)}{dt} &= -\lambda_0\, p_{j,0}(t) + \mu_1\, p_{j,1}(t) \\ \frac{dp_{j,n}(t)}{dt} &= -(\lambda_n + \mu_n)\, p_{j,n}(t) + \lambda_{n-1}\, p_{j,n-1}(t) + \mu_{n+1}\, p_{j,n+1}(t), \quad n \in \mathbb{N} \end{aligned} \tag{5}$$

to be solved with the initial conditions:

$$\lim_{t \downarrow 0} p_{j,n}(t) = \delta_{j,n} \qquad (n \in \mathbb{N}_0), \tag{6}$$

where $\delta_{j,n}$ denotes the Kronecker delta. Since the boundary at $+\infty$ is natural, the solution of (5) with condition (6) is unique and $\sum_{n=0}^{+\infty} p_{j,n}(t) = 1$ for $t \geq 0$.

In Section 2.1 we will introduce a new birth-death process related to $N(t)$, denoted as $M(t)$, whose relevant functions can be expressed in terms of those of $N(t)$. To this purpose now we introduce a compound random variable, namely

$$Z \stackrel{d}{=} [N(T) \,|\, N(0) = 0], \tag{7}$$

that describes the conditional process evaluated at a random time T exponentially distributed with mean ξ^{-1}. Hence, the probability function of (7) is given by the following mixture probabilities with exponential mixing probability density function (pdf):

$$\omega_n = P(Z = n) = \int_0^{+\infty} \xi\, e^{-\xi\, t}\, p_{0,n}(t)\, dt, \qquad n \in \mathbb{N}_0 \tag{8}$$

for $\xi > 0$. We note that ω_n can also be expressed in terms of a Laplace transform. Indeed, denoting by

$$\widehat{p}_{j,n}(s) = \int_0^{+\infty} e^{-st} p_{j,n}(t)\, dt \qquad s > 0, \quad j, n \in \mathbb{N}_0 \tag{9}$$

the Laplace transform of the transition probability $p_{j,n}(t)$, from (8) we have

$$\omega_n = \xi\, \widehat{p}_{0,n}(\xi), \qquad n \in \mathbb{N}_0. \tag{10}$$

Roughly speaking, due to (7), parameter ξ may be viewed as the (constant) sampling rate of the process $N(t)$, where Z denotes the sampled value of $N(t)$ and T represents the sampling time. Hence, the distribution of Z tends to the initial distribution of $N(t)$ when ξ grows. Formally, from (6) and (10) we have

$$\lim_{\xi \to +\infty} \omega_n = \delta_{0,n} \qquad (n \in \mathbb{N}_0).$$

A thorough analysis of the case $\xi \downarrow 0$ is performed in Section 2.3.

In addition, in Section 4 we will show that ω_n can also be viewed as the steady-state probability of the process $N(t)$ subject to catastrophes governed by a Poisson process with intensity ξ. Moreover, recalling (8), we have that ω_n is solution of the following system:

$$\begin{aligned} &(\lambda_0 + \xi)\, \omega_0 = \mu_1\, \omega_1 + \xi \\ &(\lambda_n + \mu_n + \xi)\, \omega_n = \lambda_{n-1}\, \omega_{n-1} + \mu_{n+1}\, \omega_{n+1}, \qquad n \in \mathbb{N}, \end{aligned} \tag{11}$$

obtained from (5) and (6) by setting $j = 0$.

2.1. The new birth-death process

Let $\{M(t), t \geq 0\}$ be a continuous-time birth-death process, having state-space $\mathbb{N}_0$, with 0 a reflecting state. The birth rates $\{\alpha_n,\ n \in \mathbb{N}_0\}$ and death rates $\{\beta_n,\ n \in \mathbb{N}\}$ of $M(t)$ are defined as follows:

$$
\begin{aligned}
\alpha_n &= \lambda_n \frac{R_n}{R_{n+1}} = \lambda_n \left(1 + \frac{\omega_n}{R_{n+1}}\right), \qquad n \in \mathbb{N}_0 \\
\beta_n &= \mu_n \frac{R_{n+1}}{R_n} = \mu_n \left(1 - \frac{\omega_n}{R_n}\right), \qquad n \in \mathbb{N}
\end{aligned} \tag{12}
$$

where

$$
R_k = P(Z \geq k) = \sum_{i=k}^{+\infty} \omega_i, \qquad k \in \mathbb{N}_0. \tag{13}
$$

From (12) it follows that the product of birth and death rates in state $n \in \mathbb{N}$ of the two processes is constant, i.e.

$$
\alpha_n \, \beta_n = \lambda_n \, \mu_n \quad \text{for all } n \in \mathbb{N}. \tag{14}
$$

This is reminiscent of similar transformations of birth-death processes developed in the past, where the sum of the birth and death rates of the two processes is constant (see the birth-death processes suggested by a chain sequence and the processes with similar time-dependent behaviour, studied in Lenin and Parthasarathy [9] and Lenin *et al.* [10], respectively; see also Di Crescenzo and Martinucci [28] and Hongler and Parthasarathy [29] for examples of birth-death processes for which the sum of the birth and death rates is constant). However, due to (12) in this case the sums of birth and death rates of the two processes are not necessarily identical.

A more general family of transformations, leading to the concept of ν-similarity of birth-death processes, has been investigated by Lenin *et al.* [10] and, more recently, by Poskrobko and Girejko [30]. The connection between such a notion and the transformation based on (12) will be exploited in Section 3.

By virtue of (12), the potential coefficients of the new process $M(t)$, say π_n^*, are related to the potential coefficients of $N(t)$ as

$$
\pi_0^* = 1, \qquad \pi_n^* = \frac{\alpha_0 \alpha_1 \cdots \alpha_{n-1}}{\beta_1 \beta_2 \cdots \beta_n} = \pi_n \frac{R_1}{R_n \, R_{n+1}}, \qquad n \in \mathbb{N}. \tag{15}
$$

Hence, since the endpoint $+\infty$ is natural for $N(t)$, we have that $+\infty$ is natural for $M(t)$ too. Moreover, even though $N(t)$ is either transient or recurrent, the process $M(t)$ is transient for $\xi > 0$. Intuitively, this is due to the fact that the net growth rate of $M(t)$ is larger than those of $N(t)$, since by (12) we have $\alpha_n - \beta_n > \lambda_n - \mu_n$ for all $n \in \mathbb{N}_0$, and $\alpha_n/\beta_n > \lambda_n/\mu_n$ for all $n \in \mathbb{N}$.

2.2. *Conditional probabilities*

Hereafter we are able to state the key result of the paper. We prove that, under suitable assumption, the probability distribution

$$q_{0,n}(t) = P\{M(t) = n | M(0) = 0\} \tag{16}$$

can be expressed in terms of the probability distribution of $[N(t)|N(0) = 0]$ and the distribution of Z.

Proposition 1. *If*

$$\lim_{n\to+\infty} \frac{p_{0,n}(t)}{\omega_n} = 0, \qquad \forall t > 0, \tag{17}$$

then the conditional probabilities of the process $M(t)$ having rates (12), with a reflecting condition at 0, are:

$$q_{0,n}(t) = e^{-\xi\, t}\, V_n(t) + \int_0^t \xi\, e^{-\xi\,\tau}\, V_n(\tau)\, d\tau, \qquad n \in \mathbb{N}_0,\ t > 0, \tag{18}$$

where

$$V_n(t) = \frac{\omega_n}{R_n}\Big[\frac{p_{0,n}(t)}{\omega_n} - \frac{1}{R_{n+1}} \sum_{k=n+1}^{+\infty} p_{0,k}(t)\Big], \tag{19}$$

where ω_n and R_n are introduced in (8) and (13), respectively.

Proof. The proof consists of 3 steps.
(i) First, we show the initial condition for $q_{0,n}(t)$. Making use of (6) for $j = 0$ in (19), one has $\lim_{t\downarrow 0} V_n(t) = \delta_{0,n}$ and thus from (18) we obtain $\lim_{t\downarrow 0} q_{0,n}(t) = \delta_{0,n}$.
(ii) In order to prove the normalization condition, we note that by induction we can prove:

$$\frac{1}{R_k} - \sum_{i=0}^{k-1} \frac{\omega_i}{R_i R_{i+1}} = 1, \qquad k \in \mathbb{N}. \tag{20}$$

Hence, thanks to (17) and a Stolz-Cesàro principle, from (19) it follows:

$$\sum_{n=0}^{+\infty} V_n(t) = p_{0,0}(t) + \sum_{k=1}^{+\infty} p_{0,k}(t) \left[\frac{1}{R_k} - \sum_{i=0}^{k-1} \frac{\omega_i}{R_i R_{i+1}}\right] = \sum_{k=0}^{+\infty} p_{0,k}(t) = 1.$$

Hence, due to (18), we obtain $\sum_{n=0}^{+\infty} q_{0,n}(t) = 1$.
(iii) Let us now prove that the probabilities $q_{0,n}(t)$, given in (18), satisfy the forward Kolmogorov equations:

$$\begin{aligned}\frac{dq_{0,0}(t)}{dt} &= -\alpha_0\, q_{0,0}(t) + \beta_1\, q_{0,1}(t) \\ \frac{dq_{0,n}(t)}{dt} &= -(\alpha_n + \beta_n)\, q_{0,n}(t) + \alpha_{n-1}\, q_{0,n-1}(t) + \beta_{n+1}\, q_{0,n+1}(t), \quad n \in \mathbb{N}.\end{aligned} \tag{21}$$

First of all, by setting $n = 0$ in (18) and (19), we note that

$$\frac{dq_{0,0}(t)}{dt} = e^{-\xi\, t} \frac{dV_0(t)}{dt} = \frac{e^{-\xi\, t}}{1-\omega_0}\, \frac{dp_{0,0}(t)}{dt}. \tag{22}$$

Moreover, from (12) and (18) for $t > 0$ it results:

$$\begin{aligned}\alpha_0\, q_{0,0}(t) - \beta_1\, q_{0,1}(t) &= \frac{e^{-\xi\, t}}{1-\omega_0} \Big[\lambda_0\, V_0(t) - \mu_1\,(1-\omega_0-\omega_1) V_1(t)\Big] \\ &+ \frac{1}{1-\omega_0} \int_0^t \xi\, e^{-\xi\,\tau} \Big[\lambda_0\, V_0(\tau) - \mu_1\,(1-\omega_0-\omega_1) V_1(\tau)\Big]\, d\tau.\end{aligned} \tag{23}$$

Note that from (19) one has

$$\begin{aligned}&\lambda_0\, V_0(t) - \mu_1\,(1-\omega_0-\omega_1) V_1(t) \\ &\quad = \lambda_0 p_{0,0}(t) - \mu_1 p_{0,1}(t) - \frac{\lambda_0\omega_0 - \mu_1\omega_1}{1-\omega_0}\,[1 - p_{0,0}(t)] \\ &\quad = -\frac{dp_{0,0}(t)}{dt} - \xi\,[1 - p_{0,0}(t)], \qquad \forall t > 0,\end{aligned}$$

where the last equality follows from (5) and (11). Therefore, Eq. (23) can be re-written as

$$\alpha_0\, q_{0,0}(t) - \beta_1\, q_{0,1}(t) = -\frac{e^{-\xi\, t}}{1-\omega_0}\, \frac{dp_{0,0}(t)}{dt}. \tag{24}$$

Hence, due to (22) and (24), the first of (21) holds.

Similarly, to prove the second of (21) we note that from (18) and (19) one has:

$$\begin{aligned}\frac{dq_{0,n}(t)}{dt} &= e^{-\xi\, t} \frac{dV_n(t)}{dt} = \frac{e^{-\xi\, t}}{R_n} \Big[\frac{dp_{0,n}(t)}{dt} - \frac{\omega_n}{R_{n+1}} \sum_{k=n+1}^{+\infty} \frac{dp_{0,k}(t)}{dt}\Big] \\ &= \frac{e^{-\xi\, t}}{R_n} \Big\{\frac{dp_{0,n}(t)}{dt} - \frac{\omega_n}{R_{n+1}} \big[\lambda_n\, p_{0,n}(t) - \mu_{n+1}\, p_{0,n+1}(t)\big]\Big\}, \quad n \in \mathbb{N},\end{aligned} \tag{25}$$

where the last equality follows from (5). Furthermore, from (12) and (18) for $t > 0$ one obtains

$$(\alpha_n + \beta_n)\, q_{0,n}(t) - \alpha_{n-1}\, q_{0,n-1}(t) - \beta_{n+1}\, q_{0,n+1}(t)$$
$$= e^{-\xi t} V_n^\star(t) + \int_0^t \xi\, e^{-\xi\tau} V_n^\star(\tau)\, d\tau, \qquad n \in \mathbb{N}, \tag{26}$$

where we have set

$$V_n^\star(t) = (\alpha_n + \beta_n)\, V_n(t) - \alpha_{n-1}\, V_{n-1}(t) - \beta_{n+1}\, V_{n+1}(t), \qquad t > 0. \tag{27}$$

Making use of (12) and (19), Eq. (27) becomes:

$$\begin{aligned} V_n^\star(t) = &-\frac{1}{R_n}\big[\lambda_{n-1}\, p_{0,n-1}(t) - \mu_n\, p_{0,n}(t)\big] \\ &+\frac{\omega_n}{R_n R_{n+1}}\big[\lambda_n\, p_{0,n}(t) - \mu_{n+1}\, p_{0,n+1}(t)\big] \\ &+\frac{1}{R_n^2}\big[\lambda_{n-1}\, \omega_{n-1} - \mu_n\, \omega_n\big] \sum_{k=n}^{+\infty} p_{0,k}(t) \\ &-\frac{1}{R_{n+1}^2}\big[\lambda_n\, \omega_n - \mu_{n+1}\, \omega_{n+1}\big] \sum_{k=n+1}^{+\infty} p_{0,k}(t), \quad n \in \mathbb{N}. \end{aligned} \tag{28}$$

Summing over $k = n, n+1, \ldots$ both sides of (11) one has:

$$\lambda_{n-1}\, \omega_{n-1} - \mu_n\, \omega_n = \xi\, R_n, \qquad n \in \mathbb{N},$$

so that, by virtue of (5), Eq. (28) can be re-written as:

$$\begin{aligned} V_n^\star(t) = &-\frac{1}{R_n}\frac{dp_{0,n}(t)}{dt} + \frac{\omega_n}{R_n R_{n+1}}\Big[\lambda_n p_{0,n}(t) - \mu_{n+1} p_{0,n+1}(t)\Big] \\ &+\xi\Big[\frac{1}{R_n}\sum_{k=n}^{+\infty} p_{0,k}(t) - \frac{1}{R_{n+1}}\sum_{k=n+1}^{+\infty} p_{0,k}(t)\Big], \quad n \in \mathbb{N}. \end{aligned} \tag{29}$$

Moreover, from (29) one obtains:

$$\int_0^t e^{-\xi\tau} V_n^\star(\tau)\, d\tau = \frac{e^{-\xi\, t}}{R_n}\Big[-p_{0,n}(t) + \frac{\omega_n}{R_{n+1}} \sum_{k=n+1}^{+\infty} p_{0,k}(t)\Big], \quad n \in \mathbb{N}. \tag{30}$$

Finally, recalling (25) and by using (29) and (30) in (26), one prove that the second of (21) holds. In conclusion, we have that probabilities $q_{0,n}(t)$ satisfy Eqs. (21) with a delta initial condition, so that the proof is completed. □

As a special case, let us now consider the distribution (16) for $n = 0$. Under the assumptions of Proposition 1, from (19) we have $V_0(t) = [p_{0,0}(t) - \omega_0]/R_1$, $t > 0$, so that from (8) and (18) it follows:

$$q_{0,0}(t) = \frac{1}{R_1}\left\{e^{-\xi t} p_{0,0}(t) - \int_t^{+\infty} \xi\, e^{-\xi \tau} p_{0,0}(\tau)\, d\tau\right\}, \qquad t > 0. \tag{31}$$

We note that $dq_{0,0}(t)/dt = (e^{-\xi t}/R_1)\, dp_{0,0}(t)/dt$.

Due to the assumptions of Proposition 1, from (19) and (20) one has:

$$\begin{aligned}\sum_{k=n}^{+\infty} V_k(t) &= \frac{p_{0,n}(t)}{R_n} + \sum_{k=n+1}^{+\infty} p_{0,k}(t)\Big[\frac{1}{R_k} - \sum_{i=n}^{k-1}\frac{\omega_i}{R_i R_{i+1}}\Big] \\ &= \frac{1}{R_n}\sum_{k=n}^{+\infty} p_{0,k}(t), \qquad n \in \mathbb{N}_0, \quad t > 0. \end{aligned} \tag{32}$$

By virtue of (32), one obtains the following alternative expression for $V_n(t)$:

$$V_n(t) = \frac{1}{R_n}\sum_{k=n}^{+\infty} p_{0,k}(t) - \frac{1}{R_{n+1}}\sum_{k=n+1}^{+\infty} p_{0,k}(t), \qquad n \in \mathbb{N}_0, \quad t > 0. \tag{33}$$

Given two integer-valued nonnegative random variables X and Y, we recall that X is said to be smaller than Y in the hazard rate order (denoted by $X \le_{hr} Y$) if and only if $P(X = n)/P(X \ge n) \ge P(Y = n)/P(Y \ge n)$ for all $n \in \mathbb{N}_0$ (see Section 1.B.1 of Shaked and Shanthikumar [31], for instance). Hence, from (33) we have that $V_n(t)$, $n \in \mathbb{N}_0$, is a proper probability distribution if and only if $[N(t)\,|\,N(0) = 0] \le_{hr} Z$, for $t > 0$.

The results of Proposition 1 and Eq. (32) can be used to get the conditional mean of $M(t)$. Specifically, now we show that the conditional mean of $M(t)$ can be obtained by means of a relation that is analogue to (18).

Proposition 2. *Under the assumptions of Proposition 1, the mean of the birth-death process $M(t)$ having rates (12), with a reflecting condition at* 0, *for $t > 0$ is:*

$$E_1(t) := E[M(t)|M(0) = 0] = e^{-\xi t} H(t) + \int_0^t \xi\, e^{-\xi \tau} H(\tau)\, d\tau, \tag{34}$$

where

$$H(t) = \sum_{n=1}^{+\infty} \frac{1}{R_n}\sum_{k=n}^{+\infty} p_{0,k}(t). \tag{35}$$

Proof. For $t > 0$ we have

$$E[M(t)|M(0) = 0] = \sum_{n=1}^{+\infty} n\, q_{0,n}(t) = \sum_{n=1}^{+\infty} \sum_{k=n}^{+\infty} q_{0,k}(t)$$

where, by virtue of (18) and (32),

$$\sum_{k=n}^{+\infty} q_{0,k}(t) = \frac{1}{R_n} \left[e^{-\xi\, t} \sum_{k=n}^{+\infty} p_{0,k}(t) + \int_0^t \xi\, e^{-\xi\, \tau} \sum_{k=n}^{+\infty} p_{0,k}(\tau)\, d\tau \right], \qquad (36)$$

for $n \in \mathbb{N}_0$. Hence, due to (35), Eq. (34) immediately follows. □

The procedure exploited above allows us to obtain the distribution of $M(t)$ conditional on zero initial state. Hereafter, by means of a time-reversibility relation, we are able to determine an expression for the probabilities $q_{j,0}(t)$, for $j \in \mathbb{N}_0$.

Proposition 3. *Under the assumptions of Proposition 1, for $j \in \mathbb{N}_0$ and $t > 0$ we have:*

$$\begin{aligned} q_{j,0}(t) = \frac{R_{j+1}}{R_1} \left[e^{-\xi\, t}\, p_{j,0}(t) + \int_0^t \xi\, e^{-\xi \tau} p_{j,0}(\tau)\, d\tau \right] \\ - \frac{\xi}{R_1} \widehat{p}_{j,0}(\xi) \sum_{k=j+1}^{+\infty} \left[e^{-\xi\, t}\, p_{0,k}(t) + \int_0^t \xi\, e^{-\xi \tau} p_{0,k}(\tau)\, d\tau \right], \end{aligned} \qquad (37)$$

with $\widehat{p}_{j,0}(\xi)$ defined in (9).

Proof. We note that the transition probabilities of the birth-death process $M(t)$ satisfy the following reversibility relation (cf., for instance Karlin and McGregor [32])

$$q_{j,n}(t) = \frac{\pi_n^*}{\pi_j^*}\, q_{n,j}(t), \qquad t > 0. \qquad (38)$$

Hence, recalling (19), from (18) after some calculations we obtain:

$$\begin{aligned} q_{j,0}(t) = \frac{R_{j+1}}{R_1} \left[e^{-\xi\, t}\, p_{j,0}(t) + \int_0^t \xi\, e^{-\xi \tau} p_{j,0}(\tau)\, d\tau \right] \\ - \frac{\omega_j}{\pi_j\, R_1} \sum_{k=j+1}^{+\infty} \left[e^{-\xi\, t}\, p_{0,k}(t) + \int_0^t \xi\, e^{-\xi \tau} p_{0,k}(\tau)\, d\tau \right]. \end{aligned} \qquad (39)$$

Making use of (10) and of the reversibility relation for the probabilities of $N(t)$, one has $\omega_j = \xi\, \pi_j\, \widehat{p}_{j,0}(\xi)$, so that Eq. (37) follows from (39). □

Remark 1. *The transformation procedure described above can be exploited also in the case of birth processes. Specifically, if* $\{N(t), t \geq 0\}$ *is a general birth process with state space* $\mathbb{N}_0$ *and strictly positive birth rates* $\{\lambda_n, n \in \mathbb{N}_0\}$*, then one can define a birth process* $\{M(t), t \geq 0\}$ *with birth rates* $\{\alpha_n, n \in \mathbb{N}_0\}$ *defined as in (12), i.e.*

$$\alpha_n = \lambda_n \frac{R_n}{R_{n+1}} = \lambda_n \Big(1 + \frac{\omega_n}{R_{n+1}}\Big), \qquad n \in \mathbb{N}_0, \tag{40}$$

where, according to (8),

$$\omega_n = \int_0^{+\infty} \xi \, e^{-\xi t} p_{0,n}(t)\, dt = \begin{cases} \xi \, \dfrac{1}{\xi + \lambda_0}, & n = 0 \\ \xi \, \dfrac{\lambda_0 \lambda_1 \cdots \lambda_{n-1}}{(\xi + \lambda_0)(\xi + \lambda_1) \cdots (\xi + \lambda_n)}, & n \in \mathbb{N}, \end{cases}$$

and R_n *is defined as in (13).*

The last equality in Remark 1 follows from the Laplace transform of the transition probabilities of a general birth process (see, e.g. Section 6 of [1]), Hence, similarly one can prove that Proposition 1 holds also for the birth processes $N(t)$ and $M(t)$, having transition probabilities $p_{0,n}(t)$ and $q_{0,n}(t)$, respectively.

2.3. *The special case* $\xi \downarrow 0$

The previous results refer to the case $\xi > 0$. Let us now investigate the process $M(t)$ when $\xi \downarrow 0$ in two cases: *(i)* $N(t)$ is positive recurrent, *(ii)* $N(t)$ is transient or null recurrent. Recalling the meaning of ξ, when $\xi \downarrow 0$ we expect that the distribution of Z tends to the steady-state distribution of $N(t)$, when existing.

(i) If the process $N(t)$ is positive recurrent, then it admits a steady-state distribution given by

$$\sigma_0 := \lim_{t \to +\infty} p_{j,0}(t) = \frac{1}{B}, \qquad \sigma_n := \lim_{t \to +\infty} p_{j,n}(t) = \frac{\pi_n}{B}, \quad n \in \mathbb{N} \tag{41}$$

with B defined in (3). Then, from (8) one has

$$\lim_{\xi \downarrow 0} \omega_n = \lim_{\xi \downarrow 0} [\xi \, \widehat{p}_{j,n}(\xi)] = \sigma_n, \qquad n \in \mathbb{N}_0, \tag{42}$$

so that the probability distribution of Z tends to the steady-state distribution of $N(t)$. Hence, when $\xi \downarrow 0$ the birth and death rates of $M(t)$, rather than (12), are given by

$$
\begin{aligned}
\alpha_n &= \lambda_n \frac{S_n}{S_{n+1}} = \lambda_n \Big(1 + \frac{\sigma_n}{S_{n+1}}\Big), \qquad n \in \mathbb{N}_0 \\
\beta_n &= \mu_n \frac{S_{n+1}}{S_n} = \mu_n \Big(1 - \frac{\sigma_n}{S_n}\Big), \qquad n \in \mathbb{N},
\end{aligned}
\tag{43}
$$

where

$$
S_k := \lim_{\xi \downarrow 0} R_k = \sum_{i=k}^{+\infty} \sigma_i, \qquad k \in \mathbb{N}_0. \tag{44}
$$

We remark that the process $M(t)$ having rates (43), and with a reflecting condition at 0, is transient.

The following result is a consequence of Propositions 1 and 3, and of Eqs. (42) and (44). We obtain that also in the case $\xi \downarrow 0$ the distribution of $M(t)$ can be expressed in terms of that of $N(t)$.

Proposition 4. *If $N(t)$ admits a steady–state behavior and*

$$
\lim_{n \to +\infty} \frac{p_{0,n}(t)}{\sigma_n} = 0, \qquad \forall t > 0, \tag{45}
$$

then for the birth-death process $M(t)$ having rates (43) and with a reflecting condition at 0, when $t > 0$ one has:

$$
\begin{aligned}
q_{0,n}(t) &= \frac{\sigma_n}{S_n} \Big[\frac{p_{0,n}(t)}{\sigma_n} - \frac{1}{S_{n+1}} \sum_{k=n+1}^{+\infty} p_{0,k}(t) \Big], \qquad n \in \mathbb{N}_0, \\
q_{j,0}(t) &= \frac{1}{S_1} \Big[S_{j+1}\, p_{j,0}(t) - \sigma_0 \sum_{k=j+1}^{+\infty} p_{0,k}(t) \Big], \qquad j \in \mathbb{N}_0.
\end{aligned}
\tag{46}
$$

In particular, from Proposition 4 we have

$$
q_{0,0}(t) = \frac{p_{0,0}(t) - \sigma_0}{1 - \sigma_0}, \qquad t > 0.
$$

Moreover, from the first of (46) one obtains a series form expression for the mean of the birth-death process $M(t)$ having rates (43), with a reflecting condition at 0:

$$E_1(t) := E[M(t)|M(0) = 0] = \sum_{n=1}^{+\infty} \frac{1}{S_n} \sum_{k=n}^{+\infty} p_{0,k}(t), \tag{47}$$

for $t > 0$. Such result can be also derived from (34) by taking the limit as $\xi \downarrow 0$.

(ii) If the birth-death process $M(t)$ is transient or null recurrent (i.e., it does not admit a steady state behavior), then the rates (12) tend to those of the process $N(t)$ when $\xi \downarrow 0$. Hence, in this case we cannot employ Proposition 1 to construct of a new birth-death process.

3. Connection to the ν-similarity

We recall the notion of ν-similarity (see [10] and [30]), by which a birth-death process $\widetilde{N}(t)$ is said ν-similar to the birth-death process $N(t)$ if their transition probabilities satisfy $\widetilde{p}_{j,n}(t) = c_{j,n}e^{\nu t}p_{j,n}(t)$, $t \geq 0$, for some real number ν. In this case their birth and death rates satisfy the following identities:

$$\widetilde{\lambda}_n + \widetilde{\mu}_n = \lambda_n + \mu_n - \nu, \qquad \widetilde{\lambda}_n \widetilde{\mu}_{n+1} = \lambda_n \mu_{n+1}, \qquad n \in \mathbb{N}_0. \tag{48}$$

We are now led to investigate the following problem: There exist suitable conditions such that, given a birth-death process $N(t)$, the ν-similar process $\widetilde{N}(t)$ and the process $M(t)$ obtained via positions (12) have identical birth and death rates?

Due to (12) and (48), the identities

$$\widetilde{\lambda}_n + \widetilde{\mu}_n = \alpha_n + \beta_n \qquad \text{and} \qquad \widetilde{\lambda}_n \widetilde{\mu}_{n+1} = \alpha_n \beta_{n+1}$$

are satisfied if and only if

$$\frac{\omega_n}{R_n} = \frac{\omega_{n+1}}{R_{n+1}} \qquad \text{and} \qquad \omega_n \left(\frac{\lambda_n}{R_{n+1}} - \frac{\mu_n}{R_n} \right) = -\nu. \tag{49}$$

By virtue of (13), the first of (49) is satisfied when Z has a geometric distribution over $\mathbb{N}_0$, so that the solutions to (49) have the following form:

$$\omega_n = p(1-p)^n, \qquad \lambda_n = (1-p)\left(\mu_n - \frac{\nu}{p}\right), \qquad n \in \mathbb{N}_0, \tag{50}$$

with $0 < p < 1$ and $\nu < 0$. Making use of (50) in (11), we note that the birth rates λ_n must satisfy the second order difference equation

$$(1-p)\,\lambda_{n+1} - (2-p)\,\lambda_n + \lambda_{n-1} - (\nu + \xi)\,(1-p) = 0, \qquad n \in \mathbb{N},$$

to be solved with conditions

$$\lambda_0 = -\frac{\nu\,(1-p)}{p}, \qquad \lambda_1 = \lambda_0 + \xi + \nu - \frac{\nu+\xi}{p}.$$

Hence, for $\nu < -\xi$ and $0 < p < 1$ the solution to the above problem is provided by the following transition rates:

$$\lambda_n = \frac{(-\nu-\xi)(1-p)}{p}\,n - \frac{\nu(1-p)}{p}, \qquad \mu_n = \frac{(-\nu-\xi)}{p}\,n, \qquad n \in \mathbb{N}_0, \tag{51}$$

with μ_n obtained from the second of (50). We note that the rates derived in (51) refer to an immigration-birth-death process $\{N(t), t \geq 0\}$. For $\nu < -\xi$ and $0 < p < 1$ the conditional probability of such a process is (cf., for instance, Giorno and Nobile [33]):

$$p_{0,n}(t) = \frac{1}{n!}\Big(\frac{\nu}{\nu+\xi}\Big)_n \left[\frac{p}{1-(1-p)e^{(\nu+\xi)t}}\right]^{\nu/(\nu+\xi)} \left[\frac{(1-p)\left(1-e^{(\nu+\xi)t}\right)}{1-(1-p)e^{(\nu+\xi)t}}\right]^n, \tag{52}$$

for $n \in \mathbb{N}_0$, where $(a)_n$ denotes the Pochhammer symbol, defined as $(a)_0 = 1$ and $(a)_n = a(a+1)\ldots(a+n-1)$, $n \in \mathbb{N}$. We remark that such process is positive-recurrent.

Due to the first of (50), for the new birth-death process $\{M(t), t \geq 0\}$ possessing rates (12) now we have

$$\alpha_n = \frac{(-\nu-\xi)}{p}\,n - \frac{\nu}{p}, \qquad \beta_n = \frac{(-\nu-\xi)\,(1-p)}{p}\,n, \qquad n \in \mathbb{N}_0, \tag{53}$$

with $\nu < -\xi$ and $0 < p < 1$, i.e. $M(t)$ is a transient immigration-birth-death process with conditional probability (cf. [33])

$$q_{0,n}(t) = \frac{1}{n!}\Big(\frac{\nu}{\nu+\xi}\Big)_n \left[\frac{p\,e^{(\nu+\xi)t}}{1-(1-p)e^{(\nu+\xi)t}}\right]^{\nu/(\nu+\xi)} \left[\frac{1-e^{(\nu+\xi)t}}{1-(1-p)e^{(\nu+\xi)t}}\right]^n, \tag{54}$$

for $n \in \mathbb{N}_0$. By comparing (52) and (54), it immediately follows

$$q_{0,n}(t) = (1-p)^{-n} e^{\nu\,t} p_{0,n}(t), \qquad n \in \mathbb{N}_0, \tag{55}$$

so that the birth-death processes having rates (51) and (53) satisfy the ν-similarity relation.

We conclude this section by noting that if $\nu = -\xi$ in Eq. (51), then $\lambda_n = \xi(1-p)/p$, $n \in \mathbb{N}_0$, and thus $N(t)$ is a Poisson process with intensity $\xi(1-p)/p$, with $0 < p < 1$. From Remark 1 it follows that the new birth process $M(t)$ has rates $\alpha_n = \xi/p$, $n \in \mathbb{N}_0$, i.e. it is a Poisson process with intensity ξ/p, with $0 < p < 1$. Consequently, it is not hard to see that the ν-similarity relation (55) still holds for such processes.

4. Connection to processes subject to catastrophes

In this section we give another interpretation of Proposition 1, showing that ω_n can also be seen as the steady-state probability of the process $N(t)$ subject to catastrophes, governed by a Poisson process with intensity ξ. The following results have been stimulated by Lemma 2.1 of Di Crescenzo *et al.* [34], which is concerning the M/M/1 queueing system subject to catastrophes.

We consider a birth-death process, named $\{N^c(t), t \geq 0\}$, which evolves likewise $N(t)$ but in the presence of total catastrophes. Namely, births and deaths occur with rates λ_n and μ_n, respectively, whereas catastrophes occur according to a Poisson process with rate ξ. The effect of each catastrophe is the instantaneous transition to the state 0 (cf. Figure 1).

The transition probabilities of the process $N^c(t)$, denoted as

$$p^c_{j,n}(t) = P\{N^c(t) = n | N^c(0) = j\}, \qquad j, n \in \mathbb{N}_0, \tag{56}$$

satisfy the following forward Kolmogorov equations:

$$\begin{aligned} \frac{dp^c_{j,0}(t)}{dt} &= -(\lambda_0 + \xi)\, p^c_{j,0}(t) + \mu_1\, p^c_{j,1}(t) + \xi \\ \frac{dp^c_{j,n}(t)}{dt} &= -\big(\lambda_n + \mu_n + \xi\big)\, p^c_{j,n}(t) + \lambda_{n-1}\, p^c_{j,n-1}(t) + \mu_{n+1}\, p^c_{j,n+1}(t), \quad n \in \mathbb{N} \end{aligned} \tag{57}$$

with $p^c_{j,n}(0) = \delta_{j,n}$. The probabilities $p^c_{j,n}(t)$ can be also expressed in terms of the probabilities $p_{j,n}(t)$ of the process $N(t)$. Indeed, conditioning on the age of the catastrophe process, for a birth-death process one has (see, for instance, Economou and Fakinos [35], [36], Pakes [37], Renshaw and Chen [38]):

$$p^c_{j,n}(t) = e^{-\xi t}\, p_{j,n}(t) + \xi \int_0^t e^{-\xi\tau}\, p_{0,n}(\tau)\, d\tau, \qquad n, j \in \mathbb{N}_0,\ t > 0. \tag{58}$$

Hence, the steady-state probabilities of $N^c(t)$ are given by

$$\lim_{t\to+\infty} p^c_{j,n}(t) = \xi\, \widehat{p}_{0,n}(\xi), \qquad n \in \mathbb{N}_0, \tag{59}$$

that identifies with ω_n by virtue of (10). This result confirms that the distribution (8) of Z can be viewed as the steady-state distribution of the birth-death process $N(t)$ subject to catastrophes, here denoted as $N^c(t)$. As a consequence, we are now able to provide an alternative form for (16) in terms of the conditional probabilities of $N^c(t)$.

Proposition 5. *Under the assumptions of Proposition 1, for $t > 0$ the probabilities (16) can be also expressed as:*

$$q_{0,n}(t) = \frac{\omega_n}{R_n}\Big[\frac{p^c_{0,n}(t)}{\omega_n} - \frac{1}{R_{n+1}}\sum_{k=n+1}^{+\infty} p^c_{0,k}(t)\Big], \qquad n \in \mathbb{N}_0. \tag{60}$$

Proof. From (18) and (19) it follows:

$$q_{0,n}(t) = \frac{1}{R_n}\big[e^{-\xi t}\, p_{0,n}(t) + \xi \int_0^t e^{-\xi\tau}\, p_{0,n}(\tau)\, d\tau\big]$$
$$-\frac{\omega_n}{R_n R_{n+1}} \sum_{k=n+1}^{+\infty} \Big[e^{-\xi u}\, p_{0,k}(u) + \xi \int_0^u e^{-\xi\tau}\, p_{0,k}(\tau)\, d\tau\Big]\, du, \qquad n \in \mathbb{N}_0,$$

from which, due to (58), we obtain immediately Eq. (60). □

From (60), or equivalently from (36), for $t > 0$ one has

$$\sum_{k=n}^{+\infty} q_{0,k}(t) = \frac{1}{R_n}\sum_{k=n}^{+\infty} p^c_{0,k}(t), \qquad n \in \mathbb{N}_0.$$

5. Laplace transforms and first passage time

In this section we provide some results based on the Laplace-transform approach. In particular, the latter is employed to face the first-passage-time problem for the process $M(t)$. We denote by

$$\widehat{q}_{j,n}(s) := \int_0^{+\infty} e^{-s\,t} q_{j,n}(t)\, dt, \qquad s > 0, \quad j, n \in \mathbb{N}_0 \tag{61}$$

the Laplace transform of $q_{j,n}(t)$. Hereafter we show some results analogous to Propositions 1 and 3 in terms of Laplace transforms.

Proposition 6. *Under the assumptions of Proposition 1, for $s > 0$ and $j, n \in \mathbb{N}_0$ one has:*

$$\widehat{q}_{0,n}(s) = \frac{s+\xi}{s}\,\frac{\omega_n}{R_n}\Big[\frac{\widehat{p}_{0,n}(s+\xi)}{\omega_n} - \frac{1}{R_{n+1}}\sum_{k=n+1}^{+\infty}\widehat{p}_{0,k}(s+\xi)\Big], \tag{62}$$

$$\widehat{q}_{j,0}(s) = \frac{s+\xi}{s}\,\frac{1}{R_1}\Big[R_{j+1}\,\widehat{p}_{j,0}(s+\xi) - \xi\,\widehat{p}_{j,0}(\xi)\sum_{k=j+1}^{+\infty}\widehat{p}_{0,k}(s+\xi)\Big]. \tag{63}$$

Proof. Taking the Laplace transform of (18) one has

$$\widehat{q}_{0,n}(s) = \frac{s+\xi}{s} \int_0^{+\infty} e^{-(s+\xi)t} V_n(t)\, dt, \qquad n \in \mathbb{N}_0,$$

from which Eq. (62) follows, due to (19). Moreover, by taking the Laplace transform of (37) one obtains:

$$\widehat{q}_{j,0}(s) = \frac{s+\xi}{s} \frac{1}{R_1} \Big[R_{j+1}\, \widehat{p}_{j,0}(s+\xi) - \frac{\omega_j}{\pi_j} \sum_{k=j+1}^{+\infty} \widehat{p}_{0,k}(s+\xi) \Big], \qquad j \in \mathbb{N}_0.$$

Hence, since $\omega_j = \xi\, \pi_j\, \widehat{p}_{j,0}(\xi)$, Eq. (63) immediately follows. □

In particular, by setting $n = 0$ in (62), or $j = 0$ in (63), one has:

$$\widehat{q}_{0,0}(s) = \frac{s+\xi}{s} \frac{1}{R_1} \Big[\widehat{p}_{0,0}(s+\xi) - \frac{\omega_0}{s+\xi} \Big], \qquad s > 0.$$

The previous result can be also obtained by taking the Laplace transform of both sides of (31).

Due to its relevance in applied contexts, we are now driven to introduce the first-passage time (FPT) through n starting from j for the process $M(t)$:

$$T_{j,n} = \inf\{t > 0 : M(t) = n\}, \qquad M(0) = j, \qquad j, n \in \mathbb{N}_0,\ n \neq j,$$

and denote by $g_{j,n}(t)$ its pdf. Hereafter, we evaluate the Laplace transform $\widehat{g}_{j,n}(s)$ of $g_{j,n}(t)$ in the case when the first passage occurs from above.

Proposition 7. *Let $j, n \in \mathbb{N}_0$, with $n < j$. Under the assumptions of Proposition 1, for $s > 0$ we have:*

$$\widehat{g}_{j,n}(s) = \frac{R_{j+1}\, \widehat{p}_{j,0}(s+\xi) - \xi\, \widehat{p}_{j,0}(\xi) \displaystyle\sum_{k=j+1}^{+\infty} \widehat{p}_{0,k}(s+\xi)}{R_{n+1}\, \widehat{p}_{n,0}(s+\xi) - \xi\, \widehat{p}_{n,0}(\xi) \displaystyle\sum_{k=n+1}^{+\infty} \widehat{p}_{0,k}(s+\xi)}. \tag{64}$$

Proof. The FPT pdf $g_{j,n}(t)$ is solution of the integral equation

$$q_{j,0}(t) = \int_0^t g_{j,n}(\tau)\, q_{n,0}(t-\tau)\, d\tau, \qquad j, n \in \mathbb{N}_0, \quad n < j$$

so that, taking the Laplace transform, one has $\widehat{g}_{j,n}(s) = \widehat{q}_{j,0}(s)/\widehat{q}_{n,0}(s)$. Eq. (64) thus follows, due to (63). □

Moreover, since the birth-death process $M(t)$ is transient, under the assumptions of Proposition 1, the ultimate first passage probability through the state n is given by (cf. Keilson [23]):

$$P(T_{j,n} < +\infty) = \frac{\displaystyle\sum_{k=j}^{+\infty} \frac{1}{\alpha_k \, \pi_k^*}}{\displaystyle\sum_{k=n}^{+\infty} \frac{1}{\alpha_k \, \pi_k^*}} = \frac{\displaystyle\sum_{k=j}^{+\infty} \frac{1}{\lambda_k \, \pi_k} \, R_{k+1}^2}{\displaystyle\sum_{k=n}^{+\infty} \frac{1}{\lambda_k \, \pi_k} \, R_{k+1}^2}, \qquad j, n \in \mathbb{N}_0, \; 0 \leq n < j, \tag{65}$$

where the last equality follows from (12) and (15).

We conclude this section by discussing the asymptotic behavior of the birth and death rates of $M(t)$.

Proposition 8. *For $\xi > 0$, the birth and death rates of $M(t)$ admit the following asymptotic behavior as $n \to \infty$:*

$$\alpha_n \sim \mu_{n+1} \, \frac{\widehat{p}_{n,0}(\xi)}{\widehat{p}_{n+1,0}(\xi)}, \qquad \beta_n \sim \lambda_n \, \frac{\mu_n}{\mu_{n+1}} \, \frac{\widehat{p}_{n+1,0}(\xi)}{\widehat{p}_{n,0}(\xi)}, \tag{66}$$

with $\widehat{p}_{j,n}(s)$ defined in (9).

Proof. For $\xi > 0$, recalling rates (12), due to a Stolz-Cesàro principle and Eq. (9) we obtain

$$\alpha_n \sim \lambda_n \, \frac{\widehat{p}_{0,n}(\xi)}{\widehat{p}_{0,n+1}(\xi)} = \lambda_n \, \frac{\pi_n \, \widehat{p}_{n,0}(\xi)}{\pi_{n+1} \, \widehat{p}_{n+1,0}(\xi)},$$

where the last equality is a consequence of a reversibility relation similar to (38). The first relation of (66) thus follows from (2). The second of (66) can be proved similarly. □

Due to renewal arguments, we have $\widehat{p}_{n+1,0}(\xi) = \widehat{p}_{n,0}(\xi) \, \widehat{g}_{n+1,n}(\xi)$ (see, for instance Keilson [23]). Hence, for $\xi > 0$ the right-hand-sides of (66) can be expressed in terms of the Laplace transform $\widehat{g}_{n+1,n}(\cdot)$ of the pdf of $T_{n+1,n}$, i.e.

$$\alpha_n \sim \frac{\mu_{n+1}}{\widehat{g}_{n+1,n}(\xi)}, \qquad \beta_n \sim \lambda_n \, \frac{\mu_n}{\mu_{n+1}} \, \widehat{g}_{n+1,n}(\xi).$$

Proposition 9. *If $\xi \downarrow 0$, then the birth and death rates of $M(t)$ admit the following asymptotic behavior as $n \to \infty$:*

$$\alpha_n \sim \mu_{n+1}, \qquad \beta_n \sim \lambda_n \, \frac{\mu_n}{\mu_{n+1}}. \tag{67}$$

Proof. Making use of (41) and (43), due to a Stolz-Cesàro principle we have

$$\alpha_n \sim \lambda_n \frac{\sigma_n}{\sigma_{n+1}} = \lambda_n \frac{\pi_n}{\pi_{n+1}},$$

so that the first of (67) follows from (2). The second of (67) can be shown in a similar way. □

Finally, we point out that the results shown in Propositions 8 and 9 are in strict agreement with Eq. (14) as $n \to \infty$, both when $\xi > 0$ and $\xi \downarrow 0$.

6. Analysis of special cases

Let us now make use of the previous results in order to study certain birth-death processes obtained via transformation of some classical models. We first consider the transformation when $N(t)$ is the birth-death process with constant rates. After we deal with the process with constant birth rate and linear death rate, i.e. a linear immigration-death process. In particular, we evaluate several quantities of interest by means of analytical study and suitable numerical computations. We point out that the rates of the birth-death processes $N(t)$ considered in this section do not depend on parameter ξ, differently from the cases treated in Section 3.

Moreover, the birth-death processes $M(t)$ obtained hereafter are transient processes, and thus can be usefully used to describe stochastic systems that grow rapidly due to prevalence of births on deaths, such as bacterial populations or cancer masses.

6.1. Transformation of a birth-death process with constant rates

Let $\{N(t); t \geq 0\}$ be a birth-death process with birth and death rates

$$\lambda_n = \lambda \quad (n \in \mathbb{N}_0), \qquad \mu_n = \mu \quad (n \in \mathbb{N}), \tag{68}$$

with $\lambda > 0$, $\mu > 0$. This process is well-known in queueing theory as describing the number of customers in an $M/M/1$ queue. The transition probabilities of $N(t)$, $t \geq 0$, are (see, for instance, Medhi [39], p. 116)

$$p_{j,n}(t) = e^{-(\lambda+\mu)t} \Big[\varrho^{(n-j)/2}\, I_{n-j}\Big(2t\sqrt{\lambda\mu}\Big) + \varrho^{(n-j-1)/2}\, I_{n+j+1}\Big(2t\sqrt{\lambda\mu}\Big)$$
$$+ (1-\varrho)\varrho^n \sum_{k=n+j+2}^{+\infty} \varrho^{-k/2}\, I_k\Big(2t\sqrt{\lambda\mu}\Big)\Big], \qquad j,n \in \mathbb{N}_0 \tag{69}$$

with $\varrho = \lambda/\mu$ and where

$$I_k(z) = \sum_{r=0}^{+\infty} \frac{(z/2)^{k+2r}}{r!\,(k+r)!}, \qquad k \in \mathbb{N}_0 \tag{70}$$

denotes the modified Bessel function of first kind. We recall that the process $N(t)$ admits a geometric steady-state distribution $\sigma_n = \lim_{t\to+\infty} p_{j,n}(t) = (1-\varrho)\,\varrho^n$, $n \in \mathbb{N}_0$, if and only if $\varrho < 1$. Moreover, for $j,n \in \mathbb{N}_0$ and $s > 0$ one has the Laplace transform of probabilities (69):

$$\widehat{p}_{j,n}(s) = \frac{1}{\mu[\psi_1(s)-\psi_2(s)]} \Big\{ \frac{\varrho^{(n-j+|n-j|)/2}}{[\psi_1(s)]^{|n-j|}} + \frac{\varrho^n}{[\psi_1(s)]^{n+j}} \, \frac{1-\psi_2(s)}{\psi_1(s)-1} \Big\}, \tag{71}$$

where

$$\psi_1(s), \psi_2(s) = \frac{s+\lambda+\mu \pm \sqrt{(s+\lambda+\mu)^2 - 4\lambda\mu}}{2\mu}, \qquad 0 < \psi_2(s) < 1 < \psi_1(s).$$

Hence, from (10) and (13) one has

$$\omega_n = [1-\psi_2(\xi)]\,[\psi_2(\xi)]^n, \qquad R_n = [\psi_2(\xi)]^n, \qquad n \in \mathbb{N}_0. \tag{72}$$

In Figure 2 the probabilities ω_n are shown as function of ξ for $n = 0,1,2,3$ and for some choices of λ, μ. When $\xi \downarrow 0$, one has $\psi_2(0) = 1$ if $\lambda \geq \mu$ and $\psi_2(0) = \varrho$ if $\lambda < \mu$. Hence, from the first of (72) we have $\lim_{\xi\downarrow 0} \omega_n = 0$ for $\lambda \geq \mu$, and $\lim_{\xi\downarrow 0} \omega_n = \sigma_n$ for $\lambda < \mu$, as confirmed in Figure 2.

Due to (68) and the second of (72), from (12) it follows that the birth-death process $M(t)$ is characterized by rates:

$$\alpha_n = \frac{\lambda}{\psi_2(\xi)}, \quad n \in \mathbb{N}_0, \qquad\qquad \beta_n = \mu\,\psi_2(\xi), \quad n \in \mathbb{N} \tag{73}$$

with a reflecting condition at 0. Hence, $M(t)$ is a birth-death process with constant rates too. Since $\psi_1(\xi)\psi_2(\xi) = \varrho$ one has $\alpha_n/\beta_n = \varrho/[\psi_2(\xi)]^2 = \psi_1(\xi)/\psi_2(\xi) > 1$, so that the process $M(t)$ is transient for $\xi > 0$. The transition probabilities $q_{j,n}(t)$ of $M(t)$ can be easily obtained from (69) by changing λ with $\lambda/\psi_2(\xi)$ and μ with $\mu\,\psi_2(\xi)$. Moreover, the FPT pdf for $M(t)$, $t > 0$, is:

$$g_{j,n}(t) = \frac{j-n}{t} \Big[\frac{\psi_2(\xi)}{\sqrt{\varrho}}\Big]^{j-n} e^{-(\lambda+\mu+\xi)t}\, I_{j-n}(2t\sqrt{\lambda\mu}), \qquad 0 \leq n < j. \tag{74}$$

The FPT pdf (74) exhibits an interesting property: it remains unaltered when the birth and death rates of the originating process $N(t)$ are exchanged. In Figure 3 the FPT pdf $g_{j,0}(t)$ is plotted as function of t for various choices of ξ on the left, and various choices of the initial state j on the right.

From (74) one obtains the ultimate first passage probability:

$$P(T_{j,n} < +\infty) := \int_0^{+\infty} g_{j,n}(t)\, dt = \left[\frac{\psi_2(\xi)}{\psi_1(\xi)}\right]^{j-n}, \qquad 0 \le n < j. \tag{75}$$

We note that

$$\frac{d}{d\xi} P(T_{j,n} < +\infty) = -\,(j-n) \left[\frac{\psi_2(\xi)}{\psi_1(\xi)}\right]^{j-n} \frac{2\mu\,[\psi_1(\xi) - \psi_2(\xi)]}{(\xi+\lambda+\mu)^2 - 4\lambda\mu} < 0,$$

so that $P(T_{j,n} < +\infty)$ is a decreasing function of ξ. Moreover, since

$$\frac{d}{d\mu} P(T_{j,n} < +\infty) = \,(j-n) \left[\frac{\psi_2(\xi)}{\psi_1(\xi)}\right]^{j-n} \frac{(\xi+\lambda-\mu)[\psi_1(\xi) - \psi_2(\xi)] - 4\lambda}{(\xi+\lambda+\mu)^2 - 4\lambda\mu},$$

one has that $P(T_{j,n} < +\infty)$ is a decreasing function of μ for $\mu > \xi + \lambda$.

Some computations of the ultimate first passage probabilities (75) are shown in Tables 1 and 2 when $n = 0$. In Table 1 the FPT probabilities $P(T_{j,0} < +\infty)$ are computed when j increases, for different values of ξ and for fixed λ, μ. In Table 2, $P(T_{1,0} < +\infty)$ is evaluated when μ increases, for the same values of ξ of Table 1 and for fixed λ. It is evident that $P(T_{j,0} < +\infty)$ is decreasing when ξ increases. Moreover, in Figure 4 we plot $P(T_{j,0} < +\infty)$ for $j = 1, 2$ as function of μ; we note that the FPT probabilities exhibit an unique maximum and are decreasing when $\mu > \xi + \lambda$. Some contour plots of FPT probabilities $P(T_{j,0} < +\infty)$ are shown in Figure 5 as a function of (ξ, μ).

In conclusion, we analyze the behavior of the process $M(t)$ as $\xi \downarrow 0$. From (73) we obtain:

$$\lim_{\xi \downarrow 0} \alpha_n = \max\{\lambda, \mu\}, \quad n \in \mathbb{N}_0, \qquad \lim_{\xi \downarrow 0} \beta_n = \min\{\lambda, \mu\}, \quad n \in \mathbb{N}.$$

Hence, when $\lambda \ge \mu$ the rates (73) tend to those of process $N(t)$, as $\xi \downarrow 0$. Instead, when $\lambda < \mu$ the rates are inverted as $\xi \downarrow 0$, in the sense that the birth and death rates (73) tend to the death and birth rates of $N(t)$, respectively. Finally, when $\xi \downarrow 0$ the process $M(t)$ is transient if $\lambda \ne \mu$, and null recurrent if $\lambda = \mu$.

6.2. Transformation of a linear immigration-death process

Let $\{N(t); t \ge 0\}$ be a linear immigration-death process with birth and death rates

$$\lambda_n = \nu \quad (n \in \mathbb{N}_0), \qquad \mu_n = n\,\mu \quad (n \in \mathbb{N}). \tag{76}$$

We remark that this process is well-known in queueing theory, since it describes the number of customers in an $M/M/\infty$ queue. Due to (1), the potential coefficients of $N(t)$ are $\pi_n = (\nu/\mu)^n/n!$ for $n \in \mathbb{N}_0$. The transition probabilities of $N(t)$ are given by (see, for instance, Giorno *et al.* [40]).

$$
\begin{aligned}
p_{j,n}(t) = \exp\Big\{-\frac{\nu}{\mu}\big(1-e^{-\mu\,t}\big)\Big\} \sum_{r=0}^{\min(n,j)} \binom{j}{r} \frac{1}{(n-r)!}\Big(\frac{\nu}{\mu}\Big)^{n-r} \\
\times\big(1-e^{-\mu\,t}\big)^{n+j-2r} e^{-\mu\,r\,t}, \qquad j,n\in\mathbb{N}_0.
\end{aligned}
\tag{77}
$$

For $j \in \mathbb{N}_0$ and $t > 0$ the conditional mean and variance of $N(t)$ are:

$$
\begin{aligned}
E[N(t)|N(0)=j] &= j\,e^{-\mu\,t} + \frac{\nu}{\mu}\big(1-e^{-\mu\,t}\big), \\
\mathrm{Var}[N(t)|N(0)=j] &= \Big(j\,e^{-\mu\,t} + \frac{\nu}{\mu}\Big)\big(1-e^{-\mu\,t}\big).
\end{aligned}
\tag{78}
$$

From (77) we note that $N(t)$ admits a steady-state Poisson distribution $\sigma_n = \lim_{t\to+\infty} p_{j,n}(t) = e^{-\nu/\mu}(\nu/\mu)^n/n!$ ($n \in \mathbb{N}_0$). Moreover, for $j,n \in \mathbb{N}_0$ and $s > 0$ the Laplace transform of (77) is:

$$
\begin{aligned}
\widehat{p}_{j,n}(s) = \frac{e^{-\nu/\mu}}{\mu} \sum_{r=0}^{\min(n,j)} \frac{j!}{r!} \binom{n+j-2r}{n-r} \frac{\big(\frac{\nu}{\mu}\big)^{n-r}}{\big(\frac{s}{\mu}+r\big)_{n+j-2r+1}} \\
\times\Phi\Big(\frac{s}{\mu}+r, \frac{s}{\mu}+n+j-r+1; \frac{\nu}{\mu}\Big), \qquad j,n\in\mathbb{N}_0,
\end{aligned}
\tag{79}
$$

where $(\gamma)_n$ denotes the Pochhammer symbol, defined as $(\gamma)_0 = 1$ and $(\gamma)_n = \gamma(\gamma+1)\cdots(\gamma+n-1)$ if $n \in \mathbb{N}$, and where $\Phi(a,c;x)$ is the Kummer function:

$$
\Phi(a,c;x) = 1 + \sum_{n=1}^{+\infty} \frac{(a)_n}{(c)_n} \frac{x^n}{n!}.
$$

Recalling (10) and making use of (79) we have:

$$
\omega_n = \frac{\big(\frac{\nu}{\mu}\big)^n e^{-\nu/\mu}}{\big(\frac{\xi}{\mu}+1\big)_n} \Phi\Big(\frac{\xi}{\mu}, \frac{\xi}{\mu}+1+n; \frac{\nu}{\mu}\Big), \qquad j,n\in\mathbb{N}_0. \tag{80}
$$

From (80) we note that $\lim_{\xi\downarrow 0} \omega_n = \sigma_n$ ($n \in \mathbb{N}_0$). In Figure 6 the probabilities ω_n are shown as function of ξ for $n = 0,1,2,3$ and for some choices of ν, μ.

Making use of (80) in (13), and recalling (12) we have

$$\alpha_n = \nu\,\frac{R_n}{R_{n+1}} \quad (n\in\mathbb{N}_0), \qquad\qquad \beta_n = n\,\mu\,\frac{R_{n+1}}{R_n} \quad (n\in\mathbb{N}), \tag{81}$$

with

$$R_k = e^{-\nu/\mu}\sum_{n=k}^{+\infty}\frac{\left(\frac{\nu}{\mu}\right)^n}{\left(\frac{\xi}{\mu}+1\right)_n}\,\Phi\Big(\frac{\xi}{\mu},\frac{\xi}{\mu}+1+n;\frac{\nu}{\mu}\Big), \qquad k\in\mathbb{N}_0. \tag{82}$$

From (81) we have the following asymptotic behaviors:
- $\alpha_n\sim n\,\mu$ as $n\to+\infty$ and $\lim_{n\to+\infty}\beta_n=\nu$ (for ν, μ and ξ fixed);
- $\alpha_n\sim\mu\,(n+1)$ as $\mu\to+\infty$ for all $n\in\mathbb{N}_0$ and $\lim_{\mu\to+\infty}\beta_n=n\,\nu/(n+1)$ for all $n\in\mathbb{N}$ (for ν, ξ and n fixed);
- $\alpha_n\sim\nu$ and $\beta_n\sim\mu\,n$ as $\nu\to+\infty$ (for μ,ξ and n fixed), since $\Phi(a,b;x)\sim x^{-a}$ as $x\to+\infty$ (for a,b fixed);
- $\alpha_n\sim\xi$ and $\beta_n\sim(\mu\,\nu/\xi)\,n$ as $\xi\to+\infty$ (for ν, μ and n fixed).

We note that for n increasing the birth and death rates (81) exhibit a behavior close to the death and birth rates μ_n and λ_n given in (76), respectively.

In Figure 7 the rates (81) are plotted for $\nu=0.3$, $\mu=0.2$ and $\xi=0,\,0.5,\,1.0$. According to the asymptotic behavior of $M(t)$, we have $\alpha_n\sim 0.2\,n$ and β_n approaches 0.3 as n increases. Moreover, for fixed values of n, α_n increases with ξ, whereas β_n decreases when ξ increases.

Making use of (77) and (80) and recalling that

$$\lim_{n\to+\infty}\Phi\Big(\frac{\xi}{\mu},\frac{\xi}{\mu}+1+n;\frac{\nu}{\mu}\Big)=1, \qquad \lim_{n\to+\infty}\frac{(1+\xi/\mu)_n}{n!}\,(1-e^{-\mu\,t})^n=0,$$

we can see that the limit (17) is satisfied. The assumptions of Proposition 1 thus hold. By virtue of (19) and recalling Eqs. (77), (80) and (82) for $n\in\mathbb{N}_0$ and $t>0$ we obtain:

$$V_n(t)=\frac{(\nu/\mu)^n}{(1+\xi/\mu)_n\,R_n}\,\exp\Big\{-\frac{\nu}{\mu}\,(1-e^{-\mu\,t})\Big\}\,\Phi\Big(\frac{\xi}{\mu},\frac{\xi}{\mu}+1+n;\frac{\nu}{\mu}\Big)$$
$$\times\left\{\frac{(1+\xi/\mu)_n\,(1-e^{-\mu\,t})^n}{n!\,\Phi\Big(\frac{\xi}{\mu},\frac{\xi}{\mu}+1+n;\frac{\nu}{\mu}\Big)}-\frac{e^{-\nu/\mu}}{R_{n+1}}\sum_{k=n+1}^{+\infty}\frac{1}{k!}\Big[\frac{\nu}{\mu}\big(1-e^{-\mu\,t}\big)\Big]^k\right\}. \tag{83}$$

Making use of (31), when $\xi>0$, one has:

$$q_{0,0}(t)=\frac{e^{-\xi\,t-\nu/\mu}}{R_1}\left[\exp\Big\{\frac{\nu}{\mu}e^{-\mu\,t}\Big\}-\Phi\Big(\frac{\xi}{\mu},\frac{\xi}{\mu}+1;\frac{\nu}{\mu}e^{-\mu\,t}\Big)\right], \qquad t>0 \tag{84}$$

with R_1 defined in (82). We remark that, making use of (83), the probabilities $q_{0,n}(t)$ can be evaluated from (18) via numerical integration. Figure 8 shows some plots of $q_{0,n}(t)$, obtained by using the `NIntegrate` function of MATHEMATICA®. We note that $q_{0,0}(t)$ is a decreasing function of t, whereas $q_{0,n}(t)$ ($n \in \mathbb{N}$) exhibits an unique maximum whose abscissa decreases as ξ increases.

Let us now discuss the case $\xi \downarrow 0$. From (43) it follows:

$$\alpha_n = \nu \, \frac{S_n}{S_{n+1}}, \quad n \in \mathbb{N}_0, \qquad \beta_n = n \, \mu \, \frac{S_{n+1}}{S_n}, \quad n \in \mathbb{N}, \tag{85}$$

with

$$S_k = \lim_{\xi \downarrow 0} R_k = e^{-\nu/\mu} \sum_{n=k}^{+\infty} \frac{(\nu/\mu)^n}{n!}, \qquad k \in \mathbb{N}_0. \tag{86}$$

The following asymptotic behaviors hold for rates (85):

- $\alpha_n \sim \mu\,(n+1)$ and $\beta_n \sim n\,\nu/(n+1)$ as $\mu \to +\infty$;
- $\alpha_n \sim \nu$ and $\beta_n \sim \mu\, n$ as $\nu \to +\infty$.

The rates (85) are plotted with circles in Figure 7 for $\nu = 0.3$ and $\mu = 0.2$. Moreover, they are plotted in Figure 9 as a function of μ for $\nu = 0.3$ and $n = 1, 5, 10$. According to the asymptotic behavior of $M(t)$, for large μ it holds: $\alpha_n \sim 2\mu$ for $n = 1$, $\alpha_n \sim 6\mu$ for $n = 5$ and $\alpha_n \sim 11\mu$ for $n = 10$. Instead, β_n approaches 0.15 for $n = 1$, 0.25 for $n = 5$, and 0.2727 for $n = 10$.

The rates (85) are also plotted in Figure 10 as a function of ν, for $\mu = 0.2$ and $n = 1, 5, 10$. According to the asymptotic behavior of $M(t)$, for large ν, the rates α_n increase linearly as ν increases, whereas β_n approaches 0.2 for $n = 1$, 1.0 for $n = 5$, and 2.0 for $n = 10$.

Making use of Proposition 4, one obtains the conditional probabilities of the birth-death process with rates (85), for $n \in \mathbb{N}_0$ and $t > 0$:

$$\begin{aligned} q_{0,n}(t) = {} & \frac{(\nu/\mu)^n}{n!\, S_n} \exp\Big\{-\frac{\nu}{\mu}\left(1 - e^{-\mu\, t}\right)\Big\} \\ & \times \left\{(1 - e^{-\mu\, t})^n - \frac{e^{-\nu/\mu}}{S_{n+1}} \sum_{k=n+1}^{+\infty} \frac{1}{k!} \left[\frac{\nu}{\mu}\,(1 - e^{-\mu\, t})\right]^k\right\}. \end{aligned} \tag{87}$$

In particular, from (87) for $n = 0$ and $t > 0$ one has:

$$q_{0,0}(t) = \frac{e^{-\nu/\mu}}{1 - e^{-\nu/\mu}} \left[\exp\Big\{\frac{\nu}{\mu} e^{-\mu\, t}\Big\} - 1\right],$$

that corresponds to the limit of (84) when $\xi \downarrow 0$. Making use of (77) and (47) one obtains the mean of the birth-death process with rates (85):

$$E_1(t) = \exp\Big\{-\frac{\nu}{\mu}\left(1-e^{-\mu t}\right)\Big\} \sum_{n=1}^{+\infty} \frac{1}{S_n} \sum_{k=n}^{+\infty} \frac{1}{k!}\Big[\frac{\nu}{\mu}\left(1-e^{-\mu t}\right)\Big]^k, \qquad t>0. \tag{88}$$

The probabilities (87) are plotted in Figure 11 (left) for $\nu = 0.3$ and $\mu = 0.2$. Moreover, the conditional mean (88) is shown in Figure 11 (right) for the same choice of parameters. We note that the conditional mean is an increasing function of t, indicating that the model does not admit a steady-state behavior.

Let us now come back to case $\xi > 0$ and consider the FPT problem for $M(t)$ when the first passage occurs from above. The Laplace transform $\widehat{g}_{j,n}(s)$ for $n < j$ can be determined by means of (64), with $\widehat{p}_{j,n}(s)$ given in (79) and R_k shown in (82). Moreover, due to (65) the ultimate first passage probability is:

$$P(T_{j,n} < +\infty) = \frac{\displaystyle\sum_{k=j}^{+\infty} k!\left(\frac{\mu}{\nu}\right)^k (R_{k+1})^2}{\displaystyle\sum_{k=n}^{+\infty} k!\left(\frac{\mu}{\nu}\right)^k (R_{k+1})^2}, \qquad j,n \in \mathbb{N}_0,\ 0 \le n < j, \tag{89}$$

with R_k given in (82).

Some numerical computations allow to evaluate the ultimate first passage probabilities (89), which are shown in Tables 3 and 4 when $n = 0$. In Table 3 the FPT probabilities $P(T_{j,0} < +\infty)$ are computed when j increases, for different values of ξ and for ν, μ fixed. In Table 4 the FPT probabilities $P(T_{1,0} < +\infty)$ are evaluated when μ increases, for the same values of ξ in Table 3 and for ν fixed. It is evident that $P(T_{j,0} < +\infty)$ is decreasing when ξ increases.

Concluding remarks. In this paper we have proposed a procedure able to obtain new birth-death processes for which the conditional probabilities and other functions of interest can be evaluated in closed form. Our approach allows to get new stochastic models suitable to describe random dynamics of systems subject to rapid growth. We have exploited in detail various cases, and provided interesting analytical results. Extensive numerical computations have been performed to show the role played by the models' parameters.

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j	$\xi = 0$	$\xi = 0.05$	$\xi = 0.1$	$\xi = 0.15$	$\xi = 0.2$
1	0.8	0.495879	0.381966	0.3125	0.264141
2	0.64	0.245896	0.145898	0.0976563	0.0697704
3	0.512	0.121935	0.0557281	0.0305176	0.0184292
4	0.4096	0.0604648	0.0212862	0.00953674	0.00486791
5	0.32768	0.0299832	0.00813062	0.00298023	0.00128581

Table 1: $P(T_{j,0} < +\infty)$, given in (75), for $\lambda = 0.5$ and $\mu = 0.4$.

μ	$\xi = 0$	$\xi = 0.05$	$\xi = 0.1$	$\xi = 0.15$	$\xi = 0.2$
0.1	0.2	0.158956	0.130385	0.109402	0.0934003
0.2	0.4	0.300827	0.240408	0.198964	0.168594
0.3	0.6	0.416667	0.325227	0.266667	0.225148
0.4	0.8	0.495879	0.381966	0.3125	0.264141
0.5	1.0	0.532672	0.411833	0.338851	0.28802
0.6	0.833333	0.533333	0.420204	0.349834	0.3
0.7	0.714286	0.512298	0.414184	0.35	0.303337
0.8	0.625	0.482133	0.4	0.343294	0.300827
0.9	0.555556	0.45	0.381966	0.332641	0.294614
1.0	0.5	0.419135	0.36267	0.32	0.286223

Table 2: $P(T_{1,0} < +\infty)$, given in (75), for $\lambda = 0.5$ and various choices of μ and ξ.

j	$\xi = 0$	$\xi = 0.05$	$\xi = 0.1$	$\xi = 0.15$	$\xi = 0.2$
1	0.222285	0.177844	0.146068	0.122496	0.104468
2	0.0543199	0.0369096	0.0261155	0.0190959	0.0143461
3	0.0124661	0.00744954	0.00467754	0.00305911	0.00206973
4	0.0025949	0.00139569	0.00079364	0.000472632	0.000292608
5	0.000486893	0.000239618	0.000125226	0.0000688131	0.0000394534

Table 3: $P(T_{j,0} < +\infty)$, given in (89), for $\nu = 0.3$ and $\mu = 0.2$.

μ	$\xi = 0$	$\xi = 0.05$	$\xi = 0.1$	$\xi = 0.15$	$\xi = 0.2$
0.1	0.26797	0.18112	0.133978	0.104593	0.0846308
0.2	0.222285	0.177844	0.146068	0.122496	0.104468
0.3	0.175841	0.150718	0.130757	0.114628	0.101402
0.4	0.143797	0.127825	0.11442	0.103062	0.0933512
0.5	0.121215	0.110202	0.100644	0.0922969	0.084964

Table 4: $P(T_{1,0} < +\infty)$, given in (89), for $\nu = 0.3$ and various choices of μ and ξ.

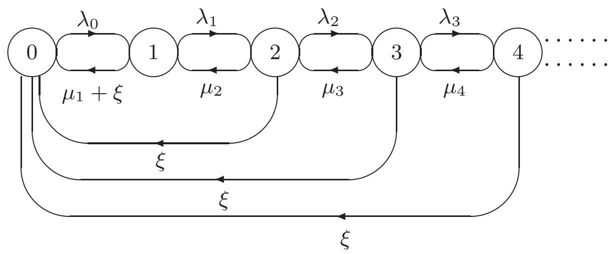
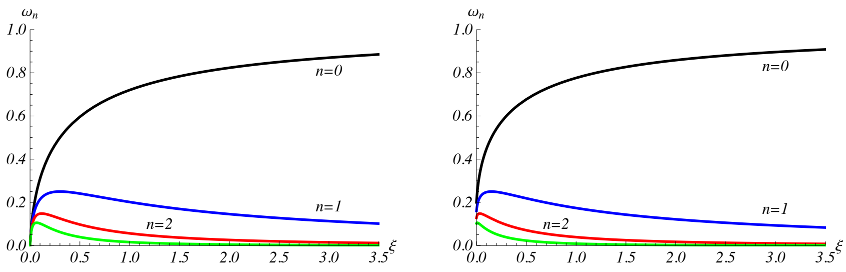
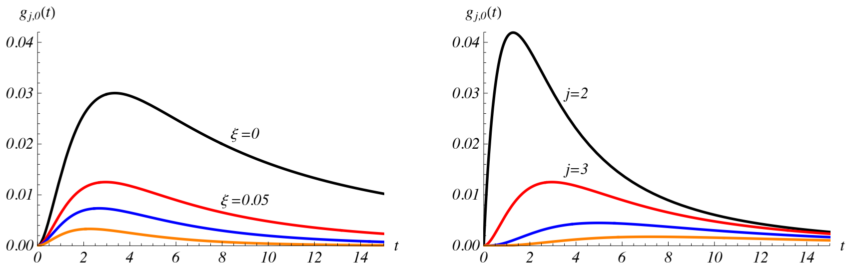


Figure 1: The state diagram of the process $N^c(t)$.



(a) $\lambda = 0.5, \mu = 0.4$ (b) $\lambda = 0.4, \mu = 0.5$

Figure 2: The probabilities ω_n, given in (72), are plotted as function of ξ for $n = 0, 1, 2, 3$ (top to bottom).



(a) $j = 3, \lambda = 0.5, \mu = 0.4$ (b) $\xi = 0.05, \lambda = 0.5, \mu = 0.4$

Figure 3: The FPT pdf $g_{j,0}(t)$ is plotted as function of t; (a) for $\xi = 0, 0.05, 0.1, 0.2$ (top to bottom), and (b) for $j = 2, 3, 4, 5$ (top to bottom).

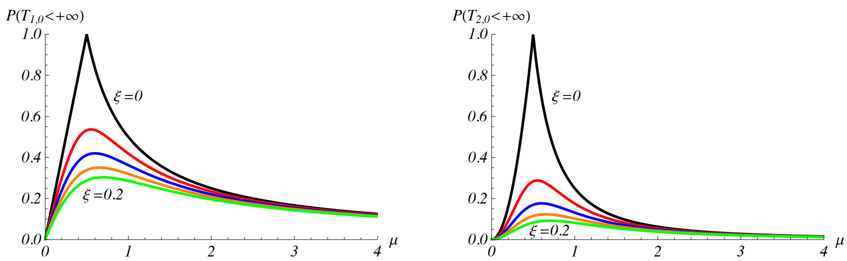


Figure 4: For $j = 1, 2$ the FPT probabilities $P(T_{j,0} < +\infty)$, given in (75), are plotted as function of μ for $\lambda = 0.5$ and for $\xi = 0, 0.05, 0.1, 0.15, 0.2$ (top to bottom at the origin).

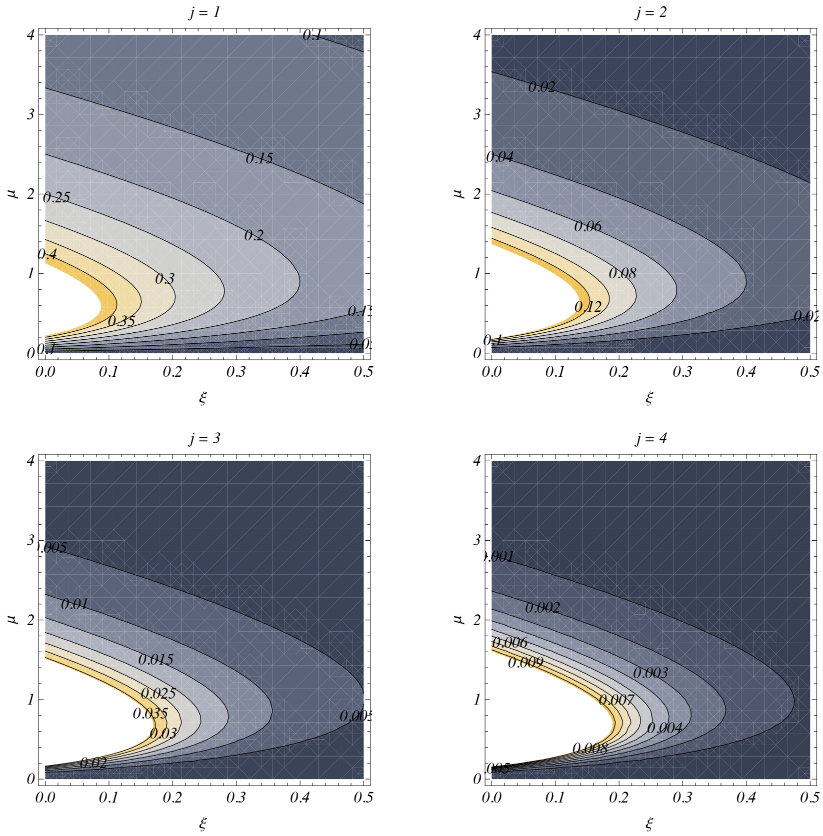
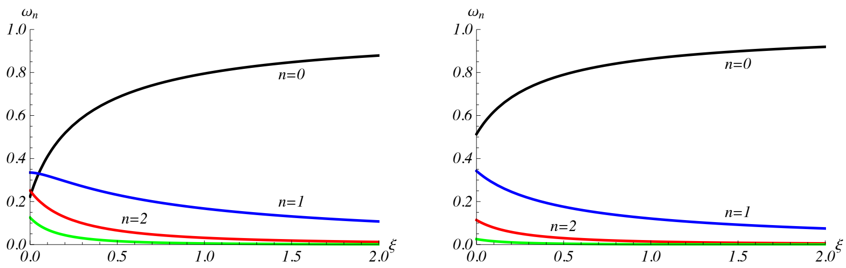


Figure 5: Contour plot of the FPT probabilities $P(T_{j,0} < +\infty)$, given in (75), as function of (ξ, μ) for $\lambda = 0.5$ and $j = 1, 2, 3, 4$.



(a) $\nu = 0.3, \mu = 0.2$ (b) $\nu = 0.2, \mu = 0.3$

Figure 6: The probabilities ω_n, given in (80), are plotted as function of ξ for $n = 0, 1, 2, 3$ (top to bottom).

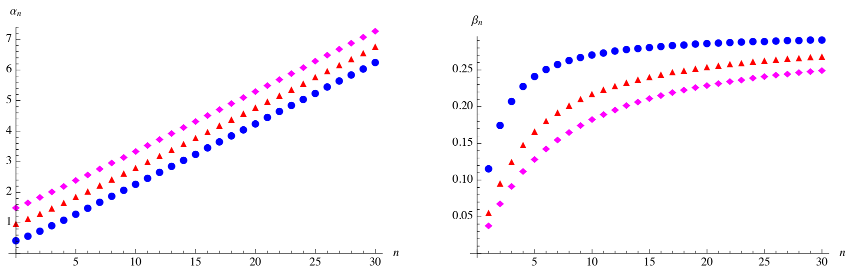
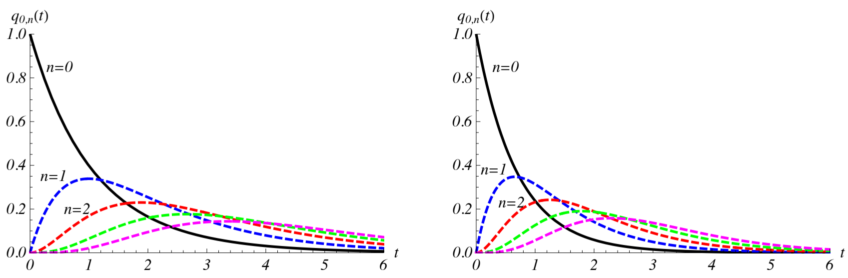


Figure 7: The birth and death rates (81) for $\nu = 0.3$, $\mu = 0.2$, with $\xi = 0$ (circle), $\xi = 0.5$ (triangle) and $\xi = 1.0$ (diamond).



(a) $\xi = 0.5$ (b) $\xi = 1.0$

Figure 8: For the birth-death process with rates (81), the probabilities $q_{0,n}(t)$ are plotted for $n = 0, 1, 2, 3, 4$ (top to bottom near the origin) with $\nu = 0.3$, $\mu = 0.2$.

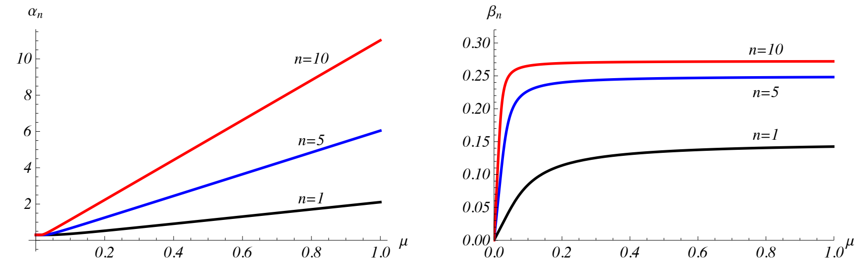


Figure 9: The birth and death rates (85) are plotted as function of μ for $\nu = 0.3$.

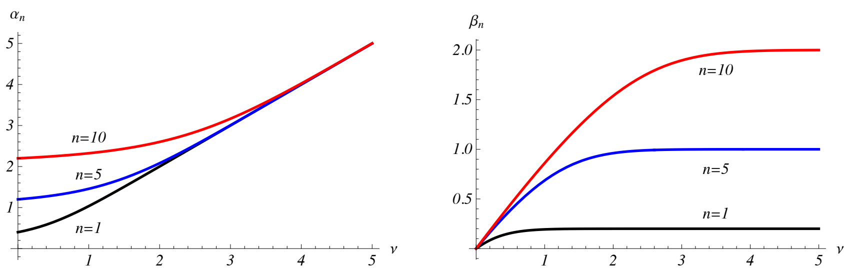


Figure 10: The birth and death rates (85) are plotted as function of ν for $\mu = 0.2$.

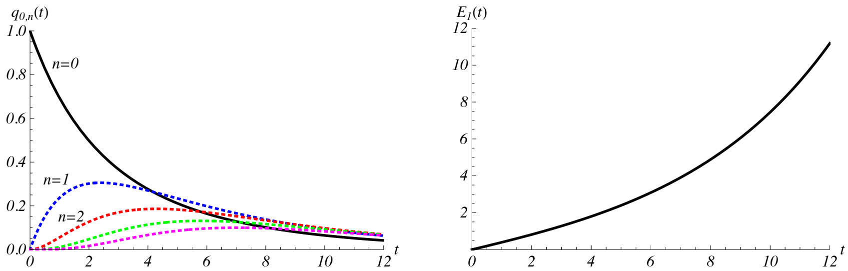


Figure 11: Probabilities $q_{0,n}(t)$ (on the left) for $n = 0, 1, 2, 3, 4$ (top to bottom near the origin) and the corresponding mean $E_1(t)$ (on the right) for the birth-death process having rates (85), with $\nu = 0.3$ and $\mu = 0.2$.