Elliptic operators with unbounded diffusion, drift and potential terms ☆

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Abstract

We prove that the realization A_p in $L^p(\mathbb{R}^N)$, $1 < p < \infty$, of the elliptic operator $A = (1 + |x|^\alpha)\Delta + b|x|^{\alpha-1}\frac{x}{|x|} \cdot \nabla - c|x|^\beta$ with domain $D(A_p) = \{u \in W^{2,p}(\mathbb{R}^N) \mid Au \in L^p(\mathbb{R}^N)\}$ generates a strongly continuous analytic semigroup $T(\cdot)$ provided that $\alpha > 2$, $\beta > \alpha - 2$ and any constants $b \in \mathbb{R}$ and $c > 0$. This generalizes the recent results in [4] and in [16]. Moreover we show that $T(\cdot)$ is consistent, immediately compact and ultracontractive.

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1. Introduction

Starting from the 1950's, the theory of linear second order elliptic operators with bounded coefficients has widely been studied. In recent years there has been a surge of activity focused on the case of unbounded coefficients. Let us recall some recent results concerning elliptic operators having polynomial coefficients.

In this paper, we are interested in studying quantitative and qualitative properties in $L^p(\mathbb{R}^N)$, $1 < p < \infty$, of the elliptic operator

$$Au(x) = q(x)\Delta u(x) + F(x) \cdot \nabla u - V(x)u(x), \quad x \in \mathbb{R}^N, \tag{1}$$

where $q(x) = (1 + |x|^\alpha)$, $F(x) = b|x|^{\alpha-2}x$, $V(x) = c|x|^\beta$, $b \in \mathbb{R}$ and $c > 0$, in the case $\alpha > 2$ and $\beta > \alpha - 2$.

Let us denote by L the operator A with $c = 0$ and illustrate the difference between the case $\alpha \in [0, 2]$ and $\alpha > 2$.

If $\alpha \in [0, 2]$ (after a modification of the drift term F near the origin, when $\alpha < 2$), it is proved in [8] that the L^p-realization L_p of L generates an analytic semigroup in $L^p(\mathbb{R}^N)$, $1 \le p \le \infty$. Moreover, if $1 < p < \infty$, then

$$D(L_p) = \{u \in L^p(\mathbb{R}^N) \cap W^{2,p}_{loc}(\mathbb{R}^N) : (1 + |x|^\alpha)^{1/2}|\nabla u|,\ (1 + |x|^\alpha)|D^2u| \in L^p(\mathbb{R}^N)\}.$$

The proof of the above result is essentially based on the a-priori estimates

$$\|(1 + |x|^\alpha)^{1/2}\nabla u\|_p \le C(\|Lu\|_p + \|u\|_p)$$
$$\|(1 + |x|^\alpha)D^2u\|_p \le C(\|Lu\|_p + \|u\|_p)$$

for $u \in C_c^\infty(\mathbb{R}^N)$.

The picture changes drastically when $\alpha > 2$. In this case G. Metafune et al. in [16] showed, if $\frac{N}{N-2+b} < p < \infty$, the generation of an analytic semigroup in $L^p(\mathbb{R}^N)$ which is contractive if and only if $p \ge \frac{N+\alpha-2}{N-2+b}$. Domain characterization and spectral properties as well as kernel estimates have been also proved.

Here the techniques are based on proving some bounds on the Green function associated to the operator L.

In [11] (resp. [4]) the generation of an analytic semigroup of the L^p-realization of the Schrödinger-type operators $(1 + |x|^\alpha)\Delta - |x|^\beta$ in $L^p(\mathbb{R}^N)$ for $\alpha \in [0, 2]$ and $\beta > 2$ (resp. $\alpha > 2$, $\beta > \alpha - 2$) is obtained. In [11,5] some estimates for the associated heat kernel are provided. Also in this case the methods for $\alpha \in [0, 2]$ and $\alpha > 2$ are completely different. This is related essentially to the fact that generation of a semigroup in $L^p(\mathbb{R}^N)$ in the case $\alpha > 2$ of the operator $(1 + |x|^\alpha)\Delta$ depends upon N, see [14], [15] and does not depend if $\alpha \le 2$, see [17]

More recently in [12] the authors showed that the operator $L = |x|^\alpha\Delta + b|x|^{\alpha-2}x \cdot \nabla - c|x|^{\alpha-2}$ generates a strongly continuous semigroup in $L^p(\mathbb{R}^N)$ if and only if $s_1 + \min\{0, 2-\alpha\} < \frac{N}{p} < s_2 + \max\{0, 2-\alpha\}$, where s_i are the roots of the equation $c + s(N-2+b-s) = 0$. Moreover the domain of the generator is also characterized.

At this point it is important to note that the techniques used in [12] are completely different from ours and lead to results which are not comparable with our case ($\beta > \alpha - 2$).

Moreover, since, in the case $\alpha > 2$, the generation of a semigroup of the operator L_p depends upon N, the L^p-realization of the operator A cannot be seen as a perturbation of L_p as one of our main results in this paper shows, see Theorem 3.

In this paper we denote by A_p the realization of A in $L^p(\mathbb{R}^N)$ endowed with the maximal domain

$$D_{p,max}(A) = \{u \in L^p(\mathbb{R}^N) \cap W^{2,p}_{loc}(\mathbb{R}^N) \,:\, Au \in L^p(\mathbb{R}^N)\}, \tag{2}$$

and assume that $\alpha > 2$, $\beta > \alpha - 2$. We note that with the assumption on β and α the operator A has unbounded coefficients at infinity and no local singularities occur.

After proving a priori estimates, we deduce that the maximal domain $D_{p,max}(A)$ of the operator A coincides with

$$D_p(A) := \{u \in W^{2,p}(\mathbb{R}^N) \,:\, Vu,\ (1+|x|^{\alpha-1})\nabla u,\ (1+|x|^{\alpha})D^2 u \in L^p(\mathbb{R}^N)\}.$$

So, we show in the main result of this paper that, for any $1 < p < \infty$, the realization A_p of A in $L^p(\mathbb{R}^N)$, with domain $D_p(A)$ generates a positive strongly continuous and analytic semigroup $(T_p(t))_{t\geq 0}$ for $p \in (1,\infty)$. This semigroup is also consistent, irreducible, immediately compact and ultracontractive.

The paper is divided as follows. In section 2 we recall the solvability of the elliptic and parabolic problems in spaces of continuous functions. In Section 3 we introduce the definition of the reverse Hölder class and recall some results given in [20] and in [4] to study the solvability of the elliptic problem in $L^p(\mathbb{R}^N)$. In section 4 we prove that the maximal domain of the operator A coincides with the weighted Sobolev space $D_p(A)$, and we state and prove the main result of this paper.

Notation. In general we use standard notations for function spaces. We denote by $L^p(\mathbb{R}^N)$ and $W^{2,p}(\mathbb{R}^N)$ the standard L^p and Sobolev spaces, respectively. For any $k \in \mathbb{N} \cup \{\infty\}$ we denote by $C^k_c(\mathbb{R}^N)$ the set of all functions $f : \mathbb{R}^N \to \mathbb{R}$ that are continuously differentiable in $\mathbb{R}^N$ up to k-th order and have compact support (say $\mathrm{supp}(f)$). The space $C_b(\mathbb{R}^N)$ is the set of all bounded and continuous functions $f : \mathbb{R}^N \to \mathbb{R}$, and we denote by $\|f\|_\infty$ its sup-norm, i.e., $\|f\|_\infty = \sup_{x\in\mathbb{R}^N} |f(x)|$. We use also the space $C_0(\mathbb{R}^N) := \{f \in C_b(\mathbb{R}^N) : \lim_{|x|\to\infty} f(x) = 0\}$. If f is smooth enough we set

$$|\nabla f(x)|^2 = \sum_{i=1}^N |D_i f(x)|^2, \qquad |D^2 f(x)|^2 = \sum_{i,j=1}^N |D_{ij} f(x)|^2.$$

For any $x_0 \in \mathbb{R}^N$ and any $r > 0$ we denote by $B(x_0, r) \subset \mathbb{R}^N$ the open ball, centered at x_0 with radius r. We simply write $B(r)$ when $x_0 = 0$. The function χ_E denotes the characteristic function of the set E, i.e., $\chi_E(x) = 1$ if $x \in E$, $\chi_E(x) = 0$ otherwise. Finally, by $x \cdot y$ we denote the Euclidean scalar product of the vectors $x, y \in \mathbb{R}^N$.

2. Solvability in $C_0(\mathbb{R}^N)$

In this short section we briefly recall some properties of the elliptic and parabolic problems associated with A in spaces of continuous functions.

Let us first consider the operator A on $C_b(\mathbb{R}^N)$ with its maximal domain

$$D_{max}(A) = \{u \in C_b(\mathbb{R}^N) \cap W^{2,p}_{loc}(\mathbb{R}^N) \quad \text{for all} \quad 1 \leq p < \infty : Au \in C_b(\mathbb{R}^N)\}.$$

It is known, cf. [2, Chapter 2, Section 2], that to the associated parabolic problem

$$\begin{cases} u_t(t,x) = Au(t,x) & x \in \mathbb{R}^N,\ t > 0, \\ u(0,x) = f(x) & x \in \mathbb{R}^N\ , \end{cases} \tag{3}$$

where $f \in C_b(\mathbb{R}^N)$, one can associate a semigroup $(T(t))_{t\geq 0}$ of bounded operators in $C_b(\mathbb{R}^N)$ such that $u(t,x) = T(t)f(x)$ is a solution of (3) in the following sense:

$$u \in C([0,+\infty) \times \mathbb{R}^N) \cap C^{1+\frac{\sigma}{2},2+\sigma}_{loc}((0,+\infty) \times \mathbb{R}^N)$$

and u solves (3) for any $f \in C_b(\mathbb{R}^N)$ and some $\sigma \in (0,1)$. Moreover, in our case the solution is unique. This can be seen by proving the existence of a Lyapunov function for A, i.e., a positive function $\varphi(x) \in C^2(\mathbb{R}^N)$ such that $\lim_{|x|\to\infty} \varphi(x) = +\infty$ and $A\varphi - \lambda\varphi \leq 0$ for some $\lambda > 0$.

Proposition 1. *Assume that $\alpha \geq 0$ and $\beta > \max\{0, \alpha - 2\}$. Let $\psi = 1 + |x|^\gamma$ where $\gamma > 2$ then there exists a constant $C > 0$ such that*

$$A\psi \leq C\psi.$$

Proof. An easy computation gives

$$\begin{aligned} A\psi &= \gamma(N+\gamma-2)(1+|x|^\alpha)|x|^{\gamma-2} + b\gamma|x|^\alpha|x|^{\gamma-2} - c(1+|x|^\gamma)|x|^\beta \\ &\leq \{\gamma(N+\gamma-2) + |b|\gamma\}(1+|x|^\alpha)|x|^{\gamma-2} - c(1+|x|^\gamma)|x|^\beta. \end{aligned}$$

Since $\beta > \alpha - 2$, it follows that there exists $C > 0$ such that

$$\{\gamma(N+\gamma-2) + |b|\gamma\}(1+|x|^\alpha)|x|^{\gamma-2} \leq c(1+|x|^\gamma)|x|^\beta + C(1+|x|^\gamma).$$

Thus, ψ is a Lyapunov function for A. $\square$

As in [4] one can prove the following result.

Proposition 2. *Assume that $N > 2$, $\alpha > 2$ and $\beta > \alpha - 2$. Then the semigroup $(T(t))$ is generated by $(A, D_{max}(A)) \cap C_0(\mathbb{R}^N)$ and maps $C_0(\mathbb{R}^N)$ into $C_0(\mathbb{R}^N)$.*

Proof. Let $f \in C_0(\mathbb{R}^N)$. Since $C_c^\infty(\mathbb{R}^N)$ is dense in $C_0(\mathbb{R}^N)$, there is a sequence $(f_n) \subset C_c^\infty(\mathbb{R}^N)$ such that $\lim_{n\to\infty} \|f_n - f\|_\infty = 0$.

On the other hand, it follows from Theorem 3 that the operator A_p with domain $D_{p,max}(A)$ generates an analytic semigroup $T_p(t)$ in $L^p(\mathbb{R}^N)$, and $D_p(A)$ is continuously embedded into $W^{2,p}(\mathbb{R}^N)$. Hence, by Theorem 2 and Sobolev's embedding theorem, $T(t)f_n = T_p(t)f_n \in D_p(A) \subset W^{2,p}(\mathbb{R}^N) \hookrightarrow C_0(\mathbb{R}^N)$ for $p > \frac{N}{2}$. Since $f_n \to f$ uniformly, it follows that $T(t)f_n \to T(t)f$ uniformly. Hence $T(t)f \in C_0(\mathbb{R}^N)$. $\square$

Remark 1. If $b > 2 - N$, then the semigroup $(T(t))$ generated by $(A, D_{max}(A)) \cap C_0(\mathbb{R}^N)$ is compact. To prove this we recall that, by [16, Proposition 2.2 (ii)], the resolvent and the minimal semigroup $(S(t))$ generated by $(L, D_{max}(L)) \cap C_0(\mathbb{R}^N)$ map $C_b(\mathbb{R}^N)$ into $C_0(\mathbb{R}^N)$ and are compact, where $L := q(x)\Delta + F(x) \cdot \nabla$. Set $v(t, x) = S(t)f(x)$ and $u(t, x) = T(t)f(x)$ for $t > 0$, $x \in \mathbb{R}^N$ and $0 \le f \in C_b(\mathbb{R}^N)$. Then the function $w(t, x) = v(t, x) - u(t, x)$ solves

$$\begin{cases} w_t(x,t) = Lw(t,x) + V(x)u(t,x), & t > 0, \\ w(0,x) = 0 & x \in \mathbb{R}^N. \end{cases}$$

So, applying [2, Theorem 4.1.3], we have $w \ge 0$ and hence $T(t) \le S(t)$. Thus, $T(t)\mathbf{1} \in C_0(\mathbb{R}^N)$, for any $t > 0$ (see [14, Proposition 2.2 (iii)]). Therefore $T(t)$ is compact for all $t > 0$ (cf. [2, Theorem 5.1.11]).

3. Solvability of $\lambda u - Au = f$ in $L^p(\mathbb{R}^N)$

In the previous section we have proved the existence and uniqueness of the elliptic and parabolic problems in $C_0(\mathbb{R}^N)$. In this section we study the solvability of the equation $\lambda u - A_p u = f$ for $\lambda > \lambda_0$, where λ_0 is a suitable positive constant.

Let $f \in L^p(\mathbb{R}^N)$ and consider the equation

$$\lambda u - Au = f. \tag{4}$$

Let $\phi = (1 + |x|^\alpha)^{b/\alpha}$, where $b \in \mathbb{R}$ is the coefficient of the drift term of A given by (1), and set $u = \frac{v}{\sqrt{\phi}}$. We note that the function ϕ is the function for which we have

$$\frac{1}{\phi}\operatorname{div}(\phi \nabla u) = \Delta u + b\frac{|x|^{\alpha-2}}{1+|x|^\alpha} x \cdot \nabla u, \quad u \in C_c^\infty(\mathbb{R}^N).$$

A simple computation gives

$$\lambda u - Au = \frac{(1+|x|^\alpha)}{\sqrt{\phi}}\left[-\Delta v + Uv + \frac{V+\lambda}{1+|x|^\alpha}v\right], \tag{5}$$

where

$$U = -\frac{1}{4}\left|\frac{\nabla\phi}{\phi}\right|^2 + \frac{1}{2}\frac{\Delta\phi}{\phi}.$$

Then solving (4) is equivalent to solve

$$-Hv = \frac{\sqrt{\phi}}{1+|x|^\alpha} f, \tag{6}$$

where H is the Schrödinger operator defined by

$$H = \Delta - U - \frac{V+\lambda}{1+|x|^\alpha}.$$

If we denote by $G(x, y)$ the Green function of H, a solution of (6) is given by

$$v(x) = \int_{\mathbb{R}^N} G(x, y) \frac{\sqrt{\phi(y)}}{1 + |y|^\alpha} f(y) dy,$$

and hence a solution of (4) should be

$$u(x) = Lf(x) := \frac{1}{\sqrt{\phi(x)}} \int_{\mathbb{R}^N} G(x, y) \frac{\sqrt{\phi(y)}}{1 + |y|^\alpha} f(y) dy. \tag{7}$$

First, we have to show that L is a bounded operator in $L^p(\mathbb{R}^N)$. For this purpose we need to estimate G.

We focus our attention to the operator H. Evaluating the potential $\mathcal{V} = U + \frac{V+\lambda}{1+|x|^\alpha}$, it follows that

$$\begin{aligned}
\mathcal{V} &= \frac{|x|^{2\alpha-2}}{(1+|x|^\alpha)^2} \left(\frac{b^2}{4} - \frac{b\alpha}{2} \right) + \frac{|x|^{\alpha-2}}{1+|x|^\alpha} \frac{b}{2} (N + \alpha - 2) + \frac{c|x|^\beta + \lambda}{1 + |x|^\alpha} \\
&= \left(\frac{1}{1+|x|^\alpha} - \frac{1}{(1+|x|^\alpha)^2} \right) |x|^{\alpha-2} \left(\frac{b^2}{4} - \frac{b\alpha}{2} \right) + \frac{|x|^{\alpha-2}}{1+|x|^\alpha} \frac{b}{2} (N+\alpha-2) + \frac{c|x|^\beta + \lambda}{1+|x|^\alpha} \\
&= \frac{|x|^{\alpha-2}}{1+|x|^\alpha} \left(\frac{b^2}{4} + b \left(\frac{N-2}{2} \right) \right) + \frac{|x|^{\alpha-2}}{(1+|x|^\alpha)^2} \left(-\frac{1}{4} b^2 + \frac{1}{2} b\alpha \right) + \frac{c|x|^\beta}{1+|x|^\alpha} + \frac{\lambda}{1+|x|^\alpha}.
\end{aligned}$$

We can choose $\lambda_0 > 0$ such that for every $\lambda \geq \lambda_0$ the potential $\mathcal{V}$ is positive. Indeed, since $\beta > \alpha - 2$ the function

$$\frac{|x|^{2\alpha-2}}{(1+|x|^\alpha)} \left(\frac{b^2}{4} - \frac{b\alpha}{2} \right) + |x|^{\alpha-2} \frac{b}{2} (N + \alpha - 2) + c|x|^\beta$$

has a nonpositive minimum μ in $\mathbb{R}^N$. So, one takes $\lambda_0 > -\mu$.

On the other hand, since $\mathcal{V}(0) = \lambda > 0$ and $\mathcal{V}$ behaves like $|x|^{\beta-\alpha}$ as $|x| \to \infty$ we have the following estimates

$$\begin{aligned}
&C_1(1 + |x|^{\beta-\alpha}) \leq \mathcal{V} \leq C_2(1 + |x|^{\beta-\alpha}) \quad \text{if } \beta \geq \alpha, \\
&C_3 \frac{1}{1+|x|^{\alpha-\beta}} \leq \mathcal{V} \leq C_4 \frac{1}{1+|x|^{\alpha-\beta}} \quad \text{if } \alpha - 2 < \beta < \alpha
\end{aligned} \tag{8}$$

for some positive constants C_1, C_2, C_3, C_4.

At this point we can use bounds of G obtained by [20] in the case of positive potentials belonging to the reverse Hölder class B_q for some $q \geq N/2$.

We recall that a nonnegative locally L^q-integrable function V on $\mathbb{R}^N$ is said to be in B_q, $1 < q < \infty$, if there exists $C > 0$ such that the reverse Hölder inequality

$$\left(\frac{1}{|B|} \int_B V^q(x) dx \right)^{1/q} \leq C \left(\frac{1}{|B|} \int_B V(x) dx \right)$$

holds for every ball B in $\mathbb{R}^N$. A nonnegative function $V \in L^\infty_{\text{loc}}(\mathbb{R}^N)$ is in B_∞ if

$$\|V\|_{L^\infty(B)} \leq C \left(\frac{1}{|B|} \int_B V(x)dx \right)$$

for every ball B in $\mathbb{R}^N$.

As regards the potential $\mathcal{V}$ we can see that it belongs to $B_{N/2}$. Indeed, using (8), one has $\mathcal{V} \in B_\infty$ and hence $\mathcal{V} \in B_{N/2}$ if $\beta \geq \alpha$. If $\beta < \alpha$, then $\mathcal{V} \in B_q$ whenever $\beta - \alpha > -\frac{N}{q}$, and since $\beta - \alpha > -2$, we have that $\mathcal{V} \in B_{N/2}$. For more details on reverse Hölder classes we refer to [21, Chapter XI], [10, Chapter 9]. So, it follows from [20, Theorem 2.7] that for any $k \in \mathbb{N}$ there is a constant $C_k > 0$ such that

$$|G(x, y)| \leq \frac{C_k}{(1 + m(x)|x - y|)^k} \cdot \frac{1}{|x - y|^{N-2}}, \quad x, y \in \mathbb{R}^N, \tag{9}$$

where the auxiliary function m is defined by

$$\frac{1}{m(x)} := \sup_{r>0} \left\{ r : \frac{1}{r^{N-2}} \int_{B(x,r)} \mathcal{V}(y)dy \leq 1 \right\}, \quad x \in \mathbb{R}^N. \tag{10}$$

In [4] a lower bound for the auxiliary function associated to the potential $\tilde{V} = \frac{|x|^\beta}{1+|x|^\alpha}$ was obtained. Since $\mathcal{V} \geq C_1 \tilde{V}$ for some positive constant C_1, we have $m(x) \geq \widetilde{m}(x)$, where $\frac{1}{\widetilde{m}(x)} := \sup_{r>0} \left\{ r : \frac{1}{r^{N-2}} \int_{B(x,r)} C_1 \tilde{V}(y)dy \leq 1 \right\}$. Replacing $\tilde{V}$ with $C_1 \tilde{V}$ in [4, Lemma 3.1, Lemma 3.2] we obtain $\widetilde{m}(x) \geq C_2 (1 + |x|)^{\frac{\beta - \alpha}{2}}$. So we have

Lemma 1. *Let $\alpha - 2 < \beta$. There exists $C = C(\alpha, \beta, N)$ such that*

$$m(x) \geq C (1 + |x|)^{\frac{\beta - \alpha}{2}}. \tag{11}$$

Finally by (9) and the previous lemma we can estimate the Green function G

Lemma 2. *Let $G(x, y)$ denote the Green function of the Schrödinger operator H and assume that $\beta > \alpha - 2$. Then*

$$G(x, y) \leq C_k \frac{1}{1 + |x - y|^k (1 + |x|)^{\frac{\beta - \alpha}{2} k}} \frac{1}{|x - y|^{N-2}}, \quad x, y \in \mathbb{R}^N \tag{12}$$

for any $k > 0$ and some constant $C_k > 0$ depending on k.

We can prove now the boundedness in $L^p(\mathbb{R}^N)$ of the operator L given by (7)

Lemma 3. *Assume that $\alpha > 2$, $N > 2$ and $\beta > \alpha - 2$. Then there exists a positive constant $C = C(\lambda)$ such that for every $0 \le \gamma \le \beta$ and $f \in L^p(\mathbb{R}^N)$*

$$\||x|^\gamma Lf\|_p \le C\|f\|_p. \tag{13}$$

Proof. Recall the function $\phi(x) = (1+|x|^\alpha)^{b/\alpha}$. Let $\Gamma(x,y) = \sqrt{\frac{\phi(y)}{\phi(x)}}\frac{G(x,y)}{1+|y|^\alpha}$, $f \in L^p(\mathbb{R}^N)$ and

$$u(x) = \int_{\mathbb{R}^N} \Gamma(x,y)f(y)dy, \quad x \in \mathbb{R}^N.$$

We have to show that

$$\||x|^\gamma u\|_p \le C\|f\|_p.$$

By setting $\Gamma_0 = \frac{G(x,y)}{1+|y|^\alpha}$, we have $\Gamma(x,y) = \left(\frac{1+|y|^\alpha}{1+|x|^\alpha}\right)^{b/(2\alpha)}\Gamma_0(x,y)$. Moreover if we set $L_0 f(x) := \int_{\mathbb{R}^N} \Gamma_0(x,y)f(y)dy$, $x \in \mathbb{R}^N$, then [4, Lemma 3.4] gives

$$\||x|^\gamma L_0 f\|_p \le C\|f\|_p. \tag{14}$$

For $x \in \mathbb{R}^N$ let us consider the regions $E_1 := \{|x-y| \le \frac{1}{2}(1+|y|)\}$ and $E_2 := \{|x-y| > \frac{1}{2}(1+|y|)\}$ and write

$$u(x) = \int_{E_1} \Gamma(x,y)f(y)dy + \int_{E_2} \Gamma(x,y)f(y)dy =: u_1(x) + u_2(x)\,.$$

In E_1 we have $1+|y| \le 1+|x|+|x-y| \le 1+|x|+\frac{1}{2}(1+|y|)$ and hence $\frac{1}{2}(1+|y|) \le 1+|x|$. Thus,

$$\frac{1+|x|}{1+|y|} \le \frac{1+|x-y|+|y|}{1+|y|} \le \frac{3}{2} \text{ and } \frac{1+|y|}{1+|x|} \le 2\,.$$

Therefore there are constants $C, \widetilde{C} > 0$ such that $\left(\frac{1+|y|^\alpha}{1+|x|^\alpha}\right)^{b/(2\alpha)} \le \widetilde{C}\left(\frac{1+|y|}{1+|x|}\right)^{b/2} \le C2^{|b|/2}$ and $\Gamma(x,y) \le C\Gamma_0(x,y)$ in E_1. So, we have

$$|u_1(x)| \le C\int_{\mathbb{R}^N} \Gamma_0(x,y)|f(y)|dy = CL_0(|f|)(x).$$

By (14) it follows that $\||x|^\gamma u_1\|_p \le C\|f\|_p$.

As regards the region E_2, we have, by Hölder's inequality,

$$\left||x|^\gamma u_2(x)\right| \le |x|^\gamma \int\limits_{E_2} \Gamma(x,y)|f(y)|dy = \int\limits_{E_2} \left(|x|^\gamma \Gamma(x,y)\right)^{\frac{1}{p'}} \left(|x|^\gamma \Gamma(x,y)\right)^{\frac{1}{p}} |f(y)|dy$$

$$\le \left(\int\limits_{E_2} |x|^\gamma \Gamma(x,y)dy\right)^{\frac{1}{p'}} \left(\int\limits_{E_2} |x|^\gamma \Gamma(x,y)|f(y)|^p dy\right)^{\frac{1}{p}}. \tag{15}$$

We propose to estimate first $\int_{E_2} |x|^\gamma \Gamma(x,y)dy$. In E_2 we have $1+|y| \le 2|x-y|$ and $1+|x| \le 1+|y|+|x-y| \le 3|x-y|$, then

$$\left(\frac{1+|y|^\alpha}{1+|x|^\alpha}\right)^{b/(2\alpha)} \le \widetilde{C}\left(\frac{1+|y|}{1+|x|}\right)^{b/2} \le C|x-y|^{|b|/2}.$$

From (12) and by the symmetry of G it follows that

$$\begin{aligned}
|x|^\gamma \Gamma(x,y) &= |x|^\gamma \left(\frac{1+|y|^\alpha}{1+|x|^\alpha}\right)^{b/(2\alpha)} \frac{G(x,y)}{1+|y|^\alpha} \\
&\le C|x|^\gamma G(x,y)|x-y|^{|b|/2} \\
&\le C\frac{1+|x|^\beta}{|x-y|^k(1+|y|)^{k\frac{\beta-\alpha}{2}}} \frac{1}{|x-y|^{N-2-|b|/2}} \\
&\le C\frac{1}{|x-y|^{k-\beta+N-2-|b|/2}} \frac{1}{(1+|y|)^{k\frac{\beta-\alpha}{2}}}, \quad y \in E_2.
\end{aligned}$$

For every $k > \beta - N + 2 + |b|/2$, taking into account that $\frac{1}{|x-y|} \le 2\frac{1}{1+|y|}$, we get

$$|x|^\gamma \Gamma(x,y) \le C\frac{1}{(1+|y|)^{k\frac{\beta-\alpha+2}{2}+N-2-\beta-|b|/2}}.$$

Since $\beta - \alpha + 2 > 0$ we can choose k such that $\frac{k}{2}(\beta-\alpha+2)+N-2-\beta-|b|/2 > N$, then there is a constant $C_1 > 0$ such that

$$\int\limits_{E_2} |x|^\gamma \Gamma(x,y)dy \le C\int\limits_{\mathbb{R}^N} \frac{1}{(1+|y|)^{\frac{k}{2}(2+\beta-\alpha)+N-2-\beta-|b|/2}}dy \le C_1.$$

Moreover by (12) as above we have

$$\begin{aligned}
|x|^\gamma \Gamma(x,y) &\le C|x|^\gamma G(x,y)|x-y|^{|b|/2} \\
&\le C\frac{1+|x|^\beta}{|x-y|^k(1+|x|)^{k\frac{\beta-\alpha}{2}}} \frac{1}{|x-y|^{N-2-|b|/2}}
\end{aligned}$$

$$\leq C \frac{1}{|x-y|^{k-\beta+N-2-|b|/2}} \frac{1}{(1+|x|)^{k\frac{\beta-\alpha}{2}}}.$$

Taking into account that $\frac{1}{|x-y|} \leq 3\frac{1}{1+|x|}$, arguing as above we obtain

$$\int_{E_2} |x|^\gamma \Gamma(x, y) dx \leq C_2 \tag{16}$$

for some constant $C_2 > 0$. Hence (15) implies

$$\left||x|^\gamma u_2(x)\right|^p \leq C_1^{p-1} \int_{E_2} |x|^\gamma \Gamma(x, y) |f(y)|^p dy. \tag{17}$$

Thus, by (17) and (16), we have

$$\||x|^\gamma u_2\|_p^p \leq C_1^{p-1} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |x|^\gamma \Gamma(x, y) \chi_{\{|x-y|>\frac{1}{2}(1+|y|)\}}(x, y) |f(y)|^p dy dx$$

$$= C_1^{p-1} \int_{\mathbb{R}^N} |f(y)|^p \left(\int_{E_2} |x|^\gamma \Gamma(x, y) dx \right) dy \leq C_1^{p-1} C_2 \|f\|_p^p. \qquad \square$$

Here and in Section 4 we will need the following covering result, see [6, Proposition 6.1].

Proposition 3. *Given a covering $\mathcal{F} = \{B(x, \rho(x))\}_{x\in\mathbb{R}^N}$ of $\mathbb{R}^N$, where $\rho : \mathbb{R}^N \to \mathbb{R}_+$ is a Lipschitz continuous function with Lipschitz constant $k < 1/2$, there exists a countable subcovering $\{B(x_n, \rho(x_n))\}_{n\in\mathbb{N}}$ of $\mathbb{R}^N$ and $\zeta = \zeta(N, k) \in \mathbb{N}$ such that at most ζ among the double balls $\{B(x_n, 2\rho(x_n))\}_{n\in\mathbb{N}}$ overlap.*

We propose now to characterize the domain $D_{p,max}(A)$.

Proposition 4. *Assume that $N > 2$, $\alpha > 2$ and $\beta > \alpha - 2$. For $1 < p < \infty$ the following holds*

$$D_{p,max}(A) = \{u \in W^{2,p}(\mathbb{R}^N) : Au \in L^p(\mathbb{R}^N)\}.$$

Proof. It suffices to prove that $D_{p,max}(A) \subset \{u \in W^{2,p}(\mathbb{R}^N) : Au \in L^p(\mathbb{R}^N)\}$. Let $u \in D_{p,max}(A)$. Then $f := Au \in L^p(\mathbb{R}^N)$. This implies that

$$\widetilde{A}u := \Delta u + b\frac{|x|^{\alpha-2}}{1+|x|^\alpha} x \cdot \nabla u - \frac{c|x|^\beta}{1+|x|^\alpha} u = \frac{f}{1+|x|^\alpha} \in L^p(\mathbb{R}^N).$$

If $\beta \leq \alpha$ then the potential $\widetilde{V}(x) := \frac{c|x|^\beta}{1+|x|^\alpha}$ is bounded and by standard regularity results for uniformly elliptic operators with bounded coefficients we deduce that $u \in W^{2,p}(\mathbb{R}^N)$.

Let us assume now that $\beta > \alpha$. Then $\widetilde{V} \in B_q$ for all $q \in (1, \infty)$. So, by [1, Theorem 1.1 and Corollary 1.3], we have that $D_{p,max}(\Delta - \widetilde{V}) = W^{2,p}(\mathbb{R}^N) \cap D_{p,max}(\widetilde{V})$ and the following estimate holds

$$\|\widetilde{V} f\|_p + \|\Delta f\|_p \le C\|\Delta f - \widetilde{V} f\|_p \tag{18}$$

for all $f \in D_{p,max}(\Delta - \widetilde{V})$ with a constant C independent of f.

Fix now $x_0 \in \mathbb{R}^N$ and $R \ge 1$. We propose to prove the following interior estimate

$$\|\Delta u\|_{L^p(B(x_0,\frac{R}{2}))} \le C\left(\|\widetilde{A}u\|_{L^p(B(x_0,R))} + \|u\|_{L^p(B(x_0,R))}\right) \tag{19}$$

with a constant C independent of u and R. To this purpose take $\sigma \in (0, 1)$ and set $\sigma' := \frac{\sigma+1}{2}$. Consider a cutoff function $\vartheta \in C_c^\infty(\mathbb{R}^N)$ such that $0 \le \vartheta \le 1$, $\vartheta(x) = 1$ for $x \in B(x_0, \sigma R)$, $\vartheta(x) = 0$ for $x \in B^c(x_0, \sigma' R)$, $\|\nabla\vartheta\|_\infty \le \frac{C}{R(1-\sigma')}$ and $\|\Delta\vartheta\|_\infty \le \frac{C}{R^2(1-\sigma')^2}$ with a constant C independent of R.

In order to simplify the notation we write $\|\cdot\|_{p,r}$ instead of $\|\cdot\|_{L^p(B(x_0,r))}$. The function $v = u\vartheta$ belongs to $D_{p,max}(\Delta - \tilde{V})$ and so by (18) we have

$$\begin{aligned}
&\|\Delta u\|_{p,\sigma R} \le \|\Delta v\|_{p,\sigma' R} \le C\|\Delta v - \widetilde{V} v\|_{p,\sigma' R}\\
&\le C\left(\|\Delta v + F\cdot\nabla v - \widetilde{V} v\|_{p,\sigma' R} + \|F\cdot\nabla v\|_{p,\sigma' R}\right)\\
&\le C\left(\|\widetilde{A}u\|_{p,\sigma' R} + 2\|\nabla\vartheta\|_\infty\|\nabla u\|_{p,\sigma' R} + \|\Delta\vartheta\|_\infty\|u\|_{p,\sigma' R} + \|F\|_\infty\|\nabla\vartheta\|_\infty\|u\|_{p,\sigma' R}\right.\\
&\qquad\left. + \|F\|_\infty\|\nabla u\|_{p,\sigma' R} + \|F\|_\infty\|\nabla\vartheta\|_\infty\|u\|_{p,\sigma' R}\right)\\
&\le C\left(\|\widetilde{A}u\|_{p,\sigma' R} + (\|\nabla\vartheta\|_\infty + \|F\|_\infty)\|\nabla u\|_{p,\sigma' R} + (\|\Delta\vartheta\|_\infty + \|\nabla\vartheta\|_\infty)\|u\|_{p,\sigma' R}\right)\\
&\le C\left(\|\widetilde{A}u\|_{p,\sigma' R} + \frac{1}{R(1-\sigma')}\|\nabla u\|_{p,\sigma' R} + \frac{1}{R^2(1-\sigma')^2}\|u\|_{p,\sigma' R}\right),
\end{aligned}$$

where $F(x) := b\frac{|x|^{\alpha-2}}{1+|x|^\alpha}x$ and C a positive constant independent of u and R, which may change from line to line. Multiplying the above estimate by $R^2(1-\sigma')^2$ and taking into account that $1-\sigma = 2(1-\sigma')$ we obtain

$$R^2(1-\sigma)^2\|\Delta u\|_{p,\sigma R} \le C\left(R^2\|\widetilde{A}u\|_{p,R} + R(1-\sigma')\|\nabla u\|_{p,\sigma' R} + \|u\|_{p,R}\right).$$

So,

$$\begin{aligned}
&\sup_{\sigma\in(0,1)}\left\{R^2(1-\sigma)^2\|\Delta u\|_{p,\sigma R}\right\}\\
&\qquad\le C\left(\sup_{\sigma\in(0,1)}\left\{R(1-\sigma)\|\nabla u\|_{p,\sigma R}\right\} + R^2\|\widetilde{A}u\|_{p,R} + \|u\|_{p,R}\right).
\end{aligned} \tag{20}$$

Thus, by [9, Theorem 7.28], for every $\gamma > 0$ there exists $\sigma_\gamma \in (0, 1)$ such that

$$\sup_{\sigma\in(0,1)}\left\{R(1-\sigma)\|\nabla u\|_{p,\sigma R}\right\}\le R(1-\sigma_\gamma)\|\nabla u\|_{p,\sigma_\gamma R}+\gamma$$

$$\le \varepsilon R^2(1-\sigma_\gamma)^2\|\Delta u\|_{p,\sigma_\gamma R}+\frac{C}{\varepsilon}\|u\|_{p,R}+\gamma$$

$$\le \varepsilon \sup_{\sigma\in(0,1)}\left\{R^2(1-\sigma)^2\|\Delta u\|_{p,\sigma R}\right\}+\frac{C}{\varepsilon}\|u\|_{p,R}+\gamma .$$

Letting $\gamma\to 0$ we deduce that

$$\sup_{\sigma\in(0,1)}\left\{R(1-\sigma)\|\nabla u\|_{p,\sigma R}\right\}\le \varepsilon \sup_{\sigma\in(0,1)}\left\{R^2(1-\sigma)^2\|\Delta u\|_{p,\sigma R}\right\}+\frac{C}{\varepsilon}\|u\|_{p,R}. \tag{21}$$

Putting (21) into (20) with a suitable choice of ε we obtain

$$\sup_{\sigma\in(0,1)}\left\{R^2(1-\sigma)^2\|\Delta u\|_{p,\sigma R}\right\}\le C\left(R^2\|\widetilde{A}u\|_{p,R}+\|u\|_{p,R}\right).$$

Hence (19) follows since $(1-\frac{1}{2})^2R^2\|\Delta u\|_{p,\frac{R}{2}}\le \sup_{\sigma\in(0,1)}\left\{R^2(1-\sigma)^2\|\Delta u\|_{p,\sigma R}\right\}$.

To prove that $u\in W^{2,p}(\mathbb{R}^N)$ we consider a covering $\{B(x_n,R/2):n\in\mathbb{N}\}$ of $\mathbb{R}^N$ such that at most ζ among the doubled balls $\{B(x_n,R):n\in\mathbb{N}\}$ overlap for some $\zeta(N)\in\mathbb{N}$, by Proposition 3. Applying (19) with the ball $B(x_n,R/2)$ we obtain

$$\begin{aligned}\|\Delta u\|_p&\le \sum_{n\in\mathbb{N}}\|\Delta u\|_{L^p(B(x_n,R/2))}\\ &\le C\sum_{n\in\mathbb{N}}\left(\|\widetilde{A}u\|_{L^p(B(x_n,R))}+\|u\|_{L^p(B(x_n,R))}\right)\\ &\le C\zeta\left(\|\widetilde{A}u\|_p+\|u\|_p\right).\end{aligned}$$

This ends the proof. □

We show now the invertibility of $\lambda-A_p$ in $D_{p,max}(A)$ for all $\lambda\ge\lambda_0$, where $\lambda_0>0$ is such that $\mathcal{V}\ge 0$ for all $\lambda\ge\lambda_0$.

Theorem 1. *Assume that $N>2$, $\alpha>2$ and $\beta>\alpha-2$. Then $[\lambda_0,\infty)\subset\rho(A_p)$ and $(\lambda-A_p)^{-1}=L$ for all $\lambda\ge\lambda_0$. Moreover there exists $C=C(\lambda)>0$ such that, for every $0\le\gamma\le\beta$ and $\lambda\ge\lambda_0$, the following holds*

$$\||\cdot|^\gamma u\|_p\le C\|\lambda u-A_pu\|_p,\quad \forall u\in D_{p,max}(A). \tag{22}$$

Proof. First we prove the injectivity of $\lambda-A_p$ for $\lambda\ge\lambda_0$. Let $u\in D_{p,max}(A)$ such that $\lambda u-A_pu=0$. We have to distinguish two cases. The first one is when $b\le 0$. In this case, by (5) we have $Hv=\Delta v-\mathcal{V}v=0$ with $v=u\sqrt{\phi}\in D_{p,max}(H)=W^{2,p}(\mathbb{R}^N)\cap D_{p,max}(\mathcal{V})$, (see [18] or [1]). Then multiplying Hv by $v|v|^{p-2}$ and integrating by part (see [13]) over $\mathbb{R}^N$, we have

$$0=\int_{\mathbb{R}^N} v|v|^{p-2}\Delta v\,dx-\int_{\mathbb{R}^N} \mathcal{V}|v|^p dx$$
$$=-(p-1)\int_{\mathbb{R}^N} |v|^{p-2}|\nabla v|^2 dx-\int_{\mathbb{R}^N} \mathcal{V}|v|^p dx.$$

Then we have $v\equiv 0$ and hence $u\equiv 0$.

The second case is when $b>0$. For this we multiply $\frac{1}{1+|x|^\alpha}(\lambda u-A_p u)$ by $u|u|^{p-2}$ and using the fact that $u\in W^{2,p}(\mathbb{R}^N)$, by Proposition 4, we have

$$0=\int_{\mathbb{R}^N} \frac{1}{1+|x|^\alpha}(\lambda u-A_p u)u|u|^{p-2}\,dx$$
$$=\int_{\mathbb{R}^N} \left(\frac{\lambda+c|x|^\beta}{1+|x|^\alpha}+\frac{b(N+\alpha-2)|x|^{\alpha-2}+b(N-2)|x|^{2\alpha-2}}{p(1+|x|^\alpha)^2}\right)|u|^p\,dx$$
$$+(p-1)\int_{\mathbb{R}^N} |\nabla u|^2|u|^{p-2}\,dx.$$

Hence, $u\equiv 0$.

Let now $f\in L^p(\mathbb{R}^N)$ and $u(x)=Lf(x)$ defined by (7). Applying Lemma 3 with $\gamma=0$, we have $u\in L^p(\mathbb{R}^N)$. Moreover $u_n:=Lf_n$ satisfies $\lambda u_n-A_p u_n=f_n$ for any $f_n\in C_c^\infty(\mathbb{R}^N)$ approximating f in $L^p(\mathbb{R}^N)$. Thus, $\lim_{n\to\infty}\|u_n-u\|_p=0$. Since, by local elliptic regularity, A_p on $D_{p,max}(A)$ is closed, it follows that $u\in D_{p,max}(A)$ and $\lambda u-A_p u=f$. Thus $\lambda-A_p$ is invertible and $(\lambda-A_p)^{-1}\in\mathcal{L}(L^p(\mathbb{R}^N))$ for all $\lambda\geq\lambda_0$.

Finally, (22) follows from (13). □

The following result shows that the resolvent in $L^p(\mathbb{R}^N)$ and $C_0(\mathbb{R}^N)$ coincides.

Theorem 2. *Assume that $N>2$, $\beta>\alpha-2$ and $\alpha>2$. Then, for all $\lambda\geq\lambda_0$, $(\lambda-A_p)^{-1}$ is a positive operator on $L^p(\mathbb{R}^N)$. Moreover, if $f\in L^p(\mathbb{R}^N)\cap C_0(\mathbb{R}^N)$, then $(\lambda-A_p)^{-1}f=(\lambda-A)^{-1}f$.*

Proof. The positivity of $(\lambda-A_p)^{-1}$ follows from Theorem 1 and the positivity of L.

For the second assertion take $f\in C_c^\infty(\mathbb{R}^N)$ and set $u:=(\lambda-A_p)^{-1}f$. Since the coefficients of A are Hölder continuous, by local elliptic regularity (cf. [9, Theorem 9.19]), we know $u\in C_{loc}^{2+\sigma}(\mathbb{R}^N)$ for some $0<\sigma<1$. On the other hand, $u\in W^{2,p}(\mathbb{R}^N)$ by Proposition 4.

If $p\geq\frac{N}{2}$ then, by Sobolev's inequality, $u\in L^q(\mathbb{R}^N)$ for all $q\in[p,+\infty)$. In particular, $u\in L^q(\mathbb{R}^N)$ for some $q>\frac{N}{2}$ (cf. [3, Corollary 9.13]) and hence $Au=-f+\lambda u\in L^q(\mathbb{R}^N)$. Moreover, since $u\in C_{loc}^{2+\sigma}(\mathbb{R}^N)$ it follows that $u\in W_{loc}^{2,q}(\mathbb{R}^N)$. So, $u\in D_{q,max}(A)\subset W^{2,q}(\mathbb{R}^N)\subset C_0(\mathbb{R}^N)$ by Proposition 4 and Sobolev's embedding theorem (cf. [3, Corollary 9.13]).

Let us now suppose that $p<\frac{N}{2}$. Take the sequence (r_n), defined by $r_n=1/p-2n/N$ and set $q_n=1/r_n$ for $n\in\mathbb{N}$. Let n_0 be the smallest integer such that $r_{n_0}\leq 2/N$ noting that $r_{n_0}>0$. Then, $u\in D_{p,max}(A)\subset L^{q_1}(\mathbb{R}^N)\cap L^p(\mathbb{R}^N)$, by the Sobolev embedding theorem. As above we obtain that $u\in D_{q_1,max}(A)\subset L^{q_2}(\mathbb{R}^N)$. Iterating this argument, we deduce that $u\in D_{q_{n_0},max}(A)$. So we can conclude that $u \in C_0(\mathbb{R}^N)$ arguing as in the previous case. Thus, $Au = -f + \lambda u \in C_b(\mathbb{R}^N)$. Again, since $u \in C^{2+\sigma}_{loc}(\mathbb{R}^N)$, it follows that $u \in W^{2,q}_{loc}(\mathbb{R}^N)$ for any $q \in (1, +\infty)$. Hence, $u \in D_{max}(A)$. So, by the uniqueness of the solution of the elliptic problem, we have $(\lambda - A_p)^{-1} f = (\lambda - A)^{-1} f$ for every $f \in C_c^\infty(\mathbb{R}^N)$. Thus the assertion follows by density. □

4. Characterization of the domain and generation of semigroups

The aim of this section is to prove that the operator A_p generates an analytic semigroup on $L^p(\mathbb{R}^N)$, for any $p \in (1, \infty)$, provided that $N > 2$, $\alpha > 2$ and $\beta > \alpha - 2$.

We characterize first the domain of the operator A_p. More precisely we prove that the maximal domain $D_{p,max}(A)$ coincides with the weighted Sobolev space $D_p(A)$ defined by

$$D_p(A) := \{u \in W^{2,p}(\mathbb{R}^N) \;:\; Vu,\ (1 + |x|^{\alpha-1})\nabla u,\ (1 + |x|^\alpha) D^2 u \in L^p(\mathbb{R}^N)\}.$$

In the following lemma we give a complete proof of the weighted gradient and second derivative estimates.

Lemma 4. *Suppose that $N > 2$, $\alpha > 2$ and $\beta > \alpha - 2$. Then there exists a constant $C > 0$ such that for every $u \in D_p(A)$ we have*

$$\|(1 + |x|^{\alpha-1})\nabla u\|_p \le C(\|A_p u\|_p + \|u\|_p)\,, \tag{23}$$

$$\|(1 + |x|^{\alpha}) D^2 u\|_p \le C(\|A_p u\|_p + \|u\|_p)\,. \tag{24}$$

Proof. Let $u \in D_p(A)$. We fix $x_0 \in \mathbb{R}^N$ and choose $\vartheta \in C_c^\infty(\mathbb{R}^N)$ such that $0 \le \vartheta \le 1$, $\vartheta(x) = 1$ for $x \in B(1)$ and $\vartheta(x) = 0$ for $x \in \mathbb{R}^N \setminus B(2)$. Moreover, we set $\vartheta_\rho(x) = \vartheta\left(\frac{x - x_0}{\rho}\right)$, where $\rho = \frac{1}{4}(1 + |x_0|)$. We apply the well-known interpolation inequality (cf. [9, Theorem 7.27])

$$\|\nabla v\|_{L^p(B(R))} \le C\|v\|^{1/2}_{L^p(B(R))} \|\Delta v\|^{1/2}_{L^p(B(R))},\quad v \in W^{2,p}(B(R)) \cap W^{1,p}_0(B(R)),\ R > 0, \tag{25}$$

to the function $\vartheta_\rho u$ and obtain for every $\varepsilon > 0$,

$$\begin{aligned}
&\|(1 + |x_0|)^{\alpha-1}\nabla u\|_{L^p(B(x_0,\rho))} \le \|(1 + |x_0|)^{\alpha-1}\nabla(\vartheta_\rho u)\|_{L^p(B(x_0,2\rho))}\\
&\quad\le C\|(1 + |x_0|)^{\alpha}\Delta(\vartheta_\rho u)\|^{\frac{1}{2}}_{L^p(B(x_0,2\rho))}\|(1 + |x_0|)^{\alpha-2}\vartheta_\rho u\|^{\frac{1}{2}}_{L^p(B(x_0,2\rho))}\\
&\quad\le C\left(\varepsilon\|(1 + |x_0|)^{\alpha}\Delta(\vartheta_\rho u)\|_{L^p(B(x_0,2\rho))} + \frac{1}{4\varepsilon}\|(1 + |x_0|)^{\alpha-2}\vartheta_\rho u\|_{L^p(B(x_0,2\rho))}\right)\\
&\quad\le C\Big(\varepsilon\|(1 + |x_0|)^{\alpha}\Delta u\|_{L^p(B(x_0,2\rho))} + \frac{2M}{\rho}\varepsilon\|(1 + |x_0|)^{\alpha}\nabla u\|_{L^p(B(x_0,2\rho))}\\
&\qquad+ \frac{\varepsilon M}{\rho^2}\|(1 + |x_0|)^{\alpha} u\|_{L^p(B(x_0,2\rho))} + \frac{1}{4\varepsilon}\|(1 + |x_0|)^{\alpha-2} u\|_{L^p(B(x_0,2\rho))}\Big)\\
&\quad\le C\Big(\varepsilon\|(1 + |x_0|)^{\alpha}\Delta u\|_{L^p(B(x_0,2\rho))} + 8M\varepsilon\|(1 + |x_0|)^{\alpha-1}\nabla u\|_{L^p(B(x_0,2\rho))}\\
&\qquad+ \left(16\varepsilon M + \frac{1}{4\varepsilon}\right)\|(1 + |x_0|)^{\alpha-2} u\|_{L^p(B(x_0,2\rho))}\Big)
\end{aligned}$$

$$
\begin{aligned}
&\le C(M)\Big(\varepsilon\|(1+|x_0|)^\alpha \Delta u\|_{L^p(B(x_0,2\rho))}+\varepsilon\|(1+|x_0|)^{\alpha-1}\nabla u\|_{L^p(B(x_0,2\rho))}\\
&\quad +\frac{1}{\varepsilon}\|(1+|x_0|)^{\alpha-2}u\|_{L^p(B(x_0,2\rho))}\Big),
\end{aligned}
$$

where $M=\|\nabla\vartheta\|_\infty+\|\Delta\vartheta\|_\infty$. Since $2\rho=\frac{1}{2}(1+|x_0|)$ we get

$$
\frac{1}{2}(1+|x_0|)\le 1+|x|\le\frac{3}{2}(1+|x_0|),\qquad x\in B(x_0,2\rho).
$$

Thus,

$$
\begin{aligned}
&\|(1+|x|)^{\alpha-1}\nabla u\|_{L^p(B(x_0,\rho))}\le\left(\frac{3}{2}\right)^{\alpha-1}\|(1+|x_0|)^{\alpha-1}\nabla u\|_{L^p(B(x_0,\rho))}\\
&\quad\le C\Big(\varepsilon\|(1+|x_0|)^\alpha \Delta u\|_{L^p(B(x_0,2\rho))}+\varepsilon\|(1+|x_0|)^{\alpha-1}\nabla u\|_{L^p(B(x_0,2\rho))}\\
&\qquad+\frac{1}{\varepsilon}\|(1+|x_0|)^{\alpha-2}u\|_{L^p(B(x_0,2\rho))}\Big)\\
&\quad\le C\Big(2^\alpha\varepsilon\|(1+|x|)^\alpha \Delta u\|_{L^p(B(x_0,2\rho))}+2^{\alpha-1}\varepsilon\|(1+|x|)^{\alpha-1}\nabla u\|_{L^p(B(x_0,2\rho))}\\
&\qquad+\frac{2^{\alpha-2}}{\varepsilon}\|(1+|x|)^{\alpha-2}u\|_{L^p(B(x_0,2\rho))}\Big).
\end{aligned}\tag{26}
$$

Let $\{B(x_n,\rho(x_n))\}$ be a countable covering of $\mathbb{R}^N$ as in Proposition 3 such that at most ζ among the double balls $\{B(x_n,2\rho(x_n))\}$ overlap.

We write (26) with x_0 replaced by x_n and sum over n, we obtain

$$
\begin{aligned}
&\|(1+|x|)^{\alpha-1}\nabla u\|_p\\
&\le\ C\zeta\left(\varepsilon\|(1+|x|)^\alpha\Delta u\|_p+\varepsilon\|(1+|x|)^{\alpha-1}\nabla u\|_p+\frac{1}{\varepsilon}\|(1+|x|)^{\alpha-2}u\|_p\right).\\
&\le C\varepsilon\|A_pu\|_p+C\varepsilon(1+|b|)\|(1+|x|)^{\alpha-1}\nabla u\|_p+C(\frac{1}{\varepsilon}+\varepsilon)\|(1+|x|^\beta)u\|_p.
\end{aligned}
$$

Choosing ε such that $\varepsilon C\zeta<\frac{1}{2(1+|b|)}$ we have

$$
\|(1+|x|)^{\alpha-1}\nabla u\|_p\le C\left(\|A_pu\|_p+\|(1+|x|^\beta)u\|_p\right)
$$

for some constant $C>0$. Furthermore, by (22), we know that $\|(1+|x|^\beta)u\|_p\le C(\|A_pu\|_p+\|u\|_p)$ for every $u\in D_p(A)\subset D_{p,max}(A)$ and some $C>0$. Hence,

$$
\|(1+|x|)^{\alpha-1}\nabla u\|_p\le C(\|A_pu\|_p+\|u\|_p).
$$

As regards the second order derivatives we recall the classical Calderón–Zygmund inequality on $B(1)$

$$
\|D^2v\|_{L^p(B(1))}\le C\|\Delta v\|_{L^p(B(1))},\quad v\in W^{2,p}(B(1))\cap W_0^{1,p}(B(1)).
$$

By rescaling and translating we obtain

$$\|D^2 v\|_{L^p(B(x_0,R))} \le C\|\Delta v\|_{L^p(B(x_0,R))} \tag{27}$$

for every $x_0 \in \mathbb{R}^N$, $R > 0$ and $v \in W^{2,p}(B(x_0, R)) \cap W_0^{1,p}(B(x_0, R))$. We observe that the constant C does not depend on R and x_0.

Then we fix $x_0 \in \mathbb{R}^N$ and choose ρ and $\vartheta_\rho \in C_c^\infty(\mathbb{R}^N)$ as above. Applying (27) to the function $\vartheta_\rho u$ in $B(x_0, 2\rho)$, we obtain

$$\begin{aligned}\|(1+|x_0|)^\alpha D^2 u\|_{L^p(B(x_0,\rho))} &\le \|(1+|x_0|)^\alpha D^2(\vartheta_\rho u)\|_{L^p(B(x_0,2\rho))}\\ &\le C\|(1+|x_0|)^\alpha \Delta(\vartheta_\rho u)\|_{L^p(B(x_0,2\rho))}.\end{aligned}$$

Arguing as above we obtain

$$\|(1+|x|)^\alpha D^2 u\|_p \le C\left(\|(1+|x|)^\alpha \Delta u\|_p + \|(1+|x|)^{\alpha-1}\nabla u\|_p + \|(1+|x|)^{\alpha-2} u\|_p\right).$$

The lemma follows from (22) and (23). □

The following result shows that $C_c^\infty(\mathbb{R}^N)$ is a core for A_p, since by Lemma 4 the norm (29) is equivalent to the graph norm of A_p. The proof is based on Theorem 1 and Lemma 4 and it is similar to the one given in [4, Lemma 4.3].

Lemma 5. *The space $C_c^\infty(\mathbb{R}^N)$ is dense in*

$$D_p(A) = \{u \in W^{2,p}(\mathbb{R}^N),\, Vu,\, (1+|x|^\alpha)D^2 u,\, (1+|x|^{\alpha-1})\nabla u \in L^p(\mathbb{R}^N)\} \tag{28}$$

endowed with the norm

$$\|u\|_{D_p(A)} := \|u\|_p + \|Vu\|_p + \|(1+|x|^{\alpha-1})|\nabla u|\|_p + \|(1+|x|^\alpha)|D^2 u|\|_p, \quad u \in D_p(A). \tag{29}$$

Now, we are ready to show the main result of this section:

Theorem 3. *Suppose that $N > 2$, $\alpha > 2$ and $\beta > \alpha - 2$. Then the operator A_p with domain $D_{p,max}(A)$ generates an analytic semigroup in $L^p(\mathbb{R}^N)$.*

Proof. Let $f \in L^p(\mathbb{R}^N)$, $\rho > 0$. Consider the operator $\widehat{A_p} := A_p - \omega$, where ω is a constant which will be chosen later. It is known that the elliptic problem in $L^p(B(\rho))$

$$\begin{cases} \lambda u - \widehat{A_p} u = f & in \;\; B(\rho), \\ u = 0 & on \;\; \partial B(\rho) \end{cases} \tag{30}$$

admits a unique solution u_ρ in $W^{2,p}(B(\rho)) \cap W_0^{1,p}(B(\rho))$ for $\lambda > 0$, (cf. [9, Theorem 9.15]).

Let us prove that $e^{\pm i\theta}\widehat{A_p}$ is dissipative in $B(\rho)$ for $0 \le \theta \le \theta_\alpha$ with suitable $\theta_\alpha \in (0, \frac{\pi}{2}]$. To this purpose observe that

$$\widehat{A_p}u_\rho = \operatorname{div}((1+|x|^\alpha)\nabla u_\rho) + (b-\alpha)|x|^{\alpha-1}\frac{x}{|x|}\cdot\nabla u_\rho - c|x|^\beta u_\rho - \omega u_\rho.$$

Set $u^* = \bar{u}_\rho|u_\rho|^{p-2}$ and recall that $q(x) = 1 + |x|^\alpha$. Multiplying $\widehat{A_p}u_\rho$ by u^* and integrating over $B(\rho)$, we obtain

$$\begin{aligned}
\int\limits_{B(\rho)} \widehat{A_p}u_\rho u^* dx = &- \int\limits_{B(\rho)} q(x)|u_\rho|^{p-4}|Re(\text{[illegible]} {}^{2}dx - \int\limits_{B(\rho)} q(x)|u_\rho|^{p-4}|Im(\bar{u}_\rho\nabla u_\rho)|^2 dx \\
&- \int\limits_{B(\rho)} \bar{u}_\rho|u_\rho|^{p-2}\nabla q(x)\nabla u_\rho dx - (p-2)\int\limits_{B(\rho)} q(x)|u_\rho|^{p-4}\bar{u}_\rho\nabla u_\rho|Re(\bar{u}_\rho\nabla u_\rho)|^2 dx \\
&+ b\int\limits_{B(\rho)} \bar{u}_\rho|u_\rho|^{p-2}|x|^{\alpha-1}\frac{x}{|x|}\nabla u_\rho dx - \int\limits_{B(\rho)} \left(c|x|^\beta + \omega\right)|u_\rho|^p dx.
\end{aligned}$$

We note here that the integration by part in the singular case $1 < p < 2$ is allowed thanks to [13]. By taking the real part of the left and the right hand side, we have

$$\begin{aligned}
&Re\left(\int\limits_{B(\rho)} \widehat{A_p}u_\rho u^* dx\right) \\
&= -(p-1)\int\limits_{B(\rho)} q(x)|u_\rho|^{p-4}|Re(\bar{u}_\rho\nabla u_\rho)|^2 dx - \int\limits_{B(\rho)} q(x)|u_\rho|^{p-4}|Im(\bar{u}_\rho\nabla u_\rho)|^2 dx \\
&\quad - \int\limits_{B(\rho)} |u_\rho|^{p-2}\nabla q(x) Re(\bar{u}_\rho\nabla u_\rho)dx + b\int\limits_{B(\rho)} |u_\rho|^{p-2}|x|^{\alpha-1}\frac{x}{|x|}Re(\bar{u}_\rho\nabla u_\rho)dx \\
&\quad - \int\limits_{B(\rho)} \left(c|x|^\beta + \omega\right)|u_\rho|^p dx. \\
&= -(p-1)\int\limits_{B(\rho)} q(x)|u_\rho|^{p-4}|Re(\bar{u}_\rho\nabla u_\rho)|^2 dx - \int\limits_{B(\rho)} q(x)|u_\rho|^{p-4}|Im(\bar{u}_\rho\nabla u_\rho)|^2 dx \\
&\quad + \int\limits_{B(\rho)} \left(\frac{(\alpha-b)(N-2+\alpha)}{p}|x|^{\alpha-2} - c|x|^\beta - \omega\right)|u_\rho|^p dx.
\end{aligned}$$

Taking now the imaginary part of the left and the right hand side, we obtain

$$\begin{aligned}
&Im\left(\int\limits_{B(\rho)} \widehat{A_p}u_\rho u^* dx\right) \\
&= -(p-2)\int\limits_{B(\rho)} q(x)|u_\rho|^{p-4} Im(\bar{u}_\rho\nabla u_\rho) Re(\bar{u}_\rho\nabla u_\rho) dx
\end{aligned}$$

$$-\int\limits_{B(\rho)} |u_\rho|^{p-2}\nabla q(x) Im(\bar{u}_\rho \nabla u_\rho)dx + b\int\limits_{B(\rho)} |u_\rho|^{p-2}|x|^{\alpha-1}\frac{x}{|x|} Im(\bar{u}_\rho \nabla u_\rho)dx.$$

We can choose $\omega > 0$ such that

$$\frac{(\alpha-b)(N-2+\alpha)}{p}|x|^{\alpha-2} - c|x|^\beta - \omega \le -\frac{|\alpha-b|(N-2+\alpha)}{p}|x|^{\alpha-2}.$$

Furthermore,

$$\begin{aligned}
-Re\left(\int\limits_{B(\rho)} \widehat{A_p}u_\rho\, u^\star dx\right) &\ge (p-1)\int\limits_{B(\rho)} q(x)|u_\rho|^{p-4}|Re(\bar{u}_\rho \nabla u_\rho)|^2 dx \\
&\quad + \int\limits_{B(\rho)} q(x)|u_\rho|^{p-4}|Im(\bar{u}_\rho \nabla u_\rho)|^2 dx + \tilde{c}\int\limits_{B(\rho)} |u_\rho|^p|x|^{\alpha-2}dx \\
&= (p-1)B^2 + C^2 + \tilde{c}D^2,
\end{aligned}$$

where $\tilde{c} = \frac{|\alpha-b|(N-2+\alpha)}{p}$ is a positive constant.

Moreover,

$$\begin{aligned}
&\left|Im\left(\int\limits_{B(\rho)} \widehat{A_p}u_\rho\, u^\star dx\right)\right| \\
&\le |p-2|\left(\int\limits_{B(\rho)} |u_\rho|^{p-4}q(x)|Re(\bar{u}_\rho \nabla u_\rho)|^2 dx\right)^{\frac{1}{2}}\left(\int\limits_{B(\rho)} |u_\rho|^{p-4}q(x)|Im(\bar{u}_\rho \nabla u_\rho)|^2 dx\right)^{\frac{1}{2}} \\
&\qquad + |\alpha-b|\left(\int\limits_{B(\rho)} |u_\rho|^{p-4}|x|^\alpha|Im(\bar{u}_\rho \nabla u_\rho)|^2\, dx\right)^{\frac{1}{2}}\left(\int\limits_{B(\rho)} |u_\rho|^p|x|^{\alpha-2}\, dx\right)^{\frac{1}{2}} \\
&= |p-2|BC + |\alpha-b|CD,
\end{aligned}$$

where we have set

$$B^2 = \int\limits_{B(\rho)} q(x)|u_\rho|^{p-4}|Re(\bar{u}_\rho \nabla u_\rho)|^2 dx$$

$$C^2 = \int\limits_{B(\rho)} q(x)|u_\rho|^{p-4}|Im(\bar{u}_\rho \nabla u_\rho)|^2 dx$$

$$D^2 = \int\limits_{B(\rho)} |x|^{\alpha-2}|u_\rho|^p dx.$$

As a result of the above estimates, we conclude

$$\left| Im\left(\int\limits_{B(\rho)} \widehat{A_p}u_\rho u^* dx\right)\right| \le l_\alpha^{-1}\left[-Re\left(\int\limits_{B(\rho)} \widehat{A_p}u_\rho u^* dx\right)\right].$$

If $\tan\theta_\alpha = l_\alpha$, then $e^{\pm i\theta}\widehat{A_p}$ is dissipative in $B(\rho)$ for $0 \le \theta \le \theta_\alpha$. From [19, Theorem I.3.9], see also [7, Theorem II.4.6], it follows that the problem (30) has a unique solution u_ρ for every $\lambda \in \Sigma_\theta$, $0 \le \theta < \theta_\alpha$ where

$$\Sigma_\theta = \{\lambda \in \mathbb{C} \setminus \{0\} : |arg\,\lambda| < \pi/2 + \theta\}.$$

Moreover, there exists a constant C_θ which is independent of ρ, such that

$$\|u_\rho\|_{L^p(B(\rho))} \le \frac{C_\theta}{|\lambda|}\|f\|_{L^p}, \quad \lambda \in \Sigma_\theta. \tag{31}$$

Let us now fix $\lambda \in \Sigma_\theta$, with $0 < \theta < \theta_\alpha$ and a radius $r > 0$. We apply the interior L^p estimates (cf. [9, Theorem 9.11]) to the functions u_ρ with $\rho > r + 1$. So, by (31), we have

$$\|u_\rho\|_{W^{2,p}(B(r))} \le C_1\left(\|\lambda u_\rho - \widehat{A_p}u_\rho\|_{L^p(B(r+1))} + \|u_\rho\|_{L^p(B(r+1))}\right) \le C_2\|f\|_p. \tag{32}$$

Using a weak compactness and a diagonal argument, we can construct a sequence $(\rho_n) \to \infty$ such that the functions (u_{ρ_n}) converge weakly in $W^{2,p}_{loc}(\mathbb{R}^N)$ to a function u which satisfies $\lambda u - \widehat{A_p}u = f$ and

$$\|u\|_p \le \frac{C_\theta}{|\lambda|}\|f\|_p, \quad \lambda \in \Sigma_\theta. \tag{33}$$

Moreover, $u \in D_{p,max}(A)$. We have now only to show that $\lambda - \widehat{A_p}$ is invertible on $D_{p,max}(A)$ for $\lambda_0 < \lambda \in \Sigma_\theta$. Consider the set

$$E = \{r > 0 : \Sigma_\theta \cap C(r) \subset \rho(\widehat{A_p})\},$$

where $C(r) := \{\lambda \in \mathbb{C} : |\lambda| < r\}$. Since, by Theorem 1, λ_0 is in the resolvent set of $\widehat{A_p}$, then $R = \sup E > 0$. On the other hand, the norm of the resolvent is bounded by $C_\theta/|\lambda|$ in $C(R) \cap \Sigma_\theta$. Consequently it cannot explode on the boundary of $C(R)$. Then $R = \infty$ and this ends the proof of the theorem. □

Let us show that $D_{p,max}(A)$ and $D_p(A)$ coincide.

Theorem 4. *Assume that $N > 2$, $\alpha > 2$ and $\beta > \alpha - 2$. Then maximal domain $D_{p,max}(A)$ coincides with $D_p(A)$.*

Proof. We have to prove only the inclusion $D_{p,max}(A) \subset D_p(A)$.

Let $\tilde{u} \in D_{p,max}(A)$ and set $f = \lambda \tilde{u} - A_p \tilde{u}$. The operator A in $B(\rho)$, $\rho > 0$, is an uniformly elliptic operator with bounded coefficients. Then the Dirichlet problem

$$\begin{cases} \lambda u - Au = f & \text{in } B(\rho) \\ u = 0 & \text{on } \partial B(\rho)\,, \end{cases} \tag{34}$$

admits a unique solution u_ρ in $W^{2,p}(B(\rho)) \cap W_0^{1,p}(B(\rho))$ (cf. [9, Theorem 9.15]). So, $\widetilde{u_\rho}$, the zero extension of u_ρ to the complement $B(\rho)^c$, belongs to $D_p(A)$. Thus, by Lemma 4 and (22), we have

$$\begin{aligned} &\|(1+|x|^{\alpha-2})\widetilde{u_\rho}\|_p + \|(1+|x|^{\alpha-1})\nabla\widetilde{u_\rho}\|_p \\ &\quad + \|(1+|x|^{\alpha})D^2\widetilde{u_\rho}\|_p + \|V\widetilde{u_\rho}\|_p \le C(\|A\widetilde{u_\rho}\|_p + \|\widetilde{u_\rho}\|_p) \end{aligned}$$

with C independent of ρ.

We observe that u_ρ is the solution of (30) with λ replaced with $\lambda - \omega$. Then arguing as in the proof of Theorem 3, by (31) and (32) for $\lambda > \omega$, we have $\|u_\rho\|_{L^p(B(\rho))} \le \frac{C_1}{\lambda-\omega}\|f\|_{L^p}$ and $\|u_\rho\|_{W^{2,p}(B(r))} \le C_2\|f\|_{L^p}$ where $r < \rho - 1$ and C_1, C_2 are positive constants which do not depend on ρ.

Using a standard weak compactness argument we can construct a sequence $\widetilde{u_{\rho_n}}$ which converges to a function u in $W^{2,p}_{loc}(\mathbb{R}^N) \cap L^p(\mathbb{R}^N)$ such that $\lambda u - Au = f$. Since the estimates above are independent of ρ, also $u \in D_p(A)$. Then $\lambda\tilde{u} - A\tilde{u} = \lambda u - Au$ and since $D_p(A) \subset D_{p,max}(A)$ and $\lambda - A$ is invertible on $D_{p,\max}(A)$ by Theorem 1, we have $\tilde{u} = u$. □

Proposition 5. *For any $f \in L^p(\mathbb{R}^N)$, $1 < p < \infty$, and any $0 < \nu < 1$ and for all $t > 0$, the function $T_p(t)f$ belongs to $C_b^{1+\nu}(\mathbb{R}^N)$. In particular, the semigroup $(T_p(t))_{t\ge 0}$ is ultracontractive.*

Proof. In Theorem 3 we have proved that A_p generates an analytic semigroup $T_p(\cdot)$ on $L^p(\mathbb{R}^N)$ and in Theorem 2 we have obtained that for $f \in L^p(\mathbb{R}^N) \cap C_0(\mathbb{R}^N)$, $(\lambda - A_p)^{-1} f = (\lambda - A)^{-1} f$. Hence this shows the coherence of the resolvents on $L^p(\mathbb{R}^N) \cap L^q(\mathbb{R}^N)$ by using a density argument. This will yield immediately that the semigroups are coherent in different L^p-spaces. One can deduce the result by using the same arguments as in the proof of [11, Proposition 2.6]. □

To end this section we study the spectrum of A_p.

Proposition 6. *Assume $N > 2$, $\alpha > 2$, $\beta > \alpha - 2$. Then, for $p \in (1, \infty)$, the resolvent operator $R(\lambda, A_p)$ is compact in $L^p(\mathbb{R}^N)$ for all $\omega_0 < \lambda \in \rho(A_p)$, where ω_0 is a suitable positive constant, and the spectrum of A_p consists of a sequence of negative real eigenvalues which accumulates at $-\infty$. Moreover, $\sigma(A_p)$ is independent of p.*

Proof. The proof is similar to the one given in [11,16]. □

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