Embedded eigenvalues of the Neumann problem in a strip with a box-shaped perturbation

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Abstract

We consider the spectral Neumann problem for the Laplace operator in an acoustic waveguide Π_l^ε formed by the union of an infinite strip and a narrow box-shaped perturbation of size $2l \times \varepsilon$, where $\varepsilon > 0$ is a small parameter. We prove the existence of the length parameter $l_k^\varepsilon = \pi k + O(\varepsilon)$ with any $k = 1, 2, 3, ...$ such that the waveguide $\Pi_{l_k^\varepsilon}^\varepsilon$ supports a trapped mode with an eigenvalue $\lambda_k^\varepsilon = \pi^2 - 4\pi^4 l^2 \varepsilon^2 + O(\varepsilon^3)$ embedded into the continuous spectrum. This eigenvalue is unique in the segment $[0, \pi^2]$, and it is absent in the case $l \neq l_k^\varepsilon$. The detection of this embedded eigenvalue is based on a criterion for trapped modes involving an artificial object, the augmented scattering matrix. The main difficulty is caused by the rather specific shape of the perturbed wall $\partial\Pi_l^\varepsilon$, namely a narrow rectangular bulge with corner points, and we discuss available generalizations for other piecewise smooth boundaries.

Résumé

Nous considérons le spectre du laplacien de Neumann dans un guide d'onde acoustique Π_l^ε formé d'une bande infinie perturbée par une cavité étroite de taille $2l \times \varepsilon$, où $\varepsilon > 0$ est un petit paramètre. Nous prouvons, pour chaque $k = 1, 2, 3, ,$ l'existence d'une longueur $l_k^\varepsilon = \pi k + O(\varepsilon)$ telle qu'il existe dans le guide d'onde $\Pi_{l_k^\varepsilon}^\varepsilon$ un mode piégé associé à une valeur propre $\lambda_k^\varepsilon = \pi^2 - 4\ pi^4 l^2 \varepsilon^2 + O(\varepsilon^3)$ plongée dans le spectre continu. Il s'agit de la seule valeur propre dans $[0, \pi^2]$, elle est de plus absente lorsque $l \neq l_k^\varepsilon$. La détection de cette valeur propre plongée est basée sur un critère pour les modes piégés impliquant un objet artificiel: la matrice de scattering augmentée. La principale difficulté vient de la forme spécifique de la perturbation du mur $\partial\Pi_l^\varepsilon$, c'est-à-dire un étroit renflement rectangulaire comportant des angles. Nous discutons également de possibles généralisations aux cas d'autres bords constants par morceaux.

Keywords:
acoustic waveguide, Neumann problem, embedded eigenvalues, continuous spectrum, box-shaped perturbation, piecewise smooth boundary, asymptotics

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1. Introduction

1.1. Formulation of the problems

In the union Π_l^ε, fig. 1, a and b, of the straight unit strip

$$\Pi = \left\{ x = (x_1, x_2) \in \mathbb{R}^2,\ x_1 \in \mathbb{R},\ x_2 \in (0,1) \right\} \tag{1.1}$$

and a rectangle of length $2l > 0$ and small width $\varepsilon > 0$,

$$\varpi_l^\varepsilon = \{ x : |x_1| < l, \quad x_2 \in (-\varepsilon, 0] \}, \tag{1.2}$$

we consider the spectral Neumann problem

$$-\Delta u^\varepsilon (x) = \lambda^\varepsilon u^\varepsilon (x), \quad x \in \Pi_l^\varepsilon = \Pi \cup \varpi_l^\varepsilon, \tag{1.3}$$

$$\partial_\nu u^\varepsilon (x) = 0,\ x \in \partial \Pi_l^\varepsilon, \tag{1.4}$$

where $\Delta = \nabla \cdot \nabla$ is the Laplace operator, $\nabla = \text{grad}$, λ^ε is the spectral parameter and $\partial_\nu = \nu \cdot \nabla$ is the directional derivative, ν stands for the unit outward normal defined everywhere at the boundary $\partial \Pi_l^\varepsilon$, except for the corner points, i.e., the vertices of the rectangle (1.2). Since a solution of the problem (1.3), (1.4) may get singularities at these points, the problem ought to be reformulated as the integral identity [43]

$$(\nabla u^\varepsilon, \nabla v^\varepsilon)_{\Pi_l^\varepsilon} = \lambda^\varepsilon (u^\varepsilon, v^\varepsilon)_{\Pi_l^\varepsilon} \ \ \forall v^\varepsilon \in H^1(\Pi_l^\varepsilon), \tag{1.5}$$

where $(\ ,\ )_{\Pi_l^\varepsilon}$ is the natural scalar product in the Lebesgue space $L^2(\Pi_l^\varepsilon)$ and $H^1(\Pi_l^\varepsilon)$ stands for Sobolev space. The symmetric bilinear form on the left-hand side of (1.5) is closed and positive in $H^1(\Pi_l^\varepsilon)$ so that the problem (1.3), (1.4) is associated [6, Ch 10] with a positive self-adjoint operator A_l^ε in $L^2(\Pi_l^\varepsilon)$ whose spectrum $\wp = \wp_{co}$ is continuous and covers the closed positive semi-axis $\overline{\mathbb{R}}_+ = [0, +\infty)$. The domain $\mathcal{D}(A_l^\varepsilon)$ of A_l^ε, of course, is contained in $H^1(\Pi_l^\varepsilon)$ but it is bigger than $H^2(\Pi_l^\varepsilon)$ due to singularities of solutions at the corner points, see, e.g., [59, Ch.2]. The discrete spectrum of the operator A_l^ε clearly is empty but its point spectrum $\wp_{po}$, consisting of embedded eigenvalues inside the continuous spectrum, can be non-empty. The main goal of our paper is to single out a particular value of the length parameter l such that the operator A_l^ε gets an eigenvalue $\lambda_l^\varepsilon \in \wp_{po}$ embedded into the continuous spectrum $\wp_{co}$. The corresponding eigenfunction $u_l^\varepsilon \in H^1(\Pi_l^\varepsilon)$ decays exponentially at infinity and is called a trapped mode, cf. [44] and [63]. We use "trapped mode" as a synonym of "eigenfunction" throughout the paper.

Our main result, formulated below in Theorem 3, roughly speaking, demonstrates that an eigenvalue λ_l^ε exists in the interval $(0, \pi^2) \subset \wp_{co}$ for $l^\varepsilon \approx \pi k$ with $k \in \mathbb{N} = \{1, 2, 3, ...\}$ only. To obtain this result, we in particular construct asymptotics of λ^ε and l^ε as $\varepsilon \to +0$.

The problem (1.3), (1.4) is a model of an acoustic waveguide with hard walls, cf. [49], but is also related in a natural way to the linear theory of surface water-waves, cf. [42]. Indeed, the velocity potential $\Phi^\varepsilon(x, z)$ satisfies the Laplace equation in the channel $\Xi_{l,d}^\varepsilon = \Pi_l^\varepsilon \times (-d, 0) \subset \mathbb{R}^3 \ni (x, z)$ of depth $d > 0$ with the Neumann condition (no normal flow) at

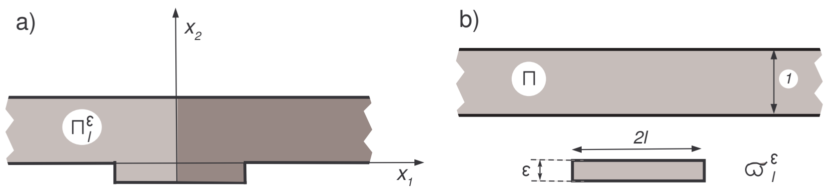

Figure 1: The waveguide with a box-shaped perturbation (a) and its fragments (b).

its vertical walls and horizontal bottom, as well as the spectral Steklov condition (the kinetic one) on the free horizontal surface

$$\partial_z \Phi^\varepsilon (x, 0) = \Lambda^\varepsilon \Phi^\varepsilon (x, 0) , \quad x \in \Pi_l^\varepsilon .$$

After factoring out the dependence on the vertical variable z,

$$\Phi^\varepsilon (x, z) = u^\varepsilon (x) \left(e^{z\lambda^\varepsilon} + e^{-(z+2d)\lambda^\varepsilon} \right) , \tag{1.6}$$

see, e.g., [29] and [44], the water-wave problem reduces to the two-dimensional Neumann problem (1.3), (1.4) for the function u^ε in (1.6) and the parameter λ^ε determined through the equation

$$\Lambda^\varepsilon = \lambda^\varepsilon \frac{1 - e^{-2d\lambda^\varepsilon}}{1 + e^{-2d\lambda^\varepsilon}} = \lambda^\varepsilon \tanh (d\lambda^\varepsilon) .$$

We will not discuss this interpretation of our problem separately.

1.2. Asymptotics of eigenvalues in the Dirichlet and mixed boundary-value problems

Imposing the Dirichlet condition

$$u^\varepsilon (x) = 0, \quad x \in \partial \Pi_l^\varepsilon , \tag{1.7}$$

instead of the Neumann condition (1.4), one creates the positive cut-off value $\lambda_\dagger = \pi^2$ of the continuous spectrum $\wp_{co}^D = [\pi^2, +\infty)$ of the Dirichlet problem (1.3), (1.7) which, for example, provides an adequate model of a quantum waveguide, cf. [30, 33]. The interval $(0, \pi^2)$ now stays below the continuous spectrum and, therefore, may contain eigenvalues composing the discrete spectrum $\wp_{di}^D \subset \wp_{po}^D$ of the problem (1.3), (1.7). As follows from a result in [18], the total multiplicity $\# \wp_{di}^D$ of the discrete spectrum $\wp_{di}^D$ is equal to 1 for a small $\varepsilon > 0$. Although the paper [18] deals with a regular (smooth) perturbation of the wall, it is possible to select two smooth shallow pockets as in fig. 2, a and b, and to extend the existence and uniqueness results in [18] for the box-shaped perturbations by means of the max-min principle, see, e.g., [6, Thm 10.2.2]. However, the corresponding asymptotic formula

$$\lambda_l^\varepsilon = \pi^2 - 4\pi^4 \varepsilon^2 l^2 + O\left(\varepsilon^3\right) , \quad \varepsilon \to 0, \tag{1.8}$$

cannot be based on these results because the change of variables which transforms Π_l^ε into Π, used in [18], requires certain smoothness properties of the boundary $\partial \Pi_l^\varepsilon$, and these are of course missing now for the waveguide in fig. 1, a. In Section 8.3, we will explain how our approach helps to justify the asymptotic formula (1.8).

Figure 2: The box-shaped perturbation entering (a) and enveloping (b) the regular perturbation.

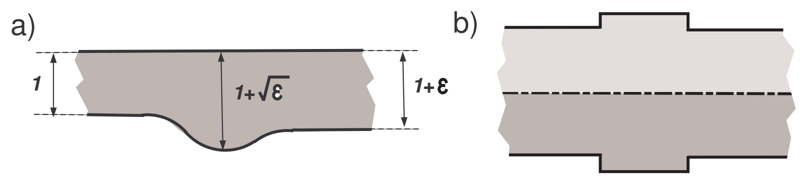

Figure 3: A non-local perturbation (a) and a symmetric box-shaped perturbation (b).

Local perturbations of quantum waveguides in $\mathbb{R}^n$, $n \geq 2$, have been intensively investigated during the last two decades and many interesting and significant results on the existence and asymptotic behavior of their discrete spectra have been published. We mention a few of them, namely [45, 28, 31, 30] for the slightly curved and twisted cylindrical waveguides, [4, 53, 60] for cranked waveguides and [34, 32] for the Laplacian perturbed by a small second-order differential operator with compactly supported coefficients. We also refer to [24] for non-local perturbations, fig. 3, a, to [18, 12, 13, 15, 16] for alternation of the Dirichlet and Neumann boundary conditions and to [5, 17, 21, 22, 19, 20, 25, 26] for other type of perturbations.

In the literature, one finds much fewer results on eigenvalues embedded into the continuous spectrum, cf. the review papers [8, 11, 44]. First of all, we describe an elegant approach [29] which is based on imposing an artificial Dirichlet condition and has been rather widely used in detecting embedded eigenvalues but only in symmetric waveguides.

Let us consider an auxiliary mixed boundary value problem and supply the Helmholtz equation (1.3) with the Neumann condition on the lower lagged wall and the Dirichlet condition on the upper straight wall, see fig. 1, a,

$$u^\varepsilon(x_1, 1) = 0, \quad x_1 \in \mathbb{R}, \qquad \partial_\nu u^\varepsilon(x) = 0, \quad x \in \partial\Pi_l^\varepsilon, \ x_2 < 1. \tag{1.9}$$

The problem (1.3), (1.9) has the continuous spectrum $\wp_{co}^M = [\pi^2/4, +\infty)$ and in Section 8.1,

Figure 4: Singular (a) and regular (b) perturbations of waveguides.

we will show the existence of only one eigenvalue

$$\lambda_l^\varepsilon = \frac{\pi^2}{4}(1 - \pi^2 l^2 \varepsilon^2) + O(\varepsilon^3 (1 + |\ln \varepsilon|)^2), \quad \varepsilon \to +0, \tag{1.10}$$

in the discrete spectrum $\wp_{di}^M \subset (0, \pi^2/4)$. Following [29], we extend the corresponding eigenfunction $u_l^\varepsilon(x_1, x_2)$ as an odd function in $x_2 - 1$ from Π_l^ε on the bigger waveguide $\widehat{\Pi}_l^\varepsilon$ drawn in fig. 3, b and obtained as the union of the strip $\mathbb{R} \times (0, 2)$ and the large box $(-l, l) \times (-\varepsilon, 2 + \varepsilon)$. Owing to the Dirichlet condition in (1.9) on the central line of $\widehat{\Pi}_l^\varepsilon$, this extension $\widehat{u}_l^\varepsilon(x_1, x_2)$ is a smooth function everywhere in $\widehat{\Pi}_l^\varepsilon$, except at the corner points, and inherits from $u_l^\varepsilon(x_1, x_2)$ the exponential decay at infinity. Clearly,

$$-\Delta \widehat{u}_l^\varepsilon(x) = \lambda_l^\varepsilon \widehat{u}_l^\varepsilon(x), \quad x \in \widehat{\Pi}_l^\varepsilon, \quad \partial_\nu \widehat{u}_l^\varepsilon(x) = 0, \qquad x \in \partial \widehat{\Pi}_l^\varepsilon, \tag{1.11}$$

and thus $\widehat{u}_l^\varepsilon$ is an eigenfunction of the Neumann problem (1.11) while the corresponding eigenvalue (1.10) belongs to the continuous spectrum $\widehat{\wp}_{co} = [0, +\infty)$ of this problem.

We emphasize that the method [29] requires the mirror symmetry of the waveguide and cannot be applied to the asymmetric waveguide Π_l^ε in fig. 1, a. The detected embedded eigenvalue λ_l^ε of the Neumann problem (1.11) is stable with respect to small symmetric perturbations of the waveguide walls but any violation of the symmetry may lead it out of the spectrum and turn it into a point of complex resonance, cf. [3] and, e.g., [56].

The intrinsic instability of embedded eigenvalues requires special techniques to detect them, as well as to describe their asymptotics. In the present paper, we use a criterion for the existence of trapped modes (see [39] and Theorem 1 below) and a concept of enforced stability of eigenvalues in the continuous spectrum (cf. [54, 56] and Section 1.4).

1.3. Reduction of the problem

In view of the natural mirror symmetry of the domain Π_l^ε about the x_2-axis, notice the apparent difference with the above-mentioned symmetry assumption in [29], we truncate the waveguide and consider the Neumann problem

$$-\Delta u_+^\varepsilon(x) = \lambda_+^\varepsilon u_+^\varepsilon(x), \quad x \in \Pi_{l+}^\varepsilon, \tag{1.12}$$

$$\partial_\nu u_+^\varepsilon(x) = 0, \qquad x \in \partial \Pi_{l+}^\varepsilon, \tag{1.13}$$

in its right half (overshaded in fig. 1, a)

$$\Pi_{l+}^\varepsilon = \{x \in \Pi_l^\varepsilon : x_1 > 0\} = \{x : x_2 \in (-\varepsilon, 0) \text{ for } x_1 \in (0, l), \ x_2 \in (0, 1) \text{ for } x_1 \geq l\}. \tag{1.14}$$

Clearly, an even extension of an eigenfunction u_+^ε of the reduced problem (1.12), (1.13) with respect to the variable x_1 becomes an eigenfunction of the original problem (1.3), (1.4). Searching for an eigenvalue

$$\lambda^\varepsilon \in \left(0, \pi^2\right), \tag{1.15}$$

we will show in Section 7 that, first, the Neumann problem (1.12), (1.13) cannot get more than one eigenvalue in the interval $(0, \pi^2)$, and second, the mixed boundary value problem in (1.14) with the Dirichlet condition at the truncation segment $\{x : x_1 = 0, \ \ x_2 \in (-\varepsilon, 1)\}$ does not have eigenvalues in $(0, \pi^2)$ at all. These mean that an eigenfunction of the original problem (1.3), (1.4) associated with the eigenvalue (1.15) is always even in the variable x_1. In this way, we will be able to describe the part $\wp_{po} \cap (0, \pi^2)$ of the point spectrum in the entire waveguide Π_l^ε. In what follows we skip the subscript l. Hence, we regard (1.5) as an integral identity serving for the problem (1.12), (1.13) in $\Pi_+^\varepsilon := \Pi_{l+}^\varepsilon$ and denote the corresponding self-adjoint operator in $L^2\left(\Pi_+^\varepsilon\right)$ by A_+^ε, cf. Section 1.1.

1.4. Enforced stability of embedded eigenvalues and the procedure of "fine tuning"

The point spectrum of a self-adjoint operator consists of isolated and embedded eigenvalues. Isolated eigenvalues compose the discrete spectrum and are stable with respect to a small perturbation of the operator. The other eigenvalues belong to the continuous spectrum and are characterized by the intrinsic instability, i.e., a small perturbation of the operator may lead them out of the spectrum and turn into points of complex resonance, cf. [3, 44] and others. The distinct properties of these two kinds of eigenvalues require for different mathematical tools for their detection in particular problems.

In the discrete spectrum variational methods, like the max-min principle, cf. [6, Thm. 10.2.2], work quite well and have provided many examples of isolated eigenvalues below the continuous spectrum, see the papers [4, 18, 24, 31, 32, 34, 45], reviews in [8, 11, 30, 44] and other publications, as well as the problem (1.3), (1.7) in Section 1.2.

Furthermore, an elegant approach proposed in [29] also permits an application of variational methods. Indeed, restricting the operator on a subspace of positive codimension may move the cutoff value of the continuous spectrum upwards and reveal points of the newly formed discrete spectrum which become embedded eigenvalues of the original operator. We have used this approach in Section 1.2 while considering the mixed boundary value problem (1.3), (1.9).

The discrete spectrum in our problem (1.3), (1.4) is empty and the waveguide Π_l^ε in fig. 1, a, does not posses a geometrical symmetry which allows to create an artificial positive cutoff value, cf. Section 1.3. Hence, to construct a box-shaped perturbation supporting a trapped mode, we engage a much more technical and involved approach proposed in [37, 38] and based on a sufficient condition [39] of exponentially decaying solutions to elliptic boundary-value problems in domains with cylindrical outlets to infinity, that is, trapped modes in waveguides of various physical nature. This approach has got successful implementations which resulted in several examples of eigenvalues in the continuous spectra of waveguides without any available geometrical and physical symmetries. These are obtained from the straight unit strip by either singular [37, 38, 55], or regular [52, 54, 56] perturbations, see fig. 4, a and b, respectively, as well as for a non-local smooth perturbation [23] in fig. 3, a. To this end, the notion of the augmented scattering matrix [39] was used together with certain traditional asymptotic methods for elliptic problems in domains with small holes and cavities, cf. [46, Ch. 4,5 and 2], or in domains with smoothly varied boundaries [40, Ch. XII, §6.5].

To fulfill the above-mentioned sufficient condition for trapped modes which becomes a criterion for our particular problem (1.3), (1.4), see Theorem 1, one has to carefully tune some geometrical parameters in order to support the enforced stability of the embedded eigenvalue, i.e., to keep it on the real axis. In the cases of singular and regular perturbation of the boundary considered in the preceeding studies, the asymptotic procedures involve the formal analyses and operator reformulations of the diffraction problems in the framework of

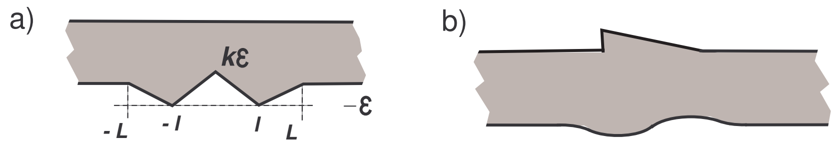

Figure 5: Other type of piecewise smooth perturbations

the perturbation theory of linear operators, cf. [35, 40]. This helps to get a smooth (the Lipschitz continuity is usually sufficient) dependence of the augmented scattering matrix on geometrical parameters describing the waveguide shape and to reduce the criterion to a nonlinear abstract equation with a contraction operator which has a unique solution due to the Banach principle, cf. Sections 2.3 and 4.1, 4.2 below.

The box-shaped perturbation (1.1) of the strip can be regarded as a combination of regular and singular perturbations, respectively outside and inside small neighborhoods of the corner points, cf. Section 3.1, but unfortunately the authors do not know any tool to reduce the problem (1.3), (1.4) (or (1.5) in the variational form) to an abstract equation in a fixed, that is, independent of ε, Banach space and to confirm the necessary properties of the augmented scattering matrix by a direct reference to general results. Instead, each step of the traditional scheme to detect an embedded eigenvalue and to prove its uniqueness requires to get a new result for the problems (1.3), (1.4) and/or (1.12), (1.13).

Although the approach and many tricks in our paper work also for other shapes like in fig. 5, we describe the novel analysis for the particular but typical shape in fig. 1, a, in order to minimize the technicalities. However, it is remarkable that the unstable embedded eigenvalue can be obtained by tuning only one geometrical parameter, namely length l of the box ϖ_l^ε.

The new technique developed in our paper can also be applied in other problems where fine tuning of geometrical parameters is needed to provide some unstable phenomena, for example, the non-reflectibility of obstacles in waveguides at prescribed frequencies. Examples of particulary shapes supporting the latter phenomenon have been presented in [9, 10] and [7, 27], however again only for regular and singular perturbations in fig. 4.

1.5. Structure of the paper

We proceed in Section 2 by introducing different waves in Π_+^ε, namely, oscillatory and exponential for $\lambda^\varepsilon \in (0, \pi^2)$ and linear in x_1 at the threshold $\lambda^\varepsilon = \pi^2$. Then on the basis of the Mandelstam energy principle, cf. [59, § 5.3] and [57], we perform the classification *incoming/outgoing* for the introduced waves and impose two physical and artificial types of radiation conditions at infinity. The corresponding diffraction problems give rise to two scattering matrices s^ε and S^ε, see Section 2.2. Due to the restriction of the boundary value problem (1.3), (1.4) on the semi-infinite waveguide (1.14), the matrix s^ε reduces to a scalar reflection coefficient but the augmented scattering matrix S^ε with the entries S_{jk}^ε is of size 2×2 because the artificial radiation conditions involve exponential waves in addition to oscillatory waves. The above-mentioned criterion for the existence of embedded eigenvalues is formulated in terms of the matrix S^ε, see Theorem 1 and its proof is completed in Theorem 7.

In Section 3, we construct formal asymptotic expansions of the augmented scattering matrix which are justified in Section 6.4. In order to detect an embedded eigenvalue in Section 4, we need the main asymptotic and first correction terms. The two-fold asymptotic nature of the box-shaped perturbation, see Section 3.1, manifests itself in different asymptotic ansätze for the diagonal entries S^ε_{11} and S^ε_{00} of the matrix S^ε. In the first case, the asymptotic procedure looks like a regular perturbation of the boundary, that is, the boundary layer phenomenon does not influence the main asymptotic term S^ε_{11} in the expansion

$$S^\varepsilon_{11} = S^0_{11} + \widehat{S}^\varepsilon_{11}, \tag{1.16}$$

with a small remainder $\widehat{S}^\varepsilon_{11}$ examined in Section 6.

In the second case, the correction term S'_{00} in the similar expansion

$$S^\varepsilon_{00} = 1 + \varepsilon S'_{00} + \widehat{S}^\varepsilon_{00} \tag{1.17}$$

results purely from the boundary layer phenomenon while the regular expansion affects higher-order terms only, see Appendix. It should be emphasized that the augmented scattering matrix is unitary and symmetric, and the main asymptotic term in the expansion

$$S^\varepsilon_{01} = S^\varepsilon_{10} = \varepsilon^{1/2} S'_{10} + \widehat{S}^\varepsilon_{10} \tag{1.18}$$

of the anti-diagonal entries can be computed by both approaches.

In Section 4, we first reduce the criterion $S^\varepsilon_{11} = -1$ from Theorem 1 to an abstract equation and then solve it with the help of the Banach contraction principle. Finally, we formulate Theorems 3 and 4 on the existence and uniqueness of the embedded eigenvalue. These assertions are proved in the next three sections. In Section 5 we present formulations of the problem (1.3), (1.4) in the Kondratiev spaces (Theorem 5) and weighted spaces with detached asymptotics (Theorem 9) as well as the operator formulation of the radiation condition at infinity that is an important step in the scheme of our analysis. At the same time, the key results for the particular box-shaped perturbation (1.2), are displayed in Sections 5.3 and 5.5 where we verify the absence of trapped modes with fast decay and clarify the dependence of majorants in a priori estimates for solutions on the small and spectral parameters, respectively, $\varepsilon \in (0, \varepsilon_0)$ and $\lambda \in (0, \pi^2)$.

In Section 6, we evaluate remainders in the asymptotic formulas (1.16)-(1.18) for the augmented scattering matrix (Theorem 12) while the boundary layer phenomenon brings additional powers of $|\ln \varepsilon|$ into the bounds in the final estimates. Other necessary properties of the scattering matrix are clarified in Section 7, where the uniqueness of the embedded eigenvalue is verified too. Again, corners of the box-shaped perturbation seriously complicate all proofs.

Final remarks are collected in Section 8, where, in particular, we discuss essential simplifications of the analysis within the discrete spectrum and certain difficulties in detecting eigenvalues near higher thresholds of the continuous spectrum. Some technical details of the asymptotic procedures are specified in Appendix.

2. Augmented scattering matrix and a criterion for trapped modes

2.1. Classification of waves.

Regarding the unit strip (1.1) as a limit of $\Pi_l^\varepsilon = \Pi \cup \varpi_l^\varepsilon$ as $\varepsilon \to +0$, we consider the Neumann problem

$$-\Delta u^\varepsilon(x) = \lambda^\varepsilon u^\varepsilon(x), \quad x \in \Pi, \quad \partial_\nu u^\varepsilon(x) = 0, \ x \in \partial\Pi, \tag{2.1}$$

with the spectral parameter (1.15) depending on ε. The functions

$$w_0^{\varepsilon\pm}(x) = (2k^\varepsilon)^{-1/2} e^{\pm ik^\varepsilon x_1}, \quad k^\varepsilon = \sqrt{\lambda^\varepsilon}, \tag{2.2}$$

which are solutions to the problem (2.1), represent oscillatory waves that are incoming or outgoing, depending on the sign in the exponent. Due to the Sommerfeld principle, see, e.g., [33, 49, 65], in the case of a compactly supported right-hand side f^ε the inhomogeneous Neumann problem

$$-\Delta u^\varepsilon(x) - \lambda^\varepsilon u^\varepsilon(x) = f^\varepsilon(x), \quad x \in \Pi_+^\varepsilon, \qquad \partial_\nu u^\varepsilon(x) = 0, \quad x \in \partial\Pi_+^\varepsilon, \tag{2.3}$$

is supplied with the radiation condition

$$u^\varepsilon(x) = c_0^\varepsilon w_0^{\varepsilon+}(x) + \widetilde{u}^\varepsilon(x), \quad \widetilde{u}^\varepsilon(x) = O(e^{-x_1\sqrt{\pi^2-\lambda^\varepsilon}}). \tag{2.4}$$

In Section 5, we will give an operator formulation of the diffraction problem (2.3), (2.4).

At the threshold

$$\lambda = \pi^2 \tag{2.5}$$

in addition to the oscillatory waves

$$w_0^{0\pm}(x) = (2\pi)^{-1/2} e^{\pm i\pi x_1}, \tag{2.6}$$

the standing wave w_1^0, independent of the longitudinal coordinate x_1, and the linear wave w_1^1 which grows when $x_1 \to +\infty$ appear, namely

$$w_1^0(x) = \cos(\pi x_2) \ \text{ and } w_1^1(x) = x_1 \cos(\pi x_2).$$

These cannot be classified by the Sommerfeld principle. However, it was observed in [59, § 5.3], that first, the waves (2.2) verify the relations

$$q_R\left(w_p^{\varepsilon\pm}, w_q^{\varepsilon\pm}\right) = \pm i\delta_{p,q}, \quad q_R\left(w_p^{\varepsilon\pm}, w_q^{\varepsilon\mp}\right) = 0, \tag{2.7}$$

and second, the linear combinations

$$w_1^{0\pm}(x) = (x_1 \mp i)\cos(\pi x_2) \tag{2.8}$$

together with the waves (2.6) fulfill formulas (2.7) at $\varepsilon = 0$ too. Here, $\delta_{p,q}$ is the Kronecker symbol, $p, q = 0$ in the first case and $p, q = 0, 1$ in the second case, and q_R is a symplectic, that is, sesquilinear and anti-Hermitian, form of the energy transfer

$$q_R(w, v) = \int_0^1 \left(\overline{v(R, x_2)}\frac{\partial w}{\partial x_1}(R, x_2) - w(R, x_2)\overline{\frac{\partial v}{\partial x_1}(R, x_2)}\right) dx_2. \tag{2.9}$$

This form emerges from Green's formula in the truncated waveguide $\Pi_+^\varepsilon(R) = \{x \in \Pi_+^\varepsilon, x_1 \in (0,R)\}$ and, therefore, does not depend on the length parameter $R > l$ for any of the introduced waves and their linear combinations. Hence, we skip the subscript R in (2.7) and (2.9).

For the waves (2.7), sign of $\operatorname{Im} q\left(w_0^{\varepsilon\pm}, w_0^{\varepsilon\pm}\right)$ coincides with the sign of the wavenumber and, therefore, indicates the propagation direction. Analogously, we, following [59, §5.6], call the wave w_1^{0+} outgoing and the wave w_1^{0-} incoming in the waveguide Π_+^ε so that the problem (2.3) with $\lambda^\varepsilon = \pi^2$ ought to be supplied with the threshold radiation condition

$$u^\varepsilon(x) = c_0^\varepsilon w_0^{0+}(x) + c_1^\varepsilon w_1^{0+}(x) + \widetilde{u}^\varepsilon(x), \quad \widetilde{u}^\varepsilon(x) = O(e^{-\sqrt{3}\pi x_1}). \tag{2.10}$$

In Section 5, we will prove that this formulation of the problem at the threshold (2.5) provides an isomorphism in its operator setting.

As was demonstrated in [59, § 5.6] and [54], the form q is closely related to the Umov-Poyting vector [62, 61] so that both the radiation conditions (2.4) and (2.10) arise from the Mandelstam (energy) principle, see [59, §5.3] and, e.g., [57]. We emphasize that in Section 5, the operator formulation of the problems (2.3), (2.4) and (2.3), (2.10) in the weighted Sobolev spaces with detached asymptotics does not rely upon this physical interpretation and is based directly on the orthogonality and normalization conditions (2.7) for all outgoing and incoming waves introduced throughout this paper.

2.2. Scattering matrices and exponential waves

In the case (1.15), the incoming wave in (2.2) in a standard way, see, e.g., [33, 65], generates the following solution of the diffraction problem (1.12), (1.13):

$$\zeta_0^\varepsilon(x) = w_0^{\varepsilon-}(x) + s_{00}^\varepsilon w_0^{\varepsilon+}(x) + \widetilde{\zeta}_0^\varepsilon(x). \tag{2.11}$$

Here, the remainder $\widetilde{\zeta}_0^\varepsilon$ decays as $O(e^{-x_1\sqrt{\pi^2-\lambda^\varepsilon}})$ and s_{00}^ε is the reflection coefficient which satisfies the normalization condition $|s_{00}^\varepsilon| = 1$ due to conservation of energy.

In a quite similar way, it was demonstrated in [59, Ch. 5], see also [52, 54], on the basis of formulas (2.7), that, for the threshold case (2.5), one can determine the following solutions generated by the incoming (minus) waves in (2.6), (2.8):

$$\zeta_p^\varepsilon(x) = w_p^{0-}(x) + s_{0p}^\varepsilon w_0^{0+}(x) + s_{1p}^\varepsilon w_1^{0+}(x) + \widetilde{\zeta}_p^\varepsilon(x). \tag{2.12}$$

Here, $p = 0, 1$, $\widetilde{\zeta}_p^\varepsilon(x) = O(e^{-x_1\sqrt{3}\pi})$ and the coefficients s_{qp}^ε form the (threshold) scattering matrix s^ε of size 2×2. Furthermore, according to the normalization and orthogonality conditions (2.7) for waves (2.6), (2.8) and the relation $w_p^{0+}(x) = \overline{w_p^{0-}(x)}$, the matrix s^ε is unitary and symmetric, i.e.,

$$(s^\varepsilon)^{-1} = (s^\varepsilon)^*, \quad s^\varepsilon = (s^\varepsilon)^\top, \tag{2.13}$$

where $\top$ stands for transposition and $(s^\varepsilon)^* = \left(\overline{s^\varepsilon}\right)^\top$ is the adjoint matrix. It should be noticed that the dependence on the small parameter ε appears in (2.11) and (2.12) due to the perturbation of the waveguide and a verification of the basic properties (2.13) of scattering matrices requires only simple algebraic calculations which can be found, e.g., in [54, §2]. We only mention here that the solution row

$$\left(\overline{\zeta_0^\varepsilon}, \overline{\zeta_1^\varepsilon}\right)\left(s^\varepsilon\right)^{-1} = \left(\overline{w_0^{0-}}, \overline{w_1^{0-}}\right)\left(s^\varepsilon\right)^{-1} + \left(\overline{w_0^{0+}}, \overline{w_1^{0+}}\right) + \left(\overline{\widetilde{\zeta}_0^\varepsilon}, \overline{\widetilde{\zeta}_1^\varepsilon}\right)\left(s^\varepsilon\right)^{-1} = \tag{2.14}$$
$$= \left(w_0^{0+}, w_1^{0-}\right)\left(s^\varepsilon\right)^{-1} + \left(w_0^{0-}, w_1^{0-}\right) + \left(\overline{\widetilde{\zeta}_0^\varepsilon}, \overline{\widetilde{\zeta}_1^\varepsilon}\right)\left(s^\varepsilon\right)^{-1}$$

has the same couple $\left(w_0^{0-}, w_1^{0-}\right)$ of incoming waves and, therefore, may differ from the row $(\zeta_0^\varepsilon, \zeta_1^\varepsilon)$ by a trapped mode only, cf. Remark 10. This observation proves the relation $s^\varepsilon = (s^\varepsilon)^{-1}$ and, thus, derives the second property (2.13) from the well-known first one.

The solutions (2.12) as well as (2.19) below are properly defined by their decompositions at infinity but a proof of their existence requires long and routine calculations presented at length in the above-cited papers but omitted here because, in view of the operator formulation in Section 5, no new idea or technique is needed to adapt the results to piecewise smooth boundaries.

The reflection coefficient s_{00}^ε in formula (2.11) ought to be regarded as a (degenerate) scattering matrix of size 1×1 in view of the only couple of waves (2.2) which are able to transport energy along the waveguide (1.14). For example, dealing with the waves

$$v_1^{\varepsilon\pm}(x) = \left(k_1^\varepsilon\right)^{-1/2} e^{\pm k_1^\varepsilon x_1} \cos\left(\pi x_2\right), \qquad k_1^\varepsilon = \sqrt{\pi^2 - \lambda^\varepsilon} \tag{2.15}$$

which are exponentially decaying $(-)$ and growing $(+)$, one readily computes that

$$q\left(v_1^{\varepsilon\pm}, v_1^{\varepsilon\pm}\right) = 0 \tag{2.16}$$

but

$$q\left(v_1^{\varepsilon+}, v_1^{\varepsilon-}\right) = -q\left(v_1^{\varepsilon-}, v_1^{\varepsilon+}\right) = 1. \tag{2.17}$$

As was observed in [59, §5.6] and mentioned above, formula (2.16) annihilates the projection on the x_1-axis of the Umov-Poyting vector [62, 61] and, therefore, the waves (2.15) cannot transport energy. In the papers [37, 38] (see also [39] for general elliptic systems), the linear combinations of exponential waves

$$w_1^{\varepsilon\pm}(x) = 2^{-1/2}\left(v_1^{\varepsilon+}(x) \mp i v_1^{\varepsilon-}(x)\right) \tag{2.18}$$

were introduced. It is remarkable that thanks to (2.16) and (2.17), the waves (2.2) and (2.18) enjoy the conditions (2.7) with $p, q = 0, 1$. Indeed,

$$q\left(w_1^{\varepsilon\pm}, w_1^{\varepsilon\pm}\right) = \frac{1}{2} q\left(v_1^{\varepsilon+} \mp i v_1^{\varepsilon-}, v_1^{\varepsilon+} \mp i v_1^{\varepsilon-}\right) = \frac{1}{2}\left(\overline{(\pm i)} - (\pm i)\right) = \pm i,$$
$$q\left(w_1^{\varepsilon\pm}, w_1^{\varepsilon\pm}\right) = \frac{1}{2} q\left(v_1^{\varepsilon+} \mp i v_1^{\varepsilon-}, v_1^{\varepsilon+} \pm i v_1^{\varepsilon-}\right) = \frac{1}{2}\left(\overline{(\pm i)} - (\mp i)\right) = 0.$$

An important observation made in [59, Ch. 5] for the threshold case and used to create the solutions (2.12), was also employed in [37, 38, 39], see also [52], for the exponential waves (2.18), namely the normalization and orthogonality conditions (2.7) based on the Umov-Poynting concept and the Mandelstam (energy) principle, are sufficient to introduce complete analogues of all the objects in the classical scattering theory dealing with traditional oscillatory waves of type (2.6). In this way, the solutions

$$Z_p^\varepsilon(x) = w_p^{\varepsilon-}(x) + S_{0p}^\varepsilon w_0^{\varepsilon+}(x) + S_{1p}^\varepsilon w_1^{\varepsilon+}(x) + \widetilde{Z}_p^\varepsilon(x), \ \widetilde{Z}_p^\varepsilon(x) = O(e^{-x_1\sqrt{4\pi^2-\lambda^\varepsilon}}), \ p = 0, 1, \tag{2.19}$$

of the problem (1.12), (1.13) with the spectral parameter (1.15) imitate the diffraction solution (2.11), that is, they describe propagation of the incoming wave $w_p^{\varepsilon-}$, oscillatory at $p = 0$ and artificial exponential at $p = 1$, its scattering by the box-shaped resonator ϖ_l^ε and its reflection with the coefficients S_{0p}^ε of $w_0^{\varepsilon+}$ and S_{1p}^ε of $w_1^{\varepsilon+}$. The conditions (2.7) also gives the unitary property to the coefficient matrix $S^\varepsilon = \left(S_{qp}^\varepsilon\right)$ called the *augmented scattering matrix* [37, 38, 39]. Moreover, since $w_p^{\varepsilon+}(x) = \overline{w_p^{\varepsilon-}(x)}$, this matrix is symmetric, see [54, §2] again and our calculation (2.14).

In Section 5, we will give an operator formulation of the problem (2.3) at $\lambda^\varepsilon \in (0, \pi^2)$ with the radiation condition

$$U^\varepsilon(x) = C_0^\varepsilon w_0^{\varepsilon+}(x) + C_1^\varepsilon w_1^{\varepsilon+}(x) + \widetilde{U}^\varepsilon(x), \quad \widetilde{U}^\varepsilon(x) = O(e^{-x_1\sqrt{4\pi^2-\lambda^\varepsilon}}). \tag{2.20}$$

We recognize this condition as artificial because the right-hand side of (2.20) involves the exponentially growing wave $w_1^{\varepsilon+}$, see (2.18) and (2.15). It should be mentioned that, first, two solutions (2.19) compose a basis in the kernel of an operator of the problem in a weighted space of functions with a certain growth at infinity (see Remark 8) and, second, the solution U^ε of the problem (2.3) with the radiation condition (2.20) contains an exponentially growing term and, therefore, differs from the solution u^ε of the problem (2.3) with the spectral parameter $\lambda^\varepsilon \in (0, \pi^2)$ and the traditional radiation condition

$$u^\varepsilon(x) = C_0^\varepsilon w_0^{\varepsilon+}(x) + \widetilde{u}^\varepsilon(x), \quad \widetilde{u}^\varepsilon(x) = O(e^{-x_1\sqrt{\pi^2-\lambda^\varepsilon}}).$$

2.3. A criterion for trapped modes

The next assertion explains the very reason for considering exponentially growing solutions of the problem with the artificial radiation conditions (2.20), namely the augmented scattering matrix S^ε is an indicator of trapped modes.

Theorem 1. *The problem (1.12), (1.13) with the spectral parameter (1.15) has a trapped mode $u^\varepsilon \in H^1\left(\Pi_+^\varepsilon\right)$, that is, an eigenfunction of the problem corresponding to its eigenvalue λ^ε, if and only if*

$$S_{11}^\varepsilon = -1. \tag{2.21}$$

In other words, the equation (2.21) provides a criterion for the existence of a trapped mode in the one-sided waveguide (1.14).

A verification of Theorem 1 can be found, e.g., in [39] and [54, Thm 2] but since the criterion (2.21) plays the central role in our analysis, we here give a condensed proof.

The unitary property of S^ε demonstrates that

$$S_{11}^\varepsilon = -1 \qquad \Rightarrow \qquad S_{10}^\varepsilon = S_{01}^\varepsilon = 0. \tag{2.22}$$

Thus, the solution (2.19) with $p = 1$ becomes a trapped mode because formulas (2.18) and (2.15) assure that

$$Z_1^\varepsilon(x) = w_1^{\varepsilon-}(x) - w_1^{\varepsilon+}(x) + \widetilde{Z}_1^\varepsilon(x) = -2^{1/2} i v_1^{\varepsilon-}(x) + \widetilde{Z}_1^\varepsilon(x) = O(e^{-x_1 k_1^\varepsilon}).$$

Hence, (2.21) is a sufficient condition. To verify the necessity, we first assume that the decomposition

$$U^\varepsilon(x) = c^\varepsilon v^{\varepsilon-}(x) + \widetilde{U}^\varepsilon(x) \tag{2.23}$$

of a trapped mode $U^\varepsilon \in H^1(\Pi_+^\varepsilon)$ has a coefficient $c^\varepsilon \neq 0$. Then U^ε becomes a linear combination of the solutions (2.19), namely, according to (2.18), we have

$$\begin{aligned} U^\varepsilon = C_0^\varepsilon Z_0^\varepsilon + C_1^\varepsilon Z_1^\varepsilon = C_0^\varepsilon w_0^{\varepsilon-} + (S_{00}^\varepsilon C_0^\varepsilon + S_{01}^\varepsilon C_1^\varepsilon) w_0^{\varepsilon+} + \\ + 2^{-1/2}(v_1^+ - i v_1^-) C_1^\varepsilon + 2^{-1/2}(v_1^+ + i v_1^-)(S_{10}^\varepsilon C_0^\varepsilon + S_{11}^\varepsilon C_1^\varepsilon) + \widetilde{U}^\varepsilon. \end{aligned}$$

Owing to the exponential decay of U^ε, coefficients of the oscillatory waves $w_0^{\varepsilon+}$ must vanish so that $C_0^\varepsilon = 0$ and $S_{01}^\varepsilon C_1^\varepsilon = 0$. Moreover, coefficients of the exponential waves $v_0^{\varepsilon+}$ and $v_0^{\varepsilon-}$, respectively, are $2^{-1/2}(S_{11}^\varepsilon + 1) C_1^\varepsilon = 0$ and $2^{-1/2}(S_{11}^\varepsilon - 1) C_1^\varepsilon = c^\varepsilon$. Recalling our assumption $c^\varepsilon \neq 0$, we see that $C_1^\varepsilon = -2^{-1/2} c^\varepsilon \neq 0$ and, therefore, (2.21) holds true.

If $c^\varepsilon = 0$ in (2.23), the trapped mode $U^\varepsilon(x)$ gains very fast decay rate $O(e^{-x_1\sqrt{4\pi^2 - \lambda^\varepsilon}})$. In Section 5.3 we will take into account the peculiar shape of the waveguide in fig. 1, a, and will show with a new argument that such trapped modes do not exist for a small ε.

Remark 2. . *The relationship between the augmented scattering matrix and the reflection coefficient in (2.11) looks as follows:*

$$s_{00}^\varepsilon = S_{00}^\varepsilon - S_{01}^\varepsilon (S_{11}^\varepsilon + 1)^{-1} S_{10}^\varepsilon, \tag{2.24}$$

see, e.g., [54, Thm 3]. Note that, in view of (2.22), the last term in (2.24) becomes null in the case $S_{11}^\varepsilon = -1$ when $s_{00}^\varepsilon = S_{00}^\varepsilon$. ⊠

3. Formal asymptotics of the augmented scattering matrix

3.1. Step-shaped perturbation of boundaries

In this section, we derive asymptotic expansions by means of a formal asymptotic analysis and postpone the justification of asymptotics to Section 6.

Perturbation of the straight waveguide drawn in fig. 1, a and in fig. 3, b, ought to be regarded as a combination of regular and singular perturbations, see, e.g., [40, Ch.XII, § 6.5] and [46, Ch. 2 and 4], respectively. For a regular perturbation of the boundary, an appropriate change of variables, which differs from the identity in magnitude $O(\varepsilon)$ only, is usually applied in order to convert the perturbed domain into the reference one. In this way, differential operators in the problem gain small perturbations but the asymptotics of solutions to the perturbed boundary-value problem can be constructed by means of standard iterative procedures, for example, decomposition in the Neumann series. However, this "rectification" procedure requires a sufficient smoothness of boundaries because, otherwise, coefficients of the obtained differential operators produce strong singularities. Clearly, the boundary $\partial\Pi_l^\varepsilon$

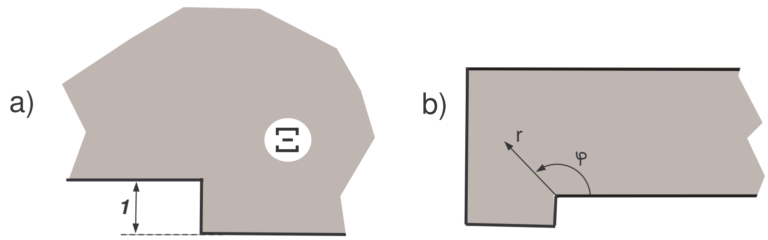

Figure 6: A distorted half-plane (a) to describe the boundary layer near the ledge (b).

in fig. 1, a, is not smooth and the corner points providing a singular perturbation of the boundary have to be inspected with other tools.

Singular perturbations of boundaries need much more delicate analysis because they need a description of asymptotics in the stretched coordinates which, for the domain $\Pi_+^\varepsilon = \Pi_{l+}^\varepsilon$, see (1.14), take the form

$$\xi = (\xi_1, \xi_2) = \varepsilon^{-1} (x_1 - l, x_2) . \tag{3.1}$$

Notice that performing the coordinate change $x \mapsto \xi$ and setting $\varepsilon = 0$ formally afterwards transform Π_+^ε into the upper half-plane $\mathbb{R}_+^2$ with a semi-infinite step, fig. 6, a,

$$\Xi = \left\{ \xi \in \mathbb{R}^2 : \xi_2 > 0 \ \text{ for } \ \xi_1 \leq 0 \text{ and } \xi_2 > -1 \ \text{ for } \ \xi_1 > 0 \right\} . \tag{3.2}$$

As a result, the singular perturbation of the waveguide wall gives rise to the boundary layer phenomenon described by the solutions to the following problem:

$$-\Delta_\xi v (\xi) = 0, \quad \xi \in \Xi, \quad \partial_{\nu(\xi)} v (\xi) = g (\xi) , \qquad \xi \in \partial\Xi. \tag{3.3}$$

The Laplace equation is obtained according to the relation $\Delta_x + \lambda = \varepsilon^{-2}\Delta_\xi + \lambda$ which singles out the Laplacian as the main asymptotic part of the Helmholtz operator while the Neumann condition is inherited directly from (1.4). The problem (3.3) with a compactly supported datum g admits a solution $v (\xi) = O(|\xi|^{-1})$ as $|\xi| \longrightarrow +\infty$ provided $\int_{\partial\Xi} g (\xi) \, ds_\xi = 0$. Otherwise, a solution must include a term growing at infinity like $c \ln |\xi|$ and loses the intrinsic decay property of a boundary layer so that traditional asymptotic procedures have to be replaced by more sophisticated approaches, see [46, Ch. 2, 4] and [36]. However, we will see that the boundary perturbation in our particular problem does not affect the main asymptotic term and, moreover, the first correction term does not include a boundary layer. At the same time, lower-order asymptotic terms involve boundary layers so that, to complete our description of the asymptotic procedure, we outline the derivation of the expansion (1.17) in Appendix which, actually, provides the simple relation

$$S_{00}^\varepsilon = 1 + o \left(\varepsilon \left(1 + |\ln \varepsilon| \right) \right) \tag{3.4}$$

of further use in Section 4.1, cf. Remark 13.

3.2. Main asymptotic terms

We search for an eigenvalue of the problem (1.12), (1.13) in the form

$$\lambda^\varepsilon = \pi^2 - \varepsilon^2\mu, \tag{3.5}$$

where the correction coefficient $\mu > 0$ will be found in Section 4. Recalling the normalization factors in (2.15) and (2.2),

$$(k_1^\varepsilon)^{-1/2} = \varepsilon^{-1/2}\mu^{-1/4} + O(\varepsilon^{1/2}), \qquad (2k^\varepsilon)^{-1/2} = (2\pi)^{-1/2} + O(\varepsilon^2), \tag{3.6}$$

we guess the following asymptotic ansätze for the entries of the augmented scattering matrix

$$S_{11}^\varepsilon = S_{11}^0 + \varepsilon S_{11}' + \widetilde{S}_{11}^\varepsilon, \quad S_{01}^\varepsilon = \varepsilon^{1/2} S_{01}^0 + \varepsilon^{3/2} S_{01}' + \widetilde{S}_{01}^\varepsilon, \tag{3.7}$$

but aim to calculate the terms S_{p1}^0 and S_{p1}' only. We will estimate the remainders $\widetilde{S}_{p1}^\varepsilon$ in Section 6.

Using definitions of the waves (2.2) and (2.15), (2.18), we take relations (3.6) and (3.7) into account and rewrite the decomposition (2.19) of the special solution Z_1^ε as follows:

$$\begin{aligned} Z_1^\varepsilon(x) = \varepsilon^{-1/2} (4\mu)^{-1/4} \cos(\pi x_2) \big(1 + i + S_{11}^0 (1-i) + \\ + \varepsilon \left(S_{11}' (1-i) + x_1\sqrt{\mu}\left(1 - i + S_{11}^0 (1+i)\right)\right) + ...\big) + \\ + \varepsilon^{1/2} (2\pi)^{-1/2} \left(S_{01}^0 + \varepsilon S_{01}' + ...\right)\left(e^{i\pi x_1} + ...\right). \end{aligned} \tag{3.8}$$

In (3.8), the Taylor formula

$$e^{k_1^\varepsilon x_1} \mp i e^{-k_1^\varepsilon x_1} = (1 \mp i) + \varepsilon x_1 \sqrt{\mu}(1 \pm i) + O\left(\varepsilon^2 x_1^2\right) \tag{3.9}$$

was applied so that the expansion (3.9) becomes meaningful under the restriction $x_1 < R$ with a fixed R, i.e., for $x \in \Pi^\varepsilon(R) = \{x \in \Pi^\varepsilon : x_1 < R\}$.

In view of the above observation, we employ the method of matched asymptotic expansions, cf. [64, 36], in the interpretation [51, 54]. Namely, we regard (3.8) as an outer expansion and introduce the inner expansion

$$Z_1^\varepsilon(x) = \varepsilon^{-1/2} Z_1^0(x) + \varepsilon^{1/2} Z_1'(x) + ... \,. \tag{3.10}$$

At the same time, multipliers of $\varepsilon^{-1/2}$ and $\varepsilon^{1/2}$ on the right-hand side of (3.8) exhibit a behavior at infinity of the terms Z_1^0 and Z_1' in (3.10) because the upper bound R for x_1 can be chosen arbitrary large. Indeed, they must satisfy the following representations at infinity

$$Z_1^0(x) = (4\mu)^{-1/4}\cos(\pi x_2)\left(1 + i + S_{11}^0(1-i)\right) + ..., \tag{3.11}$$

$$Z_1'(x) = (4\mu)^{-1/4}\cos(\pi x_2)\left(S_{11}'(1-i) + x_1\sqrt{\mu}\left(1 - i + S_{11}^0(1+i)\right)\right) + S_{01}^0 (2\pi)^{-1/2} e^{i\pi x_1} + ... \,. \tag{3.12}$$

The formal passage to $\varepsilon = 0$ transforms the waveguide (1.14) into the semi-infinite strip $\Pi_+^0 = \mathbb{R}\times(0,1)$ while due to (3.5), the Neumann problem (1.12), (1.13) converts into

$$-\Delta u^0(x) = \pi^2 u^0(x), \quad x \in \Pi_+^0, \tag{3.13}$$

$$\partial_\nu u^0(x) = 0, \quad x \in \partial\Pi_+^0. \tag{3.14}$$

This limit problem has two bounded solutions

$$u_0^0(x) = \frac{1}{2}\left(e^{-i\pi x_1} + e^{i\pi x_1}\right) = \cos(\pi x_1), \tag{3.15}$$

$$u_1^0(x) = \cos(\pi x_2). \tag{3.16}$$

Comparing these formulas with the representation (3.11), we readily see that Z_1^0 is the standing wave

$$Z_1^0(x) = (4\mu)^{-1/4}\left(1 + i + S_{11}^0(1-i)\right)\cos(\pi x_2). \tag{3.17}$$

Since $\lambda^\varepsilon = \pi^2 + O(\varepsilon^2)$, the function Z_1' also satisfies the homogeneous equation (3.13) but the Neumann condition becomes inhomogeneous because of the boundary perturbation. For $p = 1$, we have

$$-\Delta Z_p'(x) = \pi^2 Z_p'(x), \quad x \in \Pi_+^0, \qquad \partial_\nu Z_p'(x) = g_p(x), \quad x \in \partial\Pi_+^0. \tag{3.18}$$

To determine the datum g_1, we observe that the function (3.17) satisfies the Neumann condition (1.13) everywhere on $\partial\Pi^\varepsilon$ except at the lower side $\Upsilon^\varepsilon = \{x : x_1 \in (0,l), \quad x_2 = -\varepsilon\}$ of the box $\varpi_+^\varepsilon = (0,l) \times (-\varepsilon, 0]$ in (1.14). Furthermore, we obtain

$$\begin{aligned}\partial_\nu Z_1^p(x_1,-\varepsilon) &= -\partial_2 Z_1^0(x_1,-\varepsilon) = (4\mu)^{-1/4}\pi\sin(-\pi\varepsilon)\left(1+i+S_{11}^0(1-i)\right) = \\ &= -\varepsilon(4\mu)^{-1/4}\pi^2\left(1+i+S_{11}^0(1-i)\right) + O\left(\varepsilon^3\right) =: -\varepsilon G_1' + O\left(\varepsilon^3\right)\end{aligned} \tag{3.19}$$

and hence,

$$g_p(x) = \begin{cases} 0, & x \in \partial\Pi_+^0 \setminus \overline{\Upsilon^0}, \\ G_p', & x \in \Upsilon^0. \end{cases} \tag{3.20}$$

Although the Neumann datum (3.20) is not smooth and has a jump at the point $x = (l, 0)$, the problem (3.18) with $p = 1$ has a solution in $H^1_{loc}(\overline{\Pi_+^0})$ such that

$$Z_p'(x) = C_p e^{i\pi x_1} + \left(C_p^0 + x_1 C_p^1\right)\cos(\pi x_2) + \widetilde{Z}_p'(x), \quad \widetilde{Z}_p'(x) = O(e^{-x_1\sqrt{3}\pi}). \tag{3.21}$$

This fact is a direct consequence of the elliptic theory in domains with cylindrical outlets to infinity (see the key works [2, 41, 47, 48] and, e.g., the monographs [59]), but also may be derived by the Fourier method after splitting Π_+^0 into the rectangle $(0,l) \times (0,1)$ and the semi-infinite strip $(l, +\infty) \times (0,1)$. A simple explanation of how to apply the above-mentioned theory can be found in the introductory chapter 2 of the book [59], the review paper [50] and Section 5 of our paper.

The solution (3.21) is defined up to the term $c\cos(\pi x_2)$ and, therefore, the coefficient C_p^0 can be chosen arbitrarily. Other coefficients in (3.21) are computed by Green's formula in the long (R is big) rectangle $\Pi^0(R) = (0,R) \times (0,1)$. Indeed, we send R to $+\infty$ and obtain

$$\begin{aligned} 0 &= \lim_{R\to+\infty}\int_{\Pi_+^0(R)}\left(u_1^0(x)\left(\Delta+\pi^2\right)Z_1'(x) - Z_1'(x)\left(\Delta+\pi^2\right)u_1^0(x)\right)dx = \\ &= \lim_{R\to+\infty}\int_0^1 \cos(\pi x_2)\,\partial_1 Z_1'(R,x_2)\,dx_2 - \int_0^l \cos(\pi 0)\,\partial_2 Z_1'(x_1,0)\,dx_1 = \\ &= \frac{1}{2}C_1^1 + (4\pi)^{-1/4}\pi^2 l\left(1+i+S_{11}^0(1-i)\right). \end{aligned} \tag{3.22}$$

In the same way, we deal with the functions (3.15) and (3.21), that results in the equality

$$0=\lim_{R\to+\infty}\int_0^1 \left(\cos(\pi x_1)\,\partial_1 Z_1'(x)\,dx_2 - U_1'(x)\,\partial_1\cos(\pi x_1)\right)\big|_{x_1=R}\,dx_2- \tag{3.23}$$

$$-\int_0^l \cos(\pi x_1)\,\partial_2 Z_1'(x_1,0)\,dx_1 = i\pi C_1 + (4\mu)^{-1/4}\pi^2\left(1+i+S_{11}^0(1-i)\right)\int_0^l \cos(\pi x_1)\,dx_1.$$

Comparing (3.12) with (3.21), we arrive at the relations

$$(2\pi)^{-1/2}S_{01}^0 = C_1,\quad (4\mu)^{-1/4}\sqrt{\mu}\left(1-i+S_{11}^0(1+i)\right)=C_1^1,\quad S_{11}'(1-i)=C_1^0 \tag{3.24}$$

which together with our calculations (3.23) and (3.22) give us the following formulas:

$$S_{01}^0 = (4\mu)^{-1/4}(2\pi)^{1/2}\pi i\left(1+i+S_{11}^0(1-i)\right)\int_0^l \cos(\pi x_1)\,dx_1 = \tag{3.25}$$

$$= -(4\mu)^{-1/4}(2\pi)^{1/2}\left(1-i-S_{11}^0(1+i)\right)\sin(\pi l)$$

$$\sqrt{\mu}\left(1-i+S_{11}^0(1+i)\right) = -2\pi^2 l\left(1+i+S_{11}^0(1-i)\right)\Rightarrow \tag{3.26}$$

$$\Rightarrow S_{11}^0 = -\frac{\sqrt{\mu}(1-i)\,2\pi^2 l(1+i)}{\sqrt{\mu}(1+i)\,2\pi^2 l(1-i)} = -\frac{4\pi^2 l\sqrt{\mu}+i\left(4\pi^4l^2-\mu\right)}{4\pi^4 l^2+\mu}.$$

We emphasize that $\mu = 4\pi^4 l^2 \;\Rightarrow\; S_{11}^0 = -1$.

The necessary computations have been completed. It should be mentioned that to determine the correction terms S_{11}' and S_{01}' in the asymptotic ansätze (3.7), one has to make another step in our asymptotic procedure, see Appendix, but they are not so important.

4. Detection of a trapped mode

4.1. Reformulation of the criterion

The spectral and length parameters supporting a trapped mode are sought in the form

$$\mu = 4\pi^4 l^2 + \triangle\mu,\quad l=\pi k+\triangle l. \tag{4.1}$$

Here, $k\in\mathbb{N}$ is fixed but the small $\triangle\mu$ and $\triangle l$ have to be determined. If $\triangle\mu=0$ and $\triangle l=0$, the equalities $S_{11}^0=-1$ and $S_{01}^0=0$ hold due to (3.26) and (3.25).

We are going to choose the small increments $\triangle\mu$ and $\triangle l$ in (4.1) such that the criterion (2.21) for the existence of a trapped mode is satisfied. Since S_{11}^ε is complex, the criterion furnishes two equations for two real parameters $\triangle\mu$ and $\triangle l$. It is convenient to consider the other equations

$$\operatorname{Im} S_{11}^\varepsilon = 0,\qquad \operatorname{Re} S_{01}^0 = 0, \tag{4.2}$$

which, for small ε and $\triangle\mu$, $\triangle l$, are equivalent to $S_{11}^\varepsilon=-1$. Indeed, from formulas (3.26) in Section 3 and (A17), (A7) in Appendix together with the estimates (6.20) in Section 6, it follows that

$$|S_{11}^\varepsilon+1|+|S_{00}^\varepsilon-1|\le c\left(\varepsilon+|\triangle\mu|+|\triangle l|\right)^\delta,\qquad \delta\in(0,1). \tag{4.3}$$

Since S^ε is unitary and symmetric, see Section 2.2, the second assumption in (4.2) means that $S^\varepsilon_{01} = S^\varepsilon_{10} = i\sigma$ with some $\sigma \in \mathbb{R}$ and, furthermore,

$$0 = \overline{S^\varepsilon_{00}} S^\varepsilon_{01} + \overline{S^\varepsilon_{10}} S^\varepsilon_{11} = 2i\sigma + O\left(|\sigma| \left(\varepsilon + |\triangle\mu| + |\triangle l|\right)^\delta\right). \tag{4.4}$$

Hence, $\sigma = 0$ when $\triangle\mu$, $\triangle l$ and $\varepsilon > 0$ are small so that $S^\varepsilon_{01} = 0$ implies $|S^\varepsilon_{11}| = 1$ and, furthermore, $S^\varepsilon_{11} = -1$ due to (4.3), (4.2).

We have proved that (4.2) ⇒ (2.21) while the inverse implication (2.21) ⇒ (4.2) is obvious.

4.2. Solving the system of transcendental equations

By virtue of (3.26), (1.16) and (4.1), the first equation (4.2) turns into

$$\triangle\mu = -\varepsilon \left(8\pi^4 l^2 + \triangle\mu\right) \operatorname{Im} \widehat{S}^\varepsilon_{11}. \tag{4.5}$$

Formulas (A17), (1.18) and a simple algebraic calculation convert the second equation (4.2) into

$$\sin l = (4\mu)^{-1/4} (2\pi)^{-1/2} \frac{4\pi^4 l^2 + \mu}{4\pi^2 l\sqrt{\mu} + 2\mu} \varepsilon \operatorname{Re} \widehat{S}^\varepsilon_{01},$$

and thus,

$$\triangle l = \arcsin\left((-1)^k (4\mu)^{-1/4} (2\pi)^{-1/2} \frac{4\pi^2 l^2 + \mu}{2\pi^2 l + \sqrt{\mu}} \varepsilon \operatorname{Re} \widehat{S}^\varepsilon_{01}\right). \tag{4.6}$$

We search for a solution $(\triangle\mu, \triangle l)$ of the transcendental equations (4.5), (4.6) in the closed disk

$$\overline{\mathbb{B}_\varrho} = \left\{(\triangle\mu, \triangle l) \in \mathbb{R}^2 : |\triangle\mu|^2 + |\triangle l|^2 \leq \varrho^2\right\} \tag{4.7}$$

and rewrite them in the condensed form

$$(\triangle\mu, \triangle l) = T^\varepsilon (\triangle\mu, \triangle l) \quad \text{in } \overline{\mathbb{B}_\varrho}, \tag{4.8}$$

where T^ε is a nonlinear operator involving asymptotic remainders from the asymptotic expansions (1.16) and (1.18) of entries of the matrix $S^\varepsilon = S^\varepsilon (\triangle\mu, \triangle l)$. The estimates (6.20) and Proposition 14 below demonstrate that the operator is smooth in $\overline{\mathbb{B}_\varrho}$ with $\varrho \leq \varrho_0$ for some $\varrho_0 > 0$. Furthermore,

$$|T^\varepsilon (\triangle\mu, \triangle l)| \leq c_\varrho \varepsilon \left(1 + |\ln\varepsilon|\right)^2 \quad \text{for } (\triangle\mu, \triangle l) \in \overline{\mathbb{B}_\varrho}.$$

Hence, for any $\varrho \leq \varrho_0$, there exists $\varepsilon(\varrho) > 0$ such that T^ε with $\varepsilon \in (0, \varepsilon(\varrho))$ is a contraction operator in the disk (4.7). By the Banach contraction principle, the abstract equation (4.8) has a unique solution $(\triangle\mu, \triangle l) \in \overline{\mathbb{B}_\varrho}$ and the estimate $|\triangle\mu| + |\triangle l| \leq C\varepsilon \left(1 + |\ln\varepsilon|\right)^2$ is valid. This solution depends on ε and determines the spectral and length parameters (4.1) supporting a trapped mode.

4.3. The main results

Based on the performed formal calculations, we will prove the following existence and uniqueness theorems in the next three sections.

Theorem 3. *Let $k \in \mathbb{N}$. There exist ε_k, $c_k > 0$ and $\triangle\mu_k(\varepsilon)$, $\triangle l_k(\varepsilon)$ such that, for any $\varepsilon \in (0, \varepsilon_k)$, the estimate*

$$|\triangle\mu_k(\varepsilon)| + |\triangle l_k(\varepsilon)| \le c_k \varepsilon (1 + |\ln \varepsilon|)^2$$

is valid and the problem (1.3), (1.4) in the waveguide $\Pi^\varepsilon_{l(\varepsilon)} = \Pi \cup \varpi^\varepsilon_{l_k(\varepsilon)}$ with the box-shaped perturbation (1.2) of length $2l_k(\varepsilon) = 2(\pi k + \triangle l_k(\varepsilon))$ has the eigenvalue

$$\lambda^\varepsilon_k = \pi^2 - \varepsilon^2 (4\pi^4 (\pi k + \triangle l_k(\varepsilon))^2 + \triangle\mu_k(\varepsilon)). \tag{4.9}$$

The eigenvalue (4.9) is unique in the interval $(0, \pi^2)$.

Theorem 4. *Let $k \in \mathbb{N}$ and $\delta > 0$. There exist $\varepsilon^\delta_k > 0$ such that, for any $\varepsilon \in \left(0, \varepsilon^\delta_k\right)$, the waveguide Π^ε_l with the length parameter*

$$l \in [\pi(k-1) + \delta, \pi(k+1) - \delta] \tag{4.10}$$

does not support a trapped mode in the case $l \neq l_k(\varepsilon)$ where $l_k(\varepsilon)$ is taken from Theorem 3.

5. Weighted spaces with detached asymptotics

5.1. Reformulation of the problem

Let $W^1_\beta\left(\Pi^\varepsilon_+\right)$ be the Kondratiev (weighted Sobolev) space composed from functions u^ε in $H^1_{loc}\left(\overline{\Pi^\varepsilon_+}\right)$ with the finite norm

$$\left\|u^\varepsilon; W^1_\beta\left(\Pi^\varepsilon_+\right)\right\| = \left\|e^{\beta x_1} u^\varepsilon; H^1\left(\Pi^\varepsilon_+\right)\right\|, \tag{5.1}$$

where $\beta \in \mathbb{R}$ is the exponential weight index. If $\beta > 0$, the functions in $W^1_\beta\left(\Pi^\varepsilon_+\right)$ decay at infinity in the semi-infinite waveguide (1.14) but in the case $\beta < 0$, a certain exponential growth is permitted while the rate of decay/growth is ruled by β. Clearly, $W^1_0\left(\Pi^\varepsilon_+\right) = H^1\left(\Pi^\varepsilon_+\right)$. The space $C^\infty_c\left(\overline{\Pi^\varepsilon_+}\right)$ of smooth, compactly supported functions is dense in $W^1_\beta\left(\Pi^\varepsilon_+\right)$ for any β.

By a solution of the inhomogeneous problem (2.3) in $W^1_\sigma\left(\Pi^\varepsilon_+\right)$, $\sigma \in \mathbb{R}$, we, following [50, 54] and other papers, understand a function $u^\varepsilon \in W^1_\sigma\left(\Pi^\varepsilon_+\right)$ satisfying the integral identity

$$(\nabla u^\varepsilon, \nabla v^\varepsilon)_{\Pi^\varepsilon} - \lambda^\varepsilon (u^\varepsilon, v^\varepsilon)_{\Pi^\varepsilon} = F^\varepsilon(v^\varepsilon) \quad \forall v^\varepsilon \in W^1_{-\sigma}\left(\Pi^\varepsilon_+\right) \tag{5.2}$$

where $F^\varepsilon \in W^1_{-\sigma}\left(\Pi^\varepsilon_+\right)^*$ is an (anti)linear continuous functional in $W^1_{-\sigma}\left(\Pi^\varepsilon_+\right)$ and $(\ ,\ )_{\Pi^\varepsilon}$ is an extension of the Lebesgue scalar product up to a duality between appropriate weighted Lebesgue spaces. In view of (5.1), all terms in (5.2) are defined correctly so that the problem (5.2) is associated with the continuous mapping

$$W^1_\sigma\left(\Pi^\varepsilon_+\right) \ni u^\varepsilon \mapsto \mathcal{A}^\varepsilon_\sigma(\lambda^\varepsilon) u^\varepsilon = F^\varepsilon \in W^1_{-\sigma}\left(\Pi^\varepsilon_+\right)^*.$$

If $f^\varepsilon \in L^2_\sigma\left(\Pi^\varepsilon_+\right)$, that is, $e^{\sigma z} f^\varepsilon \in L^2\left(\Pi^\varepsilon_+\right)$, then the functional $v^\varepsilon \mapsto F^\varepsilon(v^\varepsilon) = (f^\varepsilon, v^\varepsilon)_{\Pi^\varepsilon_+}$ belongs to $W^1_{-\sigma}\left(\Pi^\varepsilon_+\right)^*$. Clearly, $\mathcal{A}^\varepsilon_{-\sigma}(\lambda^\varepsilon)$ is the adjoint operator for $\mathcal{A}^\varepsilon_\sigma(\lambda^\varepsilon)$.

5.2. The Fredholm property, asymptotics and the index

Let us formulate some well-known results of the theory of elliptic problems in domains with cylindrical outlets to infinity (see the key papers [41, 47, 48] and, e.g., the monograph [59, Ch. 3 and 5]). This theory mainly deals with the classical (differential) formulation of boundary value problems, however, as was observed in [50], passing to the weak formulation of type (5.2) does not meet any visible obstacle.

At the same time, on the basis of the weak formuation (5.2) of the problem (2.3), we will conclude in Section 5.5 on the dependence of constants in necessary estimates on the small parameter ε.

Theorem 5. *(see [41]) Let $\lambda^\varepsilon \in (0, \pi^2]$.*

1) The operator $\mathcal{A}^\varepsilon_\beta(\lambda^\varepsilon)$ is Fredholm if and only if

$$\beta \neq \beta_0 := 0 \text{ and } \beta \neq \beta_{\pm j} := \pm\sqrt{\pi^2 j^2 - \lambda^\varepsilon}, \quad j \in \mathbb{N}. \tag{5.3}$$

In the case $\beta = \beta_p$ with $p \in \mathbb{Z}$, the range $\mathcal{A}^\varepsilon_\beta(\lambda^\varepsilon) W^1_\beta(\Pi^\varepsilon_+)$ is not a closed subspace in $W^1_{-\beta}(\Pi^\varepsilon_+)^*$.

2) Let $\gamma \in (\beta_1, \beta_2)$ and let $u^\varepsilon \in W^1_{-\gamma}(\Pi^\varepsilon_+)$ be a solution of the problem (5.2) with the weight index $\sigma = -\gamma$ and the right-hand side $F^\varepsilon \in W^1_{-\gamma}(\Pi^\varepsilon_+)^* \subset W^1_\gamma(\Pi^\varepsilon_+)$. Then the asymptotic decomposition

$$u^\varepsilon(x) = \widetilde{u}^\varepsilon(x) + \sum_\pm \left(a^\varepsilon_\pm w^{\varepsilon\pm}_0(x) + b^\varepsilon_\pm w^{\varepsilon\pm}_1(x)\right) \tag{5.4}$$

with the waves $w^{\varepsilon\pm}_p$ in (2.2) and (2.18), and the estimate

$$\left(\left\|\widetilde{u}^\varepsilon; W^1_\gamma(\Pi^\varepsilon)\right\|^2 + \sum_\pm \left(\left|a^\varepsilon_\pm\right|^2 + \left|b^\varepsilon_\pm\right|^2\right)\right)^{1/2} \leq c_\varepsilon\left(\left\|F^\varepsilon; W^1_{-\gamma}(\Pi^\varepsilon)^*\right\| + \left\|u^\varepsilon; W^1_{-\gamma}(\Pi^\varepsilon)\right\|\right) \tag{5.5}$$

are valid where $\widetilde{u}^\varepsilon \in W^1_\gamma(\Pi^\varepsilon)$ is the asymptotic remainder, $a^\varepsilon_\pm$ and $b^\varepsilon_\pm$ are coefficients depending on F^ε and u^ε, the waves $w^{\varepsilon\pm}_0$ are given by (2.2) and $w^{\varepsilon\pm}_1$ by (2.8) for $\lambda^\varepsilon = \pi^2$ but by (2.18) for $\lambda^\varepsilon \in (0, \pi^2)$. The factor c_ε in (5.5) is independent of F^ε and u^ε but may depend on $\varepsilon \in [0, \varepsilon_0]$.

As was mentioned, $\mathcal{A}^\varepsilon_\gamma(\lambda^\varepsilon)^* = \mathcal{A}^\varepsilon_{-\gamma}(\lambda^\varepsilon)$ and hence, the kernels and co-kernels of these operators are in the relationship

$$\ker \mathcal{A}^\varepsilon_{\pm\gamma}(\lambda^\varepsilon) = \operatorname{coker}\mathcal{A}^\varepsilon_{\mp\gamma}(\lambda^\varepsilon). \tag{5.6}$$

The next theorem compares the indexes $\operatorname{Ind}\mathcal{A}^\varepsilon_{\pm\gamma}(\lambda^\varepsilon) = \dim\ker \mathcal{A}^\varepsilon_{\pm\gamma}(\lambda^\varepsilon) - \dim\operatorname{coker}\mathcal{A}^\varepsilon_{\pm\gamma}(\lambda^\varepsilon)$. Notice that the relation (5.6) assures only that

$$\operatorname{Ind}\mathcal{A}^\varepsilon_\gamma(\lambda^\varepsilon) = -\operatorname{Ind}\mathcal{A}^\varepsilon_{-\gamma}(\lambda^\varepsilon). \tag{5.7}$$

Theorem 6. *If $\gamma \in (\beta_1, \beta_2)$, see (5.3), then*

$$Ind\mathcal{A}^\varepsilon_{-\gamma}(\lambda^\varepsilon) = Ind\mathcal{A}^\varepsilon_\gamma(\lambda^\varepsilon) + 4. \tag{5.8}$$

We emphasize that the last 4 is nothing but the number of waves detached in (5.4), and the decomposition (5.4) with four coefficients $a_{\pm}^{\varepsilon}$ and $b_{\pm}^{\varepsilon}$ is the very origin of the relationship (5.8). A rigorous proof is given in [59, Thm. 3.3.3, 5.1.4 (3)].

From (5.8) and (5.7), it follows that

$$\mathrm{Ind}\mathcal{A}_{-\gamma}^{\varepsilon}(\lambda^{\varepsilon}) = -\mathrm{Ind}\mathcal{A}_{\gamma}^{\varepsilon}(\lambda^{\varepsilon}) = 2. \tag{5.9}$$

5.3. Absence of trapped modes with a fast decay rate

In this section, we prove that, for $\lambda \in (0, \pi^2]$ and $\gamma \in (\beta_1, \beta_2)$, the following formula holds:

$$\dim\ker \mathcal{A}_{\gamma}^{\varepsilon}(\lambda) = 0 \tag{5.10}$$

which, in particular, completes the proof of Theorem 1, cf. our assumption $c_{\varepsilon} \neq 0$ for trapped mode (2.23) while $c_{\varepsilon} = 0$ leads to $U^{\varepsilon} = \widetilde{U}^{\varepsilon} \in \ker \mathcal{A}_{\gamma}^{\varepsilon}(\lambda) = \{0\}$. Clearly, $\ker \mathcal{A}_{\beta}^{0}(\lambda) = 0$ for any $\beta > 0$, that is, the limit problem (1.12), (1.13) in the straight semi-infinite strip Π_{+}^{0} cannot have a trapped mode. However, as was mentioned in Section 1.4, formula (5.10) is not supported by a standard perturbation argument and, moreover, $\dim\ker \mathcal{A}_{\beta}^{\varepsilon}(\lambda^{\varepsilon}) > 0$ for some $\beta \in (0, \beta_1)$ and $\lambda^{\varepsilon} \in (0, \pi^2)$.

Theorem 7. *Let $\gamma \in (\beta_1, \beta_2)$ be fixed. There exists $\varepsilon_0 > 0$ such that, for $\varepsilon \in (0, \varepsilon_0)$ and $\lambda \in (0, \pi^2]$, the equality (5.10) is valid, i.e. no trapped mode with a fast decay at infinity exists.*

Proof. Let us assume that, for some $\lambda \in (0, \pi^2]$ and an infinitesimal positive sequence $\{\varepsilon_k\}_{k\in\mathbb{N}}$, the homogeneous problem (1.12), (1.13) has a solution $u^{\varepsilon_k} \in W_{\gamma}^{1}(\Pi_{+}^{\varepsilon})$. The restriction of u^{ε_k} on the semi-infinite strip $\Pi_{+}^{0} = \mathbb{R}\times(0,1)$ is denoted by $u_0^{\varepsilon_k}$. Under the normalization condition

$$\left\| u^{\varepsilon_k}; L^2\left(\Pi_{+}^{\varepsilon}(2l)\right)\right\| = 1, \tag{5.11}$$

we are going to perform the limit passage $\varepsilon_k \to +0$ in the integral identity

$$(\nabla u^{\varepsilon_k}, \nabla v^{\varepsilon})_{\Pi_{+}^{\varepsilon}} = \lambda^{\varepsilon}(u^{\varepsilon}, v^{\varepsilon})_{\Pi_{+}^{\varepsilon}}, \tag{5.12}$$

where v^{ε} is obtained from a test function $v \in C_c^{\infty}\left(\overline{\Pi_{+}^{0}}\right)$ by the even extension over the x_1-axis. If we prove that

(i) $u_0^{\varepsilon_k}$ converges to $u_0^0 \in W_{-\gamma}^{1}\left(\Pi_{+}^{0}\right)$ weakly in $W_{-\gamma}^{1}\left(\Pi_{+}^{0}\right)$ and, therefore, strongly in $L^2\left(\Pi_{+}^{0}(2l)\right)$;

(ii) $\left\| u^{\varepsilon_k}; L^2\left(\Pi_{+}^{\varepsilon_k} \setminus \Pi_{+}^{0}\right)\right\| \to 0$;

(iii) $(\nabla u^{\varepsilon_k}, \nabla v)_{\Pi^{\varepsilon}\setminus\Pi^{0}} \to 0$ with any smooth function v in the rectangle $[0, l] \times [-1, 1]$,

then the limit passage in (5.12) and (5.11) gives

$$\left(\nabla u_0^0, \nabla v\right)_{\Pi_{+}^{0}} = \lambda\left(u_0^0, v\right)_{\Pi_{+}^{0}} \quad \forall v \in C_c^{\infty}(\overline{\Pi_{+}^{0}}), \tag{5.13}$$

$$\left\| u_0^0; L^2\left(\Pi_{+}^{0}(2l)\right)\right\| = 1. \tag{5.14}$$

By the completion argument, the integral identity (5.13) is valid with any $v \in W_{-\gamma}^{1}\left(\Pi_{+}^{0}\right)$ and, therefore, $u_0^0 = 0$ because the limit problem in Π_{+}^{0} cannot get a non-trivial trapped mode.

Let us confirm facts $(i)-(iii)$. We write ε instead of ε_k.

First, we apply a local estimate, see e.g. [1], to the solution u^ε of the problem (1.12), (1.13) regarding λu^ε as a given right-hand side,

$$\left\| u^\varepsilon; H^2\left(\varpi'\right)\right\| \leq c\lambda \left\| u^\varepsilon; L^2\left(\varpi''\right)\right\|. \tag{5.15}$$

Here, $\varpi' = (4l/3, 5l/3) \times (0,1)$ and $\varpi'' = (l, 2l) \times (0,1)$ are rectangles such that $\varpi' \subset \varpi'' \subset \Pi^\varepsilon(2l)$ and, therefore, the right-hand side of (5.15) is less than $c\lambda$ according to (5.11).

Second, we split u^ε as follows:

$$u^\varepsilon = u_l^\varepsilon + u_\infty^\varepsilon, \quad u_l^\varepsilon = (1-\chi)\, u^\varepsilon, \quad u_\infty^\varepsilon = \chi u^\varepsilon \tag{5.16}$$

where $\chi \in C^\infty(\mathbb{R})$ is a cut-off function,

$$\chi(x_1) = 1 \text{ for } x_1 \geq \frac{5}{3} l \text{ and } \chi(x_1) = 0 \text{ for } x_1 \leq \frac{4}{3} l. \tag{5.17}$$

The components in (5.16) satisfy the integral identities

$$\begin{gathered}\left(\nabla u_l^\varepsilon, \nabla v_l\right)_{\Pi_+^\varepsilon(2l)} = \lambda\left((1-\chi)\, u^\varepsilon, v_l\right)_{\Pi_+^\varepsilon(2l)} + \left(\nabla u^\varepsilon, v_l \nabla\chi\right)_{\varpi'} - \\ -\left(u^\varepsilon \nabla\chi, \nabla v_l\right)_{\varpi'} \;\; \forall v_l \in H^1\left(\Pi_+^\varepsilon(2l)\right)\end{gathered} \tag{5.18}$$

and

$$\begin{gathered}\left(\nabla u_\infty^\varepsilon, \nabla v_\infty\right)_{\Pi_\infty(l)} - \lambda\left(u_\infty^\varepsilon, v_\infty\right)_{\Pi_\infty(l)} = \\ = F_\infty^\varepsilon(v_\infty) := \left(u^\varepsilon \nabla\chi, \nabla v_\infty\right)_{\varpi'} - \left(\nabla u^\varepsilon, v_\infty \nabla\chi\right)_{\varpi'} \quad \forall v_\infty \in W_{-\gamma}^1\left(\Pi_\infty(l)\right).\end{gathered} \tag{5.19}$$

Third, inserting $v_l = u_l^\varepsilon$ into (5.18) and taking (5.11), (5.15) into account yield

$$\left\| \nabla u_l^\varepsilon; L^2\left(\Pi_+^\varepsilon(2l)\right)\right\| \leq c. \tag{5.20}$$

The problem (5.19) needs a bit more advanced argument. It is posed in the semi-infinite strip $\Pi_\infty(l)$ independent of ε and thus, the following a priory estimate in the Kondratiev space, see [41] and, e.g., [59, Thm 5.1.4 (1)],

$$\begin{aligned}\left\| u_\infty^\varepsilon; W_{-\gamma}^1\left(\Pi_\infty(l)\right)\right\| &\leq c_1\left(\left\| F_\infty^\varepsilon; W_\gamma^1\left(\Pi_\infty(l)\right)^*\right\| + \left\| u_\infty^\varepsilon; L^2\left(\Pi_+^\varepsilon(2l) \cap \Pi_\infty(l)\right)\right\|\right) \\ &\leq c_2\left(\left\| u^\varepsilon; L^2\left(\varpi'\right)\right\| + \left\| \nabla u^\varepsilon; L^2\left(\varpi'\right)\right\| + \left\| u^\varepsilon; \Pi_+^\varepsilon(2l)\right\|\right)\end{aligned} \tag{5.21}$$

contains some constants c_m independent of ε. In this way, formulas (5.20) and (5.21), (5.15), (5.10) assure that

$$\left\| u^\varepsilon; W_{-\gamma}^1\left(\Pi_+^\varepsilon\right)\right\| \leq c. \tag{5.22}$$

Thus, the convergence in (i) occurs along a subsequence which is still denoted by $\{\varepsilon_k\}$.

The last step of our consideration uses integration in $t \in (-\varepsilon, 0)$ and $x_1 \in (0, l)$ of the Newton-Leibniz formula

$$\left|u^\varepsilon(t, x_2)\right|^2 = \int_t^{t+1/2} \frac{\partial}{\partial x_2}\left(\chi_0(x_2)\, u^\varepsilon(x_1, x_2)^2\right) dx_2$$

where $\chi_0 \in C^\infty(\mathbb{R})$ is a cut-off function,

$$\chi_0(x_2) = 1 \text{ for } x_2 < \frac{1}{6} \quad \text{and} \quad \chi_0(x_2) = 0 \text{ for } x_2 > \frac{1}{3}. \tag{5.23}$$

As a result, we derive the estimate

$$\int_{\Pi_+^\varepsilon \setminus \Pi_+^0} |u^\varepsilon(x)|^2 dx \leq c\varepsilon \int_{\Pi_+^\varepsilon(l)} \left(|\nabla u^\varepsilon(x)|^2 + |u^\varepsilon(x)|^2\right) dx \leq C\varepsilon$$

while referring to (5.22) again. This provides (ii) as well as (iii) because

$$\begin{aligned} &\left|\int_0^l \int_{-\varepsilon}^0 \nabla u^\varepsilon(x_1,x_2) \cdot \nabla v(x_1,x_2)\, dx_2 dx_1\right| \leq \\ &\leq \max_{x \in \overline{\Pi_+^\varepsilon} \setminus \Pi_+^0} |\nabla v(x)| \left(\operatorname{meas}_2\left(\Pi_+^\varepsilon \setminus \Pi^0\right)\right)^{1/2} \left\|\nabla u^\varepsilon; L^2\left(\Pi_+^\varepsilon \setminus \Pi_+^0\right)\right\| \\ &\leq c_v \varepsilon^{1/2} l^{1/2} \left\|u^\varepsilon; W_\gamma^1\left(\Pi_+^\varepsilon\right)\right\| \leq C_v \varepsilon^{1/2} \end{aligned}$$

Theorem 7 is proved. ⊠

Remark 8. *From (5.6), (5.9) and (5.10), we obtain that the kernel of the operator* $\mathcal{A}_{-\gamma}^\varepsilon(\lambda^\varepsilon)$ *is of dimension* 2. *The exponentially growing solutions* Z_0^ε *and* Z_1^ε *of the problem (1.12), (1.13) which were introduced by formula (2.19) with the waves (2.18) and, therefore, belong to* $W_{-\gamma}^1\left(\Pi_+^\varepsilon\right)$, *are linearly independent and can be regarded as a basis in* $\ker\mathcal{A}_{-\gamma}^\varepsilon(\lambda^\varepsilon)$.

5.4. Radiation conditions

Let $\lambda^\varepsilon \in (0,\pi^2)$ and $\gamma \in (\beta_1,\beta_2)$, cf. Theorem 5. The pre-image $\mathfrak{W}_\gamma^1\left(\Pi_+^\varepsilon\right)$ of the subspace $W_{-\gamma}^1\left(\Pi_+^\varepsilon\right)^*$ in $W_\gamma^1\left(\Pi_+^\varepsilon\right)^*$ for the operator $\mathcal{A}_{-\gamma}^\varepsilon(\lambda^\varepsilon)$ consists of functions in the form (5.4). Introducing the norm $\left\|u^\varepsilon; \mathfrak{W}_\gamma^1\left(\Pi_+^\varepsilon\right)\right\|$ as the left-hand side of (5.5) makes $\mathfrak{W}_\gamma^1\left(\Pi_+^\varepsilon\right)$ a Hilbert space but this Hilbert structure is of no use in our paper.

The restriction $\mathcal{B}_\gamma^\varepsilon(\lambda^\varepsilon)$ of the operator $\mathcal{A}_{-\gamma}^\varepsilon(\lambda^\varepsilon)$ onto $\mathcal{W}_\gamma^1\left(\Pi_+^\varepsilon\right) \subset W_{-\gamma}^1\left(\Pi_+^\varepsilon\right)$ inherits all the properties of $\mathcal{A}_{-\gamma}^\varepsilon(\lambda^\varepsilon)$, in particular, $\mathcal{B}_\gamma^\varepsilon(\lambda^\varepsilon)$ is a Fredholm operator with $\operatorname{Ind}\mathcal{B}_\gamma^\varepsilon(\lambda^\varepsilon) = \operatorname{Ind}\mathcal{A}_{-\gamma}^\varepsilon(\lambda^\varepsilon) = 2$, see (5.9). Thus, the restriction $\mathfrak{A}_\gamma^\varepsilon(\lambda^\varepsilon)_{out}$ of $\mathfrak{A}_\gamma^\varepsilon(\lambda^\varepsilon)$ onto the subspace

$$\mathcal{W}_\gamma^1\left(\Pi_+^\varepsilon\right)_{out} = \left\{u^\varepsilon \in \mathcal{W}_\gamma^1\left(\Pi_+^\varepsilon\right) : a_-^\varepsilon = b_-^\varepsilon = 0 \text{ in } (5.4)\right\} \tag{5.24}$$

of codimension 2 becomes of index zero.

Theorem 9. *Let* λ, γ *and* ε *be the same as in Theorem 7. Then the operator* $\mathcal{B}_\gamma^\varepsilon(\lambda^\varepsilon)$ *constitutes the isomorphism*

$$\mathcal{W}_\gamma^1\left(\Pi_+^\varepsilon\right)_{out} \approx W_{-\gamma}^1\left(\Pi_+^\varepsilon\right)^*.$$

Remark 10. *It is quite easy to demonstrate that the operator* $\mathcal{B}_\gamma^\varepsilon(\lambda^\varepsilon)$ *is a monomorphism (and hence, an epimorphism because* $\operatorname{Ind}\mathcal{B}_\gamma^\varepsilon(\lambda^\varepsilon) = 0$*). Indeed, computing the simplectic form (2.9) for* $u^\varepsilon = \widetilde{u}^\varepsilon + a^\varepsilon w_0^{\varepsilon+} + b^\varepsilon w_1^{\varepsilon+}$ *and taking (2.7) into account yield*

$$\begin{aligned} 0 &= \lim_{R\to+\infty} q_R(u^\varepsilon, u^\varepsilon) = \lim_{R\to+\infty} q_R(a^\varepsilon w_0^{\varepsilon+} + b^\varepsilon w_1^{\varepsilon+}, a^\varepsilon w_0^{\varepsilon+} + b^\varepsilon w_1^{\varepsilon+}) = \\ &= i\,|a^\varepsilon|^2 + i\,|b^\varepsilon|^2 \quad \Rightarrow \quad a^\varepsilon = b^\varepsilon = 0 \quad \Rightarrow \quad u^\varepsilon = \widetilde{u}^\varepsilon \in \ker\mathcal{A}_\gamma^\varepsilon(\lambda^\varepsilon). \end{aligned}$$

Thus, $u^\varepsilon = 0$ *because of (5.10), and the desired formula* $\ker\mathcal{B}_\gamma^\varepsilon(\lambda^\varepsilon) = \{0\}$ *is obtained.*

Since the decomposition (5.4) of a function $u^\varepsilon \in \mathfrak{W}^1_\gamma\left(\Pi^\varepsilon_+\right)_{out}$ loses the incoming waves $w_0^{\varepsilon-}$ and $w_1^{\varepsilon-}$ due to a restriction in (5.24), $\mathcal{B}^\varepsilon_\gamma(\lambda^\varepsilon)$ has to be interpreted as an operator of the problem (5.24) with the radiation condition (2.10) at $\lambda = \pi^2$ and (2.20) at $\lambda \in (0, \pi^2)$. Theorem 9 says that such a problem is uniquely solvable, while its solution in the form

$$u^\varepsilon(x) = \widetilde{u}^\varepsilon(x) + a^\varepsilon_+ w_0^{\varepsilon+}(x) + b^\varepsilon_+ w_1^{\varepsilon+}(x) \tag{5.25}$$

obeys the estimate

$$\left\|\widetilde{u}^\varepsilon; W^1_\gamma\left(\Pi^\varepsilon_+\right)\right\| + \left|a^\varepsilon_+\right| + \left|b^\varepsilon_+\right| \leq C_\varepsilon \left\|F^\varepsilon; W^1_{-\gamma}\left(\Pi^\varepsilon_+\right)\right\|. \tag{5.26}$$

5.5. Dependence of bounds on the small parameter ε

If

$$\lambda^\varepsilon \in \left[\delta, \pi^2 - \delta\right] \tag{5.27}$$

with a fixed $\delta > 0$, the coefficient in the estimate (5.5) can be chosen independently of $\varepsilon \in [0, \varepsilon(\delta)]$ with some $\varepsilon(\delta) > 0$. This fact originates in the smooth dependence of the waves (2.2) and (2.15), (2.18) on the parameter (5.27) and the following observation. By multiplying u^ε with the same cut-off function (5.17), we reduce the problem (5.2) onto $\Pi_\infty(l)$, namely, inserting $v^\varepsilon = \chi v_\infty$ with any $v_\infty \in W^1_\gamma(\Pi_\infty(l))$ as a test function, for $u^\varepsilon_\infty = \chi u^\varepsilon$ we obtain the integral identity

$$\begin{aligned}(\nabla u^\varepsilon_\infty, \nabla v_\infty)_{\Pi_\infty(l)} - \lambda^\varepsilon (u^\varepsilon_\infty, v_\infty)_{\Pi_\infty(l)} = F^\varepsilon_\infty(v_\infty) :=& \\ := F^\varepsilon(\chi v_\infty) - (\nabla u^\varepsilon, v_\infty \nabla\chi)_{\Pi_\infty(l)} + (u^\varepsilon \nabla\chi, \nabla v_\infty)_{\Pi_\infty(l)}&.\end{aligned} \tag{5.28}$$

Moreover,

$$\begin{aligned}\left\|F^\varepsilon_\infty; W^1_{-\gamma}(\Pi_\infty(l))^*\right\| \leq \left\|F^\varepsilon(\chi\cdot); W^1_{-\gamma}(\Pi_\infty(l))\right\| + c_\chi \left\|u^\varepsilon; H^1\left(\Pi_\infty(l) \cap \Pi^\varepsilon_+(2l)\right)\right\| \leq& \\ \leq c\left(\left\|F^\varepsilon; W^1_{-\gamma}\left(\Pi^\varepsilon_+\right)\right\| + \left\|u^\varepsilon; W^1_{-\gamma}\left(\Pi^\varepsilon_+\right)\right\|\right),&\end{aligned}$$

cf. the right-hand side of (5.5). Due to the restriction (5.27), λ^ε stays at a distance from the thresholds $\lambda^\dagger_0 = 0$ and $\lambda^\dagger_1 = \pi^2$ so that we may choose the same weight index γ for all legalized λ^ε.

Hence, on the basis of a perturbation argument, a general result in [41], see also [59, § 3.2], provides a common factor $c^\varepsilon = const$ in the estimate (5.5) for ingredients of the asymptotic representation (5.4) of the solution $u^\varepsilon_\infty = \chi u^\varepsilon$ to the problem (5.28) in the ε-independent domain $\Pi_\infty(l)$. Since the weight $e^{\gamma x_1}$ is uniformly bounded in $\Pi^\varepsilon_+(2l) = \Pi^\varepsilon_+ \setminus \Pi_\infty(2l)$, the evident relation

$$\left\|\widetilde{u}^\varepsilon; W^1_\gamma\left(\Pi^\varepsilon_+(2l)\right)\right\| \leq c\left\|u^\varepsilon; W^1_{-\gamma}\left(\Pi^\varepsilon_+(2l)\right)\right\| + \sum_\pm \left(\left|a^\varepsilon_\pm\right| + \left|b^\varepsilon_\pm\right|\right)$$

allows us to extend the above mentioned estimate over the whole waveguide Π^ε_+.

Similarly, in the case (5.27) the factor C^ε in (5.26) can be fixed independently of ε too.

The desired eigenvalue (3.5) is located in the vicinity of the threshold $\lambda^\dagger_1 = \pi^2$ and the above consideration becomes unacceptable. Moreover, the normalization factor $(\pi^2 - \lambda^\varepsilon)^{-1/4}$ in (2.15) is big so that the independence property of c^ε and C^ε is surely lost. Thus, our immediate objective becomes to modify our estimates in order to make them uniform for all small $\varepsilon > 0$. We emphasize that a modification of the normalization factor does not suffice because, at $\varepsilon = 0$, the waves $e^{\pm k_1^\varepsilon x_1}\cos(\pi x_2)$ in (2.15) are equal to each other.

We follow a scheme in [54, §3] and for $\lambda^\varepsilon \in (0, \pi^2)$, define the linear combinations of the waves (2.15)

$$\mathbf{w}_1^{\pm}(\lambda^\varepsilon; x) = \frac{1}{2}\cos(\pi x_2)\left(\frac{1}{k_1^\varepsilon}(e^{k_1^\varepsilon x_1} - e^{-k_1^\varepsilon x_1}) \mp i(e^{k_1^\varepsilon x_1} + e^{-k_1^\varepsilon x_1})\right), \tag{5.29}$$

cf. (2.18). A direct calculation demonstrates that the new waves (5.29) together with the old waves (2.2),

$$\mathbf{w}_0^{\pm}(\lambda^\varepsilon; x) = w_0^{\varepsilon\pm}(x) = (2k)^{-1/2} e^{\pm ik^\varepsilon x_1}, \tag{5.30}$$

still satisfy the normalization and orthogonality conditions (2.7) but additionally are in the relationships

$$\mathbf{w}_0^{\pm}(\lambda^\varepsilon; x) - w_1^{0\pm}(x) = O((\pi^2 - \lambda^\varepsilon)x_1), \quad \mathbf{w}_1^{\pm}(\lambda^\varepsilon; x) - w_0^{0\pm}(x) = O((\pi^2 - \lambda^\varepsilon)^{1/2}x_1).$$

These waves also keep the important relationship

$$\mathbf{w}_p^{-}(\lambda^\varepsilon; x) = \overline{\mathbf{w}_p^{+}(\lambda^\varepsilon; x)}, \quad p = 0, 1,$$

cf. comments to formulas (2.13).

In other words, the waves (5.29) and (5.30) turn smoothly into the waves (2.8) and (2.2). The first property of $\mathbf{w}_p^{\pm}(\lambda^\varepsilon; x)$ allows us to repeat considerations in Sections 5.4, 2.2 and compose the space $\mathbf{W}_\gamma^1(\Pi_+^\varepsilon)_{out}$ of functions satisfying the new, auxiliary, radiation condition

$$\mathbf{u}^\varepsilon(x) = \widetilde{\mathbf{u}}^\varepsilon(x) + \mathbf{a}_+^\varepsilon \mathbf{w}_0^+(\lambda^\varepsilon; x) + \mathbf{b}_+^\varepsilon \mathbf{w}_1^+(\lambda^\varepsilon; x), \quad \widetilde{\mathbf{u}}^\varepsilon \in \mathbf{W}_\gamma^1(\Pi_+^\varepsilon), \tag{5.31}$$

cf. (5.25). In this way, as was explained in Section 2.2, we determine the solutions $\mathbf{Z}_p^\varepsilon(\lambda^\varepsilon; \cdot) \in \mathbf{W}_{-\gamma}^1(\Pi_+^\varepsilon)$ of the homogeneous problem (5.2), $\sigma = -\gamma$,

$$\mathbf{Z}_p^\varepsilon(\lambda^\varepsilon; x) = \widetilde{\mathbf{Z}}_p^\varepsilon(\lambda^\varepsilon; x) + \mathbf{w}_p^-(\lambda^\varepsilon; x) + \mathbf{S}_{0p}^\varepsilon(\lambda^\varepsilon)\mathbf{w}_0^-(\lambda^\varepsilon; x) + \mathbf{S}_{1p}^\varepsilon(\lambda^\varepsilon)\mathbf{w}_1^+(\lambda^\varepsilon; x), \tag{5.32}$$
$$\widetilde{\mathbf{Z}}_p^\varepsilon(\lambda^\varepsilon; \cdot) \in \mathbf{W}_\gamma^1(\Pi_+^\varepsilon), \ p = 0, 1,$$

cf. (2.19), and introduce the auxiliary scattering matrix $\mathbf{S}^\varepsilon(\lambda^\varepsilon) = \left(\mathbf{S}_{qp}^\varepsilon(\lambda^\varepsilon)\right)_{q,\,p=0,1}$ which possesses the unitary and symmetry properties (2.13).

At the same time, the limiting property (5.30) of $\mathbf{w}_p^{\pm}(\lambda^\varepsilon; x)$ assures that, for a fixed ε, the mapping

$$\mathbf{B}_\gamma^\varepsilon(\lambda^\varepsilon)_{out} : \mathbf{W}_\gamma^1(\Pi_+^\varepsilon)_{out} \to W_{-\gamma}^1(\Pi_+^\varepsilon)^* \tag{5.33}$$

of the problem (5.2), $\sigma = -\gamma$, with the radiation condition (5.31), depends continuously on the spectral parameter $\lambda^\varepsilon \in (\pi^2 - \delta, \pi^2]$, $\delta > 0$, when the domain of the operator (5.33) is equipped with the norm

$$\left\|\mathbf{u}^\varepsilon; \mathbf{W}_\gamma^1(\Pi_+^\varepsilon)\right\| = \left\|\widetilde{\mathbf{u}}^\varepsilon; W_\gamma^1(\Pi_+^\varepsilon)\right\| + \left|\mathbf{a}_+^\varepsilon\right| + \left|\mathbf{b}_+^\varepsilon\right| \tag{5.34}$$

of a weighted space with detached asymptotics, cf. the left-hand side of (5.27).

Recalling our reasoning in Section 5.3 and the beginning of this section, we conclude that the operator (5.33) is an isomorphism while its norm and the norm of the inverse are uniformly bounded for

$$\lambda^{\varepsilon} \in \left[\pi^2 - \delta, \pi^2\right], \quad \varepsilon \in [0, \varepsilon_0]. \tag{5.35}$$

Furthermore, by the Fourier method, entries of the matrix $\mathbf{S}^{\varepsilon}(\lambda)$ can be expressed as weighted integrals of the solutions (5.32). Hence, this matrix is continuous in the variables (5.35) and the limit matrix

$$\mathbf{S}^0\left(\pi^2\right) = \operatorname{diag}\{1, -1\} \tag{5.36}$$

is nothing but the augmented scattering matrix at the threshold and its diagonal form is due to the following explicit solutions (3.15) and (3.16) in the semi-infinite strip Π^0:

$$Z_0^0(x) = (2\pi)^{-1/2}\left(e^{i\pi x_1} + e^{-i\pi x_1}\right),$$
$$Z_1^0(x_1) = \cos(\pi x_2) = \frac{1}{2i}\left((x_1 + i)\cos(\pi x_2) - (x_1 - i)\cos(\pi x_2)\right).$$

We resume the above consideration and find out a unique solution $\mathbf{u}^{\varepsilon} \in \mathbf{W}^1_{\gamma}\left(\Pi^{\varepsilon}_{+}\right)_{out} \subset W^1_{-\gamma}\left(\Pi^{\varepsilon}_{+}\right)$ of the problem (5.2) with $\sigma = -\gamma$, $F^{\varepsilon} \in W^1_{-\gamma}\left(\Pi^{\varepsilon}_{+}\right)^*$ and the artificial radiation condition (5.31). Moreover, the estimate

$$\left\|\mathbf{u}^{\varepsilon}; \mathbf{W}^1_{\gamma}\left(\Pi^{\varepsilon}_{+}\right)\right\| \leq c\left\|F^{\varepsilon}; W^1_{-\gamma}\left(\Pi^{\varepsilon}_{+}\right)^*\right\| \tag{5.37}$$

is valid, where c is independent of the parameters (5.35).

We now search for a solution $u^{\varepsilon} \in W^1_{\gamma}\left(\Pi^{\varepsilon}_{+}\right)_{out}$ of the same integral identity but with the radiation condition from Section 5.4. We set

$$u^{\varepsilon} = \mathbf{u}^{\varepsilon} + \mathbf{c}^{\varepsilon}_0 \mathbf{Z}^{\varepsilon}_0 + \mathbf{c}^{\varepsilon}_1 \mathbf{Z}^{\varepsilon}_1. \tag{5.38}$$

The unknown coefficients $\mathbf{c}^{\varepsilon}_p$ should be fixed to provide the decomposition (5.25). To this end, we insert formulas (5.31), (5.32) and (5.29), (5.30) into the right-hand side of (5.38). We then compare the resultant coefficients of the waves (2.2), (2.15) in (5.38) with those in (5.25) and arrive at the following system of linear algebraic equations for the unknowns $\mathbf{c}^{\varepsilon}_0$, $\mathbf{c}^{\varepsilon}_1$ and a^{ε}_{+}, b^{ε}_{+} :

$$a^{\varepsilon}_{+} = \mathbf{a}^{\varepsilon}_{+} + \mathbf{S}^{\varepsilon}_{01}\mathbf{c}^{\varepsilon}_1, \quad 0 = \mathbf{c}^{\varepsilon}_0, \tag{5.39}$$

$$\begin{aligned} \left(2k^{\varepsilon}_1\right)^{1/2} b^{\varepsilon}_{+} &= \left(1 - ik^{\varepsilon}_1\right)\mathbf{b}^{\varepsilon}_{+} + \left(\left(1 + ik^{\varepsilon}_1\right) + \left(1 - ik^{\varepsilon}_1\right)\mathbf{S}^{\varepsilon}_{11}\right)\mathbf{c}^{\varepsilon}_1, \\ \left(2k^{\varepsilon}_1\right)^{1/2} b^{\varepsilon}_{+} &= \left(1 + ik^{\varepsilon}_1\right)\mathbf{b}^{\varepsilon}_{+} + \left(\left(1 - \varepsilon k^{\varepsilon}_1\right) + \left(1 + ik^{\varepsilon}_1\right)\mathbf{S}^{\varepsilon}_{11}\right)\mathbf{c}^{\varepsilon}_1. \end{aligned} \tag{5.40}$$

Solving the system (5.40) with the help of Cramer's rule, a simple calculation gives the determinant

$$\left(2k^{\varepsilon}_1\right)^{3/2}\left(1 - \mathbf{S}^{\varepsilon}_{11}\right) = \left(2\sqrt{\pi^2 - \lambda^{\varepsilon}}\right)^{3/2} i\left(1 - \mathbf{S}^{\varepsilon}_{11}\right)$$

and the estimates

$$\left|b^{\varepsilon}_{+}\right| \leq c\left(\pi^2 - \lambda^{\varepsilon}\right)^{-1/2}\left|\mathbf{b}^{\varepsilon}_{+}\right|, \quad \left|\mathbf{c}^{\varepsilon}_1\right| \leq c\left|\mathbf{b}^{\varepsilon}_{+}\right|, \tag{5.41}$$

because $2 \geq \left|1 - \mathbf{S}^{\varepsilon}_{11}\right| \geq 1/2$ due to (5.36) and (5.35). In view of the first relation in (5.39), we obtain that

$$\left|a^{\varepsilon}_{+}\right| \leq c\left(\left|\mathbf{a}^{\varepsilon}_{+}\right| + \left|\mathbf{b}^{\varepsilon}_{+}\right|\right).$$

Collecting formulas (5.40), (5.41) and (5.37), (5.34) proves the inequality (5.26) as well as Theorem 9. Moreover, the following assertion is valid.

Theorem 11. *Let $\lambda^\varepsilon \in [\pi^2-\delta, \pi^2]$, $\varepsilon \in (0, \varepsilon_0]$ and $\gamma \in (\beta_1, \beta_2)$. The solution (5.25) of the problem (5.2) with $\sigma = -\gamma$ and $F^\varepsilon \in W^1_{-\gamma}\left(\Pi^\varepsilon_+\right)^*$ admits the estimate*

$$\left\|\widetilde{u}^\varepsilon; W^1_\gamma\left(\Pi^\varepsilon_+\right)\right\| + \left|a^\varepsilon_+\right| + \left(\pi^2-\lambda^\varepsilon\right)^{1/4}\left|b^\varepsilon_+\right| \leq C\left\|F^\varepsilon; W^1_{-\gamma}\left(\Pi^\varepsilon_+\right)^*\right\|, \tag{5.42}$$

where C does not depend on λ^ε, ε and F^ε.

6. Justification of the asymptotics

6.1. The global asymptotic approximation

The reformulation (4.2) of the criterion (2.20) implicates the coefficients S^ε_{11} and S^ε_{10} in the decomposition (2.19) of the special solution Z^ε_1 of the problem (1.12), (1.13) and this section is devoted to justification of the formal asymptotic expansions (3.7). We emphasize that the similar expansions (A1) of other entries in S^ε can be verified in the same way but we had actually used much simpler relations (3.4) and (4.3) in Section 4.1 only.

In Section 3, we applied the method of matched asymptotic expansions and our immediate objective becomes to compose a global approximate solution from the inner and outer expansions (3.10) and (3.12). To this end, we employ several smooth cut-off functions:

$$\begin{aligned} &X_\varepsilon(x) = 1 \text{ for } x_1 \leq l + 1/\varepsilon, && X_\varepsilon(x) = 0 \text{ for } x_1 \geq 2l + 1/\varepsilon,\\ &\chi_\infty(x) = 1 \text{ for } x_1 \geq 2l, && \chi_\infty(x) = 0 \text{ for } x_1 \leq 3l/2,\\ &\chi_\varepsilon(r) = 1 \text{ for } r \leq 2\varepsilon, && \chi_\varepsilon(r) = 0 \text{ for } r \geq 3\varepsilon, \end{aligned} \tag{6.1}$$

where $r = \left(|x_1-l|^2 + x_2^2\right)^{1/2}$. We set

$$\mathfrak{Z}^\varepsilon = \chi_\infty \mathfrak{Z}^{out} + X_\varepsilon \mathfrak{Z}^{in} - \chi_\infty X_\varepsilon \mathfrak{Z}^{mat}, \tag{6.2}$$

$$\mathfrak{Z}^{out}(x) = w^{\varepsilon-}_1(x) + S^0_{11} w^{\varepsilon+}_1(x) + \varepsilon^{1/2} S^0_{01} w^{\varepsilon+}_0(x), \tag{6.3}$$

$$\mathfrak{Z}^{in}(x) = \varepsilon^{-1/2} Z^0_1(x) + \varepsilon^{1/2}\Big(\left(1-\chi_\varepsilon(r)\right)\widehat{Z}'_1(x) + \chi_\varepsilon(r) Z'_1(l,0)\Big), \tag{6.4}$$

$$\begin{aligned} \mathfrak{Z}^{mat}(x) = {}& \varepsilon^{-1/2}(4\mu)^{-1/4}\cos(\pi x_2)\left(1+i+S^0_{11}(1-i) + \varepsilon x_1\sqrt{\mu}\left(1-i+S^0_{11}(1+i)\right)\right) + \\ &+ \varepsilon^{1/2} S^0_{01}(2\pi)^{-1/2} e^{i\pi x_1}. \end{aligned} \tag{6.5}$$

This construction needs an explanation. First, the expansions (6.3) and (6.4) of the outer and inner types are multiplied with the cut-off functions χ_∞ and X_ε whose supports overlap each other so that the sum (6.5) of the terms matched in Section 3.2, attend the global approximation twice, i.e., in $\chi_\infty \mathfrak{Z}^{out}$ and $X_\varepsilon \mathfrak{Z}^{in}$, but we compensate for this duplication by subtracting $\chi_\infty X_\varepsilon \mathfrak{Z}^{mat}$ in (6.2). Moreover, the formula $[\Delta, \chi_\infty X_\varepsilon] = [\Delta, \chi_\infty] + [\Delta, X_\varepsilon]$ for commutators demonstrates that

$$\begin{aligned} (\Delta+\lambda^\varepsilon)\mathfrak{Z}^\varepsilon = {}& \chi_\infty(\Delta+\lambda^\varepsilon)\mathfrak{Z}^{out} + X_\varepsilon(\Delta+\lambda^\varepsilon)\mathfrak{Z}^{in} - \chi_\infty X_\varepsilon(\Delta+\lambda^\varepsilon)\mathfrak{Z}^{mat} + \\ &+ [\Delta,\chi_\infty]\left(\mathfrak{Z}^{out} - \mathfrak{Z}^{mat}\right) + [\Delta, X_\varepsilon]\left(\mathfrak{Z}^{in} - \mathfrak{Z}^{mat}\right) := \\ &:= \mathcal{F}^\varepsilon = \chi_\infty \mathcal{F}^{out} + X_\varepsilon \mathcal{F}^{in} - \chi_\infty X_\varepsilon \mathcal{F}^{mat} + \mathcal{F}^{oma} + \mathcal{F}^{ima}. \end{aligned} \tag{6.6}$$

Second, the function Z_1^0 is properly defined by (3.17) in the whole waveguide but Z_1' needs an extension from Π_+^0 on Π_+^ε denoted by $\widehat{Z}_1'$ in (6.4). Since the Neumann datum (3.20) in the problem (3.18), $p=1$, has a jump at the point $(0,l)\in\partial\Pi_+^0$, the solution Z_p' gets a singular behavior near this point. A simple calculation which is based on the Kondratiev theory [41] (see also [59, Ch. 2]) and can be verified directly, shows that

$$Z_1'(x)=\pi^{-1}G_1'r(\ln r\cos\varphi-\varphi\sin\varphi)+\breve{Z}_1'(x), \tag{6.7}$$

where $(r,\varphi)\in\mathbb{R}_+\times(0,\pi)$ are the polar coordinates in fig. 6, b and $\breve{Z}_1'$ is a smooth function in the closed rectangle $\overline{\Pi_+^0(R)}$ of any fixed length R. We emphasize that the solution Z_1' has no singularities at the corner points $(0,0)$ and $(0,1)$, cf. [59, Ch. 2], but the third derivatives of $\breve{Z}_1'$ are not bounded when $r\to+0$. The extension $\widehat{Z}_1'$ in (6.4) is defined by the formula (6.7) where $\breve{Z}_1'$ is smoothly continued through the segment $\{x:x_1\in[0,l],\ x_2=0\}$.

Finally, we mention that the correction term Z_1' in (3.10) was determined in Section 3.2 up to the addendum $C_1^0\cos(\pi x_1)$ but putting $C_1^0=0$ in the expansion (3.21) uniquely defines the function Z_1' as well as its value $Z_1'(l,0)$ according to (6.7). Notice that we also must take $S_{11}'=0$ by virtue of (3.24). The extension $\widehat{Z}_1'$ of Z_1' is smooth everywhere in a neighborhood of $\overline{\Pi_+^0}$ except at the point $(l,0)$ where it inherits a singularity from (6.7). Using the partition of unity $\{1-\chi_\varepsilon,\chi_\varepsilon\}$ makes the last term in (6.4) smooth in $\overline{\Pi_+^\varepsilon}$ but produces additional discrepancies in the equation (1.3).

6.2. Estimating discrepancies

First of all, we observe that $\mathcal{F}^{out}=0$ in Π_+^0 according to the definition of waves in (2.2) and (2.18). In view of the factor χ_∞ from (6.1) the first term on the right-hand side of (6.6) vanishes. Moreover, the Taylor formulas (3.6) and (3.9) assure that

$$\left|\mathfrak{Z}^{out}(x)-\mathfrak{Z}^{mat}(x)\right|+\left|\nabla\mathfrak{Z}^{out}(x)-\nabla\mathfrak{Z}^{mat}(x)\right|\le c\varepsilon^{3/2}$$

on the rectangle $[3l/2,2l]\times[0,1]$ containing the supports of the coefficients in the commutator $[\Delta,\chi_\infty]$. Hence,

$$|\mathcal{F}^{oma}(x)|\le c\varepsilon^{3/2},\qquad \mathcal{F}^{oma}(x)=0\quad\text{for } x_1\ge 2l. \tag{6.8}$$

Let us consider the sum

$$\mathcal{F}^{inm}(x)=X_\varepsilon\mathcal{F}^{in}-X_\varepsilon\chi_\infty\mathcal{F}^{mat}. \tag{6.9}$$

Outside the finite domain $\Pi_+^\varepsilon(3l/2)$, it is equal to

$$\begin{aligned}&\varepsilon^{-1/2}X_\varepsilon(x)\left(\left(\Delta+\pi^2\right)Z_1^0(x)+\varepsilon\left(\Delta+\pi^2\right)Z_1'(x)-\chi_\infty\left(\Delta+\pi^2\right)\mathfrak{Z}^{mat}(x)\right)+\\&+\varepsilon^{-1/2}X_\varepsilon(x)\left(\lambda^\varepsilon-\pi^2\right)\left(Z_1^0(x)+\varepsilon Z_1'(x)-\chi_\infty(x)\mathfrak{Z}^{mat}(x)\right)=\\&=0+\varepsilon^{3/2}\mu(Z_1^0(x)-\chi_\infty(x)(4\mu)^{-1/4}\cos(\pi x_2)\left(1+i+S_{11}^0(1-i)\right)-\\&-\varepsilon(Z_1'(x)-\chi_\infty(x)((4\mu)^{-1/4}\cos(\pi x_2)x_1\sqrt{\mu}\left(1-i+S_{11}^0(1+i)\right)+S_{01}^0(2\pi)^{-1/2}e^{i\pi x_1}),\end{aligned} \tag{6.10}$$

where formulas (3.5) and (4.1) were taken into account. We now use the representations (3.17) and (3.21) to conclude that

$$\left|\mathcal{F}^{inm}(x)\right|\le c\varepsilon^{3/2}e^{-x_1\sqrt{3}\pi}\qquad\text{for } x_1\ge 3l/2. \tag{6.11}$$

Inside $\Pi_+^\varepsilon(3l/2)$, we have

$$\mathcal{F}^{inm} = -\varepsilon^2\mu\mathfrak{Z}^{in} + \varepsilon^{1/2}(1-\chi_\varepsilon)\left(\Delta+\pi^2\right)\widehat{Z}_1' - \varepsilon^{1/2}[\Delta,\chi_\varepsilon](\widehat{Z}_1' - Z_1'(l,0)).$$

The inequality

$$\varepsilon^2\mu\left|\mathfrak{Z}^{in}(x)\right| \le c\varepsilon^{3/2} \quad \text{in } \overline{\Pi_+^\varepsilon(3l/2)}$$

is evident. Because of the singularity $O(r|\ln r|)$ in (6.7), estimates of other two terms in (6.10) involve the weighting function

$$\rho(x) = r + (1+|\ln r|), \tag{6.12}$$

cf. the Hardy inequality (6.17) below. Since $\widehat{Z}_1' = Z_1'$ in Π_+^0 satisfies the Helmholtz equation from (3.18), we have

$$\begin{aligned}
\varepsilon^{1/2}(1-\chi_\varepsilon(r))\left(\cdot+\pi^2\right)\widehat{Z}_1'(x) = 0, &\quad x\in\Pi_+^0(3l/2),\\
\varepsilon^{1/2}\left|(1-\chi_\varepsilon(r))\left(\cdot+\pi^2\right)\widehat{Z}_1'(x)\right| \le c\varepsilon^{1/3}|x_1|(\varepsilon+r)^{-2}\rho(x), &\quad x\in\varpi_+^\varepsilon = \Pi_+^\varepsilon\setminus\Pi_+^0.
\end{aligned}$$

Observing that according to the third line in (6.1), the coefficients in the commutator $[\Delta,\chi_\varepsilon] = 2\nabla\chi_\varepsilon\cdot\nabla + \Delta\chi_\varepsilon$ gets the orders ε^{-1} and ε^{-2} respectively, but vanish outside the set $\Cap^\varepsilon = \left\{x\in\Pi_+^\varepsilon : 2\varepsilon < r < 3\varepsilon\right\}$, we obtain that

$$\begin{aligned}
\varepsilon^{1/2}[\Delta,\chi_\varepsilon]\left(\widehat{Z}_1'(x) - Z_1'(l,0)\right) = 0, &\quad x\in\Pi_+^\varepsilon(3l/2)\setminus\Cap^\varepsilon,\\
\varepsilon^{1/2}\left|[\Delta,\chi_\varepsilon(\tau)]\left(\widehat{Z}_1'(x) - Z_1'(l,0)\right)\right| &\le c\varepsilon^{1/2}\left(\varepsilon^{-1}|\ln r| + \varepsilon^{-2}r|\ln r|\right) \le\\
&\le c\varepsilon^{1/2}|x_1|(\varepsilon+r)^{-2}\rho(x),\ x\in\Cap^\varepsilon.
\end{aligned} \tag{6.13}$$

Finally, we mention that the support of the term $\mathcal{F}^{ima}$ in (6.6) belongs to the rectangle $[l+1/\varepsilon, 2l+1/\varepsilon]\times[0,1]$, see the first line of (6.1), where the remainder $\widetilde{Z}_1'(x)$ in (3.21) gains the order $O\left(e^{-\sqrt{3}\pi/\varepsilon}\right)$ and hence,

$$\left|[\Delta,X_\varepsilon]\left(\mathfrak{Z}^{in}(x) - \mathfrak{Z}^{mat}(x)\right)\right| = \varepsilon^{1/2}\left|[\Delta,X_\varepsilon]\widetilde{Z}_1'(x)\right| \le c\varepsilon^{1/2}e^{-3\sqrt{\pi}/\varepsilon}. \tag{6.14}$$

It remains to evaluate discrepancies in the Neumann condition (1.4). Since the cut-off functions X_ε and χ_∞ can be chosen depending on the longitudinal coordinate x_1 only, the approximate solution (6.2) satisfies the homogeneous Neumann condition everywhere on $\partial\Pi_+^0$ except on the sides Υ^ε and υ^ε of the rectangle (1.2). Furthermore, Z_1^0 does not depend on x_1 and Z_1' is multiplied in (6.4) with the cut-off function χ_ε in the radial variable r. Thus, $\partial_1\mathfrak{Z}^\varepsilon = 0$ on the short side υ^ε. Regarding the trace $\mathcal{G}^\varepsilon$ of $\partial_\nu\mathfrak{Z}^\varepsilon = -\partial_2\mathfrak{Z}^{as}$ on the long side Υ^ε, we take formulas (3.18)-(3.20) into account and similarly to (6.12) and (6.13), obtain the estimate

$$|\mathcal{G}^\varepsilon(x_1,-\varepsilon)| \le c\varepsilon^{3/2}(\varepsilon+r)^{-1}. \tag{6.15}$$

We emphasize that differentiation in x_2 eliminates $\ln r$ from the first term of (6.7).

6.3. Comparing the approximate and true solutions

First of all, we observe that $\mathfrak{Z}^{as}(x) = \mathfrak{Z}^{out}(x)$ as $x_1 > 2l$ by virtue of the definition (6.1) of the cut-off functions X_ε and χ_∞. Thus, in view of (2.19) and (6.2), (6.3), the difference $\mathcal{R}^\varepsilon = Z^\varepsilon - \mathfrak{Z}^\varepsilon$ loses the incoming waves $w_p^{\varepsilon-}$ and, therefore, falls into the space $W_\gamma^1\left(\Pi_+^\varepsilon\right)_{out}$. Moreover, the decomposition (5.25) of $\mathcal{R}^\varepsilon$ contains the coefficients $a_+^\varepsilon = \widehat{S}_{10}^\varepsilon$ and $b_+^\varepsilon = \widehat{S}_{11}^\varepsilon$ defined in (1.16) and (1.18). The integral identity (5.2) with $\sigma = -\gamma$ and $u^\varepsilon = \mathcal{R}^\varepsilon$ involves the functional

$$F^\varepsilon(v^\varepsilon) = (\mathcal{F}^\varepsilon, v^\varepsilon)_{\Pi_+^\varepsilon} - (\mathcal{G}^\varepsilon, v^\varepsilon)_\Upsilon, \qquad (6.16)$$

where $\mathcal{F}^\varepsilon$ is given in (6.6) and $\mathcal{G}^\varepsilon = -\partial_\nu \mathfrak{Z}^\varepsilon$. If we prove the inclusion $F^\varepsilon \in W_{-\gamma}^1\left(\Pi_+^\varepsilon\right)^*$, then the estimate (5.42) adjusted by the weight factor $\left(\pi^2 - \lambda^\varepsilon\right)^{1/4} = \varepsilon^{1/2}\mu^{1/4}$ demonstrates that

$$\left|\widehat{S}_{10}^\varepsilon\right| + \varepsilon^{1/2}\left|\widehat{S}_{11}^\varepsilon\right| \le c\left\|F^\varepsilon, W_{-\gamma}^1\left(\Pi_+^\varepsilon\right)\right\|.$$

We fix a test function $v^\varepsilon \in W_\gamma^1\left(\Pi_+^\varepsilon\right)$. The classical one-dimensional Hardy inequality

$$\int_0^l r^{-2}\left|\ln\frac{r}{l}\right|^2 |V(r)|^2\, rdr \le 4\int_0^l \left|\frac{dV}{dr}(r)\right|^2 rdr, \quad V \in C_0^\infty[0,l),$$

in a standard way, cf. [46, Ch.1, §4], leads to the estimate

$$\left\|\rho^{-1}v^\varepsilon; L^2\left(\Pi_+^\varepsilon(2l)\right)\right\|^2 \le c\left\|v^\varepsilon; H^1\left(\Pi_+^\varepsilon(2l)\right)\right\|^2 \le c_\gamma\left\|v^\varepsilon; W_\gamma^1\left(\Pi_+^\varepsilon\right)\right\|^2, \qquad (6.17)$$

where ρ is the weight factor (6.12). Moreover, introducing the new weight factor $\rho_1(x) = r\left(1 + |\ln r|\right)^2$, we derive the specific trace inequality

$$\begin{aligned}\int_{\Upsilon^\varepsilon} \rho_1^{-1}|v^\varepsilon|^2 dx_1 &= \int_{\Pi_+^\varepsilon(l)} \frac{\partial}{\partial x_2}\left(\chi_0\rho_1^{-1}|v^\varepsilon|^2\right)dx \le \\ &\le c\int_{\Pi_+^\varepsilon(l)}\left(\left|\frac{\partial v^\varepsilon}{\partial x_2}\right|\rho_1^{-1}|v^\varepsilon| + \left(1 + \frac{\partial}{\partial x_2}\rho_1^{-1}\right)|v^\varepsilon|^2\right)dx \le \\ &\le c\int_{\Pi_+^\varepsilon(l)}\left(|\nabla v^\varepsilon|^2 \rho^{-2}|v^\varepsilon|^2\right)dx \le c_\gamma\left\|v^\varepsilon; W_\gamma^1\left(\Pi_+^\varepsilon\right)\right\|^2. \end{aligned} \qquad (6.18)$$

Here, we took into account that $\left|\nabla\rho_1(x)^{-1}\right| \le c\rho(x)^{-2}$ and used the cut-off function (5.23).

The inclusion $F^\varepsilon \in W_{-\gamma}^1\left(\Pi_+^\varepsilon\right)^*$ is obvious because $\mathcal{F}^\varepsilon$ has a compact support. To estimate the norm $\left\|\mathcal{F}^\varepsilon; W_{-\gamma}^1\left(\Pi_+^\varepsilon\right)^*\right\|$, we apply the inequalities obtained in the previous section. Since $\gamma \in (0, \sqrt{3}\pi)$, the estimate (6.14) gives

$$\left|\left(\mathcal{F}^{ima}, v^\varepsilon\right)_{\Pi_+^\varepsilon}\right| \le c\varepsilon^{1/2}e^{\left(\gamma - \sqrt{3}\pi\right)/\varepsilon}\int_{l+1/\varepsilon}^{2l+1/\varepsilon}\int_0^1 e^{-\gamma x_1}|v^\varepsilon(x)|\,dx \le c\varepsilon^{3/2}\left\|v^\varepsilon; W_{-\gamma}^1\left(\Pi_+^\varepsilon\right)\right\|.$$

By the formula (6.8), we have

$$\left|\left(\mathcal{F}^{oma}, v^\varepsilon\right)_{\Pi_+^\varepsilon}\right| \le c\varepsilon^{3/2}\left\|v^\varepsilon; L^1\left(\Pi_+^\varepsilon(3l/2)\right)\right\| \le c\varepsilon^{3/2}\left\|v^\varepsilon; W_{-\gamma}^1\left(\Pi_+^\varepsilon\right)\right\|.$$

Recalling (6.9) and (6.11), (6.12), (6.13) yields

$$\left|\left(\mathcal{F}^{inm}, v^{\varepsilon}\right)_{\Pi_{+}^{\varepsilon}}\right| \leq \left(c\varepsilon^{3/2}\int_{\Pi_{\infty}(3l/2)} e^{-x_{1}\sqrt{3}\pi}\left|v^{\varepsilon}(x)\right|dx+\varepsilon^{1/2}\int_{\varpi_{+}^{\varepsilon}}\frac{\left|x_{1}\right|\rho(x)}{(\varepsilon+r)^{2}}\left|v^{\varepsilon}(x)\right|dx+\right.$$
$$\left.+\varepsilon^{1/2}\int_{\Subset^{\varepsilon}}\frac{\rho(x)}{(\varepsilon+r)^{2}}\left|v^{\varepsilon}(x)\right|dx\right) \leq$$
$$\leq c(\varepsilon^{3/2}\left(\int_{3l/2}^{+\infty} e^{2\left(\gamma-\sqrt{3}\pi\right)x_{1}}dx_{1}\right)^{1/2}\left(\int_{\Pi_{+}^{\varepsilon}(3l/2)} e^{-2\gamma x_{1}}\left|v^{\varepsilon}(x)\right|^{2}dx\right)^{1/2}+$$
$$+\varepsilon^{1/2}\left(\int_{\varpi_{+}^{\varepsilon}}\left|x_{1}\right|^{2}\frac{(\rho(x))^{4}}{(\varepsilon+r)^{4}}dx+\int_{\Subset^{\varepsilon}}\frac{(\rho(x))^{4}}{(\varepsilon+r)^{4}}dx\right)^{1/2}\left\|\rho^{-1}v^{\varepsilon};L^{1}\left(\Pi_{+}^{\varepsilon}(l)\right)\right\| \leq$$
$$\leq \varepsilon^{3/2}(1+|\ln\varepsilon|)^{2}\left\|v^{\varepsilon};W_{-\gamma}^{1}\left(\Pi_{+}^{\varepsilon}\right)\right\|.$$

Finally, from (6.15) and (6.18), we derive the following estimate of the last scalar product in (6.16):

$$\left|(\mathcal{G},v^{\varepsilon})_{\Upsilon_{\varepsilon}}\right| \leq c\varepsilon^{3/2}\left(\int_{\Upsilon_{\varepsilon}}(\varepsilon+r)^{-2}\rho_{1}dx_{1}\right)^{1/2}\left\|\rho_{1}^{-1/2}v^{\varepsilon};L^{2}\left(\Upsilon_{\varepsilon}\right)\right\| \leq c\varepsilon^{3/2}(1+|\ln\varepsilon|)^{3/2}\left\|v^{\varepsilon};W_{-\gamma}^{1}\left(\Pi^{\varepsilon}\right)\right\|.$$

Collecting the obtained inequalities, we conclude that the functional (6.16) meets the estimate

$$\left\|F^{\varepsilon};W_{-\gamma}^{1}\left(\Pi^{\varepsilon}\right)^{*}\right\| \leq c\varepsilon^{3/2}(1+|\ln\varepsilon|)^{2}. \qquad (6.19)$$

6.4. Asymptotics of the augmented scattering matrix

All cumbersome calculations and technical details have been presented in the previous sections and we are in position to formulate the main result of our paper. Since the norm $||\mathcal{R}^{\varepsilon};W_{\gamma}^{1}\left(\Pi_{+}^{\varepsilon}\right)_{out}||$ in the space with detached asymptotics (see Section 5) contains the coefficients $\widehat{S}_{10}^{\varepsilon}$ and $\widehat{S}_{11}^{\varepsilon}$ in the representation (5.25) of $\mathcal{R}^{\varepsilon}$, estimates of the asymptotic remainders in (1.18) and (1.16) follow directly from (6.19) and (5.42), (3.5).

We here fix the parameters $\mu = 4\pi^{4}l^{2}$ from (3.5) and $l = \pi k$ from (1.2) but will consider their small variations (4.1) in the next section.

Theorem 12. *Remainders in the asymptotic expansions (1.16) and (1.18) obey the estimate*

$$\left|\widehat{S}_{11}^{\varepsilon}\right| + \varepsilon^{-1/2}\left|\widehat{S}_{01}^{\varepsilon}\right| \leq c_{0}\varepsilon(1+|\ln\varepsilon|)^{2} \quad \text{for } \varepsilon \in (0,\varepsilon_{0}], \qquad (6.20)$$

where c_0 and ε_0 are some positive numbers depending on l and μ.

Remark 13. *Although the entry S_{00}^{ε} of the augmented scattering matrix was involved only in the formulas (4.1) and (4.2) which require much less information than for S_{11}^{ε}, the simple estimate*

$$\left|S_{00}^{\varepsilon}-1\right| \leq c_{0}\varepsilon(1+|\ln\varepsilon|)^{2} \quad \text{for } \varepsilon \in (0,\varepsilon_{0}] \qquad (6.21)$$

can be obtained by repeating calculations in the previous section based on the formal asymptotics (A1), (A7) word for word. We only mention that the discrepancy (A9) on the small side υ^ε of the box ϖ_+^ε can be estimated as follows:

$$(2\pi)^{1/2}\left|\sin(\pi l)\int_{-\varepsilon}^{0} v^\varepsilon(l,x_2)\,dx_2\right| \le c\varepsilon^{1/2}(1+|\ln\varepsilon|)\left\|r^{-1/2}(1+|\ln r|)^{-1}v^\varepsilon;L^2(\upsilon^\varepsilon)\right\|$$
$$\le c\varepsilon(1+|\ln\varepsilon|)\left\|v^\varepsilon;W^1_{-\gamma}\left(\Pi_+^\varepsilon\right)\right\|$$

where a weighted trace inequality of type (6.18) was applied. ■

6.5. Dependence on $\triangle l$ and $\triangle\mu$

Since our procedure to satisfy the criterion (2.21) in Theorem 1 requires solving the trascendental equations (4.5), (4.6), we need to possess the estimates (6.20) and (6.21) for the parameters $\mu+\triangle\mu$ and $l+\triangle l$ with the couple $(\triangle\mu,\triangle l)$ in the ball (4.7) of a small radius $\varrho>0$. These can be achieved by means of a standard argument of the perturbation theory for linear operators, see, e.g., the monographs [35, 40].

Let ε be fixed, small and positive. We take $l=\pi k+\triangle l$ and make the change of coordinates

$$x\to\mathbf{x}=(\mathbf{x}_1,\mathbf{x}_2)=(x_1,x_2)\,,\qquad \mathbf{x}_1=(1-\chi_k(x_1))\,x_1+\chi_k(x_1)\,(x_1-\triangle l)\,, \tag{6.22}$$

where χ_k is a smooth cut-off function,

$$\chi_k(x_1)=1 \text{ for } |x_1-\pi k|<\pi/3 \text{ and } \chi_k(x_1)=0 \text{ for } |x_1-\pi k|>2\pi/3.$$

If $\triangle l$ is small, this change is nonsingular. Moreover, it transforms Π_l^ε into $\Pi_{\pi k}^\varepsilon$ and turns the Helmholtz operator $\triangle+\pi^2-\varepsilon^2(\mu+\triangle\mu)$ into the second-order differential operator $L^\varepsilon(\triangle l,\triangle\mu;\mathbf{x},\nabla_{\mathbf{x}})$ whose coefficients depend smoothly on $\triangle\mu$ and $\triangle l$. Clearly, $L^\varepsilon(0,0;\mathbf{x},\nabla_{\mathbf{x}})=\Delta_{\mathbf{x}}+\pi^2-\varepsilon^2\mu$. Owing to the Fourier method, we can rewrite the element $S_{11}^\varepsilon=S_{11}^\varepsilon(\triangle\mu,\triangle l)$ of the augmented scattering matrix as the integral

$$S_{11}^\varepsilon(\triangle\mu,\triangle l)=\alpha_{11}^\varepsilon(\triangle\mu)\int_{Q_k} Z_{11}^\varepsilon(\triangle\mu,\triangle l;x)\,dx \tag{6.23}$$

over the rectangle $Q_k=(\pi(k+1),\pi(k+2))\times(0,1)$, where $\mathbf{x}=x$ according to (6.22). Due to the general result in the perturbation theory of linear operators, cf. [35, 40], the special solution $Z_{11}^\varepsilon(x)=Z_{11}^\varepsilon(\triangle\mu,\triangle l;x)$ rewritten in the coordinates $\mathbf{x}$ gets the smooth dependence on the couple $(\triangle\mu,\triangle l)\in\overline{\mathbb{B}_\varrho}$. The coefficient $\alpha_{11}^\varepsilon(\triangle\mu)$ on the right-hand side of (6.23) is also a smooth function whose exact form is not needed. Thus, the element (6.23) inherits the smooth dependence while the remainder $\widetilde{S}_{11}^\varepsilon(\triangle\mu,\triangle l)$ in the representation (1.16) gets the same property according to the formula (3.26) for $S_{11}^0(\triangle\mu,\triangle l)$ written in the variables (4.1). Moreover, the estimate (6.20) remains valid with a constant c_0 which may be new but is still independent of $\varepsilon\in(0,\varepsilon_0]$ and $(\triangle\mu,\triangle l)\in\overline{\mathbb{B}_\varrho}$ as well.

Similar operations apply to $S_{01}^\varepsilon(\triangle\mu,\triangle l)$ and $\widetilde{S}_{00}^\varepsilon(\triangle\mu,\triangle l)$.

Finally, recalling our examination in Section 5.5 and Theorem 12, we formulate the result.

Proposition 14. *The remainders in the asymptotic formulas (1.16), (3.26) and (1.18), (A17) satisfy the inequality*

$$\left|\nabla_{(\triangle\mu,\triangle l)}\widetilde{S}_{11}^\varepsilon(\triangle\mu,\triangle l)\right|+\varepsilon^{-1/2}\left|\nabla_{(\triangle\mu,\triangle l)}\widetilde{S}_{01}^\varepsilon(\triangle\mu,\triangle l)\right|\le C_0\varepsilon\left(1+|\ln\varepsilon|^2\right),$$

for $\varepsilon\in(0,\varepsilon_0]$ and $(\triangle\mu,\triangle l)\in\overline{\mathbb{B}_\varrho}$, while the constant C_0 depends on l and μ, but is independent of the above indicated values of ε and $\triangle\mu$, $\triangle l$.

7. The uniqueness assertions

7.1. Eigenvalues in the vicinity of the threshold π^2

Let us adapt a trick from [54, §7] for the box-shaped perturbation (1.2) and conclude with the uniqueness mentioned in Theorem 3.

Assume that there exists an infinitesimal positive sequence $\{\varepsilon_k\}_{k\in\mathbb{N}}$, such that the problem (1.12), (1.13) in the semi-infinite waveguide $\Pi^{\varepsilon}_{l_k}$ has two eigenvalues $\lambda_1^{\varepsilon_k}$ and $\lambda_2^{\varepsilon_k}$ while

$$\varepsilon_k \to +0, \quad l_k \to l_0 > 0, \qquad \lambda_j^{\varepsilon_k} = \pi^2 + \widehat{\lambda}_j^{\varepsilon_k} \in (0, \pi^2], \quad \widehat{\lambda}_j^{\varepsilon_k} \to 0, \quad j = 1, 2. \tag{7.1}$$

In what follows, we write ε instead of ε_k. The corresponding eigenfunctions u_1^ε and u_2^ε are subject to the normalization and orthogonality conditions

$$\left\| u_j^\varepsilon; L^1\left(\Pi_+^\varepsilon(2l)\right)\right\| = 1, \quad (u_1^\varepsilon, u_2^\varepsilon)_{\Pi_+^\varepsilon} = 0, \tag{7.2}$$

cf. (5.11). Repeating with evident changes to our arguments in Section 5.3, we observe that the restrictions u_{j0}^ε of u_j^ε onto Π_+^0 converge to u_{j0}^0 weakly in $W^1_{-\gamma}\left(\Pi_+^0\right)$ and strongly in $L^2\left(\Pi_+^0(2l)\right)$. Furthermore, the limits satisfy the formula (5.14) and the following integral identity, see (5.15):

$$\left(\nabla u_{j0}^0, \nabla v\right)_{\Pi_+^0} = \pi^2\left(u_{j0}^0, v\right)_{\Pi_+^0} \quad \forall v \in C_c^\infty(\overline{\Pi_+^0}). \tag{7.3}$$

Any solution in $W^1_{-\gamma}\left(\Pi_+^0\right)$ with $\gamma \in (\beta_1, \beta_2)$ of the homogeneous Neumann problem (7.3) in $\Pi_+^0 = (0, +\infty) \times (0, 1)$ is a linear combination of two bounded solutions (3.15) and (3.16), namely

$$u_j^0(x) = c_{j1}\cos(\pi x_1) + c_{j2}\cos(\pi x_2). \tag{7.4}$$

Let us prove that $c_{11} = c_{21} = 0$ in (7.4). Since the trapped mode u_j^ε has the exponential decay at infinity, Green's formula in $\Pi_\infty(3l/2)$ with u_j^ε and the bounded function $e^{\pm i x_1\sqrt{\lambda_j^\varepsilon}}$ assures that

$$\int_0^1 e^{\pm i x_1\sqrt{\lambda_j^\varepsilon}}\left(\partial_1 u_j^\varepsilon(x) \mp i\sqrt{\lambda_j^\varepsilon} u_j^\varepsilon(x)\right)\Big|_{x_1 = 3l/2} dx_2 = 0. \tag{7.5}$$

The local estimate in the rectangle ω' which had been used in (5.15) and includes the integration segment $\{3l/2\} \times (0, 1)$ in (7.5), and formulas in (7.1), (7.2) allow us to compute the limit of the left-hand side of (7.5) and obtain that

$$e^{\pm i3l\pi/2}\int_0^1\left(\frac{\partial u_{j0}^0}{\partial x_1}\left(\frac{3}{2}l, x_2\right) \pm i\pi u_{j0}^0\left(\frac{3}{2}l, x_2\right)\right) dx_2 = 0. \tag{7.6}$$

Inserting (7.4) into (7.6), we see that $c_{j1} = 0$, indeed.

Remark 15. . *If* $\widetilde{\lambda}_j^\varepsilon > 0$ *and* $\lambda_j^\varepsilon > \pi^2$ *in (7.1), one may use Green's formula in* $\Pi_\infty(3l/2)$ *with four bounded functions* $e^{\pm i x_1\sqrt{\lambda_j^\varepsilon}}$ *and* $e^{\pm i x_1\sqrt{\lambda_j^\varepsilon - \pi^2}}\cos(\pi x_2)$*. In this way one derives the equalities* $c_{j1} = c_{j2} = 0$ *(see [54, §7] for details) and concludes that a small neighborhood of the threshold* π^2 *can contain only the eigenvalues indicated in (7.1). The same reasoning shows that the problem (1.12), (1.13) cannot get an infinitesimal eigenvalue* $\lambda^\varepsilon \to +0$ *as* $\varepsilon \to +0$. ⊠

Since $c_{j1} = 0$, the limit (5.14) of the normalization conditions in (7.2) shows that $u_j^0(x) = l^{-1/2}\cos(\pi x_2)$, $j = 1,2$. Moreover, Theorem 5 (2) applied to the trapped mode $u_j^\varepsilon \in H^1(\Pi_+^\varepsilon) \subset W_{-\gamma}^1(\Pi_+^\varepsilon)$ gives the formulas

$$u_j^\varepsilon(x) = B_j^\varepsilon e^{-x_1\sqrt{\pi^2-\lambda_j^\varepsilon}}\cos(\pi x_2) + \widetilde{u}_j^\varepsilon(x), \qquad |b_j^\varepsilon| + \|\widetilde{u}_j^\varepsilon; W_\gamma^1(\Pi_+^\varepsilon)\| \le c\,\|u_j^\varepsilon; W_{-\gamma}^1(\Pi_+^\varepsilon)\|, \tag{7.7}$$

where $\gamma \in (\beta_1, \beta_2)$, c is independent of ε according to the content of Section 5.5 and the waves $w_0^{\varepsilon\pm}$ in (2.2) and $v_1^{\varepsilon+}$ in (2.15) do not appear in the expansion of the eigenfunctions u_j^ε because they decay at infinity. Since the right-hand side of (7.7) is uniformly bounded in $\varepsilon = \varepsilon_k$, $k \in \mathbb{N}$ (see Section 5.5 again), we have

$$bB_j^\varepsilon \to l^{-1/2}, \qquad \widetilde{u}_{j0}^\varepsilon \to 0 \text{ weakly in } W_\gamma^1(\Pi_+^\varepsilon)$$

along a subsequence of $\{\varepsilon_k\}_{k\in\mathbb{N}}$. Moreover, the last equality in (7.2) turns into

$$0 = (u_1^\varepsilon, u_2^\varepsilon)_{\Pi_+^\varepsilon} = \int_{\Pi_+^\varepsilon} \cos^2(\pi x_2)\, e^{-x_1\Lambda_\varepsilon} dx + (u_1^\varepsilon - \widetilde{u}_1^\varepsilon, \widetilde{u}_2^\varepsilon)_{\Pi_+^\varepsilon} + (\widetilde{u}_1^\varepsilon, u_2^\varepsilon)_{\Pi_+^\varepsilon} = \frac{1}{2}\Lambda_\varepsilon^{-1} b_1^\varepsilon \overline{b_2^\varepsilon} + O(1).$$

We multiply this relation with $\Lambda_\varepsilon = \sqrt{\pi^2 - \lambda_1^\varepsilon} + \sqrt{\pi^2 - \lambda_2^\varepsilon} \to 0$ as $\varepsilon \to 0$ and from (5.33) derive the absurd formula $o(1) = b_1^\varepsilon \overline{b_2^\varepsilon} \to l^{-2}$. Thus, there can exist at most one eigenvalue indicated in (7.1).

7.2. Absence of eigenfunctions which are odd in x_1

In Section 1.3 we have changed the original problem (1.3), (1.4) in Π^ε for the Neumann problem (1.12), (1.13) in the half of the waveguide while assuming that an eigenfunction is even in x_1. Replacing (1.13) by the mixed boundary conditions

$$\partial_\nu u^\varepsilon(x) = 0, \quad x \in \partial\Pi_+^\varepsilon, \quad x_1 > 0, \quad u_+^\varepsilon(x) = 0, \ x \in \partial\Pi_+^\varepsilon, \quad x_1 = 0, \tag{7.8}$$

we deal with the alternative, namely an eigenfunction is odd in x_1 and, therefore, vanishes at $\Gamma^\varepsilon = \{x : x_1 = 0,\ x_2 \in (-\varepsilon, 1)\}$. The variational formulation of the problem (1.12), (7.8),

$$(\nabla u_+^\varepsilon, \nabla v^\varepsilon)_{\Pi_+^\varepsilon} = \lambda_+^\varepsilon (u_+^\varepsilon, v^\varepsilon)_{\Pi_+^\varepsilon} \quad \forall v^\varepsilon \in H_0^1(\Pi_+^\varepsilon; \Gamma^\varepsilon),$$

where $H_0^1(\Pi_+^\varepsilon; \Gamma^\varepsilon)$ consists of these functions in $H^1(\Pi_+^\varepsilon)$ which vanish on Γ^ε. Evident modifications of the considerations in Sections 5 and 6 adapt all our results to the mixed boundary value problem (1.12), (7.8). The only, but important, difference is that the original solutions (3.15), (3.16) of the limit Neumann problem in $\Pi_+^0 = \mathbb{R}_+ \times (0,1)$ now turn into the following ones:

$$u_0^0(x) = i\sin(\pi x_1), \quad u_1^0(x) = x_1\cos(\pi x_2).$$

Thus, supposing that, for an infinitesimal sequence $\{\varepsilon_k\}_{k\in\mathbb{N}}$, the problem (1.12), (7.8) in $\Pi_{l_k+}^\varepsilon$ has an eigenvalue $\lambda_1^{\varepsilon_k}$ with the properties (7.1) at $j = 1$, we obtain that a non-trivial limit u_{01}^0, cf. (7.4), of the corresponding eigenfunction $u_1^{\varepsilon_k}$ becomes

$$u_{10}^0(x) = c_{11}\sin(\pi x_1) + c_{12}x_1\sin(\pi x_2).$$

Now, in contrast to Section 7.1, we may insert $u_1^{\varepsilon_k}$ into Green's formula in $\Pi_\infty(3l/2)$ together with one of the three bounded functions $e^{\pm i\sqrt{\lambda_1^{\varepsilon_k}}x_1}$ and $e^{-\sqrt{\pi^2-\lambda_1^{\varepsilon_k}}x_1}\cos(\pi x_2)$. Similarly to (7.5) and (7.6), these possibilities allow us to conclude that $c_{11}=0$, $c_{12}=0$ and, hence, $u_{10}^0=0$. Remark 15 remains true for the problem (1.12), (7.8). Thus, the above-mentioned contradiction confirms the absence of eigenvalues in a small neighborhood of the threshold $\lambda^\varepsilon=\pi^2$.

7.3. Absence of eigenvalues at a distance from the threshold

For any $\lambda\in(0,\pi^2)$, the limit Neumann problem in $\Pi_+^0=\mathbb{R}_+\times(0,1)$ has the solutions

$$
\begin{aligned}
Z_0^{00}(\lambda,x) &= (2k(\lambda))^{-1/2}\left(e^{-ik(\lambda)x_1}+e^{ik(\lambda)x_1}\right), && (7.9)\\
Z_1^{00}(\lambda,x) &= (2k_1(\lambda))^{-1/2}\left(\left(e^{k_1(\lambda)x_1}+ie^{-k_1(\lambda)x_1}\right)\cos(\pi x_2)+i\left(e^{k_1(\lambda)x_1}-ie^{-k_1(\lambda)x_1}\right)\cos(\pi x_2)=\right.\\
&= (2k_1(\lambda))^{-1/2}(1+i)\left(e^{k_1(\lambda)x_1}+e^{-k_1(\lambda)x_1}\cos(\pi x_2)\right.,
\end{aligned}
$$

where $k(\lambda)=\sqrt{\lambda}$, $k_1(\lambda)=\sqrt{\pi^2-\lambda}$ and thus, the augmented scattering matrix takes the form

$$
S^{00}=\begin{pmatrix}1&0\\0&i\end{pmatrix}. \tag{7.10}
$$

To fulfill the criterion (2.21), the perturbation ϖ_+^ε in the waveguide Π_+^ε has to turn the right-hand bottom element of the matrix (7.10) into -1 that cannot be made for any $\lambda\in[0,\pi^2-c\sqrt{\varepsilon}]$ with $c>0$ and a small ε. The latter fact may be easily verified by either constructing asymptotics in Sections 3, 6, or applying a perturbation argument as in Section 5.5. Notice that we have succeeded in Sections 3.2, 4.2 to construct the asymptotics (1.16) with the main term $S_{11}^0=-1$ because the spectral parameter (3.5) stays too close to the threshold and the augmented scattering matrix in Π_+^0 is not continuous at $\lambda=\pi^2$, cf. Section 5.5. In the case of the mixed boundary value problem (1.12), (7.8), evident changes in the solutions (7.9) give the matrix $S^{00}=\operatorname{diag}\{-1,-i\}$ instead of (7.10) but our conclusion on the absence of eigenvalues remains the same.

7.4. Final remark on uniqueness

The material of the previous three sections proves the last assertion in Theorem 3. The interval $(0,\pi^2)$ where the eigenvalue (4.9) is unique in the waveguide Π_l^ε with $l=l_k(\varepsilon)$ and a fixed $\varepsilon\in(0,\varepsilon_k)$, can be enlarged up to $(0,\pi^2+c\sqrt{\varepsilon})$, $c>0$, due to Remark 15. Moreover, enhancing our consideration in Section 7.3 by dealing with the exponential waves $e^{\pm x_1\sqrt{4\pi^2-\lambda}}\cos(2\pi x_2)$ and the augmented scattering matrix of size 3×3, cf. [54], confirms that any $\lambda\in[\pi^2+c\sqrt{\varepsilon},4\pi^2-c\sqrt{\varepsilon})$ cannot be an eigenvalue as well. We will discuss the higher thresholds π^2k^2 with $k=2,3,...$ in Section 8.2.

To confirm Theorem 4, we employ a similar reasoning. Namely, we make use of the asymptotic formulas (1.18), (A17), (6.20) and observe that S_{01}^ε cannot vanish for a small ε when the length parameter (4.10) stays outside the segment

$$
\left[\pi k-c\varepsilon(1+|\ln\varepsilon|)^2,\pi k+c\varepsilon(1+|\ln\varepsilon|)^2\right] \tag{7.11}
$$

If λ^ε belongs to (7.11), the uniqueness of the solution $(\triangle\mu,\triangle l)$ of the abstract equation (4.8), which is equivalent to the criterion in Theorem 1, follows from the Banach contraction principle.

8. Available generalizations

8.1. Eigenvalues in the discrete spectrum

As in Section 1.3 we reduce the mixed boundary value problem (1.3), (1.9) to the half (1.14) of the perturbed waveguide $\Pi^\varepsilon = \Pi \cup \varpi^\varepsilon$, cf. (1.12), (1.13):

$$\begin{aligned} -\Delta u_+^\varepsilon (x) = \lambda_+^\varepsilon u_+^\varepsilon (x), \quad x \in \Pi_+^\varepsilon, \ u_+^\varepsilon (x_1, 1) = 0, \quad x_1 > 0, \\ \partial_\nu u_+^\varepsilon (x) = 0, \quad x \in \partial\Pi_+^\varepsilon, \quad x_2 < 1. \end{aligned} \tag{8.1}$$

If $\lambda^\varepsilon \in (0, \pi^2/4)$ stays below the continuous spectrum $\wp_{co}^M = [\pi^2/4, +\infty)$ of the problem (8.1), we find no oscillatory wave but deal with the exponential waves

$$v_{1/2}^{\varepsilon\pm} (x) = \left(k_{1/2}^\varepsilon\right)^{-1/2} e^{\pm k_{1/2}^\varepsilon x_1} \cos\left(\frac{\pi}{2} x_2\right), \quad k_{1/2}^\varepsilon = \sqrt{\frac{\pi^2}{4} - \lambda^\varepsilon},$$

and similarly to (2.15), (2.18), compose the linear combinations

$$w_{1/2}^{\varepsilon\pm} (x) = 2^{-1/2} \left(v_{1/2}^{\varepsilon+} (x) \mp v_{1/2}^{\varepsilon-} (x)\right).$$

The conditions (2.16), (2.17) and (2.7) with $p, q = 1, 2$ are satisfied and we may determine the augmented scattering matrix S^ε which is now a scalar. Theorem 1 remains valid and, therefore, the equality

$$S^\varepsilon = -1 \tag{8.2}$$

states a criterion for the existence of a trapped mode, that is, an eigenfunction of the problem (8.1). Constructing asymptotics of S^ε and solving the equation (8.2) yield the relation (1.10) for an eigenvalue in the discrete spectrum of the problems (8.1) and (1.3), (1.9). Repeating arguments from Sections 5 - 7 proves estimates of the asymptotic remainders as well as the uniqueness of the eigenvalue $\lambda_+^\varepsilon \in \wp_{di}^M$, however, for any $l > 0$. The latter conclusion requires an explaination of a distinction between analysis of isolated and embedded eigenvalues belonging to the discrete $\wp_{di}$ and point $\wp_{po}$ spectra, respectively.

The main difference is caused by the application of the criterion (2.21) which in the case of the scalar S^ε changes into just one equation

$$\operatorname{Re} S^\varepsilon = -1, \tag{8.3}$$

which is equivalent to (8.2) because of the equality $|S^\varepsilon| = 1$. As a result, we may satisfy (8.3) by choosing $\triangle\mu$ and do not need the additional parameter $\triangle l$ in (4.1) which was used in Section 4 to solve the system (4.2). In other words, the absence of oscillatory waves crucially restricts a possible position of the coefficient $S_{11}^\varepsilon = S^\varepsilon$ to the unit circle in the complex plane while the entry S_{11}^ε in the previous unitary matrix S^ε of size 2×2 can step aside from $\mathbb{S}$ and a fine-tuning by means of the additional small parameter $\triangle l$ is necessary to assure the equality (2.21).

8.2. Higher thresholds

A straightforward modification of our approach may be used to construct embedded eigenvalues near the thresholds $\pi^2 k^2,\ k = 2, 3, ...$ of the continuous spectrum $\wp_{co}$ of the problem (1.12), (1.13) in Π_+^ε. At the same time, the number of oscillatory outgoing waves at the threshold $\pi^2 k^2$ equals k and, therefore, the size of the augmented scattering matrix becomes $(k+1) \times (k+1)$. In this case the fine-tuning needs at least k free parameters, cf. [54, 56], instead of only one $\triangle l$ as in Section 4. Additional parameters can be easily introduced when the perturbed wall is a broken line like in fig. 5, a, with l, L and k. The amplification of the augmented scattering matrix does not affect the existence criterion (2.21) in Theorem 1.

If the mirror symmetry with respect to the line $\{x : x_1 = 0\}$ is denied, see fig. 5, b, then we have to analyze the problem (1.3), (1.4) in the intact waveguide Π^ε where the augmented scattering matrix gets a rise in size even in the case $\lambda^\varepsilon \leq \pi^2$. In this sense, the box-shaped perturbation is optimal because it demonstrates the whole technicality but reduces the computational details to the necessary minimum. A preliminary assessment predicts that embedded eigenvalues of the problem (1.12), (1.13) in Π_+^ε do not appear near any threshold $\pi^2 k^2$ with $k > 1$ but at the moment, we are not able to verify this fact rigorously and formulate it as a hypothesis only.

8.3. The Dirichlet boundary condition

All procedures described above can be applied to detect eigenvalues of the Helmholtz equation (1.3) in the quantum waveguide Π^ε, cf. [18] and [30], with the Dirichlet condition (1.7). However, the asymptotic structures must be modified a bit due to the following examination. The correction term Z' in the inner asymptotic expansion

$$Z^\varepsilon(x) = \sin(\pi x_2) + \varepsilon Z'(x) + ...$$

must be found out from the mixed boundary value problem in the semi-infinite strip

$$\begin{aligned}&-\Delta Z'(x) = \pi^2 Z'(x),\ x \in \Pi_+^0, \qquad -\partial_1 Z'(0, x_2) = 0,\ x_2 \in (0,1),\\ &Z'(x_1, 1) = 0,\ x_1 > 0, \qquad Z'(x_1, 1) = \pi^2,\ x_1 \in (0, l), \qquad Z'(x_1, 0) = 0,\ x_1 > l.\end{aligned}$$

According to the Kondratiev theory [41], see also [59, Ch. 2], a solution to this problem admits the representation

$$Z'(x) = \left(C^0 + x_1 C^1\right) \sin(\pi x_2) + \widetilde{Z}'(x), \tag{8.4}$$

where $\widetilde{Z}'(x)$ has the decay $O\left(e^{-\sqrt{3}\pi x_1}\right)$, is smooth everywhere in $\overline{\Pi_+^0}$ except at the point $P = (l, 0)$ and behaves as follows:

$$Z'(x) = \pi\varphi + O(r), \quad r \to 0. \tag{8.5}$$

Here, $(r, \varphi) \in \mathbb{R}_+ \times (0, \pi)$ is the polar coordinate system centered at P. The singularity in (8.5) leads the function $\widetilde{Z}'$ out from the Sobolev space $H^1\left(\Pi_+^0\right)$. Nevertheless, the solution Z' still lives in an appropriate Kondratiev space with a weighted norm so that the coefficient C^1 in (8.4) can be computed by inserting $Z'(x)$ and $\sin(\pi x_2)$ into the Green formula in $\Pi_+^0(R)$.

To compensate for the singularity, one may construct a boundary layer as a solution to the Dirichlet problem in the unbounded domain (3.2) in fig. 6, a.

The above commentary exhibits all the changes in the asymptotic analysis in Section 3.2. As for the justification scheme in Section 6, it should be noted that, due to the Dirichlet condition (1.7), the inequality (6.17) of Hardy type takes the form

$$\left\| r^{-1} v^{\varepsilon}; L^2\left(\Pi_+^{\varepsilon}(2l)\right)\right\|^2 \leq c \left\| v^{\varepsilon}; H^1\left(\Pi_+^{\varepsilon}(2l)\right)\right\|^2$$

and sheds the factor $1+|\ln r|$ from the weight function (6.12). As a result, the weight factor $(1+|\ln\varepsilon|)^2$ in (1.10) and (6.20) disappears from the asymptotic remainder in (1.8) for the Dirichlet condition.

Appendix: The detailed asymptotic procedure

Let us describe the asymptotics procedure for the entries

$$S_{00}^{\varepsilon} = S_{00}^{0} + \varepsilon S_{00}' + \widetilde{S}_{00}^{\varepsilon}, \qquad S_{10}^{\varepsilon} = \varepsilon^{1/2} S_{10}^{0} + \varepsilon^{3/2} S_{10}' + \widetilde{S}_{10}^{\varepsilon} \tag{A1}$$

in the augmented scattering matrix, which were not examined in Section 3. It should be mentioned that, in contrast to the solution (3.10) examined in Section 3.2, the main terms of the asymptotic ansatz for the solution Z_0^{ε} contain the boundary layer concentrated near the ledge of the box-shaped perturbation in (1.14).

Using (3.6), (A1) and (3.9), we rewrite the decomposition (2.19) of $Z_0^{\varepsilon}(x)$ as follows:

$$\begin{aligned} Z_0^{\varepsilon}(x) = {} & (2\pi)^{-1/2}\left(e^{-i\pi x_1} + S_{00}^{0} e^{i\pi x_1} + \varepsilon S_{00}' e^{i\pi x_1} + \ldots\right) + \\ & + (4\mu)^{-1/4}\left(S_{10}^{0} + \varepsilon S_{10}' + \ldots\right)\cos(\pi x_2)\left(1 - i + \varepsilon x_1\sqrt{\mu}(1+i) + \ldots\right). \end{aligned} \tag{A2}$$

Then, in a finite part of Π_+^{ε}, we accept the asymptotic ansatz

$$Z_0^{\varepsilon}(x) = Z_0^{0}(x) + \varepsilon Z_0'(x) + \ldots . \tag{A3}$$

Referring to (A2), we fix the behavior of its terms at infinity

$$Z_0^{0}(x) = (2\pi)^{-1/2}\left(e^{-i\pi x_1} + S_{00}^{0} e^{i\pi x_1}\right) + (4\mu)^{-1/4} S_{10}^{0}(1-i)\cos(\pi x_2) + \ldots, \tag{A4}$$

$$Z_0'(x) = (2\pi)^{-1/2} S_{00}' e^{i\pi x_1} + (4\mu)^{-1/4}\cos(\pi x_2)\left(S_{10}'(1-i) + x_1\sqrt{\mu} S_{10}^{0}(1+i)\right) + \ldots . \tag{A5}$$

Comparing (A4) with (3.15), (3.16), we conclude that

$$Z_0^{0}(x) = (2\pi)^{-1/2}\left(e^{-i\pi x_1} + e^{i\pi x_2}\right) + (4\mu)^{-1/4} S_{10}^{0}(1-i)\cos(\pi x_2) \tag{A6}$$

and hence,

$$S_{00}^{0} = 1. \tag{A7}$$

Let us describe the first correction terms in (A1). The function (A6) satisfies the equation (1.12) with $\lambda^{\varepsilon} = \pi^2$ and leaves the discrepancies

$$\begin{aligned} \partial_\nu Z_0^{0}(x_1,-\varepsilon) = -\partial_2 Z_0^{0}(x_1,-\varepsilon) = -\varepsilon(4\mu)^{-1/4}\pi^2 S_{10}^{0}(1-i) + O\left(\varepsilon^3\right) = \\ = -\varepsilon G_0' + O\left(\varepsilon^3\right), \ x_1 \in (0,l) \end{aligned} \tag{A8}$$

$$\partial_\nu Z_0^{0}(l,x_2) = \partial_1 Z_0^{0}(l,x_2) = -(2\pi)^{1/2}\sin(\pi l), \quad x_2 \in (-\varepsilon, 0) \tag{A9}$$

in the boundary condition (1.13) on the big Υ^{ε} and small $\upsilon^{\varepsilon} = \{x : x_1 = l,\ x_2 \in (-\varepsilon, 0)\}$ sides of the rectangle ϖ_{+}^{ε}, respectively. The discrepancy (A8) is similar to (3.19) and appears as the datum (3.20) in the problem (3.18) with $p = 0$. To compensate for (A9), we need the boundary layer

$$V_0^0(\xi) = (2\pi)^{1/2} \sin(\pi l)\, v(\xi) \tag{A10}$$

where ξ are stretched coordinates (3.1) and v is a solution of the Neumann problem (3.3) in the unbounded domain (3.2) with the right-hand side

$$g(\xi) = \begin{cases} 0, & \xi_1 \neq 0, \\ 1, & \xi_1 = 0, \quad \xi_2 \in (-1, 0), \end{cases} \quad \text{for } \xi \in \partial\Xi.$$

This solution, of course, can be constructed by an appropriate conformal mapping, cf. [58], but we only need its decomposition at infinity

$$v(\xi) = \frac{B}{\pi} \ln \frac{1}{|\xi|} + c + O(1/|\xi|), \qquad |\xi| \to \infty. \tag{A11}$$

The constant c is arbitrary but the coefficient B can be computed by the Green formula in the truncated domain $\Xi(R) = \{\xi \in \Xi : |\xi| < R\}$ with $R \to +\infty$:

$$\begin{aligned} 0 &= \lim_{R\to+\infty} R \int_0^{\pi+\arcsin(1/R)} \frac{\partial v}{\partial \rho}(\xi)\, d\varphi + \int_0^1 \frac{\partial v}{\partial \xi_1}(0, \xi_2)\, d\xi_2 = \\ &= -\frac{B}{\pi} \int_0^{\pi} d\varphi + \int_0^1 d\xi_2 = -B + 1 \quad \Rightarrow \quad B = 1. \end{aligned} \tag{A12}$$

Here, (ρ, φ) is the polar coordinate system.

We fix $c = -\pi^{-1} \ln \varepsilon$ in (A11) and observe that

$$v(\xi) = \frac{1}{\pi} \left(\ln \frac{1}{\rho} - \ln \varepsilon \right) + O\left(\frac{1}{\rho^2}\right) = \frac{1}{\pi} \ln \frac{1}{r} + O\left(\frac{\varepsilon^2}{r^2}\right) \tag{A13}$$

in the polar coordinate system (r, φ) centered at the point $x = (l, 0) \in \partial\Pi^{\varepsilon}$.

Applying the method of matched asymptotic expansions, cf. [36, 64], [46, Ch. 2], as well as [46, Ch. 5], [58] for the ledge-shaped perturbation of domains, we consider (A3) as an outer expansion in a finite part of the waveguide while $\varepsilon V_0^0(\xi)$ becomes the main term of the inner expansion in the vicinity of the ledge. In view of (A10) and (A13), the standard matching procedure proposes Z_0' as a singular solution to the homogeneous problem (3.13), (3.14) with the following asymptotic condition at the point $x = (l, 0) \in \partial\Pi^0$:

$$Z_0'(x) = \left(\frac{2}{\pi}\right)^{1/2} \sin(\pi l) \ln \frac{1}{r} + c + O(r) \qquad \text{for } r \to +0. \tag{A14}$$

Arguing in the same way as for the function Z_1', we conclude that the problem (3.18), (A14) has a solution Z_0' which admits the representation (3.21) with $p = 0$ under the restriction $x_1 \geq R > l$ needed due to the logarithmic singularity in (A14). The coefficient C_0^0 is arbitrary but C_0 and C_0^1 can be computed again by means of Green's formula in the domain $\Pi_{+}^0(R, \delta) = \left\{x \in \Pi_{+}^0 : x_1 < R,\ r > \delta\right\}$ and the limit passages $R \to +\infty$, $\delta \to +0$, cf. (3.22), (3.23) and (A12). Dealing with u_0^0 and Z_0', we take into account the equality $\partial_2 u_0^0(x_1,0)=0$ and obtain that

$$0=\int_{\Pi(R,\delta)}\cos(\pi x_1)\left(\Delta Z_0'(x)+\pi^2 Z_0'(x)\right)dx= \tag{A15}$$
$$=\int_0^1\left(\cos(\pi x_1)\frac{\partial Z_0'}{\partial x_1}(x)+\pi\sin(\pi x_1)Z_0'(x)\right)\Big|_{x_1=R}dx_2-$$
$$-\delta\int_0^\pi\left(\cos(\pi x_1)\frac{\partial Z_0'}{\partial r}(x)-Z_0'(x)\frac{\partial}{\partial r}\cos(\pi x_1)\right)\Big|_{r=\delta}d\varphi=$$
$$=i\pi C_0+(\pi/2)^{1/2}\sin(2\pi l)+o(1)\quad\Rightarrow\quad C_0=i(2\pi)^{-1/2}\sin(2\pi l).$$

Inserting u_1^0 and Z_0' into the Green formula in $\Pi_+^0(R,\delta)$, we recall the inhomogeneous boundary condition on Υ^0 and derive that

$$0=\int_0^1\cos(\pi x_2)\,\partial_1 Z_0'(x)|_{x_1=R}\,dx_2-\int_0^{l-\delta}\cos(\pi x_2)\,\partial_2 Z_0'(x)|_{x_2=0}\,dx_1-$$
$$-\delta\int_0^\pi\left(\cos(\pi x_2)\,\partial_r Z_0'(x)-Z_0'(x)\,\partial_r\cos(\pi x_2)\right)|_{r=\delta}\,d\varphi= \tag{A16}$$
$$=\frac{1}{2}C_0^1(4\mu)^{-1/4}\pi^2 l S_{10}^0(1-i)+(2\pi)^{1/2}\sin(\pi l)+o(1).$$

We now compare coefficients in the expansions (3.21), $p=0$, and (A5). According to the calculations (A15) and (A16), we derive the formulas

$$C_0=(2\pi)^{-1/2}S_{00}'\Rightarrow S_{00}'=i\sin(2\pi l),$$
$$C_0^1=(4\mu)^{-1/4}\sqrt{\mu}S_{10}^0(1+i)\Rightarrow \tag{A17}$$
$$S_{10}^0=-\frac{(4\mu)^{1/4}2(2\pi)^{1/2}\sin(\pi l)}{\sqrt{\mu}(1+i)+2\pi^2 l(1-i)}=-(4\mu)^{1/4}(2\pi)^{1/2}\frac{\sqrt{\mu}(1-i)+2\pi^2 l(1+i)}{4\pi^4 l^2+\mu}\sin(\pi l).$$

The calculation of coefficients in the ansätze (3.7) and (A1) is completed. It is worth to underline that the expression (A17) for the main asymptotic term of $\varepsilon^{-1/2}S_{10}^\varepsilon=\varepsilon^{-1/2}S_{01}^\varepsilon$ can be derived from the cumbersome relations (3.25) and (3.26) as well.

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