Consensus in Opinion Formation Processes in Fully Evolving Environments

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Abstract

(Friedkin and Johnsen 1990) modelled opinion formation in social networks as a dynamic process which evolves in rounds: at each round each agent updates her expressed opinion to a weighted average of her innate belief and the opinions expressed in the previous round by her social neighbours. The stubbornness level of an agent represents the tendency of the agent to express an opinion close to her innate belief.

Motivated by the observation that innate beliefs, stubborness levels and even social relations can co-evolve together with the expressed opinions, we present a new model of opinion formation where the dynamics runs in a co-evolving environment. We assume that agents' stubborness and social relations can vary arbitrarily, while their innate beliefs slowly change as a function of the opinions they expressed in the past. We prove that, in our model, the opinion formation dynamics converges to a consensus if reasonable conditions on the structure of the social networks and on how the personal beliefs can change are satisfied. Moreover, we discuss how this result applies in several simpler (but realistic) settings. Finally experimental evidence is given on how the opinion formation process works in practice.

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Introduction

It is well known that how individuals form their opinions and how they express them in a social context is strongly influenced by their social relations. For example, social pressure can suggest an individual to hide her personal (unpopular) belief and publicly express an opinion that is more conform to the opinions expressed by the majority of her neighbours. On the other hand, she can be firmly tied to her personal belief and not willing to deviate from it, whatever the opinions expressed by her friends are.

Thus, the formation of the opinions in a social context can be modelled as a dynamic process, where opinions publicly expressed by individuals depend on both their personal innate beliefs and on the opinions expressed by other individuals they interact with. Understanding how opinions form, how they diffuse in a social network and how the network can influence this process is a fundamental issue both for the Artificial Intelligence and Social Science communities.

The study of opinion diffusion in social networks dates back to 70s, with the seminal paper on Threshold Model by Granovetter (Granovetter 1978) and the papers of DeGroot (Degroot 1974) and Lehrer-Wagner (Lehrer and Wagner 1981) on consensus. In the last decades this line of research received great attention and several models have been proposed for describing the process of opinion formation and diffusion (see (Easley and Kleinberg 2010; Jackson 2008; Shakarian et al. 2015) for detailed surveys). Many works model the social influence by simply considering agents that follow the majority (Berger 2001; Feldman et al. 2014; Tamuz and Tessler 2013; Mossel, Neeman, and Tamuz 2014; Auletta, Ferraioli, and Greco).

A classical simple model that takes into account both the innate beliefs and the public opinions has been proposed in (Friedkin and Johnsen 1990) (in the following denoted as the FJ model). They assume that each agent has a (innate) belief and a publicly expressed opinion and these two elements are not necessarily the same. The process of formation and expression of the opinions in a social context proceeds in rounds: at each round all the agents make an averaging between their personal beliefs and the opinions expressed by other agents with whom they have social relations. The trade-off between the innate belief and the social pressure of the opinions expressed by her neighbors is weighted by the agent's stubbornness level, that is, the scaling factor used to counterbalance the cost that the agent incurs for disagreeing with the society and the cost she incurs for disagreeing with her innate belief. (Bindel, Kleinberg, and Oren 2011) prove that this repeated averaging process can be interpreted as a best-response play in a naturally defined game that leads to a unique equilibrium. (Ghaderi and Srikant 2014) determine the convergence time of this process to a stable state. Variations of the FJ model have been studied by (Fotakis, Palyvos-Giannas, and Skoulakis 2016), that assume agents update their opinions by consulting only a small (possibly random) subset of their neighbours, and by (Mossel, Sly, and Tamuz 2014), that assume agents update their opinions according to a Bayes rule that takes into account both its own initial preferences and what is declared by her neighbours.

A discrete version of the model of FJ model with binary opinions has been proposed in (Chierichetti, Kleinberg, and Oren 2018) in the setting of *Discrete Preference Games* (see also (Ferraioli, Goldberg, and Ventre 2012)). These games have also been considered to characterize the networks where the opinion hold by a minority of agents in the initial profile may become a majority in a stable state (Auletta et al. 2015; 2017b; 2017a). (Yildiz et al. 2013), instead, discuss a discrete model where there are some stubborn agents, that never deflect from their innate beliefs but they can influence other agents.

We observe that, although the opinion formation process in the FJ model is dynamic and evolves over time, the environment where dynamics runs is essentially static and decisions that agents take in each round are based on three fundamental ingredients that are assumed to be fixed: their personal innate beliefs; their stubbornness level; their social interactions, including both the set of the social relations and weights agents put on the opinions of their neighbors.

However, our real-life experience shows that the environment is not fixed but it co-evolves together with the opinions. Indeed, we can modify our social relations, getting to know new people, reinforcing or reducing interactions with people, changing our trust on them; our stubbornness levels may also change over time, for example for the social pressure on reaching an equilibrium; even our more consolidated beliefs may change due to the prolonged interactions with other social neighbours. We remark that all these processes run simultaneously and can interwind in very complex ways.

In the last years several works were presented that study the opinion formation process in settings where opinions and the network co-evolve under the effects of the mutual influence between the agents. (Bhawalkar, Gollapudi, and Munagala 2013) introduce game-theoretic models of opinion formation in social networks and they investigate the existence and the efficiency of stable states both for the discrete and the continuous process in these models. (Bilò, Fanelli, and Moscardelli 2016) consider opinion formation games in a setting where opinions and social relationships co-evolve in a cross-influencing manner and give bounds on the price of anarchy and price of stability which only depend on the individuals' stubbornness. (Ferraioli and Ventre 2017a) study opinion formation games in a setting where the social pressure to reach an agreement makes the agents' stubbornness levels decrease with time. They characterize the graphs for which consensus is guaranteed and study the complexity of checking whether a graph satisfies such a condition.

A different but related approach to describe the coevolution of the opinions and the network structure is given by the Bounded Confidence Model of (Hegselmann and Krause 2002), where the structure of the social interaction graph reflects the affinity of the agents' opinions and it is updated during the process.

All the discussed papers consider environments where only some of the ingredients we are considering can evolve. In this paper, instead, we want to study how opinions form and are publicly expressed in a social context when the environment fully evolves and both the structure of the social interaction graph, the agents' stubbornness levels and even their innate beliefs can change over time. We remark that, at the best of our knowledge, this is the first work that considers fully evolving environments, in particular with respect to the evolution of the innate beliefs.

As in (Friedkin and Johnsen 1990) we assume that the process works in rounds and at each round agents takes an average between her innate beliefs and the opinions expressed by her neighbours. However, the dynamics is divided in epochs of finite (maybe different) length: for the whole length of an epoch each agent has a fixed belief, a stubbornness level and a set of social relations; at the beginning of each epoch each agent can change her stubbornness level, belief and social relations.

Our main contribution is the proof that in the general model, where the environment fully evolves with opinions, under mild conditions on the structure of the social graphs used in each epoch and on how the innate beliefs of the agents can change over time, the opinion formation dynamics is ergodic and converges to a consensus. We also show that if we assume the social network fixed, then it is sufficient that the graph is strongly connected and aperiodic to guarantee the convergence of the dynamics to a consensus. In order to show the versatility of our model, we present several simple but realistic settings, such as the DeGroot model (Degroot 1974) (in the following denoted as DG model) and its generalizations, and discuss how they can casted in our model by conveniently setting the length of the epochs, the way beliefs are updated or how the stubborness levels may change. Thus, our results on the convergence to consensus of the opinion formation dynamics can be extended to all these settings. Finally we present some preliminary experimental evidence on the convergence time of the dynamics.

The model

In this section we present a model of opinion formation in social networks where the environment fully co-evolves with the expressed opinions, i.e., the stubborness levels and the innate beliefs of the agents as well as their social interactions change over time.

For every integer $k \geq 1$, we define $[k] := \{1, \dots, k\}$. We say that a matrix has dimension k, for any $k \geq 1$, if it is made of k rows and k columns. For every matrix A of dimension k, we denote by A_{ij} the entry of A at the i-th row and the j-th column and by $\langle A \rangle_i$ the sum of the entries in the i-th row, i.e., $\langle A \rangle_i := \sum_{j=1}^{k} A_{ij}$. We say that A is *stochastic* if $\langle A \rangle_i = 1$, for every $i \in [k]$. We denote by $\mathbb{I}_k$ the *identity* matrix of dimension k.

We are given a set of $n \geq 2$ agents $[n]$, each one holding opinions in $[0, 1]$. The opinion formation process evolves in rounds and at each round $t \geq 0$, every agent i holds an *expressed opinion* $z_i(t) \in [0, 1]$ and an *innate belief* $g_i(t) \in [0, 1]$. We assume that $z_i(0) = g_i(0)$. For every $t \geq 1$, the opinion $z_i(t)$ expressed by agent i at time t is a function of the agent's stubborness at round t, of her innate belief $g_i(t-1)$ and of the opinions $z_j(t-1)$ expressed by her neighbors at round $t - 1$. The innate belief $g_i(t)$ at round t, instead, is a function of the opinions expressed by agent i in the previous rounds (i.e., $z_i(0), z_i(1), \dots, z_i(t-1)$). A more detailed description of how these two values are computed at each round will be given in the following.

We assume that our formation process works in a fully dynamic environment, where the relationships among the agents are modelled by a *dynamic social interaction graph* which changes over time as the agents' innate beliefs and stubborness levels. To this aim, we divide the process into *epochs*, where each epoch is the time interval in which stubborness levels, innate beliefs and social relations are fixed, but they all can change when a new epoch starts.

Every epoch $\ell \geq 0$ starts at round $\rho_\ell + 1$ and consists of a finite number of rounds $h(\ell)$. We set $\rho_0 = 0$. We denote by $R(\ell)$ the set of rounds belonging to epoch ℓ, where $R(\ell) = \{\rho_\ell + 1, \cdots, \rho_\ell + h(l)\}$ and $\rho_{\ell+1} = \rho_\ell + h(\ell)$.

For each epoch $\ell \geq 0$ and for each agent i, we denote by $b_i(\ell) \in [0, 1]$ the innate belief hold by agent i in the epoch ℓ and by ${w_{ii}}^{(\ell)} \geq 0$ her *stubbornness* in this epoch, that is the weight the agent puts on her innate belief. Moreover, for every pair of agents i, j, with $i \neq j$, we denote by ${w_{ij}}^{(\ell)} \geq 0$ the strength by which the opinion of i is influenced by j during the epoch ℓ (set $w_{ij}^{(\ell)} = 0$ if i is not influenced by j). We write ${W_i}^{(\ell)}$ to denote the summation $\sum_{j=1}^{n} {w_{ij}}^{(\ell)}$ and assume that ${W_i}^{(\ell)} > 0$.

At each round $t \in R(\ell)$ the agent i sets her expressed opinion according to the following rule

$$z_i(t) = \frac{1}{{W_i}^{(\ell)}} \left({w_{ii}}^{(\ell)} g_i(t-1) + \sum_{j \neq i} {w_{ij}}^{(\ell)} z_j(t-1) \right). \quad (1)$$

We denote by $\mathbf{z}(t) = (z_1(t), z_2(t), \ldots, z_n(t))$ and $\mathbf{g}(t) = (g_1(t), g_2(t), \ldots, g_n(t))$ the profiles of the expressed opinions and the innate beliefs of all the agents at time $t \geq 1$. For each ℓ, let $E^{(\ell)}$ be the matrix of dimension n such that ${E_{ij}}^{(\ell)} = \frac{{w_{ij}}^{(\ell)}}{{W_i}^{(\ell)}}$ for each $i \neq j$, and ${E_{ii}}^{(\ell)} = 0$ for each $i \in [n]$. Let $S^{(\ell)}$ be the matrix of dimension n such that ${S_{ij}}^{(\ell)} = 0$ for each $i \neq j$, and ${S_{ii}}^{(\ell)} = \frac{{w_{ii}}^{(\ell)}}{{W_i}^{(\ell)}}$ for $i \in [n]$. Then, (1) can be rewritten in a more compact way as

$$\begin{aligned} \mathbf{z}(t) &= S^{(\ell)} \mathbf{g}(t-1) + E^{(\ell)} \mathbf{z}(t-1) \\ &= S^{(\ell)} \mathbf{b}(\ell) + E^{(\ell)} \mathbf{z}(t-1), \end{aligned} \quad (2)$$

where the last equality follows by the assumption that during an epoch agents maintain the innate beliefs they computed at the beginning of the epoch (i.e., $g(t-1) = \mathbf{b}(\ell)$). Notice that, $S^{(\ell)} + E^{(\ell)}$ is clearly a stochastic matrix, for each $\ell \geq 0$.

In this work we look at a particular form of evolution of the innate belief profile $\mathbf{b}(\ell) = (b_1(t), b_2(t), \ldots, b_n(t))$. Indeed, we assume that the innate belief computed by agent i at the beginning of the epoch ℓ is obtained as a convex combination of all the opinions previously expressed by i and of the initial belief $b_i(0)$. More formally, for every round $t \in [0, \rho_{\ell-1} + h(\ell-1)]$, let $c_{it}^{(\ell)} \geq 0$ be the weight that agent i assigns to the opinion she expressed at round t and assume that $\sum_{t=0}^{\rho_{\ell-1}+h(\ell-1)} c_{it}^{(\ell)} = 1$. We define, $b_i(\ell) = z_i(0)$ if $\ell = 0$, while $b_i(\ell) = \sum_{t=0}^{\rho_{\ell-1}+h(\ell-1)} c_{it}^{(\ell)} z_i(t)$ otherwise. Compactly, for every epoch $\ell \geq 0$, $\mathbf{b}(\ell)$ is defined as

$$\mathbf{b}(\ell) := \begin{cases} \mathbf{z}(0) & \textit{if } \ell = 0 \\ \sum_{t=0}^{\rho_\ell} C^{(\ell,t)} \mathbf{z}(t) & \textit{if } \ell \geq 1, \end{cases} \quad (3)$$

where $C^{(\ell,t)}$ is a $n \times n$ diagonal matrix such that $C_{ii}^{(\ell,t)} = c_{it}^{\ell}$ ($C_{ij}^{(\ell,t)} = 0$ for each $j \neq i$)). Observe that, since $\rho_\ell = \rho_{\ell-1} + h(\ell-1)$, (3) can be rewritten as

$$\mathbf{b}(\ell) = C^{(\ell,0)} \mathbf{z}(0) + \sum_{r=0}^{\ell-1} \sum_{k=1}^{h(r)} C^{(\ell,\rho_r+k)} \mathbf{z}(\rho_p + k). \quad (4)$$

Before proceeding with our presentation we want to spend a few words to discuss relations between our model and the well-known models of Friedkin and Jonsen (Friedkin and Johnsen 1990) and by DeGroot (Degroot 1974). Observe that the FJ model assumes the environment is totally static. In our setting, we can see it as a process consisting of only one epoch of unbounded length. Their results cannot be derived in our model, since we assume that each epoch must have finite length. The DG model does not consider innate beliefs and stubborness and assumes the social graph fixed. At each round agents update their opinion by averaging on all the opinions announced in the previous round, including her own opinion. In our setting this model can be seen as a model with (possibly) infinite epochs of length 1 where at each round the belief is set equal to the opinion announced in the previous round.

Characterizing $\mathbf{z}(t)$. In the sequel we show that, for each $t \geq 0$, $\mathbf{z}(t)$ can be expressed in terms of $\mathbf{z}(0)$ and of a certain stochastic matrix depending on t.

Let $t \in R(\ell)$ such that $t = \rho_\ell + j$, for $j \in [h(\ell)]$. Observe that, by recursively applying (2), we have that

$$\begin{aligned} &\mathbf{z}(\rho_\ell + j) = S^{(\ell)} \mathbf{b}(\ell) + E^{(\ell)} \mathbf{z}(\rho_\ell + j - 1) \\ &= S^{(\ell)} \mathbf{b}(\ell) + E^{(\ell)} [S^{(\ell)} \mathbf{b}(\ell) + E^{(\ell)} \mathbf{z}(\rho_\ell + j - 2)] \\ &= \left(S^{(\ell)} + E^{(\ell)} S^{(\ell)} \right) \mathbf{b}(\ell) \\ &\quad + \left(E^{(\ell)} \right)^2 [S^{(\ell)} \mathbf{b}(\ell) + E^{(\ell)} \mathbf{z}(\rho_\ell + j - 3)] = \cdots \\ &= \left[S^{(\ell)} + \left(\sum_{i=1}^{j-1} \left(E^{(\ell)} \right)^i \right) S^{(\ell)} \right] \mathbf{b}(\ell) + \left(E^{(\ell)} \right)^j \mathbf{z}(\rho_\ell). \end{aligned}$$

Thus, when $\ell = 0$, since $\mathbf{b}(0) = \mathbf{z}(0) = \mathbf{z}(\rho_0)$, we have that

$$\begin{aligned} &\mathbf{z}(\rho_0 + j) \\ &= \left[S^{(0)} + \left(\sum_{i=1}^{j-1} \left(E^{(0)} \right)^i \right) S^{(0)} + \left(E^{(0)} \right)^j \right] \mathbf{z}(0). \end{aligned} \quad (5)$$

When $\ell \geq 1$, instead, by (3) and (4) we have that

$$\begin{aligned} \mathbf{z}(\rho_\ell + j) = &\left[S^{(\ell)} + \left(\sum_{i=1}^{j-1} \left(E^{(\ell)} \right)^i \right) S^{(\ell)} \right] \sum_{k=0}^{\rho_\ell} \left(C^{(\ell,k)} \mathbf{z}(k) \right) \\ &+ \left(E^{(\ell)} \right)^j \mathbf{z}(\rho_\ell). \end{aligned}$$

Let us set $A^{(\ell,j)} = \left[S^{(\ell)} + \left(\sum_{i=1}^{j-1}\left(E^{(\ell)}\right)^{i}\right)S^{(\ell)}\right]$ and $B^{(\ell,j)} = \left(E^{(\ell)}\right)^{j}$. It is not hard to check that these matrices enjoy the following property.

Lemma 1. *For every $\ell \geq 0$ and every $j = 1, \ldots, h(\ell)$, the matrix*

$$M^{(\ell,j)} = A^{(\ell,j)} + B^{(\ell,j)} \tag{6}$$

is stochastic.

Moreover, we have that

$$\begin{aligned}\mathbf{z}(\rho_\ell + j) &= \\ &A^{(\ell,j)}\left(C^{(\ell,0)}\mathbf{z}(0) + \sum_{r=0}^{\ell-1}\sum_{k=1}^{h(r)} C^{(\ell,\rho_r+k)}\mathbf{z}(\rho_r + k)\right) \\ &+ B^{(\ell,j)}\mathbf{z}(\rho_{\ell-1} + h(\ell-1)),\end{aligned} \tag{7}$$

Let $T^{(\ell,j)}$ be a matrix of dimension n defined as follows: if $\ell = 0$ then

$$T^{(\ell,j)} = A^{(\ell,j)} + B^{(\ell,j)}; \tag{8}$$

if $\ell \geq 1$ then

$$\begin{aligned}T^{(\ell,j)} = A^{(\ell,j)} &\left[C^{(\ell,0)} + \sum_{r=0}^{\ell-1}\sum_{k=1}^{h(r)} C^{(\ell,\rho_r+k)}T^{(r,k)}\right] \\ &+ B^{(\ell,j)}T^{(\ell-1,h(\ell-1))}.\end{aligned} \tag{9}$$

Next Theorem gives an interesting characterization of $\mathbf{z}(\rho_\ell + j)$ in terms of $\mathbf{z}(0)$ and $T^{(\ell,j)}$.

Theorem 1. *For every $\ell \geq 0$ and every $j = 1, \ldots, h(\ell)$, the profile of opinions at time $t = \rho_\ell + j$ can be expressed as*

$$\mathbf{z}(\rho_\ell + j) = T^{(\ell,j)}\mathbf{z}(0), \tag{10}$$

and $T^{(\ell,j)}$ is a stochastic matrix.

Proof. The proof is by induction on ℓ. Let us first prove that it holds for $\ell = 0$. To this aim, observe that $T^{(0,j)} = M^{(0,j)}$, where $M^{(0,j)}$ is defined in (6). Then, by (5), $\mathbf{z}(\rho_0 + j) = T^{(0,j)}\mathbf{z}(0)$. Moreover, by Lemma 1, it follows that $T^{(0,j)}$ is a stochastic matrix, as desired.

Consider now the case $\ell \geq 1$ and suppose, by inductive hypothesis, that $\mathbf{z}(\rho_r + k) = T^{(r,j)}\mathbf{z}(0)$ for every $r < \ell$ and every $k = 1, \ldots, h(r)$. Then, from (7), we have that

$$\begin{aligned}\mathbf{z}(\rho_\ell + j) = \Bigg[A^{(\ell,j)}&\left(C^{(\ell,0)} + \sum_{r=0}^{\ell-1}\sum_{k=1}^{h(r)} C^{(\ell,\rho_r+k)}T^{(r,k)}\right) \\ &+ B^{(\ell,j)}T^{(\ell-1,h(\ell-1))}\Bigg]\mathbf{z}(0) = T^{(\ell,j)}\mathbf{z}(0).\end{aligned} \tag{11}$$

We now show that $\left\langle T^{(\ell,j)}\right\rangle_i = 1$ for any $i \in [n]$. To this aim, let $D^{(\ell)} = C^{(\ell,0)} + \sum_{r=0}^{\ell-1}\sum_{k=1}^{h(r)}\left(C^{(\ell,\rho_r+k)}T^{(r,k)}\right)$. Observe that $\left(C^{(\ell,\rho_r+k)}T^{(r,k)}\right)_{ij} = T_{ij}^{(r,k)} \cdot C_{ii}^{(\ell,\rho_r+k)}$. Then, since $T^{(r,k)}$ is stochastic by the inductive hypothesis, it follows that $\left\langle C^{(\ell,\rho_r+k)}T^{(r,k)}\right\rangle_i = C_{ii}^{(\ell,\rho_r+k)}$, and, consequently, $\left\langle D^{(\ell)}\right\rangle_i = 1$. Then,

$$\begin{aligned}\left\langle T^{(\ell,j)}\right\rangle_i &= \sum_k \left[\left(A^{(\ell,j)}D^{(\ell)}\right)_{ik} + \left(B^{(\ell,j)}T^{(\ell-1,h(\ell-1))}\right)_{ik}\right] \\ &= \sum_k\sum_m \left(A_{im}^{(\ell,j)}D_{mk}^{(\ell)} + B_{im}^{(\ell,j)}T_{mk}^{(\ell-1,h(\ell-1))}\right) \\ &= \sum_m \left(A_{im}^{(\ell,j)}\sum_k D_{mk}^{(\ell)} + B_{im}^{(\ell,j)}\sum_k T_{mk}^{(\ell-1,h(\ell-1))}\right) \\ &= \sum_m \left(A_{im}^{(\ell,j)} + B_{im}^{(\ell,j)}\right) = \sum_m M_{im}^{(\ell,j)} = 1.\end{aligned}$$

where the third equality follows from the fact that both $D^{(\ell)}$ and $T^{(\ell-1,h(\ell-1))}$ are stochastic (for the former, this has been showed above, for the latter it follows by inductive hypothesis). The latter equality follows from Lemma 1. □

Convergence to Consensus

In this section we present the main result of this work. We prove that, under reasonable conditions related to structural properties of the social interaction graphs and to how the innate beliefs evolve, the opinion formation dynamics $\mathbf{z}(t)$ converges to a consensus.

In order to state our conditions, we consider the sequence of graphs $\mathcal{G} = \left\{G^{(t)}\right\}_{t\geq 1}$, where the graph $G^{(t)}$ encodes the social relations active during the round t of the dynamics. Indeed, if $t \in R(\ell)$, then $G^{(t)}$ is a directed graph defined over the agents in $[n]$ such that there is an arc from i to $j \neq i$ if and only if $w_{ij}^{(\ell)} > 0$, meaning that i is influenced by j during round t.

For every $x, y \in [n]$, we say that there exists an *influence path* of length k from x to y in $\mathcal{G}$ if there exists a sequence of agents $(y = \omega_k, \ldots, \omega_0 = x)$ sucht that the arc (ω_j, ω_{j-1}) exists in $G^{(j)}$ for each $j = 1, \ldots, k$. Intuitively, this means that the influence of x on y goes through the first k rounds of the dynamics and the opinions expressed by y at round k are positively influenced by the initial opinion of x.

Let $\mathcal{P}_{xy}$ be the set of all the influence paths from x to y (of any length) in $\mathcal{G}$. We say that the sequence of graphs $\mathcal{G}$ is *ergodic* if there is a positive integer t^* such that for every round $t > t^*$ and for every $x, y \in [n]$ there is a path of length t in $\mathcal{P}_{xy}$. Observe that ergodicity implies that, after a sufficently large number of rounds, the opinion of each agent is influenced from the initial opinions of all the other agents.

Moreover, for every $t \geq 0$, let $\mu(t)$ be number of epochs whose beliefs are positively influenced by the opinions expressed by the agents at round t. Then, $\mu(t) = \max_i \left|\left\{\ell\colon c_{it}^{(\ell))} > 0\right\}\right|$. Observe that, if $\mu(t)$ is finite then the opinions expressed at round t have a direct influence on the innate beliefs of only a finite number of successive epochs and their influence vanishes in the long time.

In the following we prove that the ergodicity of $\mathcal{G}$ and the finiteness of all the $\mu(t)$ are sufficient to guarantee that the opinion formation dynamics $\mathbf{z}(t)$ converges to a consensus.

Theorem 2. *If $\mathcal{G} = \left\{G^{(t)}\right\}$ is ergodic and $\mu(t)$ is finite for each $t \geq 0$, then the profile $\mathbf{z}(t)$ converges to a consensus as t goes to infinity.*

To prove the theorem we need two technical lemmas. Next lemma shows that when $\mathcal{G}$ is ergodic we can deduce useful properties on the matrices $T^{(\ell,j)}$.

Lemma 2. *Let $\mathcal{G} = \left\{G^{(t)}\right\}$ be an ergodic sequence of graphs. Then there is ℓ_0 such that for each $\ell \geq \ell_0$ and for each $j = 1, \dots, h(\ell)$, $T^{(\ell,j)}_{xy} > 0$ for each $x, y \in [n]$.*

Next lemma instead states that, under the hypothesis of Theorem 2, the matrix $T^{(\ell,j)}$ tends to a matrix whose rows are all equal to some stochastic vector π.

Lemma 3. *Let $\mathcal{G} = \left\{G^{(t)}\right\}$ be an ergodic sequence of graphs. If $\mu(t)$ is finite for each $t > 0$, then there is a stochastic vector π with n entries such that for each $j \in \{1, \dots, h(\ell)\}$ we have that $\lim_{\ell \to \infty} T^{(\ell,j)} = \pi \mathbf{1}$, where $\mathbf{1}$ is the column vector with n entries all equal to 1.*

Proof Sketch. Let ℓ_0 as defined in Lemma 2 and fix $\ell > \ell_0$, $j \in \{1, \dots, h(\ell)\}$ and $z \in [n]$. We define $L(\ell)$ as the set containing all time steps t such that the beliefs at epoch ℓ depend on the opinions at step t, i.e., $L(\ell) = \{(r, k) \colon C^{(\ell, \rho_r + k)} \neq 0\}$. We also define $L^+(\ell) = L(\ell) \cup \{(\ell - 1, h(\ell - 1)\}$.

First, observe that if there is $\pi(y)$ such that $T^{(r,k)}_{wy} = \pi(y)$ for every $(r, k) \in L^+(\ell)$ and every $w \in [n]$ then we can state that $T^{(\ell,j)}_{xy} = \pi(y)$ for every $x \in [n]$.

Suppose now that such a $\pi(y)$ does not exist. Hence there is $(\hat{r}, \hat{k}) \in L^+(\ell)$ and $\hat{w} \in [n]$ such that $T^{(\hat{r},\hat{k}))}_{\hat{w}y} < \max_{(r,k) \in L^+(\ell)} \max_w \left\{T^{(r,k)}_{wy}\right\}$. Similarly, by Lemma 2, there must be $(\check{r}, \check{k}) \in L^+(\ell)$ and $\check{w} \in [n]$ such that $T^{(\check{r},\check{k})}_{\check{w}y} > \min_{(r,k) \in L^+(\ell)} \min_w \left\{T^{(r,k)}_{wy}\right\} \geq 0$. Then, we can prove that $0 \leq \min_{(r,k) \in L^+(\ell)} \min_w \left\{T^{(\lambda,k)}_{wy}\right\} < T^{(\ell,j)}_{xy}$ $< \max_{(r,k) \in L^+(\ell)} \max_w \left\{T^{(r,k)}_{wy}\right\}$. The lemma then follows by proving that this implies that

$$\lim_{\ell \to \infty} \left(\max_{(r,k) \in L^+(\ell)} \max_w \left\{T^{(\lambda,k)}_{wy}\right\} - \min_{(r,k) \in L^+(\ell)} \min_w \left\{T^{(\lambda,k)}_{wy}\right\} \right) = 0. \qquad \square$$

Armed with these two lemmas, we are now ready to formally prove Theorem 2.

Proof of Theorem 2. From Theorem 1 and Lemma 3, it follows that for every i, when ℓ goes to infinity,

$$\mathbf{z}_i(\rho_\ell + j) = \sum_{j \in [n]} T^{(\ell,j)}_{ij} \mathbf{z}_j(0) = \sum_{j \in [n]} \pi_j \mathbf{z}_j(0) = z^*,$$

and thus the process converges to a consensus on z^*. $\square$

Notice that, since in our model the epochs have finite length, when t goes to infinity we are assuming that there are infinite epochs and each agent changes her innate beliefs an infinite number of times. This property is crucial to our proof. On the other hand, if there would be an epoch ℓ with $h(\ell) = \infty$, then our dynamics would be equivalent to run the FJ model with starting opinions $\mathbf{z}(\rho_\ell)$ and with beliefs $\mathbf{b}(\ell)$. In this case, it is known that the dynamics converge to a profile that is not necessarily a consensus.

Suppose instead that there exists some round t such that $\mu(t) = \infty$. It is easy to see that in this case the dynamics cannot converge to a consensus. In fact, suppose that a consensus is not reached within time t but it is reached in a round $t' > t$ in the same epoch. Observe that the belief update made at the beginning of the new epoch breaks the consensus. It is left open to understand whether and when, under these same assumptions, the process converges at all.

Finally, observe that the convergence to a consensus of the opinion formation dynamics can be proved even when $\mathcal{G}$ is not ergodic. Indeed, it is immediate to check that Lemma 3, and consequently Theorem 2, holds as long as the following property holds:

Property 1. There is ℓ_0 such that $T^{(\ell,j)}_{xy} > 0$ for every $\ell \geq \ell_0$, every $j = 1, \dots, h(\ell)$, and every $x, y \in [n]$.

Property 1 is a generalization of the standard assumptions for convergence of stochastic processes. There are two ways in which can Property 1 fail, leading to two different behaviors for our dynamics: in one case we can prove that the dynamics still converge but not to a consensus; in the other case the dynamics does not converge at all.

First assume that there is a pair of nodes x, y such that $T^{(\ell,j)}_{xy} = 0$ for every ℓ sufficiently large. In this case, we can partition the nodes in $N_1, \dots, N_k$ such that Property 1 holds for the subset of nodes N_i, for every i. Each partition N_i will be classified as *recurrent* if $T^{(\ell,j)}_{xy} = 0$ for every ℓ sufficiently large and every $x \in N_i$ and $y \notin N_i$, and *transient* otherwise. Note that our assumption implies that there is at least one recurrent partition (the one containing x). Observe that we can apply Theorem 2 to each recurrent partition. Hence, the opinion of agents in these partitions will converge to a consensus (even if different consensus can be reached in different recurrent partitions). As for an agent x in transient partitions, it occurs that $T^{(\ell,j)}_{xy} = 0$ for every y that is not in a recurrent partition, as ℓ goes to infinity. Hence, it follows that the opinion of x converges to a combination of the consensi of the recurrent partitions.

Suppose instead that for every x, y and for every ℓ_0 there are (ℓ, j) and (ℓ', j') with $\ell, \ell' \geq \ell_0$ such that $T^{(\ell,j)}_{xy} > 0$ but $T^{(\ell',j')}_{xy} = 0$. This implies that the opinion of x changes infinitely often, so no convergence result is possible.

Environments with fixed social graph. We conclude this section with a brief discussion related to the case where the graph of social interactions does not change among the epochs, that is $G^{(t)} = G$ for each t.

We show that in this case the condition of ergodicity of $\mathcal{G}$ is much simpler and intuitive. We say that a directed graph G is *strongly connected* if for every $x, y \in [n]$, there is at least a directed path from x to y in G. Moreover, we say that G

is *aperiodic* if the maximum common divisor of the lengths of cycles in the graph is 1 (i.e., it does not occur that all cycles have a length that is a multiple of c for some $c > 1$). It is not hard to see (e.g., through arguments in (Levin and Peres 2017, Lemma 1.7)) that if G is strongly connected and aperiodic, the sequence of graphs $\mathcal{G}$ where $G^{(t)} = G$ for each t is ergodic. Hence, we have the following theorem.

Theorem 3. *If G is strongly connected and aperiodic, and $\mu(t)$ is finite for each $t \geq 0$, then the profile $\mathbf{z}(t)$ converges to a consensus as t goes to infinity.*

Applications

In this section, we show how the convergence result for the general model presented in previous section can be used to prove convergence to consensus of opinion formation dynamics in simpler (but realistic) settings where the environment partially evolves.

For sake of presentation, in the rest of section we assume that the graph G encoding the social relations among the agents is fixed and does not change over time. As discussed in previous section, if G is strongly connected and aperiodic, then $\mathcal{G}$ is clearly ergodic.

Belief as the last opinion of the previous epoch. We already observed that the DG model (Degroot 1974) can be easily phrased in our framework, by simply assuming that stubborness levels of all the agents are constant, $h(\ell) = 1$ for every $\ell \geq 0$, and innate beliefs are updated at each round to the opinion expressed in the previous round (i.e., $\mathbf{b}((\ell) = \mathbf{z}(\rho_\ell)$, for each $\ell \geq 1$).

A natural generalization of this model would consist in assuming that innate beliefs are not updated at every round but they are maintained for a certain (finite) number of rounds. At the beginning of each epoch the innate belief of each agent is set equal to the last opinion she expressed in the previous epoch. This generalization of the DG model can be easily represented in our framework by having constant stubborness levels, allowing epochs to have any finite length, and setting $C^{(\ell,t)} = 1$ if $t = \rho_\ell$ and 0 otherwise.

By Theorem 3 our opinion formation dynamics converges in this setting to a consensus if the graph G is strongly connected and aperiodic. Actually, it turns out that for this setting Property 1 holds even if we assume that G is strongly connected (but not necessarily aperiodic) and there is at least one agent with non-zero stubborness level.

Instead to directly prove this fact, we believe it is more illustrative to show that in this setting we can give a much simpler proof of convergence to consensus. Observe that the opinion profile annuonced in each round of epoch ℓ is $\mathbf{z}(\rho_\ell + t) = E\mathbf{z}(\rho_\ell + t - 1) + S\mathbf{z}(\rho_\ell)$, where $E = E^{(0)}$ and $S = S^{(0)}$. Note that $E = E^{(\ell)}$ and $S = S^{(\ell)}$, for every ℓ, since both the graph and the stubborness levels are assumed to do not change over time. Iterating, we have $\mathbf{z}(\rho_\ell + t) = E^{(t)}\mathbf{z}(\rho_\ell) + \sum_{r=0}^{t-1} E^{(r)}S\mathbf{z}(\rho_\ell)$ and $\mathbf{z}(\rho_{\ell+1}) = Eh(\ell)\mathbf{z}(\rho_\ell) + \sum_{r=0}^{h(\ell)-1} E^{(r)}S\mathbf{z}(\rho_\ell)$. For every integer $k \geq 1$, let $T^{\{k\}} := E^{(k)} + \sum_{r=0}^{k-1} E^{(r)}S$. Then, we achieve that $\mathbf{z}(\rho_{\ell+1}) = T^{\{h(\ell)\}}\mathbf{z}(\rho_\ell)$. By iterating on all the previous epochs and using $\rho_0 = 0$, we obtain that for every epoch $\ell \geq 0$

$$\mathbf{z}(\rho_{\ell+1}) = \left(\prod_{i=\ell}^{0} T^{\{h(i)\}}\right)\mathbf{z}(0). \tag{12}$$

Now observe that, by Lemma 1, $T^{\{k\}}$ is a stochastic matrix for every $k \geq 1$. Moreover, it is immediate to see that since the graph G is strongly connected, then the matrix E is *irreducible*, i.e., for every x, y there is an integer t such that $E^t(x, y) > 0$. Hence, it immediately follows that also E^k and $T^{\{k\}}$ are irreducible, for every $k \geq 1$. Moreover, it is immediate to see that if there is at least one agent x with non-zero stubborness level, then, for every k the matrix $P = E^k + S$ is *aperiodic*, i.e., the greatest common divisor of the element in the set $\{t \geq 1 \colon P^t(x, x) > 0\}$ is 1. Consequently, $T^{\{k\}}$ is aperiodic, too. Finally, observe that, by the assumption, there is M such that $h(\ell) \leq M$ for every ℓ. Hence, even if ℓ goes to infinity, there are always finitely many different matrices $T^{\{k\}}$ involved in the product of (12). These conditions (finiteness of the number of matrices involved in the product, and the fact that all these matrices are stochastic, irreducibile and aperiodic) are sufficient to imply that $\mathbf{z}(\rho_\ell)$ tends to a consensus as ℓ goes to infinity (see, e.g., (Coppersmith and Wu 2008, Theorem 5)).

We conclude the analysis of this setting, by observing that when all epochs have the same length, i.e. $h(\ell) = h$ for every ℓ, then (12) gives that $\mathbf{z}(\rho_\ell) = \left(T^{\{h\}}\right)^\ell \mathbf{z}(0)$, i.e., the evolution of opinions can be described through the evolution of a Markov chain with transition matrix $T^{\{h\}}$. This enables the usage of the tools of Markov chain theory to analyze more in detail the evolution of opinions in this setting.

Belief as an average of the previous epoch's opinions. Another natural generalization of the DG model consists in assuming that the beliefs at epoch ℓ depend on all the opinions expressed in the epoch $\ell - 1$ (but do not depend on the opinions expressed in any epoch $r < \ell - 1$). For example, one could assume that the belief at epoch ℓ is a (discounted) average of opinions expressed in the epoch $\ell - 1$.

This model can be easily represented in our framework, by assuming, as above, constant stubborness levels and allowing $h(\ell)$ to assume any finite value. However, now we need to choose $C^{(\ell,t)}$ so that they are positive if $t \in R(\ell-1)$ and 0 otherwise.

By Theorem 3, it follows that even in this setting the opinion fornation dynamics converges to consensus when the relation graph G is strongly connected and aperiodic. However, it is not hard to verify that Property 1 still holds if G is strongly connected (but not necessarily aperiodic) and there is at least one agent with non-zero stubborness level[1].

Social pressure to consensus. Ferraioli and Ventre (2017b) introduced a generalization of the FJ model, in which the weight that agents assign to the opinions of the neighbors increases over time (and consequently the weigth on their beliefs proportionally reduce) to model settings where there is a pressure on the agents on reaching a consensus within

[1]Notice that the analysis described for the previous case does not apply to this setting and we need different techniques.

an upcoming deadline. Even if the model in Ferraioli and Ventre is quite different from ours (e.g., they assume opinions are discrete), we can import some of their ideas in our framework.

Indeed, it is sufficient to assume that the stubborness level $w_{ii}^{(\ell)}$ decreases as ℓ increases, even though the belief remains the same over time, i.e. $C^{(\ell,t)} = 0$ for every $t \neq 0$, and $C^{(\ell,0)} = 1$. Note that in this setting $\mu(0) = \infty$, and hence Theorem 2 does not necessarily imply convergence to consensus in this setting. However, if for ℓ going to infinity, the stubborness levels go to 0, then, at the limit, the model turns out to be equivalent to the DG model, and hence convergence to consensus is guaranteed under the same hypothesis as the DG model.

Agents with heterogeneous epochs. In all previous settings we assumed that epochs are homogeneous among agents and at the beginning of each epoch all the agents update their beliefs and stubborness levels. However, we show that our framework is powerful enough to allow agents with heterogeneous epochs. That is, we assume that for every agent i there is an infinite list of time steps $\left(t_i^{(1)}, t_i^{(2)}, \ldots\right)$ at which i is allowed to update her belief and, possibly, her stubborness level. Assume for sake of simplicity, that at these time steps i set her belief to the last opinion expressed in the previous epoch (but our framework allows to implement more complex dependences on history).

In order to model the dynamics in this setting, we define epochs as follows: let $\rho_0 = \min_i t_i^{(1)}$, and for each $\ell > 0$, let $\rho_\ell = \min_i \min_{r\colon t_i^{(r)} > \rho_{\ell-1}} \left\{t_i^{(r)}\right\}$. Hence, each epoch ends as soon as there is an agent that would like to update her belief. Moreover, for every epoch ℓ, we set $C_{ii}^{(\ell,\rho_\ell)} = 1$ if there is r such that $t_i^{(r)} = \rho_\ell$, and $C_{ii}^{(\ell,t^*)} = 1$ otherwise, where $\mathbf{z}(t^*)$ is the last opinion assumed as belief by i, i.e., $t^* = \max_{r\colon t_i^{(r)} < \rho_\ell} \left\{t_i^{(r)}\right\} - 1$. That is, if i is one of the agents that is supposed to change her belief at time step ρ_ℓ, then it sets the belief exactly as the last expressed opinion, otherwise it simply copies the last assumed belief.

Note that $h(\ell)$ and $\mu(\ell)$ are finite as long as each agent updates her belief at infinitely many time steps. Hence, we can conclude that this model, even if agents update their beliefs at different time steps, eventually converges to consensus if the social network is strongly connected and aperiodic, and each agent updates her belief infinitely often.

Experiments

Theorem 2 proves that in our model of opinion formation in fully evolving environments, under reasonable conditions on the structure of the social interaction graphs and on how the beliefs evolve, a large class of dynamics converge to a consensus. However, Theorem 2 does not give any bound on the convergence time.

To give some experimental evidence of how fast our dynamics is in converging we run some preliminary experiments on a very simple setting where all the epochs have the same length, the agent stubborness levels are fixed while the beliefs are updated in each epoch to the last opinion expressed in the previous epoch. Moreover, we assumed that the social interaction graph G is a clique (each agent has social interactions with all the other agents) with $w_{ij} = 1$ for each i and $j \neq i$ and it does not change over time. Observe that, as discussed in the previous section, in this setting our dynamics is modelled as a Markov chain.

We believe that even if the chosen setting is very simple it is sufficient to highlight the intrinsic properties of the model, without dealing with bottlenecks potentially introduced by a wrong or adversarial choice of some parameters (e.g., graphs contrasting the diffusion or increasing stubborness levels).

In particular, we focused on two aspects. Our first experiment evaluates how the convergence time varies with respect to n. To this aim, we run the dynamics on graphs with a number of nodes ranging from 10 to 1000 and for each size we measured how much rounds the dynamics executed before reaching a consensus. We set $\ell = \sqrt{n}$ in order to make the different runs comparable. For each configuration, we run 25 experiments by choosing the stubborness level of each agent uniformly at random in $\{1, \ldots, n+1\}$. Thus, the weight put on the belief goes from $\frac{1}{n}$ to more than $\frac{1}{2}$. Figure 1 shows our results. Notice that the convergence time is low and it

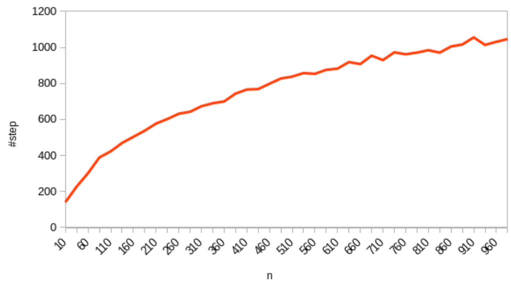

Figure 1: Converge time as function of n

increases very slowly with n. Indeed, even with about 1000 agents, the dynamics takes about 1000 rounds to converge to consensus.

The second experiment evaluates the effect of the epoch length h on the convergence time. In this case, we set $n = 100$, and make h increases from 10 to 1000. As in the previous experiment, for each configuration we run 25 experiments in which stubborness levels are randomly drawn. The results of these experiments, are showed in Figure 2. We

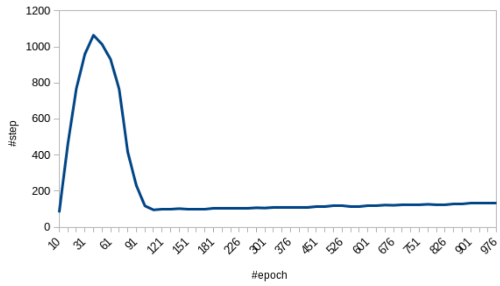

Figure 2: Converge time as function of $h(\ell)$

can observe that for small values of h the convergence time increases very quickly until it reaches its maximum for h

around 50; then it rapidly decreases and it stays quite constant for $h > 100$. Indeed, if h is sufficently large, then there is time in each epoch to reach the equilibrium of that epoch that, by our choice of parameters, occurs to be a consensus. Instead, when h is small, then more epochs are necessary.

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