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A state-dependent queueing system with asymptotic logarithmic distribution

V. Giorno[a], A.G. Nobile[a], E. Pirozzi[b]

[a]*Dipartimento di Studi e Ricerche Aziendali (Management & Information Technology) Università degli Studi di Salerno, Via Giovanni Paolo II, n. 132, 84084 Fisciano (SA), Italy, {giorno, nobile}@unisa.it*

[b]*Dipartimento di Matematica e Applicazioni, Università di Napoli Federico II, Monte S. Angelo, 80126, Napoli, Italy, epirozzi@unina.it*

Abstract

A Markovian single-server queueing model with Poisson arrivals and state-dependent service rates, characterized by a logarithmic steady-state distribution, is considered. The Laplace transforms of the transition probabilities and of the densities of the first-passage time to zero are explicitly evaluated. The performance measures are compared with those ones of the well-known $M/M/1$ queueing system. Finally, the effect of catastrophes is introduced in the model and the steady-state distribution, the asymptotic moments and the first-visit time density to zero state are determined.

Keywords: Asymptotic behavior, First-passage time, Busy period, Catastrophes
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1. Introduction

Queueing theory plays an important role in wide areas of science, technology and management. Applications of queueing can be seen in traffic modelling, business and industries, computer-communication, health sectors and medical sciences, etc. In the study of queueing systems the emphasis is often placed in obtaining steady-state performance measures, but in many applications it is necessary to know the behavior of the system not only in the asymptotic regime but also in the transient phase.

In birth-death queueing models the instantaneous arrival and departure rates, denoted by λ_n and μ_n respectively, depend on the number n of customers in the system. A systematic study of the birth-death queue with varying arrival and service rates has been carried out by Conolly, Chan, Gupta and Srinivasa Rao, Hadidi, Kyriakidis, Natvig, Parthasarathy and Servaraju, Sudhesh, Van Doorn. These authors give transient and stationary solutions for the queue length process, waiting time, busy period and output for special birth-death queues with adaptive demand and service mechanism (see [5, 11, 12, 20, 22, 23, 30, 31, 32, 34, 37, 40]). The transient analysis of the state-dependent queueing systems often presents considerable difficulties and also numerical solutions are generally difficult to get. Even in the simple $M/M/1$ queueing system, which can be described by a birth-death process with constant rates of arrivals and departures, the transient probabilities involve an infinite series of Bessel functions and their integrals (cf., for instance, [1, 13, 36]). In some queueing systems modeled as birth-death processes it is assumed that new customers enter into the system according to a Poisson process with constant rate λ, so that the PASTA property holds; moreover, the servers may not work at a constant rate, but they adapt their behavior to the state of the system by speeding up to empty the queue or by slowing down when they are overworked.

The busy period and the first-passage time (FPT) to state 0 play a relevant role in a queueing system with state-dependent arrival and service rates (cf. [10, 21, 25]). The busy period for a single-server system is the time interval between any two successive idle periods. It starts when a customer arrives to an empty system and ends when the departing customer leaves the system idle for the first time thereafter. For a single-server queueing system, a busy period is equivalent to a FPT from state 1 to state 0. The analysis of the FPT distributions and their moments is helpful for the efficient planning of the system. In particular, the FPT probability density function (pdf) from each integer positive state k to 0 is the convolution of k FPT densities from state ℓ to state $\ell - 1$ ($\ell = 1, 2, \ldots, k$). In [25], Jouini and Dallery derive closed-form expressions for FPT moments of a general birth-death process; furthermore, they compute the moments of the busy period for some Markovian queues.

Recently there has been a rapid increase in the literature of stochastic models which are subject to catastrophes; some relevant results on this topic are given in [4, 17, 18, 33, 35]. In particular, birth-death models with catastrophes have been discussed in the context of population dynamics (see, for instance, [6, 7, 29, 39]) and in queueing systems (see, for instance, [8, 14, 16, 27, 28, 37]). Whenever a catastrophe occurs in a queueing system, all the customers are destroyed immediately, the server remains inactive momentarily and it is ready for service when a new arrival occurs. For instance, queueing models with disasters can be used to analyze system breakdowns due to a reset order or computer networks with virus infections.

In [15] and [17] various functional relations are given to describe the birth-death process in the presence of catastrophes in terms of the birth-death process without catastrophes, characterized by the same birth and death rates. Furthermore, in [15] the problem of first-visit time (FVT) to state 0 and of the first occurrence of an effective catastrophe are discussed.

In this paper we investigate a single-server queueing system with Poisson arrivals (interarrival intervals of type M) and a special state-dependent service mechanism, assuming an infinite waiting-room and a first-come-first-served queueing discipline. We suppose that the queue length evolves as a birth-death process with constant arrival rates $\lambda_n = \lambda$ and with state-dependent service rates μ_n, such that $\mu_1 = \mu$ and $\mu_n = \mu\, n/(n-1)$ for $n = 2, 3, \ldots$, being n the number of customers present in the system. For this model, in Section 2 the asymptotic analysis is carried out, showing that the steady-state distribution is of logarithmic type. The Laplace transforms of the transition probabilities and of the first two moments are obtained. In Section 3, the Laplace transforms of the FPT pdf to 0 are determined; furthermore, the mean and the variance of the busy period are studied. In Section 4, we include the effect of total catastrophes in the model. Catastrophes occur with exponential rate ξ and reduce the number of customers instantaneously to 0. For the model with catastrophes, the steady-state distribution and the first two asymptotic moments are determined. The Laplace transform of FVT pdf to 0 is given. Finally, the busy period of the process with catastrophes is analyzed and the mean and the variance are provided. The obtained results for the model without (with) catastrophes are compared with the corresponding results for the $M/M/1$ queue.

2. The state of the system

We consider a single-server queueing system with infinite waiting-room and a first-come-first-served queueing discipline, described by a birth-death process $\{N(t), t \geq 0\}$ with rates

$$\lambda_n = \lambda \quad (n = 0, 1, \ldots), \qquad \mu_n = \begin{cases} \mu, & n = 1 \\ \Big(1 + \dfrac{1}{n-1}\Big)\mu, & n = 2, 3, \ldots \end{cases} \tag{1}$$

where $\lambda > 0$ and $\mu > 0$ (cf. Figure 1).

The process $N(t)$ describes the number of customers in an adaptive queueing system with constant arrival rates and state-dependent service

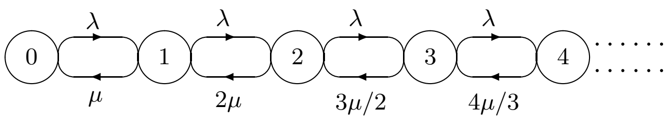


Figure 1: The state diagram of the process $N(t)$.

rates. The service rates in (1) are such that $\mu_n = \mu(1 + f_n)$, where f_n depends on the number of customers in the waiting queue. Specifically, $f_1 = 0$ and, for $n \geq 2$ one has that f_n is inversely proportional to the number $n-1$ of customers in the waiting queue, so that it decreases from 1 to 0 and $\lim_{n\to+\infty} \mu_n = \mu$.

We remark that the model with rates (1) is one of the few variants of the $M/M/1$ queue that admits a detailed exact analysis.

We denote by

$$p_{j,n}(t) = P\{N(t) = n | N(0) = j\} \qquad (n, j = 0, 1, \ldots)$$

the probability that n customers are in the system at time t, by assuming that $N(0) = j$. Then, for $j = 0, 1, \ldots$ the forward Kolmogorov equations are:

$$\begin{aligned}
\frac{dp_{j,0}(t)}{dt} &= -\lambda\, p_{j,0}(t) + \mu\, p_{j,1}(t) \\
\frac{dp_{j,1}(t)}{dt} &= \lambda\, p_{j,0}(t) - (\lambda + \mu)\, p_{j,1}(t) + 2\,\mu\, p_{j,2}(t) \\
\frac{dp_{j,n}(t)}{dt} &= \lambda\, p_{j,n-1}(t) - \Big(\lambda + \frac{n\,\mu}{n-1}\Big)\, p_{j,n}(t) + \frac{(n+1)\,\mu}{n}\, p_{j,n+1}(t) \\
&\qquad (n = 2, 3, \ldots),
\end{aligned} \tag{2}$$

with $p_{j,n}(0) = \delta_{j,n}$.

Moreover, we denote by $\{\widetilde{N}(t), t \geq 0\}$ the queueing system $M/M/1$ with arrival rates $\lambda_n = \lambda$ $(n = 0, 1, \ldots)$ and service rates $\mu_n = \mu$ $(n = 1, 2, \ldots)$ and let $\widetilde{p}_{j,0}(t) = P\{\widetilde{N}(t) = n | \widetilde{N}(0) = j\}$ be the transition probabilities of $\widetilde{N}(t)$. As well-known, the steady-state distribution of $\widetilde{N}(t)$ exists if and only if $\varrho < 1$ and one has $\widetilde{q}_n = \lim_{t\to+\infty} \widetilde{p}_{j,n}(t) = (1 - \varrho)\varrho^n$ $(n = 0, 1, \ldots)$, $E(\widetilde{N}) = \varrho/(1 - \varrho)$ and $E(\widetilde{W}) = (\mu - \lambda)^{-1}$.

In the following for the process $N(t)$ we determine the steady-state distribution and the Laplace transforms of the transition probabilities and of the first two moments. Furthermore, we compare the obtained results with those ones of the process $\widetilde{N}(t)$.

2.1. Steady-state analysis

The process $N(t)$ admits a steady-state distribution $q_n = \lim_{t\to+\infty} p_{j,n}(t)$ if and only if $\varrho = \lambda/\mu < 1$ and one has:

$$q_0 = [1 - \ln(1-\varrho)]^{-1}, \qquad q_n = \frac{\varrho^n}{n}\,[1-\ln(1-\varrho)]^{-1} \quad (n = 1, 2, \ldots). \tag{3}$$

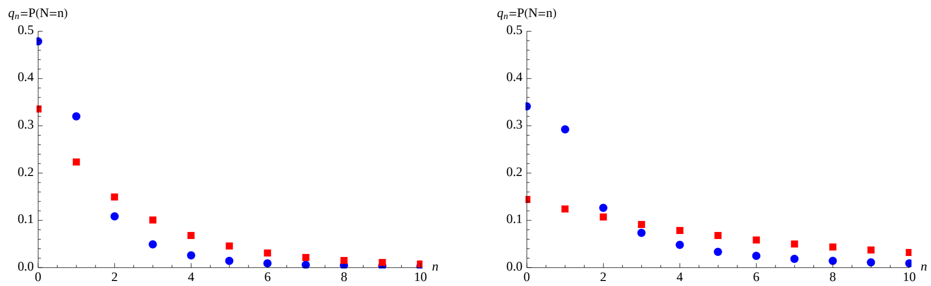
We note that

$$q_n = \left(\varrho - \frac{\varrho}{n}\right) q_{n-1} \qquad (n = 2, 3, \ldots); \tag{4}$$

hence, (3) belongs to the Sundt-Jewell class of distributions and it is called *logarithmic distribution* (cf. [38]). Making use of the inequality (cf., for instance, [2], n. 4.1.34)

$$\varrho < -\ln(1-\varrho) < \frac{\varrho}{1-\varrho} \qquad (0 < \varrho < 1), \tag{5}$$

we note that $q_0 > \widetilde{q}_0$ and $q_1 > \widetilde{q}_1$; moreover, for $n = 2, 3, \ldots$ one has $q_n > \widetilde{q}_n$ for $n < q_0/\widetilde{q}_0$ and $q_n < \widetilde{q}_n$ otherwise. These properties are shown in Figure 2, where we plot the steady-state probabilities q_n (solid circle) and $\widetilde{q}_n$ (solid square) for $\varrho = 2/3$ (on the left) and $\varrho = 6/7$ (on the right).



(a) $\varrho = 2/3$ (b) $\varrho = 6/7$

Figure 2: The probabilities q_n (solid circle) are compared with $\widetilde{q}_n$ (solid square) for $n = 0, 1, \ldots, 10$.

Furthermore, for $\varrho < 1$ one has $\widetilde{q}_n\, q_{n-1} \ge \widetilde{q}_{n-1}\, q_n$ $(n = 1, 2, \ldots)$, so that $\widetilde{N} \ge_{\ell r} N$, i.e. $\widetilde{N}$ dominates N in likelihood ratio ordering. Therefore, $\widetilde{N} \ge_{st} N$, i.e. $P(\widetilde{N} > n) \ge P(N > n)$ for $n = 0, 1, \ldots$. Since $\widetilde{N}$ dominates N in stochastic ordering, one has $E[h(\widetilde{N})] \ge E[h(N)]$ for all increasing function $h(\cdot)$ (cf. [24]).

Denoting by

$$\lambda^* = \lambda, \qquad \mu^* = \frac{1}{1-q_0}\sum_{n=1}^{+\infty} \mu_n\, q_n = \frac{\lambda}{1-q_0}$$

the effective mean arrival rate and service rate, respectively, it follows that in steady-state regime the traffic intensity for $N(t)$ is

$$\varrho^* = \frac{\lambda^*}{\mu^*} = -\frac{\ln(1-\varrho)}{1-\ln(1-\varrho)},$$

and the expected value of the duration of a service results:

$$E(S) = \frac{1}{\mu^*} = \frac{1-q_0}{\lambda}\cdot$$

Note that the traffic intensity for $N(t)$ is lower than that of $M/M/1$ queue, i.e. $\varrho^* < \varrho$; furthermore, $E(S) < E(\widetilde{S}) = 1/\mu$. Finally, in equilibrium regime the mean number of customers and the mean waiting time in the system are:

$$E(N) = \sum_{n=1}^{+\infty} n\, q_n = \frac{\varrho}{1-\varrho}\, q_0, \qquad E(W) = \frac{1}{\mu-\lambda}\, q_0 \qquad (\varrho < 1), \tag{6}$$

respectively. For $\varrho < 1$ one has $E(N) = q_0\, E(\widetilde{N})$ and $E(W) = q_0\, E(\widetilde{W})$, so that $E(N) < E(\widetilde{N})$ and $E(W) < E(\widetilde{W})$. Hence, the model (1) has asymptotically better performance measures with respect to the model $M/M/1$.

2.2. Laplace transforms of probabilities

We consider the Laplace transforms of the probabilities $p_{j,n}(t)$ with respect to time:

$$\pi_{j,n}(s) = \int_0^{+\infty} e^{-s\,t}\, p_{j,n}(t)\, dt \qquad (j,n = 0,1,\ldots;\, \mathrm{Re}\, s > 0). \tag{7}$$

Assuming that $j = 0$, we define the following generating functions:

$$G(z,t) = \sum_{n=0}^{+\infty} p_{0,n}(t)\, z^n, \qquad \varphi(z,s) = \sum_{n=0}^{+\infty} \pi_{0,n}(s)\, z^n \tag{8}$$

for $0 \le z \le 1$ and $\mathrm{Re}\, s > 0$. We transform the equations in (2), with $j = 0$, into a set of algebraic equations by employing the Laplace transforms:

$$\begin{aligned}
&(s+\lambda)\,\pi_{0,0}(s) = 1 + \mu\,\pi_{0,1}(s)\\
&(s+\lambda+\mu)\,\pi_{0,1}(s) = \lambda\,\pi_{0,0}(s) + 2\,\mu\,\pi_{0,2}(s)\\
&[(s+\lambda)\,n\,(n-1) + \mu\, n^2]\,\pi_{0,n}(s) = \lambda\, n\,(n-1)\,\pi_{0,n-1}(s)\\
&\qquad\qquad + (n-1)\,(n+1)\,\mu\,\pi_{0,n+1}(s) \quad (n = 2,3,\ldots).
\end{aligned} \tag{9}$$

The solution of system (9) can be obtained by applying the method introduced by Bailey [3] and used in [9] to analyze various queueing models. Indeed, multiplying the last equation in (9) by z^n, adding over $n = 2, 3, \ldots$ and making use of the first two equations, we obtain the following second order partial differential equation:

$$[-\lambda\, z^3 + (s + \lambda + \mu)\, z^2 - \mu\, z]\, \frac{\partial^2 \varphi(z, s)}{\partial z^2} + (-2\lambda\, z^2 + \mu\, z + \mu)\, \frac{\partial \varphi(z, s)}{\partial z} = \mu\,(z + 1)\,\pi_{0,1}(s) \tag{10}$$

to solve with the conditions:

$$\lim_{z\to 0} \varphi(z, s) = \pi_{0,0}(s) = \frac{1 + \mu\,\pi_{0,1}(s)}{s + \lambda}, \qquad \lim_{z\to 1} \varphi(z, s) = \frac{1}{s}\cdot \tag{11}$$

The function $\pi_{0,1}(s)$ can be determined from the consideration that $\varphi(z, s)$ must converge for $0 < z < 1$, provided $\mathrm{Re}\, s > 0$.

In the sequel we denote by

$$\psi_i(s) = \frac{s + \lambda + \mu \pm \sqrt{(s + \lambda + \mu)^2 - 4\,\lambda\,\mu}}{2\,\lambda} \qquad [i = 1, 2;\; \psi_1(s) > \psi_2(s)],$$
$$\alpha(s) = \frac{s + \lambda + 2\,\mu - 3\,\lambda\,\psi_1(s)}{\lambda\,[\psi_1(s) - \psi_2(s)]}. \tag{12}$$

The functions $\psi_i(s)$ $(i = 1, 2)$ are solutions of the equation $\lambda\, z^2 - (s + \lambda + \mu)\, z + \mu = 0$ and one has $\psi_1(s) > 1$ and $0 < \psi_2(s) < 1$. We note that $\psi_2(s)$ coincides with the Laplace transform of the busy period of the $M/M/1$ queueing system. Furthermore, for $s > 0$ the inequality $-2 < \alpha(s) < -1$ holds.

Proposition 1. *The solution of (10) with the conditions (11) is:*

$$\varphi(z, s) = \frac{1}{s} + \frac{\lambda}{s}\, \frac{H(z, s)}{\lambda - (s + \lambda)\, H(0, s)}, \tag{13}$$

where

$$H(z, s) = \frac{\lambda}{\mu} \Big\{ \psi_1(s)\, \frac{z - \psi_2(s)}{\psi_1(s) - z}\, F\Big(1, \alpha(s) + 2; \alpha(s) + 3; -\frac{z - \psi_2(s)}{\psi_1(s) - z}\Big)$$
$$- \psi_2(s)\, \frac{\alpha(s) + 1}{\alpha(s) + 3}\, \frac{z - \psi_2(s)}{\psi_1(s) - z}\, F\Big(1, \alpha(s) + 3; \alpha(s) + 4; -\frac{z - \psi_2(s)}{\psi_1(s) - z}\Big)$$
$$- \psi_1(s)\, \frac{1 - \psi_2(s)}{\psi_1(s) - 1} \Big\}, \tag{14}$$

with

$$F(a,b;c;x)=\sum_{n=0}^{+\infty}\frac{(a)_n\,(b)_n}{(c)_n}\,\frac{x^n}{n!} \tag{15}$$

denoting the Gauss hypergeometric function. Furthermore, one has:

$$\pi_{0,0}(s)=\frac{\lambda-sH(0,s)}{s\,[\lambda-(s+\lambda)H(0,s)]}, \tag{16}$$

$$\pi_{0,1}(s)=\frac{\lambda^2}{s\,\mu\,[\lambda-(s+\lambda)H(0,s)]}. \tag{17}$$

Proof. Setting $\beta(z,s)=\partial\varphi(z,s)/\partial z$, Eq. (10) leads to

$$z[\lambda\,z^2-(s+\lambda+\mu)\,z+\mu]\,\frac{\partial\beta(z,s)}{\partial z}+(2\lambda\,z^2-\mu\,z-\mu)\,\beta(z,s)=-\mu\,(z+1)\,\pi_{0,1}(s),$$

whose general solution is:

$$\beta(z,s)=\frac{z\,[z-\psi_1(s)]^{\alpha(s)}}{[z-\psi_2(s)]^{\alpha(s)+3}}\left[C(s)-\frac{\mu}{\lambda}\,\pi_{0,1}(s)\int^z\frac{u+1}{u^2}\,\frac{[u-\psi_2(s)]^{\alpha(s)+2}}{[u-\psi_1(s)]^{\alpha(s)+1}}\,du\right], \tag{18}$$

with $C(s)$ arbitrary real function. From (8) it follows that

$$\beta(z,s)=\sum_{n=1}^{+\infty}nz^{n-1}\,\pi_{0,n}(s)\le\frac{1}{s}\sum_{n=1}^{+\infty}nz^{n-1}=\frac{1}{s\,(1-z)^2}$$

must converge for $0<z<1$, provided $\mathrm{Re}\,s>0$. Since $\alpha(s)+3>1$, the zero of denominator in (18) is $\psi_2(s)$, so that the numerator must vanish when $z=\psi_2(s)$. Therefore, from (18) one has

$$\beta(z,s)=\frac{\mu}{\lambda}\,\pi_{0,1}(s)\,\frac{z\,[z-\psi_1(s)]^{\alpha(s)}}{[z-\psi_2(s)]^{\alpha(s)+3}}\int_z^{\psi_2(s)}\frac{u+1}{u^2}\,\frac{[u-\psi_2(s)]^{\alpha(s)+2}}{[u-\psi_1(s)]^{\alpha(s)+1}}\,du. \tag{19}$$

Then, recalling that $\beta(z,s)=\partial\varphi(z,s)/\partial z$ and making use of the second condition in (11), one has:

$$\varphi(z,s)=\frac{1}{s}+\frac{\mu}{\lambda}\,\pi_{0,1}(s)\,H(z,s), \tag{20}$$

where

$$H(z,s)=\int_z^1\frac{x\,[x-\psi_1(s)]^{\alpha(s)}}{[x-\psi_2(s)]^{\alpha(s)+3}}\,dx\int_{\psi_2(s)}^x\frac{u+1}{u^2}\,\frac{[u-\psi_2(s)]^{\alpha(s)+2}}{[u-\psi_1(s)]^{\alpha(s)+1}}\,du. \tag{21}$$

In Appendix A we prove that (21) leads to (14). Moreover, taking the limit as $z \downarrow 0$ in (14), by virtue of (A.9) one has:

$$H(0,s) = -\frac{\lambda}{\mu}\left\{\frac{\psi_1(s)-\psi_2(s)}{\psi_1(s)-1}+\frac{[\psi_2(s)]^2}{\psi_1(s)[\alpha(s)+3]}F\Big(1,\alpha(s)+3;\alpha(s)+4;\frac{\psi_2(s)}{\psi_1(s)}\Big)\right\}. \tag{22}$$

In order to determine $\pi_{0,1}(s)$, we take the limit as $z \downarrow 0$ in (20). Recalling the first of (11) one has:

$$\frac{1}{s}+\frac{\mu}{\lambda}\,\pi_{0,1}(s)\,H(0,s) = \frac{1+\mu\,\pi_{0,1}(s)}{s+\lambda},$$

from which (17) follows. Substituting (17) in the first of (9), one is led to (16). Finally, by virtue of (17), from (20) one obtains (13). □

Remark 1. *For $\lambda < \mu$, the asymptotic probability generating function is:*

$$G(z) = \lim_{t\to+\infty} G(z,t) = \frac{1-\ln(1-\lambda\,z/\mu)}{1-\ln(1-\lambda/\mu)} \qquad (0 \le z \le 1). \tag{23}$$

Proof. For $\lambda < \mu$, from (12) follows that $\psi_1(0) = \mu/\lambda$, $\psi_2(0) = 1$ and $\alpha(0) = -1$. Furthermore, recalling that (cf. [19], p. 1006, n. 9.121.6)

$$F(1,1;2;x) = -\frac{\ln(1-x)}{x}, \qquad F(1,2;3;x) = -\frac{2\,[x+\ln(1-x)]}{x^2},$$

from (14) one obtains:

$$\lim_{s\to 0} H(z,s) = \ln\Big(\frac{\mu-\lambda}{\mu-\lambda\,z}\Big). \tag{24}$$

Therefore, (23) follows from (13) by noting that

$$G(z) = \lim_{s\to 0}\,[s\,\varphi(z,s)] = 1+\lambda\,\lim_{s\to 0}\frac{H(z,s)}{\lambda-(s+\lambda)\,H(0,s)} \qquad (\lambda<\mu).$$

□

Eq. (23) can be also obtained making use of the asymptotic distribution (3).

We now derive the Laplace transforms $\pi_{0,n}(s)$ $(n = 2,3,\ldots)$ by expanding the function $\varphi(z,s)$ in power series of z.

Proposition 2. *For $n = 2, 3, \dots$ one has:*

$$\pi_{0,n}(s) = \frac{\pi_{0,1}(s)}{n\,[\psi_1(s)]^{n-1}} \Big\{ 1 + \frac{\psi_1(s) - \psi_2(s)}{\psi_1(s)} \sum_{r=0}^{n-1} (-1)^r \binom{n}{r+1} \frac{r\,[\alpha(s)+1]}{\alpha(s)+r+2} \times F\Big(1, \alpha(s)+2; \alpha(s)+r+3; \frac{\psi_2(s)}{\psi_1(s)}\Big) \Big\}, \tag{25}$$

with $\pi_{0,1}(s)$ given in (17).

Proof. From (8) and (13), for $n = 1, 2, \dots$ we note that

$$\pi_{0,n}(s) = \frac{1}{n!} \frac{\partial^n \varphi(z,s)}{\partial z^n}\Big|_{z=0} = \frac{1}{n!} \frac{\lambda}{s\,[\lambda - (s+\lambda)\,H(0,s)]} \frac{\partial^n H(z,s)}{\partial z^n}\Big|_{z=0}$$
$$= \frac{1}{n!} \frac{\mu\,\pi_{0,1}(s)}{\lambda} \frac{\partial^n H(z,s)}{\partial z^n}\Big|_{z=0}, \tag{26}$$

where $H(z,s)$ is defined in (14). Denoting by

$$C_{n,k+1}(s) = \frac{1}{n!} \frac{\partial^n}{\partial z^n} \Big(\frac{z - \psi_2(s)}{\psi_1(s) - z} \Big)^{k+1}\Big|_{z=0} \qquad (k = 0, 1, \dots), \tag{27}$$

and recalling (14), from (26) one has:

$$\pi_{0,n}(s) = \pi_{0,1}(s) \Big\{ \psi_1(s) \sum_{k=0}^{+\infty} (-1)^k \frac{\alpha(s)+2}{\alpha(s)+k+2}\, C_{n,k+1}(s) - \psi_2(s) \frac{\alpha(s)+1}{\alpha(s)+3} \sum_{k=0}^{+\infty} (-1)^k \frac{\alpha(s)+3}{\alpha(s)+k+3} C_{n,k+1}(s) \Big\} \quad (n = 1, 2, \dots). \tag{28}$$

In Appendix B we prove that

$$C_{0,k+1}(s) = \Big[-\frac{\psi_2(s)}{\psi_1(s)} \Big]^{k+1} \qquad (k = 0, 1, \dots),$$
$$C_{n,k+1}(s) = \Big[-\frac{\psi_2(s)}{\psi_1(s)} \Big]^{k+1} \frac{k+1}{n\,[\psi_1(s)]^n} \sum_{r=0}^{\min(k,n-1)} \binom{k}{r} \binom{n}{r+1} \Big[\frac{\psi_2(s) - \psi_1(s)}{\psi_2(s)} \Big]^{r+1} \qquad (k = 0, 1, \dots, n = 1, 2 \dots). \tag{29}$$

Therefore, substituting (29) in (28), after some cumbersome calculations, for $n = 1, 2, \dots$ one has:

$$\pi_{0,n}(s) = \pi_{0,1}(s) \frac{\psi_1(s) - \psi_2(s)}{n\,[\psi_1(s)]^n} \Big\{ \sum_{r=0}^{n-1} \binom{n}{r+1} \Big[\frac{\psi_2(s) - \psi_1(s)}{\psi_2(s)} \Big]^r \sum_{j=r}^{+\infty} \binom{j}{r} \Big[\frac{\psi_2(s)}{\psi_1(s)} \Big]^j$$

$$+\sum_{r=0}^{n-1} r \binom{n}{r+1} \Big[\frac{\psi_2(s)-\psi_1(s)}{\psi_2(s)}\Big]^r \sum_{j=r}^{+\infty} \frac{\alpha(s)+1}{\alpha(s)+j+2} \binom{j}{r} \Big[\frac{\psi_2(s)}{\psi_1(s)}\Big]^j \Big\}. \qquad (30)$$

Finally, recalling that

$$\sum_{j=r}^{+\infty} \binom{j}{r} x^j = \frac{x^r}{(1-x)^{r+1}}, \qquad \sum_{j=r}^{+\infty} \binom{j}{r} \frac{x^j}{j} = \frac{x^r}{r\,(1-x)^r},$$

$$\sum_{j=r}^{+\infty} \binom{j}{r} \frac{x^j}{a+j+2} = \frac{x^r}{a+r+2}\, F(r+1, a+r+2; a+r+3; x),$$

from (30) one obtains (25). □

Starting from (16), (17) and (25) one can check that $\sum_{n=0}^{+\infty} \pi_{0,n}(s) = 1/s$. Furthermore, for $\lambda < \mu$, the steady-state probabilities (3) can be also derived from (16), (17) and (25) making use of Tauberian theorem.

2.3. Laplace transforms of the moments

For $k = 1, 2, \ldots$, let

$$E[N^k(t)|N(0)=0] = \sum_{n=1}^{+\infty} n^k\, p_{0,n}(t), \quad \mathcal{M}_k(s) = \int_0^{+\infty} e^{-s\,t} E[N^k(t)|N(0)=0]\; dt$$

the conditional moments of $N(t)$ and their Laplace transforms, respectively.

Proposition 3. *The Laplace transforms of first two moments of the process $N(t)$ are:*

$$\mathcal{M}_1(s) = \frac{\mu\,\pi_{0,1}(s)}{\lambda\,[\psi_1(s)-1]^2} \Big\{ \frac{\psi_1(s)-\psi_2(s)}{\psi_2(s)} - \frac{s}{\lambda[\alpha(s)+3]\,[\psi_1(s)-1]}$$
$$\times F\Big(1, \alpha(s)+2; \alpha(s)+3; -\frac{1-\psi_2(s)}{\psi_1(s)-1}\Big) \Big\}, \qquad (31)$$

$$\mathcal{M}_2(s) = \frac{2\,\mu\,\pi_{0,1}(s)}{\lambda\,[\psi_1(s)-1]^2} \Big\{ \frac{2[\lambda\,\psi_1(s)+\mu\,\psi_2(s)-2\,\mu]+s[\psi_1(s)-\psi_2(s)]}{2\,s\,\psi_2(s)}$$
$$-\frac{2(\lambda-\mu)+s}{2\,\lambda\,[\alpha(s)+3][\psi_1(s)-1]}\, F\Big(1, \alpha(s)+2; \alpha(s)+3; -\frac{1-\psi_2(s)}{\psi_1(s)-1}\Big) \Big\}, \qquad (32)$$

with $\pi_{0,1}(s)$ given in (17).

Proof. By virtue of (8), one has:

$$\mathcal{M}_1(s) = \frac{\partial \varphi(z,s)}{\partial z}\Big|_{z=1}, \qquad \mathcal{M}_2(s) = \mathcal{M}_1(s) + \frac{\partial^2 \varphi(z,s)}{\partial z^2}\Big|_{z=1}. \tag{33}$$

By noting that

$$\begin{aligned}\frac{\partial}{\partial z} F\Big(1, \alpha(s)+k; \alpha(s)+k+1; -\frac{z-\psi_2(s)}{\psi_1(s)-z}\Big) &= \frac{\alpha(s)+k}{z-\psi_2(s)} \\ \times \Big\{1 - \frac{\psi_1(s)-\psi_2(s)}{\psi_1(s)-z}\, F\Big(1, \alpha(s)+k; \alpha(s)+k+1; -\frac{z-\psi_2(s)}{\psi_1(s)-z}\Big)\Big\},\end{aligned} \tag{34}$$

for $k = 1, 2, \ldots$, from (13) and (14) one obtains:

$$\begin{aligned}\frac{\partial \varphi(z,s)}{\partial z} = \frac{\mu\, \pi_{0,1}(s)}{\lambda\, [\psi_1(s)-z]} \Big\{ 1 - \frac{s}{\lambda\, [\psi_1(s)-\psi_2(s)]\, [\psi_1(s)-z]} \\ \times \Big[\frac{\psi_1(s)}{\alpha(s)+2}\, F\Big(1, \alpha(s)+2; \alpha(s)+3; -\frac{z-\psi_2(s)}{\psi_1(s)-z}\Big) \\ - \frac{\psi_2(s)}{\alpha(s)+3}\, F\Big(1, \alpha(s)+3; \alpha(s)+4; -\frac{z-\psi_2(s)}{\psi_1(s)-z}\Big)\Big]\Big\}.\end{aligned} \tag{35}$$

Hence, by virtue of (A.9), relation (31) immediately follows from (33) and (35). Furthermore, by using (34), one has

$$\begin{aligned}\frac{\partial^2 \varphi(z,s)}{\partial z^2} = \frac{\mu\, \pi_{0,1}(s)}{\lambda\, [\psi_1(s)-z]^2} \Big\{ 1 + \frac{s}{\lambda\, [z-\psi_2(s)]} + \frac{s}{\lambda\, [\psi_1(s)-z][\psi_1(s)-\psi_2(s)]} \\ \times \Big[\psi_1(s) \Big(\frac{2}{\alpha(s)+2} - \frac{\psi_1(s)-\psi_2(s)}{z-\psi_2(s)} \Big) F\Big(1, \alpha(s)+2; \alpha(s)+3; -\frac{z-\psi_2(s)}{\psi_1(s)-z}\Big) \\ - \psi_2(s) \Big(\frac{2}{\alpha(s)+3} - \frac{\psi_1(s)-\psi_2(s)}{z-\psi_2(s)} \Big) F\Big(1, \alpha(s)+3; \alpha(s)+4; -\frac{z-\psi_2(s)}{\psi_1(s)-z}\Big)\Big]\Big\},\end{aligned}$$

from which, by virtue of (A.9), relation (32) follows. □

In particular, for $\lambda < \mu$ from (31) and (32) one obtains the asymptotic moments:

$$E(N) = \lim_{s\to 0} [s\, \mathcal{M}_1(s)] = \frac{\varrho}{1-\varrho}\, q_0, \qquad E(N^2) = \lim_{s\to 0} [s\, \mathcal{M}_2(s)] = \frac{\varrho}{(1-\varrho)^2}\, q_0,$$

that follow also by using (3).

Propositions 2 and 3 will be used in Section 4 to obtain the steady-state distribution and its moments for the process $N(t)$ in the presence of random catastrophes.

3. First-passage time to zero state and busy period

Let us define the first-passage time of $N(t)$ to zero state as:

$$\mathcal{T}_{j,0} = \inf\{t \geq 0 : N(t) = 0\} \qquad N(0) = j \quad (j = 1, 2, \ldots)$$

and we denote by

$$g_{j,0}(t) = \frac{d}{dt} P\{\mathcal{T}_{j,0} < t\}, \qquad \gamma_{j,0}(s) = \int_0^{+\infty} e^{-s\,t}\, g_{j,0}(t)\, dt$$

the related FPT pdf and its Laplace transform, respectively. We remark that $g_{1,0}(t)$ identifies the busy period density of the process $N(t)$.

Proposition 4. *The Laplace transform of the busy period density is*

$$\gamma_{1,0}(s) = \frac{\lambda}{\lambda - s\, H(0, s)} \tag{36}$$

and, for $j = 2, 3, \ldots$ one has:

$$\begin{aligned} \gamma_{j,0}(s) = \Big[\frac{\mu}{\lambda \psi_1(s)}\Big]^{j-1} \frac{\lambda}{\lambda - s\, H(0, s)} \Bigg\{ 1 + \frac{\psi_1(s) - \psi_2(s)}{\psi_1(s)} \sum_{r=1}^{j-1} (-1)^r \binom{j}{r+1} \\ \times \frac{r\,[\alpha(s) + 1]}{\alpha(s) + r + 2} F\Big(1, \alpha(s) + 2; \alpha(s) + r + 3; \frac{\psi_2(s)}{\psi_1(s)}\Big) \Bigg\}. \end{aligned} \tag{37}$$

Furthermore, for $j = 1, 2, \ldots$ the probability of ultimate absorption at zero is:

$$\begin{aligned} P\{\mathcal{T}_{j,0} < +\infty\} &= \int_0^{+\infty} g_{j,0}(t)\, dt \\ &= \begin{cases} 1, & \lambda \leq \mu \\ \dfrac{(\mu/\lambda)^j}{1 - \mu/\lambda + (\mu/\lambda)^2} \Big[\dfrac{\mu}{\lambda} + j\Big(1 - \dfrac{\mu}{\lambda}\Big)\Big], & \lambda > \mu. \end{cases} \end{aligned} \tag{38}$$

Proof. The transition probabilities of a general birth-death process satisfy the following relation (cf., for instance, [26]):

$$p_{j,n}(t) = \frac{u_n}{u_j}\, p_{n,j}(t) \qquad (j, n = 0, 1, \ldots), \tag{39}$$

where

$$u_0 = 1, \qquad u_k = \frac{\lambda_0 \lambda_1 \cdots \lambda_{k-1}}{\mu_1 \mu_2 \cdots \mu_k} \qquad (k = 1, 2, \ldots)$$

are the potential coefficients of the birth-death process. Note that (39) is a form of reversibility which allows to turn the forward Kolmogorov equations for the transition probabilities in the backward Kolmogorov equations and vice versa.

Specifically, for the process $N(t)$ with rates (1) one has:

$$u_0 = 1, \qquad u_k = \frac{1}{k}\Big(\frac{\lambda}{\mu}\Big)^k \qquad (k = 1, 2, \ldots).$$

The FPT pdf $g_{j,0}(t)$ is solution of the integral equation

$$p_{j,0}(t) = \int_0^t g_{j,0}(\tau)\, p_{0,0}(t-\tau)\, d\tau \qquad (j = 1, 2, \ldots),$$

so that, taking the Laplace transform, by virtue of (39), one has:

$$\gamma_{j,0}(s) = \frac{\pi_{j,0}(s)}{\pi_{0,0}(s)} = j\,\Big(\frac{\mu}{\lambda}\Big)^j\, \frac{\pi_{0,j}(s)}{\pi_{0,0}(s)} \qquad (j = 1, 2, \ldots). \tag{40}$$

For $j = 1$, substituting (16) and (17) in (40), relation (36) follows. Similarly, for $j = 2, 3, \ldots$ making use of (16) and (25) in (40) one obtains (37).

To prove (38), we note that $P\{T_{j,0} < +\infty\} = \lim_{s\to 0} \gamma_{j,0}(s)$. Furthermore, since

$$\frac{\psi_1(s) - \psi_2(s)}{\psi_1(s) - 1} = \frac{1-\psi_2(s)}{s\,\psi_2(s)}\Big\{\mu - \lambda\,[\psi_2(s)]^2\Big\},$$

from (22) one has:

$$\lambda - s\,H(0,s) = \lambda + \frac{\lambda}{\mu}\,\frac{1-\psi_2(s)}{\psi_2(s)}\Big\{\mu - \lambda\,[\psi_2(s)]^2\Big\}$$
$$+ \frac{\lambda\, s\, \psi_2^2(s)}{\mu\psi_1(s)[\alpha(s)+3]} F\Big(1, \alpha(s)+3; \alpha(s)+4; \frac{\psi_2(s)}{\psi_1(s)}\Big). \tag{41}$$

Recalling (12), one has:

- if $\lambda < \mu$ then $\psi_1(0) = \mu/\lambda, \quad \psi_2(0) = 1, \quad \alpha(0) = -1;$
- if $\lambda = \mu$ then $\psi_1(0) = \psi_2(0) = 1, \quad \alpha(0) = -3/2;$
- if $\lambda > \mu$ then $\psi_1(0) = 1, \quad \psi_2(0) = \mu/\lambda, \quad \alpha(0) = -2.$

Therefore, from (41) it follows

$$\lim_{s\to 0}[\lambda - s\,H(0,s)] = \begin{cases} \lambda, & \lambda \le \mu \\ \dfrac{\lambda^2-\lambda\,\mu+\mu^2}{\mu}, & \lambda>\mu, \end{cases}$$

so that, from (36) and (37) one immediately obtains (38). □

We note that if $\lambda > \mu$ one has

$$P\{\mathcal{T}_{1,0}<+\infty\}<1, \qquad \frac{P\{\mathcal{T}_{j+1,0}<+\infty\}}{P\{\mathcal{T}_{j,0}<+\infty\}} = \frac{1+j\,(1-\mu/\lambda)}{1+j\,(1-\mu/\lambda)(\lambda/\mu)}<1,$$

so that $P\{\mathcal{T}_{j,0}<+\infty\}$ decreases as j increases. In particular, the busy period of $N(t)$ ends with probability 1 if and only if $\lambda \le \mu$ and there is non-zero probability that the busy period, when $\varrho > 1$, is infinitely large.

Let $\widetilde{\mathcal{T}}_{j,0}$ be the random variable describing the FPT to zero state for the $M/M/1$ queueing system $\widetilde{N}(t)$. The Laplace transform of the FPT density $\widetilde{g}_{j,0}(t)$ is $\widetilde{\gamma}_{j,0}(s) = [\psi_2(s)]^j$, so that $P(\widetilde{\mathcal{T}}_{j,0}<+\infty)=1$ for $\lambda\le\mu$ and $P(\widetilde{\mathcal{T}}_{j,0}<+\infty)=(\mu/\lambda)^j$ if $\lambda>\mu$. Hence, if $\varrho\le 1$ one has $P(\mathcal{T}_{j,0}<+\infty)=P(\widetilde{\mathcal{T}}_{j,0}<+\infty)=1$, whereas $P(\widetilde{\mathcal{T}}_{j,0}<+\infty)<P(\mathcal{T}_{j,0}<+\infty)$ if $\varrho>1$. Moreover, since the rates of $N(t)$ and of $\widetilde{N}(t)$ are such that $\lambda_n=\lambda$ and $\mu_n\ge\mu$, one has (cf. [41]):

$$\mathcal{T}_{j,0} \le_{st} \widetilde{\mathcal{T}}_{j,0} \qquad j=1,2,\ldots), \tag{42}$$

so that $E[\mathcal{T}_{j,0}^k]\le E[\widetilde{\mathcal{T}}_{j,0}^k]$ for $k=1,2,\ldots$, provided that the moments exist.

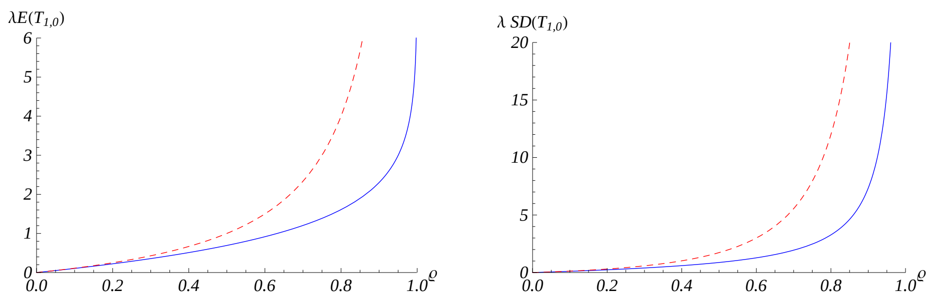


Figure 3: On the left we compare $\lambda E(\mathcal{T}_{1,0})$ (solid curve) and $\lambda E(\widetilde{\mathcal{T}}_{1,0})$ (dashed curve); on the right we plot $\lambda\,\mathrm{SD}(\mathcal{T}_{1,0})$ (solid curve) and $\lambda\,\mathrm{SD}(\widetilde{\mathcal{T}}_{1,0})$ (dashed curve).

Proposition 5. *For $\lambda < \mu$ the mean and the variance of the busy period are:*

$$E(\mathcal{T}_{1,0}) = -\frac{1}{\lambda}\ln(1-\varrho)$$
$$\mathrm{Var}(\mathcal{T}_{1,0}) = \frac{4\varrho^2 + 2\varrho(1-\varrho)\ln(1-\varrho) + (1-\varrho)^2[\ln(1-\varrho)]^2 - 2\varrho\,\mathrm{PolyLog}(2,\varrho)}{\lambda^2(1-\varrho)^2}, \tag{43}$$

where

$$\mathrm{PolyLog}(2,x) = \sum_{k=1}^{+\infty}\frac{x^k}{k^2}.$$

Proof. Since the interarrival intervals are exponentially distributed with mean $1/\lambda$, one has:

$$E(\mathcal{T}_{1,0}) = \frac{(1-q_0)}{\lambda\, q_0},$$

from which the first of (43) follows by virtue of (3). Furthermore, from (36) it follows:

$$E(\mathcal{T}_{1,0}^2) = \frac{d^2\gamma_{1,0}(s)}{ds^2}\Big|_{s=0} = \frac{2}{\lambda^2}\lim_{s\to 0}\Big\{[H(0,s)]^2 + \lambda\,\frac{dH(0,s)}{ds}\Big\}. \tag{44}$$

From (24) one has $\lim_{s\to 0} H(0,s) = \ln(1-\varrho)$. Moreover, as proved in Appendix C, it results:

$$\lim_{s\to 0}\frac{dH(0,s)}{ds} = \frac{2\varrho + (1-\varrho)\ln(1-\varrho) - \mathrm{PolyLog}(2,\varrho)}{\mu(1-\varrho)^2}. \tag{45}$$

Hence, making use (45) in (44) it follows:

$$E(\mathcal{T}_{1,0}^2) = \frac{2\,[(1-\varrho)^2\ln^2(1-\varrho) + 2\,\varrho^2 + \varrho(1-\varrho)\ln(1-\varrho) - \varrho\mathrm{PolyLog}(2,\varrho)]}{\lambda^2(1-\varrho)^2},$$

from which the second of (43) immediately follows. □

For $\varrho < 1$, the mean and variance of the busy period of the $M/M/1$ queueing system $\widetilde{N}(t)$ are $E(\widetilde{\mathcal{T}}_{1,0}) = [\mu(1-\varrho)]^{-1}$ and $\mathrm{Var}(\widetilde{\mathcal{T}}_{1,0}) = (1+\varrho)/[\mu^2(1-\varrho)^3]$, respectively. Making use of inequalities (5), we note that $E(\mathcal{T}_{1,0}) < E(\widetilde{\mathcal{T}}_{1,0})$ and $\mathrm{Var}(\mathcal{T}_{1,0}) < \mathrm{Var}(\widetilde{\mathcal{T}}_{1,0})$, so that mean and variance of the busy period for the process $N(t)$ are less than those of $M/M/1$ queue. In Figure 3 we compare the mean and the standard deviation (SD) for the busy periods of processes $N(t)$ and $\widetilde{N}(t)$.

4. Effect of catastrophes on the system

We consider a birth-death process $\{M(t), t \geq 0\}$ in the presence of catastrophes, such that births and deaths occur with rates (1). We assume that catastrophes occur according to a Poisson process with rate ξ, the effect of each catastrophe being the instantaneous transition to the state 0 (cf. Figure 4).

4.1. The state of the system

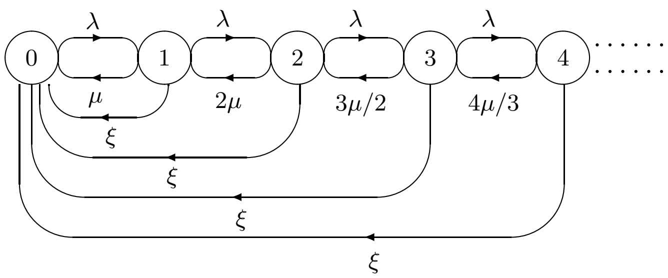


Figure 4: The state diagram of the process $M(t)$.

The transition probabilities of the process $M(t)$

$$r_{j,n}(t) = P\{M(t) = n | M(0) = j\} \qquad (j, n = 0, 1, 2, \ldots) \tag{46}$$

satisfy the following forward Kolmogorov equations:

$$\begin{aligned}
\frac{dr_{j,0}(t)}{dt} &= -(\lambda + \xi)\, r_{j,0}(t) + \mu\, p_{j,1}(t) + \xi \\
\frac{dr_{j,1}(t)}{dt} &= \lambda\, r_{j,0}(t) - (\lambda + \mu + \xi)\, r_{j,1}(t) + 2\,\mu\, r_{j,2}(t) \\
\frac{dr_{j,n}(t)}{dt} &= \lambda\, r_{j,n-1}(t) - \Big(\lambda + \frac{n\,\mu}{n-1} + \xi\Big)\, r_{j,n}(t) + \frac{(n+1)\,\mu}{n}\, r_{j,n+1}(t), \\
&\qquad (n = 2, 3, \ldots).
\end{aligned} \tag{47}$$

with $r_{j,n}(0) = \delta_{j,n}$. The probabilities $r_{j,n}(t)$ can be also expressed in terms of the probabilities $p_{j,n}(t)$ of the process $N(t)$. Indeed, conditioning on the age of the catastrophe process, for a birth-death process one has (see, for instance, [17], [18], [33], [35]):

$$r_{j,n}(t) = e^{-\xi t}\, p_{j,n}(t) + \xi \int_0^t e^{-\xi \tau}\, p_{0,n}(\tau)\, d\tau \qquad (n, j = 0, 1, \ldots, t > 0). \tag{48}$$

Hence, the steady-state probabilities of $M(t)$ are given by

$$\omega_n = \lim_{t\to\infty} r_{j,n}(t) = \xi\, \pi_{0,n}(\xi) \qquad (n = 0, 1, \ldots), \tag{49}$$

so that the asymptotic moments of the process $M(t)$ are:

$$E(M^k) = \sum_{n=1}^{+\infty} n^k\, \omega_n = \xi \sum_{n=1}^{+\infty} n^k \pi_{0,n}(\xi) = \xi\, \mathcal{M}_k(\xi) \qquad (k = 1, 2, \ldots), \tag{50}$$

where $\mathcal{M}_k(s)$ denotes the Laplace transform of the kth-order conditional moment of $N(t)$.

Making use of (16), (17) and (25) in (49), one obtains the following expressions:

$$\begin{aligned}
\omega_0 &= P\{M = 0\} = \frac{\lambda - \xi H(0,\xi)}{\lambda - (\xi + \lambda) H(0,\xi)},\\
\omega_1 &= P\{M = 1\} = \frac{\lambda^2}{\mu\,[\lambda - (\xi + \lambda) H(0,\xi)]},\\
\omega_n &= P\{M = n\} = \frac{\omega_1}{n\,[\psi_1(\xi)]^{n-1}} \left\{ 1 + \frac{\psi_1(\xi) - \psi_2(\xi)}{\psi_1(\xi)} \sum_{r=0}^{n-1} (-1)^r \binom{n}{r+1} \right.\\
&\quad \left. \times \frac{r\,[\alpha(\xi)+1]}{\alpha(\xi)+r+2}\, F\Big(1, \alpha(\xi)+2; \alpha(\xi)+r+3; \frac{\psi_2(\xi)}{\psi_1(\xi)}\Big) \right\} \quad (n = 2, 3, \ldots),
\end{aligned} \tag{51}$$

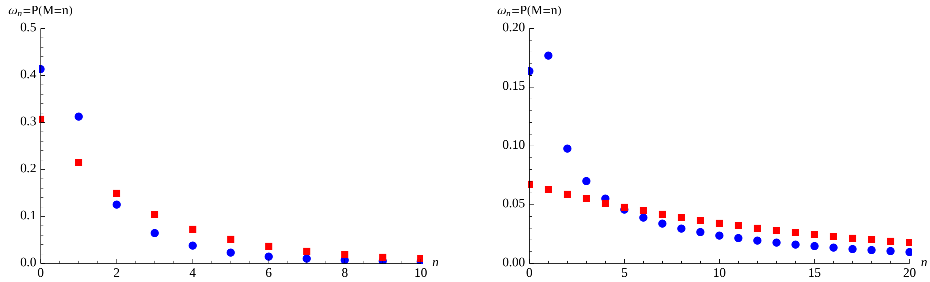
with $\psi_i(s)$ $(i = 1, 2)$ and $\alpha(s)$ defined in (12) and $H(0, s)$ given in (22). We note that the steady-state distribution of $M(t)$ always exists, differently from that of the process $N(t)$ without catastrophes, in which it exists only if $\varrho < 1$. Furthermore, making use of (31) and (32) in (50) one obtains:

$$\begin{aligned}
E(M) &= \frac{\mu\,\omega_1}{\lambda\,[\psi_1(\xi) - 1]^2} \left\{ \frac{\psi_1(\xi) - \psi_2(\xi)}{\psi_2(\xi)} - \frac{\xi}{\lambda[\alpha(\xi)+3]\,[\psi_1(\xi) - 1]} \right.\\
&\quad \left. \times F\Big(1, \alpha(\xi)+2; \alpha(\xi)+3; -\frac{1 - \psi_2(\xi)}{\psi_1(\xi) - 1}\Big) \right\},\\
E(M^2) &= \frac{2\,\mu\,\omega_1}{\lambda\,[\psi_1(\xi) - 1]^2} \left\{ \frac{2[\lambda\,\psi_1(\xi) + \mu\,\psi_2(\xi) - 2\,\mu] + \xi[\psi_1(\xi) - \psi_2(\xi)]}{2\,\xi\,\psi_2(\xi)} \right.\\
&\quad \left. - \frac{2(\lambda - \mu) + \xi}{2\,\lambda\,[\alpha(\xi)+3][\psi_1(\xi) - 1]}\, F\Big(1, \alpha(\xi)+2; \alpha(\xi)+3; -\frac{1 - \psi_2(\xi)}{\psi_1(\xi) - 1}\Big) \right\}.
\end{aligned}$$

Now, let $\widetilde{M}(t)$ be the process $M/M/1$ in the presence of catastrophes to zero state, that occur according to a Poisson process with rate ξ. The steady-state probabilities of $\widetilde{M}(t)$ are (cf., for instance, [14]):

$$\widetilde{\omega}_n = P\{\widetilde{M} = n\} = \left[1 - \frac{\lambda}{\mu}\psi_2(\xi)\right]\left[\frac{\lambda}{\mu}\psi_2(\xi)\right]^n \qquad (n = 0, 1, \ldots). \tag{52}$$

In Figure 5 we compare (51) and (52) for some choices of the parameters; specifically, Figure 5(a) refers to the case $\varrho < 1$, whereas Figure 5(b) refers to $\varrho > 1$.



(a) $\lambda = 0.6, \mu = 0.7, \xi = 0.05$ (b) $\lambda = 0.7, \mu = 0.6, \xi = 0.01$

Figure 5: The probabilities ω_n (solid circle) are compared with $\widetilde{\omega}_n$ (solid square) for $n = 0, 1, \ldots, 10$ (on the left) and for $n = 0, 1, \ldots, 20$ (on the right).

In Figures 6 and 7 we compare the asymptotic mean and variance for the process $M(t)$ with the related quantities for the $\widetilde{M}(t)$ process (cf., for instance, [14]):

$$E(\widetilde{M}) = \frac{1}{\psi_1(\xi) - 1}, \qquad \mathrm{Var}(\widetilde{M}) = \frac{\psi_1(\xi)}{(\psi_1(\xi) - 1)^2}.$$

In particular, in Figure 6 we assume that $\lambda = 0.6$ and $\mu = 0.7$; in the

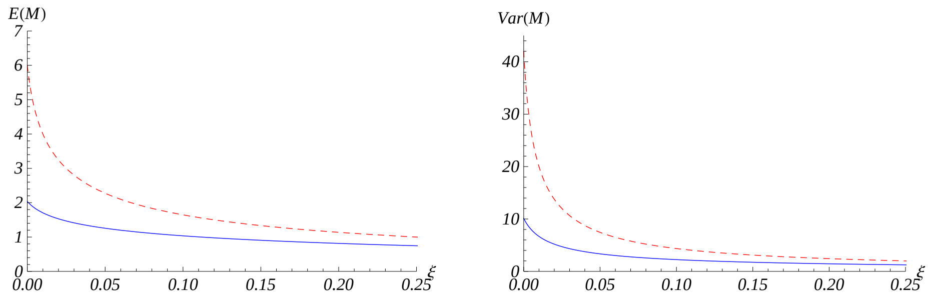


Figure 6: $E(M)$ and $\mathrm{Var}(M)$ (solid curves) are compared with $E(\widetilde{M})$ and $\mathrm{Var}(\widetilde{M})$ (dashed curves) for $\lambda = 0.6$ and $\mu = 0.7$.

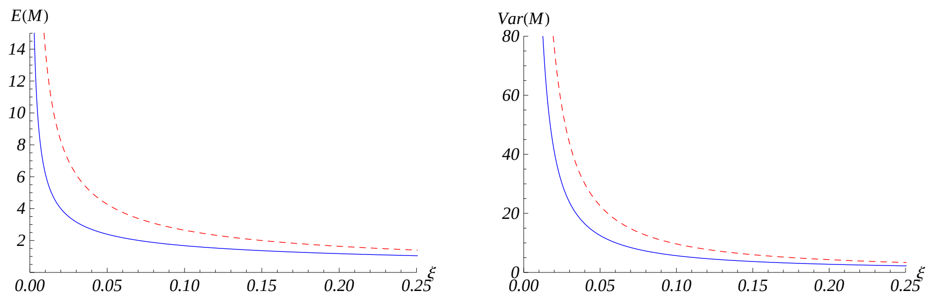


Figure 7: As in Figure 6 with $\lambda = 0.7$ and $\mu = 0.6$.

absence of catastrophes $E(M) = 2.03672$, $\mathrm{Var}(M) = 10.1088$ and $E(\widetilde{M}) = 6$, $\mathrm{Var}(\widetilde{M}) = 42$. Instead, in Figure 7 we assume that $\lambda = 0.7$ and $\mu = 0.6$; in this case, in the absence of catastrophes, the stationary distribution does not exist, so that when $\xi \downarrow 0$ mean and variance diverge.

4.2. First-visit time to zero state

Let

$$\Theta_{j,0} = \inf\{t \geq 0 : M(t) = 0\}, \qquad M(0) = j > 0$$

be the first-visit time (FVT) of $M(t)$ to 0 starting from the initial state j and let $h_{j,0}(t) = dP(\Theta_{j,0} \leq t)/dt$ be its pdf. For $j = 1, 2, \ldots$ the random variable $\Theta_{j,0}$ has the same distribution as $\min(\mathcal{T}_{j,0}, Z)$, where Z is an exponentially distributed random variable, with mean $1/\xi$, independent of $\mathcal{T}_{j,0}$. Therefore, one has:

$$h_{j,0}(t) = e^{-\xi t}\, g_{j,0}(t) + \xi\, e^{-\xi t} \left[1 - \int_0^t g_{j,0}(\tau)\, d\tau\right] \qquad (j = 1, 2, \ldots), \tag{53}$$

where $g_{j,0}(t)$ is the pdf of $\mathcal{T}_{j,0}$. Denoting by $\eta_{j,0}(s)$ the Laplace transform of $h_{j,0}(t)$ and making use of (53), for $j = 1, 2, \ldots$ one obtains:

$$\eta_{j,0}(s) = \frac{\xi}{s + \xi} + \frac{s}{s + \xi}\, \gamma_{j,0}(s + \xi), \tag{54}$$

where $\gamma_{j,0}(s)$ is the Laplace transform of $g_{j,0}(t)$, given in (36) and (37). We note that $P(\Theta_{j,0} < +\infty) = 1$, i.e. the first-visit time of $M(t)$ to zero state occurs with probability 1, whereas for $N(t)$ such an event has probability equal to 1 only if $\lambda < \mu$.

Let now $\widetilde{\Theta}_{j,0}$ be the FVT of $\widetilde{M}(t)$ to 0 starting from the initial state j. Since $\widetilde{\Theta}_{j,0}$ has the same distribution as $\min(\widetilde{\mathcal{T}}_{j,0}, Z)$, where Z is an exponentially distributed random variable, with mean $1/\xi$, recalling (42) it follows:

$$\Theta_{j,0} \le_{st} \widetilde{\Theta}_{j,0} \qquad (j = 1, 2, \ldots), \tag{55}$$

i.e. the FVT to zero state for $M(t)$ is smaller than the FVT for the $M/M/1$ with catastrophes in the stochastic order. Consequently, $E[\Theta_{j,0}^k] \le E[\widetilde{\Theta}_{j,0}^k]$ for $k = 1, 2, \ldots$.

Recalling (36), the Laplace transform of the busy period of $M(t)$ is

$$\eta_{1,0}(s) = \frac{1}{s+\xi}\Big[\xi + \frac{\lambda\, s}{\lambda - (s+\xi)\, H(0, s+\xi)}\Big], \tag{56}$$

with $H(0,s)$ given in (22). Moreover, by virtue of (56), we obtain the mean and the variance of the busy period in the presence of catastrophes:

$$E(\Theta_{1,0}) = \frac{H(0,\xi)}{\xi\, H(0,\xi) - \lambda},$$
$$\mathrm{Var}(\Theta_{1,0}) = \frac{1}{[\lambda - \xi\, H(0,\xi)]^2}\Big[H^2(0,\xi) + 2\,\lambda\, \frac{d}{d\xi} H(0,\xi)\Big]. \tag{57}$$

In Figures 8 and 9 we compare the mean and the standard deviation of the busy period for the process $M(t)$ with the related quantities for the $\widetilde{M}(t)$ process (cf., for instance, [14]):

$$E(\widetilde{\Theta}_{1,0}) = \frac{1}{\psi_1(\xi) - 1}, \qquad \mathrm{Var}(\widetilde{\Theta}_{1,0}) = \frac{\psi_1(\xi)}{(\psi_1(\xi) - 1)^2}.$$

In particular, in Figure 8 we set $\lambda = 0.6$ and $\mu = 0.7$; in the absence of catastrophes $E(\Theta_{1,0}) = 3.24318$, $\mathrm{SD}(\Theta_{1,0}) = 8.15072$ and $E(\widetilde{\Theta}_{1,0}) = 10$, $\mathrm{SD}(\widetilde{\Theta}_{1,0}) = 36.0555$. In Figure 9 we have $\lambda > \mu$.

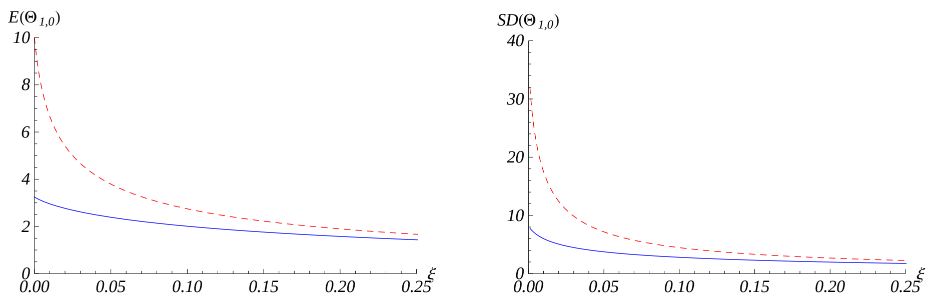


Figure 8: $E(\Theta_{1,0})$ and $\mathrm{SD}(\Theta_{1,0})$ (solid curves) are compared with $E(\widetilde{\Theta}_{1,0})$ and $\mathrm{SD}(\widetilde{\Theta}_{1,0})$ (dashed curves) for $\lambda = 0.6$ and $\mu = 0.7$.

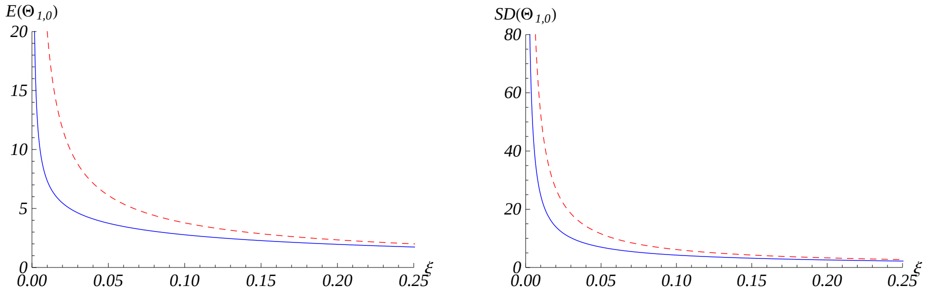


Figure 9: As in Figure 8 with $\lambda = 0.7$ and $\mu = 0.6$.

5. Conclusions

In the present paper we consider an adaptive single-server queueing model with constant arrival rates and state-dependent service rates (1). For this model, the asymptotic analysis is performed showing that the steady-state distribution is of logarithmic type. Moreover, the Laplace transforms of the transition probabilities are determined. Furthermore, the effect of total catastrophes, occurring according to a Poisson process with constant rate, is included and the asymptotic behaviour of the obtained model is studied. The analysis of the busy period for the system without and with catastrophes is carried out. The performance measures are compared with those of the $M/M/1$ system.

Appendix A. Evaluation of $H(z, s)$

In this appendix, starting from (21), we prove (14). By performing in (21) the change of variables $w = [u - \psi_2(s)]/[\psi_1(s) - u]$ and $y = [x - \psi_2(s)]/[\psi_1(s) - x]$ and recalling that (cf. [19], p. 315, n. 194.1)

$$\int_0^x \frac{w^{\xi-1}}{(1+\beta\, w)^{\nu}}\, dw = \frac{x^{\xi}}{\xi}\, F(\nu, \xi; \xi+1; -\beta\, x) \quad (\mathrm{Re}\, \xi > 0, |\arg(1+\beta\, x)| < \pi),$$

one obtains:

$$H(z,s) = -\frac{1}{\alpha(s)+3}\, \frac{1}{[\psi_1(s) - \psi_2(s)]^2} \int_{[z-\psi_2(s)]/[\psi_1(s)-z]}^{[1-\psi_2(s)]/[\psi_1(s)-1]} \zeta(y,s)\, dy, \quad \text{(A.1)}$$

where we have set

$$\zeta(y,s) = \Big[\psi_2(s) + \psi_1(s) y\Big] \Bigg\{ \frac{\psi_1(s)[\psi_1(s)-1]}{\psi_2(s)}\, F\Big(1, \alpha(s)+3; \alpha(s)+4; -\frac{\psi_1(s)}{\psi_2(s)} y\Big)$$

$$
\begin{aligned}
&+\frac{\psi_1(s)\,[\psi_1(s)-\psi_2(s)]}{[\psi_2(s)]^2}\,F\Big(2,\alpha(s)+3;\alpha(s)+4;-\frac{\psi_1(s)}{\psi_2(s)}\,y\Big)\\
&-[\psi_1(s)-1]\,F\Big(1,\alpha(s)+3;\alpha(s)+4;-y\Big)\\
&-[\psi_1(s)-\psi_2(s)]\,F\Big(2,\alpha(s)+3;\alpha(s)+4;-y\Big)\Big\}. \qquad (A.2)
\end{aligned}
$$

We now note that the Gauss hypergeometric functions satisfy the following recurrence relations (cf. [19], p. 1010, n. 9.137.11, n. 9.137.12 and n. 9.137.18):

$$
\begin{aligned}
&c\,F(a,b;c;w)-c\,F(a,b+1;c;w)-a\,w\,F(a+1,b+1;c+1;w)=0,\\
&c\,F(a,b;c;w)-c\,F(a+1,b;c;w)+b\,w\,F(a+1,b+1;c+1;w)=0, \quad (A.3)\\
&c\,F(a,b;c;w)-(c-a)\,F(a,b;c+1;w)-a\,F(a+1,b;c+1;w)=0.
\end{aligned}
$$

Furthermore, since $\psi_1(s)\psi_2(s)=\mu/\lambda$ and $\psi_1(s)+\psi_2(s)=(s+\lambda+\mu)/\lambda$, one has:

$$
\begin{aligned}
&[\psi_1(s)-1]-[\alpha(s)+2]\,[\psi_1(s)-\psi_2(s)]=\frac{s}{\lambda},\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (A.4)\\
&[\psi_1(s)-1]-[\alpha(s)+2]\,\frac{\psi_1(s)-\psi_2(s)}{\psi_2(s)}=0.
\end{aligned}
$$

Hence, making use of (A.3) and (A.4) in (A.2), after some calculations, one obtains:

$$
\begin{aligned}
\zeta(y,s)=&[\alpha(s)+3]\,\frac{\psi_1(s)-\psi_2(s)}{y+1}\\
&+\frac{s}{\lambda}\Big\{-\psi_2(s)\,F(1,\alpha(s)+3;\alpha(s)+4;-y)\\
&\qquad+\psi_1(s)\,\frac{\alpha(s)+3}{\alpha(s)+2}\,F(1,\alpha(s)+2;\alpha(s)+3;-y)\Big\}. \qquad (A.5)
\end{aligned}
$$

Therefore, one can calculate (A.1) by using (A.5). Indeed, recalling that

$$
\int_0^x F(1,\beta;\beta{+}1;-y)\;dy=\frac{-x\,F(1,\beta;\beta+1;-x)+\beta\ln(1+x)}{\beta-1}\qquad(\mathrm{Re}\,x\geq-1),
$$

and using the identities:

$$
\begin{aligned}
&\frac{\psi_1(s)}{\alpha(s)+1}-\frac{\psi_2(s)}{\alpha(s)+2}=-\frac{\lambda}{s}\,[\psi_1(s)-\psi_2(s)]^2,\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (A.6)\\
&\frac{1}{[\alpha(s)+1]\,[\alpha(s)+2]}=-\frac{\lambda^2}{s\,\mu}\,[\psi_1(s)-\psi_2(s)]^2,
\end{aligned}
$$

from (A.1) one has:

$$H(z,s)=\frac{\lambda}{\mu}\Big\{\frac{z-\psi_2(s)}{\psi_1(s)-z}\,K(z,s)-\frac{1-\psi_2(s)}{\psi_1(s)-1}\,K(1,s)\Big\}, \tag{A.7}$$

where we have set:

$$\begin{aligned}K(z,s)&=\psi_1(s)\,F\Big(1,\alpha(s)+2;\alpha(s)+3;-\frac{z-\psi_2(s)}{\psi_1(s)-z}\Big)\\&\quad-\psi_2(s)\,\frac{\alpha(s)+1}{\alpha(s)+3}\,F\Big(1,\alpha(s)+3;\alpha(s)+4;-\frac{z-\psi_2(s)}{\psi_1(s)-z}\Big).\end{aligned} \tag{A.8}$$

Furthermore, from (A.3) it results:

$$F(1,\alpha(s)+2;\alpha(s)+3;y)\,=1+y\,\frac{\alpha(s)+2}{\alpha(s)+3}\,F(1,\alpha(s)+3;\alpha(s)+4;y). \tag{A.9}$$

Hence $K(1,s)=\psi_1(s)$, so that (14) follows from (A.7) and (A.8).

Appendix B. Evaluation of $C_{n,k+1}(s)$

In this appendix, starting from (27) we prove (29). Since $0<z<\psi_1(s)$ one has:

$$\frac{z-\psi_2(s)}{\psi_1(s)-z}=\frac{z-\psi_2(s)}{\psi_1(s)\Big[1-\dfrac{z}{\psi_1(s)}\Big]}=\sum_{j=0}^{+\infty}\gamma_j(s)\,z^j$$

where

$$\gamma_0(s)=-\frac{\psi_2(s)}{\psi_1(s)},\qquad \gamma_j(s)=\frac{\psi_1(s)-\psi_2(s)}{\psi_1(s)}\,\Big[\frac{1}{\psi_1(s)}\Big]^j\quad(j=1,2,\ldots).$$

Recalling the power series raised to powers (cf. [19], p. 17, n. 0.314) one has

$$\Big(\frac{z-\psi_2(s)}{\psi_1(s)-z}\Big)^{k+1}=\Big(\sum_{j=0}^{+\infty}\gamma_j(s)\,z^j\Big)^{k+1}=\sum_{j=0}^{+\infty}C_{j,k+1}(s)\,z^j\qquad(k=0,1,\ldots), \tag{B.1}$$

with

$$\begin{aligned}&C_{0,k+1}(s)=[\gamma_0(s)]^{k+1},\\&C_{n,k+1}(s)=\frac{1}{n\,\gamma_0(s)}\sum_{j=1}^{n}[j(k+1)-n+j]\,\gamma_j(s)\,C_{n-j,k+1}(s)\quad(n=1,2,\ldots).\end{aligned}$$

from which, proceeding by induction on n, one can prove that (29) hold.

Appendix C. Derivation of (45)

In this appendix, starting from (22) we prove (45). From (22) one obtains:

$$\begin{aligned}\frac{dH(0,s)}{ds} &= -\frac{\lambda}{\mu}\,\frac{d}{ds}\Big[\frac{\psi_1(s)-\psi_2(s)}{\psi_1(s)-1}\Big]\\ &\quad -\frac{\lambda}{\mu}\,\frac{d}{ds}\Big\{\frac{[\psi_2(s)]^2}{\psi_1(s)[\alpha(s)+3]}\Big\}F\Big(1,\alpha(s)+3;\alpha(s)+4;\frac{\psi_2(s)}{\psi_1(s)}\Big)\\ &\quad -\frac{\lambda}{\mu}\,\frac{[\psi_2(s)]^2}{\psi_1(s)[\alpha(s)+3]}\,\frac{d}{ds}F\Big(1,\alpha(s)+3;\alpha(s)+4;\frac{\psi_2(s)}{\psi_1(s)}\Big). \qquad \text{(C.1)}\end{aligned}$$

Recalling that $\varrho<1$, from (12) one has $\psi_1(0)=\mu/\lambda$, $\psi_2(0)=1$, $\alpha(0)=-1$,

$$\frac{d\psi_1(s)}{ds}\Big|_{s=0}=\frac{1}{\lambda(1-\varrho)},\qquad \frac{d\psi_2(s)}{ds}\Big|_{s=0}=-\frac{1}{\mu(1-\varrho)},\qquad \frac{d\alpha(s)}{ds}\Big|_{s=0}=-\frac{1}{\mu(1-\varrho)^2}.$$

Therefore, for $\varrho<1$ we have

$$\begin{aligned}\lim_{s\to 0}\frac{d}{ds}\Big[\frac{\psi_1(s)-\psi_2(s)}{\psi_1(s)-1}\Big] &= \frac{\varrho}{\mu(1-\varrho)^2},\\ \lim_{s\to 0}\frac{d}{ds}\Big\{\frac{[\psi_2(s)]^2}{\psi_1(s)[\alpha(s)+3]}\Big\} &= \frac{\varrho(6\varrho-5)}{4\mu(1-\varrho)^2}.\end{aligned}$$

Furthermore, recalling (15), one has:

$$F\Big(1,\alpha(s)+3;\alpha(s)+4;\frac{\psi_2(s)}{\psi_1(s)}\Big)=\sum_{k=0}^{+\infty}\frac{\alpha(s)+3}{\alpha(s)+3+k}\Big[\frac{\psi_2(s)}{\psi_1(s)}\Big]^k,$$

so that

$$\begin{aligned}&\lim_{s\to 0}F\Big(1,\alpha(s)+3;\alpha(s)+4;\frac{\psi_2(s)}{\psi_1(s)}\Big)=F(1,2,3;\varrho)=-\frac{2}{\varrho^2}\,[\varrho+\ln(1-\varrho)],\\ &\lim_{s\to 0}\frac{d}{ds}F\Big(1,\alpha(s)+3;\alpha(s)+4;\frac{\psi_2(s)}{\psi_1(s)}\Big)=\frac{1}{\lambda\varrho(1-\varrho)^2}\Big[\varrho\,(4\varrho-9)\\ &\qquad +(8\varrho-7)\ln(1-\varrho)+2\,\mathrm{PolyLog}(2,\varrho)\Big].\end{aligned}$$

Finally, taking the limit in (C.1) as $s\to 0$, one obtains (45).

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