Existence and nonexistence of positive solutions of p-Kolmogorov equations perturbed by a Hardy potential

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A B S T R A C T

In this article, we establish the phenomenon of existence and nonexistence of positive weak solutions of parabolic quasi-linear equations perturbed by a singular Hardy potential on the whole Euclidean space depending on the controllability of the given singular potential. To control the singular potential we use a weighted Hardy inequality with an optimal constant, which was recently discovered in Hauer and Rhandi (2013). Our results in this paper extend the ones in Goldstein et al. (2012) concerning a linear Kolmogorov operator significantly in several ways: firstly, by establishing existence of positive global solutions of singular parabolic equations involving nonlinear operators of p-Laplace type with a nonlinear convection term for $1 < p < \infty$, and secondly, by establishing nonexistence *locally in time* of positive weak solutions of such equations without using any growth conditions.

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1. Introduction and main results

The aim of this article is to establish the phenomenon of existence and nonexistence of positive weak solutions of *p-Kolmogorov equations perturbed by a Hardy-type potential*

$$\frac{\partial u}{\partial t} - K_p u = V\ |u|^{p-2}\, u \quad \text{on } \mathbb{R}^d \times]0,T[, \tag{1.1}$$

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depending whether $\lambda \le \left(\frac{|d-p|}{p}\right)^p$ or $\lambda > \left(\frac{|d-p|}{p}\right)^p$ for $1 < p < \infty$, $d \ge 2$, and the potential $V \in L^\infty_{\mathrm{loc}}(\mathbb{R}^d \setminus \{0\})$ satisfies

$$0 \le V(x) \le \tfrac{\lambda}{|x|^p} \quad \text{for a.e. } x \in \mathbb{R}^d. \tag{1.2}$$

Here, we call a real-valued measurable function u on $\mathbb{R}^d \times (0,T)$ positive if $u(x,t) \ge 0$ for a.e. $x \in \mathbb{R}^d$ and a.e. $t \in (0,T)$ and the operator

$$K_p u := \operatorname{div}\left(|\nabla u|^{p-2} \nabla u\right) + \rho^{-1} |\nabla u|^{p-2} \nabla u \nabla \rho \tag{1.3}$$

is the p-*Kolmogorov operator* for the particular density function

$$\rho(x) := N\, e^{-\frac{1}{p}(x^t A x)^{p/2}} \tag{1.4}$$

for every $x \in \mathbb{R}^d$, where A is a real symmetric positive definite $(d \times d)$-matrix and N some normalisation constant such as the integral $\int_{\mathbb{R}^d} \rho(x)\,\mathrm{d}x = 1$. The operator K_p was first introduced in [17] and we note that the case $A = 0$ corresponds to the density function $\rho \equiv 1$. In this case, one does not normalise and the phenomenon of existence and nonexistence of positive solutions of Eq. (1.1) on bounded and unbounded domains has been well-studied in the past (see, for instance, [15,2,19]). Thus, it is the task of this article, to investigate the case A is a real symmetric positive definite $(d \times d)$-matrix. Furthermore, we denote by $\mathrm{d}\mu$ the finite Borel-measure on $\mathbb{R}^d$ given by

$$\mathrm{d}\mu = \rho\,\mathrm{d}x,$$

for $1 \le q \le \infty$ and any open subset D of $\mathbb{R}^d$, let $L^q(D,\mu)$ and $W^{1,q}(D,\mu)$ denote the standard Lebesgue and first Sobolev space with respect to the measure $\mathrm{d}\mu$ and $W^{1,q}_0(D,\mu)$ the closure of $C^\infty_c(D)$ in $W^{1,q}(D,\mu)$. Under these assumptions, the second and third authors of this article established in [21] the following Hardy inequality with a remainder term.

Lemma 1.1 *([21]). Let $d \ge 2$, $1 < p < \infty$ and A be a real symmetric positive definite $(d \times d)$-matrix. Then*

$$\left(\tfrac{|d-p|}{p}\right)^p \int_{\mathbb{R}^d} \tfrac{|u|^p}{|x|^p}\,\mathrm{d}\mu \le \int_{\mathbb{R}^d} |\nabla u|^p\,\mathrm{d}\mu + \left(\tfrac{|d-p|}{p}\right)^{p-1} \operatorname{sign}(d-p) \int_{\mathbb{R}^d} |u|^p\, \frac{(x^t A x)^{p/2}}{|x|^p}\,\mathrm{d}\mu \tag{1.5}$$

for all $u \in W^{1,p}(\mathbb{R}^d,\mu)$ with optimal constant $\left(\frac{|d-p|}{p}\right)^p$.

In contrast to the case $A \equiv 0$ (cf., for instance, [15] or [26] and the references therein), our weighted Hardy inequality (1.5) admits the remainder term

$$\left(\tfrac{|d-p|}{p}\right)^{p-1} \operatorname{sign}(d-p) \int_{\mathbb{R}^d} |u|^p\, \frac{(x^t A x)^{p/2}}{|x|^p}\,\mathrm{d}\mu. \tag{1.6}$$

This term has, in fact, a great impact on the existence of weak solutions of Eq. (1.1) in the degenerate case $2 < p < d$, while for establishing nonexistence of positive solutions this term does not cause any problems. It is somehow surprising that in the case $p > d$, the remainder term (1.6) becomes negative and so provides further estimates in $L^p(\mathbb{R}^d,\mu)$. We note that one does not find much in the literature about Hardy type inequalities in the case $p > d \ge 2$.

In this article, we make use of the following notion of *weak solutions*, which seems to be natural for parabolic equations of p-Laplace type with singular potentials (cf. [10,8,9] or [18] for $p = 2$ and [19] by J. Goldstein and Kombe).

Definition 1.2. Let $V \in L^\infty_{\mathrm{loc}}(\mathbb{R}^d \setminus \{0\},\mu)$ be positive. If $p \ne 2$, then for given $u_0 \in L^2_{\mathrm{loc}}(\mathbb{R}^d,\mu)$ we call u a *weak solution* of Eq. (1.1) with initial value $u(0) = u_0$ provided

$$u \in C([0,T); L^2_{\mathrm{loc}}(\mathbb{R}^d \setminus \{0\},\mu)) \cap L^p(0,T; W^{1,p}_{\mathrm{loc}}(\mathbb{R}^d \setminus \{0\},\mu)),$$

for all open sets $\mathcal{K}$ with compact closure in $\mathbb{R}^d \setminus \{0\}$, (abbreviated by $\mathcal{K} \Subset \mathbb{R}^d \setminus \{0\}$)

$$u(t) \to u_0 \quad \text{in } L^2(\mathcal{K}, \mu) \text{ as } t \to 0+, \tag{1.7}$$

for all $0 \le t_1 < t_2 < T$, and all $\varphi \in W^{1,2}(t_1, t_2; L^2(\mathcal{K}, \mu)) \cap L^p(t_1, t_2; W_0^{1,p}(\mathcal{K}, \mu))$,

$$(u, \varphi)_{L^2_\mu(\mathcal{K})}\Big|_{t_1}^{t_2} + \int_{t_1}^{t_2} \int_{\mathcal{K}} \Big\{-u\,\tfrac{d\varphi}{dt} + |\nabla u|^{p-2}\,\nabla u \nabla\varphi\Big\}\,\mathrm{d}\mu\,\mathrm{d}t = \int_{t_1}^{t_2} \int_{\mathcal{K}} V(x)\,|u|^{p-2}\,u\,\varphi\,\mathrm{d}\mu\,\mathrm{d}t. \tag{1.8}$$

If $p = 2$, then for given $u_0 \in L^2_{\mathrm{loc}}(\mathbb{R}^d, \mu)$, we call u a *weak solution* of Eq. (1.1) provided $u \in C([0,T); L^2_{\mathrm{loc}}(\mathbb{R}^d \setminus \{0\}, \mu))$ satisfies for every $\mathcal{K} \Subset \mathbb{R}^d \setminus \{0\}$, initial condition (1.7), for every open ball $B(0,r)$ centred at $x = 0$ with radius $r > 0$ and every $0 \le t_1 < t_2 < T$, one has that $V\,u \in L^1(t_1, t_2, L^1(B(0,r), \mu))$ and

$$(u, \varphi)_{L^2(\mathbb{R}^d,\mu)}\Big|_{t_1}^{t_2} + \int_{t_1}^{t_2} \int_{\mathbb{R}^d} u\,\Big\{-\tfrac{d\varphi}{dt} - K_2\varphi\Big\}\,\mathrm{d}\mu\,\mathrm{d}t = \int_{t_1}^{t_2} \int_{\mathbb{R}^d} V(x)\,u\,\varphi\,\mathrm{d}\mu\,\mathrm{d}t$$

for all $\varphi \in W^{1,2}(t_1, t_2; L^2(\mathbb{R}^d, \mu)) \cap L^2(t_1, t_1; W^{2,2}(\mathbb{R}^d, \mu))$ with $\varphi(\cdot, t)$ having compact support.

Our two main results of this article read as follows. We begin with the existence result.

Theorem 1.3. *Let $d \ge 2$ and A be a real symmetric positive definite $(d \times d)$-matrix. Then the following statements hold true.*

1. *Let either $\frac{2d}{d+2} < p \le 2$ and $p \ne d$ if $d \ge 2$ or $d < p < \infty$. If $\lambda \le \big(\frac{|d-p|}{p}\big)^p$, then for every $T > 0$, and every positive $u_0 \in L^2(\mathbb{R}^d, \mu)$, there is a weak solution $u \in C([0,T]; L^2(\mathbb{R}^d, \mu))$ of Eq. (1.1) with initial value $u(0) = u_0$.*
2. *If $1 < p < \frac{2d}{d+2}$, then for every $\lambda > 0$ and every positive $u_0 \in L^2(\mathbb{R}^d, \mu)$, there is a strong solution of Eq. (1.1) in $L^2(\mathbb{R}^d, \mu)$.*

Note, the notion of *strong solutions* of Eq. (1.1) is given in Definition 2.1 in Section 2. We want to point out that by using a Galerkin method, one can, in particular, establish the existence of sign-changing solution of Eq. (1.1) with a right-hand side $f \in L^2(0,T; L^2(\mathbb{R}^d, \mu))$ or $f \in L^{p'}(0,T; W^{-1,p'}(\mathbb{R}^d, \mu))$ for $\lambda \le \big(\frac{|d-p|}{p}\big)^p$, where $W^{-1,p'}(\mathbb{R}^d, \mu)$ denotes the dual space of $W^{1,p}(\mathbb{R}^d, \mu)$. In addition, taking initial values $u_0 \in W^{1,p}(\mathbb{R}^d, \mu)$ or $\lambda < \big(\frac{|d-p|}{p}\big)^p$ provides more regularity on the weak solutions of (1.1) (similarly, as in [15] or [2]). However, in order to lose not the focus on the phenomenon of existence and nonexistence provided by the optimality of the Hardy constant $\big(\frac{|d-p|}{p}\big)^p$, we state here only our results on the existence and nonexistence of positive very weak solutions of Eq. (1.1).

Our nonexistence results read as follows.

Theorem 1.4. *Let $d \ge 2$, A be a real symmetric positive definite $(d \times d)$-matrix and $V = \frac{\lambda}{|x|^p}$. Then the following statements hold true.*

(i) *For $d = 2$ let $1 < p < 2$, and for $d \ge 3$ let $\frac{2d}{d+2} \le p \le 2$. If $\lambda > \big(\frac{d-p}{p}\big)^p$ and if u_0 is a positive nontrivial element of $L^2_{\mathrm{loc}}(\mathbb{R}^d, \mu)$, then for any $T > 0$, Eq. (1.1) has no positive weak solution.*

(ii) *Let $d \ge 2$, $p > 2$, and $p \ne d$. If $\lambda > \big(\frac{|d-p|}{p}\big)^p$, and if $u_0 \in L^2_{\mathrm{loc}}(\mathbb{R}^d, \mu)$ is positive and for some $r > 0$,*

$$\operatorname*{ess\,inf}_{x \in B(0,r)} u_0(x) \ge \delta > 0, \tag{1.9}$$

then for any $T > 0$, Eq. (1.1) has no positive weak solution.

The intimate relation between Hardy's inequality and the nonexistence results of positive solutions of parabolic equations with a singular potential was discovered in [5] by Baras and the first author of this article. More precisely, they established the following result:

Theorem (*Baras–Goldstein, [5]*). *Let $\Omega = (0,R)$ for some $1 \le R \le +\infty$ if $d = 1$ and if $d \ge 2$, let Ω be a domain in $\mathbb{R}^d$ with $B(0,1) \subseteq \Omega$ and a smooth boundary $\partial\Omega$. If $\lambda \le \left(\frac{d-2}{2}\right)^2$, then for every positive $u_0 \in L^1(\Omega) \setminus \{0\}$ and every $T > 0$, problem*

$$\frac{\partial u}{\partial t} - \Delta u = \frac{\lambda}{|x|^2} u \quad \text{in } \Omega \times (0,T), \qquad u = 0 \text{ on } \quad \partial\Omega \times (0,T), \qquad u(\cdot,0) = u_0 \quad \text{in } \Omega \tag{1.10}$$

has a positive weak solution. If $\lambda > \left(\frac{d-2}{2}\right)^2$ and if $u_0 \in L^1(\Omega) \setminus \{0\}$ is positive, then for any $T > 0$ problem (1.10) has no positive weak solution.

A few years later, Cabré and Martel discovered in [9] a second and more intuitive proof of the Baras–Goldstein result [5]. They proved in [9] that indeed the existence and nonexistence of positive solutions of problem (1.10) is largely determined by the generalised eigenvalue of $-\Delta - \lambda\,|x|^{-2}$ given by

$$\sigma(\lambda\,|x|^{-2}, \Omega) = \inf_{0 \not\equiv \varphi \in C_c^\infty(\Omega)} \frac{\int_\Omega |\nabla\varphi|^2 \,d\mu - \int_\Omega \frac{\lambda}{|x|^2} |\varphi|^2 \,d\mu}{\int_\Omega |\varphi|^2 \,d\mu}.$$

In [19], the first author and Kombe showed that the method introduced in [9] can be very useful to establish nonexistence (locally in time) of positive solutions of singular nonlinear diffusion equations associated with either the p-Laplace operator or other fast-diffusion operators. The existence and the qualitative behaviour of positive solutions of singular parabolic problems associated with the p-Laplace operator has been intensively studied, for instance, in the articles [2,15]. In particular, by using a separation of variables method, Garcia Azorero and Peral Alonso established in [15] in the degenerate case $2 < p < d$ nonexistence (locally in time) of positive solutions of a parabolic p-Laplace equations perturbed by the potential $V = \lambda\,|x|^{-p}$ on a bounded domain with zero Dirichlet boundary conditions. In [13,14], Galaktionov employed the zero counting method (Sturm's first theorem, cf. [30]) to show that the assumption that the weak solutions of these equations need to be positive can be omitted, but with the restriction that the initial datum u_0 is assumed to be continuous and $u_0(0) > 0$.

Recently, the first and the third author discovered in [18, Theorem 3.4] together with G.R. Goldstein, the weighted Hardy inequality (1.5) for $p = 2$ and by employing the Cabré–Martel approach [9], they established existence and nonexistence of positive *global* weak solutions of Eq. (1.1) for the potential $V = \lambda\,|x|^{-2}$ depending whether $\lambda \le \left(\frac{d-2}{2}\right)^2$ or $\lambda > \left(\frac{d-2}{2}\right)^2$. Moreover, in order to establish nonexistence (globally in time) of positive weak solution of (1.1), the additional assumption that the solutions satisfy the exponential growth condition

$$\|u(t)\|_{L^2(\mathbb{R}^d,\mu)} \le M\, \|u_0\|_{L^2(\mathbb{R}^d,\mu)}\, e^{\omega\, t} \tag{1.11}$$

for all $t \ge 0$ is needed in [18].

The results of this article complement the known literature in the following way. We establish existence of positive solutions of Eq. (1.1) for $1 < p < \frac{2d}{d+2}$, $\frac{2d}{d+2} < p \le 2$, $p \ne d$, $p > d \ge 2$ and nonexistence results for all $1 < p \le 2$ with $p \ne d$ and $p > d \ge 2$. Until now, only the case $p = 2$ has been considered in [18, Theorems 1.3 & 2.1]. In this work, we improve the results in [18] by proving nonexistence of positive *locally in time* weak solutions of Eq. (1.1) for $p = 2$. In contrast to [18], our proofs in this article provide a way to establish nonexistence of positive solutions of (1.1) without using the exponential growth condition (1.11). In addition, by comparing the results in [15], we provide a different proof to establish nonexistence locally in time of positive weak solutions of (1.4) in the degenerate case $2 < p < \infty$ and $p \ne d$.

In this article, we neither prove existence of positive solutions of Eq. (1.1) for the critical case $p = \frac{2d}{d+2}$ nor for $2 < p < d$. The reason for this is that in the critical case $p = \frac{2d}{d+2}$, there is a lack of compactness and in the case $2 < p < d$, the remainder term in the weighted Hardy inequality (1.5), does not allow us to derive uniform L^q-*a priori* bounds for some suitable $q \geq 1$.

2. Proof of existence of positive weak solutions

This section is dedicated to the proof of Theorem 1.3. Here, we proceed in two steps. In the first step, we establish for every positive $u_0 \in L^2(\mathbb{R}^d, \mu)$ and $m \geq 1$, the existence of positive solutions u_m of Eq. (1.1) when the potential $V \in L^\infty_{\text{loc}}(\mathbb{R}^d \setminus \{0\})$ is replaced by the *truncated potential*

$$V_m(x) := \min\{V(x), m\} \tag{2.1}$$

and when the initial value u_0 is replaced by the *truncated initial value*

$$u_{0,m}(x) := \min\{u_0, m\}. \tag{2.2}$$

Then, the corresponding solution u_m admits more regularity and the sequence (u_m) is monotone increasing. Using this fact together with Hardy's inequality (1.5) for $\lambda \leq \left(\frac{|d-p|}{p}\right)^p$, we show in the second step that the limit function

$$u := \lim_{m\to\infty} u_m$$

is a positive weak solution of the singular Eq. (1.1) with initial value u_0.

2.1. Existence of strong approximate solutions

Suppose that the potential $V \in L^\infty_{\text{loc}}(\mathbb{R}^d \setminus \{0\})$ is positive. Then for every $u \in L^2(\mathbb{R}^d, \mu)$, let

$$\varphi(u) = \begin{cases} \frac{1}{p}\displaystyle\int_{\mathbb{R}^d} |\nabla u|^p \,\mathrm{d}\mu & \text{if } u \in W^{1,p}(\mathbb{R}^d, \mu), \\ +\infty & \text{if otherwise,} \end{cases} \tag{2.3}$$

and

$$\varphi_V(u) = \begin{cases} \frac{1}{p}\displaystyle\int_{\mathbb{R}^d} V\,|u|^p \,\mathrm{d}\mu & \text{if } V\,|u|^p \in L^1(\mathbb{R}^d, \mu), \\ +\infty & \text{if otherwise.} \end{cases} \tag{2.4}$$

The functionals $\varphi : L^2(\mathbb{R}^d, \mu) \to [0, +\infty]$ and $\varphi_V : L^2(\mathbb{R}^d, \mu) \to [0, +\infty]$ are convex, proper, lower semi-continuous on $L^2(\mathbb{R}^d, \mu)$ and have dense domains $D(\varphi) := \{u \in L^2(\mathbb{R}^d, \mu) \,|\, \varphi(u) < +\infty\}$ and $D(\varphi_V)$ in $L^2(\mathbb{R}^d, \mu)$. Moreover, the subgradient $\partial\varphi$ in $L^2(\mathbb{R}^d, \mu)$ is single-valued, has domain

$$D(\partial\varphi) = \left\{ u \in W^{1,p}(\mathbb{R}^d, \mu) \cap L^2(\mathbb{R}^d, \mu) \;\middle|\; \begin{array}{l} \text{there exists } h_u \in L^2(\mathbb{R}^d, \mu) \text{ s.t. for all } v \in C_c^\infty(\mathbb{R}^d), \\ \displaystyle\int_{\mathbb{R}^d} |\nabla u|^{p-2}\,\nabla u \nabla v \,\mathrm{d}\mu = \int_{\mathbb{R}^d} h_u\, v \,\mathrm{d}\mu \end{array} \right\},$$

$$\partial\varphi(u) = h_u \quad \text{for every } u \in D(\partial\varphi),$$

and an integration by parts shows that $-\partial\varphi$ describes the realisation in $L^2(\mathbb{R}^d, \mu)$ of the *p-Kolmogorov operator* K_p defined in (1.3).

Similarly, the subgradient $\partial\varphi_V$ in $L^2(\mathbb{R}^d, \mu)$ is single-valued and given by

$$D(\partial\varphi_V) = \left\{ u \in L^2(\mathbb{R}^d, \mu) \;\middle|\; \begin{array}{l} V\,|u|^p \in L^1(\mathbb{R}^d, \mu), \text{ there exists } h_u \in L^2(\mathbb{R}^d, \mu) \text{ s.t.} \\ \displaystyle\int_{\mathbb{R}^d} V\,|u|^{p-2}\, u\, v \,\mathrm{d}\mu = \int_{\mathbb{R}^d} h_u\, v \,\mathrm{d}\mu \;\forall\, v \in C_c^\infty(\mathbb{R}^d) \end{array} \right\},$$

$$\partial\varphi_V(u) = h_u \quad \text{for every } u \in D(\partial\varphi_V).$$

Let $m \geq 1$. Then, we begin by establishing the existence of positive solutions of Eq. (1.1) *for the truncated potential* V_m *given by* (2.1):

$$\frac{du_m}{\mathrm{d}t} + \partial\varphi(u_m) = \partial\varphi_{V_m}(u_m) \quad \text{on } (0,T) \tag{2.5}$$

in $L^2(\mathbb{R}^d,\mu)$ satisfying $u_m(0) = u_{0,m}$ for the truncated initial value $u_{0,m}$ given by (2.2). The approximate solutions u_m of Eq. (2.5) is, generally, more regular than the weak solution u of (1.1). In fact, we show that solutions of Eq. (2.5) are *strong* in the following sense.

Definition 2.1. For given $u_0 \in L^2(\mathbb{R}^d,\mu)$ and positive $V \in L^\infty(\mathbb{R}^d \setminus \{0\})$, we call a function $u \in C([0,T]; L^2(\mathbb{R}^d,\mu))$ a *strong solution* of Eq. (2.5) in $L^2(\mathbb{R}^d,\mu)$ with initial value u_0 if u satisfies $u(0) = u_0$ in $L^2(\mathbb{R}^d,\mu)$,

$$u \in L^p(0,T;W^{1,p}(\mathbb{R}^d,\mu)) \cap W^{1,2}(\delta,T;L^2(\mathbb{R}^d,\mu))$$

for every $\delta > 0$, and for a.e. $t \in (0,T)$, $u(t) \in D(\partial\varphi_V)$ and

$$\int_{\mathbb{R}^d} \frac{du}{\mathrm{d}t}(t)\, v \,\mathrm{d}\mu + \int_{\mathbb{R}^d} |\nabla u(t)|^{p-2}\, \nabla u(t)\nabla v \,\mathrm{d}\mu = \int_{\mathbb{R}^d} \partial\varphi_V(u(t))\, v \,\mathrm{d}\mu$$

for all $v \in W^{1,p}(\mathbb{R}^d,\mu) \cap L^2(\mathbb{R}^d,\mu)$.

Our result on the existence of strong solutions of Eq. (2.5) for the truncated potential V_m is given in the following proposition. Note that we make neither further restriction on $1 < p < \infty$ nor on the positive potential $V \in L^\infty_{\mathrm{loc}}(\mathbb{R}^d \setminus \{0\})$.

Proposition 2.2. *Let* $V \in L^\infty_{\mathrm{loc}}(\mathbb{R}^d \setminus \{0\})$ *be positive,* $T > 0$*, and* $m \geq 1$ *an integer. Then, for every positive* $u_0 \in L^2(\mathbb{R}^d,\mu)$*, there is at least one positive strong solution* $u_m \in L^\infty(\mathbb{R}^d \times (0,T))$ *of Eq.* (2.5) *with initial value* $u_m(0) = u_{0,m}$ *satisfying*

$$u_m(x,t) \leq u_{m+1}(x,t) \tag{2.6}$$

for all $t \in [0,T]$*, a.e.* $x \in \mathbb{R}^d$*.*

For the proof of Proposition 2.2 we employ a method due to Fujita [12]. The same method has already been employed in [15, Section 6.] and [2, Section 2.] to study Eq. (1.1) with density function $\rho \equiv 1$.

The following compactness result is quite interesting and helpful, for instance, to establish further auxiliary inequalities (see (2.8)), which we use to establish existence of strong solutions of Eq. (2.5) and weak solutions of Eq. (1.1). Note that the density function ρ defined in (1.4) is, in general, not radial, compactness results concerning weighted Sobolev spaces with radial weight functions are well-studied (see, for instance, [1] or [3]).

Theorem 2.3. *Let* $d \geq 2$*,* $1 < p < \infty$ *and* A *be a real symmetric positive definite* $(d \times d)$*-matrix. Then the embedding from* $W^{1,p}(\mathbb{R}^d,\mu)$ *into* $L^p(\mathbb{R}^d,\mu)$ *is compact.*

Since Theorem 2.3 is not the central object of this paper, we post its proof to Appendix of this article. As an immediate consequence of this compactness result, we obtain the following useful Poincaré inequality:

Corollary 2.4. *Let* $d \geq 2$*,* $1 < p < \infty$ *and* A *be a real symmetric positive definite* $(d \times d)$*-matrix. Then, there is a constant* $C > 0$ *such that*

$$\|u - \overline{u}\|_{L^p(\mathbb{R}^d,\mu)} \leq C\, \|\nabla u\|_{L^p(\mathbb{R}^d,\mu)^d} \tag{2.7}$$

for all $u \in W^{1,p}(\mathbb{R}^d,\mu)$*, where* $\overline{u} = \int_{\mathbb{R}^d} u \,\mathrm{d}\mu$*.*

Note that the proof of Corollary 2.4 follows immediately by using standard arguments. Hence we omit its proof. By the triangle inequality and since $L^q(\mathbb{R}^d,\mu)$ is continuously embedded into $L^1(\mathbb{R}^d,\mu)$ for any $1\le q\le\infty$, we can conclude the following result, which we state for later reference.

Corollary 2.5. *Let $d\ge 2$, $1<p<\infty$ and A be a real symmetric positive definite $(d\times d)$-matrix. Then for $1\le q\le\infty$, there is a constant $C>0$ such that*

$$\|u\|_{L^p(\mathbb{R}^d,\mu)} \le C\left(\|\nabla u\|_{L^p(\mathbb{R}^d,\mu)^d} + \|u\|_{L^q(\mathbb{R}^d,\mu)}\right) \tag{2.8}$$

for all $u\in W^{1,p}(\mathbb{R}^d,\mu)\cap L^q(\mathbb{R}^d,\mu)$.

For the proof of Proposition 2.2, we employ weak comparison principles. The following one will be useful also later by establishing nonexistence (locally in time) of positive weak solutions. Here, for a given open subset D of $\mathbb{R}^d$ and $T>0$, we denote by $W^{1,p}_0(D,\mu)$ the closure of $C^\infty_c(D)$ in $W^{1,p}(D,\mu)$ and set $D_T=D\times(0,T)$ and $\mathcal{P}D_T=(\partial D\times(0,T))\cup(D\times\{0\})$.

Lemma 2.6. *Let $D\subseteq\mathbb{R}^d$ be a bounded open subset with a Lipschitz continuous boundary and $f:D\times\mathbb{R}\to\mathbb{R}$ be a Carathéodory function satisfying $f(x,0)=0$ for a.e. $x\in D$ and there is a constant $L>0$ such that $|f(x,u)-f(x,\hat{u})|\le L\,|u-\hat{u}|$ for all u, $\hat{u}\in\mathbb{R}$ and a.e. $x\in D$. Suppose that*

$$u,v\in C([0,T];L^2(D,\mu))\cap W^{1,2}(\delta,T;L^2(D,\mu))\cap L^p(0,T;W^{1,p}(D,\mu))$$

for any $0<\delta<T$ and satisfy

$$\begin{aligned}\int_D\left[\frac{du}{\mathrm{d}t}(t)-\frac{dv}{\mathrm{d}t}(t)\right]\varphi\,\mathrm{d}\mu+\int_D\left[|\nabla u(t)|^{p-2}\nabla u(t)-|\nabla v(t)|^{p-2}\nabla v(t)\right]\nabla\varphi\,\mathrm{d}\mu\\ +\int_D[f(x,u(t))-f(x,v(t))]\,\varphi\,\mathrm{d}\mu\le 0\,,\end{aligned} \tag{2.9}$$

for all positive $\varphi\in W^{1,p}_0(D,\mu)$ and a.e. $t\in(0,T)$. Then

$$\operatorname*{ess\,sup}_{(x,t)\in D_T} e^{-Lt}(u-v)(x,t)\le \operatorname*{ess\,sup}_{(x,t)\in\mathcal{P}D_T} e^{-Lt}\big[u-v\big]^+(x,t). \tag{2.10}$$

Proof. To prove the assertion of this lemma, we employ the truncation method of Stampacchia (cf. the proof of Théorème X.3 in [7, p. 211]). Suppose that

$$k=\operatorname*{ess\,sup}_{(x,t)\in\mathcal{P}D_T} e^{-Lt}\big[u-v\big]^+(x,t)\quad\text{is finite,}$$

where we denote by $[v]^+:=\max\{0,v(x,t)\}$ the positive part of a measurable function v defined on D_T, and set

$$w(x,t)=u(x,t)-v(x,t)-ke^{Lt}\quad\text{for a.e. }(x,t)\in D_T.$$

Since the function $s\mapsto[s]^+$ is Lipschitz-continuous on $\mathbb{R}$, we have by [7, Corollaire VIII.10] that $w^+\in W^{1,2}(\delta,T;L^2(D,\mu))\cap L^p(0,T,W^{1,p}(D,\mu))$. Since by hypothesis, $w^+=0$ on $\partial D\times(0,T)$, we have by [23, Lemma 3.3], that $w^+\in L^p(0,T;W^{1,p}_0(D,\mu))$. We denote by $\mathbb{1}_{\{u-v>ke^{Lt}\}}$ the characteristic function of the set $\big\{(x,t)\in D_T\mid u(x,t)-v(x,t)>ke^{Lt}\big\}$, and we set $\varphi(t)=\frac{1}{2}\|w^+(t)\|^2_{L^2(D,\mu)}$ for all $t\in[0,T]$. Then, $\varphi\in W^{1,2}(\delta,T)$ for all $0<\delta<T$, $\varphi\in C[0,T]$, $\varphi(0)=0$, $\varphi\ge 0$ on $[0,T]$, and for a.e. $t\in(0,T)$,

$$\begin{aligned}\varphi'(t)&=\int_D\left[\frac{du}{\mathrm{d}t}(t)-\frac{du}{\mathrm{d}t}(t)\right]w^+(t)\,\mathrm{d}\mu-\int_D k\,L\,e^{Lt}\,w^+(t)\,\mathrm{d}\mu\\ &\le-\int_D\left[|\nabla u(t)|^{p-2}\nabla u(t)-|\nabla v(t)|^{p-2}\nabla v(t)\right]\left[\nabla u(t)-\nabla v(t)\right]\mathbb{1}_{\{u-v>ke^{Lt}\}}\mathrm{d}\mu\end{aligned}$$

$$
\begin{aligned}
&- \int_D [f(x,u(t)) - f(x,v(t))]\, w^+(t)\, \mathrm{d}\mu - \int_D k\, L\, e^{Lt}\, w^+(t)\, \mathrm{d}\mu \\
&\le L \int_{\{u-v>ke^{Lt}\}} [u(t)-v(t)]\, w^+(t)\, \mathrm{d}\mu - \int_D k\, L\, e^{Lt}\, w^+(t)\, \mathrm{d}\mu \\
&\le 0.
\end{aligned}
$$

Thus, $\varphi(t) \equiv 0$, proving that inequality (2.10) holds. □

Now, we turn to the proof of Proposition 2.2.

Proof of Proposition 2.2. Let $u_0 \in L^2(\mathbb{R}^d, \mu)$ be positive and for $m \ge 1$, let $u_{0,m}$ be given by (2.2). Then, as a 1*st Step*, we begin by constructing iteratively a sequence $(y_{m,k})_{k\ge 1}$ of positive strong solutions $y_{m,k}$ of the equations

$$
\frac{dy_{m,k}}{\mathrm{d}t} + \partial\varphi(y_{m,k}) = V_m y_{m,k-1}^{p-1} \quad \text{on } (0,T) \tag{2.11}
$$

in $L^2(\mathbb{R}^d, \mu)$ with initial value $y_{m,k}(0) = u_{0,m}$ satisfying

$$
y_{m,k} \in L^\infty(\mathbb{R}^d \times (0,T)) \tag{2.12}
$$

and

$$
y_{m,k}(x,t) \le y_{m,k+1}(x,t) \le L_m := M + m \tag{2.13}
$$

for all $t \in [0,T]$, a.e. $x \in \mathbb{R}^d$ and all integers $k \ge 1$, where for given $T > 0$, we choose $M > 0$ such that $T = M/(L_m^{p-1}\, m + m)$.

We begin by constructing the function $y_{m,1}$. For this, we need the following two auxiliary functions w_m and v_m. One easily verifies that the function

$$
w_m(x,t) := t\,(m\, L_m^{p-1} + m) + m \tag{2.14}
$$

for all $t \ge 0$ and $x \in \mathbb{R}^d$, is the unique strong solution of the equation

$$
\frac{dw_m}{\mathrm{d}t} + \partial\varphi(w_m) = m\, L_m^{p-1} + m \quad \text{on } (0,T)
$$

in $L^2(\mathbb{R}^d, \mu)$ with initial value $w_m(0) = m$. Further, by [6, Théorème 3.1], there is a unique strong solution v_m of

$$
\frac{dv_m}{\mathrm{d}t} + \partial\varphi(v_m) = 0 \quad \text{on } (0,T) \tag{2.15}
$$

in $L^2(\mathbb{R}^d, \mu)$ with initial value $v_m(0) = u_{0,m}$. Since $0 \le v_m(0) = u_m^0 \le m = w_m(0)$ and since $0 \le m\, L_m^{p-1} + m$, the weak comparison principle (Lemma 2.6) implies

$$
0 \le v_m \le w_m
$$

for all $t \in [0,T]$ and a.e. $x \in \mathbb{R}^d$. By [6, Théorème 3.6], there is a unique strong solution $y_{m,1}$ of equation

$$
\frac{dy_{m,1}}{\mathrm{d}t} + \partial\varphi(y_{m,1}) = V_m\, v_m^{p-1} \quad \text{on } (0,T)
$$

in $L^2(\mathbb{R}^d, \mu)$ with initial value $y_{m,1}(0) = u_{0,m}$. Since $T = M\,(L_m^{p-1}\, m + m)^{-1}$ and since $w_m \le M + m = L_m$ on $\mathbb{R}^d \times (0,T)$, one has that

$$
0 \le V_m (v_m)^{p-1} \le m\, L_m^{p-1} + m
$$

a.e. on $\mathbb{R}^d \times (0,T)$. Hence the weak comparison principle yields

$$
0 \le v_m(x,t) \le y_{m,1}(x,t) \le w_m(x,t)
$$

for a.e. $x \in \mathbb{R}^d$ and all $t \in [0,T]$. Now, iteratively, for every $k \geq 2$, there is a unique strong solution $y_{m,k}$ of Eq. (2.11) with initial value $y_{m,k}(0) = u_{0,m}$. Since

$$0 \leq V_m\,(v_m)^{p-1} \leq V_m\,(y_{m,k-2})^{p-1} \leq V_m\,(y_{m,k-1})^{p-1} \leq m\, L_m^{p-1} + m$$

a.e. on $\mathbb{R}^d \times (0,T)$ for all $k \geq 2$, where we set $y_{m,0} = v_m$ if $k = 2$, the weak comparison principle implies

$$0 \leq v_m(x,t) \leq y_{m,k-1}(x,t) \leq y_{m,k}(x,t) \leq w_m(x,t) \leq L_m$$

for a.e. $x \in \mathbb{R}^d$ and for all $t \in [0,T]$. Thus, every $y_{m,k}$ satisfies (2.12) and (2.13).

Step 2: Next, we show that the following *a priori*-estimates hold:

$$\|y_{m,k}(t)\|_{L^2(\mathbb{R}^d,\mu)} \leq \|u_{0,m}\|_{L^2(\mathbb{R}^d,\mu)} + \int_0^t \left\|V_m\, w_m^{p-1}(s)\right\|_{L^2(\mathbb{R}^d,\mu)} \,\mathrm{d}s, \tag{2.16}$$

$$\int_0^t \int_{\mathbb{R}^d} |\nabla y_{m,k}(s)|^p \,\mathrm{d}\mu\,\mathrm{d}s \leq \|u_{0,m}\|^2_{L^2(\mathbb{R}^d,\mu)} + \tfrac{3}{2}\left[\int_0^t \left\|V_m\, w_m^{p-1}(s)\right\|_{L^2(\mathbb{R}^d,\mu)} \,\mathrm{d}s\right]^2 \tag{2.17}$$

and if $u_0 \in W^{1,p}(\mathbb{R}^d,\mu) \cap L^2(\mathbb{R}^d,\mu)$ then

$$\begin{aligned}\int_0^t \left\|\frac{dy_{m,k}}{ds}(s)\right\|^2_{L^2(\mathbb{R}^d,\mu)} \,\mathrm{d}s + \tfrac{2}{p}\int_{\mathbb{R}^d} |\nabla y_{m,k}(t)|^p \,\mathrm{d}\mu \\ \leq \tfrac{2}{p}\int_{\mathbb{R}^d} |\nabla u_{0,m}|^p \,\mathrm{d}\mu + \int_0^t \left\|V_m\, w_m^{p-1}(s)\right\|^2_{L^2(\mathbb{R}^d,\mu)} \mathrm{d}s\end{aligned} \tag{2.18}$$

and

$$\begin{aligned}\int_0^t s\left\|\frac{dy_{m,k}}{ds}(s)\right\|^2_{L^2(\mathbb{R}^d,\mu)} \,\mathrm{d}s + \tfrac{t\,2}{p}\int_{\mathbb{R}^d} |\nabla y_{m,k}(t)|^p \,\mathrm{d}\mu \\ \leq \tfrac{2}{p}\,\|u_{0,m}\|^2_{L^2(\mathbb{R}^d,\mu)} + \tfrac{3}{p}\left[\int_0^t \left\|V_m\, w_m^{p-1}(s)\right\|_{L^2(\mathbb{R}^d,\mu)} \,\mathrm{d}s\right]^2 \\ + \int_0^t s\left\|V_m\, w_m^{p-1}(s)\right\|^2_{L^2(\mathbb{R}^d,\mu)} \mathrm{d}s\end{aligned} \tag{2.19}$$

for all $k \geq 1$ and all $t \in [0,T]$. To see that the estimates (2.16) and (2.17) hold, we multiply Eq. (2.11) by $y_{m,k}$ with respect to the $L^2(\mathbb{R}^d,\mu)$ inner product and subsequently integrate over $(0,t)$, for $t \in (0,T]$. Then, by Cauchy–Schwarz's inequality and since $0 \leq y_{m,k-1} \leq w_m$, we obtain that

$$\begin{aligned}\tfrac{1}{2}\,\|y_{m,k}(t)\|^2_{L^2(\mathbb{R}^d,\mu)} + \int_0^t \int_{\mathbb{R}^d} |\nabla y_{m,k}(s)|^p \,\mathrm{d}\mu\,\mathrm{d}s \\ \leq \tfrac{1}{2}\,\|u_{0,m}\|^2_{L^2(\mathbb{R}^d,\mu)} + \int_0^t \left\|V_m\, w_m^{p-1}(s)\right\|_{L^2(\mathbb{R}^d,\mu)} \|y_{m,k}(s)\|_{L^2(\mathbb{R}^d,\mu)}\, ds,\end{aligned}$$

from where by [6, Lemme A.5] inequality (2.16) follows. Inserting inequality (2.16) into the latter one and applying Young's inequality, we see that inequality (2.17) holds. By [16, Lemma 7.6], if $u_0 \in W^{1,p}(\mathbb{R}^d,\mu)$ $\cap\, L^2(\mathbb{R}^d,\mu)$, then $u_{0,m} \in W^{1,p}(\mathbb{R}^d,\mu) \cap L^2(\mathbb{R}^d,\mu)$ with

$$\nabla u_{0,m} = \nabla u_0\, \mathbb{1}_{\{u_0<m\}}.$$

Hence, multiplying Eq. (2.11) by $2\,\frac{dy_{m,k}}{\mathrm{d}t}$ with respect to the $L^2(\mathbb{R}^d,\mu)$ inner product and subsequently integrating over $(0,t)$ for $t \in (0,T)$ and applying [6, Lemme 3.3], Cauchy–Schwarz's and Young's inequalities, we see that inequality (2.18) holds. Similarly, by multiplying Eq. (2.11) with $2\,t\,\frac{dy_{m,k}}{\mathrm{d}t}$ and applying inequality (2.18), we see that estimate (2.19) holds.

Step 3: The sequence $(y_{m,k})$ constructed in *Step* 1 consists of positive measurable functions $y_{m,k}$ on $\mathbb{R}^d \times [0,T]$, satisfies monotonicity property (2.13) and is uniformly bounded on $\mathbb{R}^d \times [0,T]$. Thus the limit function

$$u_m(x,t) := \sup_{k\geq 1} y_{m,k}(x,t) = \lim_{k\to\infty} y_{m,k}(x,t) \tag{2.20}$$

exists for a.e. $x \in \mathbb{R}^d$ and every $t \in [0,T]$ and satisfies $0 \leq u_m \leq L_m$ on $\mathbb{R}^d \times [0,T]$. We show that the limit function

$$u_m \in C([0,T]; L^2(\mathbb{R}^d,\mu)) \cap L^\infty(\mathbb{R}^d \times (0,T))$$

and satisfies $u_m(0) = u_{0,m}$ in $L^2(\mathbb{R}^d,\mu)$. By Lebesgue's monotone convergence theorem [29, Theorem 1.26], the function u_m is measurable on $\mathbb{R}^d \times [0,T]$. Moreover, one has that

$$\|y_{m,k}(t)\|_{L^2(\mathbb{R}^d,\mu)} \leq \|y_{m,k+1}(t)\|_{L^2(\mathbb{R}^d,\mu)}$$

for every $t \in [0,T]$ and every $k \geq 1$, and

$$\|y_{m,k}(t)\|_{L^2(\mathbb{R}^d,\mu)} \nearrow \|u_m(t)\|_{L^2(\mathbb{R}^d,\mu)}$$

as $k \to \infty$ for all $t \in [0,T]$. Since

$$\sup_{k\geq 1} \sup_{t\in[0,T]} \|y_{m,k}(t)\|_{L^2(\mathbb{R}^d,\mu)} \leq L_m$$

and $y_{m,k} \in C([0,T]; L^2(\mathbb{R}^d,\mu))$ for every $k \geq 1$, Dini's Theorem and the uniform convexity of $L^2(\mathbb{R}^d,\mu)$ imply that $u_m \in C([0,T]; L^2(\mathbb{R}^d,\mu))$ and

$$\lim_{k\to\infty} y_{m,k} \to u_m \quad \text{in } C([0,T]; L^2(\mathbb{R}^d,\mu)). \tag{2.21}$$

In particular, since $y_{m,k}(0) = u_{0,m}$ for all $k \geq 1$, we have that $u_m(0) = u_{0,m}$.

Step 4: First, we consider the case $u_0 \in W^{1,p}(\mathbb{R}^d,\mu) \cap L^2(\mathbb{R}^d,\mu)$. Due to the *a priori*-estimates (2.16)–(2.18) and by inequality (2.8), the sequence $(y_{m,k})$ is bounded in the space $W^{1,2}(0,T; L^2(\mathbb{R}^d,\mu)) \cap L^p(0,T; W^{1,p}(\mathbb{R}^d,\mu))$. Since this space is reflexive and by limit (2.21), we obtain that $u_m \in W^{1,2}(0,T; L^2(\mathbb{R}^d,\mu)) \cap L^p(0,T; W^{1,p}(\mathbb{R}^d,\mu))$ and there is a subsequence of $(y_{k,m})$, which we denote, for simplicity, again by $(y_{m,k})$ such that

$$\lim_{k\to\infty} y_{m,k} = u_m \quad \text{weakly in } W^{1,2}(0,T; L^2(\mathbb{R}^d,\mu)) \cap L^p(0,T; W^{1,p}(\mathbb{R}^d,\mu)). \tag{2.22}$$

Since $0 \leq V_m y_{m,k-1}^{p-1} \leq m\, L_m^{p-1}$, we have for a.e. $t \in (0,T)$ that $y_{m,k-1}(t) \in D(\partial\varphi_{V_m})$, $\partial\varphi_{V_m}(y_{m,k-1}(t)) = V_m y_{m,k-1}^{p-1}(t)$ and by limit (2.20),

$$\lim_{k\to\infty} \partial\varphi_{V_m}(y_{m,k-1}(t)) = \partial\varphi_{V_m}(u_m(t)) \quad \text{strongly in } L^2(\mathbb{R}^d,\mu).$$

Using again the fact that $0 \leq V_m y_{m,k-1}^{p-1} \leq m\, L_m^{p-1}$, we see that

$$\left\| V_m y_{m,k-1}^{p-1}(t) \right\|_{L^2(\mathbb{R}^d,\mu)} \leq m\, L_m^{p-1}$$

for a.e. $t \in (0,T)$. Thus

$$\lim_{k\to\infty} \partial\varphi_{V_m}(y_{m,k-1}) = \partial\varphi_{V_m}(u_m) \quad \text{strongly in } L^2(0,T; L^2(\mathbb{R}^d,\mu)).$$

Moreover, for a.e. $t \in (0,T)$, $y_{m,k}(t) \in D(\partial\varphi)$ with

$$\partial\varphi(y_{m,k}(t)) = \partial\varphi_{V_m}(y_{m,k-1}) - \frac{dy_{m,k}}{dt}(t)$$

in $L^2(\mathbb{R}^d,\mu)$. Therefore

$$\lim_{k\to\infty} \partial\varphi(y_{m,k}(t)) = \partial\varphi_{V_m}(u_m) - \frac{du_m}{\mathrm{d}t}(t) \quad \text{weakly in } L^2(0,T;L^2(\mathbb{R}^d,\mu)). \tag{2.23}$$

Since for $\partial\varphi$ the associated operator on $L^2(0,T;L^2(\mathbb{R}^d,\mu))$ is maximal monotone by [6, Exemple 2.3.3], the two limits (2.23) and (2.21) imply by [6, Proposition 2.5] that for a.e. $t\in(0,T)$, $u_m(t)\in D(\partial\varphi)$ and

$$\partial\varphi(u_m(t)) = \partial\varphi_{V_m}(u_m(t)) - \frac{du_m}{\mathrm{d}t}(t).$$

This shows that $u_m \in L^\infty(\mathbb{R}^d\times(0,T))$ is a positive strong solution of Eq. (2.5) with initial value $u_m(0) = u_{0,m}$. Furthermore, sending $k\to\infty$ in the inequalities (2.16), (2.17) and (2.19), and by using the limits (2.21) and (2.22) yields

$$\|u_m(t)\|_{L^2(\mathbb{R}^d,\mu)} \le \|u_{0,m}\|_{L^2(\mathbb{R}^d,\mu)} + \int_0^t \left\|V_m\, w_m^{p-1}(s)\right\|_{L^2(\mathbb{R}^d,\mu)}\,\mathrm{d}s, \tag{2.24}$$

$$\int_0^t\int_{\mathbb{R}^d} |\nabla u_m(s)|^p\,\mathrm{d}\mu\,\mathrm{d}s \le \|u_{0,m}\|^2_{L^2(\mathbb{R}^d,\mu)} + \tfrac{3}{2}\left[\int_0^t \left\|V_m\, w_m^{p-1}(s)\right\|_{L^2(\mathbb{R}^d,\mu)}\,\mathrm{d}s\right]^2 \tag{2.25}$$

and

$$\int_0^t s\left\|\frac{du_m}{ds}(s)\right\|^2_{L^2(\mathbb{R}^d,\mu)}\mathrm{d}s \le \tfrac{2}{p}\|u_{0,m}\|^2_{L^2(\mathbb{R}^d,\mu)} + \tfrac{3}{p}\left[\int_0^t \left\|V_m\, w_m^{p-1}(s)\right\|_{L^2(\mathbb{R}^d,\mu)}\,\mathrm{d}s\right]^2 + \int_0^t s\left\|V_m\, w_m^{p-1}(s)\right\|^2_{L^2(\mathbb{R}^d,\mu)}\mathrm{d}s. \tag{2.26}$$

It is left to show that the sequence $(u_m)_{m\ge1}$ satisfies (2.6). If for every integer $m\ge1$, w_m is given by (2.14), then $w_m\le w_{m+1}$ and if v_m denotes the unique strong solution of problem (2.15) with initial value $v_m(0)=u_{0,m}$, then by the weak comparison principle, $0\le v_m\le v_{m+1}$ for a.e. $x\in\mathbb{R}^d$ and all $t\in[0,T]$. Since $M>0$ can always be chosen such that $T = M\,m^{-1}\,L_m^{1-p} =: T_m$ for every $m\ge1$, we have that $0\le y_{m,1}\le y_{m+1,1}\le w_{m+1}$, and by iteration, $0\le y_{m,k}\le y_{m+1,k}\le w_{m+1}$ for a.e. $x\in\mathbb{R}^d$ and all $t\in[0,T_{m+1}]$. Therefore, if we send $k\to+\infty$ in inequality

$$0\le y_{m,k}\le y_{m+1,k}\le w_{m+1} \tag{2.27}$$

for fixed $m\ge1$, then we find that $0\le u_m\le u_{m+1}\le w_{m+1}$ for a.e. $x\in\mathbb{R}^d$ and all $t\in[0,T_{m+1}]$. This shows that the claim of this propositions holds if the positive initial value $u_0\in W^{1,p}(\mathbb{R}^d,\mu)\cap L^2(\mathbb{R}^d,\mu)$.

Now, let $u_0\in L^2(\mathbb{R}^d,\mu)$ be positive. Standard mollifying and truncation techniques show that $C_c^\infty(\mathbb{R}^d)\subseteq W^{1,p}(\mathbb{R}^d,\mu)\cap L^2(\mathbb{R}^d,\mu)$ is dense in $L^2(\mathbb{R}^d,\mu)$ and, in particular, we may assume that there is a sequence $(u_0^{(j)})$ of positive functions $u_0^{(j)}\in W^{1,p}(\mathbb{R}^d,\mu)\cap L^2(\mathbb{R}^d,\mu)$ such that $u_0^{(j)}$ converges to u_0 in $L^2(\mathbb{R}^d,\mu)$ as $j\to\infty$. Fix $m\ge1$. Then $u_{0,m}^{(j)} := \min\{u_0^{(j)},m\}$ converges to $u_{0,m}=\min\{u_0,m\}$ in $L^2(\mathbb{R}^d,\mu)$ as $j\to\infty$. For every $j\ge1$, there is a sequence $(y_{m,k}^{(j)})$ of strong solutions $y_{m,k}^{(j)}$ of (2.11) satisfying initial value $y_{m,k}^{(j)}(0)=u_{0,m}^{(j)}$ and a sequence $(u_m^{(j)})$ of strong solutions $u_m^{(j)}$ of Eq. (2.5) with initial value $u_m^{(j)}(0)=u_{0,m}^{(j)}$ such that limit

$$u_m^{(j)}(x,t) = \sup_{k\ge1} y_{m,k}^{(j)}(x,t) = \lim_{k\to\infty} y_{m,k}^{(j)}(x,t)$$

and monotonicity property (2.27) hold a.e. on $\mathbb{R}^d\times(0,T)$ for $y_{m,k}^{(j)}$ instead of $y_{m,k}$. Thus sending $k\to\infty$ in (2.27) for fixed $j\ge1$, we obtain

$$0\le u_m^{(j)}\le u_{m+1}^{(j)}\le w_{m+1} \tag{2.28}$$

for every $j \geq 1$ and every $m \geq 1$. By the inequalities (2.24) and (2.26), the sequence $(u_m^{(j)})$ is bounded in $W^{1,2}(\delta, T; L^2(\mathbb{R}^d, \mu))$ for every $\delta \in (0, T)$. Hence $(u_m^{(j)})$ is bounded and equicontinuous in $C([\delta, T]; L^2(\mathbb{R}^d, \mu))$ for every $\delta \in (0, T)$. By Arzelà–Ascoli, there is a function $u_m \in C((0, T]; L^2(\mathbb{R}^d, \mu))$ and by a diagonal sequence argument, there is a subsequence $(u_m^{(k_j)})$ of $(u_m^{(j)})$ such that

$$\lim_{j\to\infty} u_m^{(k_j)} = u_m \quad \text{strongly in } C([\delta, T]; L^2(\mathbb{R}^d, \mu)) \tag{2.29}$$

for every $\delta \in (0, T)$. In particular, for every $t \in (0, T]$,

$$\lim_{j\to\infty} u_m^{(k_j)}(t) = u_m(t) \quad \text{a.e. on } \mathbb{R}^d. \tag{2.30}$$

Thus, by (2.28), and since $0 \leq V_m \leq m$, we have for all $t \in (0, T]$ that $u_m(t) \in D(\partial\varphi_{V_m})$ with $\partial\varphi_{V_m}(u_m(t)) = V_m\,|u_m(t)|^{p-2}\,u_m(t)$ and

$$\lim_{j\to\infty} \varphi_{V_m}(u_m^{(k_j)}) = \varphi_{V_m}(u_m) \quad \text{strongly in } L^2(0, T; L^2(\mathbb{R}^d, \mu)). \tag{2.31}$$

Moreover, for a.e. $t \in (0, T)$, $u_m^{(k_j)}(t) \in D(\partial\varphi)$ with

$$\partial\varphi(u_m^{(k_j)}(t)) = \partial\varphi_{V_m}(u_m^{(k_j)}(t)) - \frac{du_m^{(k_j)}}{\mathrm{d}t}(t)$$

in $L^2(\mathbb{R}^d, \mu)$ and

$$\lim_{k\to\infty} \partial\varphi(u_m^{(k_j)}) = \partial\varphi_{V_m}(u_m) - \frac{du_m}{\mathrm{d}t}(t) \quad \text{weakly in } L^2(\delta, T; L^2(\mathbb{R}^d, \mu)) \tag{2.32}$$

for every $\delta \in (0, T)$. Since for $\partial\varphi$ the associated operator on $L^2(0, T; L^2(\mathbb{R}^d, \mu))$ is maximal monotone, the two limits (2.29) and (2.32) imply that for a.e. $t \in (0, T)$, $u_m(t) \in D(\partial\varphi)$ and

$$\partial\varphi(u_m(t)) = \partial\varphi_{V_m}(u_m(t)) - \frac{du_m}{\mathrm{d}t}(t).$$

On the other hand, by limit (2.30) and by (2.28)

$$\lim_{j\to\infty} V_m \left|u_m^{(k_j)}\right|^{p-2} u_m^{(k_j)} = V_m\,|u_m|^{p-2}\,u_m \quad \text{strongly in } L^{p'}(0, T; L^{p'}(\mathbb{R}^d, \mu))$$

and hence, in particular,

$$\lim_{j\to\infty} V_m \left|u_m^{(k_j)}\right|^{p-2} u_m^{(k_j)} = V_m\,|u_m|^{p-2}\,u_m \quad \text{strongly in } L^{p'}(0, T; (W^{1,p}(\mathbb{R}^d, \mu))' + L^2(\mathbb{R}^d, \mu)).$$

By the estimates (2.24) and (2.25), $(u_m^{(k_j)})$ is bounded in $L^p(0, T; W^{1,p}(\mathbb{R}^d, \mu) \cap L^2(\mathbb{R}^d, \mu))$. Therefore, $(\frac{du_m^{(k_j)}}{dt})$ is bounded in $L^{p'}(0, T; (W^{1,p}(\mathbb{R}^d, \mu))' + L^2(\mathbb{R}^d, \mu))$ and so $(u_m^{(k_j)})$ is bounded in the reflexive space

$$\mathcal{V} := W^{1,p'}(0, T; (W^{1,p}(\mathbb{R}^d, \mu))' + L^2(\mathbb{R}^d, \mu)) \cap L^p(0, T; W^{1,p}(\mathbb{R}^d, \mu) \cap L^2(\mathbb{R}^d, \mu)).$$

Therefore, there is a $u \in \mathcal{V}$ and a subsequence of $(u_m^{(k_j)})$, which we denote again by $(u_m^{(k_j)})$ such that $u_m^{(k_j)}$ converges to u weakly in $\mathcal{V}$. Since $\mathcal{V}$ is continuously embedded into $C([0, T]; L^2(\mathbb{R}^d, \mu))$ (cf. [24, Remarque 1.2, Chapitre 2]), $u \in C([0, T]; L^2(\mathbb{R}^d, \mu))$ and $u_{0,m}^{(j)} = u_m^{(k_j)}(0)$ converges weakly to $u(0)$ in $L^2(\mathbb{R}^d, \mu)$ as $j \to \infty$. Hence and since $u_{0,m}^{(j)}$ converges to $u_{0,m}$ in $L^2(\mathbb{R}^d, \mu)$ as $j \to \infty$, we have that $u_m(0) = u_{0,m}$ in $L^2(\mathbb{R}^d, \mu)$. In addition, sending $j \to \infty$ in (2.28) for appropriate subsequences $(u_m^{(k_j)})$ and $(u_{m+1}^{(k_j')})$, we see that the sequence (u_m) satisfies the monotonicity property (2.6). This completes the proof of this proposition. □

2.2. Proof of Theorem 1.3

With these preliminaries, we can outline now the proof of Theorem 1.3.

2.2.1. Proof of claim (1) of Theorem 1.3

We begin to outline claim (1) of Theorem 1.3. As in the previous proof, we proceed here in several steps.

Step 1: For given $m \geq 1$ and positive $u_0 \in L^2(\mathbb{R}^d, \mu)$, let u_m be a positive strong solution of Eq. (2.5) with initial value $u_m(0) = u_{0,m}$. Then we begin by showing that (u_m) satisfies the following *a priori*-estimates:

$$\begin{aligned} &\|u_m(t)\|_{L^2(\mathbb{R}^d,\mu)} \\ &\quad \leq \left[\|u_{0,m}\|^{2-p}_{L^2(\mathbb{R}^d,\mu)} \exp\left(\left(1-\tfrac{p}{2}\right) t\right) + (2-p)\left(\tfrac{d-p}{p}\right)^{p-1} \Lambda_A^{\frac{p}{2}} \int_0^t \exp\left(\left(1-\tfrac{p}{2}\right)(t-s)\right) \mathrm{d}s \right]^{\frac{1}{2-p}} \end{aligned} \tag{2.33}$$

if $1 < p < 2$,

$$\|u_m(t)\|_{L^2(\mathbb{R}^d,\mu)} \leq \|u_{0,m}\|_{L^2(\mathbb{R}^d,\mu)} \exp\left(\frac{1+2\left(\frac{d-p}{p}\right)^{p-1} \Lambda_A^{\frac{p}{2}}}{2}\, t \right) \tag{2.34}$$

if $p = 2 < d$, and if $p > d \geq 2$, one has

$$\|u_m(t)\|_{L^2(\mathbb{R}^d,\mu)} \leq \|u_{0,m}\|_{L^2(\mathbb{R}^d,\mu)} \tag{2.35}$$

and

$$\int_0^t \|u_m(s)\|^p_{L^p(\mathbb{R}^d,\mu)}\, \mathrm{d}s \leq \left(\tfrac{p-d}{p}\right)^{1-p} \lambda_A^{-\frac{p}{2}}\, \tfrac{1}{2} \|u_{0,m}\|^2_{L^2(\mathbb{R}^d,\mu)} \tag{2.36}$$

where λ_A and Λ_A denote the smallest and largest eigenvalues of the matrix A. To see this, multiply Eq. (2.5) by u_m with respect to the $L^2(\mathbb{R}^d, \mu)$ inner product and subsequently integrate over $(0, t)$, for $t \in (0, T]$. First, consider the case $1 < p \leq 2$ and $p \neq d$. Then, by using Hardy's inequality (1.5) and Hölder's inequality,

$$\|u_m(t)\|^2_{L^2(\mathbb{R}^d,\mu)} \leq \|u_{0,m}\|^2_{L^2(\mathbb{R}^d,\mu)} + 2\left(\tfrac{d-p}{p}\right)^{p-1} \Lambda_A^{\frac{p}{2}} \int_0^t \|u_m(s)\|^p_{L^2(\mathbb{R}^d,\mu)}\, \mathrm{d}s,$$

from where we can deduce the inequalities (2.33) and (2.34) by using a nonlinear generalisation of Gronwall's inequality (cf. [27, Theorem 1, p.360]). Now, consider the case $p > d$. Then the reminder term (1.6) in Hardy's inequality (1.5) has a negative sign. Hence

$$\tfrac{1}{2} \|u_m(t)\|^2_{L^2(\mathbb{R}^d,\mu)} \leq \tfrac{1}{2} \|u_{0,m}\|^2_{L^2(\mathbb{R}^d,\mu)} - \left(\tfrac{|d-p|}{p}\right)^{p-1} \lambda_A^{\frac{p}{2}} \int_0^t \|u_m(s)\|^p_{L^p(\mathbb{R}^d,\mu)}\, \mathrm{d}s$$

from where we can deduce the inequalities (2.35) and (2.36).

Step 2: By Proposition 2.2, each strong solution u_m of Eq. (2.5) satisfies (2.6). Thus by Lebesgue's monotone convergence theorem, the function

$$u(x,t) := \sup_{m\geq 1} u_m(x,t) = \lim_{m\to\infty} u_m(x,t)$$

for all $t \in [0, T]$ and a.e. $x \in \mathbb{R}^d$ is measurable on $\mathbb{R}^d \times (0, T)$. In addition, since for every $t \in [0, T]$,

$$|u_m(x,t)|^2 \leq |u_{m+1}(x,t)|^2 \nearrow |u(x,t)|^2\,,$$

for a.e. $x \in \mathbb{R}^d$, Lebesgue's monotone convergence theorem and by the bound (2.33)–(2.35) combined with the boundedness of the sequences $(u_{0,m})$, we obtain that for all $t \in [0, T]$, $u(t) \in L^2(\mathbb{R}^d, \mu)$,

$$\|u_m(t)\|_{L^2(\mathbb{R}^d,\mu)} \leq \|u_{m+1}(t)\|_{L^2(\mathbb{R}^d,\mu)}$$

for every $m \geq 1$ and

$$\lim_{m\to\infty} \|u_m(t)\|_{L^2(\mathbb{R}^d,\mu)} = \|u(t)\|_{L^2(\mathbb{R}^d,\mu)}\,.$$

Now, we can apply Dini's theorem, to deduce that $u \in C([0,T];L^2(\mathbb{R}^d,\mu))$ and

$$\lim_{m\to\infty} u_m = u \quad \text{in } C([0,T];L^2(\mathbb{R}^d,\mu)). \tag{2.37}$$

Moreover, since $u_m(0) \to u(0)$ in $L^2(\mathbb{R}^d,\mu)$ and $u_m(0) = u_{0,m} \to u_0$ in $L^2(\mathbb{R}^d,\mu)$, it follows that $u(0) = u_0$.

Step 3: Next, we show that (u_m) satisfies the following *a priori*-estimates on the annuli $R_k := \{x \in \mathbb{R}^d \mid 2^{-k} < |x| < 2^k\}$ in $\mathbb{R}^d$ for any fixed $k \geq 1$:

$$\int_{R_k} \tfrac{1}{2} u_m^2(t)\,\mathrm{d}x + \tfrac{1}{2\,p'} \int_0^t \int_{R_k} |\nabla u_m(s)|^p \,\mathrm{d}x\,\mathrm{d}s \leq \int_{R_{k+1}} \tfrac{1}{2} u_{0,m}^2 \zeta^p \,\mathrm{d}x + C \int_0^t \|u_m(s)\|^p_{L^p(\mathbb{R}^d,\mu)}\,\mathrm{d}s \tag{2.38}$$

for some constant $C > 0$ depending on p, 2^k, $\nabla\xi$ and $\sup_{R_{k+1}} \rho^{-1}$. Fix $k \geq 1$ and choose a cutoff function $\zeta \in C_c^\infty(R_{k+1})$ satisfying $0 \leq \zeta \leq 1$ in R_{k+1} and $\zeta \equiv 1$ on R_k. If ρ denotes the density function given by (1.4), then the weak gradient

$$\nabla(u_m \zeta^p \rho^{-1}) = \nabla u_m\, \zeta^p \rho^{-1} + p\, u_m \zeta^{p-1} \nabla\zeta\, \rho^{-1} - u_m \zeta^p \tfrac{\nabla\rho}{\rho^2}.$$

Thus, multiplying Eq. (2.5) by $u_m\, \zeta^p \rho^{-1}$ with respect to the $L^2(\mathbb{R}^d,\mu)$ inner product and subsequently integrating over $(0,t)$ for $t \in (0,T]$ yields

$$\begin{aligned}
&\int_{R_{k+1}} \tfrac{1}{2} u_m^2(t) \zeta^p \,\mathrm{d}x + \int_0^t \int_{R_{k+1}} |\nabla u_m(s)|^p\, \zeta^p \,\mathrm{d}x\,\mathrm{d}s + p \int_0^t \int_{R_{k+1}} |\nabla u_m(s)|^{p-2}\, \nabla u_m(s) \nabla\zeta\, \zeta^{p-1} u_m(s)\,\mathrm{d}x\,\mathrm{d}s \\
&\quad = \int_{R_{k+1}} \tfrac{1}{2} u_{0,m}^2 \zeta^p \,\mathrm{d}x + \int_0^t \int_{R_{k+1}} |\nabla u_m(s)|^{p-2}\, \nabla u_m(s) \nabla\rho\, \zeta^p \tfrac{u_m(s)}{\rho} \,\mathrm{d}x\,\mathrm{d}s + \int_0^t \int_{R_{k+1}} \frac{u_m^p(s)}{|x|^p} \zeta^p \,\mathrm{d}x\,\mathrm{d}s
\end{aligned}$$

and so by applying Hölder's and Young's inequality, we see that inequality (2.38) holds.

Step 4: Since $u_{0,m}$ converges to u_0 in $L^2(\mathbb{R}^d,\mu)$, the *a priori*-estimates (2.33)–(2.36) imply that (u_m) is bounded in $C([0,T];L^2(\mathbb{R}^d,\mu))$ for $1 < p \leq 2$ and $p \neq d$ and bounded in $C([0,T];L^p(\mathbb{R}^d,\mu))$ for $p > d \geq 2$. Thus by *a priori*-estimate (2.38) on the annuli (R_k), the sequence (u_m) is bounded in $L^p(0,T;W^{1,p}(R_k))$ for every $k \geq 1$. Now, fix $k \geq 1$. Since the space $L^p(0,T;W^{1,p}(R_k))$ is reflexive and by limit (2.37), it follows that $u \in L^p(0,T;W^{1,p}(R_k))$ and there is a subsequence (u_{k_m}) of (u_m) such that

$$\lim_{m\to\infty} u_{k_m} = u \quad \text{weakly in } L^p(0,T;W^{1,p}(R_k)). \tag{2.39}$$

Furthermore, the operator

$$\mathcal{A}_0 u := -\rho^{-1} \operatorname{div}\left(\rho\, |\nabla u|^{p-2}\, \nabla u\right) - |x|^{-p}\, |u|^{p-2}\, u$$

maps bounded sets of $L^p(0,T;W^{1,p}(R_k))$ into bounded sets of $L^{p'}(0,T;W^{-1,p'}(R_k))$, where we denote by $W^{-1,p'}(R_k)$ the dual space of $W_0^{1,p}(R_k)$. Hence the sequence (u_{k_m}) is bounded in the space $W^{1,p'}(0,T;W^{-1,p'}(R_k))$. Thus by the uniqueness of the limit (2.37), we have that $u \in W^{1,p'}(0,T;W^{-1,p'}(R_k))$ and there is a subsequence of (u_{k_m}), which we denote, for convenience, again by (u_{k_m}) such that

$$\lim_{m\to\infty} \tfrac{du_{k_m}}{dt} = \tfrac{du}{dt} \quad \text{weakly in } L^{p'}(0,T;W^{-1,p'}(R_k)). \tag{2.40}$$

If $\frac{2d}{d+2} < p \leq 2$ and $p \neq d$, then by Rellich–Kondrachov (cf. [7, Théorème IX.16]), the embedding from $W^{1,p}(R_k)$ into $L^2(R_k)$ is compact. If $p > d \geq 2$, we use that the embedding from $W^{1,p}(R_k)$ into $L^p(R_k)$ is compact. Hence by the lemma of Lions–Aubin (cf. [25, Théorème 5.1]), the limit function u given by (2.37) belongs to $L^p(0,T;L^p(R_k))$ and there is a subsequence of (u_{k_m}) such that

$$\lim_{m\to\infty} u_{k_m} = u \quad \text{strongly in } L^p(0,T;L^q(R_k)) \tag{2.41}$$

for $q=2$ if $\frac{2d}{d+2}<p\le 2$, $p\neq d$, and $q=p$ if $p>d$. Therefore, since $\rho>0$ and since ρ is bounded on R_k,

$$\lim_{m\to\infty}\frac{|u_{k_m}|^{p-2}\,u_{k_m}}{|x|^p}=\frac{|u|^{p-2}\,u}{|x|^p}\quad\text{strongly in } L^{p'}(0,T;L^{p'}(R_k,\mu)). \tag{2.42}$$

Furthermore, by *a priori*-estimate (2.38) on the annulus R_k, the sequence (∇u_{k_m}) is bounded in $L^p(0,T;L^p(R_k,\mu)^d)$. Thus there is a function $\chi_k\in L^{p'}(0,T;L^{p'}(R_k,\mu)^d)$ and another subsequence of (u_{k_m}) such that

$$\lim_{m\to\infty}|\nabla u_{k_m}|^{p-2}\,\nabla u_{k_m}=\chi_k\quad\text{weakly in } L^{p'}(0,T;L^{p'}(R_k,\mu)^d). \tag{2.43}$$

Since $R_k\subseteq R_{k+1}$ such that $\bigcup_{k\ge 1}R_k=\mathbb{R}^d\setminus\{0\}$ and since the limits (2.39), (2.41)–(2.43) and (2.40) on R_k also hold on R_{k-1}, a diagonal sequence argument shows that $u\in L^p(0,T;W^{1,p}_{\text{loc}}(\mathbb{R}^d\setminus\{0\},\mu))$, such that $\frac{du}{dt}\in L^{p'}(0,T;W^{-1,p'}(\mathbb{R}^d\setminus\{0\},\mu))$ and there is a subsequence (u_{k_m}) of (u_m) such that

$$\lim_{m\to\infty}u_{k_m}=u\quad\text{weakly in } L^p(0,T;W^{1,p}_{\text{loc}}(\mathbb{R}^d\setminus\{0\},\mu)), \tag{2.44}$$

$$\lim_{m\to\infty}u_{k_m}=u\quad\text{strongly in } L^p(0,T;L^q_{\text{loc}}(\mathbb{R}^d\setminus\{0\},\mu)), \tag{2.45}$$

where $q=2$ if $\frac{2d}{d+2}<p\le 2$, $p\neq d$, and $q=p$ if $p>d$,

$$\lim_{m\to\infty}\frac{|u_{k_m}|^{p-2}\,u_{k_m}}{|x|^p}=\frac{|u|^{p-2}\,u}{|x|^p}\quad\text{strongly in } L^{p'}(0,T;L^{p'}_{\text{loc}}(\mathbb{R}^d\setminus\{0\},\mu)), \tag{2.46}$$

$$\lim_{m\to\infty}\tfrac{du_{k_m}}{dt}=\tfrac{du}{dt}\quad\text{weakly in } L^{p'}(0,T;W^{-1,p'}_{\text{loc}}(\mathbb{R}^d\setminus\{0\},\mu)). \tag{2.47}$$

Moreover, there is a function $\chi\in L^{p'}(0,T;L^{p'}_{\text{loc}}(\mathbb{R}^d\setminus\{0\},\mu)^d)$ such that

$$\lim_{m\to\infty}|\nabla u_{k_m}|^{p-2}\,\nabla u_{k_m}=\chi\quad\text{weakly in } L^{p'}(0,T;L^{p'}_{\text{loc}}(\mathbb{R}^d\setminus\{0\},\mu)^d). \tag{2.48}$$

Step 5: In this part of the proof, we prove that

$$\chi=|\nabla u|^{p-2}\,\nabla u\quad\text{a.e. on }\mathbb{R}^d\times(0,T).$$

Fix $k\ge 1$. Then, multiplying (2.5) by $v\in L^p(0,T;W^{1,p}_0(R_{k+1},\mu))\cap L^2(0,T;L^2(R_{k+1},\mu))$ and subsequently integrating over $(0,T)$ yields

$$\Big\langle\!\Big\langle \tfrac{du_{k_m}}{ds},v\Big\rangle\!\Big\rangle+\int_0^T\int_{R_{k+1}}|\nabla u_{k_m}|^{p-2}\,\nabla u_{k_m}\nabla v\,\mathrm{d}\mu\,\mathrm{d}s=\int_0^t\int_{R_{k+1}}\frac{u_{k_m}^{p-1}(s)}{|x|^p}v\,\mathrm{d}\mu\,\mathrm{d}s,$$

where $\langle\!\langle\cdot,\cdot\rangle\!\rangle$ denotes the duality pairing between the spaces $L^{p'}(0,T;W^{-1,p'}(R_{k+1},\mu))$ and $L^p(0,T;W^{1,p}_0(R_{k+1},\mu))$. Now, sending $m\to+\infty$ in this equation and using the limits (2.46)–(2.48), we obtain

$$\langle\!\langle \tfrac{du}{ds},v\rangle\!\rangle+\int_0^T\int_{R_{k+1}}\chi\nabla v\,\mathrm{d}\mu\,\mathrm{d}s=\int_0^T\int_{R_{k+1}}\frac{u^{p-1}}{|x|^p}v\,\mathrm{d}\mu\,\mathrm{d}s$$

for all $v\in L^p(0,T;W^{1,p}_0(R_{k+1},\mu))\cap L^2(0,T;L^2(R_{k+1},\mu))$. Note that

$$\langle\!\langle \tfrac{du}{ds},u\zeta^p\rangle\!\rangle=\tfrac{1}{2}\int_{R_{k+1}}u^2(T)\,\zeta^p\,\mathrm{d}\mu-\tfrac{1}{2}\int_{R_{k+1}}u^2(0)\,\zeta^p\,\mathrm{d}\mu. \tag{2.49}$$

This formula is easily proved by either approximating $u\zeta^p$ by convolution (similarly as described in [31, Section 1.5, p.264]) or by using the Steklov average together with Lemma A.2. Thus taking $v=u\zeta^p$, in the last equation and applying formula (2.49) yields

$$\tfrac{1}{2}\int_{R_{k+1}} u^2(T)\,\zeta^p\,\mathrm{d}\mu + \int_0^T\int_{R_{k+1}} \chi\nabla u\,\zeta^p \mathrm{d}\mu\,\mathrm{d}s = \tfrac{1}{2}\int_{R_{k+1}} u^2(0)\,\zeta^p\,\mathrm{d}\mu - p\int_0^T\int_{R_{k+1}} \chi\nabla\zeta\,\zeta^{p-1}\,u\,\mathrm{d}\mu\,ds$$
$$+ \int_0^T\int_{R_{k+1}} \frac{u^{p-1}}{|x|^p} v\,\mathrm{d}\mu\,\mathrm{d}s. \tag{2.50}$$

On the other hand, multiplying Eq. (2.5) by $u_{k_m}\,\zeta^p$ with respect to the $L^2(R_{k+1},\mu)$ inner product and then integrating over $(0,T)$ gives

$$\int_0^T\int_{R_{k+1}} |\nabla u_{k_m}|^p\ \zeta^p \mathrm{d}\mu\,\mathrm{d}s = \tfrac{1}{2}\int_{R_{k+1}} u_m^2(0)\,\zeta^p\,\mathrm{d}\mu - \tfrac{1}{2}\int_{R_{k+1}} u_{k_m}^2(T)\,\zeta^p\,\mathrm{d}\mu$$
$$- p\int_0^T\int_{R_{k+1}} |\nabla u_{k_m}|^{p-2}\,\nabla u_{k_m}\nabla\zeta\,\zeta^{p-1}\,u_m\,\mathrm{d}\mu\,\mathrm{d}s$$
$$+ \int_0^T\int_{R_{k+1}} V_m\,|u_{k_m}|^p\,\zeta^p\,\mathrm{d}\mu\,ds. \tag{2.51}$$

Sending $m\to\infty$ in this equation and using the limits (2.37), (2.45), (2.46) and (2.48) and subsequently comparing the resulting limit with Eq. (2.50) yields

$$\lim_{m\to\infty}\int_0^T\int_{R_{k+1}} |\nabla u_{k_m}|^p\ \zeta^p \mathrm{d}\mu\,\mathrm{d}s = \int_0^T\int_{R_{k+1}} \chi\nabla u\,\zeta^p \mathrm{d}\mu\,\mathrm{d}s. \tag{2.52}$$

Now, we use a method due to Leray and Lions [24]. For every $m\geq 1$, let

$$H_{k_m}(x,t) := \Big(|\nabla u_{k_m}|^{p-2}\,\nabla u_{k_m} - |\nabla u|^{p-2}\,\nabla u, \nabla u_{k_m} - \nabla u\Big)_{\mathbb{R}^d}$$

for a.e. $(x,t)\in\mathbb{R}_{k+1}\times(0,T)$. Every H_{k_m} is a positive measurable function on $R_{k+1}\times(0,T)$ and the limits (2.44), (2.48) and (2.52) imply that $H_{k_m}\zeta^p$ converges to 0 in $L^1(R_{k+1},\mu)$. Since we have chosen $\zeta\equiv 1$ on R_k, it follows

$$\lim_{m\to\infty} H_{k_m} = 0 \quad \text{in } L^1(R_k\times(0,T)). \tag{2.53}$$

There is a subsequence of (u_{k_m}) and there is a measurable subset $\mathcal{Z}\subseteq R_k\times(0,T)$ of Lebesgue measure zero such that for all $(x,t)\in R_k\times(0,T)\setminus\mathcal{Z}$, $H_{k_m}(x,t)$ is finite and $H_{k_m}(x,t)$ converges to 0 in $\mathbb{R}$ as $m\to+\infty$. By Hölder's and Young's inequalities,

$$\begin{aligned}|\nabla u_{k_m}(x,t)|^p &\leq H_{k_m}(x,t) + |\nabla u(x,t)|^{p-2}\,\nabla u(x,t)\nabla u_{k_m}(x,t) + |\nabla u_{k_m}|^{p-2}\,\nabla u_{k_m}\nabla u(x,t) - |\nabla u|^p\\ &\leq H_{k_m}(x,t) + c\,|\nabla u(x,t)|^p + \tfrac{1}{2}\,|\nabla u_{k_m}(x,t)|^p\,,\end{aligned}$$

for some constant $c=c(p,\eta,\beta)>0$ and so

$$\tfrac{1}{2}\,|\nabla u_{k_m}(x,t)|^p \leq H_{k_m}(x,t) + c\,|\nabla u(x,t)|^p \tag{2.54}$$

for every $m\geq 1$. Thus, for every $(x,t)\in R_k\times(0,T)\setminus\mathcal{Z}$, the sequence $(\nabla u_m(x,t))$ is bounded in $\mathbb{R}^d$ and so there is an $\xi(x,t)\in\mathbb{R}^d$ and a subsequence of (u_{k_m}) such that $\nabla u_{k_m}(x,t)\to\xi(x,t)$ in $\mathbb{R}^d$ as $m\to+\infty$. On one side, $H_{k_m}(x,t)$ converges to 0 as $m\to+\infty$, but on the other side,

$$\lim_{m\to\infty} H_{k_m}(x,t) = \big(|\xi(x,t)|^{p-2}\,\xi(x,t) - |\nabla u(x,t)|^{p-2}\,\nabla u(x,t)\big)\big(\xi(x,t) - \nabla u(x,t)\big).$$

Hence

$$\big(|\xi(x,t)|^{p-2}\,\xi(x,t) - |\nabla u(x,t)|^{p-2}\,\nabla u(x,t)\big)\big(\xi(x,t) - \nabla u(x,t)\big) = 0$$

and so the strict convexity of $x \mapsto |x|^p$ on $\mathbb{R}^d$ implies that $\xi(x,t) = \nabla u(x,t)$ in $\mathbb{R}^d$. Since we can identify the limit of the sequence $(\nabla u_{k_m}(x,t))$ as $m \to \infty$ with $\nabla u(x,t)$ for every $(x,t) \in R_k \times (0,T) \setminus \mathcal{Z}$ and since $H_{k_m}(x,t)$ converges to 0 as $m \to \infty$ for every $(x,t) \in R_k \times (0,T) \setminus \mathcal{Z}$, we have thereby shown that $\nabla u_{k_m}(x,t)$ converges $\nabla u(x,t)$ for all $(x,t) \in R_k \times (0,T) \setminus \mathcal{Z}$. Integrating inequality (2.54) over a measurable subset $E \subseteq R_k \times (0,T)$ yields

$$\tfrac{1}{2} \int_E |\nabla u_m|^p \, \mathrm{d}\mu \mathrm{d}s \leq \int_E H_{k_m} + c \int_E |\nabla u|^p \, \mathrm{d}\mu \mathrm{d}s$$

and so by limit (2.53), the sequence $(|\nabla u_{k_m}|^p)$ is equi-integrable in $L^1(R_k \times (0,T))$. Thus Vitali's convergence theorem implies that

$$\lim_{m\to\infty} \nabla u_m = \nabla u \quad \text{strongly in } L^p(R_k \times (0,T))^d$$

and so by limit $\chi = |\nabla u|^{p-2} \nabla u$ a.e. on $R_k \times (0,T)$. Since $k \geq 1$ was arbitrary and since $\bigcup_{k\geq 1} R_k = \mathbb{R}^d \setminus \{0\}$, we have thereby shown that $\chi = |\nabla u|^{p-2} \nabla u$ a.e. on $\mathbb{R}^d \times (0,T)$. Using again a diagonal sequence arguments shows that there is a subsequence of (u_{k_m}) such that

$$\lim_{m\to\infty} u_{k_m} = u \quad \text{strongly in } L^p(0,T; W^{1,p}_{\mathrm{loc}}(\mathbb{R}^d \setminus \{0\}, \mu)). \tag{2.55}$$

Now, let $\mathcal{K} \Subset \mathbb{R}^d \setminus \{0\}$ and $0 \leq t_1 < t_2 < T$. Multiplying Eq. (2.5) with $\varphi \in W^{1,2}(t_1,t_2; L^2(\mathcal{K},\mu)) \cap L^p(t_1,t_2; W^{1,p}_0(\mathcal{K},\mu))$ and subsequently integrating over (t_1,t_2) yields

$$\int_{\mathbb{R}^d} u_{k_m} \varphi \, \mathrm{d}\mu \Big|_{t_1}^{t_2} + \int_{t_1}^{t_2} \int_{\mathbb{R}^d} \Big\{ u_{k_m} \tfrac{d\varphi}{dt} + |\nabla u_{k_m}|^{p-2} \nabla u_{k_m} \nabla \varphi \Big\} \mathrm{d}\mu \, \mathrm{d}s = \int_{t_1}^{t_2} \int_{\mathbb{R}^d} \frac{|u_{k_m}|^{p-2} u_{k_m}}{|x|^p} \varphi \, \mathrm{d}\mu \, \mathrm{d}s.$$

Sending $m \to \infty$ in this equation and using the limits (2.37), (2.55) and (2.46) shows that u is a weak solution of Eq. (1.1) and by *Step* 2, $u \in C([0,T]; L^2(\mathbb{R}^d,\mu))$, u is positive and $u(0) = u_0$. This completes the proof of claim (1) of Theorem 1.3.

2.2.2. Proof of claim (2) of Theorem 1.3

Here, we show that claim (2) of Theorem 1.3 holds.

Proof of Theorem 1.3 (Continued). Suppose that $1 < p < \frac{2d}{d+2}$ and $\lambda \in \mathbb{R}$. Then, we note first that Hölder's inequality yields

$$\begin{aligned}
\int_{\mathbb{R}^d} \left(\tfrac{|\lambda|}{|x|^p} \right)^{\frac{2}{2-p}} \mathrm{d}\mu &\leq |\lambda|^{\frac{2}{2-p}} \sigma(S_{d-1}) \int_0^{+\infty} (r^p)^{-\frac{2}{2-p} + \frac{d-p}{p}} e^{-\frac{\lambda_A^{p/2}}{p} r^p} r^{p-1} \, \mathrm{d}r \\
&\leq |\lambda|^{\frac{2}{2-p}} \sigma(S_{d-1}) \left(\frac{p}{\lambda_A^{p/2}} \right)^{-\frac{2}{2-p} + \frac{d-p}{p}} \lambda_A^{-p/2} \int_0^{+\infty} t^{1 - \frac{2}{2-p} + \frac{d-p}{p} - 1} e^{-t} \, dt =: M_0,
\end{aligned} \tag{2.56}$$

where $\lambda_A > 0$ is the lowest eigenvalue of the positive definite matrix A. Since

$$1 - \frac{2}{2-p} + \frac{d-p}{p} > 0 \quad \text{if and only if } p < \tfrac{2d}{d+2},$$

the integral

$$\int_0^{+\infty} t^{1 - \frac{2}{2-p} + \frac{d-p}{p} - 1} e^{-t} \, \mathrm{d}t$$

is finite and hence M_0 is finite. Now, we proceed again in several steps.

Step 1: Let $u_0 \in L^2(\mathbb{R}^d,\mu)$ be positive. Then for every $m \geq 1$, the approximate solution u_m of (2.5) satisfies the following *a priori*-estimates:

$$\|u_m(t)\|_{L^2(\mathbb{R}^d,\mu)} \leq \left[\|u_{0,m}\|^{2-p}_{L^2(\mathbb{R}^d,\mu)} \exp\left(\left(1-\tfrac{p}{2}\right)\, t\right) + (2-p)\, M_0 \int_0^t \exp\left(\left(1-\tfrac{p}{2}\right)(t-s)\right) \mathrm{d}s\right]^{\frac{1}{2-p}}, \quad (2.57)$$

$$\int_0^t \int_{\mathbb{R}^d} |\nabla u_m(s)|^p \,\mathrm{d}\mu\,\mathrm{d}s \leq \|u_{0,m}\|^2_{L^2(\mathbb{R}^d,\mu)} + \int_0^t M_0\, \|u_m(s)\|^p_{L^2(\mathbb{R}^d,\mu)}\,\mathrm{d}s \quad (2.58)$$

and if $u_0 \in W^{1,p}(\mathbb{R}^d,\mu) \cap L^2(\mathbb{R}^d,\mu)$,

$$\int_0^t s\,\left\|\frac{du_m}{ds}(s)\right\|^2_{L^2(\mathbb{R}^d,\mu)} \mathrm{d}s + \tfrac{t}{p}\int_{\mathbb{R}^d} |\nabla u_m(t)|^p\,\mathrm{d}\mu + \tfrac{1}{p}\int_0^t\int_{\mathbb{R}^d} V_m\,|u_m|^p\,\mathrm{d}\mu\mathrm{d}s$$
$$\leq \tfrac{1}{p}\int_0^t\int_{\mathbb{R}^d} |\nabla u_m(s)|^p\,\mathrm{d}\mu\,\mathrm{d}s + \tfrac{M_0\,t}{p}\,\|u_m(s)\|^p_{L^2(\mathbb{R}^d,\mu)}\,\mathrm{d}s. \quad (2.59)$$

To see this, multiply Eq. (2.5) by u_m with respect to the $L^2(\mathbb{R}^d,\mu)$ inner product and subsequently integrate over $(0,t)$, for $t \in (0,T]$. Then, by (1.2) and estimate (2.56),

$$\|u_m(t)\|^2_{L^2(\mathbb{R}^d,\mu)} + \int_0^t\int_{\mathbb{R}^d} |\nabla u_m(s)|^p\,\mathrm{d}\mu\,\mathrm{d}s \leq \|u_{0,m}\|^2_{L^2(\mathbb{R}^d,\mu)} + \int_0^t M_0\,\|u_m(s)\|^p_{L^2(\mathbb{R}^d,\mu)}\,\mathrm{d}s$$

from where we can deduce the inequalities (2.57) and (2.58) by using a nonlinear generalisation of Gronwall's inequality. If $u_0 \in W^{1,p}(\mathbb{R}^d,\mu)\cap L^2(\mathbb{R}^d,\mu)$, then $u_{0,m} \in W^{1,p}(\mathbb{R}^d,\mu)\cap L^2(\mathbb{R}^d,\mu)$. Hence multiplying Eq. (2.5) by $s\,\frac{du_m}{ds}(s)$ with respect to the $L^2(\mathbb{R}^d,\mu)$ inner product, subsequently integrating over $(0,t)$, for $t \in (0,T]$ and then applying [6, Lemme 3.3], the fact that the potential V satisfies (1.2) and estimate (2.56) yields inequality (2.59). Now, proceeding as in *Step* 4 of the proof of Proposition 2.2, we see that the claim (2) of Theorem 1.3 holds. Since this part of the proof of Theorem 1.3 coincides with *Step* 4 of the proof of Proposition 2.2, we omit the details of the proof. □

3. Nonexistence of positive solutions

This section is dedicated to the proof of Theorem 1.4. We make use of several lemmata. Thus we divide this section into two subsections.

3.1. Preliminaries for the proof of Theorem 1.4

We begin this subsection with the following lemma, which generalises [19, Proposition A.1].

Lemma 3.1. *Let $D \subseteq \mathbb{R}^d$ be a bounded domain and let $1 \leq p < d$. If the function $M \in L^{d/p}(D,\mu)$, then for every $\varepsilon \in (0,1)$, there is a constant $C(\varepsilon) > 0$ such that*

$$\int_D M\,|\phi|^p\,\mathrm{d}\mu \leq \tfrac{\varepsilon}{1-\varepsilon}\int_D |\nabla\phi|^p\,\mathrm{d}\mu + C(\varepsilon)\int_D |\phi|^p\,\mathrm{d}\mu \quad \textit{for all } \phi \in W^{1,p}_0(D,\mu). \quad (3.1)$$

Proof. Let $(M_n)_{n\geq 1}$ be the sequence defined by $M_n(x) := \min\{M(x), n\}$ for a.e. $x \in D$, and every $n \geq 1$. Then, $M_n(x) \to M(x)$ as $n \to +\infty$ for a.e. $x \in D$ and $|M_n| \leq |M(x)|$ for a.e. $x \in D$ and all $n \geq 1$. Thus and since by hypothesis, $M \in L^{d/p}(D,\mu)$, we have by Lebesgue's dominated convergence theorem (see Théorème IV.2 in [7]) that

$$M_n \to M \quad \text{in } L^{d/p}(D,\mu) \text{ as } n \to +\infty. \quad (3.2)$$

We fix $\phi \in C_c^1(D)$. Then by Hölder's inequality, for every $n \geq 1$,

$$\begin{aligned}\int_D M\, |\phi|^p \,\mathrm{d}\mu &\leq \int_D |M - M_n|\, |\phi|^p \,\mathrm{d}\mu + n \int_D |\phi|^p \,\mathrm{d}\mu \\ &\leq \left(\int_D |M - M_n|^{\frac{d}{p}} \,\mathrm{d}\mu\right)^{\frac{p}{d}} \|\rho\|_{L^\infty(D)}^{\frac{d-p}{d}} \left(\int_D |\phi|^{\frac{d\,p}{d-p}} \,\mathrm{d}x\right)^{\frac{d-p}{d}} + n \int_D |\phi|^p \,\mathrm{d}\mu. \end{aligned} \tag{3.3}$$

Since ϕ is assumed to have a compact support, the function ϕ belongs in particular to $C_c^1(\mathbb{R}^d)$. Thus by the Sobolev–Gagliardo–Nirenberg inequality (see Théorème IX.9 in [7]), there is a constant $C = C(p, d) > 0$ such that

$$\left(\int_D |\phi|^{\frac{d\,p}{d-p}} \,\mathrm{d}x\right)^{\frac{(d-p)p}{dp}} \leq C \int_D |\nabla\phi|^p \,\mathrm{d}x \leq C \left\|\rho^{-1}\right\|_{L^\infty(D)} \int_D |\nabla\phi|^p \,\mathrm{d}\mu.$$

Inserting this inequality into estimate (3.3), gives

$$\int_D M\, |\phi|^p \,\mathrm{d}\mu \leq \left(\int_D |M - M_n|^{\frac{d}{p}} \,\mathrm{d}\mu\right)^{\frac{p}{d}} \|\rho\|_{L^\infty(D)}^{\frac{d-p}{d}}\, C \left\|\rho^{-1}\right\|_{L^\infty(D)} \int_D |\nabla\phi|^p \,\mathrm{d}\mu + n \int_D |\phi|^p \,\mathrm{d}\mu.$$

Due to the limit (3.2), for every given $\varepsilon \in (0, 1)$, there is a $n(\varepsilon) \geq 1$ such that

$$\left(\int_D \left|M - M_{n(\varepsilon)}\right|^{\frac{d}{p}} \,\mathrm{d}\mu\right)^{\frac{p}{d}} \leq \tfrac{\varepsilon}{(1-\varepsilon)\,C} \|\rho\|_{L^\infty(D)}^{-\frac{d-p}{d}} \left\|\rho^{-1}\right\|_{L^\infty(D)}^{-1},$$

and hence

$$\int_D M\, |\phi|^p \,\mathrm{d}\mu \leq \tfrac{\varepsilon}{1-\varepsilon} \int_D |\nabla\phi|^p \,\mathrm{d}\mu + n(\varepsilon) \int_D |\phi|^p \,\mathrm{d}\mu.$$

Since $C_c^1(D)$ lies dense in $W_0^{1,p}(D, \mu)$, the claim of this lemma holds with $C(\varepsilon) = n(\varepsilon)$. □

The next lemma generalises [32, Remark 2.1.4, p.158].

Lemma 3.2. *Let $p \neq 2$, and let $V \in L^\infty_{\mathrm{loc}}(\mathbb{R}^d \setminus \{0\})$ be positive. If u is a weak solution of Eq. (1.1) and if $g : \mathbb{R} \to \mathbb{R}$ is Lipschitz-continuous, then for every $\phi \in C_c^1(\mathbb{R}^d \setminus \{0\})$ and every $0 \leq t_1 < t_2 < T$,*

$$\begin{aligned}&\int_{\mathbb{R}^d} \int_0^{u(t_2)} g(s)\,ds\,\phi \,\mathrm{d}\mu - \int_{\mathbb{R}^d} \int_0^{u(t_1)} g(s)\,ds\,\phi \,\mathrm{d}\mu + \int_{t_1}^{t_2} \int_{\mathbb{R}^d} |\nabla u|^p\, g'(u)\,\phi \,\mathrm{d}\mu \,\mathrm{d}t \\ &+ \int_{t_1}^{t_2} \int_{\mathbb{R}^d} |\nabla u|^{p-2}\, \nabla u\, \nabla\phi\, g(u) \,\mathrm{d}\mu \,\mathrm{d}t = \int_{t_1}^{t_2} \int_{\mathbb{R}^d} V\, |u|^{p-2}\, u\, g(u)\,\phi \,\mathrm{d}\mu \,\mathrm{d}t. \end{aligned} \tag{3.4}$$

Proof. We fix $\phi \in C_c^1(\mathbb{R}^d \setminus \{0\})$, and for fixed $0 < t < t + h < T$, we take $t_1 = t$, $t_2 = t + h$, and multiply Eq. (1.8) by h^{-1}. Then,

$$\begin{aligned}&\int_{\mathbb{R}^d} h^{-1} \left(u(t+h) - u(t)\right) \phi \,\mathrm{d}\mu + \int_t^{t+h} \int_{\mathbb{R}^d} h^{-1} \left(|\nabla u|^{p-2}\, \nabla u\right) \nabla\phi \,\mathrm{d}\mu \,\mathrm{d}t \\ &\quad = \int_t^{t+h} \int_{\mathbb{R}^d} h^{-1} \left(V\, |u|^{p-2}\, u\right) \phi \,\mathrm{d}\mu \,\mathrm{d}t. \end{aligned}$$

Let u_h be the Steklov average of u (cf. Definition A.1 in Appendix). Then by Fubini's theorem, since $\frac{\partial u_h}{\partial t}(t) = h^{-1}\left(u(t+h) - u(t)\right)$, and by definition of the Steklov average, the last equation can be rewritten as

$$\int_{\mathbb{R}^d} \tfrac{\partial u_h}{\partial t}(t)\,\phi \,\mathrm{d}\mu + \int_{\mathbb{R}^d} \left(|\nabla u|^{p-2}\, \nabla u\right)_h (t)\, \nabla\phi \,\mathrm{d}\mu = \int_{\mathbb{R}^d} \left(V\, |u|^{p-2}\, u\right)_h (t)\,\phi \,\mathrm{d}\mu.$$

By the hypothesis and by Theorem 2.1.11 in [33], for any $t \in (0,T)$, $g(u_h(t))\,\phi \in W^{1,p}_0(\mathcal{K},\mu)$ with distributional partial derivatives

$$\tfrac{\partial}{\partial x_i}\left(g(u_h(t))\,\phi\right) = g'(u_h(t))\left(\tfrac{\partial u}{\partial x_i}\right)_h(t)\,\phi + g(u_h(t))\,\tfrac{\partial \phi}{\partial x_i},$$

where $\mathcal{K} \Subset \mathbb{R}^d\backslash\{0\}$ is chosen such that the support of ϕ is contained in $\mathcal{K}$. Thus, we can replace ϕ by $g(u_h(t))\,\phi$ in the last equation. Now, we integrate the resulting equation over $]t_1,t_2[$ for fixed $0 \le t_1 < t_2 < T$ and apply first Fubini's theorem and then the fundamental theorem of calculus with respect to $\mathrm{d}t$. We obtain that

$$\begin{aligned}
&\int_{\mathbb{R}^d}\left(\int_0^{u_h(t_2)} g(s)\,ds\right)\phi\,\mathrm{d}\mu - \int_{\mathbb{R}^d}\left(\int_0^{u_h(t_1)} g(s)\,ds\right)\phi\,\mathrm{d}\mu\\
&\quad + \int_{t_1}^{t_2}\int_{\mathbb{R}^d}\left\{\left(|\nabla u|^{p-2}\,\nabla u\right)_h(t)\,g'(u_h(t))(\nabla u)_h(t)\,\phi + \left(|\nabla u|^{p-2}\,\nabla u\right)_h(t)\,g(u_h(t))\,\nabla\phi\right\}\mathrm{d}\mu\,\mathrm{d}t\\
&= \int_{t_1}^{t_2}\int_{\mathbb{R}^d}\left(V\,|u|^{p-2}\,u\right)_h(t)\,g(u_h(t))\,\phi\,\mathrm{d}\mu\,\mathrm{d}t.
\end{aligned}$$

Sending $h \to 0+$ in the last equation and using Lemma A.2 leads to Eq. (3.4). □

Lemma 3.3. *Let $D \subseteq \mathbb{R}^d$ be an open and bounded set with $\partial D \in C^2$, and let $d(x)$ denote the distance of a point $x \in D$ to the boundary ∂D. If $u \in L^1(D,\mu)$ and if there is a constant $c > 0$ such that $u(x) \ge c\,d(x)$ for a.e. $x \in D$, then*

$$\log(u\,d) \in L^p(D,\mu) \quad \textit{for all } p > 1. \tag{3.5}$$

For the proof of Lemma 3.3, we have been partially inspired by the proof of [20, Theorem 1.1].

Proof. We set $D = D_1\dot\cup D_2$ with $D_1 = \{x \in D \mid ud < 1\}$ and $D_2 = \{x \in D \mid ud \ge 1\}$. Since D is bounded, the diameter $\mathrm{diam}(D) := \sup\{|x-y| \mid x,y \in D\}$ of D is bounded. Then,

$$\int_D |\log(u\,d)|^p\,\mathrm{d}\mu = \int_{D_1} |\log(u\,d)|^p\,\mathrm{d}\mu + \int_{D_2} |\log(u\,d)|^p\,\mathrm{d}\mu. \tag{3.6}$$

We have $D_2 = D_{2,1}\dot\cup D_{2,2}$ with $D_{2,1} = D_2 \cap \{ud \ge e^{p-1}\}$ and $D_{2,2} = D_2 \cap \{ud < e^{p-1}\}$. Hence

$$\int_{D_2} |\log(u\,d)|^p\,\mathrm{d}\mu = \mathcal{I}_{2,1} + \mathcal{I}_{2,2}$$

with

$$\mathcal{I}_{2,1} = \int_{D_{2,1}} |\log(u\,d)|^p\,\mathrm{d}\mu \quad\text{and}\quad \mathcal{I}_{2,2} = \int_{D_{2,2}} |\log(u\,d)|^p\,\mathrm{d}\mu.$$

We show that $\mathcal{I}_{2,1}$ and $\mathcal{I}_{2,2}$ are both finite. Indeed,

$$\mathcal{I}_{2,2} = \int_{D_{2,2}} (\log(u\,d))^p\,\mathrm{d}\mu \le \mu(D)\,(p-1)^p \quad\text{is finite.}$$

To see that $\mathcal{I}_{2,1}$ is also finite, we assume that $\mu(D_{2,1}) > 0$, since otherwise there would be nothing to show. The mapping $s \mapsto (\log s)^p$ is concave on the interval $]e^{p-1},+\infty[$. Thus by Jensen's inequality for concave functions and since $d(x) \le \mathrm{diam}(D)$ for every $x \in D$, we obtain that

$$\mathcal{I}_{2,1} = \int_{D_{2,1}} (\log(u\,d))^p\,\mathrm{d}\mu \le \mu(D)\left(\log\tfrac{\mathrm{diam}(D)}{\mu(D_{2,1})}\int_{D_{2,1}} u\,\mathrm{d}\mu\right)^p.$$

Since $u \in L^1(D,\mu)$, the right hand-side of this inequality is finite and so the second integral on the right hand-side in Eq. (3.6) is finite. It remains to verify that also the first integral on the right hand-side in

Eq. (3.6) is finite. For this, we note that $\log(ud) < 0$ on D_1. Thus and since by hypothesis, there is a $c > 0$ such that $u(x) \geq c\, d(x)$ for a.e. $x \in D_1$, we have that

$$\begin{aligned}\int_{D_1} |\log(u\,d)|^p \,\mathrm{d}\mu &= \int_{D_1} (-\log(u\,d))^p \,\mathrm{d}\mu \\ &\leq \int_{D_1} (-\log(c\,d^2))^p \,\mathrm{d}\mu \\ &\leq C_p \left(|\log c|^p \,\mu(D_1) + 2^p \int_D |\log d|^p \,\mathrm{d}\mu \right).\end{aligned}$$

Since we have assumed that D has a C^2 boundary, $\log d \in L^p(D)$ for every $p > 1$. Thus and since $\rho > 0$ is bounded on D, we obtain that $\log d \in L^p(D, \mu)$ and so by the last estimate, the claim of this lemma holds. □

Lemma 3.4. *Let $D \subseteq \mathbb{R}^d$ be a bounded domain with $\partial D \in C^2$. If $v \in C^{2,1}(\overline{D} \times (0,T))$ is a positive nontrivial solution of the boundary value problem*

$$\begin{cases} v_t - K_2 v = 0 & \text{in } D \times (0,T), \\ v = 0 & \text{on } \partial D \times (0,T), \end{cases}$$

then for every $t \in (0,T)$, there is a constant $C(t) > 0$ such that

$$v(x,t) \geq C(t)\, d(x) \quad \text{for all } x \in D. \tag{3.7}$$

Proof. By the weak maximum principle, v attains its minimum at the boundary $\partial D \times (0,T)$. Since D has a C^2 boundary ∂D, then the boundary ∂D satisfies a uniform interior sphere condition (cf. [11, Proposition B.2]). Thus at every point $z \in \partial D$ there is an open ball $B(y,r)$ centered at some $y \in D$ with some radius $r > 0$ such that $B(y,r) \subseteq D$ and $\overline{B}(y,r) \cap \partial D = \{z\}$ and by [28, Theorem 6], the outer normal derivative $\frac{\partial v}{\partial \nu}(\cdot,t) < 0$ on ∂D for every $t \in (0,T)$. Since for every $t \in (0,T)$, $\frac{\partial v}{\partial \nu}(\cdot,t)$ is continuous on the compact set ∂D, $\frac{\partial v}{\partial \nu}(\cdot,t)$ attains a maximum on ∂D and so $\nu_0(t) := \max\bigl\{\frac{\partial v}{\partial \nu}(x,t) \mid x \in \partial D\bigr\} < 0$. For the rest of this proof, we fix $t \in (0,T)$.

By [11, Proposition B.3], there is a $\delta \in (0,1)$ such that for every $z \in \partial D$, the open ball $B_{z,\delta} := B(z - \nu(z)\frac{\delta}{2}, \frac{\delta}{2}) \subseteq D_\delta := \{x \in D \mid d(x) < \delta\}$ and $\partial B_{z,\delta} \cap \partial D = \{z\}$. Here $\nu(z)$ denotes the unit outward normal vector to ∂D at z. Furthermore, for every $x \in D_\delta := \{x \in D \mid d(x) < \delta\}$, there exists a unique $z = z(x) \in \partial D$ such that $|x - z(x)| = d(x)$ holds. Thus every $x \in D_\delta$ can be written as $x = z(x) - \nu(z(x))d(x)$ for a unique $d(x) \in (0,\delta)$. By hypothesis,

$$\sum_{i,j=1}^{d} \left\| \tfrac{\partial^2 v(\cdot,t)}{\partial x_i \partial x_j} \right\|_{C(\overline{D})} + 1 =: C(t) \quad \text{is finite.}$$

We calculate the Taylor expansion of $v(\cdot,t)$ at $z(x) \in \partial D$ for every $x \in D_\delta$. Then for every $x \in D_\delta$, there is a $\theta(x) \in (0,1)$ such that

$$\begin{aligned} v(x,t) &= v(z(x) - \nu(z(x))d(x), t) \\ &= -\langle \nabla v(z(x),t), \nu(z(x)) \rangle_{\mathbb{R}^d}\, d(x) + \tfrac{1}{2} \langle \nu(z(x))^t H_v(\theta(x)\nu(z(x))), \nu(z(x)) \rangle_{\mathbb{R}^d}\, d(x)^2 \\ &= -\tfrac{\partial v}{\partial \nu}(z(x),t)\, d(x) + \tfrac{1}{2} \langle \nu(z(x))^t H_v(\theta(x)\nu(z(x))), \nu(z(x)) \rangle_{\mathbb{R}^d}\, d(x)^2 \\ &\geq (-\nu_0(t))\, d(x) - C(t)\, d^2(x), \end{aligned}$$

where H_v denotes the Hessian of v. We take $\delta_0 = \min\bigl\{\delta, \frac{(-\nu_0(t))}{2\,C(t)}\bigr\}$. Then, for all $x \in D_{\delta_0}$, $(-\nu_0(t)) - C(t)\, d(x) > \frac{(-\nu_0(t))}{2}$. Therefore

$$v(x,t) \geq \tfrac{(-\nu_0(t))}{2}\, d(x) \quad \text{for all } x \in D_{\delta_0}.$$

On the other hand, by the strong maximum principle [28, Theorem 5], $v(t) > 0$ on the compact set $D \setminus D_{\delta_0}$. Thus and since $D \setminus D_{\delta_0}$ has a positive and uniform distance to the boundary of D, there is a constant $C_2(t) > 0$ such that

$$v(x,t) \ge C_2(t)\, d(x) \quad \text{for all } x \in D \setminus D_{\delta_0}.$$

Therefore, if we set $C_3(t) = \min\{C_2(t), \frac{(-\nu_0(t))}{2}\}$, then $C_3(t) > 0$ and for this constant $v(t)$ satisfies inequality (3.7) on D. □

The following weak comparison principle for positive weak solutions is a refined version of [18, Proposition 4.1].

Lemma 3.5. *Consider the case $p = 2$. Let $u_0 \in L^2(\mathbb{R}^d, \mu)$ and $V \in L^\infty_{\text{loc}}(\mathbb{R}^d \setminus \{0\})$ be both positive, D be an open and bounded subset of $\mathbb{R}^d$ with a Lipschitz-continuous boundary, and $g \in L^\infty(D)$ be such that*

$$0 \le g(x) \le V(x) \quad \text{for a.e. } x \in D. \tag{3.8}$$

If for given $m \ge 1$, $v_m \in W^{1,2}(\delta, T; L^2(D, \mu)) \cap L^2(0, T; W^{1,2}_0(D, \mu))$ ($\delta \in (0, T)$), is the unique positive strong solution of

$$\begin{cases} \frac{\partial v_m}{\partial t} - K_2 v = g(x) v & \text{in } D \times (0,T), \\ v = 0 & \text{on } \partial D \times (0,T), \\ v(0) = u_{0,m} & \text{in } D, \end{cases} \tag{3.9}$$

and if u is a positive weak solution of Eq. (1.1) for $p = 2$ with initial value $u(0) = u_0$, then $0 \le v_m \le u$ almost everywhere on $D \times (0,T)$.

Proof. Note first that by [6, Proposition 3.12], for every positive $u_0 \in L^2(\mathbb{R}^d, \mu)$ and $m \ge 1$, problem (3.9) admits a unique strong solution v_m. In addition, by the weak maximum principle (see Lemma 2.6), the solution v_m is positive.

Now, for given $\varepsilon \in (0, T)$, let $\varphi \in W^{1,2}(0, T; L^2(D, \mu)) \cap L^2(0, T; W^{1,2}_0(D, \mu))$ be positive such that $\varphi(\cdot, T - \varepsilon) \equiv 0$. If we extend φ by zero on $(\mathbb{R}^d \setminus D) \times (0, T)$, the extension has compact support in $\mathbb{R}^d$ for almost every $t \in (0,T)$ and belongs to $W^{1,2}(0, T; L^2(\mathbb{R}^d, \mu)) \cap L^2(0, T; W^{1,2}_0(\mathbb{R}^d, \mu))$. Thus every weak solution u of Eq. (1.1) on $\mathbb{R}^d \times (0, T)$ is, in particular, a weak solution of Eq. (1.1) on $D \times (0, T)$. Thus

$$\int_0^{T-\varepsilon} \int_D u \left\{ -\varphi_s - K_2 \varphi \right\} \mathrm{d}\mu\, ds = (u(0), \varphi)_{L^2(D,\mu)} + \int_0^{T-\varepsilon} \int_D V(x)\, u\, \varphi \,\mathrm{d}\mu\, ds. \tag{3.10}$$

Next, the strong solution v_m satisfies

$$\left(\tfrac{\partial v_m}{\partial t}(t), \phi\right)_{L^2(D,\mu)} = (K_2 v_m(t), \phi)_{L^2(D,\mu)} + (g(x) v_m(t), \phi)_{L^2(D,\mu)}$$

for almost every $t \in (0,T)$, and all $\phi \in L^2(D, \mu)$. If we take $\phi = \varphi$ in this equation, integrate over $(0, T - \varepsilon)$, and apply integration by parts once with respect to ds and twice with respect to dμ, then

$$\int_0^{T-\varepsilon} \int_D v_m \left\{ -\varphi_s - K_2 \varphi \right\} \mathrm{d}\mu\, ds = (v_m(0), \varphi)_{L^2(D,\mu)} + \int_0^{T-\varepsilon} \int_D g(x)\, v_m\, \varphi \,\mathrm{d}\mu\, ds. \tag{3.11}$$

Using both Eqs. (3.10) and (3.11), we obtain for $(v_m - u)$ that

$$\begin{aligned} \int_0^{T-\varepsilon} \int_D (v_m - u) \left\{ -\varphi_s - K_2\varphi - g(x)\varphi \right\} \mathrm{d}\mu\, ds &= (v_m(0) - u(0), \varphi)_{L^2(D,\mu)} \\ &\quad + \int_0^{T-\varepsilon} \int_D [g(x) - V(x)]\, u\, \varphi \,\mathrm{d}\mu\, ds. \end{aligned}$$

By hypothesis, $v_m(0) = u_{0,m} \le u_0 = u(0)$, since we have assumed that $g(x)$ satisfies condition (3.8), and since u is positive, we can deduce from the last equation that

$$\int_0^{T-\varepsilon}\int_D (v_m - u)\Big\{-\varphi_s - K_2\varphi - g(x)\varphi\Big\}\,\mathrm{d}\mu\,ds \le 0 \tag{3.12}$$

for all positive $\varphi \in W^{1,2}(0,T-\varepsilon;L^2(D,\mu)) \cap L^2(0,T-\varepsilon;W_0^{1,2}(D,\mu))$.

Now, let $\psi \in C([0,T-\varepsilon];C_c^\infty(D))$ be positive and consider the parabolic problem

$$\begin{cases} z_t - K_2 z = g(x)z + \psi & \text{in } D\times(0,T-\varepsilon),\\ z = 0 & \text{on } \partial D\times(0,T-\varepsilon),\\ z(0) = 0 & \text{in } D.\end{cases} \tag{3.13}$$

By [6, Proposition 3.12], problem (3.13) has a unique strong solution

$$z \in W^{1,2}(0,T-\varepsilon;L^2(D,\mu)) \cap C([0,T-\varepsilon];W_0^{1,2}(D,\mu))$$

and this solution is positive by the weak maximum principle. If we set $\varphi_0(x,s) = z(x,t-s)$ for a.e. $x \in D$ and all $s \in [0,T-\varepsilon]$, then φ_0 is a strong solution of

$$\begin{cases} -\varphi_{0\,t} - K_2\varphi_0 - g(x)\varphi_0 = \psi & \text{in } D\times(0,T-\varepsilon),\\ \varphi_0 = 0 & \text{on } \partial D\times(0,T-\varepsilon),\\ \varphi_0(T-\varepsilon) = 0 & \text{in } D.\end{cases}$$

Inserting φ_0 into inequality (3.12) shows that for all positive $\psi \in C([0,T-\varepsilon];C_c^\infty(D))$,

$$\int_0^{T-\varepsilon}\int_D (v-u)\,\psi\,\mathrm{d}\mu\,\mathrm{d}t \le 0.$$

The set $C([0,T-\varepsilon];C_c^\infty(D))$ lies dense in $L^2(0,T-\varepsilon;L^2(D,\mu))$. Thus by an approximation argument, we can take $\psi = [v-u]^+$ in the last inequality and hence we obtain that $v \le u$ a.e. on $D\times(0,T-\varepsilon)$. Since $\varepsilon > 0$ has been arbitrary, the claim of this lemma holds. □

The statement of the next lemma, is well-known in the case $\rho \equiv 1$ (cf. [7, Remarque 18. in IX.4, p. 171] or [4, Remark 2.6 on p. 1019]).

Lemma 3.6. *Let $d \ge 2$, D be an open subset of $\mathbb{R}^d$, and let $1 \le p < d$. Then,*

$$W_0^{1,p}(D,\mu) = W_0^{1,p}(D\setminus\{a\},\mu) \quad \textit{for every } a \in D.$$

Proof. In this proof, we follow the idea of [4, Lemma 2.4 and Remark 2.6]. Let $(\rho_n)_{n\ge1}$ denote a standard mollifier (see [7, p. 70]): that is, for every $n \ge 1$, $\rho_n \in C_c^\infty(\mathbb{R}^d)$, the support $\operatorname{supp}(\rho_n) \subseteq \overline{B}(0,\frac{1}{n})$, $\rho_n \ge 0$, and $\int_{\mathbb{R}^d}\rho_n\,\mathrm{d}x = 1$. We fix $a \in D$ and set

$$\psi(x) = \int_{\mathbb{R}^d}\rho_5(y)\,\mathbb{1}_{B(a,5/4)}(x-y)\,\mathrm{d}y \quad \text{for all } x \in \mathbb{R}^d,$$

where $\mathbb{1}_{B(a,5/4)}$ denotes the indicator function over the open ball $B(a,\frac{5}{4})$ of centre $x = a$ and radius $r = \frac{5}{4}$. Then, it is not hard to verify that $\psi \in C_c^\infty(\mathbb{R}^d)$, $\psi \ge 0$, $\psi \equiv 1$ on $B(a,1)$ and $\psi \equiv 0$ on $\mathbb{R}^d\setminus B(a,2)$. For every $n \ge 1$, we set $\psi_n(x) = \psi(nx)$ for every $x \in \mathbb{R}^d$. Then, $\psi_n \in C_c^\infty(\mathbb{R}^d)$, $\psi_n \ge 0$, $\psi \equiv 1$ on $B(a,\frac{1}{n})$, and $\psi \equiv 0$ on $\mathbb{R}^d\setminus B(a,\frac{2}{n})$. Since

$$\|\psi_n\|_{L^p(\mathbb{R}^d)} = n^{-\frac{d}{p}}\,\|\psi\|_{L^p(\mathbb{R}^d)} \quad \text{and} \quad \left\|\frac{\partial\psi_n}{\partial x_i}\right\|_{L^p(\mathbb{R}^d)} = n^{-\frac{d-p}{p}}\left\|\frac{\partial\psi}{\partial x_i}\right\|_{L^p(\mathbb{R}^d)}$$

for every $i = 1,\dots,d$, and since by hypothesis, $p < d$, we have that $\psi_n \to 0$ in $W^{1,p}(\mathbb{R}^d)$ as $n \to +\infty$.

Obviously, it suffices to show that $W_0^{1,p}(D,\mu) \subseteq W_0^{1,p}(D\setminus\{a\},\mu)$ since the other implication is clear. Since $W_0^{1,p}(D,\mu)$ is the closure of the set $C_c^1(D)$ in $W^{1,p}(D,\mu)$, we need to show that for every $\varphi \in C_c^1(D)$ and for every $\varepsilon > 0$, there is $\theta \in C_c^1(D\setminus\{a\})$ such that $\|\varphi-\theta\|_{W^{1,p}(D,\mu)} \le \varepsilon$. To do so, we fix $\varphi \in C_c^1(D)$ and $\varepsilon > 0$. Since D is open and $a \in D$, there is an $r > 0$ such that the open ball $B(a,r)$ of centre $x = a$ and radius r is contained in D. By the first step of this proof, there is an index $n_r \ge 1$ such that $\psi_n \in C_c^\infty(B(a,r))$ for every $n \ge n_r$. Then, for every $n \ge n_r$, $\theta_n := \varphi(1-\psi_n) \in C_c^1(D\setminus\{a\})$ and

$$\|\varphi-\theta_n\|_{W^{1,p}(D,\mu)} = \|\varphi\psi_n\|_{W^{1,p}(D,\mu)} \le C\,\|\psi_n\|_{W^{1,p}(\mathbb{R}^d)}\,,$$

where the constant $C \ge 0$ depends on φ and $\|\rho\|_{L^\infty(\mathrm{supp}(\varphi))}$ but is independent of ψ_n. Since we can choose $n \ge n_r$ large enough such that $\|\psi_n\|_{W^{1,p}(\mathbb{R}^d)} \le \frac{\varepsilon}{C+1}$, the claim of this lemma holds. $\square$

3.2. Proof of Theorem 1.4

With the prerequisites of the preceding subsection in mind, we finally turn to the proof of Theorem 1.4.

Proof of Theorem 1.4. First, we study the case $1 < p < 2$ if $d = 2$, and $\frac{2d}{d+2} \le p < 2$ if $d \ge 3$. In this part of the proof, the authors follow partially the idea given in [19]. Let $\lambda > \left(\frac{d-p}{p}\right)^p$. We argue by contradiction and hence we suppose that there is a positive nontrivial $u_0 \in L^2_{\mu,loc}(\mathbb{R}^d\setminus\{0\},\mu)$ such that there is a $T > 0$ and a positive very weak solution u of Eq. (1.1) with initial value $u(0) = u_0$. We fix some bounded domain $D \subseteq \mathbb{R}^d$ containing $x = 0$. Let $\varphi \in C_c^1(D\setminus\{0\})$, and for every integer $k \ge 1$ and every $s \in \mathbb{R}$, let $g_k(s) := (s+\frac{1}{k})^{1-p}$ if $s \ge 0$ and $g_k(s) = k^{p-1}$ if $s < 0$. Then, by Lemma 3.2 for $g = g_k$, $\phi = |\varphi|^p$, $t_1 = 0$, and $t_2 = t$,

$$\begin{aligned}\int_D \frac{\left(u(t)+\frac{1}{k}\right)^{2-p}}{2-p}\,|\varphi|^p\,\mathrm{d}\mu - \int_D \frac{\left(u(0)+\frac{1}{k}\right)^{2-p}}{2-p}\,|\varphi|^p\,\mathrm{d}\mu + (1-p)\int_0^t\int_D \frac{|\nabla u(s)|^p\,|\varphi|^p}{\left(u(s)+\frac{1}{k}\right)^p}\,\mathrm{d}\mu\,ds\\ +p\int_0^t\int_D \frac{|\nabla u(s)|^{p-2}\nabla u(s)\,|\varphi|^{p-2}\varphi}{\left(u(s)+\frac{1}{k}\right)^{p-1}}\nabla\varphi\,\mathrm{d}\mu\,ds = \int_0^t\int_D \frac{\lambda}{|x|^p}\frac{u^{p-1}(s)}{\left(u(s)+\frac{1}{k}\right)^{p-1}}\,|\varphi|^p\,\mathrm{d}\mu\,ds.\end{aligned} \tag{3.14}$$

By Young's inequality,

$$\begin{aligned}p\int_0^t\int_D |\nabla u(s)|^{p-2}\,\nabla u(s)\,\left(u(s)+\tfrac{1}{k}\right)^{1-p}\,\nabla\varphi\,|\varphi|^{p-2}\,\varphi\,\mathrm{d}\mu\,ds\\ \le (p-1)\int_0^t\int_D |\nabla u(s)|^p\,\left(u(s)+\tfrac{1}{k}\right)^{-p}\,|\varphi|^p\,\mathrm{d}\mu\,ds + t\int_D |\nabla\varphi|^p\,\mathrm{d}\mu,\end{aligned}$$

and since $(u(0)+\frac{1}{k})^{2-p}\,|\varphi|^p \ge 0$ a.e. on D, we can deduce from equality (3.14) that

$$\int_0^t\int_D \tfrac{\lambda}{|x|^p}u^{p-1}(s)\,\left(u(s)+\tfrac{1}{k}\right)^{1-p}\,|\varphi|^p\,\mathrm{d}\mu\,ds \le t\int_D |\nabla\varphi|^p\,\mathrm{d}\mu + \tfrac{1}{2-p}\int_D \left(u(t)+\tfrac{1}{k}\right)^{2-p}\,|\varphi|^p\,\mathrm{d}\mu. \tag{3.15}$$

For almost every $(x,s) \in D\times(0,t)$, we have that

$$0 \le \tfrac{\lambda}{|x|^p}\,\frac{u^{p-1}(x,s)\,|\varphi(x)|^p}{\left(u(x,s)+\frac{1}{k}\right)^{p-1}} \nearrow \tfrac{\lambda}{|x|^p}\,|\varphi(x)|^p \quad\text{and}\quad \left(u(x,s)+\tfrac{1}{k}\right)^{2-p}\,|\varphi(x)|^p \searrow u^{2-p}(x,s)\,|\varphi(x)|^p$$

as $k \to +\infty$. Thus, by Beppo-Levi's theorem, sending $k \to +\infty$ in inequality (3.15) gives

$$t\int_D \tfrac{\lambda}{|x|^p}\,|\varphi|^p\,\mathrm{d}\mu - t\int_D |\nabla\varphi|^p\,\mathrm{d}\mu \le \tfrac{1}{2-p}\int_D u^{2-p}(t)\,|\varphi|^p\,\mathrm{d}\mu.$$

First, consider the case $d = 2$ and $1 < p < 2$. Then, $(2-p)\frac{2}{p} < 2$, and so $\frac{p}{(2-p)} > 1$. Thus by Hölder's inequality,

$$\int_D u(t)^{\frac{(2-p)2}{p}}\,\mathrm{d}\mu \le \mu(D)^{\frac{2p-2}{p}}\,\|u\|_{L^2(D,\mu)}^{\frac{p}{2-p}}\,.$$

Since $u(t) \in L^2(D,\mu)$, the last estimate implies that $u^{2-p} \in L^{2/p}(D,\mu)$.

Next, consider the case $d \geq 3$ and let $\frac{2d}{d+2} \leq p < 2$. Then, $(2-p)\frac{d}{p} \leq 2$ and since $u(t) \in L^2(D,\mu)$, we obtain again by Hölder's inequality that $u^{2-p} \in L^{d/p}(D,\mu)$.

Therefore in both cases, $d = 2$ and $1 < p < 2$ or $d \geq 3$ and $\frac{2d}{d+2} \leq p < 2$, Lemma 3.1 implies that for any $\varepsilon \in (0,1)$, there is constant $C(\varepsilon) > 0$ such that

$$\int_D \tfrac{\lambda}{|x|^p} \, |\varphi|^p \, \mathrm{d}\mu - \int_D |\nabla\varphi|^p \, \mathrm{d}\mu \leq \tfrac{\varepsilon}{1-\varepsilon} \int_D |\nabla\varphi|^p \, \mathrm{d}\mu + C(\varepsilon) \int_D |\varphi|^p \, \mathrm{d}\mu. \tag{3.16}$$

If the matrix A is positive definite, then there are

$$\lambda_A,\ \Lambda_A > 0 \quad \text{such that } \lambda_A \, |x|^2 \leq x^t A x \leq \Lambda_A \, |x|^2 \ \text{ for all } x \in \mathbb{R}^d. \tag{3.17}$$

Furthermore, by Lemma 3.6 and since $1 < p < 2 \leq d$, the set $C_c^1(D \setminus \{0\})$ is dense in $W_0^{1,p}(D,\mu)$. Therefore, by estimate (3.16), and since $\varphi \in C_c^1(D \setminus \{0\})$ has been arbitrary, we obtain that

$$\begin{aligned} &\inf \frac{\int_D |\nabla\varphi|^p \, \mathrm{d}\mu - [(1-\varepsilon)\lambda]^{\frac{p-1}{p}} \int_D |\varphi|^p \, \frac{(x^t A x)^{p/2}}{|x|^p} \, \mathrm{d}\mu - (1-\varepsilon)\int_D \frac{\lambda}{|x|^p} \, |\varphi|^p \, \mathrm{d}\mu}{\int_D |\varphi|^p \, \mathrm{d}\mu} \\ &\geq -\Lambda_A^{p/2} \, [(1-\varepsilon)\lambda]^{\frac{p-1}{p}} - C(\varepsilon)\,(1-\varepsilon) > -\infty, \end{aligned} \tag{3.18}$$

where the infimum is taken over all $\varphi \in W_0^{1,p}(D,\mu)$ with $\|\varphi\|_{L^p(D,\mu)} > 0$.

But for every

$$0 < \varepsilon < 1 - \lambda^{-1}\left(\tfrac{d-p}{p}\right)^p \quad \text{we have that } (1-\varepsilon)\lambda > \left(\tfrac{d-p}{p}\right)^p.$$

Thus and since $0 \in D$, (3.18) obviously contradicts to the optimality of the constant $\left(\frac{d-p}{p}\right)^p$ in Hardy's inequality (1.5). Therefore the assumption is false and hence claim (i) of Theorem 1.4 is true for $1 < p < 2$ if $d = 2$ and $\frac{2d}{d+2} \leq p < 2$ if $d \geq 3$.

We turn to the case $p = 2$ and $d \geq 3$. In this case, we follow partially an idea given in [9]. Again, we argue by contradiction. Let $\lambda > \left(\frac{d-2}{2}\right)^2$ and let u_0 be a positive nontrivial element of $L^2(\mathbb{R}^d,\mu)$. We suppose that there is a $T > 0$ and there is a positive weak solution u of Eq. (1.1) with initial value $u(0) = u_0$. Let $(D_l)_{l \geq 1}$ be a sequence of bounded domains D_l of $\mathbb{R}^d$ satisfying $0 \in D_l \subseteq D_{l+1}$, $\overline{D_l} \subseteq \mathbb{R}^d$, with boundary ∂D_l of class C^∞, and $\bigcup_{l\geq 1} D_l = D$. Then by hypothesis, there is an $D_{l_0} \subseteq \mathbb{R}^d$ such that $u_0 \neq 0$ almost everywhere on D_{l_0}. We set $D = D_{l_0}$ and fix $\varphi \in C_c^1(D)$. For every integer $n \geq 1$, let u_m be the unique positive strong solutions of Eq. (2.5) with initial value $u_m(0) = u_{0,m}$. Further, let v_m be the unique positive strong solutions of (3.9) with $g(x) \equiv 0$. Then by the weak comparison principle (see Lemmas 2.6 and 3.5), for all $m \geq 1$,

$$0 \leq v_1 \leq v_m \leq u_m \leq u \quad \text{almost everywhere on } D \times (0,T). \tag{3.19}$$

Since the boundary of D is smooth, v_1 is infinitely differentiable in $D \times (0,T)$ and v_1, $\frac{\partial v_1}{\partial x_i}$ and $\frac{\partial^2 v_1}{\partial x_i \partial x_j}$ are continuous up to the boundary of D (cf. [22, Theorem 12.1 in Chapter III & Theorem 1.1 in Chapter V]). Since v_1 is positive and since $v_1(0) \neq 0$ a.e. on D, the strong maximum principle (cf. [28, Chapter III, Theorem 5]) implies that for every $t \in (0,T)$, $v_1(t) > 0$ on the compact subset $\mathrm{supp}(\varphi)$ of D. Thus for all $t \in (0,T)$, there is a constant $C_0(t) > 0$ such that for all $n \geq 1$,

$$u(t) \geq u_m(t) \geq v_m(t) \geq v_1(t) \geq C_0(t) > 0 \quad \text{on } \mathrm{supp}(\varphi).$$

Thus, for every $m \geq 1$, we may multiply Eq. (2.5) with truncated potential V_m by $u_m^{-1} \, |\varphi|^2$ with respect to the $L^2(D,\mu)$ inner product, and subsequently integrate over the interval (t_0,t) with respect to $\mathrm{d}s$ for any fixed $0 < t_0 < t < T$. Then, we obtain

$$\int_D \log\Big(\tfrac{u_n(t)}{u_n(t_0)}\Big)\;|\varphi|^2\,\mathrm{d}\mu - \int_{t_0}^t\int_D |\nabla u_n(s)|^2\;(u_n(s))^{-2}\;|\varphi|^2\,\mathrm{d}\mu\,ds$$
$$+\,2\int_{t_0}^t\int_D \nabla u_n(s)\nabla\varphi\,(u_n(s))^{-1}\,\varphi\,\mathrm{d}\mu\,ds = (t-t_0)\int_D \min\{n, \tfrac{\lambda}{|x|^2}\}\;|\varphi|^2\,\mathrm{d}\mu. \tag{3.20}$$

By Young's inequality,

$$2\int_{t_0}^t\int_D \nabla u_n(s)\nabla\varphi\,(u_n(s))^{-1}\,\varphi\,\mathrm{d}\mu\,ds \le (t-t_0)\int_D |\nabla\varphi|^2\,\mathrm{d}\mu + \int_{t_0}^t\int_D |\nabla u_n(s)|^p\;(u_n(s))^{-2}\;|\varphi|^2\,\mathrm{d}\mu\,ds.$$

Thus, we can deduce from equality (3.20) that

$$\int_D \min\{n, \tfrac{\lambda}{|x|^2}\}\;|\varphi|^2\,\mathrm{d}\mu - \int_D |\nabla\varphi|^2\,\mathrm{d}\mu \le \tfrac{1}{t-t_0}\int_D \log\Big(\tfrac{u_n(t)}{u_n(t_0)}\Big)\;|\varphi|^2\,\mathrm{d}\mu. \tag{3.21}$$

Furthermore, by Lemma 3.4, for every $t \in (0,T)$, there is another constant $C_1(t) > 0$ such that $v_1(t) \ge C_1(t)\,d(x)$ for all $x \in D$ and so by (3.19), we have that for all $n \ge 1$,

$$u(x,t) \ge u_n(x,t) \ge C_1(t)\,d(x) \quad \text{for almost every } x \in D.$$

Thus

$$\int_D \log\Big(\tfrac{u_n(t)}{u_n(t_0)}\Big)\;|\varphi|^2\,\mathrm{d}\mu \le \int_D \log\Big(\tfrac{u(t)}{C_1(t_0)\,d}\Big)\;|\varphi|^2\,\mathrm{d}\mu = \int_D \log\Big(\tfrac{u(t)\,d}{C_1(t_0)\,d^2}\Big)\;|\varphi|^2\,\mathrm{d}\mu$$

and hence sending $n \to +\infty$ in inequality (3.21) yields to

$$\int_D \tfrac{\lambda}{|x|^2}\;|\varphi|^2\,\mathrm{d}\mu - \int_D |\nabla\varphi|^2\,\mathrm{d}\mu \le \tfrac{1}{t-t_0}\int_D \log\Big(\tfrac{u(t)\,d}{C_1(t_0)\,d^2}\Big)\;|\varphi|^2\,\mathrm{d}\mu.$$

By Lemma 3.3, $\log(u\,d) \in L^p(D)$ for all $p > 1$ and since D has smooth boundary,

$$\log(C_1(t_0)\,d^2) = \log(C_1(t_0)) + 2\,\log d \in L^p(D,\mu)$$

for all $p > 1$. Thus $\log\big(\tfrac{u(t)\,d}{C_1(t_0)\,d^2}\big) \in L^{d/2}(D,\mu)$. By Lemma 3.1, for every $\varepsilon \in (0,1)$ there is a $C(\varepsilon) > 0$ such that

$$\int_D \tfrac{\lambda}{|x|^2}\;|\varphi|^2\,\mathrm{d}\mu - \int_D |\nabla\varphi|^2\,\mathrm{d}\mu \le \tfrac{\varepsilon}{1-\varepsilon}\int_D |\nabla\varphi|^2\,\mathrm{d}\mu + C(\varepsilon)\int_D |\varphi|^2\,\mathrm{d}\mu.$$

Now, we proceed as in the proof of claim (i) and reach a contradiction to the optimality of $\big(\tfrac{d-2}{2}\big)^2$ in Hardy's inequality (1.5).

Let $p > 2$, $d \ne p$, and let $\lambda > \big(\tfrac{|d-p|}{p}\big)^p$ and let $u_0 \in L^2_{\mu,loc}(\mathbb{R}^d)$ be positive and satisfying (1.9) for some $r > 0$. We argue by contradiction and so we assume, there is a $T > 0$, for which Eq. (1.1) has a positive weak solution u on $[0,T)$ with initial value $u(0) = u_0$. For the above given $r > 0$, we fix $\varphi \in C^1_c(B(0,r)\setminus\{0\})$ and for every integer $k \ge 1$ and every $s \in \mathbb{R}$, we set $g_k(s) = (s + \frac{1}{k})^{1-p}$ if $s \ge 0$ an $g_k(s) = k^{p-1}$ if $s < 0$. Then, by Lemma 3.2 for $t_1 = 0$, $t_2 = t > 0$, and $\phi = |\varphi|^p$, we obtain that

$$\int_{B(0,r)} \frac{\big(u(0)+\frac{1}{k}\big)^{2-p}}{p-2}\,|\varphi|^p\,\mathrm{d}\mu - \int_{B(0,r)} \frac{\big(u(t)+\frac{1}{k}\big)^{2-p}}{p-2}\,|\varphi|^p\,\mathrm{d}\mu$$
$$+\,(1-p)\int_0^t\int_{B(0,r)} \frac{|\nabla u(s)|^p|\varphi|^p}{\big(u(s)+\frac{1}{k}\big)^p}\,\mathrm{d}\mu\,\mathrm{d}s + p\int_0^t\int_{B(0,r)} \frac{|\nabla u(s)|^{p-2}\nabla u(s)\,|\varphi|^{p-2}\varphi}{\big(u(s)+\frac{1}{k}\big)^{p-1}}\nabla\varphi\,\mathrm{d}\mu\,\mathrm{d}s$$
$$= \int_0^t\int_{B(0,r)} \tfrac{\lambda}{|x|^p}\,\frac{u^{p-1}(s)}{\big(u(s)+\frac{1}{k}\big)^{p-1}}\;|\varphi|^p\,\mathrm{d}\mu\,\mathrm{d}s. \tag{3.22}$$

By Young's inequality

$$p \int_0^t \int_{B(0,r)} |\nabla u(s)|^{p-2} \nabla u(s) \, \big(u(s) + \tfrac{1}{k}\big)^{1-p} \, \nabla\varphi \, |\varphi|^{p-2} \varphi \, \mathrm{d}\mu \, ds$$
$$\leq (p-1) \int_0^t \int_{B(0,r)} |\nabla u(s)|^p \, \big(u(s) + \tfrac{1}{k}\big)^{-p} \, |\varphi|^p \, \mathrm{d}\mu \, ds + t \int_{B(0,r)} |\nabla\varphi|^p \, \mathrm{d}\mu,$$

and since $(u(t) + \frac{1}{k})^{2-p} \, |\varphi|^p$ is positive, we can deduce from (3.22) that

$$\int_0^t \int_{B(0,r)} \tfrac{\lambda}{|x|^p} u^{p-1}(s) \, \big(u(s) + \tfrac{1}{k}\big)^{1-p} \, |\varphi|^p \, \mathrm{d}\mu \, ds \leq t \int_{B(0,r)} |\nabla\varphi|^p \, \mathrm{d}\mu + \int_{B(0,r)} \tfrac{(u(0)+\frac{1}{k})^{2-p}}{p-2} \, |\varphi|^p \, \mathrm{d}\mu.$$

We send $k \to +\infty$ in this inequality. Then, by Beppo-Levi's convergence theorem, we obtain that

$$\int_{B(0,r)} \tfrac{\lambda}{|x|^p} \, |\varphi|^p \, \mathrm{d}\mu - \int_{B(0,r)} |\nabla\varphi|^p \, \mathrm{d}\mu \leq \tfrac{1}{t\,(p-2)} \int_{B(0,r)} u^{2-p}(0) \, |\varphi|^p \, \mathrm{d}\mu. \tag{3.23}$$

Thus, since $\mathrm{supp}(\varphi) \subseteq B(0,r) \setminus \{0\}$, and by hypothesis (1.9), we have that

$$\int_{B(0,r)} \tfrac{\lambda}{|x|^p} \, |\varphi|^p \, \mathrm{d}\mu - \int_{B(0,r)} |\nabla\varphi|^p \, \mathrm{d}\mu \leq \tfrac{\delta^{2-p}}{t\,(p-2)} \int_{B(0,r)} |\varphi|^p \, \mathrm{d}\mu. \tag{3.24}$$

Since the matrix A is positive definite, inequality (3.17) holds. Thus and since inequality (3.24) holds for all $\varphi \in C^1_c(B(0,r) \setminus \{0\})$, we can conclude that

$$\inf \frac{\int_{B(0,r)} |\nabla\varphi|^p \, \mathrm{d}\mu - \lambda^{\frac{p-1}{p}} \int_{B(0,r)} |\varphi|^p \, \frac{(x^t A x)^{p/2}}{|x|^p} \, \mathrm{d}\mu - \int_{(B(0,r))} \frac{\lambda}{|x|^p} \, |\varphi|^p \, \mathrm{d}\mu}{\int_{B(0,r)} |\varphi|^p \, \mathrm{d}\mu} \geq -\tfrac{\delta^{2-p}}{t\,(2-p)} - \Lambda_A^{p/2} \, \lambda^{\frac{p-1}{p}},$$

where the infimum is taken over all $\varphi \in C^1_c(B(0,r) \setminus \{0\})$ with $\|\varphi\|_{L^p_\mu(B(0,r))} > 0$.

If $2 < p < d$, then by Lemma 3.6, the set $C^1_c(B(0,r) \setminus \{0\})$ lies dense in $W^{1,p}_{\mu,0}(B(0,r))$ and if $p > d \geq 2$, then the set $C^1_c(B(0,r) \setminus \{0\})$ lies dense in $W^{1,p}_{\mu,0}(B(0,r) \setminus \{0\})$. Therefore in both cases, the last inequality contradicts to the optimality of the constant $\big(\frac{|d-p|}{p}\big)^p$ in Hardy's inequality (1.5). $\square$

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Appendix

A.1. Steklov averages and an integration by parts

In this subsection of the appendix, we recall some well-known facts about Steklov averages for the purpose of proving an integration by parts formula (Lemma 3.2).

For any open subset D of $\mathbb{R}^d$ and any $T > 0$, we denote by D_T the cylinder $D \times (0,T)$. For $q, r \geq 1$, we denote by $L^{q,r}(D_T)$ the *parabolic Lebesgue space* $L^r(0,T;L^q(D))$. The space $L^{q,r}(D_T)$ is equipped by the norm

$$\|u\|_{L^{q,r}(D_T)} := \left(\int_0^T \left(\int_D |u(x)|^q \, \mathrm{d}x\right)^{\frac{r}{q}}\right)^{1/r} \qquad \text{for all } u \in L^{q,r}(D_T).$$

Definition A.1. Let D be an open subset of $\mathbb{R}^d$ and let $T > 0$. Then, for $v \in L^1(D_T)$, $h > 0$, $t \in (0, T)$, and for a.e. $x \in D$, we define the *Steklov mean value* of v (also called *Steklov average*) by

$$v_h(x,t) := \begin{cases} \frac{1}{h} \int_t^{t+h} v(x,s)\, ds & \text{if } t \in (0, T-h), \text{ and} \\ 0 & \text{if } t > T-h. \end{cases}$$

The following lemma is a more detailed version of Lemma 4.7 in [22, p. 85]. Nevertheless, this lemma is quite standard (cf. [22,10]) and so we omit its proof.

Lemma A.2. *Let $D \subseteq \mathbb{R}^d$ be an open set. Then the following assertions hold true.*

(i) *For $v \in L^{q,r}(D_T)$, let $\tilde{v}(t) := v(t)$ if $t \in [0,T]$ and $\tilde{v}(t) = 0$ if $t \in \mathbb{R} \setminus [0,T]$. Further for every $0 < h < \delta < T$, let $\rho_h(t) = h^{-1}\, \mathbb{1}_{[-1,0]}(h^{-1}t)$ for every $t \in \mathbb{R}$. Then,*

$$\int_{\mathbb{R}} \rho_h(t)\, \mathrm{d}t = 1, \quad \rho_h \geq 0, \qquad \lim_{h \to 0+} \int_{\mathbb{R}\setminus(-r,r)} \rho_h\, \mathrm{d}t = 0 \quad \textit{for all } 0 < r < T, \tag{A.1}$$

and

$$(\tilde{v} * \rho_h)(t) = \int_{\mathbb{R}} \tilde{v}(s)\, \rho_h(t-s)\, ds = v_h(t) \quad \textit{for every } t \in (0, T-\delta). \tag{A.2}$$

(ii) *If $v \in L^{q,r}(D_T)$, then for every $0 < \delta < T$,*

$$v_h \to v \quad \textit{in } L^{q,r}(D_{T-\delta}) \textit{ as } h \to 0+\,.$$

(iii) *If $v \in C([0,T]; L^q(D))$, then $v_h(t)$ can be defined in $t = 0$ by $v_h(x,0) = \frac{1}{h}\int_0^{0+h} v(x,s)\, ds$ $(x \in D)$ for all $h > 0$, and for every $0 < \varepsilon < T$ and every $t \in [0, T-\varepsilon)$,*

$$v_h \to v \quad \textit{in } C([0,T-\delta]; L^q(D)) \textit{ as } h \to 0+\,.$$

(iv) *If $v \in V^p(D_T) := C([0,T]; L^2(D)) \cap L^p(0,T; W^{1,p}(D))$, then*

$$v_h \in W^{1,2}([0,T-\delta]; L^2(D)), \quad \textit{and} \quad \tfrac{\partial v_h}{\partial t}(t) = h^{-1}(v(t+h) - v(t))$$

for every $t \in (0, T-\delta)$ and every $0 < h < \delta < T$,

In the standard references (as, e.g., [22,10]) the second and third claims of Lemma A.2 are often employed, but in general without any proof. Thus we give its proof here.

Proof. First, we prove that the sequence $(\rho_h)_{h>0}$ defined in (i) satisfies the properties (A.1). Since $\mathbb{1}_{[-1,0]}(h^{-1}t) = \mathbb{1}_{[-h,0]}(t)$ for all $t \in \mathbb{R}$,

$$\int_{\mathbb{R}} \rho_h(t)\, \mathrm{d}t = h^{-1} \int_{-h}^{0} \mathbb{1}\, \mathrm{d}t = 1.$$

Further, for every $0 < r < T$, we have that $(\mathbb{R} \setminus (-r,r)) \cap [-h, 0] = \emptyset$ for every $0 < h < r$. Hence

$$\int_{\mathbb{R}\setminus]-r,r[} \rho_h(t)\, \mathrm{d}t = h^{-1} \int_{\mathbb{R}\setminus]-r,r[} \mathbb{1}_{[-h,0]}\, \mathrm{d}t = 0.$$

To see that formula (A.2) holds, we fix $t \in]0, T-\delta[$, and note that $\mathbb{1}_{[-1,0]}(\frac{t-s}{h}) = \mathbb{1}_{[t,t+h]}(s)$ for every $s \in \mathbb{R}$. Hence,

$$(\tilde{v} * \rho_h)(t) = h^{-1} \int_{\mathbb{R}} \tilde{v}(s)\, \mathbb{1}_{[-1,0]}\left(\tfrac{t-s}{h}\right)\, ds = h^{-1} \int_{\mathbb{R}} v(s)\, \mathbb{1}_{[t,t+h]}(s)\, ds = v_h(t).$$

Now, let $v \in L^{q,r}(D_T)$ and fix $0 < h < \delta < T$. Then, for every $t \in]0, T-\delta[$,

$$v_h(t) = h^{-1} \int_t^{t+h} v(s)\, ds = \int_{\frac{t}{h}}^{\frac{t+h}{h}} v(h\,r)\, dr = \int_0^1 v\left(h\, \left(s + \tfrac{t}{h}\right)\right)\, ds,$$

where we applied in the first equality the substitution $s \mapsto r(s) = h^{-1}\,s$ and in the second equality the substitution $r \mapsto s(r) = r - \frac{t}{h}$. Hence for every $t \in (0, T-\delta)$,

$$\begin{aligned} \|v_h(t) - v(t)\|_{L^q(D)} &= \left\| \int_0^1 \left(v\left(h\,\left(s+\tfrac{t}{h}\right)\right) - v(t)\right)\, ds \right\|_{L^q(D)} \\ &\le \int_0^1 \left\|v\left(h\,\left(s+\tfrac{t}{h}\right)\right) - v(t)\right\|_{L^q(D)}\, ds. \end{aligned} \tag{A.3}$$

If $v \in C([0,T]; L^q(D))$, then v is uniformly continuous on $[0, T-\delta]$ with values in $L^q(D)$. Thus for every $\varepsilon > 0$, there is a $\tilde{\delta} > 0$ such that for all $0 < h < \tilde{\delta}$, and all $s \in [0,1]$, all $t \in [0, T-\delta]$,

$$\left\|v\left(h\,\left(s+\tfrac{t}{h}\right)\right) - v(t)\right\|_{L^q(D)} < \varepsilon,$$

and so by estimate (A.3), for all $0 < h < \tilde{\delta}$,

$$\sup_{t \in [0, T-\delta]} \|v_h(t) - v(t)\|_{L^q(D)} < \varepsilon.$$

This shows that claim (iii) holds. On the other hand, by estimate (A.3) and by Hölder's inequality,

$$\begin{aligned} \left(\int_0^{T-\delta} \|v_h(t) - v(t)\|^r_{L^q(D)}\, \mathrm{d}t\right)^{1/r} &\le \left(\int_0^{T-\delta} \left(\int_0^1 \|v(h\cdot s + t) - v(t)\|_{L^q(D)}\, ds\right)^r \mathrm{d}t\right)^{1/r} \\ &\le \left(\int_0^{T-\delta} \int_0^1 \|v(h\cdot s+t) - v(t)\|^r_{L^q(D)}\, ds\, \mathrm{d}t\right)^{1/r}. \end{aligned}$$

By Tonelli's theorem and again Hölder's inequality,

$$\begin{aligned} \left(\int_0^{T-\delta} \int_0^1 \|v(h\cdot s+t) - v(t)\|^r_{L^q(D)}\, \mathrm{d}s\, \mathrm{d}t\right)^{1/r} &= \left(\int_0^1 \int_0^{T-\delta} \|v(h\cdot s+t) - v(t)\|^r_{L^q(D)}\, \mathrm{d}t\, ds\right)^{1/r} \\ &\le \int_0^1 \left(\int_0^{T-\delta} \|v(h\cdot s+t) - v(t)\|^r_{L^q(D)}\, \mathrm{d}t\right)^{1/r} ds. \end{aligned}$$

If $v \in L^{q,r}(D_T)$, then for a.e. $s \in (0,1)$,

$$\left(\int_0^{T-\delta} \|v(h\cdot s+t) - v(t)\|^r_{L^q(D)}\, \mathrm{d}t\right)^{1/r} \to 0 \quad \text{as } h \to 0+\,,$$

and by Minkowski's inequality, for a.e. $s \in (0,1)$ and all $0 < h < \delta$,

$$\begin{aligned} \left(\int_0^{T-\delta} \|v(h\cdot s+t) - v(t)\|^r_{L^q(D)}\, \mathrm{d}t\right)^{1/r} &\le \left(\int_0^{T-\delta} \left(\|v(h\cdot s+t)\|_{L^q(D)} + \|v(t)\|_{L^q(D)}\right)^r \mathrm{d}t\right)^{1/r} \\ &\le \left(\int_0^{T-\delta} \|v(h\cdot s+t)\|^r_{L^q(D)}\, \mathrm{d}t\right)^{1/r} + \|v\|_{L^{q,r}(D_{T-\delta})} \end{aligned}$$

$$
\begin{aligned}
&= \left(\int_{h\cdot s}^{h\cdot s+T-\delta} \|v(\tau)\|_{L^q(D)}^r \, d\tau \right)^{1/r} + \|v\|_{L^{q,r}(D_{T-\delta})} \\
&\le \; 2 \cdot \|v\|_{L^{q,r}(D_T)} \, .
\end{aligned}
$$

Therefore, by Lebesgue's dominated convergence theorem, we obtain that claim (ii) holds. To see that claim (iv) holds, let $0 < h < \delta$ and fix $\xi \in C_c^1(0, T-\delta)$. Then by Fubini's theorem,

$$
\begin{aligned}
\int_0^{T-\delta} h^{-1} \int_t^{t+h} v(s) \, \tfrac{d\xi}{\mathrm{d}t}(t) \, ds \, \mathrm{d}t &= h^{-1} \int_0^{T+h-\delta} \int_{\max\{0,s-h\}}^{\min\{s,T-\delta\}} v(s) \, \tfrac{d\xi}{\mathrm{d}t}(t) \, \mathrm{d}t \, ds \\
&= h^{-1} \int_0^h \int_0^s v(s) \, \tfrac{d\xi}{\mathrm{d}t}(t) \, \mathrm{d}t \, ds + h^{-1} \int_h^{T-\delta} \int_0^s v(s) \, \tfrac{d\xi}{\mathrm{d}t}(t) \, \mathrm{d}t \, ds \\
&\quad + h^{-1} \int_{T-\delta}^{T+h-\delta} \int_{s-h}^{T-\delta} v(s) \, \tfrac{d\xi}{\mathrm{d}t}(t) \, \mathrm{d}t \, ds.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\int_0^{T-\delta} v_h(t) \, \tfrac{d\xi}{\mathrm{d}t}(t) \, \mathrm{d}t &= \int_0^{T-\delta} h^{-1} \int_t^{t+h} v(s) \, \tfrac{d\xi}{\mathrm{d}t}(t) \, ds \, \mathrm{d}t \\
&= h^{-1} \int_0^h \int_0^s v(s) \, \tfrac{d\xi}{\mathrm{d}t}(t) \, \mathrm{d}t \, ds + h^{-1} \int_h^{T-\delta} \int_0^s v(s) \, \tfrac{d\xi}{\mathrm{d}t}(t) \, \mathrm{d}t \, ds \\
&\quad + h^{-1} \int_{T-\delta}^{T+h-\delta} \int_{s-h}^{T-\delta} v(s) \, \tfrac{d\xi}{\mathrm{d}t}(t) \, \mathrm{d}t \, ds \\
&= h^{-1} \int_0^h v(s)(\xi(s) - \xi(0)) \, ds + h^{-1} \int_h^{T-\delta} v(s)(\xi(s) - \xi(s-h)) \, ds \\
&\quad + h^{-1} \int_{T-\delta}^{T+h-\delta} v(s)(\xi(T-\delta) - \xi(s-h)) \, ds \\
&= h^{-1} \int_0^{T-\delta} v(s)\xi(s) \, ds - h^{-1} \int_h^{T-\delta} v(s)\xi(s-h) \, ds \\
&\quad - h^{-1} \int_{T-\delta}^{T+h-\delta} v(s)\xi(s-h) \, ds \\
&= h^{-1} \int_0^{T-\delta} v(s)\xi(s) \, ds - h^{-1} \int_h^{T+h-\delta} v(s)\xi(s-h) \, ds \\
&= h^{-1} \int_0^{T-\delta} v(s)\xi(s) \, ds - h^{-1} \int_0^{T-\delta} v(r+h)\xi(r) \, dr \\
&= - \int_0^{T-\delta} h^{-1}(v(s+h) - v(s)) \, \xi(s) \, ds.
\end{aligned}
$$

Therefore, claim (iv) holds true and this completes the proof of Lemma A.2. □

A.2. Proof of Theorem 2.3

Let D be an open set in $\mathbb{R}^d$ for $d \ge 1$. We call a real-valued measurable function $\omega : D \to [0, \infty]$ a *weight function*. For $1 \le p < \infty$ we denote by $L^p(D, \omega \mathrm{d}x)$ the set of all equivalent classes of measurable functions u having finite integral

$$
\|u\|_{L^p(D,\omega \mathrm{d}x)}^p := \int_D |u(x)|^p \, \omega(x) \, dx,
$$

where two measurable functions belong to the same equivalent class if they coincide a.e. on D. Then $L^p(D, \omega \mathrm{d}x)$ equipped with the norm $\|\cdot\|_{L^p(D,\omega \mathrm{d}x)}$ is a Banach space. Furthermore, we define the weighted

Sobolev space $W^{1,p}(D, \omega \mathrm{d}x)$ as the set of all $u \in L^p(D, \omega \mathrm{d}x)$ such that all weak partial derivatives $\frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_d}$ belong again to $L^p(D, \omega \mathrm{d}x)$. We equip $W^{1,p}(D, \omega \mathrm{d}x)$ with the norm

$$\|u\|^p_{W^{1,p}(D,\omega \mathrm{d}x)} := \|u\|^p_{L^p(D,\omega \mathrm{d}x)} + \|\nabla u\|^p_{L^p(D,\omega \mathrm{d}x)}$$

for every $u \in W^{1,p}(D, \omega \mathrm{d}x)$.

For the proof of Theorem 2.3, we need the following classical result of a *sufficient condition* for compact embeddings in the non-weighted case:

Lemma A.3 *([1, Theorem 6.47]). Let D be an open set in $\mathbb{R}^d$ having the following properties:*

1. *There exists a sequence $\{D^*_N\}_{N=1}^\infty$ of open subsets of D such that $D^*_N \subseteq D^*_{N+1}$ and such that for each N the embedding*

$$W^{1,p}(D^*_N) \hookrightarrow L^p(D^*_N)$$

   is compact.
2. *There is a continuously differentiable function $\Phi : U \to D$, where $U \subseteq D \times \mathbb{R}$ is an open set containing $D \times \{0\}$, and which satisfies $\Phi(x, 0) = x$ for every $x \in D$ and if $D_N = D \setminus D^*_N$ then*
   (a) *$D_N \times [0, 1] \subseteq U$ for each N,*
   (b) *$\Phi(\cdot, t)$ is injective on D for every fixed t,*
   (c) *there is a constant $M \geq 0$ such that $\left|\frac{\partial}{\partial t}\Phi(x, t)\right| \leq M$ for all $(x, t) \in U$.*
3. *The function $d_N(t) = \sup_{x\in D_N} |\det D_x\phi(x, t)|^{-1}$ satisfies*
   (a) *$\lim_{N\to\infty} d_N(1) = 0$,*
   (b) *$\lim_{N\to\infty} \int_0^1 d_N(t)\, dt = 0$.*

Then the embedding

$$W^{1,p}(D) \hookrightarrow L^p(D)$$

is compact.

Moreover, we make use of the following sufficient condition for compact embeddings in the weighted case.

Lemma A.4 *([3, Theorem 4.3]). Let D be an open set in $\mathbb{R}^d$ and ω a weight function such that the sets $D_0 := \{x \in D \mid \omega(x) = 0\}$ and $D_0 := \{x \in D \mid \omega(x) = +\infty\}$ are both closed subsets of D with Lebesgue measure zero. Further suppose the ω is bounded from below and above by positive constants on every compact subsets $K \subseteq D \setminus (D_0 \cup D_\infty)$. If the subgraph*

$$D_\omega := \left\{(x, z) \in D \times \mathbb{R} \,\middle|\, 0 < z < \omega(x)\right\}$$

is such that the embedding

$$W^{1,p}(D_\omega) \hookrightarrow L^p(D_\omega) \tag{A.4}$$

is compact, then the embedding

$$W^{1,p}(D, \omega \mathrm{d}x) \hookrightarrow L^p(D, \omega \mathrm{d}x)$$

is compact.

Proof of Theorem 2.3. Since A is a real symmetric positive definite $(d \times d)$-matrix, there is orthogonal $(d \times d)$-matrix B and eigenvalues $\lambda_1 > 0, \ldots, \lambda_d > 0$ of A such that $B^t AB = \Lambda$, where Λ is a $(d \times d)$-diagonal matrix with entries $\lambda_1, \ldots, \lambda_d$. By using the isomorphism $\Psi_1 : \mathbb{R}^{d+1} \to \mathbb{R}^{d+1}$ defined by $\Phi_1(x, z) = (B^t x, z)$, we see that the subgraph D_ρ is linear isomorphic to the subgraph

$$\left\{ (y, z) \in \mathbb{R}^d \times \mathbb{R} \,\middle|\, 0 < \qquad\qquad d)\, e^{-\frac{1}{p}\left(\sum_{i=1}^{d} \lambda_i y_i^2\right)^{p/2}} \right\},$$

wherein we denote the vector $B^t x = y = (y_1, \ldots, y_d)$. Since all eigenvalues λ_i are strictly positive, we have that

$$\|y\| := \left(\sum_{i=1}^{d} \lambda_i y_i^2 \right)^{1/2} \tag{A.5}$$

for every $y \in \mathbb{R}^d$, defines a norm on $\mathbb{R}^d$. Now, if we set $r = \|y\|$ and $\theta := y/\|y\|$ for every $y \in \mathbb{R}^d \setminus \{0\}$ and $r = 0$ and $\theta = 0$ if $y = 0$, then the mapping $\Psi_2 : [0, \infty) \times S_{d-1} \times \mathbb{R} \to \mathbb{R}^{d+1}$ defined by $\Psi_2(r, \theta, z) := (r\theta, z)$ is another linear isomorphism of $\mathbb{R}^{d+1}$, where $S_{d-1} := \{y \in \mathbb{R}^d \mid \|y\| = 1\}$ denotes the unit sphere in $\mathbb{R}^d$ with respect to the norm $\|\cdot\|$ defined in (A.5). By using this isomorphic transformation, we see that subgraph

$$D_\rho = \left\{ (r, \theta, z) \in [0, \infty) \times S_{d-1} \times \mathbb{R} \,\middle|\, 0 < z < g(r) \right\},$$

where the function $g(r) := C(A, d)\, e^{-\frac{1}{p} r^p}$ for every $r \geq 0$ and equality of the two sets means isomorphic. For the rest of the proof, we follow partially the idea of the proof of [3, Lemma 5.3]. To do so, consider for every positive integer N the set

$$(D_\rho)_N := \left\{ (r, \theta, z) \in D_\rho \,\middle|\, r \geq N \right\}.$$

Since the function g belongs to $C^1([0, \infty))$, is non-increasing, and has bounded derivative, we see that the set $(D_\rho)_N^* := D_\rho \setminus (D_\rho)_N$ is bounded and has the cone property (see [1, Definition 4.3, p.66]). Hence the embedding

$$W^{1,p}((D_\rho)_N^*) \hookrightarrow L^p((D_\rho)_N^*)$$

is compact. Moreover, we have that $(D_\rho)_N^* \subseteq (D_\rho)_{N+1}^*$ for all N. Now, consider the mapping $\Phi : U \to D_\rho$ defined by

$$\Phi(r, \theta, z, t) := \left(r - t, \theta, \frac{g(r - t)}{g(r)} z \right)$$

for every $(r, \theta, z, t) \in U := \left\{ (r, \theta, z, t) \,\middle|\, (r, \theta, z,) \in D_\rho,\ 0 \leq t < r \right\}$. Obviously, the open set U contains the sets $D_\rho \times \{0\}$ and $(D_\rho)_N \times [0, 1]$ for every N. Since g is strictly positive and of class C^1, the mapping Φ is continuously differentiable on U. By construction of Φ and by g, one sees that $\Phi(\cdot, \cdot, \cdot, t)$ is injective on D_ρ for every fixed t. In addition,

$$\frac{\partial}{\partial t} \Phi(r, \theta, z, t) = \left(-1, 0, \frac{-g'(r - t)}{g(r)} z \right)$$

for every $(r, \theta, z, t) \in U$. Thus and since g' is bounded on $[0, \infty)$, we have that

$$\left| \frac{\partial}{\partial t} \Phi(r, \theta, z, t) \right| \leq \left(1 + \|g'\|_{L^\infty([0,\infty))}^2 \right)^{1/2}$$

for every $(r, \theta, z, t) \in U$. Moreover,

$$\det D_{(r,\theta,z)} \Phi(r, \theta, z, t) = \frac{g(r - t)}{g(r)}$$

on U and so

$$d_N(t) = \sup_{(r,\theta,z)\in(D_\rho)_N} \left|\frac{g(r)}{g(r-t)}\right| = \sup_{r\geq N} e^{\frac{1}{p}((r-t)^p - r^p)}.$$

For $1 < p < \infty$, we have that $p(r-t)^{p-1}t \leq r^p - (r-t)^p$ for every $r \geq t$, and so $(r-t)^p - r^p \leq (-p)(r-t)^{p-1}t$ for every $r \geq t$. Hence

$$d_N(t) \leq e^{-(N-t)^{p-1}t}$$

for every $t \in [0,1]$. Thus Φ satisfies the conditions in Lemma A.3 and so the embedding (A.4) is compact with $\omega = \rho$. Therefore and by Lemma A.4 we can conclude that the embedding from $W^{1,p}(\mathbb{R}^d, \mu)$ into $L^p(\mathbb{R}^d, \mu)$ is compact. This completes the proof of this theorem. $\square$

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