Reasoning about Consensus when Opinions Diffuse through Majority Dynamics

Vincenzo Auletta[1], **Diodato Ferraioli**[1], **Gianluigi Greco**[2],
[1] University of Salerno, Italy
[2] University of Calabria, Italy
auletta@unisa.it, dferraioli@unisa.it, gianluigi.greco@unical.it

Abstract

Opinion diffusion is studied on social graphs where agents hold binary opinions and where social pressure leads them to conform to the opinion manifested by the majority of their neighbors. Within this setting, questions related to whether a minority/majority can spread the opinion it supports to all the other agents are considered. It is shown that, no matter of the underlying graph, there is always a group formed by a half of the agents that can annihilate the opposite opinion. Instead, the influence power of minorities depends on certain features of the given graph, which are NP-hard to be identified. Deciding whether the two opinions can coexist in some stable configuration is NP-hard, too.

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1 Introduction

Consider the following prototypical scenario. The members of a department are organizing a social dinner, and they have to decide whether to go to a restaurant or to a pizzeria. Initially, each of them holds an opinion on her ideal choice. They are neither strategic [Osborne and Rubinstein, 1994; Gottlob *et al.*, 2005] nor inclined to use voting rules or other mechanisms to find an agreement [Brandt *et al.*, 2016; Endriss, 2017]. Rather, we know that, at a certain point, they will exchange their viewpoints and each of them will be affected by a social pressure leading to adapt her opinion to the one manifested by the majority of her friends. So, we ask: Would be they capable to reach a *consensus* for some/all profiles of their initial opinions? Can a *minority* have enough social "power" to influence all other agents? Is it any easier for a *majority* to guide convergence towards consensus?

Our goal is to analyze the above kinds of questions under the lens of algorithm design and computational complexity, by focusing on the setting where agents (e.g., the members of the department) hold binary opinions (e.g., restaurant vs pizzeria), where at each time instant precisely one agent can change her opinion (*asynchronous model*), and where social relationships are encoded as a graph. Indeed, while the study of opinion diffusion has been attracting much attention in the literature (e.g., [Grandi *et al.*, 2015; Cholvy, 2016; Brill *et al.*, 2016; Grandi *et al.*, 2017; Bredereck and Elkind, 2017; Acar *et al.*, 2017]), our knowledge about problems related to consensus—in the setting of interest—was basically confined so far to the results by Mossel *et al.* [2014] and Feldman *et al.* [2014], roughly stating that, in expander graphs, a sufficiently large majority will always influence all other agents.

In the paper we fill this gap. In more details, we first analyze in Section 3 the question of whether there exists some profile for which a fraction α of the agents that already agree on some given opinion can spread that opinion to all the remaining agents. We show that the answer is positive *on any social graph* and for any given value of $\alpha \geq 1/2$. In particular, a (rather elaborated) polynomial-time algorithm computing the desired initial configurations is exhibited. The result is not only the main technical achievement of the paper, but it is also conceptually interesting and somehow surprising. Indeed, one might be inclined to believe that the topology of the social graphs (in particular, for values of α close to $1/2$) should play some role w.r.t. the possibility of reaching consensus on some given opinion. To the contrary, our result says that there always exists a majority formed by a half of the agents that can annihilate the opposite opinion.

Moreover, we evidence in Section 4 that the value $\alpha = 1/2$ defines a sharp boundary for the consensus problem. Indeed, we show that a minority (i.e., $\alpha < 1/2$) that can spread its opinion to all the agents exists only in certain graphs. In fact, a result of this kind was holds for the problem of assessing the existence of a minority that will become a majority [Auletta *et al.*, 2015; 2017a; 2017b]. However, in their setting the graphs enjoying the desired property admit a (computationally) simple characterization, while in our setting their characterization is NP-hard. Furthermore, we show that even just deciding whether there is a minority that can double the number of agents with its opinion is NP-hard.

Finally, we address the question of identifying those graphs in which consensus is the only possible stable outcome. In fact, assessing whether the two opinions can/cannot coexist in some stable configuration has been left as an open research issue in [Bredereck and Elkind, 2017]. Moreover, this problem has a natural graph-theoretic interpretation in terms of the existence of *non-trivial cuts* that are *locally stable*, whose complexity was, so far, open in the literature [Ferraioli and Ventre, 2017]. In Section 5 we solve this problem, by showing that checking whether a cut of this kind exists is NP-hard.

The exposition of the above results is self-contained,[1] with Section 2 presenting basic notions and notations.

2 Formal Framework for Opinion Diffusion

Let $G = (N, E)$ be an undirected graph encoding the interactions of a set N of agents. We assume that G is connected—otherwise apply our results on each connected component.

A *configuration* for G is a function $\mathsf{c} : N \to \{1, 0\}$; its intended meaning is that agent $x \in N$ with $\mathsf{c}(x) = 1$ (respectively, $\mathsf{c}(x) = 0$) holds opinion *black* 1 (respectively, *white* 0). For any set $S \subseteq N$ of agents, we define $S_{1/\mathsf{c}}$ (shortly S_1, if the configuration c is clearly understood) as the set of all agents with opinion 1. The set $S_{0/\mathsf{c}}$ (shortly S_0) is defined analogously. For each agent $x \in N$, the set $\{y \mid \{x, y\} \in E\}$ of her neighbors is denoted by $\delta(x)$. Agent $x \in N$ is *stable* in c if her opinion agrees with the opinion held by a (non-strict) majority of her neighbors; that is, either $|\delta(x)_0| \leq |\delta(x)_1|$ and $\mathsf{c}(x) = 1$; or $|\delta(x)_1| \leq |\delta(x)_0|$ and $\mathsf{c}(x) = 0$. A configuration c is *stable* if all agents in N are stable.

We consider an asynchronous model where, at each time instant, precisely one agent which is not stable changes her opinion. Accordingly, a *dynamics* for G is modeled as a sequence of configurations $\mathsf{c}_0, ..., \mathsf{c}_k$ such that c_{h+1}, for each $h \in \{0, ..., k-1\}$, is obtained from c_h by flipping the opinion of an agent that is not stable in c_h. The dynamics will be also shortly denoted as $\mathsf{c}_0 \rightsquigarrow \mathsf{c}_k$, whenever we are just interested in the initial and final configurations only.

Let $\mathsf{op} \in \{1, 0\}$ be an opinion. Given an initial configuration c, let us denote by $\max_{\mathsf{op}}(\mathsf{c})$ the configuration resulting from the dynamics $\mathsf{c} \rightsquigarrow \max_{\mathsf{op}}(\mathsf{c})$ defined as follows: First, as long as there is an agent $x \in N$ that is not stable and whose opinion is not op, then flip her opinion. Second, as long as there is an agent $x \in N$ that is not stable and whose opinion is op, then flip her opinion. It is known that $\max_{\mathsf{op}}(\mathsf{c})$ is stable and that $|N_{\mathsf{op}/\max_{\mathsf{op}}(\mathsf{c})}| \geq |N_{\mathsf{op}/\mathsf{c}'}|$ for each dynamics $\mathsf{c} \rightsquigarrow \mathsf{c}'$ with $\mathsf{c}' \neq \max_{\mathsf{op}}(\mathsf{c})$ [Bredereck and Elkind, 2017].

3 From Majority to Consensus

Let $\mathsf{op} \in \{1, 0\}$ be an opinion. Define $\forall\mathsf{op}$ as the "*consensus*" configuration where all agents hold opinion op, and consider the following problem defined on graphs and parameterized w.r.t. a fixed rational number α such that $0 < \alpha < 1$:

CONSENSUS[α]: Given an undirected graph $G = (N, E)$, compute a configuration c for G such that *(i)* $|N_{\mathsf{op}/\mathsf{c}}| \leq \lceil \alpha |N| \rceil$ and *(ii)* $\max_{\mathsf{op}}(\mathsf{c}) = \forall\mathsf{op}$, or check that there is no configuration enjoying *(i)* and *(ii)*.

In words, CONSENSUS[α] asks whether consensus on op can be reached in G from a configuration where the fraction of agents that initially holds opinion op is at most α. In the following, for simplicity, we assume that G is given and we omit it in our notation; moreover (and w.l.o.g), we fix $\mathsf{op} = 1$.

Our main result is that, for each undirected graph G, whenever the fraction α covers a majority of the agents, i.e., $\alpha \geq 1/2$, a configuration enjoying *(i)* and *(ii)* always exist and can be computed in polynomial-time. In particular, we propose (in Section 3.4) a polynomial time algorithm to solve CONSENSUS[1/2]. The algorithm explores the space of *binary partitions* $\mathcal{P}$ of the agent set N, that are pairs $\mathcal{P} = (A, B)$ where $A, B \subseteq N$ are such that $A \cap B = \emptyset$ and $A \cup B = N$. In the following, the fact that a set $X \subseteq N$ belongs to $\{A, B\}$ is shortly denoted by $X \in \mathcal{P}$; moreover, we define $\bar{X} = N \setminus X$.

[1] Details available at `https://tinyurl.com/ycj3lh3c`.

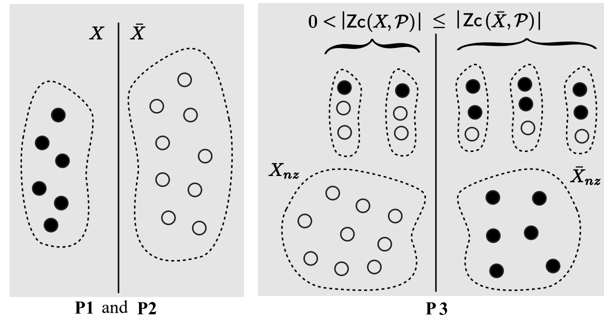

Figure 1: Configurations leading to consensus on "nice" partitions.

Our approach is founded on the observation (see Section 3.1) that there are certain "nice" partitions from which solutions to CONSENSUS[1/2] can be easily computed. However, it is not obvious how to single out these partitions. In fact, we propose an algorithm that proceeds as follows: it starts from an arbitrary *balanced* partition $\mathcal{P} = (A, B)$, i.e., such that $-1 \leq |A| - |B| \leq 1$, and incrementally modifies the current partition via update procedures (described in Section 3.2 and Section 3.3) until a "nice" partition is found.

3.1 Nice Partitions

We next introduce three kinds of "nice" partitions from which solutions to CONSENSUS[1/2] can be easily computed.

To this aim, let $\mathcal{P}$ be a partition of N. For any agent $x \in N$, let $\mathcal{P}_x$ (resp., $\bar{\mathcal{P}}_x$) denote the set of $\mathcal{P}$ to which x belongs (resp., does not belong), and let us define the *utility* of x in $\mathcal{P}$ as the value $\mathsf{u}(x, \mathcal{P}) = |\delta(x) \cap \mathcal{P}_x| - |\delta(x) \cap \bar{\mathcal{P}}_x|$. Moreover, for any two disjoint sets of agents A' and B', not necessarily forming a partition, we define $E(A', B')$ as the set of edges $e \in E$ such that $e \cap A' \neq \emptyset$ and $e \cap B' \neq \emptyset$.

Observe that, whenever each "side" of $\mathcal{P}$ consists of all the agents that have the same specific opinion, then $\mathsf{u}(x^*, \mathcal{P}) > 0$ holds if, and only if, agent x^* is stable (with her opinion associated) with the side $\mathcal{P}_{x^*}$. The first kind of "nice" partition we consider assumes the existence of a stable agent x^* and, in addition, it relates the utility of all pairs of agents x and y from "opposite" sides with the cardinality of $E(\{x\}, \{y\})$.

Definition 1. $\mathcal{P}$ is a **P1**-*partition* if there is a set $X \in \mathcal{P}$ such that $|X| \leq \lceil \frac{N}{2} \rceil$ and the following conditions hold:

- $\mathsf{u}(x, \mathcal{P}) + \mathsf{u}(y, \mathcal{P}) \leq -2|E(\{x\}, \{y\})|$ holds for each pair of agents $x \in X$ and $y \in \bar{X}$;
- there is $x^\star \in X$ with $\mathsf{u}(x^\star, \mathcal{P}) > 0$. □

To define the second kind of "nice" partition, for each side X of a partition $\mathcal{P}$, we denote by $G[X]$ the subgraph of G induced by X. Let $\mathsf{Zc}(X, \mathcal{P})$ be the set of all connected components of $G[X]$ such that $\mathsf{u}(y, \mathcal{P}) = 0$ for each component $C \in \mathsf{Zc}(X, \mathcal{P})$ and each agent y of C. That is, $\mathsf{Zc}(X, \mathcal{P})$ is the set of connected components of $G[X]$ such that all agents within these components have the same number of neighbors within the component and on the other side of the partition.

Definition 2. $\mathcal{P}$ is a **P2**-*partition* if there is a set $X \in \mathcal{P}$ such that $|X| \leq \lceil \frac{N}{2} \rceil$ and the following hold:

- every $x \in \bar{X}$ such that $\mathsf{u}(x, \mathcal{P}) = k > 0$ has at least $\lfloor (k+2)/2 \rfloor$ neighbors $y \in \bar{X}$ with $\mathsf{u}(y, \mathcal{P}) < 0$;
- $\mathsf{Zc}(\bar{X}, \mathcal{P}) = \emptyset$. □

Let $\bar{\mathsf{c}}(\mathcal{P})$ be the configuration that, for a **P1** or a **P2** partition $\mathcal{P}$, assigns opinion 1 (resp., 0) to all the agents in X (resp., $\bar{X}$). We show that $\bar{\mathsf{c}}(\mathcal{P})$ leads to consensus (see Figure 1).

Lemma 1. *If $\mathcal{P}$ is a* **P1***-partition, then* $\max_1(\bar{\mathsf{c}}(\mathcal{P})) = \forall 1$.

Proof. For each $y \in \bar{X}$, it holds that $\mathsf{u}(y, \mathcal{P}) \leq -\mathsf{u}(x^\star, \mathcal{P}) - 2|E(\{x^\star\}, \{y\})| < 0$, where $x^\star$ and X are as in Definition 1. Hence, in the configuration $\bar{\mathsf{c}}(\mathcal{P})$ all agents in $\bar{X}$ are not stable and they will change their opinion. Indeed, note that, while the dynamics is running, $|\delta(y)_0| - |\delta(y)_1| \leq \mathsf{u}(y, \mathcal{P})$ holds, for each $y \in \bar{X}$ that has not yet changed her opinion. □

Lemma 2. *If $\mathcal{P}$ is a* **P2***-partition, then* $\max_1(\bar{\mathsf{c}}(\mathcal{P})) = \forall 1$.

Proof Sketch. Let $X \in \mathcal{P}$ be as in Definition 2. Note that agents in $S = \{y \in \bar{X} \mid \mathsf{u}(y, \mathcal{P}) < 0\}$ are not stable in $\bar{\mathsf{c}}(\mathcal{P})$ and they flip their opinion. Let c^* be the configuration reached from $\bar{\mathsf{c}}(\mathcal{P})$ after all agents in S flipped their opinion, and let $\mathcal{P}^\star = (A^\star, B^\star)$, with $A^\star = X \cup S$ and $B^\star = \bar{X} \setminus S$.

Now, observe that $\mathsf{u}(x, \mathcal{P}^\star) \leq \mathsf{u}(x, \mathcal{P})$ for each $x \in B^\star$. Moreover, if $\mathsf{u}(x, \mathcal{P}) > 0$ or $\mathsf{u}(x, \mathcal{P}) = 0$, then x has neighbors in S, and $\mathsf{u}(x, \mathcal{P}^\star) < 0$. Hence, $\mathsf{Zc}(B^\star, \mathcal{P}^\star) = \emptyset$ and $\mathsf{u}(x, \mathcal{P}^\star) \leq 0$ for each $x \in B^\star$. Thus, for each $y_0 \in B^\star$ there is a path $y_0, \ldots, y_h$ in $B^\star$, with $h \geq 0$, where $\mathsf{u}(y_h, \mathcal{P}^\star) < 0$ and $\mathsf{u}(y_i, \mathcal{P}^\star) = 0$ for $i = 0, \ldots, h-1$. Such agents flip their opinion if selected in the reverse order $y_h, \ldots, y_0$. □

We can also define a third kind of "nice" partition. For any $x \in N$, let $\mathsf{toZc}(x, \mathcal{P})$ be the set of all components $C \in \mathsf{Zc}(\bar{\mathcal{P}}_x, \mathcal{P}_x)$ containing an agent adjacent to x.

Definition 3. $\mathcal{P}$ is a **P3**-*partition* if it is balanced and there is a set $X \in \mathcal{P}$ such that the following hold:

- $\mathsf{u}(x, \mathcal{P}) + \mathsf{u}(y, \mathcal{P}) \leq -2|E(\{x\}, \{y\})|$ holds for each pair of agents $x \in X$ and $y \in \bar{X}$;
- $\mathsf{u}(x, \mathcal{P}) \leq 0$ holds for each $x \in N$;
- $|\mathsf{Zc}(\bar{X}, \mathcal{P})| \geq |\mathsf{Zc}(X, \mathcal{P})| > 0$;
- for each $D \in \mathsf{Zc}(\bar{X}, \mathcal{P})$ there is $x \in X$ with $D \in \mathsf{toZc}(x, \mathcal{P})$ and $\mathsf{u}(x, \mathcal{P}) < -2|\mathsf{toZc}(x, \mathcal{P})|$. □

Assume that $\mathcal{P}$ is a **P3**-partition. In this case, we cannot reach consensus by starting from $\bar{\mathsf{c}}(\mathcal{P})$ but we need a more complex construction. Let X_{nz} (resp., $\bar{X}_{nz}$) denote the set of agents that do not belong to any component in $\mathsf{Zc}(X, \mathcal{P})$ (resp., $\mathsf{Zc}(\bar{X}, \mathcal{P})$). Consider the configuration $\tilde{\mathsf{c}}(\mathcal{P})$ shown in Figure 1 and built as follows. All agents in $\bar{X}_{nz}$ (resp., X_{nz}) hold opinion 1 (resp., 0). Moreover, for each component $C \in \mathsf{Zc}(X, \mathcal{P})$, precisely one arbitrarily chosen agent $x_C \in C$ holds opinion 1 and all the remaining agents of C hold opinion 0. For each component $D \in \mathsf{Zc}(\bar{X}, \mathcal{P})$, precisely one agent $y_D \in D$ holds opinion 0 and all the remaining agents of D hold opinion 1, where $y_D \in \bar{X}$ is an agent connected to some $x_D \in X$ such that $\mathsf{u}(x_D, \mathcal{P}) < -2|\mathsf{toZc}(x_D, \mathcal{P})|$. Note that, by definition of **P3**-partition, y_D is well-defined.

Lemma 3. *If $\mathcal{P}$ is a* **P3***-partition, then* $\max_1(\tilde{\mathsf{c}}(\mathcal{P})) = \forall 1$.

Proof Sketch. Let $X \in \mathcal{P}$ be as in Definition 3. Let x be an agent in some $C \in \mathsf{Zc}(X, \mathcal{P})$. Note that all neighbors of x are either confined in the component C or they occur in $\bar{X}_{nz}$, since (by Definition 3) we have $\mathsf{u}(y, \mathcal{P}) < 0$ if y is adjacent to x. Now, in $\tilde{\mathsf{c}}(\mathcal{P})$ if $x \neq x_C$, then x has $|\delta(x) \cap \bar{X}| + 1$ neighbors with opinion 1 and $|\delta(x) \cap X| - 1$ neighbors with opinion 0. Since $\mathsf{u}(x, \mathcal{P}) = 0$, $|\delta(x) \cap \bar{X}| + 1 > |\delta(x) \cap X| - 1$. Hence, x can flip her opinion. Moreover, for each $D \in \mathsf{Zc}(\bar{X}, \mathcal{P})$, consider the agent $x_D \in X$. Note that x_D has in $\tilde{\mathsf{c}}(\mathcal{P})$ at least $|\delta(x) \cap \bar{X}| - |\mathsf{toZc}(x_D, \mathcal{P})|$ neighbors having opinion 1 and at most $|\delta(x) \cap X)| + |\mathsf{toZc}(x_D, \mathcal{P})|$ neighbors with opinion 0. Therefore, $\mathsf{u}(x_D, \mathcal{P}) < -2|\mathsf{toZc}(x_D, \mathcal{P})|$ implies that $|\delta(x) \cap \bar{X}| - |\mathsf{toZc}(x_D, \mathcal{P})| > |\delta(x) \cap X| + |\mathsf{toZc}(x_D, \mathcal{P})|$, and x_D can flip her opinion.

Now, let c^* be the profile derived from $\tilde{\mathsf{c}}(\mathcal{P})$ after all agents $x \neq x_C$ occurring in $C \in \mathsf{Zc}(X, \mathcal{P})$ and all agents x_D with $D \in \mathsf{Zc}(\bar{X}, \mathcal{P})$ have changed their opinion. For each $D \in \mathsf{Zc}(\bar{X}, \mathcal{P})$, y_D can change her opinion, because y_D has at least $|\delta(y_D) \cap \bar{X}| + 1$ neighbors with opinion 1 in c^* and at most $|\delta(y_D) \cap X| - 1$ neighbors with opinion 0 (and we have $\mathsf{u}(y_D, \mathcal{P}) = 0$). Finally, consider the configuration derived from c^* after all agents y_D with $D \in \mathsf{Zc}(\bar{X}, \mathcal{P})$ have changed their opinion. In this configuration, the only agents with opinion 0 are agents x in X_{nz} with $x \neq x_D$. For these agents, the fact that they can change their opinion can be shown with the arguments used in the proof of Lemma 2. □

These lemmas imply that if $\mathcal{P}$ is a **P1** or **P2** (resp., **P3**) partition, then $|N_{1/\bar{\mathsf{c}}(\mathcal{P})}| \leq \lceil 1/2 |N| \rceil$ (resp., $|N_{1/\tilde{\mathsf{c}}(\mathcal{P})}| \leq \lceil 1/2 |N| \rceil$) and consensus is reached from these configurations. This is formalized in the following corollary. Hereinafter, we say that a partition is *nice* (for G) if it is a **P1**, **P2** or **P3** partition.

Corollary 1. *Assume that, for any graph G, a nice partition exists and can be computed in polynomial time. Then,* CONSENSUS[1/2] *can be solved in polynomial time, too.*

3.2 Step 1: Remove Pivotal Agents

In the light of Corollary 1, our goal is to design an algorithm that, given as input a graph G, is able to compute (in polynomial time) a nice partition of G. To this end, a crucial role is played by the identification of certain agents, called *pivotal*.

Definition 4. An agent $x \in N$ is *pivotal* in the partition $\mathcal{P}$ if there is an agent $y \in \bar{\mathcal{P}}_x$, called a *witness* of x, such that $\mathsf{u}(x, \mathcal{P}) + \mathsf{u}(y, \mathcal{P}) > -2|E(\{x\}, \{y\})|$. □

Intuitively, pivotal agents are obstructions to the current partition $\mathcal{P}$ to being a **P1** and **P3** partitions. To remove these obstructions we exploit the property that, if we swap a pivotal agent with her witness, then the number of edges crossing the two sides of the partition increases. Hence, by successive swaps, we eventually remove all pivotal agents. To formalize this property, if $x \in N$ is pivotal and y is its witness, define $\text{SWAP}(x, \mathcal{P})$ as the function returning the partition $\mathcal{P}'$ such that $\mathcal{P}'_x = \bar{\mathcal{P}}_x \cup \{x\} \setminus \{y\}$. When a pivotal agent has multiple witnesses, the one to be swapped can be chosen arbitrarily.

Lemma 4. *Let x be a pivotal agent in the partition $\mathcal{P}$, and let $\mathcal{P}' = \text{SWAP}(x, \mathcal{P})$. Then, $|E(\mathcal{P}'_x, \bar{\mathcal{P}}'_x)| > |E(\mathcal{P}_x, \bar{\mathcal{P}}_x)|$. Moreover, if $\mathcal{P}$ is balanced, then $\mathcal{P}'$ is balanced, too.*

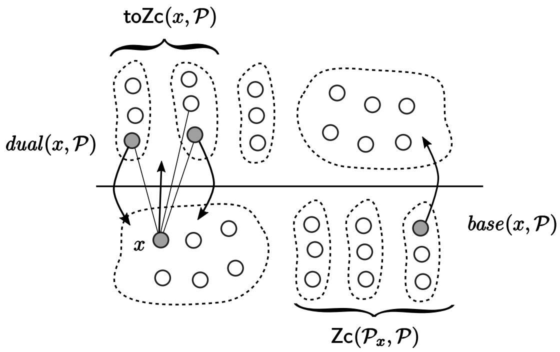

Figure 2: Illustration of the function REMOVEC1.

Proof. Consider the witness y of x. Let $\Delta = |E(\mathcal{P}_x \setminus \{x\}, \bar{\mathcal{P}}_x \setminus \{y\})|$ and note that: $|E(\mathcal{P}_x, \bar{\mathcal{P}}_x)| = \Delta + |\delta(x) \cap \bar{\mathcal{P}}_x| + |\delta(y) \cap \mathcal{P}_x| - |E(\{x\},\{y\})|$ and $|E(\mathcal{P}'_x, \bar{\mathcal{P}}'_x)| = \Delta + |\delta(x) \cap \mathcal{P}_x| + |\delta(y) \cap \bar{\mathcal{P}}_x| + |E(\{x\},\{y\})|$. Hence, $|E(\mathcal{P}'_x, \bar{\mathcal{P}}'_x)| - |E(\mathcal{P}_x, \bar{\mathcal{P}}_x)|$ coincides with the value $\mathsf{u}(x,\mathcal{P}) + \mathsf{u}(y,\mathcal{P}) + 2|E(\{x\},\{y\})| > 0$. To conclude, note that balancedness of $\mathcal{P}'$ is immediate whenever $\mathcal{P}$ is balanced. □

However, removing all pivotal agents (by swapping them with their witnesses till none exists) is not yet sufficient to end up with a nice partition. Indeed, further obstructions can exist. In particular, the following result straightforwardly derives by inspecting Definitions 1–4.

Hereinafter, let $\#\mathsf{Zc}(\mathcal{P}) = |\cup_{X\in\mathcal{P}} \mathsf{Zc}(X,\mathcal{P})|$.

Fact 1. *Let $\mathcal{P}$ be a balanced partition with no pivotal agents and which is not nice. Then,*

(i) $\mathsf{u}(x,\mathcal{P}) \le 0$*, for each* $x \in N$*;*
(ii) for each $X \in \mathcal{P}$*, it holds that* $|\mathsf{Zc}(X,\mathcal{P})| > 0$*;*
(iii) for each $X \in \mathcal{P}$ *with* $|\mathsf{Zc}(X,\mathcal{P})| \ge \#\mathsf{Zc}(\mathcal{P})/2$*, there is* $D \in \mathsf{Zc}(X,\mathcal{P})$ *such that* $\mathsf{u}(x,\mathcal{P}) \ge -2|\mathsf{toZc}(x,\mathcal{P})|$*, for each* $x \in X$ *with* $D \in \mathsf{toZc}(x,\mathcal{P})$*.* □

A component D satisfying condition *(iii)* is termed *critical*, and the set of all critical components is denoted by $\mathsf{cr}(\mathcal{P})$. Fact 1 suggests that, to obtain a nice partition, we must modify a given balanced partition by removing all critical components while keeping it balanced and with no pivotal agents.

3.3 Step 2: Remove Critical Components

Let $\mathcal{P}$ be a balanced partition without pivotal agents and which is not nice. An agent $x \in N$ is *critical* if there is a component in $\mathsf{toZc}(x,\mathcal{P})$ that is critical. We distinguish two kinds of critical agents as follows.

Definition 5. *A critical agent $x \in N$ is of kind*

C1: *if* $|\mathsf{toZc}(x,\mathcal{P})| > 1$ *and* $\mathsf{u}(x,\mathcal{P}) \ge -2|\mathsf{toZc}(x,\mathcal{P})|$*, or if* $|\mathsf{toZc}(x,\mathcal{P})| = 1$ *and* $\mathsf{u}(x,\mathcal{P}) > -2|\mathsf{toZc}(x,\mathcal{P})|$*;*
C2: *if* $|\mathsf{toZc}(x,\mathcal{P})| = 1$ *and* $\mathsf{u}(x,\mathcal{P}) = -2|\mathsf{toZc}(x,\mathcal{P})|$*.* □

We remove critical components involving agents x of kind **C1** by swapping certain agents between the two sides of $\mathcal{P}$—see Figure 2. Formally, let us define REMOVEC1$(x,\mathcal{P})$ as the function returning the partition $\mathcal{P}'$ such that $\mathcal{P}'_x = \bar{\mathcal{P}}_x \cup base(x,\mathcal{P}) \cup \{x\} \setminus dual(x,\mathcal{P})$, where $dual(x,\mathcal{P})$ is the set including an arbitrarily chosen agent y of C connected to x, for each $C \in \mathsf{toZc}(x,\mathcal{P})$, and where $base(x,\mathcal{P})$ is a set containing up to $|\mathsf{toZc}(x,\mathcal{P})| - 1$ nodes, each one being chosen from a distinct component of $\mathsf{Zc}(\mathcal{P}_x,\mathcal{P})$.

function REMOVEC2: $(x,\mathcal{P}) \mapsto \mathcal{P}'$
  let C be the unique component in $\mathsf{toZc}(x,\mathcal{P})$
  let D be an arbitrary component in $\mathsf{Zc}(\mathcal{P}_x,\mathcal{P})$
  let $x_0,\ldots,x_n$ be a path s.t. $x_0 \in C$, $x_n \in D$, and $x_1 = x$
  $\mathcal{P}^0 \leftarrow \mathcal{P}$ and $i \leftarrow 0$
  repeat
    if $x_{i+1} \in \mathcal{P}^i_{x_i}$ **then** $\mathcal{P}^{i+1} \leftarrow \mathcal{P}^i$
    else let $\mathcal{P}^{i+1}$ be such that $\mathcal{P}^{i+1}_{x_i} = \bar{\mathcal{P}}^i_{x_i} \cup \{x_i\} \setminus \{x_{i+1}\}$
    $i \leftarrow i+1$
  until *either* $\mathcal{P}^i$ is nice,
    or $\mathcal{P}^i$ has a pivotal or critical agent of kind **C1**,
    or $\#\mathsf{Zc}(\mathcal{P}^i) < \#\mathsf{Zc}(\mathcal{P}^0)$,
    or $|\mathsf{cr}(\mathcal{P}^i)| < |\mathsf{cr}(\mathcal{P}^0)|$
  return $\mathcal{P}^i$
end function

Figure 3: Processing a critical agent of kind **C2**.

The crucial observation is now that the application of REMOVEC1 can be related with the monotonic behavior of certain values, analogously to what we did in Lemma 4 (for pivotal agents and considering the number of crossing edges).

Lemma 5. *Let $\mathcal{P}$ be a balanced partition with no pivotal agents and which is not nice. Let $x \in N$ be a critical agent in $\mathcal{P}$ of kind* **C1**. *Then, either* $\mathcal{P}' = $ REMOVEC1$(x,\mathcal{P})$ *is nice, or it is balanced and one of the following conditions holds:*

- $|E(\mathcal{P}'_x, \bar{\mathcal{P}}'_x)| > |E(\mathcal{P}_x, \bar{\mathcal{P}}_x)|$*;*
- $|E(\mathcal{P}'_x, \bar{\mathcal{P}}'_x)| = |E(\mathcal{P}_x, \bar{\mathcal{P}}_x)|$ *and* $\#\mathsf{Zc}(\mathcal{P}') < \#\mathsf{Zc}(\mathcal{P})$*.*

Proof Sketch. For each $z \in N$, if $y \in \bar{\mathcal{P}}_z$ is connected to z, then $\mathsf{u}(z,\mathcal{P}) + \mathsf{u}(y,\mathcal{P}) \le -2|E(\{z\},\{y\})| = -2$. Since the utility of each agent in $base(x,\mathcal{P}) \cup dual(x,\mathcal{P})$ equals to 0, we get $E(base(x,\mathcal{P}), dual(x,\mathcal{P}))=\emptyset$, $\mathsf{u}(x,\mathcal{P}) \ne 0$, agents in $base(x,\mathcal{P})$ are not connected with each other, as they belong to different components, and, by the definition of $\mathsf{Zc}(\mathcal{P}_x,\mathcal{P})$, x is not connected to any of the agents in $base(x,\mathcal{P})$. By these observations and by using arguments similar to those in Lemma 4, we can state that $|E(\mathcal{P}'_x, \bar{\mathcal{P}}'_x)| - |E(\mathcal{P}_x, \bar{\mathcal{P}}_x)| = \mathsf{u}(x,\mathcal{P}) + \sum_{y\in dual(x,\mathcal{P})} \mathsf{u}(y,\mathcal{P}) + 2|\mathsf{toZc}(x,\mathcal{P})|$. In particular, since $\mathsf{u}(y,\mathcal{P}) = 0$ holds for each $y \in dual(x,\mathcal{P})$, we derive $|E(\mathcal{P}'_x, \bar{\mathcal{P}}'_x)| - |E(\mathcal{P}_x, \bar{\mathcal{P}}_x)| = \mathsf{u}(x,\mathcal{P}) + 2|\mathsf{toZc}(x,\mathcal{P})|$.

Assume now that $\mathcal{P}'$ is a balanced but not nice. If $\mathsf{u}(x,\mathcal{P}) > -2|\mathsf{toZc}(x,\mathcal{P})|$, then $|E(\mathcal{P}'_x, \bar{\mathcal{P}}'_x)| > |E(\mathcal{P}_x, \bar{\mathcal{P}}_x)|$. Instead, if $\mathsf{u}(x,\mathcal{P}) = -2|\mathsf{toZc}(x,\mathcal{P})|$, then $|E(\mathcal{P}'_x, \bar{\mathcal{P}}'_x)| = |E(\mathcal{P}_x, \bar{\mathcal{P}}_x)|$ and $|\mathsf{toZc}(x,\mathcal{P})| > 1$. For the latter case, $|\cup_{X\in\mathcal{P}'} \mathsf{Zc}(X,\mathcal{P}')| < |\cup_{X\in\mathcal{P}} \mathsf{Zc}(X,\mathcal{P})|$ holds. Indeed, moving agents in $base(x,\mathcal{P})$ to the other side destroys $|base(x,\mathcal{P})|$ components in $\mathcal{P}_x$, while creating at most the same number of components in the other side. Moreover, swapping x with agents in $dual(x,\mathcal{P})$ causes replacing $|dual(x,\mathcal{P})| = |\mathsf{toZc}(x,\mathcal{P})| > 1$ components in $\bar{\mathcal{P}}_x$ with at most one component including x.

Suppose now that $\mathcal{P}'$ is not balanced. Then $|\mathsf{Zc}(\mathcal{P}_x,\mathcal{P})| < |\mathsf{toZc}(x,\mathcal{P})| - 1$, so that $\bar{\mathcal{P}}'_x$ has no zero component. Then, $\mathcal{P}'$ is a **P2** partition; indeed, nodes in $\bar{\mathcal{P}}'_x$ with positive utility must be neighbors of agents from $dual(x,\mathcal{P})$. □

Critical agents x of kind **C2** are processed via the function REMOVEC2$(x,\mathcal{P})$ described in Figure 3. Its salient properties are formalized below—as for Lemma 4 and Lemma 5, note the relationship we established between the application of the function and the monotonic behavior of certain values.

Lemma 6. *Let $\mathcal{P} = (A, B)$ be a balanced partition with no pivotal agents, and no critical agents of kind* **C1**, *and which is not nice. Let $x \in N$ be critical of kind* **C2**. *Then, either $\mathcal{P}' =$ REMOVEC2$(x, \mathcal{P})$ is nice, or it is balanced, $|E(\mathcal{P}'_x, \bar{\mathcal{P}}'_x)| = |E(\mathcal{P}_x, \bar{\mathcal{P}}_x)|$ and one of the following conditions holds:*

- $\#\mathsf{Zc}(\mathcal{P}') < \#\mathsf{Zc}(\mathcal{P})$;
- $\#\mathsf{Zc}(\mathcal{P}') = \#\mathsf{Zc}(\mathcal{P})$ *and* $|\mathsf{cr}(\mathcal{P}')| < |\mathsf{cr}(\mathcal{P})|$;
- $\#\mathsf{Zc}(\mathcal{P}') = \#\mathsf{Zc}(\mathcal{P})$, $|\mathsf{cr}(\mathcal{P}')| = |\mathsf{cr}(\mathcal{P})|$ *and $\mathcal{P}'$ has either a pivotal agent or a critical agent of kind* **C1**.

Moreover, REMOVEC2 *converges in at most $|N|$ iterations.*

Proof Sketch. W.l.o.g., assume that $x_0 \in A$ and $x_1 \in B$. Note that $\mathsf{u}(x_0, \mathcal{P}^0) = 0$ while $\mathsf{u}(x_1, \mathcal{P}^0) = -2$, since x_1 is critical of kind **C2**. Hence, swapping x_0 with x_1 does not modify the size of the cut. Moreover, the swap produces a balanced partition $\mathcal{P}^1$. Note also that moving x_0 to B destroys one component of $\mathsf{Zc}(A, \mathcal{P}^0)$ and does not add components to the other side; to this end, in particular, observe that $\mathsf{u}(x_0, \mathcal{P}^1) = -2$. Instead, moving x_1 to A might create one fresh component at most (precisely including x_1). So, $|\cup_{X \in \mathcal{P}^1} \mathsf{Zc}(X, \mathcal{P}^1)| \leq |\cup_{X \in \mathcal{P}'} \mathsf{Zc}(X, \mathcal{P}^0)|$. In fact, existing components are not affected by the swap and we clearly have $|\mathsf{cr}(\mathcal{P}^1)| \leq |\mathsf{cr}(\mathcal{P}^0)|$. Therefore, if no exit condition is satisfied for $i = 0$, then the above relationships are actually equalities, and there are no pivotal agents and no critical agents of kind **C1**. In fact, in this case, the fresh component where x_1 occurs, say D_1, is critical. Now, observe that not in every iteration the current partition is changed. Let i be the first index such that $\mathcal{P}^i \neq \mathcal{P}^1$. Then, x_i still occurs in the critical component D_1 (as we are traversing a path) and x_{i+1} is therefore a critical agent of kind **C2**.

At this point, it is easily seen that the arguments used for the pair x_0, x_1 can be used for x_i, x_{i+1}, too. And the reasoning can be repeated (formally by induction) till some step i^* where one exit condition is satisfied. To conclude, we claim that this step exists. Indeed, $x_n \in D$, where D is a component of $\mathcal{P}^0$. In particular, if no exit condition were satisfied before the n-th step, then D would be not affected by any swap, and thus it would be a component in $\mathcal{P}^{n-1}$. In this case, however, x_{n-1} would belong to a critical component of A. But this is impossible, since x_{n-1} and x_n are neighbors: indeed, in $\mathcal{P}^{n-1}$, we have that $\mathsf{u}(x_{n-1}, \mathcal{P}^{n-1}) + \mathsf{u}(x_n, \mathcal{P}^{n-1}) = 0 > -2E(x_{n-1}, x_n)$, and thus x_{n-1} is a pivotal agent.

Concerning the complexity, note that $|N|$ iterations are required to go through the entire path between x_0 and x_n. □

3.4 Putting It All Together

All ingredients are now in place to illustrate Algorithm 1, which we designed to solve CONSENSUS[1/2].

The algorithms takes in input the graph G and starts by computing a balanced partition. At each iteration, it first removes all pivotal agents—via the repeated application of SWAP—and, subsequently, it removes a critical agent (if any exists). Observe that the application of functions REMOVEC1 and REMOVEC2 is potentially creating nice distributions or pivotal agents. This is why Algorithm 1 removes only one critical agent in the main loop. Eventually, the main loop terminates when the current partition is nice, and then the algorithm returns the configuration associated to that partition as we discussed in Section 3.1.

Algorithm 1 Solving CONSENSUS[1/2] on input $G = (N, E)$

```
compute a balanced partition P of N
done ← false
while not done do
    while there is a pivotal agent x in P do P ← SWAP(x, P)
    if there is a critical agent x of kind C1 in P then
        P ← REMOVEC1(x, P)
    else if there is a critical agent x of kind C2 in P then
        P ← REMOVEC2(x, P)
    end if
    if P is nice then done ← true
end while
if P is a P1 or P2-partition then return c̄(P) else return c̃(P)
```

Theorem 1. *After at most* poly$(|N|)$ *iterations Algorithm 1 returns a solution to* CONSENSUS[1/2].

Proof. For a partition $\mathcal{P} = (A, B)$, let $\Phi_1(\mathcal{P}) = |E(A, B)|$, $\Phi_2(\mathcal{P}) = |\cup_{X \in \mathcal{P}} \mathsf{Zc}(X, \mathcal{P})|$, and $\Phi_3(\mathcal{P}) = |\mathsf{cr}(\mathcal{P})|$. Observe that $\Phi_1(\mathcal{P}) \leq |N|^2$ and $\Phi_3(\mathcal{P}) \leq \Phi_2(\mathcal{P}) \leq |N|/2$. Moreover, by Lemma 4, Lemma 5, and Lemma 6, after every two iterations, at least one of these functions decreases and, if Φ_k decreases, $k = 2, 3$, then $\Phi_{k'}$ for $k' < k$ does not increase. Finally, if these functions cannot be further minimized (after at most $|N|^4/2$ iterations), then a nice partition is obtained.

Correctness then follows from Corollary 1. □

4 From Minority to Consensus

In this section we address the question of whether a consensus on some opinion $\mathsf{op} \in \{1, 0\}$ can be reached starting from a configuration where op is supported by a minority.

We show that the answer to the question depends on the structure of the social graph and, more interestingly, recognizing such graphs is NP-hard. Inspired by similar results in earlier literature [Kempe *et al.*, 2005; Chen, 2008], the following proof exhibits a reduction from the well-known VERTEX COVER problem [Garey and Johnson, 1979].

Theorem 2. *For every* $0 < \alpha < \frac{1}{2}$, CONSENSUS$[\alpha]$ *is* NP-*hard.*

Proof Sketch. Let $G = (N, E)$ be a graph and $k > 0$ be a natural number. Consider the problem of deciding whether G admits a vertex cover of cardinality at most k, that is a set $S \subseteq N$ of nodes with $|S| \leq k$ and such that $\{i, j\} \cap S \neq \emptyset$, for each $\{i, j\} \in E$. This is NP-hard even if $|\{j \mid \{i, j\} \in E| = 3$ for each $i \in N$ [Greenlaw and Petreschi, 1995].

Define $\bar{G} = (\bar{N}, \bar{E})$ as the graph obtained by including into G, for each $i \in N$, 2 fresh nodes each of them connected to i only. Note that G has a vertex cover S with $|S| \leq k$ if, and only if, there exists a dynamics $\mathsf{c} \rightsquigarrow \forall 1$ such that $|\bar{N}_{1/\mathsf{c}}| \leq k$. Since $|\bar{N}| = 3|N|$, this immediately shows that consensus$[\alpha]$ is NP-hard, for every α such that $\lceil \alpha |N| \rceil = \lceil \bar{\alpha} |N| \rceil$ where $\bar{\alpha} = \frac{k}{(3|N|)} \leq \frac{1}{3}$. We now adapt the reduction to show that hardness holds for each $0 < \alpha < 1/2$.

We distinguish two cases. If $\lceil \alpha |N| \rceil > \lceil \bar{\alpha} |N| \rceil$, then we build a graph $\bar{G}^+ = (\bar{N}^+, \bar{E}^+)$ as follows: arbitrarily pick

$u \in \bar{N} \setminus N$, and add to $\bar{G}$ vertices u' and u'' both connected to u and to each other and $m \geq 1$ copies of a gadget consisting of a four nodes clique with each gadget node connected to u'. Note that G has a vertex cover S with $|S| \leq k$ iff there is a dynamics $\mathsf{c} \rightsquigarrow \forall 1$ s.t. $|\bar{N}^+_{1/\mathsf{c}}| \leq k+2m+1$. The result follows by choosing m such that $\lceil \alpha(3|N|+4m+2) \rceil = k + 2m + 1$.

Assume finally that $\lceil \alpha|N| \rceil < \lceil \bar{\alpha}|N| \rceil$. In this case, we build a graph $\bar{G}^- = (\bar{N}^-, \bar{E}^-)$ as follows: arbitrarily pick $u \in \bar{N} \setminus N$, and add to $\bar{G}$ vertex u' connected to u, $m + 3 - (m \bmod 2)$ nodes arranged in a ring and connected with u' and with a new node u''. Then, G has a vertex cover S with $|S| \leq k$ iff there is a dynamics $\mathsf{c} \rightsquigarrow \forall 1$ with $|\bar{N}^-_{1/\mathsf{c}}| \leq k + 3$. The result eventually follows by setting m such that $\lceil \alpha(3|N| + m + 4 + (m \bmod 2)) \rceil = k + 3$. □

An interesting generalization of the above result is that it is still NP-hard to decide whether there is a dynamics leading to a stable configuration where the opinion initially held by a minority of $\alpha|N|$ agents, with $\alpha < \frac{1}{2}$, is spread over (just) $2\alpha|N|$ agents. The precise formulation is given in Theorem 3. Its proof is not an extension of Theorem 2 and, in fact, it is inspired by the reduction in [Auletta *et al.*, 2015, Theorem 4] to the NP-hard problem 2P3N-3SAT [Yoshinaka, 2005].

Theorem 3. *For every $\delta \in (0, 1)$ and for every $\varepsilon \leq \frac{1}{2} - \frac{\delta}{4}$, it is* NP*-hard to decide whether there is a configuration* c *such that $|N_{1/\mathsf{c}}| = \lceil \varepsilon n \rceil$ and $\max_1(\mathsf{c}) \geq (2 + \delta) \lceil \varepsilon n \rceil$.*

5 Stable Configurations without Consensus

For any graph G, consensus configurations ($\forall 1$ and $\forall 0$) are clearly stable. An interesting question, addressed in this section, is then whether G admits further stable configurations, in which case we would say that G is *plural*.

Note that G is plural if it admits a *non-trivial* partition $\mathcal{P}$, i.e., a partition $\mathcal{P} = (A, B)$ with $A \neq \emptyset$ and $B \neq \emptyset$, which has moreover *local minimum cut-size*, that is, $|E(A', B')| \geq |E(A, B)|$ holds for each partition $\mathcal{P}' = (A', B')$ obtained by moving one node from one side to the other (formally, either $A' \subsetneq A$ and $|A'| = |A| - 1$, or $B' \subsetneq B$ and $|B'| = |B| - 1$). The complexity of recognizing whether a graph admits a partition of this kind was, so far, an open problem [Ferraioli and Ventre, 2017]. The result below answers that question, by exhibiting an involved NP-hardness reduction.

Theorem 4. *Deciding if a graph is plural is* NP*-complete.*

Proof Sketch. Membership in NP is trivial. For the hardness, let $\phi = c_1 \wedge \cdots \wedge c_m$ be a Boolean formula such that, for each $i \in \{1, ..., m\}$, the clause c_i is the disjunction of 3 literals, i.e., $c_i = \ell_{1,i} \vee \ell_{2,i} \vee \ell_{3,i}$. Based on ϕ, we define the graph $\hat{G} = (\hat{N}, \hat{E})$ over nodes $\hat{N} = \{p_1, ..., p_{6m+2}, q_1, q_2\} \cup \bigcup_{i=1}^{m} \{\sigma_{1,i}, ..., \sigma_{8,i}\}$, where $\sigma_{1,i}, ..., \sigma_{8,i}$ are the satisfying truth assignments (viewed as nodes) for c_i. Moreover, for each pair $\sigma_{j,i}$ and $\sigma_{j',i'}$, there is an edge connecting them in $\hat{E}$ if, and only if, the assignments are compatible; finally, each node $\{p_1, ..., p_{6m+2}\}$ is connected with all the other nodes in $\hat{N}$ but q_1 and q_2. Let $n = m + 6m + 2$ and note that any clique in $\hat{G}$ has size n at most. In particular, a maximum clique of size n exists if, and only if, ϕ is satisfiable. Moreover, note that $|\hat{N}| = 8m + 6m + 2 + 2 = 2n$ and that each node in $\hat{G}$ has at most $2n - 2$ neighbors.

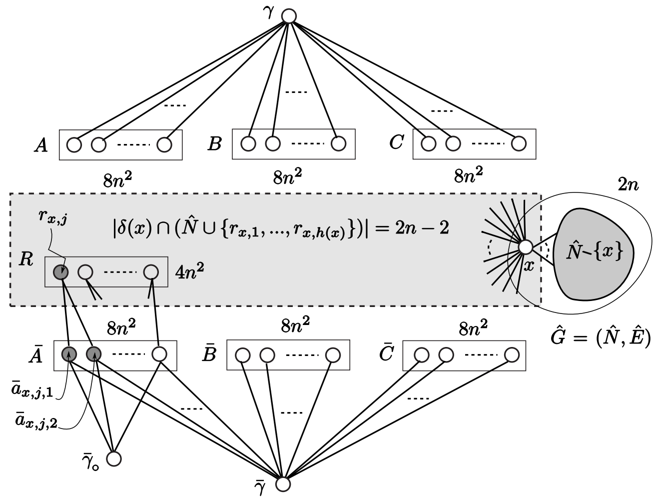

Figure 4: Construction in the proof of Theorem 4.

Having the graph $\hat{G}$, we now build the social graph $G = (N, E)$ over $N = \hat{N} \cup A \cup B \cup C \cup \bar{A} \cup \bar{B} \cup \bar{C} \cup R \cup \{\gamma, \bar{\gamma}, \bar{\gamma}_\circ\}$. Edges include those in $\hat{E}$ plus those illustrated, according to an intuitive notation, in Figure 4. Note, in particular, that for each $x \in \hat{N}$ and each $h \in \{1, ..., 2n\}$, R contains the agent r_{x_j} which is connected to the agents $\bar{a}_{x,j,1}$ and $\bar{a}_{x,j,2}$ in $\bar{A}$. Moreover, x is connected (in addition to the edges in E) to all agents in $N \setminus R$ and to the agents $r_{x_1}, ..., r_{x_{h(x)}}$, where $h(x)$ is defined in a way that x is adjacent to $2n - 2$ nodes of $\hat{N} \cup R$. On this graph, we claim that a stable configuration c with $N_0 \neq \emptyset$ and $N_0 \neq N$ exists iff $\hat{G}$ has a clique of size n.

("only-if"). Assume, w.l.o.g., that $\mathsf{c}(\gamma) = 1$. If $|\hat{N}_1| \neq n$, then the dominant opinion in $\hat{N}$ spreads to all agents outside $\hat{N}$. From plurality of $\hat{N}$ and $\mathsf{c}(\gamma) = 1$, we derive $(A \cup B \cup C \cup \{\gamma\})_0 = \emptyset$ and $(\bar{A} \cup \bar{B} \cup \bar{C} \cup \{\bar{\gamma}, \bar{\gamma}_\circ\} \cup R)_1 = \emptyset$. Now, let x be a node in $\hat{N}_1$. Note that $|\delta(x)_1| \leq |A \cup B \cup C| + n - 1$, since $|\hat{N}_1| = n$. Moreover, $|\delta(x)_0| \geq |\bar{A} \cup \bar{B} \cup \bar{C}| + (2n-2) - (n-1)$, since x is adjacent to $2n - 2$ nodes in $\hat{N} \cup R$. By stability of c, $|\delta(x)_1| \geq |\delta(x)_0|$ implies $|\delta(x)_1| = |A \cup B \cup C| + n - 1$. So, agents in $\hat{N}_1$ form a clique in $\hat{G}$.

("if") Assume that $\hat{G}$ has a clique, say C, of size n. Consider the configuration c such that: $(A \cup B \cup C \cup \{\gamma\})_0 = \emptyset$, $(\bar{A} \cup \bar{B} \cup \bar{C} \cup \{\bar{\gamma}, \bar{\gamma}_\circ\} \cup R)_1 = \emptyset$, $\hat{N}_1 = C$ and $\hat{N}_0 = \hat{N} \setminus C$. Each agent $x \in N \setminus \hat{N}$ is clearly stable. Consider $x \in \hat{N}_1$. Since $\hat{N}_1$ is a clique, we have $|\delta(x)_1| = |A \cup B \cup C| + n - 1$. However, $|\delta(x)| = |A \cup B \cup C| + |\bar{A} \cup \bar{B} \cup \bar{C}| + 2n - 2$ which means that $|\delta(x)_0| = |\bar{A} \cup \bar{B} \cup \bar{C}| + n - 1$. Hence, $|\delta(x)_1| = |\delta(x)_0|$ and x is stable. Finally, consider $x \in \hat{N}_0$. Since $\hat{N}_b$ is a clique with $|\hat{N}_1| = n$ and since we know that is no larger clique in $\hat{G}$, we have $|\delta(x) \cap \hat{N}_1| \leq n - 1$. So, $|\delta(x)_0| \geq |\bar{A} \cup \bar{B} \cup \bar{C}| + n - 1$, while $|\delta(x)_1| \leq |A \cup B \cup C| + n - 1$. It follows that x is stable, too. □

The following is an immediate consequence.

Corollary 2. *Deciding whether* $\forall 1$ *and* $\forall 0$ *are the only stable configurations is* coNP*-complete.*

6 Discussion and Conclusion

In the paper we addressed a number of questions related to whether consensus can be achieved in settings where opinions of the agents are affected by social influence phenomena. We have shown that a configuration always exists (and can be computed in polynomial-time) for which the opinion of the majority diffuse to all the agents. Moreover, we exhibited hardness results for spreading the opinion of the minority and for checking the existence of "plural" configurations.

An interesting avenue for further research is to identify special classes of graphs where it is easy to answer similar questions, by looking at structural and topological restrictions.

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