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On combinatorial structures arising in group theory

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ABSTRACT

This thesis explores two central themes in contemporary group theory. The first concerns graphs naturally arising from groups, showing how their combinatorial features can illuminate significant aspects of the group's algebraic structure. The second addresses embedding problems, with a focus on verbal subgroups and subgroups defined by commutator conditions, highlighting how such embeddings reflect deeper structural constraints. Together, these perspectives illustrate the interplay between combinatorial methods and algebraic properties in the study of groups.

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Introduction

Groups can be studied not only through their algebraic structure, but also via graphs that encode their internal relations. By representing elements or subgroups as vertices and prescribing edges according to specific group-theoretic rules, one obtains a wide variety of graphs whose combinatorial properties reflect, and often illuminate, the structure of the underlying group.

The association of graphs to groups goes back to the 19th century when Cayley [13] introduced a graph that encodes the abstract structure of a group. Let G be a group and S a subset of $G \setminus \{1\}$ such that $S = S^{-1}$. Then the Cayley graph $\Gamma(G : S)$ of G with respect to S is the graph whose set of vertices is $V = G$ and two vertices $x, y \in V$ are connected by an edge if and only if $xy^{-1} \in S$. The Cayley graph is nowadays a central tool in combinatorial and geometric group theory, as it is possible to grasp information about a group studying the combinatorial properties of its Cayley graph. Later, in the middle of the 20th century, a new family of graphs associated to a group has been considered. More precisely, given a group property P and a group G one can consider the graph $\Gamma_P(G)$ whose set of vertices is G and two vertices x and y of G are adjacent if and only if the subgroup generated by x and y, namely $\langle x, y\rangle$, is a P-group.

In the literature, a first example is the commuting graph of a group G, $\Gamma_{\mathrm{comm}}(G)$, where two vertices x and y of G are connected by an edge if and only if the $\langle x, y\rangle$ is abelian, that is, if x and y commute. This graph was implicitly defined by Brauer and Fowler in 1955 [8]. In that paper, they showed that, if there are at least two conjugacy classes of involutions, then any two involutions have distance at most 3 by a path in the commuting graph which avoids the identity. Of particular importance is the study of connectivity and diameter in group-related graphs. The first step is to detect the set of universal vertices of the graph; namely, those vertices that are adjacent to all the other vertices of the graph. Obviously, in the commuting graph, this set is precisely the center of the group G, namely $Z(G)$. The second step is to consider the subgraph in which all universal vertices have been removed, in this way a new graph appears, for which it is not an easy task, in general, to establish whether this is connected or not, and, in the affirmative case, which is its diameter. Thus, the graph $\Delta_{\mathrm{comm}}(G)$, which is the subgraph of the commuting graph induced by $G \setminus Z(G)$, was considered.

In this direction, Segev and Seitz in [49, Theorem 8] gave a bound on the diameter of $\Delta_{\mathrm{comm}}(G)$ regarding classic simple non-abelian groups; Iranmanesh and Jafaradeh in [37] studied the connectivity and the diameter of $\Delta_{\mathrm{comm}}(G)$ of alternating and symmetric groups, giving again a bound for these classes. However, in 2013, Giudici and Parker in [35, Theorem 1.1] showed that without additional hypothesis on G it is impossible to find a universal bound to the diameter of $\Delta_{\mathrm{comm}}(G)$. Thus, in this direction, Morgan and Parker in [42, 43] showed that if G has trivial center then the diameter of $\Delta_{\mathrm{comm}}(G)$ is at most 10 and, moreover, if G is also soluble the bound is brought down to 8. Other authors contributed on this topic, establishing some conditions under which the bound can be improved, see for instance [6, 11, 36].

Another line of research is investigated, asking which properties does the group G satisfy given a precise value k to the diameter of the connected components of $\Delta_{\mathrm{comm}}(G)$. Recall that a group G is called commutative-transitive (CT-group) if $[x, y] = 1$ and $[y, z] = 1$ imply $[x, z] = 1$ for all $x, y, z \in G \smallsetminus \{1\}$, where $[x, y]$ denotes the commutator $x^{-1}y^{-1}xy$. Obviously, asking for G to be a CT-group is equivalent to ask for $\Delta_{\mathrm{comm}}(G)$ to have all the connected components of diameter 1. In 1998 Wu, in [57, Theorem 9], characterized locally finite CT-groups completing the discussion for the case $k = 1$.

We investigate the case $k = 2$. To simplify the notation, in this specific case, we refer to diameter of $\Delta_{\mathrm{comm}}(G)$ intending, when this is disconnected, the maximum of the diameters of its connected components. Consider a group G such that $\Delta_{\mathrm{comm}}(G)$ has diameter 2. We first address the disconnected case, supposing G is not a simple group. Indeed, we characterize non-simple groups with trivial center for which $\Delta_{\mathrm{comm}}(G)$ is disconnected of diameter 2.

Theorem A. *Let G be a non-simple finite group with trivial center. Then $\Delta_{\mathrm{comm}}(G)$ is disconnected of diameter 2 if and only if G is a Frobenius group with non-abelian kernel or non-abelian complement.*

We follow our investigation focusing on the connected case, distinguishing between the sub-case in which the group is decomposable and the one in which it is indecomposable. In Theorem B, we characterize decomposable groups for which $\Delta_{\mathrm{comm}}(G)$ is connected of diameter 2.

Theorem B. *Let $G = H \times K$ be a group. Then $\Delta_{\mathrm{comm}}(G)$ is connected of diameter 2 if and only if $\Delta_{\mathrm{comm}}(H)$ or $\Delta_{\mathrm{comm}}(K)$ is connected of diameter 2.*

We also discuss some necessary conditions for an indecomposable group to have $\Delta_{\mathrm{comm}}(G)$ connected of diameter 2.

Another graph we analyzed is the nilpotent graph of a group G, which is the simple undirected graph where the vertices correspond to the elements of

G, and two distinct vertices x and y are adjacent if and only if the subgroup $\langle x, y\rangle$ generated by x and y is nilpotent. In particular, here we focus on finite groups. The concept of the nilpotent graph first emerged implicitly in the series of papers [21–23], where the authors examine groups whose nilpotent graphs consist of complete connected components. Later, in [1], the authors studied the properties of the complement of the nilpotent graph. More recently, some specific properties of this graph have been explored in [53].

Recent work has focused on the properties of neighborhoods in various graphs defined on groups (see [2, 3, 41]). We start our analysis by focusing on the properties of neighborhoods within the nilpotent graph. For each element $x \in G$, the nilpotent neighborhood of x, denoted by $\mathrm{Nil}_G(x)$, is the set of elements $y \in G$ such that the subgroup $\langle x, y\rangle$ is nilpotent. Understanding the structure of these neighborhoods is a crucial step in classifying groups based on their nilpotent graph. As shown in [1, Lemma 3.3], the subset $\mathrm{Nil}_G(x)$ is not a subgroup in general. A still open question is the classification of groups for which $\mathrm{Nil}_G(x)$ is a subgroup of G for every $x \in G$. Such groups, which following [1] we call $\mathfrak{n}$-*groups*, exhibit special properties in relation to their nilpotent subgroups. While the classification of simple $\mathfrak{n}$-groups has been achieved in earlier works, such as [1], the study of soluble $\mathfrak{n}$-groups remains a relatively unexplored area.

We investigate soluble $\mathfrak{n}$-groups, establishing a full classification of those $\mathfrak{n}$-groups for which every nilpotent neighborhood is a nilpotent subgroup: indeed we prove that the class of Frobenius groups having a nilpotent Frobenius complement is the only class of soluble groups for which this condition holds.

Theorem C. *Let G be a finite soluble group with trivial center. Then $\mathrm{Nil}_G(x)$ is a nilpotent subgroup of G for every non-trivial element $x \in G$ if and only if G is a Frobenius group with nilpotent Frobenius complement.*

Moreover, we study the connectivity and diameter of the nilpotent graph. As mentioned before, the first step is to identify the set of universal vertices. When G is a finite group, the set of universal vertices in the nilpotent graph for G coincides with the hypercenter $Z_\infty(G)$ of G (see Proposition 2.1 in [1]). Thus, we consider the graph $\Delta_{\mathrm{nil}}(G)$, which is defined as the induced subgraph of the nilpotent graph on the set $G \setminus Z_\infty(G)$. We show that this graph can be computed via $G/Z_\infty(G)$.

Theorem D. *For any group G the number of connected components of $\Delta_{\mathrm{nil}}(G)$ equals the number of connected components of $\Delta_{\mathrm{nil}}(G/Z_\infty(G))$, and there is a correspondence between the connected components of $\Delta_{\mathrm{nil}}(G)$ and $\Delta_{\mathrm{nil}}(G/Z_\infty(G))$ that maps connected components of diameter 1 to connected components of diameter 0 or 1 and preserves the diameter of connected components whose diameter is greater than 1.*

For soluble groups, we show that $\Delta_{\mathrm{nil}}(G)$ is disconnected exactly when $G/Z_\infty(G)$ is Frobenius or 2-Frobenius.

Theorem E. *Let G be a non-nilpotent soluble group. Then $\Delta_{\mathrm{nil}}(G)$ is disconnected if and only if $G/Z_\infty(G)$ is a Frobenius group or a 2-Frobenius group.*

We see that the structure of the quotient group $G/Z_\infty(G)$ plays a pivotal role in determining the connectivity of $\Delta_{\mathrm{nil}}(G)$. In fact, we obtain this result starting from the relationship between $\Delta_{\mathrm{nil}}(G)$ and $\Delta_{\mathrm{comm}}(G)$. The graphs $\Delta_{\mathrm{nil}}(G)$ and $\Delta_{\mathrm{comm}}(G)$ share the same vertex set if and only if the center of G coincides with the hypercenter of G, in which case $\Delta_{\mathrm{comm}}(G)$ is a subgraph of $\Delta_{\mathrm{nil}}(G)$.

It makes sense to ask when these two graphs are equal. In fact, these two graphs coincide if and only if G is an A-group, i.e., a group whose Sylow subgroups are abelian, as proved in Theorem F.

Theorem F. *Let G be a group. Then $\Delta_{\mathrm{nil}}(G)$ coincides with $\Delta_{\mathrm{comm}}(G)$ if and only if G is an A-group.*

We prove that if $\Delta_{\mathrm{nil}}(G)$ is connected, then its diameter is bounded. In fact, building upon results of Morgan and Parker in [42] and [43], one may show that the diameter of any connected component of $\Delta_{\mathrm{nil}}(G)$ is at most 10. (We note that this result appears as [10, Proposition 7.6], but we include it for completeness). This bound can be reduced to 8 when G is soluble and $\Delta_{\mathrm{nil}}(G)$ is connected. When G is soluble and $\Delta_{\mathrm{nil}}(G)$ is disconnected, we can reduce it even further, as shown in Theorem G.

Theorem G. *Let G be a group. Then the following are true:*

(*i*) *the connected components of $\Delta_{\mathrm{nil}}(G)$ have diameter at most* 10*;*

(*ii*) *if G is soluble and $\Delta_{\mathrm{nil}}(G)$ is connected, then $\Delta_{\mathrm{nil}}(G)$ has diameter at most* 8*;*

(*iii*) *if G is soluble and $\Delta_{\mathrm{nil}}(G)$ is disconnected, then one connected component of $\Delta_{\mathrm{nil}}(G)$ has diameter at most* 5 *and the remaining connected components have diameter at most* 2.

Regarding $\Delta_{\mathrm{nil}}(G)$, we focus on the soluble case. When G is soluble, we present an example of a soluble group where $\Delta_{\mathrm{nil}}(G)$ is connected and has diameter 7. At this time, we do not have any examples where diameter 8 is obtained, so there is a gap between the bound on the diameter of $\Delta_{\mathrm{nil}}(G)$ and the examples we can obtain. The same situation occurs in the disconnected case: we do not know if the bound 5 is the best possible.

We show that tighter bounds on the diameter of $\Delta_{\mathrm{nil}}(G)$ can be obtained by imposing additional conditions on G; in such cases, the desired bounds are achieved. For example, assuming that G is an A-group, we prove that $\operatorname{diam}(\Delta_{\mathrm{nil}}(G)) \leq 6$ follows directly from Theorem G and a result of Carleton and Lewis. When G is a semidirect product of a normal cyclic subgroup by an abelian one, we show that $\operatorname{diam}(\Delta_{\mathrm{nil}}(G)) \leq 4$ (see Proposition 2.2.15). When G is soluble and $\operatorname{Fit}(G)$ is cyclic, we show that $\operatorname{diam}(\Delta_{\mathrm{nil}}(G)) \leq 5$ (see Proposition 2.2.16). Moreover, when G is a $\{p, q\}$-group, we show that $\operatorname{diam}(\Delta_{\mathrm{nil}}(G)) \leq 6$ (see Theorem 2.2.17).

Regarding the study of graphs defined on groups it is not uncommon to also consider directed graphs. In particular, we study a directed graph which encodes information about the lattice of normal subgroups of the associated group. The *directed normalizing graph* of a group G is the directed simple graph $\vec{\Gamma}_{\mathrm{norm}}(G)$ whose vertices are all elements of G, and there is a directed edge from a vertex x to a vertex y if the subgroup $\langle x \rangle$ is normal in the subgroup $\langle x, y \rangle$.

As our aim remains the study of connectivity, we start by studying the set $\operatorname{Univ}(G)$ of all universal vertices of $\vec{\Gamma}_{\mathrm{norm}}(G)$ for a group G, with the intention of detecting it and hence removing it. In general, this is far from being an easy task, and even worse when the graph is directed. Thus, our first objective shifts into providing information about the set $\operatorname{Univ}(G)$. It is easy to see that this, in general, is not a subgroup (see Example 2.3.5). We point out that, in the context of directed graphs, the term *universal vertex* refers to a *bidirectional universal vertex*, that is, a vertex x such that $x \leftrightarrow y$ for every $y \in G$. Groups whose elements are all universal in $\vec{\Gamma}_{\mathrm{norm}}(G)$ are precisely Dedekind groups, that is groups whose subgroups are all normal (see Theorem 2.3.12). Moreover, groups with only one universal vertex have been characterized (see Corollary 2.3.4).

Secondly, we move to the directed graph $\vec{\Delta}_{\mathrm{norm}}(G)$, obtained from $\vec{\Gamma}_{\mathrm{norm}}(G)$ by removing all bidirectional universal vertices. In particular, we study this graph when G is a decomposable group, establishing conditions under which $\vec{\Delta}_{\mathrm{norm}}(G)$ is strongly connected and giving a sharp upper bound on its diameter.

Theorem H. *Let $G = H \times K$ be a direct product of non-Dedekind groups. Then the graph $\vec{\Delta}_{\mathrm{norm}}(G)$ is strongly connected of diameter at most 3 provided that one of the following conditions holds:*

(*i*) $\operatorname{Univ}(G) = \operatorname{Univ}(H) \times \operatorname{Univ}(K)$*;*

(*ii*) $(h, 1) \in \operatorname{Univ}(G)$ *if and only if* $h \in Z(H)$*;*

(*iii*) $(1, k) \in \operatorname{Univ}(G)$ *if and only if* $k \in Z(K)$*.*

Next, we focus on finite soluble groups G having trivial center, with the aim of characterizing when $\vec{\Delta}_{\mathrm{norm}}(G)$ is strongly disconnected. We present the characterization in Theorem I.

Theorem I. *Let G be a finite soluble group with trivial center. Then $\vec{\Delta}_{\mathrm{norm}}(G)$ is strongly disconnected if and only if one of the following holds:*

- (i) *G is a Frobenius group;*
- (ii) *G is a 2-Frobenius group with $K \lhd KH \lhd G$, KH a Frobenius group and G/K a Frobenius group with kernel KH/K such that $p \nmid r-1$ for all $p \in \pi(H)$ and for all $r \in \pi(K)$.*

As a consequence of the main result in [43], we give a general bound on the diameter of $\vec{\Delta}_{\mathrm{norm}}(G)$, proving Theorem J.

Theorem J. *Let G be a finite soluble group with trivial center.*

- (i) *If $\vec{\Delta}_{\mathrm{norm}}(G)$ is strongly connected, then the diameter of $\vec{\Delta}_{\mathrm{norm}}(G)$ is at most* 8.
- (ii) *If $\vec{\Delta}_{\mathrm{norm}}(G)$ is strongly disconnected, then the number of strongly connected components is* $|\mathrm{Fit}(G)| + 1$*; moreover, one strongly connected component has diameter at most* 6 *and all other strongly connected components have diameter at most* 2.

It is worth mentioning that tighter bounds on the diameter of $\vec{\Delta}_{\mathrm{norm}}(G)$ can be obtained by imposing additional conditions on G. For instance, we prove that $\mathrm{diam}(\vec{\Delta}_{\mathrm{norm}}(G)) \leq 4$ provided that either G is cyclic-by-abelian (see Proposition 2.3.36), or the Fitting subgroup of G has prime index (see Proposition 2.3.38).

Finally, we also study the undirected normalizing graph, denoted by $\Gamma_{\mathrm{norm}}(G)$, which is the undirected graph induced by $\vec{\Gamma}_{\mathrm{norm}}(G)$. This graph has appeared in the literature very recently. More specifically, in [31, Theorem 1] Farrell and Parker classify when the subgraph induced by $\Gamma_{\mathrm{norm}}(G)$ on $G \setminus \{1\}$ is connected, also giving a sharp upper bound on its diameter provided the group is soluble with trivial center. Our main result in this direction shows that G is nilpotent of class at most 3 whenever $\Gamma_{\mathrm{norm}}(G)$ is complete (see Theorem 2.3.14).

The last graph we consider in this work is the verbal graph associated with a group. The main purpose is to introduce a general line of research in the context of graphs associated with groups that can be developed and investigated further. Thus, we present some considerations and basic results concerning the verbal graph. Let G be a group and $w(x, y)$ a word in two variables. The *verbal graph* of G related to w, or w-graph of G, denoted by $\vec{\Gamma}_w(G)$, is the directed simple graph whose set of vertices is the set of elements of G and a directed edge is drawn from an element $g \in G$ to an element $h \in G$ if and only if $w(g, h) = 1$. This graph has been defined first by Detomi, Lucchini and Nemmi in [28], but it has only been studied for particular choices of $w(x, y)$, like the commutator word

$w(x, y) = [x, y]$ which gives rise to the commuting graph and the n-th Engel word $w(x, y) = [x,_n y]$ which gives rise to the n-th Engel graph. Obviously, the verbal graph is strongly dependent on the choice of the word w. We focus on the set of universal vertices of this graph, detecting some properties, we characterize when such a graph is complete and we classify groups which have associated undirected verbal graphs with small clique numbers for commutator words.

In the final part of the thesis we focus on the study of words and embedding properties. The work has been motivated by the fact that the study of verbal subgroups within a group is well-known for being an effective tool to obtain structural information about a group. For instance, in Chapter 2, we show how the usage of words can be helpful in generalizing and understanding deeper group-theoretic problems. Therefore, conditions that allow the classification of words in a free group are of paramount importance. In the first section of Chapter 3, a hierarchy among words is introduced, generalizing the concept of concise words.

Let $w = w(x_1, \ldots, x_n)$ be a group-word in the variables $x_1, \ldots, x_n$. For any group G and arbitrary $g_1, \ldots, g_n \in G$, the elements $w(g_1, \ldots, g_n)$ are called the w-values in G. We write G_w to denote the set of all w-values in G and $w(G)$ to denote the verbal subgroup of G corresponding to w, which is the subgroup generated by all w-values. A word w is called concise if the verbal subgroup $w(G)$ is finite in each group G such that G_w is finite. One of the most studied problems is to establish which words are concise. In the sixties, Hall conjectured that every word is concise, but his conjecture was refuted in 1989 by Ivanov [39]. However, many words of common use are known to be concise. For instance, Hall proved in an unpublished work that non-commutator and lower central words are concise, and Turner-Smith in [54] showed that derived words are concise. More generally, multilinear commutator words have been shown to be concise by Wilson in [56, Theorem 1]. More examples of concise words can be found in [27] and in [32]. Later, aiming to enhance the comprehension of words within a free group, the concept of semiconcise words was introduced in [26], where a word is called semiconcise if the subgroup $[w(G), G]$ is finite in each group G such that G_w is finite. Of course concise words are semiconcise. In [26, Proposition 4.2] it is proved that if w is a semiconcise word and z is any variable not appearing in w, then the word $[w, z]$ is also semiconcise. Moreover, in [26, Proposition 4.4] the authors proved that there exists a word which is not semiconcise.

In this context we give the following definition. Let w be a group-word and n a positive integer. The word w is said to be $\frac{1}{n}$-*concise* if for any group G the finiteness of G_w implies that the subgroup

$$[w(G), \underbrace{G, \ldots, G}_{n-1 \text{ times}}]$$

is finite. Notice that for $n = 1, 2$ we obtain the definitions of concise and semiconcise word, respectively. Moreover, a word w is 0-*concise* if the finiteness of G_w for any group G implies that there exists a positive integer n (depending on the group G) such that the subgroup

$$[w(G), \underbrace{G, \ldots, G}_{n-1 \text{ times}}]$$

is finite. Obviously every $\frac{1}{n}$-concise word is $\frac{1}{m}$-concise for all $m \geq n$, and every $\frac{1}{n}$-concise word is 0-concise; so this introduces a sort of hierarchy on words. At the time of publication of our work on this topic in the "Journal of Group Theory" in [19], it was not known if there existed $\frac{1}{m}$-concise words that are not $\frac{1}{n}$-concise for some $m > n$. Still, we proved, in Corollary 3.1.10, that there exists a word which is not $\frac{1}{n}$-concise for every positive integer n. As a consequence, we provided an example of a word which is not 0-concise. However, later, further research on classification of words by Zozaya, proved that there exists a word that is semiconcise but not concise and, moreover, that every 0-concise word (and consequently every $\frac{1}{m}$-concise word) is also semiconcise [58].

Going further, in Chapter 3, we contribute to the study of FC-groups. A group G is called an FC-group if every element of G has finitely many conjugates, or equivalently if for each $x \in G$ the centralizer $C_G(x)$ of x has finite index in G. Finite groups and abelian groups provide immediate examples of FC-groups. We refer to [17, 51, 52] for a survey on the topic. The property of having finitely many conjugates can be used to study embedding properties of a subgroup of a group. A subgroup H of a group G is said to be FC-embedded in G if g^H is finite for all $g \in G$; furthermore H is said to be BFC-embedded in G if g^H is finite for all $g \in G$ and the number of elements in g^H is bounded by a constant that does not depend on the choice of g. This concept was both introduced and brought in the realm of group words and verbal subgroups in [33]. More precisely, given a group-word w, a group G is said to be FC(w)-group if the set of conjugates g^{G_w} is finite for all $g \in G$; moreover G is a BFC(w)-group if g^{G_w} is finite for all $g \in G$ and the number of elements in g^{G_w} is bounded by a constant that does not depend on the choice of g. Choosing $w = x_1$, one obtains the notions of FC-group and BFC-group.

Our contribution in this context follows.

Theorem K. *Let w be a $\frac{1}{m+1}$-concise word for some positive integer m, and let G be an* FC(w)*-group. Then the subgroup*

$$[w(G), \underbrace{G, \ldots, G}_{m \text{ times}}]$$

is FC*-embedded in G.*

Theorem L. *Let w be a $\frac{1}{m+1}$-concise word for some positive integer m, and let G be a* $\mathrm{BFC}(w)$*-group. Then the subgroup*

$$[w(G), \underbrace{G, \dots, G}_{m \text{ times}}]$$

is BFC*-embedded in G.*

These results are somewhat sharp. Indeed, we provide a group-word w for which there exists a $\mathrm{BFC}(w)$-group G such that the subgroup

$$[w(G), \underbrace{G, \dots, G}_{n-1 \text{ times}}]$$

is not FC-embedded in G, demonstrating that the hypothesis on the word w in Theorem K and Theorem L is necessary. We point out that different but related research directions in the context of profinite groups have been considered in [29, 50].

To better understand the behavior of these subgroups, we focus on the concept of perfectly embedded subgroups of a group, where, given subgroups H and K of a group G with $H \leq K$, we say that H is *perfectly embedded* in K if $[H, K] = H$. We explore three complementary directions. First, we consider the class of all groups G in which every normal subgroup H is perfectly embedded in K, in the extremal cases $K = G$ and $K = H$. We also show that the case $K = N_G(H)$, which would seem equally natural to be studied, is actually too restrictive because every non-trivial group contains a subgroup that is not perfectly embedded in its normalizer (see Proposition 3.2.8). Second, we prove that the groups in which the trivial subgroup is the unique perfectly embedded normal subgroup are precisely the hypocentral groups (see Theorem 3.2.9). Finally, we focus on the subclass of abelian normal subgroups, again considering the two extremal classes of groups: those in which all such subgroups are perfectly embedded, and those in which the trivial subgroup is the unique perfectly embedded abelian normal subgroup.

During the whole work, the use of GAP [34] was very helpful, not only in providing examples but also as a practical tool that supported the development of the theoretical results. Thus, at the end of the work, we provide an appendix containing the most useful GAP programs we used.

The thesis is organized as follows. In Chapter 1, we introduce the necessary preliminaries and background material required throughout the thesis. Chapter 2 is devoted to our work on graphs defined on groups, where we develop and analyze the main results. In Chapter 3, we address the problems related to group words and embeddings and Chapter 4 consists of the appendix.

Chapter 1

Preliminaries

1.1 Basic definitions on groups

This section collects the necessary group-theoretic background and notation used throughout the thesis, with a focus on concepts central to the themes of the work.

1.1.1 Relevant subgroups

We first start by recalling the definitions and most important properties of relevant subgroups of a group. Let G be a group. We denote by $Z(G)$ the center of the group G, i.e. the set of all elements that commute with every other element of the group. We denote by G' the derived subgroup of G, i.e., the subgroup generated by all commutators $[g, h] = g^{-1}h^{-1}gh$, with $g, h \in G$. Related to these two fundamental subgroups, there are the notions of the hypercenter and the hypocenter of a group, which provide important tools for analyzing the internal structure of a group. Both concepts are defined via central series, but from dual perspectives: the hypercenter is constructed as the union of the ascending upper central series, while the hypocenter is defined as the intersection of the descending lower central series. The hypercenter captures how much of the group is "central" in an ascending sense, identifying the largest normal subgroup for which the quotient has trivial center. In contrast, the hypocenter reflects the "residual non-nilpotence" of the group, as it is the smallest normal subgroup such that the corresponding quotient is residually nilpotent.

If G is any group and α an ordinal, the terms $Z_\alpha(G)$ of the upper central series are defined by the rules $Z_0(G) = \{1\}$ and

$$Z_{\alpha+1}(G)/Z_\alpha(G) = Z\Big(G/Z_\alpha(G)\Big)$$

together with the completeness condition

$$Z_\lambda(G) = \bigcup_{\alpha<\lambda} Z_\alpha(G),$$

for each limit ordinal λ. Equivalently, $Z_{\alpha+1}(G)$ is the full preimage of the center of $G/Z_\alpha(G)$ under the canonical projection from G to $G/Z_\alpha(G)$. Since the ascending chain of subgroups of G cannot continue strictly beyond $|G|$, there is some ordinal δ such that

$$Z_\delta(G) = Z_{\delta+1}(G),$$

and this terminal subgroup $Z_\delta(G)$ is called the *hypercenter* of G. Notice that $Z_\delta(G)$ does not need to coincide with G, in fact when $Z_\delta(G) = G$ the group G is said to be *hypercentral.* The hypercenter $Z_\delta(G)$ is the largest normal subgroup of G such that the quotient $G/Z_\delta(G)$ has a trivial center. If G is a finite group we denote the hypercenter by $Z_\infty(G)$, which is is clearly a nilpotent subgroup of G. Of course if we restrict our attention to finite groups then the class of hypercentral groups coincides with the class of nilpotent groups.

Dually we define the hypocenter of a group. If G is any group and α an ordinal, the terms G_α of the lower central series are defined by the rules $G_0 = G$ and

$$G_{\alpha+1} = [G, G_\alpha],$$

together with the completeness condition

$$G_\lambda = \bigcap_{\alpha<\lambda} G_\alpha$$

for each limit ordinal λ, where $[G, G_\alpha]$ denotes the subgroup generated by all commutators $[g, h]$ with $g \in G$ and $h \in G_\alpha$. This descending chain of subgroups of G will stop at some ordinal, say ω. Thus $G_\omega = G_{\omega+1}$. This terminal subgroup G_ω is called the *hypocenter* of G. When $G_\omega = \{1\}$, one says that G is *hypocentral.* If ω is the first infinite ordinal, then G_ω is the smallest normal subgroup of G such that the quotient G/G_ω is residually nilpotent; that is, such that every non-identity element has a non-identity homomorphic image in a nilpotent group. If G is finite we denote the hypocenter by G_∞.

Together, these two constructions offer complementary perspectives on how close a group is to being abelian or nilpotent, and they often appear in the study of soluble and nilpotent groups.

Another subgroup that will have a key role in the work is the Fitting subgroup of a group, which plays a central role in the structure theory of finite groups, particularly in the study of soluble groups. It captures the largest normal nilpotent part of a group and serves as a useful tool in analyzing its internal composition.

Let G be a group. The *Fitting subgroup* of G, denoted by $\mathrm{Fit}(G)$, is the subgroup generated by all normal nilpotent subgroups of G. If G is finite then $\mathrm{Fit}(G)$ is nilpotent, and evidently it is the unique largest normal nilpotent subgroup.

The Fitting subgroup possesses several fundamental properties that make it a central object of study in the theory of finite groups. These properties not only reveal its internal structure but also highlight its influence on the structure and behavior of the group as a whole, especially in the case in which the group is finite and soluble. The Fitting subgroup of a group G is always a characteristic subgroup of G and, if G is nilpotent, then $\mathrm{Fit}(G) = G$. For finite soluble groups, it is possible to say something more. If G is a non-trivial soluble group then $\mathrm{Fit}(G)$ contains the smallest non-trivial term of the derived series and hence it cannot be trivial. Furthermore, it is possible to prove the following.

Theorem 1.1.1. *Let G be a finite soluble group. Then the following statements hold:*

- (*i*) *if $1 \neq N \trianglelefteq G$, then N contains a non-trivial normal abelian subgroup of G, and $N \cap \mathrm{Fit}(G) \neq 1$;*
- (*ii*) $C_G(\mathrm{Fit}(G)) = Z(\mathrm{Fit}(G))$.

Proof. For a proof, see 5.4.4 in [44]. □

We highlight that from property (*ii*) it follows that the Fitting subgroup of G contains the center of G. The hypothesis of finiteness and solubility are of course essential. In fact, in general, the Fitting subgroup does not always have these properties. Indeed, if a group is infinite its Fitting subgroup need not be nilpotent and if a group is not soluble then its Fitting subgroup does not always contain its own centralizer and it could also be trivial.

A very useful result is the following, that we state with proof for completeness.

Proposition 1.1.2. *Let G be a finite soluble group. If $\mathrm{Fit}(G)$ is cyclic then $G/\mathrm{Fit}(G)$ is abelian.*

Proof. Let $F = \mathrm{Fit}(G)$. Notice that since G is a finite soluble group we have $C_G(F) = Z(F) = F$. Thus $N_G(F)/C_G(F) = G/F$ is isomorphic to a subgroup of $\mathrm{Aut}(F)$ which is abelian because F is cyclic; therefore G/F is abelian. □

Of course there are much more properties that can be proved regarding the Fitting subgroup of a group, we only stated the most fundamental ones.

The last notable subgroup we highlight is the socle of a group, which gathers the “indecomposable” normal building blocks that cannot be broken down further by passing to normal subgroups. The *socle* of a group G, denoted by $\mathrm{Soc}(G)$, is the subgroup generated by all minimal normal subgroups of G:

$$\mathrm{Soc}(G) = \Big\langle M \mid M \trianglelefteq G,\ M \text{ minimal}\Big\rangle.$$

Thus, $\mathrm{Soc}(G)$ is characteristic in G and equals the product of all minimal normal subgroups of G.

For a finite group G the socle of G is always non-trivial. In particular, its description is clear: every minimal normal subgroup of G is characteristically simple, i.e. it does not contain any proper non-trivial characteristic subgroups, thus it is either an elementary abelian p-group or a direct power T^r of a non-abelian simple group T. Therefore we have

$$\mathrm{Soc}(G) \cong \prod_i C_{p_i}^{r_i} \times \prod_j T_j^{s_j},$$

where p_i is a prime, r_i, s_j are non-negative integers, C_{p_i} is the cyclic group of order p_i and T_j is a non-abelian simple group, for all i, j.

The socle of a group completes our list of notable subgroups. However, before turning our attention to other preliminary notions, we fix the following notation. If a group G has an element x whose order is divisible by a prime p, we denote by x_p the p-part of x. Moreover, for a prime p we denote by $O_p(G)$ the largest normal p-subgroup of G. Lastly, the symmetric group of degree n, the alternating group of degree n, the dihedral group of order $2n$, the quaternion group of order 2^n will be denoted by S_n, A_n, D_{2n} and Q_{2^n}, respectively.

1.1.2 Frobenius and 2-Frobenius Groups

In this section we recall the basic definitions and fundamental properties of Frobenius groups and 2-Frobenius groups. Throughout this section, all groups are finite, and G will denote a generic finite group. A finite group G is called a *Frobenius group* if there exists a proper non-trivial subgroup $H < G$ satisfying

$$H \cap H^g = \{1\},$$

for all $g \in G \setminus H$. Such an H is called a *Frobenius complement* and the subset

$$K = \Big(G \setminus \bigcup_{g \in G} H^g\Big) \cup \{1\}$$

is a normal subgroup, called the *Frobenius kernel.* Moreover, it is possible to prove that $G = K \rtimes H$.

The smallest example of a Frobenius group is the symmetric group S_3. It is possible to prove that the dihedral group D_{2n} is a Frobenius group if and only if

n is odd and $n > 1$. The following result highlights key properties of Frobenius groups that are particularly useful for identifying when a semidirect product forms a Frobenius group.

Theorem 1.1.3. *Assume that $G = K \rtimes H$ is the semidirect product of a normal subgroup K and a complement H. Then the following conditions are equivalent:*

- (i) *G is a Frobenius group with complement H;*
- (ii) *for every non-identity element $k \in K$, the centralizer $C_G(k) \subseteq K$;*
- (iii) *for every non-identity element $k \in K$, the centralizer $C_H(k) = \{1\}$;*
- (iv) *for every non-identity element $h \in H$, the centralizer $C_G(h) \subseteq H$.*

Proof. For a proof, see Theorem 6.4 in [38]. □

An important result that focuses on the action of the complement over the kernel is the following.

Theorem 1.1.4. *Let G be a finite group and $H < G$.*

- (i) *If G is a Frobenius group with complement H, then the action of G on the right cosets of H yields a faithful representation of G as a transitive non-regular permutation group in which no non-trivial element has more than one fixed point.*
- (ii) *Let G be a transitive but non-regular permutation group in which no non-trivial element has more than one fixed point. Then G is a Frobenius group. The Frobenius kernel consists of 1 and all elements of G with no fixed points.*

Proof. For a proof, see 8.5.6 in [44]. □

Frobenius groups have been extensively studied due to their rich structure and numerous applications. Below, we present a collection of properties, which not only illustrate the depth of their structure but also provide practical tools for recognizing Frobenius groups.

Theorem 1.1.5. *Let G be a Frobenius group with kernel K and complement H. Then the following statements hold:*

- (i) *the center of G is trivial;*
- (ii) *if $L \lhd G$, then either $L \leq K$ or $K \leq L$;*
- (iii) *every complement of K in G is a Frobenius complement, and all Frobenius complements are conjugate in G;*

(*iv*) $\gcd\big(|K|,|H|\big) = 1$, *moreover* $|H|$ *divides* $|K| - 1$;

(*v*) $Z(H) \neq \{1\}$;

(*vi*) *if* $|G : K|$ *is even, then* K *is abelian and* $|K|$ *is odd;*

(*vii*) *if* G *is finite then it has a unique Frobenius kernel, the Fitting subgroup of* G, *namely* $\mathrm{Fit}(G)$;

(*viii*) *if* $U \leq G$ *with* $U \cap K = \{1\}$, *then* $U \leq H^g$ *for some* $g \in G$.

Proof. For a proof, see chapter 6 of [38]. □

Probably, one of the most important properties of a Frobenius group is stated in (*i*) of the following Theorem.

Theorem 1.1.6. *Let* G *be a Frobenius group with kernel* K *and complement* H, *then:*

(*i*) K *is nilpotent;*

(*ii*) *the Sylow* p*-subgroups of* H *are cyclic if* $p > 2$, *and cyclic or generalized quaternion if* $p = 2$.

Proof. For a proof, see 10.5.6 in [44]. □

Lastly, for the sake of convenience and completeness, we state the following Lemma with proof.

Lemma 1.1.7. *Let* $G = KH$ *be a Frobenius group, with kernel* K *and complement* H. *Then* $N_G(\langle h \rangle) \subseteq H$ *for all* $h \in H \setminus \{1\}$.

Proof. Let $h \in H \setminus \{1\}$ and $g \in N_G(\langle h \rangle)$. Then h^g belongs to $H^g \cap H \neq \{1\}$. If $g \in G \setminus H$ then $H \cap H^g = \{1\}$. Therefore $g \in H$. □

In the broader study of Frobenius groups, an important role is covered by 2-Frobenius groups which appear naturally in various classification problems in finite group theory.

Definition 1.1.8. *A group* G *is a* 2-Frobenius group *if there exist three subgroups* K, H, L *of* G *such that* K *and* KH *are normal in* G, KH *is a Frobenius group with kernel* K *and complement* H *and* G/K *is a Frobenius group with kernel* KH/K *and complement* L/K. *Moreover, we set* $X = \bigcup_{g \in G} H^g$.

In this setup, both quotient groups HK/K and G/K are Frobenius groups, and G is said to have a 2-step Frobenius structure. An example of a 2-Frobenius group is the symmetric group S_4. As a consequence of point (vii) in Theorem 1.1.5 we have $K = \mathrm{Fit(G)}$ and $HK/K = \mathrm{Fit}(G/K)$.

A 2-Frobenius group is always soluble, and more properties, in addition with some useful results, are listed below.

Proposition 1.1.9. *Let G be a 2-Frobenius group as in Definition 1.1.8. Then the following statements hold:*

(i) *$N_G(H)$ is a Frobenius group with Frobenius kernel H;*

(ii) *$N_L(H)$ is cyclic;*

(iii) *$L = KN_L(H)$;*

(iv) *L is a Hall subgroup of G.*

Proof. For a proof of (i), (ii), (iii) and (iv) see [16, Lemma 2.1, Lemma 2.3, Lemma 2.4 and Lemma 2.5], respectively. ☐

Lemma 1.1.10. *Let G be a 2-Frobenius group as in Definition 1.1.8. Then:*

(i) *H is cyclic of odd order;*

(ii) *for all $g \in G \setminus HK$ such that $o(g) = p$, with p prime, $C_K(g) \neq \{1\}$;*

(iii) *$\bigcup_{g\in G} H^g = \bigcup_{k\in K} H^k$.*

Proof. For a proof of (i) and (ii), see [16, Lemma 2.2 and Lemma 3.8]. For (iii) we argue as follows. Let $g \in G$. Since KH is normal in G we have $H^g \subseteq HK$ and so $\bigcup_{g\in G} H^g \subseteq KH$. Moreover, KH is a Frobenius group with kernel K and thus $\bigcup_{g\in G} H^g \subseteq \bigcup_{g\in KH} H^g = \bigcup_{k\in K} H^k$. The other inclusion is trivial. ☐

1.1.3 Words and verbal subgroups

Group words and verbal subgroups play a fundamental role in group theory, particularly in the study of group identities and varieties of groups.

Let F be the free group in the variables $x_1, x_2, \ldots, x_n$, with $n \in \mathbb{N}$. A *group-word* $w(x_1, \ldots, x_n)$ is an element of F. This word is called a *commutator word* if it belongs to the derived subgroup F'. As a consequence, we can describe commutator words efficiently, in the following way. Let $w(x_1, x_2, \ldots, x_n) = z_1^{\varepsilon_1} \ldots z_r^{\varepsilon_r}$ be a group-word, with $r \geq n$, $\varepsilon_i \in \{-1, 1\}$ and $z_i \in \{x_1, \ldots, x_n\}$ for all i. Then w is a commutator word if

$$\sum_{i:z_i=x_j} \varepsilon_i = 0$$

for all $j \in \{1, \dots, n\}$. If a word is not a commutator word it is said to be a non-commutator word.

For any group G, a word $w(x_1, \dots, x_n)$ can be interpreted as a map from G^n in G, that maps every n-tuple $(g_1, \dots, g_n)$ of elements of G to the element of G obtained by substituting the group elements in the place of the variables, or in other words by evaluating $w(g_1, \dots, g_n)$. The set

$$G_w = \{w(g_1, g_2, \dots, g_n) \mid g_i \in G, 1 \leq i \leq n\}$$

is called the set of w-values of G. It is possible to generalize the above definition to a non-empty set of words W defining the set of W-values as

$$G_W = \bigcup_{w \in W} G_w.$$

In general this set is not a subgroup of G. The subgroup generated by all W-values of G is the *verbal subgroup* of G associated with W, which is denoted with $W(G)$. When W is a singleton, say $W = \{w\}$, we denote the verbal subgroup of G associated with W by $w(G)$ and we call it, for simplicity, the verbal subgroup of G associated with the word w. Verbal subgroups are fully-invariant subgroups of G. The converse, in general, is false, but it holds in the case in which G is a free group, as proved by B.H. Neumann.

Theorem 1.1.11. *Let G be a free group. If H is a fully-invariant subgroup of G then H is verbal.*

Proof. For a proof, see 2.3.1 in [44]. □

Philip Hall defined the dual concept to verbal subgroups which is the one of marginal subgroup. Let W be a set of words in the variables $x_1, \dots x_n$. The marginal subgroup of G associated with W, denoted by $W^*(G)$, is the collection of all $a \in G$ such that for all $w \in W$, for all $g_1, \dots, g_n \in G$ and each $i \in \{1, \dots, n\}$ it happens

$$w(g_1, ..., g_{i-1}, ag_i, g_{i+1}, ..., g_n) = w(g_1, ..., g_{i-1}, g_i a, g_{i+1}, ..., g_n) = w(g_1, ..., g_n).$$

As in the previous case, if the set W is a singleton, say $W = \{w\}$, we denote the marginal subgroup of G associated with W by $w^*(G)$. It turns out that this collection is, indeed, a characteristic subgroup, but, in general, it is not fully-invariant. For example one can consider the word $w(x, y) = x^{-1}y^{-1}xy = [x, y]$. Then $w(G)$ is the subgroup generated by all the commutators $[g, h]$, with $g, h \in G$, and thus $w(G) = G'$. Moreover, if $a \in w^*(G)$ and $g \in G$ then $[g, a] = [g, 1a] = [g, 1] = 1$, for all $g \in G$. Thus $a \in Z(G)$. Conversely, if $a \in Z(G)$, then $[g_1, g_2 a] = [g_1, g_2]$, for all $g_1, g_2 \in G$. This shows that $w^*(G) = Z(G)$ for this word w.

These two subgroups are somehow related to each other. In fact, it is possible to prove the following result.

Proposition 1.1.12. *Let G be a group and W a non-empty set of words. Then $W(G) = \{1\}$ if and only if $W^*(G) = G$.*

Proof. For a proof, see 2.3.2 in [44]. □

These subgroups provide insight into the structure of G in relation to the word w. Thus, understanding group words and their associated verbal and marginal subgroups lays the groundwork for many advanced results in group theory, both in abstract and applied contexts.

Lastly, we recall how words can be used to define many classes of groups. Let W a non-empty set of words. The variety determined by W is denoted by $\mathcal{B}(W)$ and it is the class of all groups such that $W^*(G) = G$.

This definition can be given equivalently, using Proposition 1.1.12, changing the condition $W^*(G) = G$ into $W(G) = \{1\}$. As an example for $W = \{[x_1, x_2]\}$ we obtain the class of abelian groups, while for $W = \{[x_1, x_2], x_1^p\}$, with p a prime, we obtain the class of elementary abelian p-groups.

1.2 Graphs associated with groups

Graphs are fundamental objects in discrete mathematics and computer science, used to model relationships between entities. In this thesis, we will assume basic familiarity with graph theory but recall here the most relevant definitions and notation that will be used throughout the work. A graph is an ordered pair $\Gamma = (V, E)$, where V is a set whose elements are called *vertices* (or *nodes*), and E is a set of pairs of distinct elements from V, called *edges*. Each edge represents a connection between two different vertices. Depending on whether the edges are directed or not, we distinguish between directed and undirected graphs.

1.2.1 Undirected graphs

An *undirected graph* is a graph $\Gamma = (V, E)$ in which the edges are unordered pairs of vertices, meaning that if $\{u, v\} \in E$, then the edge connects u and v symmetrically, with no assigned direction. When $\{u, v\} \in E$ we say that u and v are adjacent vertices in Γ. We will often denote this with $u \sim v$. We often work with simple graphs, which are graphs without any additional structure such as loops or multiple edges.

A simple graph is *complete* if every pair of distinct vertices is adjacent. Moreover, a graph is said to be *bipartite* if its vertex set V can be partitioned into two disjoint subsets V_1 and V_2, such that every edge connects a vertex from V_1 to a vertex from V_2, i.e. there are no edges between vertices within the same subset. These appear frequently in applications like matching problems and scheduling.

In many situations, it is useful to study a smaller portion of a larger graph. For example, one might be interested in analyzing the local structure around a specific set of vertices, or focusing on a particular region of interest within a complex network. To formalize this idea, we recall the notion of a subgraph, which allows us to consider subsets of vertices and the edges connecting them while preserving the overall graph structure. Let $\Gamma = (V, E)$ be a graph. A graph $\Delta = (V', E')$ is called a *subgraph* of Γ if $V' \subseteq V$ and $E' \subseteq E \cap \{\{u, v\} \mid u, v \in V'\}$. In other words, Δ is a subgraph of Γ if its vertex set is a subset of the vertex set of Γ, and its edge set consists of edges from Γ that connect only vertices in V'. A common and particularly important type of subgraph is the one that includes all edges from the original graph that connect pairs of vertices in the chosen subset. This leads to the following notion. Let $\Gamma = (V, E)$ be a graph and $U \subseteq V$ a subset of vertices. The *induced subgraph* on U, denoted by $\Gamma[U]$, is the graph whose vertex set is U and whose edge set consists of all edges $\{u, v\} \in E$ such that $u, v \in U$. That is,

$$\Gamma[U] = (U, \{\{u, v\} \in E \mid u, v \in U\}).$$

Induced subgraphs preserve the adjacency relations of the original graph within the subset of vertices and are often used to study properties such as cliques, connectivity, or local neighborhoods.

A *clique* in a graph is a subset of vertices such that every two distinct vertices in the subset are connected by an edge. In other words, the induced subgraph on the clique is complete. Recall that for an undirected graph Γ the *clique number* of Γ, denoted by $\omega(\Gamma)$, is the maximum cardinality of a clique in Γ,

$$\omega(\Gamma) = \max\{|C| \mid C \subseteq V(\Gamma) : C \text{ is a clique in } \Gamma\}.$$

If Γ contains no edges, then $\omega(\Gamma) = 1$.

To analyze local properties around individual vertices, we often refer to the concept of a neighborhood. Let $\Gamma = (V, E)$ be a graph and $v \in V$ a vertex. The (closed) *neighborhood* of v, denoted by $N(v)$, is the set of all vertices adjacent to v together with v itself, that is,

$$N(v) = \{u \in V \mid \{u, v\} \in E\} \cup \{v\}.$$

The elements of $N(v) \setminus \{v\}$ are called *neighbors* of v. When $N(v) = V$ we say that v is a *universal* vertex, that is a vertex which is adjacent to every other vertex in the graph. Notice that in a complete graph, every vertex is universal. On the contrary, a vertex is said to be *isolated* if it is not adjacent to any other vertex in the graph. Equivalently, its neighborhood is the empty set.

Connectivity plays a central role in many graph-theoretic questions. A *path* in a graph $\Gamma = (V, E)$ is a k-tuple of vertices $P = (v_1, v_2, \ldots, v_k)$, with $v_i \in V$

for all i and $k \in \mathbb{N}$, $k \geq 2$ such that each consecutive pair $\{v_i, v_{i+1}\} \in E$. The path is called *simple* if all the vertices are distinct. Such a path is said to start at the vertex v_1 and end at vertex v_k. To indicate a path $P = (v_1, v_2, \ldots, v_k)$ we will often omit the parentheses and write the sequence $v_1, v_2, \ldots, v_k$ or write $v_1 \sim v_2 \sim \cdots \sim v_k$. The *length* of a path $P = (v_1, v_2, \ldots, v_k, v_{k+1})$, is the number of edges it contains, we denote this with $\operatorname{len}(P) = k$. We will often refer to a path of length k as a path composed by k steps. A *cycle* is a path that starts and ends at the same vertex, with all intermediate vertices being distinct. Thus, a cycle has the form $C = (v_1, v_2, \ldots, v_k, v_1)$, where $k \geq 3$ and all v_i are distinct, for $1 \leq i \leq k$. A graph $\Gamma = (V, E)$ is *connected* if for every pair of vertices $u, v \in V$, there exists a path that starts at u and ends at v. If this condition is not met, the graph is said to be *disconnected.* Let $\Gamma = (V, E)$ be a connected graph. The *diameter* of Γ, denoted by $\operatorname{diam}(\Gamma)$, is the greatest distance between any two vertices in Γ. That is,

$$\operatorname{diam}(\Gamma) = \sup\{d(u, v) \mid u, v \in V\},$$

where $d(u, v)$ denotes the length of the shortest path connecting the vertices u and v in Γ. If the graph has a set of vertices of finite cardinality and it is connected, then its diameter would obviously be a natural number. When a graph is not connected, it can be broken into smaller pieces called connected components. A *connected component* of a graph is a maximal connected subgraph. That is, it is a subset of the graph's vertices such that any two vertices in the subset are connected by a path and no additional vertices from the graph can be added to the subset without losing the property of connectivity. When not specified if a graph Γ is connected or not, by $\operatorname{diam}(\Gamma)$ we denote the maximum of the diameters of the connected components of Γ.

1.2.2 Directed graphs

Throughout this work, we adopt the following notational convention: plain symbols denote undirected graphs, while symbols bearing an arrow accent denote directed graphs. This way we make sure to distinguish between directed graphs and undirected graphs even when not specified.

A *directed graph*, or *digraph*, is a graph $\vec{\Gamma} = (V, E)$ where V is a set of vertices, and $E \subseteq V \times V$ is a set of ordered pairs of vertices. The pair $(u, v) \in E$ represents a directed edge from vertex u to vertex v. If $(u, v) \in E$ we often refer to u as the tail, or source, of the arc, and to v as the head, endpoint or target and we often denote this by $u \to v$. If there is also a directed edge from v to u we denote it by $u \leftrightarrow v$. However, we say that u and v are adjacent in $\vec{\Gamma}$ if $(u, v) \in E$ or $(v, u) \in E$. Notice that from a directed graph one can always obtain an undirected graph, by replacing the directed edges with undirected edges (avoiding repetitions). In other words if $\vec{\Gamma} = (V, E)$ is a directed graph we can define $\Gamma = (V, \bar{E})$ where $\bar{E} = \{\{u, v\} \mid (u, v) \in E\}$, the undirected graph induced by $\vec{\Gamma}$.

In analogy with the case of undirected graphs, we recall different notions in the context of directed graphs. A directed graph $\vec{\Gamma} = (V, E)$ is called *simple* if two things occur: there are no loops, i.e. for all $v \in V$ we have $(v, v) \notin E$ and there is at most one arc in each direction between any two distinct vertices. Moreover, a simple directed graph $\vec{\Gamma} = (V, E)$ is *complete* if for every pair of distinct vertices $u, v \in V$ we have $(u, v) \in E$ and $(v, u) \in E$.

As before, in order to study smaller portions of graphs, we recall the definitions of subgraphs and induced subgraphs of a digraph. A directed graph $\vec{\Delta} = (V', E')$ is a *subgraph* of $\vec{\Gamma} = (V, E)$ if $V' \subseteq V$ and $E' \subseteq E \cap (V' \times V')$. Let $\vec{\Gamma} = (V, E)$ be a directed graph and $U \subseteq V$. The *induced subgraph* on U, denoted by $\vec{\Gamma}[U]$, is

$$\vec{\Gamma}[U] = \Big(U,\ E \cap (U \times U)\Big),$$

i.e. it has vertex set U and includes all arcs of $\vec{\Gamma}$ whose sources and targets lie in U. When looking at elements adjacent to a specific vertex v, as in the undirected case, it is useful to define a neighborhood of this vertex. In this case, it can be advantageous to distinguish from elements that are endpoints of arcs where v is the source and from elements that are sources of arcs directed to v. Let $\vec{\Gamma} = (V, E)$ be a directed graph and let $v \in V$. The (closed) *out-neighborhood*, or forward neighborhood, and the (closed) *in-neighborhood*, or backward neighborhood of v are, respectively,

$$N^+(v) = \{\, w \in V : (v, w) \in E\} \cup \{v\}$$

and

$$N^-(v) = \{\, u \in V : (u, v) \in E\} \cup \{v\}.$$

The elements of $N^+(v) \setminus \{v\}$ and $N^-(v) \setminus \{v\}$ are called *out-neighbors* of v and *in-neighbors* of v, respectively.

Particular kinds of vertices of directed graphs are defined by the notions of sink and source. Let $\vec{\Gamma} = (V, E)$ be a directed graph. A vertex $v \in V$ is called a *sink* if there are no edges in E with v as a source, while a vertex $v \in V$ is called a *source* if there are no edges in E with v as an endpoint. A vertex is called isolated if it is both a sink and a source. The same notions can be extended to subsets of the vertex set. Thus, we say that a subset of vertices $S \subseteq V$ is called a *sink set* if there are no edges from any vertex in S to any vertex in $V \setminus S$, while subset of vertices $S \subseteq V$ is called a *source set* if there are no edges from any vertex in $V \setminus S$ to any vertex in S.

Just as we distinguish between out-neighborhoods and in-neighborhoods, we likewise distinguish between out-universal and in-universal vertices. Let $\vec{\Gamma} = (V, E)$ be a directed graph. A vertex $v \in V$ is called *out-universal*, or universal forward, if it has an arc to every other vertex, i.e. $N^+(v) = V$. Meanwhile, a vertex $v \in V$ is called *in-universal*, or universal backward, if it receives an arc from every other vertex, i.e. $N^-(v) = V$. When these two notions are merged together, we obtain the following. A vertex $v \in V$ is called *bidirectional universal* if it is both in-universal and out-universal, i.e. $N^+(v) = N^-(v) = V$. In the context of directed graphs we will often refer to bidirectional universal vertices as universal vertices.

A *directed path* in a directed graph $\vec{\Gamma} = (V, E)$ is a k-tuple of vertices $P = (v_1, v_2, \ldots, v_k)$, with $v_i \in V$ for all i and $k \in \mathbb{N}$, $k \geq 2$ such that for each $i = 1, \ldots, k-1$, there is a directed edge from v_i to v_{i+1}. A path is called *simple* if it contains no repeated vertices. The *length* of a path $P = (v_1, v_2, \ldots, v_k, v_{k+1})$ is the number of edges it contains. A *directed cycle* is a directed path that starts and ends at the same vertex, with all intermediate vertices being distinct. In a directed graph we will often use the words path and cycle to denote respectively a directed path and a directed cycle. We now give dual definitions to connectedness and diameter in the context of directed graphs. A directed graph $\vec{\Gamma} = (V, E)$ is said to be *strongly connected* if for every pair of vertices $u, v \in V$, there exist a directed path from u to v. Let $\vec{\Gamma} = (V, E)$ be a strongly connected directed graph. For any two vertices $u, v \in V$ such that $u \neq v$, we define the *directed distance*

$$\vec{d}(u, v) = \min\Big\{\ell \geq 1 : \text{there exists a directed path of length } \ell \text{ from } u \text{ to } v\Big\}.$$

In other words, the directed distance from u to v is the length of the shortest path connecting u to v. Since $\vec{\Gamma}$ is strongly connected, $\vec{d}(u, v)$ is well defined for all ordered pairs (u, v). Notice that, in general, $\vec{d}(u, v) \neq \vec{d}(v, u)$. The *diameter* of the directed graph $\vec{\Gamma} = (V, E)$, denoted by $\operatorname{diam}(\vec{\Gamma})$, is

$$\operatorname{diam}(\vec{\Gamma}) = \sup\Big\{\vec{d}(u, v) \mid u, v \in V, u \neq v\Big\}.$$

Notice that if a directed graph $\vec{\Gamma}$ is strongly connected then the undirected graph induced by $\vec{\Gamma}$ is connected and its diameter will be at most the diameter of $\vec{\Gamma}$. A *strongly connected component* is a maximal subgraph that is strongly connected. That is, it is a subgraph such that any two vertices in the subgraph are connected by a path, and no additional vertices from the graph can be added without losing the property of strong connectivity. As in the undirected case, when not specified if a digraph $\vec{\Gamma}$ is strongly connected or not, by diameter of $\vec{\Gamma}$ we denote the maximum of the diameters of the strongly connected components of $\vec{\Gamma}$.

1.2.3 Notation and basic properties of graphs defined on groups

The construction of graphs from groups has become an active area of research in modern algebra. Given a group G, one can define a graph by selecting a subset of G as the vertex set and introducing edges based on a specific rule that reflects the algebraic structure of G. These graphs are useful tools for visualizing and analyzing group properties.

Let G be a group and $\mathcal{P}$ a group property. One way to obtain an undirected graph from G is the following. Construct $\Gamma_{\mathcal{P}}(G) = (V, E)$ as follows. Consider as the set of vertices V the set of all elements of the group, i.e. $V = G$. Connect with an edge two elements $g, h \in G$ if $\langle g, h \rangle$ has the property $\mathcal{P}$.

- If $\mathcal{P}$ is the property of commutativity we call the graph $\Gamma_{\mathcal{P}}(G)$ the *commuting graph* of G and we denote it with the symbol $\Gamma_{\mathrm{comm}}(G)$;
- if $\mathcal{P}$ is the property of nilpotency we call the graph $\Gamma_{\mathcal{P}}(G)$ the *nilpotent graph* of G and we denote it with the symbol $\Gamma_{\mathrm{nil}}(G)$;
- if $\mathcal{P}$ is the property of solubility we call the graph $\Gamma_{\mathcal{P}}(G)$ the *soluble graph* of G and we denote it with the symbol $\Gamma_{\mathrm{sol}}(G)$;
- if $\mathcal{P}$ is the property of supersolubility we call the graph $\Gamma_{\mathcal{P}}(G)$ the *supersoluble graph* of G and we denote it with the symbol $\Gamma_{\mathrm{sup}}(G)$.

Of course it is possible to consider even more properties and thus other graphs. Changing the rule in which edges are created one can also give birth to directed graphs. Let G be a group. Construct the directed *normalizing graph* of G, denoted by $\vec{\Gamma}_{\mathrm{norm}}(G) = (V, E)$, as follows. Set $V = G$ and $(g, h) \in E$ if $\langle g \rangle$ is normal in $\langle g, h \rangle$. Let $w = w(x, y)$ be a two variables word. Construct the *verbal graph* or *w-graph* of G, denoted by $\vec{\Gamma}_w(G)$, as follows. Set $V = G$ and $(g, h) \in E$ if $w(g, h) = 1$. From these directed graphs one can obviously consider the induced undirected graphs, that we denote, respectively for the normalizing and the verbal graph, with $\Gamma_{\mathrm{norm}}(G)$ and $\Gamma_w(G)$.

As mentioned before, in the case in which the graph is directed, we indicate this by over-scripting with an arrow, thus it will always be clear if the graph is directed or not, even when it has not been specified.

Often we will consider subgraphs of these graphs, therefore we want to introduce a general notation that will specify which graph we are looking at. When the vertex set of the graph is the whole group G, we denote the graph by using the letter Γ, as before. If we remove all universal vertices, we denote the resulting graph using the letter Δ, such as $\Delta_{\mathrm{comm}}(G)$, $\Delta_{\mathrm{nil}}(G)$, $\vec{\Delta}_{\mathrm{norm}}(G)$, $\vec{\Delta}_w(G)$ and so on.

While the commuting graph and the nilpotent graph has been considered before, the normalizing graph is fairly new and the verbal graph, in its general form, is actually studied for the first time by us. Thus, for the first two graphs we address some well-known important properties that will be useful later.

Proposition 1.2.1. *Let G be a group with commuting graph $\Gamma_{\text{comm}}(G)$. Then the set of universal vertices of $\Gamma_{\text{comm}}(G)$ is $Z(G)$.*

Proof. Let $x \in G$. Then $x \in Z(G)$ if and only if x commutes with y for all $y \in G$. □

A similar result, although in the finite case, holds for the nilpotent graph.

Proposition 1.2.2. *Let G be a finite group with nilpotent graph $\Gamma_{\text{nil}}(G)$. Then the set of universal vertices of $\Gamma_{\text{nil}}(G)$ is $Z_\infty(G)$.*

Proof. See Proposition 2.1 in [1]. □

This notation provides a flexible framework for defining and studying various graphs associated with groups. By adjusting the vertex set and the rule of adjacency, a wide range of algebraic properties can be analyzed.

Chapter 2

Graphs defined on groups

2.1 The commuting graph

We recall that the *commuting graph* of a group G, denoted by $\Gamma_{\mathrm{comm}}(G)$, is the simple and undirected graph whose vertices are the elements of G and two distinct vertices are adjacent if they commute, or, equivalently, if they generate an abelian subgroup of G. Then the graph $\Delta_{\mathrm{comm}}(G)$ denotes the subgraph of $\Gamma_{\mathrm{comm}}(G)$ induced by $G \setminus Z(G)$, i.e. the subgraph of the commuting graph from which the universal vertices have been removed.

Here we investigate the properties of $\Delta_{\mathrm{comm}}(G)$ in the case where every connected component has diameter at most 2. To simplify the notation we will refer to diameter of $\Delta_{\mathrm{comm}}(G)$ intending, when this is disconnected, the maximum of the diameters of its connected components.

Consider a group G such that $\Delta_{\mathrm{comm}}(G)$ has diameter 2. We first address the disconnected case, supposing G is not a simple group. Indeed, we characterize non-simple finite groups with trivial center having $\Delta_{\mathrm{comm}}(G)$ disconnected of diameter 2.

Theorem A. *Let G be a non-simple finite group with trivial center. Then $\Delta_{\mathrm{comm}}(G)$ is disconnected of diameter 2 if and only if G is a Frobenius group with non-abelian kernel or non-abelian complement.*

Proof. Since G is not simple, if $\Delta_{\mathrm{comm}}(G)$ is disconnected, then by the main Theorem of [24] it follows that G is a Frobenius group, say with kernel K and complement H. Moreover, the connected components of G are precisely the kernel K and each of the conjugates H^g, with $g \in G$. Furthermore, if both K and H are abelian then $\Delta_{\mathrm{comm}}(G)$ would have diameter 1. Thus, in order to occur diameter 2 at least one between K and H has to be non-abelian.

Conversely, suppose that G is a Frobenius group with non-abelian kernel K or non-abelian complement H. Since $C_G(k) \leq K$ for any $k \in K \setminus \{1\}$ and $C_G(h) \leq H$

for any $h \in H \setminus \{1\}$ we have $\Delta_{\text{comm}}(G)$ disconnected. Moreover, since one of K and H is not abelian $\Delta_{\text{comm}}(G)$ cannot have diameter 1. Noticing that both $Z(K)$ and $Z(H)$ are non-trivial it follows that $\Delta_{\text{comm}}(G)$ has diameter exactly 2. □

We follow our investigation focusing on the connected case, distinguishing between the sub-case in which the group is decomposable and the one in which it is indecomposable.

Before starting we notice first that the property of $\Delta_{\text{comm}}(G)$ of being connected of diameter 2 can be equivalently expressed in the following way.

Proposition 2.1.1. *Let G be a group. Then $\Delta_{\text{comm}}(G)$ is connected of diameter 2 if and only if $C_G(x) \cap C_G(y) \neq Z(G)$ for all $x, y \in G$.*

Proof. Let $\Delta_{\text{comm}}(G)$ be connected of diameter 2. Notice first that G cannot be abelian. Let $x, y \in G$. If x, y are both central then $C_G(x) \cap C_G(y) = G \neq Z(G)$ since G is not abelian. Without loss of generality suppose x is non-central. If x, y commute then $x \in C_G(x) \cap C_G(y) \setminus Z(G)$. Suppose $[x, y] \neq 1$. Then also y is not central, and thus there exists a vertex $z \in G \setminus Z(G)$ such that $x \sim z \sim y$ is a path in $\Delta_{\text{comm}}(G)$ connecting x and y. Thus, $z \in C_G(x) \cap C_G(y) \setminus Z(G)$.

Conversely, suppose $C_G(x) \cap C_G(y) \neq Z(G)$ for all $x, y \in G$. Observe that G cannot be abelian. Let $a, b \in G \setminus Z(G)$ be two distinct vertices of $\Delta_{\text{comm}}(G)$. We will show that a, b are connected by a path of at most 2 steps. If a, b commute then they are adjacent in $\Delta_{\text{comm}}(G)$. If they do not commute, then by hypothesis, there exists $z \in C_G(a) \cap C_G(b) \setminus Z(G)$. Thus $a \sim z \sim b$ is a path in $\Delta_{\text{comm}}(G)$ and we are done. □

Focusing on decomposable groups it is possible to classify which groups have a connected commuting graph of diameter 2.

Theorem B. *Let $G = H \times K$ be a group. Then $\Delta_{\text{comm}}(G)$ is connected of diameter 2 if and only if $\Delta_{\text{comm}}(H)$ or $\Delta_{\text{comm}}(K)$ is a connected of diameter 2.*

Proof. Suppose first that one between $\Delta_{\text{comm}}(H)$ or $\Delta_{\text{comm}}(K)$ is connected of diameter 2. Without loss of generality suppose it is H. Let $(s, t), (u, v) \in G$. Then, by Proposition 2.1.1 there exists $a \in C_H(s) \cap C_H(u) \setminus Z(H)$. Thus, $(a, 1)$ commutes with (s, t) and (u, v) in G and $(a, 1) \notin Z(G)$. This shows that for every $(s, t), (u, v) \in G$ we have $C_G((u, v)) \cap C_G((s, t)) \neq Z(G)$. Thus, by Proposition 2.1.1 $\Delta_{\text{comm}}(G)$ is connected and it has diameter 2.

Conversely, assume by contradiction nor $\Delta_{\text{comm}}(H)$ or $\Delta_{\text{comm}}(K)$ is connected of diameter 2. Then by Proposition 2.1.1 there exist $s, t \in H$ and $u, v \in K$ such that $C_H(s) \cap C_H(t) = Z(H)$ and $C_K(u) \cap C_K(v) = Z(K)$. Let $(a, b) \in C_G((s, u)) \cap C_G((t, v))$. Then $(a, b)(s, u) = (s, u)(a, b)$ and so $as = sa$. Moreover, $(a, b)(t, v) = (t, v)(a, b)$ and so $at = ta$. Since $a \in H$ it follows that $a \in C_H(s) \cap$

$C_H(t)$ and thus $a \in Z(H)$. In the same way we obtain $b \in Z(K)$. Therefore $C_G((s,u)) \cap C_G((t,v)) = Z(G)$ and so, by Proposition 2.1.1, $\Delta_{\text{comm}}(G)$ is not connected of diameter 2, which is a contradiction. This concludes the proof. $\square$

Corollary 2.1.2. *The class of groups with* $\Delta_{\text{comm}}(G)$ *connected of diameter* 2 *is closed under direct products.*

In general, a group G with $\Delta_{\text{comm}}(G)$ connected of diameter 2 is not soluble. Indeed, it is sufficient to consider $H = A_5$, the alternating group of degree 5, and the group $K = \text{SmallGroup}(32, 49)$ and $G = H \times K$. Since $\Delta_{\text{comm}}(K)$ is connected of diameter 2, applying Theorem B, we obtain that G is a non-soluble group with $\Delta_{\text{comm}}(G)$ connected of diameter 2.

We now turn our attention to indecomposable groups, specifically those with trivial center. We start by stating this easy result.

Proposition 2.1.3. *Let* G *be a group with trivial center and* $\Delta_{\text{comm}}(G)$ *connected of diameter* 2*. Then* G *cannot be* 2*-generated.*

Proof. Since $\Delta_{\text{comm}}(G)$ is connected of diameter 2, by Proposition 2.1.1 we have $C_G(x) \cap C_G(y) \neq \{1\}$ for all $x, y \in G$. Suppose by contradiction that G is 2-generated, then there exist $a, b \in G$ such that $\langle a, b\rangle = G$. However, we have $C_G(a) \cap C_G(b) \neq \{1\}$, and so $C_G(\langle a, b\rangle) \neq \{1\}$. Therefore G has non-trivial center, a contradiction. $\square$

Corollary 2.1.4. *Let* G *be a group with trivial center and* $\Delta_{\text{comm}}(G)$ *connected of diameter* 2*. Then* G *cannot be simple.*

In order to describe such groups, it can be useful to find properties satisfied by centralizers of elements of the group, since these represent the neighborhood of such elements in the commuting graph.

Proposition 2.1.5. *Let* G *be a group with trivial center and* $\Delta_{\text{comm}}(G)$ *connected of diameter* 2*. If* $C_G(x)$ *has exactly* n *distinct cyclic subgroups of prime order* $H_1 = \langle h_1\rangle, \ldots, H_n = \langle h_n\rangle$*, then* $G = \bigcup_{i=1}^{n} C_G(h_i)$*.*

Proof. Notice first that there exists $t \in \mathbb{N}$ such that x^t has prime order. Therefore, there exists $j \in \{1, \ldots, n\}$ such that $\langle h_j\rangle = \langle x^t\rangle$ and $C_G(x) \subseteq C_G(h_j)$. Let $y \in G \setminus C_G(x)$. Since $C_G(x) \cap C_G(y) \neq 1$, there exists $z \in C_G(x)$ such that $[y, z] = 1$. Moreover, for any $m \in \mathbb{N}$ we have $[y, z^m] = 1$. In particular, we can choose m such that $\langle z^m\rangle = \langle h_i\rangle$ is a group of prime order. Therefore, there exists $i \in \{1, \ldots, n\}$ such that y commutes with h_i and so $y \in C_G(h_i)$. This concludes the proof. $\square$

Actually, this covering property characterizes these groups.

Proposition 2.1.6. *Let G be a group with trivial center. Then $\Delta_{\mathrm{comm}}(G)$ is connected of diameter 2 if and only if for any $x \in G$ holds $G = \bigcup_{i=1}^{n} C_G(h_i)$, where $h_1, \dots h_n$ are the generators of all the distinct cyclic subgroups of prime order of $C_G(x)$.*

Proof. One direction follows from Proposition 2.1.5. For the other direction, let $a, b \in G \setminus \{1\}$ and consider $C_G(a)$. Let $a_1, \dots, a_m$ be the generators of all the distinct cyclic subgroups of prime order of $C_G(a)$. Then $G = \bigcup_{i=1}^{m} C_G(a_i)$. Thus, there exists $j \in \{1, \dots, m\}$ such that $b \in C_G(a_j)$. Therefore, $a_j \in C_G(a) \cap C_G(b)$ and so the $\Delta_{\mathrm{comm}}(G)$ is connected of diameter 2. ☐

Proposition 2.1.7. *Let G be a group with trivial center and $\Delta_{\mathrm{comm}}(G)$ connected of diameter 2, and let $x \in G$. If $C_G(x)$ is abelian then there exist three distinct cyclic subgroups of prime order H_1, H_2 and H_3 of $C_G(x)$ such that $x \notin H_i$, for all i.*

Proof. Assume by contradiction that the statement is false. Then there exist at most two distinct cyclic subgroups of $C_G(x)$ of prime order H_1 and H_2 such that $x \notin H_i$ for $i = 1, 2$. Let $H_1 = \langle h_1 \rangle$ and $H_2 = \langle h_2 \rangle$. Since $\Delta_{\mathrm{comm}}(G)$ is connected of diameter 2, by Proposition 2.1.1 we have $C_G(x) \cap C_G(y) \neq \{1\}$ for all $y \in G$. If $y \in C_G(x)$ then $C_G(x) \subseteq C_G(y)$ because $C_G(x)$ is abelian. If $y \notin C_G(x)$ then there exists $t \in G$ such that $\langle t \rangle \leq C_G(x) \cap C_G(y)$ is of prime order and if $\langle t \rangle$ contains x, then $\langle x \rangle = \langle t \rangle \leq C_G(y)$, which cannot occur. Thus, in both cases, we have $H_1 \cap C_G(x) \cap C_G(y) \neq \{1\}$ or $H_2 \cap C_G(x) \cap C_G(y) \neq \{1\}$. Therefore, $h_1 \in C_G(x) \cap C_G(y)$ or $h_2 \in C_G(x) \cap C_G(y)$ for all $y \in G$. It follows that $G = C_G(h_1) \cup C_G(h_2)$, which yields $C_G(h_1) = G$ or $C_G(h_2) = G$ and so $h_1 \in Z(G)$ or $h_2 \in Z(G)$, which is a contradiction. ☐

Corollary 2.1.8. *Let G be a group with trivial center and $\Delta_{\mathrm{comm}}(G)$ connected of diameter 2, and let $x \in G \smallsetminus \{1\}$. If $|\pi(C_G(x))| \leq 2$ then $C_G(x)$ cannot be cyclic.*

Proof. Suppose by contradiction that $C_G(x)$ is cyclic. Since $|C_G(x)|$ is divided by at most two primes, $C_G(x)$ has at most two distinct cyclic subgroups of prime order. This contradicts Proposition 2.1.7. ☐

Proposition 2.1.9. *Let G be a group with trivial center and $\Delta_{\mathrm{comm}}(G)$ connected of diameter 2 and let $x \in G$. Then there exist at least two distinct cyclic subgroups of $C_G(x)$ of prime order H_1 and H_2 such that $x \notin H_i$. If these are exactly two, then $H_i \cap Z(C_G(x)) = \{1\}$, for $i = 1, 2$.*

Proof. We argue by contradiction. Let $C = C_G(x)$. Suppose first that there exists no cyclic subgroup of $C_G(x)$ of prime order that does not contain x. Then x has prime order. By Proposition 2.1.1 we have $C_G(x) \cap C_G(y) \neq \{1\}$ for all $y \in G$

and thus $x \in C_G(x) \cap C_G(y)$ for all $y \in G$. This yields $x \in Z(G)$, which is a contradiction.

Suppose that there exists only one cyclic subgroup $\langle h \rangle$ of C of prime order such that $x \notin \langle h \rangle$. Then C contains at most 2 distinct cyclic subgroups of prime order, depending on whether x has prime order or not. In either case, applying Proposition 2.1.6, it follows that $G = C_G(x) \cup C_G(h)$, which yields $C = G$ or $C_G(h) = G$, and so $Z(G) \neq \{1\}$, a contradiction.

Suppose now that there exist exactly two subgroups of C of prime order $H_1 = \langle h_1 \rangle$ and $H_2 = \langle h_2 \rangle$ such that $x \notin H_i$ for $i = 1, 2$ and without loss of generality, suppose that $H_1 \cap Z(C) \neq \{1\}$. Since $H_1 \cap Z(C)$ is a non-trivial subgroup of H_1, which is a group of prime order, we have $H_1 = H_1 \cap Z(C)$ and so $H_1 \subseteq Z(C)$. Let $y \in G \setminus C$. By Proposition 2.1.1 we have $C_G(y) \cap C \neq \{1\}$ and so it follows that $[y, h_1] = 1$ or $[y, h_2] = 1$. Since $H_1 \subseteq Z(C)$ we have $C_G(x) \subseteq C_G(h_1)$ and thus we have $G = C_G(h_1) \cup C_G(h_2)$, which contradicts the hypothesis $Z(G) = \{1\}$.

□

Lemma 2.1.10. *Let G be a finite group and H_1, H_2, H_3 be proper subgroups of G, such that $|G : H_1| \leq |G : H_2| \leq |G : H_3|$ and $G = H_1 \cup H_2 \cup H_3$. Then $|G| \leq 12|H_1 \cap H_2 \cap H_3|$.*

Proof. By Theorem 5 in [15] we have $|G : H_1| = 2$ and $|G : H_2| = 2$. Thus, $H_1 \cap H_2$ has index 4 in G. Since $G \setminus (H_1 \cup H_2) \subseteq H_3$ and $|H_1 \cup H_2| = |H_1| + |H_2| - |H_1 \cap H_2| = \frac{1}{2}|G| + \frac{1}{2}|G| - \frac{1}{4}|G| = \frac{3}{4}|G|$, it follows that $|H_3| > \frac{1}{4}|G|$ (equality cannot hold since the intersection between subgroups is always non-empty) and therefore H_3 has at most index 3 in G. The result follows. □

Lemma 2.1.11. *Let G be a finite group and H_1, H_2, H_3, H_4 proper subgroups of G, such that $|G : H_i| \leq |G : H_{i+1}|$ for all i and $G = H_1 \cup H_2 \cup H_3 \cup H_4$. Then:*

(i) $|G| \leq 35|H_1 \cap H_2|$;

(ii) $|G| \leq 105|H_1 \cap H_2 \cap H_3|$.

Proof. Since G is covered by 4 subgroups we have $|H_1| > \frac{1}{4}|G|$. Thus, $|G : H_1| \leq 3$. Observe that the larger the index of H_1 in G, the smaller the portion of G that H_1 covers. Consequently, a larger index for H_1 would force the remaining subgroups H_j for $2 \leq j \leq 4$ to have indices that are bounded by a smaller constant. Hence, the worst case occurs for $|G : H_1| = 2$. If H_1 has index 2 in G then $|H_2| > \frac{1}{6}|G|$ and thus $|G : H_2| \leq 5$. As before, we only consider the worst case, thus we assume $|G : H_2| = 2$. Then $H_1 \cap H_2$ has index 4 and so $H_3 \cup H_4$ has to contain at least $\frac{1}{4}|G|$ elements of G. Therefore, we have $|H_3| > \frac{1}{8}|G|$ and so $|G : H_3| \leq 7$. Since $|G| \leq |G : H_1| \cdots |G : H_j||H_1 \cap \cdots \cap H_j|$, the result follows. □

Direct computation in GAP [34], gives a complete list of groups with trivial center such that $\Delta_{\mathrm{comm}}(G)$ is connected of diameter 2 up to order 2000. The procedure in which these groups are identified is described in Section 4.2 of the appendix. We now list the parameters that identify the groups in the `SmallGroup` library, separated by orders. In particular, only order $486, 1458, 1536, 1944$ occur.

Order of the group	**Parameter that identifies the group in GAP**
486	176, 177, 178, 179, 236, 238
1458	982, 985, 988, 993, 1008, 1010, 1020, 1030, 1181, 1188 1194, 1195, 1197, 1200, 1203, 1207, 1208, 1210, 1213, 1216, 1219, 1225, 1231, 1235, 1238, 1241, 1454, 1459, 1641, 1693, 1700, 1707, 1710, 1716, 1721, 1726, 1731, 1733, 1736, 1739, 1742, 1782, 1789, 1790
1536	408527488, 408527495, 408527499, 408527503, 408527504, 408527505, 408527506, 408527507, 408527508, 408544473, 408544476, 408544483, 408544487, 408544492, 408544493, 408544494, 408544495, 408544496
1944	2451, 2452, 2454, 2462, 2468, 2469, 2470, 2471, 2472, 2473, 3592, 3596, 3601

Thanks to this computation it is possible to prove the following result.

Proposition 2.1.12. *Let G be a group with trivial center and $\Delta_{\mathrm{comm}}(G)$ connected of diameter 2. Then, for every $x \in G$, either $|C_G(x)| \geq 16$, or $C_G(x)$ is isomorphic to $C_2 \times C_2 \times C_2$, $C_4 \times C_2$, D_8 or D_{12} .*

Proof. Let $C = C_G(x)$. If $|C| < 30$, by Corollary 2.1.8 it follows that C cannot be cyclic. Moreover, $x \in Z(C)$ and so $C_G(x)$ has non-trivial center. Thus, we only need to focus on non-cyclic groups with non-trivial center. Proposition 2.1.7 shows that C cannot have order less than 8. By Proposition 2.1.9 we have $C \ncong Q_8$. We now argue by contradiction.

Suppose $C \cong C_3 \times C_3$. Then C has exactly 4 distinct subgroups of prime order. Then, by Proposition 2.1.5 and by Lemma 2.1.11 we have $|G| \leq 35|K_1 \cap K_2|$, where $K_1 = C_G(k_1)$ and $K_2 = C_G(k_2)$ with k_1, k_2 two different elements of order 3 of C which generate different subgroups. Notice now that if $y \in G$ commutes with both k_1 and k_2 we have that y commutes with all elements of C, and thus $y \in C$. We can conclude that $|K_1 \cap K_2| = 9$. Thus $|G| \leq 315$. Since no group of order less than or equal to 315 has trivial center and $\Delta_{\mathrm{comm}}(G)$ connected of diameter 2 we have a contradiction.

Suppose $C \cong C_6 \times C_2$. Then C has exactly 4 distinct subgroups of prime order. Thus, by Proposition 2.1.5 and by Lemma 2.1.11 we have $|G| \leq 105|K_1 \cap K_2 \cap K_3|$, where $K_1 = C_G(k_1)$, $K_2 = C_G(k_2)$, $K_3 = C_G(k_3)$ with k_1, k_2, k_3 three different elements of prime order of C. Let $y \in G$. If y commutes at the same time with k_1, k_2 and k_3 we have $y \in C$, thus $K_1 \cap K_2 \cap K_3 = C$, and so $|G| \leq 1260$. Since computation showed that no group of order less than or equal to 1260 with trivial center and $\Delta_{\text{comm}}(G)$ connected of diameter 2 has C as centralizer of an element, we have a contradiction.

Suppose $C \cong C_3 \rtimes C_4$. Then C has exactly 2 distinct subgroups of prime order, one of which lies in the center of C. Thus, Proposition 2.1.9 yields a contradiction. □

We believe that Proposition 2.1.12 can be improved, proving that the best possible bound on the order of any centralizer of an element of the group G is actually 16 and that the other cases cannot happen. The bound 16 is actually realized by the group SmallGroup(1536,408527488), which has an element whose centralizer has order 16. However, we point out that, even if improved, the single condition on the orders of the centralizers of Proposition 2.1.12 is not sufficient to say that a group with trivial center and $\Delta_{\text{comm}}(G)$ connected needs to have $\Delta_{\text{comm}}(G)$ of diameter 2. Indeed, the group SmallGroup$(432, 736)$ is a group with trivial center, such that $\Delta_{\text{comm}}(G)$ is connected of diameter 3 and $|C_G(x)| \geq 16$ for any $x \in G$.

Proposition 2.1.13. *Let G be a finite group with $\Delta_{\text{comm}}(G)$ connected of diameter 2. If $\mathrm{Fit(G)} = 1$ then $\mathrm{Soc}(G)$ is not 2-generated.*

Proof. Let M be the Socle of G. Suppose by contradiction that M is 2-generated, say by a and b. By hypothesis and Proposition 2.1.1 we have $C_G(a) \cap C_G(b) \neq \{1\}$. It follows that $C_G(M) \neq \{1\}$. Since the Fitting subgroup of G is trivial it follows that M is the direct product of non-abelian simple groups, thus, it has trivial center. Since $C_G(M)$ is a normal subgroup of G it follows that it contains a minimal normal subgroup of G, say L. Therefore, L is contained in M and centralizes M, which leads to a contradiction. □

In general, an indecomposable group G with trivial center and $\Delta_{\text{comm}}(G)$ connected of diameter 2 is not soluble. In particular, we show an example of such a group G which is perfect. We wish to acknowledge that the following example was developed with the help of Pablo Spiga.

Example 2.1.14. Let $H = {A_5}^{20}$. Due to Corollary 4.2 in [4] it follows that H is not 2-generated. Let $p \neq 2, 3, 5$ be a prime and F_p the field of order p. Consider V the vector space over F_p of dimension $|H|$ and let I be the subset of $\mathbb{N}$ consisting of all natural numbers less than or equal to $|H|$. Let $H = \{h_i \mid i \in I\}$. Let B_V be the canonic base of V: $B_V = \{e_h \mid h \in H\} = \{e_{h_i} \mid i \in I\}$. Consider

$$W = \left\{ \sum_{h \in H} a_h e_h : \sum_{h \in H} a_h = 0 \right\}.$$

This is a subspace of V of dimension $\dim V - 1$. In fact, a base of W is given by $B_W = \{e_{h_i} - e_{h_{i+1}} \mid i \in I \setminus \{|H|\}\}$, which are the row vectors of the following matrix.

$$\begin{pmatrix} 1 & -1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -1 \end{pmatrix}$$

Consider the group $G = W \rtimes H$, where the action of H is the one of right multiplication on the indices of the base vectors: $x \in H : (e_h)_{h \in H} \to (e_{hx})_{h \in H}$. Notice first that G is perfect. Indeed, we have $[G, G] = [W, W][W, H] \rtimes [H, H]$. However, H is perfect, since it is the direct product of perfect groups, so $[H, H] = H$. Moreover, W is abelian, thus $[W, W] = \{1\}$. It is sufficient to show that $[W, H] = W$. Notice first that for $w \in W$ and $h \in H$ we have $[w, h] = -w + w^h$. Thus

$$[W, H] = \langle \{-w + w^h \mid w \in W, h \in H\} \rangle.$$

Let $w = e_x - e_y$, with $x, y \in H$ and $x \neq y$ and let $h \in H$ such that $o(h) = n$. Then consider

$$[e_x - e_y, h] + [e_{xh} - e_{yh}, h] + \cdots + [e_{xh^{n-1}} - e_{yh^{n-1}}, h] = -2w.$$

Thus $w \in [W, H]$, so $B_W \subseteq [W, H]$ and therefore $[W, H] = W$.

Furthermore, $Z(G) = \{1\}$. Let $x \in Z(G)$. Then there exist $w \in W$ and $g \in H$ such that $x = (w, g)$. Since x commutes with every element of W we have $v^g = v$, for all $v \in W$, which yields $g = 1$. Thus $x = w = (\sum_{h \in H} a_h e_h)$. However, x also commutes with every element of H, thus $a_h = a_{h'}$ for all $h, h' \in H$. Therefore we have $\sum_{h \in H} a_h = |H| a_1 = 0$. Since p does not divide $|H|$ it follows that $a_1 = 0$ and so that $w = 0$. Hence $Z(G)$ is trivial.

Lastly, $\Delta_{\mathrm{comm}}(G)$ is connected of diameter 2. We claim that for all $h_1, h_2 \in H$ we have $C_W(h_1) \cap C_W(h_2) \neq \{0\}$. In fact, notice that $C_W(h_1) \cap C_W(h_2) = C_W(\langle h_1, h_2 \rangle)$ and $K = \langle h_1, h_2 \rangle < H$, since H is not 2-generated. Furthermore, H acts on the set of indices by regular right action. Then the coset xK is preserved under the action of K, for any $x \in H$. Let $x \in H \setminus K$. Consider now the following vector: $w = \sum_{h \in H} a_h e_h$ such that $a_h = 1$ if $h \in K$, $a_h = -1$ if $h \in xK$ and $a_h = 0$ otherwise. Since $|K| = |xK|$ we have $w \in W$. Moreover, w is fixed under the action of K and thus it centralizes K. This proves the claim. Let now $g_1, g_2 \in G$. Then $g_1 = (w_1, h_1)$ and $g_2 = (w_2, h_2)$. By claim there exists $v \in W \setminus \{0\}$ such that $v \sim h_1$ and $v \sim h_2$. Therefore $g_1 v = (w_1, h_1)(v, 1) = (v + w_1, h_1) = (v, 1)(w_1, h_1) = v g_1$ and $g_2 v = (w_2, h_2)(v, 1) = (v + w_2, h_2) = (v, 1)(w_2, h_2) = g_2 v$. Thus v commutes with g_1 and g_2.

As seen before, in general, a decomposable group with $\Delta_{\mathrm{comm}}(G)$ connected of diameter 2 is not soluble. Thanks to Example 2.1.14 we can say that, in general, such a group can also be perfect. For instance, one can take $G = H \times K$, with H the alternating group of degree 5 and K as the group in the Example 2.1.14.

We now turn our attention to the nilpotent graph. However, as we will see, the commuting graph will still have an important role in the study of the above mentioned graph.

2.2 The nilpotent graph

The *nilpotent graph* of a group G is the simple undirected graph where the vertices correspond to the elements of G, and two distinct vertices x and y are adjacent if and only if the subgroup $\langle x, y\rangle$ generated by x and y is nilpotent. The results presented in this section are collected in [18].

2.2.1 Nilpotent graphs whose neighborhoods are cliques

For each element $x \in G$, the nilpotent neighborhood of x, denoted by $\mathrm{Nil}_G(x)$, is the set of elements $y \in G$ such that the subgroup $\langle x, y\rangle$ is nilpotent. Understanding the structure of these neighborhoods is a crucial step in classifying groups based on their nilpotent graph. As shown in [1, Lemma 3.3], the subset $\mathrm{Nil}_G(x)$ is not a subgroup in general. A still open question is the classification of groups for which $\mathrm{Nil}_G(x)$ is a subgroup of G for every $x \in G$. Such groups, which following [1] we call $\mathfrak{n}$-*groups*, exhibit special properties in relation to their nilpotent subgroups. While the classification of simple $\mathfrak{n}$-groups has been achieved in earlier works, such as [1], the study of soluble $\mathfrak{n}$-groups remains a relatively unexplored area.

Here, we investigate soluble $\mathfrak{n}$-groups, establishing a full classification of those $\mathfrak{n}$-groups for which every nilpotent neighborhood is a nilpotent subgroup: indeed we prove that the class of Frobenius groups having a nilpotent Frobenius complement is the only class of soluble groups for which this condition holds.

We will start by stating some preliminary results concerning the closed neighborhood of an element in the nilpotent graph. We begin with the following lemma.

Lemma 2.2.1. *Fix an element $x \in G$ of order $p_1^{\alpha_1} \cdots p_k^{\alpha_k}$ and let x_i be the p_i-part of x where $p_1, \dots, p_k$ are distinct primes and α_i are positive integers. Then*

$$\mathrm{Nil}_G(x) = \bigcap_{i=1}^{k} \mathrm{Nil}_G(x_i).$$

Proof. For an element $c \in \mathrm{Nil}_G(x)$, the subgroup $\langle x, c\rangle$ is nilpotent and as a subgroup, so is $\langle x_i, c\rangle$. Hence, $c \in \mathrm{Nil}_G(x_i)$ for every $i = 1, \dots, k$. We obtain $c \in \bigcap_{i=1}^{k} \mathrm{Nil}_G(x_i)$, and so, $\mathrm{Nil}_G(x) \subseteq \bigcap_{i=1}^{k} \mathrm{Nil}_G(x_i)$.

Conversely, assume $c \in \mathrm{Nil}_G(x_i)$ for every $i = 1, \dots, k$, and write $c = c_1 \cdots c_k c'$, where c_i is the p_i-part of c and c' is its $\{p_1, \dots p_k\}'$-part. Then the subgroups $\langle x_i, c_j\rangle$ and $\langle x_i, c'\rangle$ are nilpotent for every $i, j = 1, \dots, k$, as subgroups of $\langle x_i, c\rangle$. This implies that $\langle x_i, c_i\rangle$ is a p_i-subgroup. In addition, x_i and c_j will commute when $i \neq j$, as well as, x_i and c'. It follows that

$$\langle x, c\rangle = \langle x_1, \dots, x_k, c_1, \dots, c_k, c'\rangle = \langle x_1, c_1\rangle \times \cdots \times \langle x_k, c_k\rangle \times \langle c'\rangle$$

which is a direct product of nilpotent groups, so it is nilpotent. □

An immediate consequence of Lemma 2.2.1 is the following result, which coincides with Lemma 3.4 of [1].

Corollary 2.2.2. *Let G be a group. Then G is an $\mathfrak{n}$-group if and only if $\mathrm{Nil}_G(y)$ is a subgroup of G for every element $y \in G$ of prime power order.*

In [1], the authors provide some classes of $\mathfrak{n}$-groups. For instance, they prove that every A-group is such a group, where we recall that an A-group is a group in which all sylow subgroups are abelian. As a consequence, A-groups may be characterized as follows:

Theorem 2.2.3. *A group G is an A-group if and only if $\mathrm{Nil}_G(x) = C_G(x)$ for every element $x \in G$ of prime power order.*

Proof. When G is an A-group, the result follows by Lemma 3.5 of [1]. Conversely, suppose $\mathrm{Nil}_G(x) = C_G(x)$ for every element $x \in G$ of prime power order. Let P be a Sylow p-subgroup of G, and consider the elements $x, y \in P$. Then $\langle x, y\rangle$ is nilpotent, which implies $y \in \mathrm{Nil}_G(x) = C_G(x)$. Therefore, x commutes with y and P is abelian. □

We now show that the class of $\mathfrak{n}$-groups is closed under taking direct products.

Lemma 2.2.4. *Let A and B be groups such that $\mathrm{Nil}_A(a)$ and $\mathrm{Nil}_B(b)$ are subgroups of A and B respectively, for every pair of elements $a \in A$ and $b \in B$. Then $\mathrm{Nil}_{A\times B}(x)$ is a subgroup of $A \times B$ for every element $x \in A \times B$.*

Proof. Consider elements $x, y \in A \times B$. Then there exist elements $a, a' \in A$ and $b, b' \in B$ such that $x = ab$ and $y = a'b'$. Now, $\langle x, y\rangle$ is nilpotent if and only if $\langle ab, a'b'\rangle = \langle a, a'\rangle \times \langle b, b'\rangle$ is nilpotent. This implies that $\mathrm{Nil}_{A\times B}(ab) = \mathrm{Nil}_A(a) \times \mathrm{Nil}_B(b)$. The result is now clear. $\square$

We next show that any Frobenius group whose Frobenius complement is an $\mathfrak{n}$-group is again an $\mathfrak{n}$-group.

Lemma 2.2.5. *Let G be a Frobenius group with Frobenius complement H. If H is an $\mathfrak{n}$-group, then G is an $\mathfrak{n}$-group. In particular, if H is nilpotent, then G is an $\mathfrak{n}$-group such that $\mathrm{Nil}_G(x)$ is nilpotent for every $x \in G \setminus \{1\}$.*

Proof. It suffices to prove the first statement, as the second will then follow directly.

Suppose that $x \in K$, where K is the Frobenius kernel of G. Since K is nilpotent, it follows that $K \subseteq \mathrm{Nil}_G(x)$. For the reverse inclusion, observe that if $y \notin K$, then x and y have coprime orders, as G is a Frobenius group. Hence, the subgroup $\langle x, y\rangle$ is nilpotent if and only if x and y commute. This contradicts the Frobenius property, and thus $\mathrm{Nil}_G(x) \subseteq K$. Therefore, $\mathrm{Nil}_G(x) = K$.

Now assume that x belongs to some conjugate of H, say H_1. We show that $\mathrm{Nil}_G(x) = \mathrm{Nil}_{H_1}(x)$. Let $k \in K$ and $h \in H_1$ such that $\langle kh, x\rangle$ is nilpotent. By way of contradiction assume that $k \neq 1$. Hence kh belongs to a conjugate of H_1 distinct from H_1. As a consequence, the nilpotent subgroup $\langle kh, x\rangle$ contains a non trivial element of K, which should commute with x since K and H_1 have coprime orders. This contradiction yields $k = 1$ and $\mathrm{Nil}_G(x) \subseteq H_1$, implying that $\mathrm{Nil}_G(x) = \mathrm{Nil}_{H_1}(x)$. Since H_1 is an $\mathfrak{n}$-group, the result follows. $\square$

We denote by $O_\pi(G)$ the π-radical of G, that is, the maximum normal π-subgroup of G. Next, $O^\pi(G)$ denotes the π-residual of G, namely, the smallest normal subgroup of G for which the quotient is a π-group. We are now ready to prove the following.

Theorem C. *Let G be a finite soluble group with trivial center. Then $\mathrm{Nil}_G(x)$ is a nilpotent subgroup of G for every non-trivial element $x \in G$ if and only if G is a Frobenius group with nilpotent Frobenius complement.*

Proof. Firstly, assume that $\mathrm{Nil}_G(x)$ is a nilpotent subgroup of G for every non-trivial element $x \in G$. Let π be the set of all primes $p \in \pi(G)$ such that $O_p(G)$ is not trivial. Since G is soluble, the set π is not empty. For a prime $p \in \pi$, consider an element $x_p \in O_p(G) \setminus \{1\}$.

Then $O^{p'}(G)$ is contained in $\mathrm{Nil}_G(x_p)$ and therefore it is nilpotent. As a consequence $O^{p'}(G)$ is a Sylow p-subgroup, implying that for every prime $p \in \pi$ there exists a unique Sylow p-subgroup of G. Set K to be the subgroup of G generated by all Sylow p-subgroups of G for $p \in \pi$. Note that K coincides with the Fitting subgroup of G because any minimal normal subgroup of G is contained in the Fitting subgroup of G. By the Schur-Zassenhaus Theorem, there exists a subgroup H of G such that $G = KH$ and $H \cap K = \{1\}$.

To prove that G is a Frobenius group with Frobenius kernel K and Frobenius complement H, it is sufficient to show that for any $1 \neq x \in K$ the centralizer $C_G(x)$ is contained in K. Indeed, assume that there exists an element $1 \neq x \in K$ such that $C_G(x) \not\leq K$. Then there exist $k \in K$ and $h \in H \setminus \{1\}$ such that $[kh, x] = 1$. As a consequence $kh \in \mathrm{Nil}_G(x)$. Since $K \leq \mathrm{Nil}_G(x)$ and $\mathrm{Nil}_G(x)$ is a subgroup, we have $K\langle h\rangle \leq \mathrm{Nil}_G(x)$. Now, the fact that H and K have coprime orders implies that h centralizes K, which is the Fitting subgroup of G. Therefore $h \in C_G(K) \leq K$ leads to a contradiction.

Conversely, if G is a Frobenius group with nilpotent complement, then by Lemma 2.2.5 the desired conclusion follows. ☐

We have seen that A-groups and Frobenius groups whose Frobenius complements are soluble $\mathfrak{n}$-groups are $\mathfrak{n}$-groups as are their direct products. Recall that by modding out by the hypercenter, we may consider centerless soluble groups. Thus, a future research line could be to understand whether any centerless soluble group G that is an $\mathfrak{n}$-group is a subgroup of a direct product $A \times F$ where A is an A-group and F is the direct product of Frobenius groups whose Frobenius complements are $\mathfrak{n}$-groups.

2.2.2 Connectivity of $\Delta_{\mathrm{nil}}(G)$

Here, we consider the graph $\Delta_{\mathrm{nil}}(G)$, which is defined as the induced subgraph of the nilpotent graph on the set $G \setminus Z_\infty(G)$. First, we show that this graph can be computed via $G/Z_\infty(G)$.

Proposition 2.2.6. *Let G be a group and $x, y \in G \setminus Z_\infty(G)$ such that $xZ_\infty(G) \neq yZ_\infty(G)$. Then x and y are adjacent in $\Delta_{\mathrm{nil}}(G)$ if and only if $xZ_\infty(G)$ and $yZ_\infty(G)$ are adjacent in $\Delta_{\mathrm{nil}}(G/Z_\infty(G))$.*

Proof. If x and y generate a nilpotent subgroup, then also the subgroup generated by $xZ_\infty(G)$ and $yZ_\infty(G)$ is clearly nilpotent.

Now assume that, modulo $Z_\infty(G)$, x and y generate a nilpotent subgroup H of G. Thus there exists a positive integer c such that

$$[\underbrace{H, ..., H}_{c \text{ times}}] \leq Z_\infty(G).$$

Hence, for a suitable t we have

$$[\underbrace{H, ..., H}_{c \text{ times}}, \underbrace{H, ..., H}_{t \text{ times}}] \quad \leq \quad [Z_\infty(G), \underbrace{G, ..., G}_{t \text{ times}}] = 1.$$

Therefore, H is nilpotent. □

Theorem D. *For any group G the number of connected components of $\Delta_{\mathrm{nil}}(G)$ equals the number of connected components of $\Delta_{\mathrm{nil}}(G/Z_\infty(G))$, and there is a correspondence between the connected components of $\Delta_{\mathrm{nil}}(G)$ and $\Delta_{\mathrm{nil}}(G/Z_\infty(G))$ that maps connected components of diameter 1 to connected components of diameter 0 or 1 and preserves the diameter of connected components whose diameter is greater than 1.*

Proof. We first observe that when x and y are adjacent in $\Delta_{\mathrm{nil}}(G)$, by Proposition 2.2.6 either $xZ_\infty(G) = yZ_\infty(G)$ or $xZ_\infty(G)$ and $yZ_\infty(G)$ are adjacent in $\Delta_{\mathrm{nil}}(G/Z_\infty(G))$.

Suppose $d \geq 2$. We show that there exists a path of length d between x and y in $\Delta_{\mathrm{nil}}(G)$ if and only if there is a path of length d between $xZ_\infty(G)$ and $yZ_\infty(G)$ in $\Delta_{\mathrm{nil}}(G/Z_\infty(G))$. To see this suppose $x, y \in G \setminus Z_\infty(G)$ such that the distance between x and y is d. Since $d \geq 2$, $xZ_\infty(G) \neq yZ_\infty(G)$ and there exist $x_1, \ldots, x_d{+}1 \in G \backslash Z_\infty(G)$ such that $x_1 = x, x_d{+}1 = y$ and x_i is adjacent to x_{i+1} for every $i \in \{1, \ldots, d\}$. Then $x_1Z_\infty(G), \ldots, x_{d+1}Z_\infty(G)$ is a path in $\Delta_{\mathrm{nil}}(G/Z_\infty(G))$ of length at most d. Actually, for every $i \in \{1, \ldots d\}$, $x_iZ_\infty(G) \neq x_{i+1}Z_\infty(G)$. For, assume that $x_jZ_\infty(G) = x_{j+1}Z_\infty(G)$ for some $j \in \{1, \ldots, d\}$. Then, by Proposition 2.2.6, removing an element from $\{x_1, \ldots, x_{d+1}\}$ we find a path in $\Delta_{\mathrm{nil}}(G)$ connecting x and y of length less than d, which is a contradiction. This proves the observation.

It follows that x and y are in the same connected component of $\Delta_{\mathrm{nil}}(G)$ if and only if $xZ_\infty(G)$ and $yZ_\infty(G)$ are in the same connected component of $\Delta_{\mathrm{nil}}(G/Z_\infty(G))$. This gives the bijection between connected components.

Notice that a connected component of diameter 1 in $\Delta_{\mathrm{nil}}(G)$ can correspond to either a connected component that is a complete graph or one that has diameter 1 in $\Delta_{\mathrm{nil}}(G/Z_\infty(G))$. For connected components of bigger diameter, we see that the diameter is preserved. □

We obtain the following corollary for nilpotent graphs that are connected.

Corollary 2.2.7. *Let G be a group. Then $\Delta_{\mathrm{nil}}(G)$ is connected if and only if $\Delta_{\mathrm{nil}}(G/Z_\infty(G))$ is connected.*

For soluble groups, we can determine precisely when this occurs. We obtain this characterization from the relationship between $\Delta_{\mathrm{nil}}(G)$ and the graph $\Delta_{\mathrm{comm}}(G)$. First, we show the following result.

Proposition 2.2.8. *If G is a Frobenius group or a 2-Frobenius group, then $\Delta_{\mathrm{nil}}(G)$ is disconnected.*

Proof. First assume that G is a Frobenius group. As the center of G is trivial, any non-trivial element is a vertex in $\Delta_{\text{nil}}(G)$. Denote by K the Frobenius kernel of G. Since K is nilpotent, $K \setminus \{1\}$ is contained in a connected component of G. Actually, $K \setminus \{1\}$ is a connected component of $\Delta_{\text{nil}}(G)$. Indeed, assume by way of contradiction that there exist an element $x \in K \setminus \{1\}$ and an element $y \in G \setminus K$ such that $H = \langle x, y \rangle$ is nilpotent. Since x and y have coprime order, they commute. Thus $y \in C_G(x)$ which is a contradiction.

Now assume that G is a 2-Frobenius group, and let F_1, F_2 be subgroups of G such that $F_1 = \text{Fit}(G)$ and $F_2/F_1 = \text{Fit}(G/F_1)$. Then F_1 is the Frobenius kernel of F_2 and F_2/F_1 is the Frobenius kernel of G/F_1. It is sufficient to prove that an element in $F_2 \setminus F_1$ cannot be adjacent to any non-trivial element outside $F_2 \setminus F_1$ in $\Delta_{\text{nil}}(G)$. Consider an element $x \in F_2 \setminus F_1$. Since F_1 is the Frobenius Kernel of F_2, then the previous argument implies that for every element $y \in F_1 \setminus \{1\}$ the elements x and y are not adjacent in $\Delta_{\text{nil}}(G)$. Assume by contradiction that there exists an element $z \in G \setminus F_2$ such that x and z are adjacent in $\Delta_{\text{nil}}(G)$. Then xF_1 and zF_1 are also adjacent in $\Delta_{\text{nil}}(G/F_1)$. Thus there exists an element $t \in G \setminus F_1$ such that tF_1 centralizes both xF_1 and zF_1. This is a contradiction since G/F_1 is a Frobenius group. ☐

Now, we show that for a soluble group with trivial center the converse of the previous result is also true. Recall that Parker shows in [43] that the commuting graph of a soluble group G with a trivial center is disconnected if and only if G is a Frobenius group or a 2-Frobenius group.

Proposition 2.2.9. *Let G be a group with trivial center.*

- (i) *$\Delta_{\text{nil}}(G)$ is connected if and only if $\Delta_{\text{comm}}(G)$ is connected. Moreover, if $\Delta_{\text{nil}}(G)$ is connected of diameter k, then the* $\text{diam}(\Delta_{\text{comm}}(G)) \leq 2k$.
- (ii) *When G is soluble, $\Delta_{\text{nil}}(G)$ is disconnected if and only if G is a Frobenius group or a 2-Frobenius group.*

Proof. First observe that $\Delta_{\text{nil}}(G)$ and $\Delta_{\text{comm}}(G)$ have the same vertex set since $Z(G)$ is trivial. To prove (i), notice that if $\Delta_{\text{comm}}(G)$ is connected, then clearly $\Delta_{\text{nil}}(G)$ is connected also. On the other hand, assume that $\Delta_{\text{nil}}(G)$ is connected. Suppose $x, y \in G \setminus \{1\}$, then there exist $k \in \mathbb{N}$ and $x_1, \ldots, x_{k-1} \in G \setminus \{1\}$ such that $\langle x_i, x_{i+1} \rangle$ is nilpotent, with $x_0 = x$, $x_k = y$, and $0 \leq i \leq k-1$. Therefore, for every integer $i \in \{0, \ldots, k-1\}$ there is a non-trivial central element z_i in $\langle x_i, x_{i+1} \rangle$. Observe that $x = x_0$, z_0, x_1, z_1, x_2, $\ldots$, z_{k-1}, x_k is a path of length at most $2k$ in $\Delta_{\text{comm}}(G)$. Thus x and y are connected by a path in $\Delta_{\text{comm}}(G)$ of length at most $2k$.

In order to prove (ii), by Theorem 1.1 (i) of [43], $\Delta_{\text{comm}}(G)$ is disconnected if and only if G is a Frobenius group or a 2-Frobenius group. Now, applying (i), we obtain the desired result. ☐

It is worth mentioning that if we suppose $Z(G) \neq Z_\infty(G)$, then $\Delta_{\mathrm{nil}}(G)$ and $\Delta_{\mathrm{comm}}(G)$ are always different because they have a different set of vertices. We now extend Proposition 2.2.9 (*ii*) to the case where $Z(G)$ is non-trivial.

For soluble groups, we now show that $\Delta_{\mathrm{nil}}(G)$ is disconnected exactly when $G/Z_\infty(G)$ is Frobenius or 2-Frobenius.

Theorem E. *Let G be a non-nilpotent soluble group. Then $\Delta_{\mathrm{nil}}(G)$ is disconnected if and only if $G/Z_\infty(G)$ is a Frobenius group or a 2-Frobenius group.*

Proof. First assume that $\Delta_{\mathrm{nil}}(G)$ is disconnected, that is, there exist elements $x, y \in G \setminus Z_\infty(G)$ which are not connected by any path in $\Delta_{\mathrm{nil}}(G)$. By Proposition 2.2.9 (*ii*), since $G/Z_\infty(G)$ has trivial center it is sufficient to show that $\Delta_{\mathrm{nil}}(G/Z_\infty(G))$ is disconnected to obtain the conclusion.

By way of contradiction, assume that $\Delta_{\mathrm{nil}}(G/Z_\infty(G))$ is connected. Notice that $xZ_\infty(G) \neq yZ_\infty(G)$ otherwise $\langle x, y\rangle Z_\infty(G)$ would be cyclic, which yields $\langle x, y\rangle$ nilpotent. This is impossible by the choice of x and y. Therefore there exists a path in $\Delta_{\mathrm{nil}}(G/Z_\infty(G))$ connecting $xZ_\infty(G)$ and $yZ_\infty(G)$, that is, there exist $n \in \mathbb{N}$ and $x_1, \ldots, x_n \in G \setminus Z_\infty(G)$ such that $\langle x_i Z_\infty(G), x_{i+1}Z_\infty(G)\rangle$ is nilpotent, with $x_0 = x$, $x_{n+1} = y$ and $0 \leq i \leq n$. By Proposition 2.2.6 $x_i Z_\infty(G)$ and $x_{i+1}Z_\infty(G)$ are adjacent in $\Delta_{\mathrm{nil}}(G/Z_\infty(G))$ if and only if x_i and x_{i+1} are adjacent in $\Delta_{\mathrm{nil}}(G)$, therefore $x_0, \ldots, x_{n+1}$ realizes a path in $\Delta_{\mathrm{nil}}(G)$ connecting x and y, which is a contradiction.

Now, assume that $G/Z_\infty(G)$ is either a Frobenius group or a 2-Frobenius group. Then $\Delta_{\mathrm{nil}}(G/Z_\infty(G))$ is disconnected by Proposition 2.2.9, and there exist $xZ_\infty(G), yZ_\infty(G) \in G/Z_\infty(G)$ which are not connected. By contradiction assume that $\Delta_{\mathrm{nil}}(G)$ is connected. Therefore there exist $n \in \mathbb{N}$ and $x_1, \ldots, x_n \in G \setminus Z_\infty(G)$ such that $\langle x_i, x_{i+1}\rangle$ is nilpotent, with $x_0 = x$, $x_{n+1} = y$ and $0 \leq i \leq n$. By Proposition 2.2.6 $x_i Z_\infty(G)$ and $x_{i+1}Z_\infty(G)$ are adjacent in $\Delta_{\mathrm{nil}}(G/Z_\infty(G))$ if and only if x_i and x_{i+1} are adjacent in $\Delta_{\mathrm{nil}}(G)$. This implies that $x_0 Z_\infty(G), \ldots, x_{n+1}Z_\infty(G)$ realizes a path in $\Delta_{\mathrm{nil}}(G/Z_\infty(G))$ connecting $xZ_\infty(G)$ and $yZ_\infty(G)$, which is a contradiction. $\square$

We note that Theorem E shares a certain similarity with Theorem 1.2 of [6].

2.2.3 Diameter of $\Delta_{\mathrm{nil}}(G)$

In this subsection we will provide some upper bounds for the diameter $\mathrm{diam}(\Delta_{\mathrm{nil}}(G))$ of the graph $\Delta_{\mathrm{nil}}(G)$. Recall that a group G is said to be an *AC*-group if every non-trivial element of G has abelian centralizer. soluble *AC*-groups have been classified by Schmidt in [48].

Proposition 2.2.10. *If G is a soluble, non-nilpotent AC-group, then $\Delta_{\mathrm{nil}}(G)$ is disconnected.*

Proof. By [48, Satz 5.12], either G has an abelian normal subgroup of prime index, $G/Z(G)$ is Frobenius or 2-Frobenius, or G is nilpotent. Assume that G has an abelian normal subgroup A of prime index. Clearly, A is the Fitting subgroup of G and $C_G(x) = A$ for every element $x \in A \setminus \{1\}$. Indeed, $C_G(x) < G$ and $|G : A|$ is prime. Hence G is a Frobenius group and Proposition 2.2.8 implies that $\Delta_{\text{nil}}(G)$ is disconnected.

If $G/Z(G)$ is Frobenius or 2-Frobenius, then $Z(G) = Z_\infty(G)$ and Theorem E implies that $\Delta_{\text{nil}}(G)$ is disconnected. Since G is not nilpotent, we are done. □

As we have seen in the previous subsection, it can be useful to understand the interplay between the nilpotent and the commuting graph. Thus, it makes sense to ask when these graphs are equal. We now show that A-groups are the only groups where $\Delta_{\text{nil}}(G)$ and $\Delta_{\text{comm}}(G)$ coincide, which is completely coherent with Theorem 2.2.3.

Theorem F. *Let G be a group. Then $\Delta_{\text{nil}}(G)$ coincides with $\Delta_{\text{comm}}(G)$ if and only if G is an A-group.*

Proof. Suppose that $\Delta_{\text{nil}}(G)$ coincides with $\Delta_{\text{comm}}(G)$. Clearly the center of G coincides with the hypercenter of G. Let P be a Sylow p-subgroup of G. If P is contained in the center of G, then it is abelian. Thus, assume that P is not contained in the center of G and let $x, y \in P \setminus Z(G)$. Since $\langle x, y\rangle$ is nilpotent, it follows that x and y are adjacent in $\Delta_{\text{nil}}(G)$. Thus, by the hypothesis, they also commute and therefore P is abelian.

Conversely, assume that G is an A-group. As a consequence, any nilpotent subgroup of G is abelian. We now show that the hypercenter of G coincides with the center $Z(G)$ of G: if $x \in Z_\infty(G)$ then for every $y \in G$ we have $\langle x, y\rangle$ is nilpotent, thus $[x, y] = 1$ and $x \in Z(G)$. Hence $\Delta_{\text{nil}}(G)$ and the commuting graph of G have the same set of vertices. Moreover any two vertices x and y are adjacent in $\Delta_{\text{nil}}(G)$ if and only if they are adjacent in $\Delta_{\text{comm}}(G)$. This concludes the proof. □

From Theorem F and [11, Theorem 1.1] it follows that for an A-group G the diameter of $\Delta_{\text{nil}}(G)$ is at most 6.

Now we prove the following theorem, focusing on each instance of the result separately.

Theorem G. *Let G be a group. Then the following are true:*

(*i*) *the connected components of $\Delta_{\text{nil}}(G)$ have diameter at most 10;*

(*ii*) *if G is soluble and $\Delta_{\text{nil}}(G)$ is connected, then $\Delta_{\text{nil}}(G)$ has diameter at most 8;*

(*iii*) *if* G *is soluble and* $\Delta_{\text{nil}}(G)$ *is disconnected, then one connected component of* $\Delta_{\text{nil}}(G)$ *has diameter at most* 5 *and the remaining connected components have diameter at most* 2*.*

As a consequence of Theorem D and Theorem E we may assume that G is neither Frobenius nor 2-Frobenius. Going further, we can apply a result of Morgan and Parker to obtain items (*i*) and (*ii*) of Theorem G. One should compare this result with Proposition 7.6 of [10].

Corollary 2.2.11. *Let* G *be a non-nilpotent group. Then the connected components of* $\Delta_{\text{nil}}(G)$ *have diameter at most* 10*. Moreover, if* G *is soluble and* $\Delta_{\text{nil}}(G)$ *is connected, then* $\operatorname{diam}(\Delta_{\text{nil}}(G)) \leq 8$*.*

Proof. By Theorem D, we have $\operatorname{diam}(\Delta_{\text{nil}}(G)) = \operatorname{diam}(\Delta_{\text{nil}}(G/Z_\infty(G)))$. Since $G/Z_\infty(G)$ has trivial center, Theorem 1.1 of [42] implies that the commuting graph of $G/Z_\infty(G)$ has diameter at most 10. Observing that the commuting graph of $G/Z_\infty(G)$ is a subgraph of $\Delta_{\text{nil}}(G/Z_\infty(G))$, then $\operatorname{diam}(\Delta_{\text{nil}}(G)) \leq 10$ as $\operatorname{diam}(\Delta_{\text{nil}}(G)) = \operatorname{diam}(\Delta_{\text{nil}}(G/Z_\infty(G)))$.

Now assume that G is soluble, using the same argument as before and applying Theorem 1.1 (ii) of [43], we obtain the desired result. □

In [43], Parker provided an example of a soluble group H such that $\Delta_{\text{comm}}(H)$ has diameter 8. However, $\Delta_{\text{nil}}(H)$ has diameter at most 7, because any two vertices of $\Delta_{\text{nil}}(H)$ belonging in $\operatorname{Fit}(H)$ are adjacent. This leaves open the question of whether there exists a soluble group G such that $\operatorname{diam}(\Delta_{\text{nil}}(G)) = 8$. Finally, we find the bound on the diameter of the connected components of $\Delta_{\text{nil}}(G)$ when G is soluble and $\Delta_{\text{nil}}(G)$ is disconnected, which proves item (*iii*) of Theorem G.

Proposition 2.2.12. *Let* G *be a soluble group and suppose that* $\Delta_{\text{nil}}(G)$ *is disconnected. Then the diameter of one connected component is at most* 5 *and the other connected components have diameters at most* 2*.*

Proof. Without loss of generality, we may assume that $Z_\infty(G) = 1$, thus G is either a Frobenius group or a 2-Frobenius group. Let F_1 be the Fitting subgroup of G and F_2/F_1, the Fitting subgroup of G/F_1. We know that F_2 is a Frobenius group in either case, and if H is a Frobenius complement for F_2, then $H \setminus \{1\}$ is a connected component of $\Delta_{\text{nil}}(G)$. We know $Z(H) > 1$ and we know every non-identity element in $Z(H)$ will be adjacent to every element in $H \setminus \{1\}$, so the induced subgraph with vertex set $H \setminus \{1\}$ is a connected component of $\Delta_{\text{nil}}(G)$ with diameter at most 2. Since every element in $F_2 \setminus F_1$ is conjugate to some element in $H \setminus \{1\}$, this accounts for all of the elements in $F_2 \setminus F_1$.

If $G = F_2$, then the remaining connected component has vertex set $F_1 \setminus \{1\}$ and it is a complete graph. Thus, we may assume that we are in the case where

G is a 2-Frobenius group, and the remaining connected component has vertex set $(G \setminus F_2) \cup (F_1 \setminus \{1\})$. We prove that every element in $G \setminus F_2$ has distance at most 2 from a point in $F_1 \setminus \{1\}$, and this will prove the conclusion. By the Frattini argument, we have that $G = F_1 N_G(H)$, and since every element in $F_2 \setminus F_1$ is conjugate to $H \setminus \{1\}$, it follows that every element in $G \setminus F_2$ is conjugate to an element in $N_G(H) \setminus H$. Thus, if $x \in G \setminus F_2$, then without loss of generality, we may assume that $x \in N_G(H) \setminus H$. We know that $N_G(H)$ is a Frobenius group with Frobenius complement C that is cyclic. Thus, without loss of generality, we may assume that $x \in C$. Now, we know that C does not act Frobeniously on F_1, so there exists $y \in C \setminus \{1\}$ so that $C_{F_1}(y) > 1$. This gives our path of distance at most 2 from x to a point in F_1. ☐

Now, we show that we can obtain a tighter bound on the diameter of $\Delta_{\mathrm{nil}}(G)$ when we impose extra conditions on G. Recall that the Fitting subgroup of a soluble group will be non-trivial, and so, the elements in $\mathrm{Fit}(G) \setminus Z_\infty(G)$ will lie in a single connected component of $\Delta_{\mathrm{nil}}(G)$. We now study this connected component. We also show that if $Z(G)$ is trivial, all of the primes dividing $|G|$ divide $|\mathrm{Fit}(G)|$, and $\Delta_{\mathrm{nil}}(G)$ is connected, then $\Delta_{\mathrm{nil}}(G)$ has diameter at most 5.

Proposition 2.2.13. *Let G be a soluble group with trivial center. Then:*

- *(i)* $\mathrm{Fit}(G) \setminus \{1\}$ *is contained in a connected component $\mathcal{X}$ of $\Delta_{\mathrm{nil}}(G)$;*
- *(ii) if x is a non-trivial element of G such that $(o(x), |\mathrm{Fit}(G)|) \neq 1$, then $x \in \mathcal{X}$ and there exists an element $w \in \mathrm{Fit}(G) \setminus \{1\}$ such that $d(x, w) \leq 2$;*
- *(iii) if $\pi(G) = \pi(\mathrm{Fit}(G))$ and $\Delta_{\mathrm{nil}}(G)$ is connected, then* $\mathrm{diam}(\Delta_{\mathrm{nil}}(G)) \leq 5$.

Proof. As $\mathrm{Fit}(G)$ is a non-trivial, nilpotent subgroup of G, it is clear that $\mathrm{Fit}(G) \setminus \{1\}$ is contained in a connected component $\mathcal{X}$ of $\Delta_{\mathrm{nil}}(G)$, and (i) is clear. Now, take an element $x \in G \setminus \{1\}$ and a prime p such that p divides $(o(x), |\mathrm{Fit}(G)|)$. Of course x is adjacent to its p-part, x_p. Let P be a Sylow p-subgroup of G containing x_p and observe that $P \setminus \{1\}$ is a clique of $\Delta_{\mathrm{nil}}(G)$. Moreover, as p divides $|\mathrm{Fit}(G)|$, there exists a non-trivial element $w \in P \cap \mathrm{Fit}(G)$. Thus, $d(x, w) \leq 2$ since both x and w are adjacent to x_p. Consider elements $x, y \in G \setminus \{1\}$. If $\pi(G) = \pi(\mathrm{Fit}(G))$, then applying conclusion (ii), we can find $w_1, w_2 \in \mathrm{Fit}(G) \setminus \{1\}$ so that $d(x, w_1) \leq 2$ and $d(y, w_2) \leq 2$. Since w_1 and w_2 will be adjacent in $\Delta_{\mathrm{nil}}(G)$, the result follows. ☐

We next show that if $\Delta_{\mathrm{nil}}(G)$ is connected and $|G : \mathrm{Fit}(G)|$ is a prime, then $\Delta_{\mathrm{nil}}(G)$ has diameter at most 3.

Proposition 2.2.14. *Let G be a non-nilpotent group and $F = \mathrm{Fit}(G)$ such that $|G : F| = p$ with p prime. If $\Delta_{\mathrm{nil}}(G)$ is connected, then* $\mathrm{diam}(\Delta_{\mathrm{nil}}(G)) \leq 3$.

Proof. Assume first that the center of G is trivial. Since any two distinct elements of F are adjacent, it is sufficient to prove that for every element $x \in G \setminus \{1\}$ there exists $w \in F \setminus \{1\}$ such that $d(x, w) = 1$. Assume $x \notin F$. If $x^p \neq 1$, then $x^p \in F \setminus \{1\}$ and $d(x, x^p) = 1$. Thus, suppose that $x^p = 1$. If p does not divide $|F|$, then $G = F\langle x\rangle$, and it is not a Frobenius group since $\Delta_{\text{nil}}(G)$ is connected. Therefore there exists an element $w \in F \setminus \{1\}$ such that $[x, w] = 1$. If p divides $|F|$, then there exists P a Sylow p-subgroup of G such that $x \in P$ and $P \cap F \neq \{1\}$. It follows that there exists $w \in F \setminus \{1\}$ such that $w \neq x$ and $w \in P$, so $\langle x, w\rangle$ is nilpotent and $d(x, w) = 1$.

Now assume that the center of G is not trivial. Then, by Theorem D, it follows that $\Delta_{\text{nil}}(G/Z_\infty(G))$ is connected and $\text{diam}(\Delta_{\text{nil}}(G)) = \text{diam}\Delta_{\text{nil}}(G/Z_\infty(G))$. Notice that $F/Z_\infty(G))$ has prime index in $G/Z_\infty(G)$. The statement is now clear. ☐

This bound is the best possible, in fact the group $G =$ SmallGroup (54,5) has Fitting Subgroup of order 27 and $\Delta_{\text{nil}}(G)$ is connected of diameter 3.

Next, we show that if G is non-nilpotent and has the form $N \rtimes A$ with $Z(G) = Z_\infty(G)$ where N is cyclic and A is abelian, then $\Delta_{\text{nil}}(G)$ has diameter at most 4.

Proposition 2.2.15. *Let $G = N \rtimes A$ a non-nilpotent group with N cyclic and A abelian. If $\Delta_{\text{nil}}(G)$ is connected and $Z(G) = Z_\infty(G)$, then $\Delta_{\text{nil}}(G)$ has diameter at most* 4.

Proof. From $\Delta_{\text{nil}}(G)$ connected and Proposition 2.2.10, it follows that G is not an *AC*-group. Therefore, there exists a noncentral element $x \in G$ such that $C_G(x)$ is non-abelian. By Theorem 1.2 (a) of [55], $\Delta_{\text{comm}}(G)$ is connected and by Theorem 1.2 (b) of [55], it has diameter at most 4. Therefore, also $\Delta_{\text{nil}}(G)$ has diameter at most 4. ☐

We point out that this bound is sharp. Indeed the group $G = \langle x\rangle \rtimes \langle y\rangle$ with $x^{15} = y^4 = 1$ and $x^y = x^8$ satisfies the hypothesis of the previous result and $\Delta_{\text{nil}}(G)$ is connected of diameter 4.

We now show that if $\text{Fit}(G)$ is cyclic, then $\Delta_{\text{nil}}(G)$ has diameter at most 5.

Proposition 2.2.16. *Let G be a soluble group with trivial center such that $\Delta_{\text{nil}}(G)$ is connected. If the Fitting subgroup of G is cyclic, then the diameter of $\Delta_{\text{nil}}(G)$ is at most* 5.

Proof. Fix an element $x \in G \setminus \{1\}$ and write $F = \text{Fit}(G)$. It is sufficient to prove that there exists an element $f \in F \setminus \{1\}$ such that x is connected to f in at most 2 steps.

If $(o(x), |F|) \neq 1$, then by Proposition 2.2.13 (*ii*) the result follows. Suppose $(o(x), |F|) = 1$, and let p be a prime that divides $o(x)$. Denote by x_p the p-part of

x. Obviously x and x_p are adjacent in $\Delta_{\mathrm{nil}}(G)$. Let P be a Sylow p-subgroup of G such that $x_p \in P$. By Proposition 1.1.2 G/F is abelian.

Thus, FP is normal in G and by the Frattini argument, it follows that $G = N_G(P)F$. If $N_G(P) \cap F = \{1\}$, then G has a cyclic-by-abelian factorization, and by Proposition 2.2.15 the diameter of $\Delta_{\mathrm{nil}}(G)$ is at most 4. Suppose $N_G(P) \cap F \neq \{1\}$. Notice that $N_G(P) \cap F$ is normal in $N_G(P)$, and thus, $N_G(P) \cap F \subseteq \mathrm{Fit}(N_G(P))$. Moreover, $P \subseteq \mathrm{Fit}(N_G(P))$. Therefore, there exists $f \in F \setminus \{1\}$ such that $\langle x_p, f\rangle$ is nilpotent. The conclusion is now clear. □

We now show that if G is a $\{p, q\}$-group, then $\Delta_{\mathrm{nil}}(G)$ has diameter at most 6.

Theorem 2.2.17. *Let G be a non-nilpotent $\{p, q\}$-group with trivial center. If $\Delta_{\mathrm{nil}}(G)$ is connected, then* $\mathrm{diam}(\Delta_{\mathrm{nil}}(G)) \leq 6$.

Proof. It is sufficient to show that for any element $x \in G \setminus \{1\}$ the distance between x and any element of $F = \mathrm{Fit}(G)$ is at most 3. If $\pi(F) = \pi(G)$, then by Proposition 2.2.13 (*iii*) the result follows. Therefore, without loss of generality, we may assume that q does not divide $|F|$, and so, F is a p-group.

Consider an element $x \in G \setminus \{1\}$ and $P \in \mathrm{Syl}_p(G)$. Assume first that P is normal in G; that is, $P = O_p(G) = F$. If p divides $o(x)$, then x is adjacent in $\Delta_{\mathrm{nil}}(G)$ to its p-part, say x_p, which lies in F. If p does not divide $o(x)$, then x is a q-element and there exists $Q \in \mathrm{Syl}_q(G)$ such that $x \in Q$. Moreover, notice that $G = FQ$ which is not a Frobenius group due to Proposition 2.2.8 and the fact that $\Delta_{\mathrm{nil}}(G)$ is connected. Thus, there exist elements $w \in Q \setminus \{1\}$ and $f \in F \setminus \{1\}$ such that $[w, f] = 1$. Therefore, x is connected to an element of F in at most 2 steps.

Assume now that $O_p(G) = F < P$. If p divides $o(x)$, then x is adjacent to its p-part, say x_p, which is adjacent to a non-identity element of F. If p does not divide $o(x)$, then x is a q-element. If p divides $|C_G(x)|$ then there exists an element $w \in C_G(x)$ such that w is a p-element, and so, x is connected to an element of F in at most 2 steps. Therefore, we may assume that p does not divide $|C_G(x)|$, and thus, that there exists $Q \in \mathrm{Syl}_q(G)$ such that $C_G(x) \leq Q$. If there exist elements $f \in F \setminus \{1\}$ and $y \in Q \setminus \{1\}$ such that $[f, y] = 1$, then x is connected to f in at most 2 steps. Thus, we may assume for every $f \in F \setminus \{1\}$ and for every $y \in Q \setminus \{1\}$ that we have $[f, y] \neq 1$. Now, we wish to show that FQ is a Frobenius group. Let $v \in Q \setminus \{1\}$ and $z \in C_{FQ}(v)$. There exist $f \in F$ and $g \in Q$ such that $z = fg$. We have

$$1 = [z, v] = [fg, v] = [f, v]^g[g, v].$$

Notice that $[g, v] \in Q$, $[f, v] = f^{-1}f^v \in F$ and so also $[f, v]^g \in F$. Therefore, $[f, v] = 1$ which yields $f = 1$ and $z \in Q$. Thus, $C_{FQ}(v) \leq Q$ and FQ is a Frobenius group. Denote $K/F = O_q(G/F)$. Obviously K/F is a subgroup of FQ/F. By Theorem 9.3.1 of [44], we have $C_G(K/F) \leq K$, and thus, $C_{G/F}(K/F) \leq K/F$.

If q is odd then the fact that FQ is a Frobenius group implies that Q is cyclic and $FQ/F \cong Q$ is contained in $C_{G/F}(K/F)$, therefore $C_{G/F}(K/F) \leq K/F \leq FQ/F \leq C_{G/F}(K/F)$, which implies that $FQ = K$ is normal in G. Obviously Q is a Sylow q-subgroup of FQ; therefore by the Frattini argument, it follows that $G = N_G(Q)FQ = N_G(Q)F$. Since $\Delta_{\mathrm{nil}}(G)$ is connected, by Proposition 2.2.8 G is not a 2-Frobenius group, and so, G/F is not a Frobenius group. Thus $N_G(Q)$ is not a Frobenius group, so there exist elements $y \in C_{N_G(Q)}(x) \setminus \{1\}$ and $g \in C_{N_G(Q)}(y) \setminus Q$, with $o(g) = p$ such that for every $w \in F$ we have $x \sim y \sim g \sim w$.

Therefore x is connected to any non-identity element of F in at most 3 steps. If $q = 2$, then we may assume that Q is not cyclic, otherwise we can argue as before. Thus, the fact that FQ is a Frobenius group implies that Q is generalized quaternion. Again

$$C_{G/F}(K/F) \leq K/F;$$

moreover $\dfrac{G/F}{C_{G/F}(K/F)}$ is isomorphic to a subgroup of $\mathrm{Aut}(K/F)$. Suppose that K/F is cyclic or generalized quaternion of order at least 16, it follows that $\mathrm{Aut}(K/F)$ is a 2-group and $P/F \leq C_{G/F}(K/F) \leq K/F$, therefore $P/F = 1$ and so $P = F$ which is a contradiction. Then $Q \cong Q_8$ and $K/F \cong Q$. Since $\mathrm{Aut}(Q_8)$ is the symmetric group of degree 4, it follows that $p = 3$ and $G/QF \cong C_3$ and so $G/F \cong SL_2(3)$. Therefore, there exist $u \in F$ and $v \in O_2(G)$ such that $x \sim v \sim u$. $\square$

This concludes our investigation on the nilpotent graph.

2.3 The normalizing graph

This section deals with a directed graph which encodes information about the lattice of normal subgroups of the associated group. The *directed normalizing graph* of a group G is the directed simple graph $\vec{\Gamma}_{\mathrm{norm}}(G)$ whose vertices are all elements of G, and there is a directed edge from a vertex x to a vertex y if the subgroup $\langle x \rangle$ is normal in the subgroup $\langle x, y \rangle$. The results presented in this section are collected in [20].

2.3.1 Universal vertices

As seen previously, the study of the connectivity of a graph only makes sense after detecting and hence removing the set of all universal vertices. In general, this is far from being an easy task, and even worse when the graph is directed. Thus, our first aim is to provide information about the set $\mathrm{Univ}(G)$, which is the set of universal bidirectional vertices of the directed normalizing graph.

We start by asking how the neighborhood of an element looks like. We set $N^+(x) = \{y \in G \mid \langle x \rangle \trianglelefteq \langle x, y \rangle\}$ and $N^-(x) = \{y \in G \mid \langle y \rangle \trianglelefteq \langle x, y \rangle\}$.

Proposition 2.3.1. *Let G be a group and let $x \in G$. Then:*

(i) $N^+(x) = N_G(\langle x \rangle)$*;*

(ii) $\{a \in G \mid \langle a \rangle \trianglelefteq G\} \cup C_G(x) \subseteq N^-(x)$.

Proof. Statement (i) follows from the fact that $\langle x \rangle$ is normal in $\langle x, y \rangle$ if and only if y normalizes $\langle x \rangle$.

We now prove statement (ii). Let $a \in G$ such that $\langle a \rangle$ is normal in G. Then $\langle a \rangle$ is normal in $\langle a, x \rangle$ and so $a \to x$. Thus $a \in N^-(x)$. If $a \in C_G(x)$ then a commutes with x and thus $a \to x$. ☐

Notice that, in general, the set $N^-(x)$ is not a subgroup of G and also that the inclusion in (ii) can be strict. As an example take the symmetric group S_3 and the transposition $x = (12)$. Then $N^-(x) = \{1, (12), (123), (132)\}$, which is not a subgroup of S_3. Furthermore, consider the symmetric group S_4 and $x = (34)$. Then $(243) \in N^-(x)$ but $\langle (243) \rangle$ is not normal in S_4 and $(234) \notin C_{S_4}(x)$.

Linked to both sets of neighborhoods one can consider the sets of universal forward and of universal backward vertices. Denote $\mathrm{Univ}^-(G) = \{x \in G \mid x \to y, \text{ for every } y \in G\}$ the set of universal backward vertices, and $\mathrm{Univ}^+(G) = \{x \in G \mid y \to x, \text{ for every } y \in G\}$ the set of universal forward vertices.

Proposition 2.3.2. *Let G be a group and let $x \in G$. Then:*

(i) $\mathrm{Univ}^-(G) = \{a \in G \mid \langle a \rangle \trianglelefteq G\}$*;*

(ii) $\mathrm{Univ}^+(G) = \bigcap_{a \in G} N_G(\langle a \rangle)$.

Proof. We first prove (i), noticing that $x \in \mathrm{Univ}^-(G)$ if and only if $N^+(x) = G$, so, by (i) of Proposition 2.3.1, if and only if $N_G(\langle x \rangle) = G$, that is $\langle x \rangle \trianglelefteq G$.

We now prove (ii). An element $x \in G$ belongs to $\bigcap_{a \in G} N_G(\langle a \rangle)$ if and only if it normalizes every $a \in G$ and thus if and only if $x \in \mathrm{Univ}^+(G)$. ☐

Notice that $\mathrm{Univ}^+(G)$ is always a subgroup of G. In fact, it is the Baer norm of G, which is the intersection of the normalizers of all subgroups of G. It was introduced by Baer in [5]. Several properties are satisfied by this subgroup. For example, it has been shown that this is always a characteristic subgroup of G which is contained in the second term of the upper central series of G. Meanwhile, the set $\mathrm{Univ}^-(G)$ is not a subgroup of G, as one can see in Example 2.3.5.

We denote by $\mathrm{Univ}(G) = \mathrm{Univ}^-(G) \cap \mathrm{Univ}^+(G)$ the set of all bidirectional universal vertices. When not specified, we will refer to bidirectional universal vertices as universal vertices.

The following result is a direct consequence of a result in [47]. We write $Z_2(G)$ for the second center of a group G.

Proposition 2.3.3. *Let G be a group. Then $Z(G) \subseteq \mathrm{Univ}(G) \subseteq Z_2(G)$.*

Proof. First notice that every element of $Z(G)$ commutes with every element of G and thus it is trivially universal. If $x \in \mathrm{Univ}(G)$ then $x \in \mathrm{Univ}^+(G)$ which is contained in $Z_2(G)$ by Theorem in [47]. This concludes the proof. $\square$

Corollary 2.3.4. *Let G be a group. Then $\mathrm{Univ}(G) = \{1\}$ if and only if $Z(G) = \{1\}$.*

We observe that in general $Z(G) \neq \mathrm{Univ}(G) \neq Z_2(G)$; moreover, $\mathrm{Univ}(G)$ need not be a subgroup of G.

Example 2.3.5. In fact, let $G = \langle x, y \mid x^8 = y^2 = 1, x^y = x^5\rangle$. Then $Z(G) = \{1, x^2, x^4, x^6\}$ has order 4. Consider $x^2y \in G \setminus Z(G)$. We have $H = \langle x^2y\rangle = \{1, x^2y, x^4, x^6y\}$ which is normal in G. Furthermore, $(x^ny)^{x^2y} = x^{5n}y = (x^ny)^5 \in \langle x^ny\rangle$. Therefore, $x^2y \in \mathrm{Univ}(G) \setminus Z(G)$. Moreover, since $y^x = x^6y \notin \langle y\rangle$, it follows that $x, y \in G \setminus \mathrm{Univ}(G)$ and so $\mathrm{Univ}(G) \neq G = Z_2(G)$. It is not difficult to see that $\mathrm{Univ}(G) = Z(G) \cup \{x^2y, x^4y\}$ and thus $|\mathrm{Univ}(G)| = 6$ and it is not a subgroup of G. In general, the fact that $\mathrm{Univ}(G)$ is not a subgroup of G also shows that $\mathrm{Univ}^-(G)$ need not be a subgroup of G.

Now we go further in our investigation about universal vertices.

Lemma 2.3.6. *Let G be a group and $x \in \mathrm{Univ}(G)$. Then:*

(i) $x^m \in \mathrm{Univ}(G)$ *for any integer m;*

(ii) $x^g \in \mathrm{Univ}(G)$ *for any $g \in G$;*

(iii) if x has finite order then for any prime $p \in \pi(o(x))$ there exists an element of $\mathrm{Univ}(G)$ *having order p;*

(iv) if x has order 2, then $Z(G) \neq \{1\}$.

Proof. Let m be an integer and $g \in G$. Then there exists a suitable integer n such that $(x^m)^g = (x^g)^m = x^{nm} = (x^m)^n \in \langle x^m\rangle$. This proves (i). Furthermore, conjugating g by x gives a power of g and thus we have also $g^{x^m} \in \langle g\rangle$.

We now prove (ii). Since $\langle x\rangle$ is normal in G we have $x^g = x^m$ for some integer m. Thus, x^g is a power of a universal vertex and so by statement (i) we have $x^g \in \mathrm{Univ}(G)$.

Statement (iii) is an immediate consequence of (i).

Lastly, we prove statement (iv). Let $x \in \mathrm{Univ}(G)$ of order 2. In particular, for any $g \in G$ we have that g normalizes $\langle x\rangle$, and thus it centralizes x. It follows that $x \in Z(G)$. $\square$

From the previous result it follows that $\mathrm{Univ}(G)$ is a union of conjugacy classes.

Lemma 2.3.7. *Let G be a finite group and let $x, y \in \mathrm{Univ}(G)$ of coprime order such that $[x, y] = 1$. Then $xy \in \mathrm{Univ}(G)$.*

Proof. Let $g \in G$. From $x, y \in \mathrm{Univ}(G)$ we have $x, y \in N_G(\langle g\rangle)$. It follows that $xy \in N_G(\langle g\rangle)$. Consider now $(xy)^g$. We have

$$(xy)^g = x^g y^g = x^n y^m,$$

for some integers n and m. Since $(o(x), o(y)) = 1$ there exists an integer k such that $k \equiv n \pmod{o(x)}$ and $k \equiv m \pmod{o(y)}$. Thus $x^n y^m = x^k y^k$. Since x and y commute we have $x^k y^k = (xy)^k$. Therefore g normalizes $\langle xy\rangle$ and $xy \in \mathrm{Univ}(G)$. $\square$

It is possible to generalize statement (*iv*) of Lemma 2.3.6 showing that every element of prime order of $\mathrm{Univ}(G)$ is central.

Proposition 2.3.8. *Let G be a finite group and let $x \in \mathrm{Univ}(G)$. If x has prime order, then $x \in Z(G)$.*

Proof. By Lemma 2.3.6 we can assume that x has order p, where p is an odd prime. Let $C = C_G(\langle x\rangle)$. Suppose by contradiction that $x \notin Z(G)$. Then C is a proper subgroup of G. Since $\langle x\rangle \trianglelefteq G$, we have $N_G(\langle x\rangle)/C = G/C$ and it is isomorphic to a non-trivial subgroup of $\mathrm{Aut}(\langle x\rangle)$ which is cyclic of order $p-1$. Thus G/C is cyclic. Let $d = |G/C| = q_1^{\alpha_1} \dots q_s^{\alpha_s}$, with q_i different primes and $\alpha_i \in \mathbb{N}$ for all i. Consider $z \in G \setminus C$. Then $o(zC)$ divides d. Suppose t is a prime that divides $o(z)$ but not $o(zC)$ and say t^ℓ the maximum power of t that divides $o(z)$. Then $z^{t^\ell}C \neq C$, since t and $o(zC)$ are coprime. Thus, there exists $y \in G \setminus C$ of order $q_1^{r_1} \dots q_s^{r_s}$, with $r_i \in \mathbb{N}$ for all i. However, x normalizes $\langle y\rangle$ and so there exists a homomorphism from $\langle x\rangle$ to $\mathrm{Aut}(\langle y\rangle)$. Since $|\mathrm{Aut}(\langle y\rangle)| = q_1^{r_1-1}(q_1-1) \dots q_s^{r_s-1}(q_s-1)$ we have $(p, |\mathrm{Aut}(\langle y\rangle)|) = 1$. Therefore, the only possibility is that x centralizes y, which is a contradiction. $\square$

We point out that from Proposition 2.3.8 and Lemma 2.3.6 (*i*) it is possible to obtain an alternative proof of Corollary 2.3.4 for finite groups, which avoids using the result in [47].

The following result helps us to describe the set $\mathrm{Univ}(G)$ when G is a decomposable finite group.

Proposition 2.3.9. *Let $G = H \times K$ be a finite group. Then:*

(*i*) $\mathrm{Univ}^+(G) \subseteq \mathrm{Univ}^+(H) \times \mathrm{Univ}^+(K)$;

(*ii*) $\mathrm{Univ}^-(G) \subseteq \mathrm{Univ}^-(H) \times \mathrm{Univ}^-(K)$;

(iii) $\mathrm{Univ}(G) \subseteq \mathrm{Univ}(H) \times \mathrm{Univ}(K)$*;*

(iv) *if* $(|H|, |K|) = 1$ *then* $\mathrm{Univ}(G) = \mathrm{Univ}(H) \times \mathrm{Univ}(K)$*;*

(v) *if* $Z(H) = \mathrm{Univ}(H)$ *and* $Z(K) = \mathrm{Univ}(K)$ *then* $\mathrm{Univ}(G) = \mathrm{Univ}(H) \times \mathrm{Univ}(K)$*.*

Proof. We first prove statement (i). Let $g \in \mathrm{Univ}^+(G)$. Then there exist $h \in H$ and $k \in K$ such that $g = (h, k)$. Let $h_1 \in H$ and $k_1 \in K$. Then $(h_1, k_1) \in G$, thus $(h_1, k_1)^g = (h_1, k_1)^n = (h_1^n, k_1^n)$, for some integer n. However, $(h_1, k_1)^g = (h_1, k_1)^{(h,k)} = (h_1^h, k_1^k)$. Therefore $h_1^h = h_1^n$, $k_1^h = k_1^n$ and so $h \in \mathrm{Univ}^+(H)$ and $k \in \mathrm{Univ}^+(K)$.

For statement (ii) we argue similarly. Let $g \in \mathrm{Univ}^-(G)$. Then there exist $h \in H$ and $k \in K$ such that $g = (h, k)$. Let $h_1 \in H$ and $k_1 \in K$. Then $(h_1, k_1) \in G$, thus $(h, k)^{(h_1,k_1)} = (h, k)^n = (h^n, k^n)$ for some integer n. However, $(h, k)^{(h_1,k_1)} = (h^{h_1}, k^{k_1})$. Therefore $h^{h_1} = h^n$ and $k^{k_1} = k^n$ and so $h \in \mathrm{Univ}^-(H)$ and $k \in \mathrm{Univ}^-(K)$.

Statement (iii) follows from (i) and (ii).

We now prove (iv). By (iii), we only need to prove one inclusion. Let $h \in \mathrm{Univ}(H)$ and $k \in \mathrm{Univ}(K)$. Let $g = (h_1, k_1) \in G$. Then $(h, k)^g = (h^{h_1}, k^{k_1}) = (h^n, k^m)$ for some integers n and m. However, since $o(h)$ and $o(k)$ are coprime, there exists $x \in Z$ such that $x \equiv n \pmod{o(h)}$ and $x \equiv m \pmod{o(k)}$. Therefore $(h, k)^g = (h^n, k^m) = (h^x, k^x) = (h, k)^x$ and $(h, k) \in \mathrm{Univ}^-(G)$. Moreover, $g^{(h,k)} = (h_1^s, k_1^t)$ for some integers s and t. As before there exists an integer y such that $(h_1^s, k_1^t) = (h_1, k_1)^y$. Thus $(h, k) \in \mathrm{Univ}^+(G)$ and we are done.

Lastly, we prove (v). By (iii) we only need to prove one inclusion. By hypothesis $\mathrm{Univ}(H) \times \mathrm{Univ}(K) = Z(H) \times Z(K) = Z(G) \subseteq \mathrm{Univ}(G)$. □

In general the inclusions in $(i), (ii)$ and (iii) of Proposition 2.3.9 can be strict. For instance, consider $G = Q_8 \times Q_8$. Then $\mathrm{Univ}(G) = Z(G)$ which has order 4, while Q_8 is a Dedekind group, and thus $\mathrm{Univ}(Q_8) = Q_8$.

The next two results will be very useful in the study of the diameter of the graph $\vec{\Delta}_{\mathrm{norm}}(G)$, for a decomposable group G.

Lemma 2.3.10. *Let* $G = H \times K$*, where* H *is a non-abelian finite group. Let* $x \in \mathrm{Univ}(H) \setminus Z(H)$ *and* $y \in H$*, and let* n *be an integer such that* $y^x = y^n \neq y$*. If there exists* $k \in K$ *such that* $o(k) = o(y)$ *then* $(x, 1) \notin \mathrm{Univ}(G)$*.*

Proof. By hypothesis, $(y, k)^{(x,1)} = (y^n, k)$. Since $o(k) = o(y)$ there are no integers m such that $m \equiv n \pmod{o(y)}$ and $m \equiv 1 \pmod{o(k)}$, and thus $(y^n, k) \notin \langle (y, k) \rangle$. Therefore $(x, 1) \notin \mathrm{Univ}(G)$. □

Lemma 2.3.11. *Let* $G = H \times K$ *be a group. Then:*

(*i*) *if* $(a,1) \in \mathrm{Univ}(G)$ *and* $(1,b) \in \mathrm{Univ}(G)$*, then* $(a,b) \in \mathrm{Univ}^+(G)$*;*

(*ii*) *if* $a \in \mathrm{Univ}(H)$*, then* $(a,1) \in \mathrm{Univ}^-(G)$*;*

(*iii*) *if* $b \in \mathrm{Univ}(K)$*, then* $(1,b) \in \mathrm{Univ}^-(G)$*.*

Proof. Statement (*i*) follows from the fact that $(a,1) \in \mathrm{Univ}^+(G)$, $(1,b) \in \mathrm{Univ}^+(G)$ and $\mathrm{Univ}^+(G)$ is a subgroup of G.

We now prove (*ii*). Let $a \in \mathrm{Univ}(H)$ and $(c,d) \in G$. Then $(a,1)^{(c,d)} = (a^c,1) = (a^n,1)$, for some integer n. Thus, $(a,1)^{(c,d)} = (a,1)^n \in \langle (a,1) \rangle$ and so $(a,1) \in \mathrm{Univ}^-(G)$.

The statement (*iii*) is proven in the same way as (*ii*).

□

2.3.2 Completeness

Let G be a group. In this subsection, we focus on the graphs $\vec{\Gamma}_{\mathrm{norm}}(G)$ and the undirected graph induced by it, namely $\Gamma_{\mathrm{norm}}(G)$. The graph $\Gamma_{\mathrm{norm}}(G)$ has appeared in the literature very recently. More specifically, in [31, Theorem 1] Farrell and Parker classify when the subgraph induced by $\Gamma_{\mathrm{norm}}(G)$ on $G \setminus \{1\}$ is connected, also giving a sharp upper bound on its diameter provided the group is soluble with trivial center. The aim of this section is to understand when these graphs are complete. In particular, we characterize when $\vec{\Gamma}_{\mathrm{norm}}(G)$ is complete for any group G and, using completeness of $\Gamma_{\mathrm{norm}}(G)$, we provide a sufficient condition for a group to be nilpotent.

Theorem 2.3.12. *A group G is Dedekind if and only if $\vec{\Gamma}_{\mathrm{norm}}(G)$ is complete.*

Proof. If G is a Dedekind group, then for every $x \in G$ we have $\langle x \rangle$ is normal in G. Therefore for every $x, y \in G$ we have $\langle x \rangle$ is normal in $\langle x, y \rangle$. Thus $\vec{\Gamma}_{\mathrm{norm}}(G)$ is complete.

On the other hand, assume that $\vec{\Gamma}_{\mathrm{norm}}(G)$ is complete. Let H be a subgroup of G. For every $h \in H$ and for every $g \in G$ we have $\langle h \rangle$ is normal in $\langle h, g \rangle$. Thus $h^g \in H$, so H is normal in G. □

Let now consider the undirected graph $\Gamma_{\mathrm{norm}}(G)$ whose vertices are the element of the group G and where x and y are adjacent in $\Gamma_{\mathrm{norm}}(G)$ if and only if either $x \to y$ or $y \to x$ in $\vec{\Gamma}_{\mathrm{norm}}(G)$. Recall that the supersolubility graph of a group G is the undirected simple graph obtained taking as vertices all elements of G and drawing an edge between two elements $x, y \in G$ if and only if the subgroup $\langle x, y \rangle$ is supersoluble.

Proposition 2.3.13. *Let G be a group. Then $\Gamma_{\mathrm{norm}}(G)$ is a subgraph of the supersolubility graph of G.*

Proof. If x and y are adjacent in $\Gamma_{\text{norm}}(G)$ then $\langle x\rangle$ is normal in $\langle x,y\rangle$ or $\langle y\rangle$ is normal in $\langle x,y\rangle$. Without loss of generality suppose that $\langle x\rangle$ is normal in $\langle x,y\rangle$. Then $1 \le \langle x\rangle \le \langle x,y\rangle$ is a normal series with cyclic factors. Therefore x and y generate a supersoluble group and thus they are adjacent in the supersolubility graph of G. □

For a finite group G it immediately follows from Proposition 2.3.13 and [12, Theorem 4.8] that G is supersoluble provided $\Gamma_{\text{norm}}(G)$ is complete. Actually it is possible to say something more.

Theorem 2.3.14. *Let G be a finite group with $\Gamma_{\text{norm}}(G)$ complete. Then:*

- (i) *G is nilpotent of nilpotency class at most 3;*
- (ii) *all involutions of G commute.*

Proof. We first prove (i). Let $x,y \in G$. Since $\Gamma_{\text{norm}}(G)$ is complete we have $\langle x\rangle \trianglelefteq \langle x,y\rangle$ or $\langle y\rangle \trianglelefteq \langle x,y\rangle$. Without loss of generality suppose $\langle x\rangle \trianglelefteq \langle x,y\rangle$. Moreover, we have $\langle y\rangle \trianglelefteq \langle xy,y\rangle$ or $\langle xy\rangle \trianglelefteq \langle xy,y\rangle = \langle x,y\rangle$. Therefore, it follows that $\langle x,y\rangle = \langle x\rangle\langle y\rangle$ or $\langle x,y\rangle = \langle x\rangle\langle xy\rangle$. In either case, $\langle x,y\rangle$ is a product of cyclic normal subgroups and thus it is nilpotent of class at most 2. Thus, applying the main result in [40] the proof is completed.

We now prove (ii). Let g,h be two involutions of G. Since $\Gamma_{\text{norm}}(G)$ is complete it follows that g normalizes $\langle h\rangle$ or h normalizes $\langle g\rangle$. Since both h and g have order 2, we are done. □

In general a nilpotent group of class 2 does not have a complete normalizing graph, as the dihedral group of order 8 shows.

We also point out that the completeness of $\Gamma_{\text{norm}}(G)$ does not imply that $\vec{\Gamma}_{\text{norm}}(G)$ is complete. As a counterexample, one can take G as in Example 2.3.5. In fact, $\text{Univ}^+(G)$ has index 2 in G and so by Corollary 2.3.17 it follows that $\Gamma_{\text{norm}}(G)$ is complete.

Lemma 2.3.15. *Let G be a group and $x \in G \setminus \text{Univ}^+(G)$. If for all $y \in G \setminus \text{Univ}^+(G)$ there exists $c \in C_G(x)$ such that $c^{-1}y \in \text{Univ}^+(G)$ then $\langle x\rangle$ is normal in G.*

Proof. We prove that any $y \in G$ normalizes $\langle x\rangle$. This is obviously true if $y \in \text{Univ}^+(G)$. Now assume $y \in G \setminus \text{Univ}^+(G)$. By hypothesis, there exists $c \in C_G(x)$ such that $c^{-1}y \in \text{Univ}^+(G)$. Thus $x^y = x^{c^{-1}y} \in \langle x\rangle$. Therefore, $\langle x\rangle$ is normal in G. □

The following provides a sufficient condition for $\Gamma_{\text{norm}}(G)$ to be complete.

Proposition 2.3.16. *If G is a group such that for any $x, y \in G \setminus \mathrm{Univ}^+(G)$ there exists $c \in C_G(x)$ such that $c^{-1}y \in \mathrm{Univ}^+(G)$ then $\Gamma_{\mathrm{norm}}(G)$ is complete.*

Proof. Let $x, y \in G$. If either x or y belongs to $\mathrm{Univ}^+(G)$ then x and y are adjacent in $\Gamma_{\mathrm{norm}}(G)$. Otherwise apply Lemma 2.3.15 to obtain the result. $\square$

Corollary 2.3.17. *If* $\mathrm{Univ}^+(G)$ *has prime index in a group G then* $\Gamma_{\mathrm{norm}}(G)$ *is complete.*

Proof. Since every $x \in G \setminus \mathrm{Univ}^+(G)$ is a generator of G modulo $\mathrm{Univ}^+(G)$, then any $y \in G \setminus \mathrm{Univ}^+(G)$ is a power of x modulo $\mathrm{Univ}^+(G)$. Thus Proposition 2.3.16 applies and the result follows. $\square$

We point out that the hypothesis of Proposition 2.3.16 implies that any element outside $\mathrm{Univ}^+(G)$ generates a normal subgroup of G by Lemma 2.3.15. However the converse of Proposition 2.3.16 does not hold. Indeed, if $G = \mathrm{SmallGroup}(64, 28)$ then $\Gamma_{\mathrm{norm}}(G)$ is complete and there exists $x \in G \setminus \mathrm{Univ}^+(G)$ such that $\langle x \rangle$ is not normal in G.

2.3.3 Strong connectivity and diameter

In this subsection, we focus on the directed normalizing graph from which its universal vertices have been removed, i.e. $\vec{\Delta}_{\mathrm{norm}}(G)$. In general, it could be difficult to predict whether this graph is strongly connected or not. For instance, if you consider the 2-group G as in Example 2.3.5, then $\vec{\Delta}_{\mathrm{norm}}(G)$ is strongly disconnected. Thus, we firstly focus on decomposable groups, establishing some conditions that ensure a strong connectivity of $\vec{\Delta}_{\mathrm{norm}}(G)$ and give us a bound on its diameter. Later, we focus on groups with trivial center, since, due to Corollary 2.3.4, for this class of groups G we have $\mathrm{Univ}(G) = \{1\}$ and thus, the vertex set of $\vec{\Delta}_{\mathrm{norm}}(G)$ is $G \setminus \{1\}$. In this case, for soluble groups, we characterize when $\vec{\Delta}_{\mathrm{norm}}(G)$ is strongly connected and establish some bounds on its diameter.

Lemma 2.3.18. *Let $G = H \times K$ be a direct product of two non-Dedekind groups. If $(a, b) \in G \smallsetminus \mathrm{Univ}(G)$ and either $(a, 1) \notin \mathrm{Univ}(G)$ or $(1, b) \notin \mathrm{Univ}(G)$ then for every $(c, d) \in G \smallsetminus \mathrm{Univ}(G)$ there is a directed path connecting (a, b) to (c, d) in at most 3 steps.*

Proof. If $(1, d) \notin \mathrm{Univ}(G)$ then we can consider the path $(a, b) \to (a, 1) \to (1, d) \to (c, d)$. If $(1, d) \in \mathrm{Univ}(G)$ and $(c, 1) \in \mathrm{Univ}(G)$ then by (i) of Lemma 2.3.11 we have $(c, d) \in \mathrm{Univ}^+(G)$ and thus $(a, b) \to (c, d)$. Finally, suppose $(1, d) \in \mathrm{Univ}(G)$ and $(c, 1) \notin \mathrm{Univ}(G)$. Since K is not a Dedekind group by Theorem 2.3.12 there exists $x \in K \smallsetminus \mathrm{Univ}(K)$. Since $(1, d) \in \mathrm{Univ}(G)$ then $(1, x)$ is normalized by (c, d). Thus, we can consider the path $(a, b) \to (a, 1) \to (1, x) \to (c, d)$. $\square$

We are now ready to prove the following theorem.

Theorem H. *Let $G = H \times K$ be a direct product of non-Dedekind groups. Then the graph $\vec{\Delta}_{\mathrm{norm}}(G)$ is strongly connected of diameter at most 3 provided that one of the following conditions holds:*

(*i*) $\mathrm{Univ}(G) = \mathrm{Univ}(H) \times \mathrm{Univ}(K)$*;*

(*ii*) $(h, 1) \in \mathrm{Univ}(G)$ *if and only if* $h \in Z(H)$*;*

(*iii*) $(1, k) \in \mathrm{Univ}(G)$ *if and only if* $k \in Z(K)$*.*

Proof. From H and K being non-Dedekind groups it follows that G is not a Dedekind group. Let $(a, b), (c, d) \in G \smallsetminus \mathrm{Univ}(G)$.

First assume (*i*). Since $(a, b) \notin \mathrm{Univ}(G)$, we have $(a, 1) \notin \mathrm{Univ}(G)$ or $(1, b) \notin \mathrm{Univ}(G)$. Thus Lemma 2.3.18 gives the result.

Now suppose condition (*ii*) or (*iii*) holds. By Lemma 2.3.18 we can assume that $(a, 1), (1, b) \in \mathrm{Univ}(G)$. If both $(c, 1)$ and $(1, d)$ are elements of $\mathrm{Univ}(G)$, then by (*i*) of Lemma 2.3.11 we have $(a, b) \to (c, d)$. Without loss of generality assume that $(c, 1) \notin \mathrm{Univ}(G)$. Since K is a non-Dedekind group, by Theorem 2.3.12 there exists $y \in K$ such that $(1, y) \in G \setminus \mathrm{Univ}(G)$. If (*ii*) holds, then $(a, 1) \in \mathrm{Univ}(G)$ implies that $a \in Z(H)$. Therefore $(a, b) \to (c, 1) \to (c, d)$. If (*iii*) holds, then $(1, b) \in \mathrm{Univ}(G)$ implies that $b \in Z(K)$. Therefore $(a, b) \to (1, y) \to (c, 1) \to (c, d)$.

□

The hypothesis on H and K being non-Dedekind is necessary in Theorem H.

Example 2.3.19. Let $G = C_3 \times S_3$. Then

$$\mathrm{Univ}(G) = Z(G) = Z(C_3) \times Z(S_3) = C_3 \times \{1\} = \mathrm{Univ}(C_3) \times \mathrm{Univ}(S_3).$$

Then let $g = (x, (12))$, with $x \in C_3$. Let $h = (y, z) \in G$ such that $g \to h$. Then $g^h = (x^y, (12)^z) = (x, (12)^z) = g^n = (x^n, (12)^n)$, with $n \in \mathbb{Z}$. Since (12) has order 2 it follows that $(12)^z = (12)$, thus z centralizes (12) and so $h = (y, (12))$ or $h = (y, 1)$. However, $(y, 1) \in \mathrm{Univ}(G)$ and thus it is not a vertex of $\vec{\Delta}_{\mathrm{norm}}(G)$. Therefore $h = (y, (12))$ and this proves that the set $\{(x, (12)) \mid x \in C_3\}$ is a sink and thus the graph $\vec{\Delta}_{\mathrm{norm}}(G)$ is strongly disconnected.

It is also possible to find examples where H and K are both non-abelian. Indeed, if you consider $G = Q_8 \times D_{10}$, it follows that $\vec{\Delta}_{\mathrm{norm}}(G)$ is strongly disconnected and $\mathrm{Univ}(G) = Q_8 \times \{1\} = \mathrm{Univ}(Q_8) \times \mathrm{Univ}(D_{10})$.

Moreover, the bound in Theorem H is the best possible, as the following example shows.

Example 2.3.20. Let $G = S_3 \times S_3$. Notice first that $\mathrm{Univ}(S_3) = Z(S_3) = \{1\}$, thus by (v) of Proposition 2.3.9 we have $\mathrm{Univ}(G) = \{1\} = \mathrm{Univ}(S_3) \times \mathrm{Univ}(S_3)$. Consider $x_1 = ((12), (23)) \in G$ and $x_2 = ((23), (12)) \in G$. Every x_i has order 2, thus $N_G(\langle x_i \rangle) = C_G(x_i)$. Notice first that x_1 and x_2 are not adjacent. Moreover, suppose $h = (a, b) \in G \setminus \{x_1\}$ such that $x_1 \to h$. Then $(12)^a = (12)$ and $(23)^b = (23)$, thus $a \in \{1, (12)\}$ and $b \in \{1, (23)\}$. Therefore, $h = ((12), 1)$ or $h = (1, (23))$. In both cases, we do not have $h \to x_2$ and therefore the diameter of $\vec{\Delta}_{\mathrm{norm}}(G)$ is 3.

Notice that in general $Z(G) = Z(H) \times Z(K) \neq \mathrm{Univ}(G) \neq \mathrm{Univ}(H) \times \mathrm{Univ}(K)$. In fact, let H be the group in Example 2.3.5 and $K = \langle z \rangle \rtimes Q_8$, $z^3 = 1$, $z^i = z^2$ and $z^j = z^2$. Then $G = H \times K$ has center of order 8, while $|\mathrm{Univ}(G)| = 12$ and $|\mathrm{Univ}(H) \times \mathrm{Univ}(K)| = 24$.

We can say more in the case $K = H$.

Proposition 2.3.21. *Let $G = H \times H$, where H is a non-abelian group. Then $\vec{\Delta}_{\mathrm{norm}}(G)$ is strongly connected of diameter at most* 3. *Moreover,* $\mathrm{Univ}(G) = Z(G)$.

Proof. If $\mathrm{Univ}(H) = Z(H)$ then from Theorem 2.3.12 it follows that H is not a Dedekind group. Moreover, by (v) of Proposition 2.3.9 we have $\mathrm{Univ}(G) = \mathrm{Univ}(H) \times \mathrm{Univ}(H)$. Thus, G satisfies condition (i) of Theorem H, and we are done.

Suppose now $\mathrm{Univ}(H) \neq Z(H)$. We claim that condition (ii) of Theorem H is satisfied. Indeed let $(h, 1) \in \mathrm{Univ}(G)$. Then $h \in \mathrm{Univ}(H)$. Assume by contradiction that $h \in \mathrm{Univ}(H) \setminus Z(H)$. Since h is non-central there exists $y \in H$ such that $y^h = y^n \neq y$ for some integer n. Applying Lemma 2.3.10, the claim follows. If H is not a Dedekind group then the result follows by Theorem H. Suppose now H is a Dedekind group. Let $(a, b), (c, d) \in G \setminus \mathrm{Univ}(G)$. Then, using again condition (ii) of Theorem H, we can assume $(a, 1) \notin \mathrm{Univ}(G)$. If $(1, d) \notin \mathrm{Univ}(G)$ then $(a, b) \leftrightarrow (a, 1) \leftrightarrow (1, d) \leftrightarrow (c, d)$ is a path of length at most 3. If $(1, d) \in \mathrm{Univ}(G)$ then $d \in Z(H)$ and for any $y \in H \setminus Z(H)$ we have $(1, y) \notin \mathrm{Univ}(G)$ and $(a, b) \leftrightarrow (a, 1) \leftrightarrow (1, y) \to (c, d)$ is the path that connects (a, b) to (c, d) in at most 3 steps.

Finally, we prove that $\mathrm{Univ}(G) = Z(G)$. By Proposition 2.3.3 one inclusion is obvious. Let $(a, b) \in \mathrm{Univ}(G)$. Let $h \in H$. Then $(h, h)^{(a,b)} \in \langle (h, h) \rangle$. Thus, we have $h^a = h^b$. Since ab^{-1} centralizes every $h \in H$, there exists $z \in Z(H)$ such that $b = az$. On the other hand, $(a, b) \in \mathrm{Univ}(G)$ implies that there exists an integer m such that $(a, b)^{(h,1)} = (a^m, b^m) = (a^m, (az)^m)$ and $(a, b)^{(h,1)} = (a^h, b) = (a^m, az)$. Thus, $(a^m, (az)^m) = (a^m, az)$. Therefore, $m \equiv 1 \pmod{\mathrm{mcm}(o(a), o(z))}$ and so $m \equiv 1 \pmod{o(a)}$. From this it follows that $a^h = a^m = a$. Thus, $a \in Z(H)$. Since $b = az$, it follows that $b \in Z(H)$ and, therefore, that $(a, b) \in Z(G)$. □

The bound in Proposition 2.3.21 is the best possible, as Example 2.3.20 shows.

We investigate now the strong connectivity of $\vec{\Delta}_{\text{norm}}(G)$ for a group G with trivial center. Before looking in depth into the main results we first focus on the relationship between the directed normalizing graph of a group and the directed normalizing graph of quotients.

Let G be a group and N a normal subgroup of G. Obviously if we take two elements $x, y \in G$ such that $x^{-1}y \in G \backslash N$ and $x \to y$ in $\vec{\Gamma}_{\text{norm}}(G)$ then $xN \to yN$ in $\vec{\Gamma}_{\text{norm}}(G/N)$. It could be useful to find conditions under which adjacency between two elements in $\vec{\Gamma}_{\text{norm}}(G/N)$ gives adjacency in $\vec{\Gamma}_{\text{norm}}(G)$.

Proposition 2.3.22. *Let G be a group and N a normal subgroup of G. If $xN \to yN$ in $\vec{\Gamma}_{\text{norm}}(G/N)$ and $\langle x, y\rangle \cap N = \{1\}$ then $x \to y$ in $\vec{\Gamma}_{\text{norm}}(G)$.*

Proof. Assume that yN normalizes $\langle xN\rangle$ in G/N. This means that $(xN)^{yN} = x^m N$ for some integer m. Thus $y^{-1}xy = x^m n$, and so $n = y^{-1}xyx^{-m} \in \langle x, y\rangle \cap N$. Therefore $n = 1$ and the result follows. □

We start our investigation on connectivity of $\vec{\Delta}_{\text{norm}}(G)$ by pointing out some connections with other graphs.

Theorem 2.3.23. *Let G be a finite group with trivial center. If $\Delta_{\text{nil}}(G)$ is connected then $\vec{\Delta}_{\text{norm}}(G)$ is strongly connected.*

Proof. Let $x, y \in G \setminus \{1\}$ such that x and y are adjacent in $\Delta_{\text{nil}}(G)$. Then $\langle x, y\rangle$ is nilpotent, and thus its center is non-trivial. Consider $z \in Z(\langle x, y\rangle) \setminus \{1\}$. Then z commutes both with x and y and we have $x \leftrightarrow z \leftrightarrow y$. It follows that $\vec{\Delta}_{\text{norm}}(G)$ is strongly connected. □

Corollary 2.3.24. *Let G be a finite soluble group with trivial center. If $\vec{\Delta}_{\text{norm}}(G)$ is strongly disconnected, then G is a Frobenius group or a 2-Frobenius group.*

Proof. Assume that $\vec{\Delta}_{\text{norm}}(G)$ is strongly disconnected. By Theorem 2.3.23 it follows that $\Delta_{\text{nil}}(G)$ is disconnected and Theorem E implies that G is a Frobenius or a 2-Frobenius group. □

Now we show that $\vec{\Delta}_{\text{norm}}(G)$ is always strongly disconnected when G is a Frobenius group.

Proposition 2.3.25. *Let G be a Frobenius group. Then $\vec{\Delta}_{\text{norm}}(G)$ is strongly disconnected.*

Proof. Let $G = K \rtimes H$, with K the Frobenius kernel of G and H a Frobenius complement of G. By Lemma 1.1.7 for any $h \in H \setminus \{1\}$ we have $N_G(\langle h\rangle) \subseteq H$ and thus no element of $G \setminus \{1\}$ outside H normalizes any cyclic subgroup of H.

Therefore, the induced subgraph of $\vec{\Delta}_{\text{norm}}(G)$ on H is a source, no arrow goes from a vertex of H to a vertex outside H, and thus $\vec{\Delta}_{\text{norm}}(G)$ is strongly disconnected. □

We now start our investigation on 2-Frobenius groups, with the goal of characterizing when $\vec{\Delta}_{\text{norm}}(G)$ for such a group is strongly disconnected. The next two results will be very useful in the sequel.

Lemma 2.3.26. *Let G be a finite group and K a normal nilpotent subgroup of G. If $h \in G \setminus K$ is an element of prime order p and there exists $r \in \pi(K)$ such that $p \mid r-1$, then there exist $k \in Z(K) \setminus \{1\}$ such that $k \to h$ in $\vec{\Gamma}_{\text{norm}}(G)$.*

Proof. Let $V \leq Z\big(O_r(K)\big) \leq Z(K) \leq K$ be a minimal normal subgroup of $Z(K)\langle h\rangle$. Assume by contradiction that V is not cyclic. Then $|V| = r^m$ with $m > 1$. The minimality of V implies that h does not normalize any non-trivial proper subgroup of V.

Let $v \in V \setminus \{1\}$. Then $W = \langle v, v^h, v^{h^2}, \ldots, v^{h^{p-1}}\rangle \leq V$ and it is normal in $Z(K)\langle h\rangle$. Thus, $V = W$. If V has order r^p then $w = v \cdot v^h \cdot v^{h^2} \cdots v^{h^{p-1}}$ is non-trivial and it is centralized by h, a contradiction. Therefore there exists a natural number $2 \leq \ell \leq p-1$ such that $|V| = r^\ell$.

Let $\mathcal{C}$ be the set of all non-trivial cyclic subgroups of V. As $p \mid r-1$, we have $r \equiv 1 \pmod{p}$. Obviously there are $r^\ell - 1$ non-trivial elements in V and for every non-trivial element x of V the subgroup $\langle x\rangle$ is a cyclic subgroup with $r-1$ generators. It follows that

$$|\mathcal{C}| = \frac{r^\ell - 1}{r-1} = r^{\ell-1} + r^{\ell-2} + \cdots + r + 1 \equiv \ell \pmod{p}.$$

Moreover, consider the action of $\langle h\rangle$ on $\mathcal{C}$ by conjugation. Thus, every orbit of the action has size p. So

$$|\mathcal{C}| \equiv 0 \pmod{p}.$$

It follows that $\ell \equiv 0 \pmod{p}$, a contradiction.

Thus V is cyclic. Then, h normalizes V and we have a connection between an element of $Z(K) \setminus \{1\}$ and h. □

Corollary 2.3.27. *Let G be a Frobenius group with kernel K and cyclic complement H. If there exist a prime $p \in \pi(H)$ and a prime $r \in \pi(K)$ such that $p \mid r-1$, then there exists a path in $\vec{\Delta}_{\text{norm}}(G)$ connecting a vertex in $Z(K)$ to any vertex in H^g in at most 2 steps for all $g \in G$.*

Proof. Let G be a Frobenius group. By Lemma 2.3.26 there exist elements $k \in Z(K) \setminus \{1\}$ and $h \in H \setminus \{1\}$ such that $k^h = k^n$, for some integer n. Let $g \in G$. Obviously $h^g \in H^g \setminus \{1\}$. Then $k^g \in Z(K) \setminus \{1\}$ and

$$(k^g)^{h^g} = (k^h)^g = (k^n)^g = (k^g)^n,$$

thus h^g normalizes $\langle k^g \rangle$. Since H^g is cyclic we have $h^g \to \bar{h}$ for all $\bar{h} \in H^g \setminus \{1\}$, and so $k^g \to h^g \to \bar{h}$ is a path that connects k^g to $\bar{h}$ for all $\bar{h} \in H^g \setminus \{1\}$. Thus, a vertex in $Z(K)$ is connected to every vertex in H^g in at most 2 steps and we are done. $\square$

In the following, for an element x of a group G whose order is divisible by a prime p, we denote by $x_{[p]}$ the power of x of order p.

Proposition 2.3.28. *Let G be a 2-Frobenius group as in Definition 1.1.8. Then:*

- *(i) for any $g \in G \setminus HK$ there exists $k \in K$ such that $g \leftrightarrow g_{[p]} \leftrightarrow k$;*
- *(ii) $G \setminus X$ lies in a strongly connected component of $\vec{\Delta}_{\mathrm{norm}}(G)$;*
- *(iii) for any $x \in X \setminus \{1\}$ there exists an element $y_{[p]} \in G \setminus HK$ of prime power order such that $x \to y_{[p]}$.*

Proof. To prove (i), let $g \in G \setminus HK$. Then $g \leftrightarrow g_{[p]}$. Of course $g_{[p]}$ is outside HK, otherwise $g_{[p]}K$ would lie in the kernel but also in the complement of the Frobenius group G/K. By Lemma 1.1.10 $C_K(g_{[p]}) \neq \{1\}$, therefore there exists an element $k \in K \setminus \{1\}$ such that $g_{[p]} \leftrightarrow k$.

To show (ii), let $g \in G \setminus X$ such that $g \notin K$. By (i), there exists $k \in K \setminus \{1\}$ such that $g \leftrightarrow g_{[p]} \leftrightarrow k$. Since K has non-trivial center, there exists $z \in Z(K)$ such that $k_1 \leftrightarrow z \leftrightarrow k$ for all $k_1 \in K$. Thus there exists a path that connects every element of $G \setminus X$ with every other element of $G \setminus X$, and item (ii) is proved.

As regards item (iii), let $g \in G \setminus HK$. Then $H^g K = HK$, so there exists $k \in K$ such that $H^g = H^k$ by Lemma 1.1.10. As a consequence $y = gk^{-1} \in N_G(H) \setminus HK$. In particular, there exists $y_{[p]}$ of y in $G \setminus HK$, which normalizes H, too. As H is cyclic, all its subgroups are characteristic in H, so for any $h_1 \in H$ we have $h_1 \to y_{[p]}$. By definition, any element of X belongs to some conjugate of H. Thus, taking a suitable conjugate of $y_{[p]}$, the result follows. $\square$

For a group G, we denote by $\Delta_{\mathrm{norm}}(G)$ the undirected graph induced by $\vec{\Delta}_{\mathrm{norm}}(G)$.

Proposition 2.3.29. *Let G be a 2-Frobenius group as in Definition 1.1.8. Then $\vec{\Delta}_{\mathrm{norm}}(G)$ is strongly disconnected if and only if $p \nmid r - 1$ for all $p \in \pi(H)$ and for all $r \in \pi(K)$.*

Proof. First assume that $p \nmid r-1$ for all $p \in \pi(H)$ and for all $r \in \pi(K)$. Then $\Delta_{\mathrm{norm}}(HK)$ is strongly disconnected by Proposition 10 of [31]. It immediately follows that no arrows goes from any vertex in K to a vertex in any conjugate H^g. Finally, by contradiction assume that for some $g \in G \setminus HK$ there exists $x \in X \setminus \{1\}$ with $g \to x$. Then $\langle gK \rangle$ is normalized by xK in G/K. Since gK lies in a complement of G/K, Lemma 1.1.7 forces $xK = K$, hence $x \in K$, which is a contradiction. Therefore X is a source set, and $\vec{\Delta}_{\mathrm{norm}}(G)$ is strongly disconnected.

Conversely, assume that $\vec{\Delta}_{\mathrm{norm}}(G)$ is strongly disconnected. Then, it suffices to show that $\Delta_{\mathrm{norm}}(HK)$ is strongly disconnected, as the result will follow from Proposition 10 of [31]. Thus, by way of contradiction assume that $\Delta_{\mathrm{norm}}(HK)$ is strongly connected. Then for any $x_1, x_2 \in X \setminus \{1\}$ there is a path connecting x_1 to x_2. Indeed, Lemma 1.1.7 implies the existence of $k_2 \in K$ such that $k_2 \to x_2$. Moreover, by items (i) and (iii) of Proposition 2.3.28 there exist $g_{[p]} \in G \setminus HK$ of prime order and $k_1 \in K \setminus \{1\}$ such that $x_1 \to g_{[p]} \leftrightarrow k_1$. As the center of K is nontrivial, there exists $z \in Z(K) \setminus \{1\}$ such that $x_1 \to g_{[p]} \to k_1 \to z \to k_2 \to x_2$. This, together with item (ii) of Proposition 2.3.28, implies that $\vec{\Delta}_{\mathrm{norm}}(G)$ is strongly connected, which is a contradiction. □

We can now characterize when $\vec{\Delta}_{\mathrm{norm}}(G)$ is strongly disconnected for a soluble group G with trivial center, proving the following theorem.

Theorem I. *Let G be a finite soluble group with trivial center. Then $\vec{\Delta}_{\mathrm{norm}}(G)$ is strongly disconnected if and only if one of the following holds:*

- (i) *G is a Frobenius group;*
- (ii) *G is a 2-Frobenius group with $K \lhd KH \lhd G$, KH a Frobenius group and G/K a Frobenius group with kernel KH/K such that $p \nmid r-1$ for all $p \in \pi(H)$ and for all $r \in \pi(K)$.*

Proof. If G is a Frobenius group or a 2-Frobenius group as in (ii), then $\vec{\Delta}_{\mathrm{norm}}(G)$ is strongly disconnected by Proposition 2.3.25 and Proposition 2.3.29, respectively.

Conversely, if $\vec{\Delta}_{\mathrm{norm}}$ is strongly disconnected, then G is a Frobenius or a 2-Frobenius group by Corollary 2.3.24. Suppose G is a 2-Frobenius group, with $K \lhd KH \lhd G$, KH a Frobenius group and G/K a Frobenius group with kernel KH/K. Then, by Proposition 2.3.29, $\vec{\Delta}_{\mathrm{norm}}$ strongly disconnected implies that for all $p \in \pi(H)$ and for all $r \in \pi(K)$ we have $p \nmid r-1$. This concludes the proof. □

Now we focus on the diameter of $\vec{\Delta}_{\mathrm{norm}}(G)$ when it is strongly connected.

Proposition 2.3.30. *Let G be a 2-Frobenius group as in Definition 1.1.8. If $\vec{\Delta}_{\mathrm{norm}}(G)$ is strongly connected then* $\mathrm{diam}(\vec{\Delta}_{\mathrm{norm}}(G)) \leq 6$.

Proof. From Proposition 2.3.29 there exist $p \in \pi(H)$ and $r \in \pi(K)$ such that $p \mid r-1$.

Let $x, y \in G \setminus \{1\}$. If $x, y \in K$, then $x \leftrightarrow z \leftrightarrow y$ for any non-trivial $z \in Z(K)$.

If $x \in K$ and $y \in X$, then by Corollary 2.3.27 for any vertex $z \in Z(K)$ there exists a path connecting z to y in at most two steps. Thus $x \to z \to w \to y$ for a suitable w and we are done. Moreover, by Proposition 2.3.28 (iii) and Lemma 1.1.10 there exist vertices $g_{[p]} \in G \setminus HK$, $k_2 \in K$ and $z_1 \in Z(K)$ such that $y \to g_{[p]} \to k_2 \to z_1 \to x$.

If $x \in K$ and $y \in G \setminus HK$, by Lemma 1.1.10 there exist vertices $y_{[p]} \in G \setminus HK$, $k \in K$ and $z \in Z(K)$ such that $y \leftrightarrow y_{[p]} \leftrightarrow k \leftrightarrow z \leftrightarrow x$.

If $x, y \in X$, then applying Lemma 2.3.26 and Proposition 2.3.28 (iii) there exist vertices $z \in Z(K)$, $k \in K$, $g_{[p]} \in G \setminus HK$, $y_1 \in X$ such that $x \to g_{[p]} \to k \to z \to y_1 \to y$.

Let $x \in X$ and $y \in G \setminus HK$. On the one hand, Proposition 2.3.28 and Lemma 1.1.10 implies the existence of vertices $g_{[p]}, y_{[q]} \in G \setminus HK$, $k, k_1 \in K$, $z \in Z(K)$ such that $x \to g_{[p]} \to k \to z \leftrightarrow k_1 \leftrightarrow y_{[q]} \leftrightarrow y$. On the other hand, Lemma 2.3.26 ensures the existence of a vertex $k_2 \in Z(K)$ and $x_1 \in X$ such that $k_2 \to x$. Thus $y \to y_{[q]} \to k_1 \to k_2 \to x_1 \to x$.

If $x, y \in G \setminus HK$ then by Proposition 2.3.28 there exist $k, k_1 \in K$ and $z \in Z(K)$ such that $x \leftrightarrow x_{[p]} \leftrightarrow k \leftrightarrow z \leftrightarrow k_1 \leftrightarrow y_{[q]} \leftrightarrow y$. ☐

Exploring further the relations between different graphs arising from the same group one can easily notice that if x and y are adjacent in $\Delta_{\mathrm{comm}}(G)$ then $x \leftrightarrow y$ in $\vec{\Gamma}_{\mathrm{norm}}(G)$. This allows us to prove the following.

Proposition 2.3.31. *Let G be a finite soluble A-group with trivial center and $\vec{\Delta}_{\mathrm{norm}}(G)$ strongly connected. Then* $\mathrm{diam}(\vec{\Delta}_{\mathrm{norm}}(G)) \leq 6$.

Proof. Since $\vec{\Delta}_{\mathrm{norm}}(G)$ is strongly connected, G is not Frobenius. If G is 2-Frobenius, then the result follows by Proposition 2.3.30. If G is not 2-Frobenius, then $\Gamma_{\mathrm{comm}}(G)$ is connected of diameter at most 6 by [11, Theorem 1.1]. Therefore $\vec{\Delta}_{\mathrm{norm}}(G)$ is strongly connected too, and the result follows. ☐

We now turn our attention to the proof of the following result.

Theorem J. *Let G be a finite soluble group with trivial center.*

- (i) *If $\vec{\Delta}_{\mathrm{norm}}(G)$ is strongly connected, then the diameter of $\vec{\Delta}_{\mathrm{norm}}(G)$ is at most* 8.
- (ii) *If $\vec{\Delta}_{\mathrm{norm}}(G)$ is strongly disconnected, then the number of strongly connected components is* $|\mathrm{Fit}(G)| + 1$*; moreover, one strongly connected component has diameter at most* 6 *and all other strongly connected components have diameter at most* 2.

We will prove the items in Theorem J separately. Instance (*i*) follows from Theorem 1.1 of [43]. This provides a general bound on the diameter of $\vec{\Delta}_{\text{norm}}(G)$ for a soluble group G with trivial center.

Theorem 2.3.32. *Let G be a finite soluble group with trivial center. If $\vec{\Delta}_{\text{norm}}(G)$ is strongly connected then the diameter of $\vec{\Delta}_{\text{norm}}(G)$ is at most 8.*

Proof. Since $\vec{\Delta}_{\text{norm}}(G)$ is strongly connected, G is not Frobenius by Proposition 2.3.25. If G is 2-Frobenius, then $diam(\vec{\Delta}_{\text{norm}}(G)) \leq 6$ by Proposition 2.3.30. Now assume that G is not 2-Frobenius. Then $\Gamma_{\text{comm}}(G)$ is strongly connected and $\operatorname{diam}(\Gamma_{\text{comm}}(G)) \leq 8$ by Theorem 1.1 [43]. Then $\operatorname{diam}(\vec{\Delta}_{\text{norm}}(G)) \leq \operatorname{diam}(\Gamma_{\text{comm}}(G))$ and the result follows.

□

It is still unknown whether this bound is sharp. However, we point out that in [31, Section 6] Farrell and Parker provided an example of a soluble group G for which $\Delta_{\text{norm}}(G)$ is connected of diameter 6. Since G is neither a Frobenius group nor a 2-Frobenius group, then $\vec{\Delta}_{\text{norm}}(G)$ is strongly connected and therefore its diameter is at least 6.

We investigate now the case in which $\vec{\Delta}_{\text{norm}}(G)$ is strongly disconnected, finding the number of strongly connected components and a bound on their diameters, proving item (*ii*) of Theorem J.

Theorem 2.3.33. *Let G be a finite soluble group with trivial center and suppose that $\vec{\Delta}_{\text{norm}}(G)$ is strongly disconnected. Then the number of strongly connected components is $|\mathrm{Fit}(G)| + 1$; moreover, one strongly connected component has diameter at most 6 and all other strongly connected components have diameters at most 2.*

Proof. By Theorem I, G is a Frobenius or a 2-Frobenius group. Suppose first that $G = KH$ is a Frobenius group. Then no arrows goes from a vertex in K to a vertex in any H^g and by Lemma 1.1.7 no arrows goes form a vertex in any H^g to a vertex in K. Moreover, K is nilpotent and therefore its center is non-trivial, thus the subgraph induced by K is a strongly connected component of G. By Lemma 1.1.7 and the fact that the center of a Frobenius complement is non-trivial it follows that any of the conjugates of H gives rise to a strongly connected component. Finally, since $K = \mathrm{Fit}(G)$, there are $|\mathrm{Fit}(G)| + 1$ strongly connected components of diameter at most 2. Now suppose that G is a 2-Frobenius group as in Definition 1.1.8. By Proposition 2.3.28 the subgraph induced by $G \setminus X$ lies in a strongly connected component and there exists a path from any vertex in X to some $g \in G \setminus X$. Therefore, there is no arrow from any element of $G \setminus X$ to $X \setminus \{1\}$, as $\vec{\Delta}_{\text{norm}}(G)$ is strongly disconnected. Thus the subgraph induced by $G \setminus X$ coincides with a strongly connected component. One can easily see that any conjugate of H is a strongly connected component of diameter 1. As $K = \mathrm{Fit}(G)$, there are exactly $|\mathrm{Fit}(G)| + 1$ strongly connected components. Using Proposition 2.3.28 (i) there is a path from any element of $G \setminus HK$ to any vertex in $Z(K)$ of length at most 3. Thus, we can connect any two elements of $G \setminus HK$ in at most 6 steps. $\square$

We point out that Theorem 2.3.12 also characterizes when $\vec{\Gamma}_{\mathrm{norm}}(G)$ is strongly connected of diameter 1. It is not difficult to describe also when $\vec{\Delta}_{\mathrm{norm}}(G)$ is strongly disconnected with all its strongly connected components of diameter 1, in the case in which G is a soluble group with trivial center.

Proposition 2.3.34. *Let G be a finite soluble group with trivial center. If $\vec{\Delta}_{\mathrm{norm}}(G)$ is strongly disconnected with strongly connected components of diameter 1, then G is a Frobenius group with Dedekind Frobenius kernel and Dedekind Frobenius complement.*

Proof. By Theorem I it follows that G is either a Frobenius or a 2-Frobenius group. Suppose, by contradiction, that G is a 2-Frobenius group as in Definition 1.1.8. Then by Proposition 2.3.28, $G \setminus X$ lies in a strongly connected component of $\vec{\Delta}_{\mathrm{norm}}(G)$. Therefore, $a \leftrightarrow b$ for any $a, b \in G \setminus X$, which implies that aK normalizes $\langle bK \rangle$. However this is a contradiction by Lemma 1.1.7. Therefore G is a Frobenius group, whose complement and kernel are Dedekind by Theorem 2.3.12. $\square$

The bound in Proposition 2.3.30 can be improved under certain conditions.

Proposition 2.3.35. *Let G be a 2-Frobenius group as in Definition 1.1.8 and let $\vec{\Delta}_{\mathrm{norm}}(G)$ be strongly connected. If $\pi(K) = \pi(L)$ or for any prime $p \in \pi(L) \setminus \pi(K)$ there exists a prime $r \in \pi(K)$ such that $p \mid r - 1$ then* $\mathrm{diam}(\vec{\Delta}_{\mathrm{norm}}(G)) \leq 5$.

Proof. Following the proof of Proposition 2.3.30, we only need to show that x, y are at directed distance at most 5 when $x \in X$ and $y \in G \setminus HK$ and when $x, y \in G \setminus HK$. In both cases it suffices to prove that for any $g \in G \setminus HK$ there exists $z \in Z(K)$ such that z reaches g in at most 2 steps.

Let $g \in G \setminus HK$. If there exists a prime p such that $p \mid o(g)$ and $p \mid |K|$, then for any Sylow p-subgroup P of G containing $g_{[p]}$ we have $z \leftrightarrow g_{[p]} \leftrightarrow g$, where z is a non-trivial element of $Z(P) \cap Z(O_p(G)) \leq Z(K)$.

Assume now that $(o(g), |K|) = 1$. By hypothesis, if p is a prime dividing $o(g)$ then there exists a prime r such that $p \mid r - 1$. Consider $K\langle g \rangle$. By Lemma 2.3.26, there exists an element $z \in Z(K) \setminus \{1\}$ such that z reaches g in at most 2 steps. This concludes the proof. $\square$

In the following, we show that the upper bound on the diameter of $\vec{\Delta}_{\text{norm}}(G)$ can be improved for some classes of groups. Recall that a group G is a cyclic-by-abelian if G has a cyclic normal subgroup N such that the quotient G/N is abelian.

Proposition 2.3.36. *Let G be a finite cyclic-by-abelian group with trivial center and $\vec{\Delta}_{\text{norm}}(G)$ strongly connected. Then $\operatorname{diam}(\vec{\Delta}_{\text{norm}}(G)) \leq 4$.*

Proof. Let $N = \langle c \rangle$ be the cyclic normal subgroup of G such that G/N is abelian and let $g \in G$. Since N is cyclic, every subgroup of N is normal in G and so $N \subseteq \operatorname{Univ}^-(G)$. Thus, it suffices to show that g can reach an element of $N \setminus \{1\}$ in at most 3 steps. If $g \in N$ we are done, thus suppose that $g \notin N$.

If $(|N|, o(g)) \neq 1$, take a prime p dividing both $|N|$ and $o(g)$. Thus, there exists a Sylow p-subgroup P of G containing g_p. Now, for a $z \in Z(P)$ and a suitable positive integer m we have $g \to g_p \to z \to c^m$, with $1 \neq c^m \in N \cap P$.

Now assume $(|N|, o(g)) = 1$ and let p be a prime dividing $o(g)$. Then, there exists a Sylow p-subgroup P of G containing g_p. Since PN/N is normal in G/N, PN is normal in G. Thus, by Frattini's argument $G = NN_G(P)$. If $N \cap N_G(P) = \{1\}$ then $N_G(P)$ is abelian. Moreover, since $\vec{\Delta}_{\text{norm}}(G)$ is strongly connected, by Proposition 2.3.25 it follows that G is not a Frobenius group, and so there exists $h \in N$ such that $C_{N_G(P)}(h) \neq \{1\}$. Let $t \in C_{N_G(P)}(h) \setminus \{1\}$. Thus, we have $g \to g_p \to t \to h$. Now assume $N \cap N_G(P) \neq \{1\}$ and $1 \neq u \in N \cap N_G(P)$. Since $N \cap N_G(P)$ is normal in $N_G(P)$, we have $N \cap N_G(P) \subseteq \operatorname{Fit}(N_G(P))$. Moreover, $P \subseteq \operatorname{Fit}(N_G(P))$, so $\langle g_p, u \rangle$ is nilpotent. Thus we have $g \to g_p \to z \to u$, where z is any non-trivial element in $Z(\langle g_p, u \rangle)$. ☐

Corollary 2.3.37. *Let G be a finite soluble group with trivial center and $\vec{\Delta}_{\text{norm}}(G)$ strongly connected. If $\operatorname{Fit}(G)$ is cyclic then $\operatorname{diam}(\vec{\Delta}_{\text{norm}}(G)) \leq 4$.*

Proof. By Proposition 2.3.36, it suffices to prove that G/F is abelian. This is true due to Proposition 1.1.2.

☐

The same bound holds when the Fitting subgroup of G has prime index.

Proposition 2.3.38. *Let G be a finite soluble group with trivial center and $\vec{\Delta}_{\text{norm}}(G)$ strongly connected. If $|G : \operatorname{Fit}(G)|$ is a prime number then $\operatorname{diam}(\vec{\Delta}_{\text{norm}}(G)) \leq 4$.*

Proof. Let $F = \operatorname{Fit}(G)$ and $|G : F| = p$. We will prove that there exist paths connecting every element of $G \setminus \{1\}$ to any element of $Z(F) \setminus \{1\}$ and vice versa in at most 2 steps.

Let $x \in G \setminus \{1\}$. If $x \in F$ we are done. Assume $x \in G \setminus F$. If $x^p \neq 1$ then $x^p \in F \setminus \{1\}$ and so we have the path $x \leftrightarrow x^p \leftrightarrow z$, for all $z \in Z(F)$.

If $x^p = 1$ then x lies in Sylow p-subgroup of G, say P. If p divides $|F|$ then we have $Z(O_p(G)) \leq P$. Thus there exists $z_1 \in Z(O_p(G)) \leq Z(F)$ such that $x \leftrightarrow z_1$. If p does not divide $|F|$ then $G = F \rtimes \langle x \rangle$. Since $\vec{\Delta}_{\mathrm{norm}}(G)$ is strongly connected, due to Proposition 2.3.25 it follows that G is not a Frobenius group and thus there exists a non-trivial element $y \in \langle x \rangle$ that centralizes a non-trivial element of F, say f. Hence x itself commutes with f and so we have the path $x \leftrightarrow f \leftrightarrow z$ for all $z \in Z(F) \setminus \{1\}$. This concludes the proof. □

The bound in Proposition 2.3.38 is sharp. Indeed the group G = SmallGroup(384,591) has Fitting subgroup of order 128 and index 3, and $\vec{\Delta}_{\mathrm{norm}}(G)$ is strongly connected of diameter 4.

This concludes the discussion on the normalizing graph and thus we are ready to consider the last graph of this work.

2.4 The verbal graph

In this section, we discuss the verbal graph associated with a group. The main purpose is to introduce a general line of research in the context of graphs associated with groups that can be developed and investigated further. Thus, here we present some considerations and basic results concerning the verbal graph. Let G be a group and $w(x, y)$ a word in two variables. The *verbal graph* of G related to w, or *w-graph* of G, denoted by $\vec{\Gamma}_w(G)$, is the directed simple graph whose set of vertices is the set of elements of G and an arrow is drawn from an element $g \in G$ to an element $h \in G$ if and only if $w(g, h) = 1$. This graph has been defined first by Detomi, Lucchini and Nemmi in [28], but it has only been studied for particular choices of $w(x, y)$, like the commutator word $w(x, y) = [x, y]$ which gives rise to the commuting graph and the n-th Engel word $w(x, y) = [x,_n y]$ which gives rise to the n-th Engel graph. Obviously, the verbal graph is strongly dependent on the choice of the word w.

2.4.1 Universal vertices

Our primary objective is to obtain a deeper understanding of the set of bidirectional universal vertices of $\vec{\Gamma}_w(G)$ for a group G. We start our investigation by looking at the neighborhood of an element $g \in G$. We define $N_w^+(g) = \{h \in G \mid w(g, h) = 1\}$ and $N_w^-(g) = \{h \in G \mid w(h, g) = 1\}$ which are, respectively, the forward neighborhood and the backward neighborhood of g. Moreover, if $w(x, y)$ is a word in which both variables x and y appear, let denote by s_x and s_y the algebraic sum of the exponents in w of x and y, respectively.

Lemma 2.4.1. *Let G be a group, $w(x, y)$ a commutator word and $g, h \in G$. If g, h commute then $w(g, h) = 1 = w(h, g)$.*

Proof. Suppose g, h commute. Then $w(g, h) = g^{s_x} h^{s_y} = w(h, g)$. Since w is a commutator word we have $s_x = 0 = s_y$ and thus $w(g, h) = 1 = w(h, g)$. $\square$

Proposition 2.4.2. *Let G be a group, let w be a commutator word and let $g \in G$. Then $C_G(g)w^*(G) \subseteq N_w^+(g) \cap N_w^-(g)$.*

Proof. Let $h \in C_G(g)w^*(G)$. There exist $c \in C_G(g)$ and $b \in w^*(G)$ such that $h = cb$. Then $w(g, h) = w(g, cb) = w(g, c)$. Since g and c commute, by Lemma 2.4.1 we have $w(g, c) = 1$ and so $h \in N^+(g)$. The same argument applies for $w(h, g)$. Thus, we have the result. $\square$

The inclusion in Proposition 2.4.2 can be strict and in general, both $N_w^+(g)$ and $N_w^-(g)$ are not subgroups, as the following example shows.

Example 2.4.3. Consider $G = S_3$, the symmetric group of degree 3, and the word $w(x, y) = [x^3, y^3]$. It is easy to see that $w^*(G) = \{1\}$. Consider $g = (123)$. Then $C_G(g)w^*(G) = C_G(g) = \langle g \rangle$. However, $w(g, h) = w(1, h) = 1 = w(h, 1) = w(h, g)$ for any $h \in G$. Therefore $N^+(g) \cap N^-(g) = G$. Moreover, consider the words $v_1(x, y) = [x^2, y]$ and $v_2 = [x, y^2]$ and $a = (12) \in G$. Then $N_{v_1}^-(a) = \{1, (12), (13), (23)\} = N_{v_2}^+(a)$ which is not a subgroup of G.

Related to both sets of neighborhoods one can consider the sets of forward, backward and bidirectional universal vertices for a group G. Denote by $\mathrm{Univ}_w^-(G) = \{x \in G \mid x \to y, \text{ for every } y \in G\}$ the set of universal backward vertices, $\mathrm{Univ}_w^+(G) = \{x \in G \mid y \to x, \text{ for every } y \in G\}$ the set of universal forward vertices and

$$\mathrm{Univ}_w(G) = \mathrm{Univ}_w^+ \cap \mathrm{Univ}_w^- = \{g \in G \mid w(g, h) = 1 = w(h, g) \text{ for all } h \in G\}$$

the set of bidirectional universal vertices.

It is easy to characterize when $\vec{\Gamma}_w(G)$ is complete in terms of the marginal subgroup of the group G.

Theorem 2.4.4. *Let G be a group and w a word. Then $\vec{\Gamma}_w(G)$ is complete if and only if $w^*(G) = G$.*

Proof. The graph $\vec{\Gamma}_w(G)$ is complete if and only if for any $g, h \in G$ we have $w(g, h) = 1$. This is equivalent to the verbal subgroup being trivial, i.e. $w(G) = \{1\}$. However, this happens if and only if $w^*(G) = G$. $\square$

Recall that if W is a set of words the class of all groups G such that $W^*(G) = G$ is called the variety determined by W and it is denoted with $\mathcal{B}(W)$. As usual, if $W = \{w\}$ is a singleton we denote $\mathcal{B}(W)$ with $\mathcal{B}(w)$. As a corollary of Theorem 2.4.4 we obtain the following.

Corollary 2.4.5. *Let $w(x, y)$ and $v(x, y)$ be group words. Then $\mathcal{B}(w)$ and $\mathcal{B}(v)$ are equal if and only if $\vec{\Gamma}_w(G)$ is complete whenever $\vec{\Gamma}_v(G)$ is complete and vice versa for any group G.*

For a group G, we denote by $\Gamma_w(G)$ the undirected graph induced by $\vec{\Gamma}_w(G)$. It is quite natural to ask if it is true that the graphs $\vec{\Gamma}_w(G)$ and $\vec{\Gamma}_v(G)$ (or $\Gamma_w(G)$ and $\Gamma_v(G)$) are isomorphic, provided that the words $w(x, y)$ and $v(x, y)$ determine the same variety. This is not the case as shown in Example 2.4.7. We start proving the following.

Proposition 2.4.6. *Let $w(x, y)$ be a group word. If $(s_x, s_y) = 1$ then $\mathcal{B}(w)$ contains only the trivial subgroup.*

Proof. Let G be a group. Since s_x and s_y are coprime, there exist $\alpha, \beta \in \mathbb{Z}$ such that $1 = \alpha s_x + \beta s_y$. Let $g \in G$. Then $w(g^\alpha, g^\beta) = g$. Therefore $w(G) = G$ and so $w^*(G) = \{1\}$. It follows that G belongs to $\mathcal{B}(w)$ if and only if $G = \{1\}$. $\square$

Example 2.4.7. Consider $w(x, y) = xy$ and $w(x, y) = xy^2$. By Proposition 2.4.6 we have $\mathcal{B}(w) = \mathcal{B}(v)$. However, if $G = C_2$ then $\vec{\Gamma}_w(G)$ is not isomorphic to $\vec{\Gamma}_v(G)$ and nor $\Gamma_w(G)$ is isomorphic $\Gamma_v(G)$.

Let G be a group. In case $\vec{\Gamma}_w(G)$ is complete, as Theorem 2.4.4 shows, we have $w^*(G) = G = \mathrm{Univ}_w(G)$. Moreover, if you consider the word $w(x, y) = [x, y]$ then the marginal subgroup of G is $w^*(G) = Z(G)$ which is precisely the set of universal vertices of the graph. Thus, it is interesting to compare the subgroup $w^*(G)$ and the set $\mathrm{Univ}_w(G)$ of all universal vertices of the graph.

Theorem 2.4.8. *Let G be a group and w a word. Then* $\mathrm{Univ}_w(G)$ *is a normal set. Moreover, it is the union of cosets of $w^*(G)$.*

Proof. Let $g \in \mathrm{Univ}_w(G)$ and $h, k \in G$. Then $w(g^k, h) = w(g, h^{k^{-1}})^k = 1^k = 1$ and $w(h, g^k) = w(h^{k^{-1}}, g)^k = 1^k = 1$. Moreover, let $h \in \mathrm{Univ}_w(G)$ and let $a \in w^*(G)$. Then we have $w(ha, g) = w(h, g) = 1 = w(g, h) = w(g, ha)$. Thus, we obtain $ha \in \mathrm{Univ}_w(G)$ for any $a \in w^*(G)$ and so we have $hw^*(G) \subseteq \mathrm{Univ}_w(G)$. Thus

$$\mathrm{Univ}_w(G) = \bigcup_{h \in \mathrm{Univ}_w(G)} hw^*(G).$$

$\square$

Corollary 2.4.9. *Let G be a group and let w be a word. Then $w^*(G) \subseteq \mathrm{Univ}_w(G)$ if and only if $1 \in \mathrm{Univ}_w(G)$.*

Proof. If $w^*(G) \subseteq \mathrm{Univ}_w(G)$ then obviously $1 \in \mathrm{Univ}_w(G)$. Conversely, if $1 \in \mathrm{Univ}_w(G)$ by Theorem 2.4.8 we have $w^*(G)1 = w^*(G) \subseteq \mathrm{Univ}_w(G)$. $\square$

Corollary 2.4.10. *Let G be a group and let w be a word. If w is a commutator word then $w^*(G) \subseteq \mathrm{Univ}_w(G)$.*

Proof. By Corollary 2.4.9 it is sufficient to notice that if w is a commutator word then $1 \in \mathrm{Univ}_w(G)$. It is easy to see that this is the case. $\square$

In general, the inclusion in Corollary 2.4.10 is strict. In fact, if you consider S_3, the symmetric group of degree 3, and the word $w(x, y) = [x^3, y^3]$ we have $w^*(G) = \{1\}$, while $\mathrm{Univ}_w(G)$ also contains the 3-cycles.

If w is a non-commutator word, then $\mathrm{Univ}_w(G)$ can be the empty set. It is sufficient to take C_2, the cyclic group of order 2 and the word $w(x, y) = xy$.

Proposition 2.4.11. *Let w be a word. If $w^*(G) = \mathrm{Univ}_w(G)$ for any group G then w is a commutator word.*

Proof. We prove the contrapositive statement. Let w be a non-commutator word. It follows that at least one between s_x and s_y is not 0. Without loss of generality suppose $s_x \neq 0$. Then, we consider a non-trivial group G of order coprime to s_x. Thus, it follows that there exists an element $g \in G$ such that $g^{s_x} \neq 1$. Therefore, $w(g, 1) = g^{s_x} \neq 1$ and so $1 \notin \mathrm{Univ}_w(G)$. Since $1 \in w^*(G)$, we have the result. $\square$

Proposition 2.4.12. *Let G be a group and $w(x, y)$ a word. Then $1 \in \mathrm{Univ}_w(G)$ if and only if $exp(G)$ divides (s_x, s_y).*

Proof. If 1 is universal then $w(1, g) = w(g, 1) = 1$ for every $g \in G$. Therefore $g^{s_x} = g^{s_y} = 1$ for every $g \in G$ and so $o(g)$ divides (s_x, s_y) for every $g \in G$, thus $exp(G)$ divides (s_x, s_y).

Conversely, if $exp(G)$ divides (s_x, s_y) then $g^{s_x} = g^{s_y} = 1$. Thus $w(1, g) = w(g, 1) = 1$ and 1 is a bidirectional universal vertex. $\square$

Proposition 2.4.13. *Let $w(x, y) = x^n y^m x^t$ be a word, with m, n, t integers. Then $\mathrm{Univ}_w(G) = G$ if and only if G is periodic with finite exponent and $exp(G)$ divides $(n + t, m)$.*

Proof. Suppose $\mathrm{Univ}_w(G) = G$. Then for every $g \in G$ we have $w(1, g) = 1$ and $w(g, 1) = 1$, thus $g^{n+t} = 1$ and $g^m = 1$. Therefore $o(g)$ divides $(n + t, m)$. Conversely, suppose that $exp(G)$ divides $(n + t, m)$. Then for every $g, h \in G$ we have $w(g, h) = g^n h^m g^t = g^{n+t} = 1$ and similarly $w(h, g) = 1$. Thus $g \in \mathrm{Univ}_w(G)$ for every $g \in G$. $\square$

Proposition 2.4.14. *Let* $w(x, y) = x^n y^m x^t$ *be a word, with* m, n, t *integers. Then* $\mathrm{Univ}_w(G) = \varnothing$ *or* $\mathrm{Univ}_w(G) = G$.

Proof. Notice that w cannot be a commutator word. Suppose that $\mathrm{Univ}_w(G) \neq G$. Then by Proposition 2.4.13 $exp(G)$ does not divide $(n+t, m)$, thus there exists $h \in G$ such that $o(h)$ does not divide $(n+t, m)$. Argue by contradiction and suppose that there exists an element $g \in \mathrm{Univ}_w(G)$. Then $w(1, g) = 1$ and $w(g, 1) = 1$, thus $g^{n+t} = 1$ and $g^m = 1$. Therefore $o(g)$ divides $(n + t, m)$. Furthermore, $w(g, h) = 1$ and $w(h, g) = 1$. Thus $g^n h^m g^t = 1$ and $h^n g^m h^t = 1$, so we have $h^m = g^{-n-t} = 1$ and $h^{n+t} = g^{-n} = 1$. A contradiction. $\square$

For any word w, Proposition 2.4.14 allows us to describe the set of all bidirectional universal vertices for an abelian group.

Proposition 2.4.15. *If* G *is an abelian group then* $\mathrm{Univ}_w(G) = G$ *or* $\mathrm{Univ}_w(G) = \varnothing$*. In particular if* $\mathrm{Univ}_w(G) = \varnothing$ *then* w *is a non-commutator word.*

Proof. If w is a commutator word then by Lemma 2.4.1 $\mathrm{Univ}_w(G) = G$. If w is a non-commutator word then notice that for an abelian group the word $w(x, y) = x^{h_1} y^{k_1} \cdots x^{h_s} y^{k_s}$ is equivalent to the word $v(x, y) = x^{s_x} y^{s_y}$. Therefore, by Proposition2.4.14 the result follows.

In particular, if w is a commutator word then $\mathrm{Univ}_w(G) \neq \varnothing$ since $1 \in \mathrm{Univ}_w(G)$. $\square$

In general, for a group G, it is not true that if w is a non-commutator word then $\mathrm{Univ}_w(G)$ is G or the empty set. Indeed, if you consider S_3, the symmetric group of degree 3, and the word $w(x, y) = (x^2 y^2 xy)^2$ it is easy to see that $\mathrm{Univ}_w(G)$ is the subgroup generated by a 3-cycle.

It is possible to describe the set $\mathrm{Univ}_w(G)$ for a decomposable group G.

Proposition 2.4.16. *Let* $G = A \times B$ *be a group and* w *a word. Then* $\mathrm{Univ}_w(G) = \mathrm{Univ}_w(A) \times \mathrm{Univ}_w(B)$.

Proof. Let $u \in \mathrm{Univ}_w(A)$, $v \in \mathrm{Univ}_w(b)$ and $g \in G$. Let $a \in A$ and $b \in B$ such that $g = ab$. Then $w(uv, ab) = w(u, a)w(v, b) = 1$ and $w(ab, uv) = w(a, u)w(b, v) = 1$. Thus uv is universal in G. Let $h \in \mathrm{Univ}_w(G)$. Then there exist $c \in A$ and $d \in B$ such that $h = cd$. Then $w(cd, ab) = 1 = w(c, a)w(d, b)$. Notice that $w(c, a) \in A$ and $w(d, b) \in B$. Thus $w(c, a) = 1 = w(d, b)$. $\square$

Let G be a group and $g, h \in G$. Obviously, if $g \to h$ in $\vec{\Gamma}_w(G)$ then we also have $gw^*(G) \to hw^*(G)$ in $\vec{\Gamma}_w(G/w^*(G))$. It can be useful to find conditions under which the converse also holds.

Proposition 2.4.17. *Let G be a group, $g, h \in G$ and w a group word such that $w^*(G) \cap w(G) = \{1\}$. If $gw^*(G) \to hw^*(G)$ in $\vec{\Gamma}_w(G/w^*(G))$ then $g \to h$ in $\vec{\Gamma}_w(G)$.*

Proof. Suppose that $gw^*(G) \to hw^*(G)$ in $\vec{\Gamma}_w(G/w^*(G))$, then $w(g,h) \in w^*(G)$. Moreover, $w(g,h) \in w(G)$ and thus $w(g,h) \in w^*(G) \cap w(G)$. Therefore $w(g,h) = 1$ and so $g \to h$ in $\vec{\Gamma}_w(G)$. $\square$

Recall that a word is said to be concise if $w(G)$ is finite whenever the set G_w of w-values of G is finite.

Proposition 2.4.18. *Let G be an infinite group and w a concise word. If G_w is finite and $C_G(w(G)) \subseteq w^*(G)$ then* $\mathrm{Univ}_w(G)$ *is infinite or* $\mathrm{Univ}_w(G) = \varnothing$.

Proof. $G/C_G(w(G))$ embeds into $\mathrm{Aut}(w(G))$ which is finite. Thus, $C_G(w(G))$ is infinite and so $w^*(G)$ is. If w is a commutator word then by Proposition 2.4.10 $w^*(G) \subseteq \mathrm{Univ}_w(G)$ and we have the result. If w is a non-commutator word and $\mathrm{Univ}_w(G)$ is not empty then there exists $x \in \mathrm{Univ}_w(G)$. By Theorem 2.4.8 the result follows. $\square$

As mentioned above, the verbal graph generalizes the commuting graph, since this can be seen as the verbal graph obtained for the word $w(x,y) = [x,y]$. For a group G, we denote the directed subgraph of $\vec{\Gamma}_w(G)$ induced by the set $G \setminus \mathrm{Univ}_w(G)$, by $\vec{\Delta}_w(G)$. It is easy to prove the following.

Proposition 2.4.19. *Let G be a group and w a commutator word. Then $\Gamma_{\mathrm{comm}}(G)$ is a subgraph of $\Gamma_w(G)$. Moreover, if $Z(G) =$* $\mathrm{Univ}_w(G)$ *and $\Delta_{\mathrm{comm}}(G)$ is connected, then $\vec{\Delta}_w(G)$ is strongly connected and* $\mathrm{diam}(\vec{\Delta}_w(G)) \leq \mathrm{diam}(\Delta_{\mathrm{comm}}(G))$.

Proof. If g, h commute then by Lemma 2.4.1 we have $w(g,h) = w(h,g) = 1$.

Moreover, suppose that $Z(G) = \mathrm{Univ}_w(G)$ and $\Delta_{\mathrm{comm}}(G)$ is connected. Then for any $g, h \in G \setminus Z(G)$ there exists a path that connects g to h, and so there exist $g_1, \ldots, g_m \in G \setminus Z(G)$ such that $[g, g_1] = 1$, $[g_i, g_{i+1}] = 1$, $[g_m, h] = 1$ for all $i = 1, \ldots, m-1$. Thus, by Lemma 2.4.1, we have $g \leftrightarrow g_1 \leftrightarrow \cdots \leftrightarrow g_m \leftrightarrow h$ in $\vec{\Delta}_w(G)$. This concludes the proof. $\square$

Thanks to Proposition 2.4.19, for a group G with $Z(G) = \mathrm{Univ}_w(G)$ we can use the existing results about the commuting graph to establish strong connectivity and bound the diameter of the graph $\vec{\Delta}_w(G)$.

For decomposable groups it is possible to prove the following.

Theorem 2.4.20. *Let w be a commutator word and $G = H \times K$ a group such that $w^*(H) \neq H$ and $w^*(K) \neq K$. Then the graph $\vec{\Delta}_w(G)$ is strongly connected of diameter at most* 3.

Proof. From $w^*(H) \neq H$ and $w^*(K) \neq K$ it follows that $w^*(G) \neq G$. Let $(a,b),(c,d) \in G \smallsetminus \mathrm{Univ}_w(G)$. Since $(a,b) \notin \mathrm{Univ}_w(G)$, by Proposition 2.4.18 we have $(a,1) \notin \mathrm{Univ}_w(G)$ or $(1,b) \notin \mathrm{Univ}_w(G)$. Without loss of generality suppose $(a,1) \notin \mathrm{Univ}_w(G)$ and let $(c,d) \in G \smallsetminus \mathrm{Univ}_w(G)$. By Proposition 2.4.18 we have $(1,d) \notin \mathrm{Univ}_w(G)$ or $(1,c) \notin \mathrm{Univ}_w(G)$. Suppose $(1,d) \notin \mathrm{Univ}_w(G)$. Then, by Lemma 2.4.1 we can consider the path $(a,b) \to (a,1) \to (1,d) \to (c,d)$.

Suppose now $(1,d) \in \mathrm{Univ}_w(G)$ and $(c,1) \notin \mathrm{Univ}_w(G)$. Since $w^*(K) \neq K$ by Theorem 2.4.4 there exists $x \in K \smallsetminus \mathrm{Univ}_w(K)$. Since $(1,d) \in \mathrm{Univ}_w(G)$ then $w((1,x),(c,d)) = w(1,c)w(x,d) = 1$. Thus, by Lemma 2.4.1, we can consider the path $(a,b) \to (a,1) \to (1,x) \to (c,d)$.

□

This bound is the best possible, as it is realized for the group $G = S_3 \times S_3$ with the word $w(x,y) = [x,y]$.

The hypothesis $w^*(H) \neq H$ and $w^*(K) \neq K$ in Theorem 2.4.20 are necessary. In fact, for the word $w(x,y) = [x,y]$ and the group $G = C_3 \times S_3$ we have $\vec{\Delta}_w(G)$ strongly disconnected.

For non-commutator words this does not hold in general. Indeed, for any non-trivial group G there exists the non-commutator word $w(x,y) = xy$ that yields $\vec{\Delta}_w(G)$ strongly and weakly disconnected. In general, it is possible to prove the following.

Proposition 2.4.21. *Let G be a non-trivial group and $w(x,y)$ a non commutator word. If s_x or s_y are coprime to $exp(G)$ then $\vec{\Delta}_w(G)$ is strongly disconnected.*

Proof. Suppose s_x is coprime to $exp(G)$. Let $g \in G \setminus \{1\}$. We have $w(g,1) = g^{s_x}$, and so $w(g,1) \neq 1$, since $g^{s_x} = 1$ would yield to $o(g)|s_x$ and so $o(g)|(s_x, exp(G))$, which is a contradiction. Therefore, 1 is a source vertex and thus there exists no path connecting an element of G with 1. In the case in which s_y is coprime to $exp(G)$ we can prove in the same way that 1 is a sink vertex. The result follows. □

2.4.2 Clique number

We now focus on the study of the clique number of the verbal graph, classifying the groups that have a small clique number. In the following, for a group G, we consider the undirected graph induced by the verbal graph, namely $\Gamma_w(G)$. We first address when the graph $\Gamma_w(G)$ is bipartite for a group G.

Proposition 2.4.22. *Let G be a group and w a word. If $\Gamma_w(G)$ is bipartite then $|G| = 2$ or w is a non-commutator word.*

Proof. Assume that w is a commutator word. If G has at least an element x of order greater than or equal to 3 then the subgroup $\langle x \rangle$ gives rise to a clique in

$\Gamma_w(G)$ and thus the graph is not bipartite. Therefore, every non-identity element in G has to have order 2. Thus, G is abelian and $\Gamma_w(G)$ is bipartite, so G must have order 2. $\square$

It is easy to find groups with bipartite graph for a non-commutator word. For example consider the group S_3 and the word $w(x, y) = x^5y^6$. Then the identity element is adjacent to every other vertex in $\Gamma_w(G)$ and two non-trivial elements are not adjacent. Therefore, the graph is bipartite. The idea behind this example allows us to construct an infinite amount of examples, that shows how the properties of the graph is completely determined by the word w. Consider a finite group G. Consider now the word $w(x, y) = x^{exp(G)-1}y^{exp(G)}$. Then the identity is adjacent to every other vertex and two non-trivial elements are not adjacent. Therefore G has a bipartite graph.

Since the identity element, for commutator words, is always a universal vertex, one can study the same problem for the subgraph induced by $\Gamma_w(G)$ on the set $G \setminus \{1\}$, namely $\Gamma^*_w(G)$. This is the graph obtained from $\Gamma_w(G)$ removing the identity element.

Proposition 2.4.23. *Let G be a group and w a word. If $\Gamma^*_w(G)$ is bipartite then G is isomorphic to C_2, C_3 or S_3 or w is a non-commutator word.*

Proof. Assume that w is a commutator word. If G has at least an element x of order greater than or equal to 4 then the subgroup $\langle x\rangle \setminus \{1\}$ gives rise to a clique in $\Gamma^*_w(G)$ and thus the graph is not bipartite. Therefore every non-identity element in G has to have order 2 or 3. If $|G| = 2, 3$ the statement is trivial. Assume $|G| \geq 4$ and consider Q a Sylow 3-subgroup of G. If $|Q| \geq 9$ then it would contain an abelian subgroup of order 9. Thus, $|Q| = 3$. Consider P a Sylow 2-subgroup of G. It is abelian, therefore $|P| = 2$. Thus, it follows that G is isomorphic to S_3. $\square$

Proposition 2.4.24. *Let G be a group and w a commutator word. Then the clique number of $\Gamma_w(G)$ is 2 if and only if $|G| = 2$.*

Proof. If G has at least an element x of order greater than or equal to 3 then the subgroup $\langle x\rangle$ gives rise to a clique in $\Gamma_w(G)$. Therefore, every non-identity element in G has to have order 2. Thus G is abelian and $\Gamma_w(G)$ has clique number 2, therefore $|G| = 2$. The converse is trivial. $\square$

Notice that, in general, if w is a non-commutator word then requesting the graph to have a small clique number does not imply anything on the group. Indeed, consider the word $w(x, y) = xy$. Then for every group G the graph $\Gamma_w(G)$ has every element of order 2 and the identity element that are isolated vertices and all other elements are connected only to their inverse. Thus, this graph has clique number 2 for any non-trivial group G.

Proposition 2.4.25. *Let G be a p-group of order greater than p, with p a prime, and let w be a commutator word. Then the clique number of $\Gamma_w(G)$ is greater than or equal to p^2.*

Proof. It follows from the fact that G has an abelian subgroup of order p^2. $\square$

Proposition 2.4.26. *Let G be a 2-group and w a commutator word. If $|G| \geq 16$ the clique number of $\Gamma_w(G)$ is greater than or equal to 5.*

Proof. Assume by contradiction that the clique number of $\Gamma_w(G)$ is less than 5. Thus, G has elements of order 2 or 4. Notice that there exists an element x of order 4, otherwise G would be abelian. If x lies in the center of G then there exists a clique composed by 5 elements. Therefore the center of G has order 2 and for every element y of G having order 4 we have $y^2 \in Z(G)$. Thus $G/Z(G)$ is abelian. Consider $P = \langle x \rangle$. Obviously $C_G(P) = P$. Then $N_G(P)/C_G(P)$ is isomorphic to a subgroup of $Aut(P)$ which is C_2. Therefore $|N_G(P)| = 8$. Also $C_{G/Z(G)}(N_G(P)/Z(G)) \leq C_{G/Z(G)}(P/Z(G)) \leq N_G(P)/Z(G)$. Thus $G/Z(G) = N_G(P)/Z(G)$, therefore $|G| = 8$, a contradiction. $\square$

Proposition 2.4.27. *Let G be a group of order $p^m q^n$ with p, q primes such that $p < q$ and let w be a commutator word. If $n > 1$ the clique number of $\Gamma_w(G)$ is greater than or equal to q^2.*

Proof. Consider a Sylow q-subgroup of G and apply Proposition 2.4.25. $\square$

Proposition 2.4.28. *Let G be a group and w a commutator word. If the clique number of $\Gamma_w(G)$ is 3 then G is isomorphic to the cyclic group C_3 or the symmetric group S_3.*

Proof. If there is an element of order greater than or equal to 4 then the clique number of $\Gamma_w(G)$ is at least 4. Therefore, every non-identity element of G has to have order 2 or 3. Suppose $|G| = 2^m 3^n$. Notice first that $n \neq 0$, otherwise G would be a 2-group and thus, applying Proposition 2.4.25 we would contradict the hypothesis. Moreover, applying Proposition 2.4.27, we have $n = 1$. If $m = 0$ then G is isomorphic to the cyclic group of order 3. Suppose $m \geq 1$. If $m > 1$ then a Sylow 2-subgroup of G has order at least 4 and thus, by Proposition 2.4.25, the clique number of the graph would be greater than 3. Therefore, $m = 1$, so G has order 6 and cannot be abelian, thus it can only be S_3. $\square$

For both C_3 and S_3 the word $w(x, y) = [x, y]$ yields a verbal graph with clique number 3. However, of course, in general, there could be commutator words for which S_3 has clique number greater than 3. As an example, consider the word $w(x, y) = [x, y]^6$. Then S_3 has a complete graph.

Proposition 2.4.29. *Let G be a group and w a commutator word. If the clique number of $\Gamma_w(G)$ is 4 then G is isomorphic to V_4, S_3, Q_8, D_8, A_4 or S_4.*

Proof. If there is an element of order greater than 4 then the clique number of $\Gamma_w(G)$ is at least 5. Therefore, every non-identity element of G has to have order 2, 3 or 4. Suppose $|G| = 2^n 3^m$. If $m = 0$ then by Proposition 2.4.26 we have G isomorphic to V_4, Q_8 or D_8. Suppose $m \geq 1$. By Proposition 2.4.26 we have $n = 1$ or $n = 2$ or $n = 3$ and by Proposition 2.4.27 we have $m = 1$. Therefore, G has order 6, 12 or 24 and cannot be abelian. If $|G| = 6$ then it is isomorphic to S_3. Notice now that G does not have elements of order 6 and thus if $|G| = 12$ then G is isomorphic to A_4, while if $|G| = 24$ then G is isomorphic to S_4. ☐

We conclude this section by pointing out that for all groups listed in Proposition 2.4.29 there exists a word w that realizes clique number 4 on the verbal graph associated with them. Indeed, if you consider the word $w(x, y) = [x^2, y]$ then the clique number of $\Gamma_w(S_3)$ is 4 and if you consider the word $w(x, y) = [x, y]$ then the clique number of the graph of $\Gamma_w(G)$ with G isomorphic to V_4, A_4, Q_8, D_8 or S_4 is again 4.

Chapter 3

Words and embeddings

The study of verbal subgroups within a group is well-known to be an effective tool to obtain structural information about a group. Therefore, conditions that allow the classification of words in a free group are of paramount importance. One of the most studied problems is to establish which words are concise. In order to further develop the classification of words, in the first section, a hierarchy among words is introduced, generalizing the concept of concise words. Moreover, we present some results regarding embeddings properties of specific commutator subgroups for a class of generalized FC-groups. In the second section, to better understand the behavior of these subgroups, we focus on the concept of a perfectly embedded subgroup of a group, where, given a group G and H and K subgroups of G with $H \leq K$, we say that H is *perfectly embedded* in K if $[H, K] = H$.

3.1 A generalization of concise words

Let $w = w(x_1, \dots, x_n)$ be a group-word in the variables $x_1, \dots, x_n$. We recall that w is called concise if the verbal subgroup $w(G)$ is finite in each group G such that G_w is finite. Later, aiming to enhance the comprehension of words within a free group, the concept of semiconcise words was introduced in [26], where a word is called semiconcise if the subgroup $[w(G), G]$ is finite in each group G such that G_w is finite. Of course concise words are semiconcise. In [26, Proposition 4.2] it is proved that if w is a semiconcise word and z is any variable not appearing in w, then the word $[w, z]$ is also semiconcise.

In this context we give the following definition. Let w be a group-word and n a positive integer. The word w is said to be $\frac{1}{n}$*-concise* if for any group G the finiteness of G_w implies that the subgroup

$$[w(G), \underbrace{G, \dots, G}_{n-1 \text{ times}}]$$

is finite. Moreover, we say that the word w is 0-*concise* if the finiteness of G_w for any group G implies that there exist a positive integer n (depending on the group G) such that the subgroup

$$[w(G), \underbrace{G, \dots, G}_{n-1 \text{ times}}]$$

is finite. Obviously every $\frac{1}{n}$-concise word is $\frac{1}{m}$-concise for every $m \geq n$, and every $\frac{1}{n}$-concise word is 0-concise; so this introduces what we can call a hierarchy on words.

Going further, if w is a $\frac{1}{n}$-concise word, one may investigate embedding properties of the subgroup

$$[w(G), \underbrace{G, \dots, G}_{n-1 \text{ times}}].$$

For subsets S and T of a group G we write S^T to denote the set of conjugates $\{s^t | s \in S, t \in T\}$. A subgroup H of a group G is said to be FC-embedded in G if g^H is finite for all $g \in G$; furthermore H is said to be BFC-embedded in G if g^H is finite for all $g \in G$ and the number of elements in g^H is bounded by a constant that does not depend on the choice of g. Given a group-word w, a group G is called an FC(w)-group if the set of conjugates g^{G_w} is finite for all $g \in G$; moreover G is called a BFC(w)-group if g^{G_w} is finite for all $g \in G$ and the number of elements in g^{G_w} is bounded by a constant that does not depend on the choice of g. The above definitions are quite natural as they generalize the notions of FC-group and BFC-group, which correspond to the case $w = x_1$.

3.1.1 Unbounded case

The aim of this subsection is to prove Theorem K, which gives us a tool that allows us to prove that not all words are 0-concise. We start with some preliminary results. For a subset S of a group G we denote $S^* = S \cup S^{-1}$. Note that if S is a normal set then S^* is normal too. Clearly, any conjugate of a w-value is again a w-value, and so G_w and G_w^* are normal sets.

Lemma 3.1.1. *Let $w = w(x_1, \dots, x_n)$ be a group-word and set*

$$v = [w(x_1, \dots, x_n), x_{n+1}, \dots, x_{n+m}].$$

Then, for every $h \in G_v$ there exist $w_1, \dots, w_k \in G_w^*$ such that $h = w_1 \cdots w_k$, with $k \leq 2^m$.

Proof. We argue by induction on m. The case $m = 1$ was proven in [26, Lemma 2.3]. Assume that the assertion is true for m. Consider

$$\begin{aligned} h &= [w(g_1, \dots, g_n), g_{n+1}, \dots, g_{n+m}, g_{n+m+1}] \\ &= [[w(g_1, \dots, g_n), g_{n+1}, \dots, g_{n+m}], g_{n+m+1}] \,. \end{aligned}$$

By induction hypothesis there exist $w_1, \dots, w_k \in G_w^*$ such that $h = [w_1 \cdots w_k, g_{n+m+1}]$, with $k \leq 2^m$. Thus we have

$$\begin{aligned} h &= [w_1 \cdots w_k, g_{n+m+1}] \\ &= (w_1 \cdots w_k)^{-1}(w_1 \cdots w_k)^{g_{n+m+1}} \\ &= w_k^{-1} \cdots w_1^{-1} w_1^{g_{n+m+1}} \cdots w_k^{g_{n+m+1}}. \end{aligned}$$

Note that $w_i^j \in G_w^*$, with $i \in \{1, \dots, k\}, j \in \{-1, g_{n+m+1}\}$. Hence there exist $w'_1, \dots, w'_{k'} \in G_w^*$ such that $h = w'_1 \cdots w'_{k'}$, with $k' \leq 2k \leq 2^{m+1}$. ☐

Lemma 3.1.2. *Let $w = w(x_1, \dots, x_n)$ be a group-word and set*

$$v = [w(x_1, \dots, x_n), x_{n+1}, \dots, x_{n+m}].$$

If G is an FC(w)*-group, then it is an* FC(v)*-group.*

Proof. Let $g \in G$ and $h \in G_v$. By Lemma 3.1.1 there exist $w_1, \dots, w_k \in G_w^*$ such that $h = w_1 \cdots w_k$, with $k \leq 2^m$. By [25, Proposition 2.9(i)] G is an FC(w^{-1})-group. Note that $G_{w^{-1}} = {G_w}^{-1}$, so it follows that $g^{G_w^*}$ is finite. Let $g^{G_w^*} = \{g^{b_1}, \dots, g^{b_s}\}$ and put $A = \{b_1, \dots, b_s\}$. By [26, Lemma 2.2] we have that $g^h = g^{w_1 \cdots w_k} = g^{a_1 \cdots a_k}$, with $a_1, \dots, a_k \in A$. Therefore $|g^{G_v}| \leq |A|^{2^m}$ and so it is finite. ☐

Lemma 3.1.3. *Let $w = w(x_1, \dots, x_n)$ be a $\left(\frac{1}{m+1}\right)$-concise word, with m positive integer, and set*

$$v = [w(x_1, \dots, x_n), x_{n+1}, \dots, x_{n+m}].$$

Let G be an FC(w)*-group and B a finite subset of G_v^*. Then, for any $g \in G$, there exists a positive integer e such that $b^e \in Z(\langle g, B\rangle)$ for all $b \in B$.*

Proof. Let $B = \{b_1, \dots, b_r\}$ and $g \in G$. For any $b_i \in B$ there exist elements $g_{i_1}, \dots, g_{i_{n+m}} \in G$ such that

$$b_i = [w(g_{i_1}, \dots, g_{i_n}), g_{i_{n+1}}, \dots, g_{i_{n+m}}]^{\varepsilon_i},$$

with $\varepsilon_i \in \{1, -1\}$. Put

$$J = \langle g, g_{i_j} | 1 \leq i \leq r, 1 \leq j \leq n+m\rangle.$$

By [25, Lemma 2.7(i)] the set $(J/Z(J))_w$ is finite. As w is $(\frac{1}{m+1})$-concise, the subgroup

$$[w(J/Z(J)), \underbrace{J/Z(J), \dots, J/Z(J)}_{m \text{ times}}] = [w(J), \underbrace{J, \dots, J}_{m \text{ times}}]Z(J)/Z(J)$$

is finite. Thus $v(J)$ has finite order modulo $Z(J)$, say e. Since $B \subseteq v(J)$, it follows that $b_i^e \in Z(J)$ for all i. As $\langle g, B\rangle \leq J$, the result follows. ☐

We are now in a position to prove Theorem K.

Theorem K. *Let w be a $\left(\frac{1}{m+1}\right)$-concise word, with m positive integer, and let G be an* FC(w)*-group. Then*

$$[w(G), \underbrace{G, \dots, G}_{m \text{ times}}]$$

is FC*-embedded in G.*

Proof. Set $v = [w(x_1, \dots, x_n), x_{n+1}, \dots, x_{n+m}]$. Then G is an FC(v)-group by Lemma 3.1.2. Let $g \in G$. By [26, Lemma 2.1], we can choose $b_1, \dots, b_r \in G_v^*$ such that $g^{G_v^*} = \{g^{b_1}, \dots, g^{b_r}\}$. Write $B = \{b_1, \dots, b_r\}$. Define the order $<$ on the set of all (formal) products of the form $b_{i_1} \cdots b_{i_j}$, with $1 \le i_k \le r$ and $j \ge 1$, as follows. Put

$$b_{i_1} \cdots b_{i_j} < b_{i'_1} \cdots b_{i'_{j'}}$$

if and only if one of the following conditions is satisfied: $j < j'$ or $j = j'$ and there is a positive integer $l \le j$ such that $i_l < i'_l$ and $i_k = i'_k$ for all $k > l$. Let h be an arbitrary element of $v(G)$. Then $h = h_1 \cdots h_j$, where each $h_i \in G_v^*$. By [26, Lemma 2.2] for all $k \in \{1, \dots, j\}$, there exist an integer $i_k \in \{1, \dots, r\}$ such that $g^h = g^{b_{i_1} \cdots b_{i_j}}$. Clearly, we can choose $b_{i_1} \cdots b_{i_j}$ to be the smallest (respect to $<$) product of elements from B such that $g^h = g^{b_{i_1} \cdots b_{i_j}}$. Now we show that $i_1 \ge i_2 \ge \cdots \ge i_j$. Suppose to the contrary that $i_k < i_{k+1}$ for some k. Then

$$\begin{aligned} g^h &= g^{b_{i_1} \cdots b_{i_{k-1}} b_{i_k} b_{i_{k+1}} b_{i_{k+2}} \cdots b_{i_j}} \\ &= g^{b_{i_1} \cdots b_{i_{k-1}} c b_{i_k} b_{i_{k+2}} \cdots b_{i_j}}, \end{aligned}$$

where $c = b_{i_k} b_{i_{k+1}} b_{i_k}^{-1} \in G_v^*$. In view of [26, Lemma 2.2], we have

$$g^{b_{i_1} \cdots b_{i_{k-1}} c} = g^{b_{i'_1} \cdots b_{i'_{k-1}} b_{i'_{k+1}}}$$

for some $1 \le i'_1, \dots, i'_{k-1}, i'_{k+1} \le r$ so that

$$g^h = g^{b_{i'_1} \cdots b_{i'_{k-1}} b_{i'_{k+1}} b_{i_k} b_{i_{k+2}} \cdots b_{i_j}}.$$

This contradicts the choice of the product $b_{i_1} \cdots b_{i_j}$ because

$$b_{i_1} \cdots b_{i_{k-1}} b_{i_k} b_{i_{k+1}} b_{i_{k+2}} \cdots b_{i_j} > b_{i'_1} \cdots b_{i'_{k-1}} b_{i'_{k+1}} b_{i_k} b_{i_{k+2}} \cdots b_{i_j}.$$

Thus $g^h = g^{b_{i_1} \cdots b_{i_j}}$ with $i_1 \ge i_2 \ge \cdots \ge i_j$. Equivalently we can write

$$g^h = g^{b_r^{er} \cdots b_1^{e1}}$$

for some non-negative integers $e_r, \ldots, e_1$. By Lemma 3.1.3, there exists a positive integer e such that $b_i^e \in Z(\langle g, B\rangle)$ for all i. Hence we may assume that $e_i < e$ for all i, because if we write $g^h = g^{b_r^{er} \cdots b_j^{e+t} \cdots b_1^{e1}}$, with t non-negative integer, we have

$$\begin{aligned} g^h &= g^{b_r^{er} \cdots b_j^{e+t} \cdots b_1^{e1}} \\ &= g^{b_r^{er} \cdots b_j^{e} b_j^{t} \cdots b_1^{e1}} \\ &= g^{b_j^{e} b_r^{er} \cdots b_j^{t} \cdots b_1^{e1}} \\ &= \left(g^{b_j^e}\right)^{b_r^{er} \cdots b_j^{t} \cdots b_1^{e1}} \\ &= g^{b_r^{er} \cdots b_j^{t} \cdots b_1^{e1}}, \end{aligned}$$

by the fact that $b_j^e \in Z(\langle g, B\rangle)$. Thus $|g^{v(G)}| < e^r$, so $g^{v(G)}$ is finite for all $g \in G$. We conclude therefore that $v(G)$ is FC-embedded in G. □

3.1.2 Bounded case

In this section we prove Theorem L, which is the bounded version of Theorem K. In order to do so we first prove some preliminary results. Recall that for a non-empty set I, a filter over I is a set $\mathcal{F} \subseteq \mathcal{P}(I)$, where $\mathcal{P}(I)$ denotes the set of all subsets of I, satisfying the following conditions:

(i) $\varnothing \notin \mathcal{F}$, $I \in \mathcal{F}$;

(ii) if $X, Y \in \mathcal{F}$, then $X \cap Y \in \mathcal{F}$;

(iii) if $X \in \mathcal{F}$ and $X \subseteq Y \subseteq I$, then $Y \in \mathcal{F}$.

The filter $\mathcal{F}$ is principal if there exists a non-empty set $Y \subseteq I$ such that

$$\mathcal{F} = \{X \subseteq I | Y \subseteq X\},$$

and non-principal otherwise. An example of a non-principal filter over an (infinite) set I is the so-called cofinite filter

$$\mathcal{F} = \{X \subseteq I | I \smallsetminus X \text{ is finite}\}.$$

A filter $\mathcal{U}$ over I is called an ultrafilter if, for every $X \subseteq I$, either $X \in \mathcal{U}$ or $I \smallsetminus X \in \mathcal{U}$. This is equivalent to saying that $\mathcal{U}$ is a maximal filter over I. Also, $\mathcal{U}$ is a non-principal ultrafilter if and only if it contains the cofinite filter (see [30, Proposition 1.4]). Given an ultrafilter $\mathcal{U}$ over I and a family $\{G_i\}_{i \in I}$ of groups, the ultraproduct modulo $\mathcal{U}$ is the quotient set of the Cartesian product $\Pi_{i\in I} G_i$ with respect to the equivalence relation defined as follows: the tuples $(g_i)_{i\in I}$ and $(h_i)_{i \in I}$ of the Cartesian product are equivalent modulo $\mathcal{U}$ if and only if $\{i \in I | g_i = h_i\} \in \mathcal{U}$.

Thus the ultraproduct modulo $\mathcal{U}$ can be seen as the quotient of the unrestricted direct product of groups G_i by the subgroup consisting of all tuples $(g_i)_{i\in I}$ such that $\{i \in I | g_i = 1\} \in \mathcal{U}$.

Recall that the width of a group-word w in a group G is the supremum of the minimum length of all decompositions of an element g in $w(G)$ as a product of elements of G_w^* as g ranges over $w(G)$. Clearly a word w has finite width at most $k \in \mathbb{N}$ if and only if any product of $k+1$ elements or more of G_w^* can be expressed as a product of at most k elements of G_w^*.

Proposition 3.1.4. *Let $r \geq 1$. Suppose that w is a $\left(\frac{1}{m+1}\right)$-concise word, with m positive integer, and G is a group in which w takes precisely r values. Then the order of*

$$[w(G), \underbrace{G, \dots, G}_{m \text{ times}}]$$

is $\{m, r\}$-bounded.

Proof. Assuming w involves n variables, write $w = w(x_1, \dots, x_n)$, and set

$$\begin{aligned} v &= v(x_1, \dots, x_n, x_{n+1}, \dots, x_{n+m}) \\ &= [w(x_1, \dots, x_n), x_{n+1}, \dots, x_{n+m}]. \end{aligned}$$

Then

$$v(G) = [w(G), \underbrace{G, \dots, G}_{m \text{ times}}].$$

By way of contradiction, suppose there exists a family of groups $\mathcal{G} = \{G_i\}_{i\in\mathbb{N}}$ with the property that $|(G_i)_w| \leq r$ for all $i \in \mathbb{N}$ but

$$\lim_{i\to\infty} |v(G_i)| = \infty.$$

Consider a non-principal ultrafilter $\mathcal{U}$ over $\mathbb{N}$, and let Q be the ultraproduct modulo $\mathcal{U}$ of $\mathcal{G}$. Then, by the fact that for a given integer r, the property that a given word takes at most r values in a group can be expressed as a sentence in the first-order language of groups and [26, Lemma 3.1], we have $|Q_w| \leq r$, because $\mathbb{N} \in \mathcal{U}$. As w is $\left(\frac{1}{m+1}\right)$-concise, it follows that $v(Q)$ is finite. In particular, v has finite width, say k, in Q. Now the fact that for a given positive integer k, the property that a given word has finite width at most k in a group can be expressed as a sentence in the first-order language of groups and [26, Lemma 3.1] yield that there exist $X \in \mathcal{U}$ such that v has finite width at most k in G_i for all $i \in X$. Hence every element of $v(G_i)$ can be written as a product of at most k elements of $(G_i)_v^*$. Moreover from $|(G_i)_w| \leq r$, we get $|(G_i)_v| \leq (2r)^{2^m}$ for all $i \in \mathbb{N}$, by Lemma 3.1.1. Therefore, $|v(G_i)| \leq (2r)^{2^m k}$ for all $i \in X$. As noted above, $\mathcal{U}$ contains the cofinite filter over $\mathbb{N}$, so $X \cap Y \in \mathcal{U}$ for every cofinite subset Y of $\mathbb{N}$. In particular, $X \cap Y$ is non-empty. Therefore every cofinite subset of $\mathbb{N}$ contains some element i for which $|v(G_i)| \leq (2r)^{2^m k}$. This is incompatible with the assumption that $|v(G_i)|$ goes to infinity because $\lim_{i\to\infty} |v(G_i)| = \infty$ implies that for every $M \in \mathbb{N}$ there exist an index $\nu_M \in \mathbb{N}$ such that $|v(G_i)| > M$ for every $i > \nu_M$, but if we choose $M = (2r)^{2^m k}$ and consider the set $L = \{1, 2, \dots, \nu_M\}$ we have that $X = \mathbb{N} \smallsetminus L$ is a cofinite subset of $\mathbb{N}$ and therefore it contains some element j, which is greater than ν_M, for which $|v(G_j)| \leq (2r)^{2^m k}$, a contradiction. □

Lemma 3.1.5. *Let $w = w(x_1, \dots, x_n)$ be a group-word and set*

$$v = [w(x_1, \dots, x_n), x_{n+1}, \dots, x_{n+m}].$$

If G is a BFC(w)*-group such that $|g^{G_w}| \leq r$ for all $g \in G$, then G is a* BFC(v)*-group and g^{G_v} has $\{m, n, r\}$-bounded order for all $g \in G$.*

Proof. Let $g \in G$ and $h \in G_v$. By Lemma 3.1.1 there exist $w_1, \dots, w_k \in G_w^*$ such that $h = w_1 \cdots w_k$, with $k \leq 2^m$. By [25, Proposition 2.9(ii)] G is an BFC(w^{-1})-group and $g^{G_{w^{-1}}}$ has $\{n, r\}$-bounded order. Note that $G_{w^{-1}} = {G_w}^{-1}$, so it follows that $g^{G_w^*}$ is finite with $\{n, r\}$-bounded order. Let $g^{G_w^*} = \{g^{b_1}, \dots, g^{b_s}\}$ and put $A = \{b_1, \dots, b_s\}$. Note that s is an $\{n, r\}$-bounded integer. By [26, Lemma 2.2] we have that $g^h = g^{w_1 \cdots w_k} = g^{a_1 \cdots a_k}$, with $a_1, \dots, a_k \in A$. Therefore $|g^{G_v}| \leq |A|^{2^m} = s^{2^m}$. Therefore g^{G_v} has $\{m, n, r\}$-bounded order □

Lemma 3.1.6. *Let $w = w(x_1, \dots, x_n)$ be a $\left(\frac{1}{m+1}\right)$-concise word, with m positive integer, and set*

$$v = [w(x_1, \dots, x_n), x_{n+1}, \dots, x_{n+m}].$$

Let G be an BFC(w)*-group such that $|g^{G_w}| \leq r$ for all $g \in G$, and let B be a finite subset of G_v^*. Then, for any $g \in G$, there exists an $\{m, n, r, |B|\}$-bounded positive integer e such that $b^e \in Z(\langle g, b\rangle)$ for all $b \in B$.*

Proof. Following the proof of Lemma 3.1.3, by [25, Lemma 2.7(ii)] the set $(J/Z(J))_w$ is finite of $\{m, n, r, |B|\}$-bounded order and, by Proposition 3.1.4, the number e is $\{m, n, r, |B|\}$-bounded. □

Now, we are ready to prove Theorem L.

Theorem L. *Let w be a $\left(\frac{1}{m+1}\right)$-concise word, with m positive integer, and let G be a* BFC(w)*-group. Then*

$$[w(G), \underbrace{G, \dots, G}_{m \text{ times}}]$$

is BFC*-embedded in G.*

Proof. Set $v = [w(x_1, \dots, x_n), x_{n+1}, \dots, x_{n+m}]$. We have that G is a BFC(w)-group, therefore it exists a positive integer r such that $|g^{G_w}| \leq r$ for any $g \in G$. Then, by Lemma 3.1.5, G is a BFC(v)-group and g^{G_v} has $\{m, n, r\}$-bounded order for any $g \in G$. Let $g \in G$ and choose $b_1, \dots, b_s \in G_v^*$ such that $g^{G_v^*} = \{g^{b_1}, \dots, g^{b_s}\}$. Write $B = \{b_1, \dots, b_s\}$. Define the order $<$ on the set of all (formal) products of the form $b_{i_1} \cdots b_{i_j}$, with $1 \leq i_k \leq s$ and $j \geq 1$, as in the proof of Theorem K.

Let $h \in v(G)$. As in the proof of Theorem K, we can write

$$g^h = g^{b_s^{e_s} \cdots b_1^{e_1}}$$

for some non-negative integers $e_s, \dots, e_1$. Since s is $\{m, n, r\}$-bounded by Lemma 3.1.5, it follows, from Lemma 3.1.6, that there exists an $\{m, n, r\}$-bounded positive integer e such that $b_i^e \in Z(\langle g, B \rangle)$ for all i. Hence we may assume that $e_i < e$ for all i, by the same argument shown in the proof of Theorem K, and so $|g^{v(G)}| < e^s$. Thus $g^{v(G)}$ is finite of $\{m, n, r\}$-bounded order for all $g \in G$. We conclude therefore that $v(G)$ is BFC-embedded in G. □

3.1.3 Examples

Every $\frac{1}{n}$-concise word is a $\frac{1}{m}$-concise word for every $m \geq n$. The next result shows another way to obtain $\frac{1}{m}$-concise words.

Proposition 3.1.7. *Let $w = w(x_1, \dots, x_n)$ be a group-word, and set*

$$v = [w(x_1, \dots, x_n), x_{n+1}].$$

If w is $\frac{1}{m+1}$-concise, with m positive integer, then v is $\frac{1}{m+1}$-concise.

Proof. Let G be a group and assume that G_v is finite. Since $v(G)'$ is finite (see [32, Proposition 1]), we may assume that $v(G)$ is abelian. It follows that every subgroup of $v(G)$ is finitely generated. Let $K = \langle k_1, \dots, k_\ell \rangle$ be a finitely generated subgroup of G such that $v(G) = v(K)$. Consider now a generator of the group

$$[v(G), \underbrace{G, \dots, G}_{m \text{ times}}].$$

It is of the form $[a, g_1, \dots, g_m]$, with $a \in v(G)$ and $g_i \in G$. In particular it lies in the subgroup

$$[v(G), \langle g_1 \rangle, \dots, \langle g_m \rangle].$$

Put $g_i = k_{\ell+i}$ and $H = \langle k_1, \dots, k_\ell, k_{\ell+1}, \dots, k_{\ell+m} \rangle$. Note that $v(H) = v(G)$. Also we have that $|H : C_H(v(H))|$ is finite, because H_v is a normal set, so every

$h \in H_v$ has only finitely many conjugates in H, and $|H : C_H(v(H))| = |H : \bigcap_{h \in H_v} C_H(h)| \leq \Pi_{h \in H_v} |H : C_H(h)|$. Now put

$$L = [v(H), \underbrace{H, \ldots, H}_{m-1 \text{ times}}].$$

From the fact that $C_H(v(H)) \subseteq C_H(L)$ we have that $|H : C_H(L)|$ is also finite. We claim that $|L : L \cap Z(H)|$ is also finite. In fact note that the set

$$\{[g, k_i] | g \in H_w, i = 1, \ldots, \ell + m\}$$

is finite because it is a subset of G_v, and therefore, by [26, Lemma 4.1], H_w is contained in finitely many right cosets of $w(H) \cap Z(H)$. Hence $(H/w(H) \cap Z(H))_w$ is finite and so $(H/Z(H))_w$ is finite. Since w is $\frac{1}{m+1}$-concise, we obtain that

$$[w(H), \underbrace{H, \ldots, H}_{m \text{ times}}]Z(H)/Z(H) \cong L/L \cap Z(H)$$

is finite. From this it follows that

$$[L, H] = [v(H), \underbrace{H, \ldots, H}_{m \text{ times}}]$$

is finite by [45, Corollary p. 103]. Indeed, to use this result, we consider the four subgroups $H, L, L \cap Z(H), C_H(L)$ of the group H. The subgroup L is normal in H because it is a subgroup of $v(H)$ which is abelian and fully-invariant in H. The subgroup $C_H(L)$ is normal in H because it is a centralizer of a normal subgroup and $L \cap Z(H)$ is normal in H because it is an intersection of normal subgroups. Moreover $[H, L \cap Z(H)] = 1 = [L, C_H(L)]$ trivially. So we have that

$$[L, H] = [v(G), \underbrace{H, \ldots, H}_{m \text{ times}}]$$

is finite. In particular, $[v(G), \langle g_1 \rangle, \ldots, \langle g_m \rangle]$ is finite. Thus $[a, g_1, \ldots, g_m]$ is periodic. Therefore

$$[v(G), \underbrace{G, \ldots, G}_{m \text{ times}}]$$

is a finitely generated periodic abelian group, and so it is finite. That proves that v is $\frac{1}{m+1}$-concise. □

According to [39], for any odd integer $n > 10^{10}$ and any prime number $p > 5000$, the word $v(x, y) = [[x^{pn}, y^{pn}]^n, y^{pn}]^n$ is not concise. Indeed, Ivanov constructed a 2-generator torsion-free group A whose center is cyclic and $A/Z(A)$ is infinite of exponent $p^2 n$, such that v takes only two values in A and the non-trivial value is a generator of $Z(A)$. In [9, Section 4], the authors considered a modification of Ivanov's example, namely the wreath product

$$G = A\mathrm{wr}B, \tag{3.1}$$

where $B = \langle b \rangle$ is a cyclic group of order 2. Taking

$$w(x, y) = v(x^2, y^2), \tag{3.2}$$

they showed that $|G_w| \leq 4$ and $b^{w(G)}$ is infinite. In [26, Section 4] they showed that $b^{[w(G),G]}$ is also infinite. In a similar way we now prove that $b^{[w(G),G,\ldots,G]}$, where G is repeated m times, is also infinite for every positive integer m. This implies that w is not $\frac{1}{m}$-concise for every m. In order to show this, we will need the following.

Lemma 3.1.8. *Let G be a group and let $h, b \in G$ such that $b^2 = 1$ and $[h, h^b] = 1$. Then for every non negative integer m the following equality holds*

$$[h,{}_{m+1}b] = h^{(-1)^m 2^m b} h^{(-1)^{m+1} 2^m}.$$

Proof. We argue by induction on m. Let $m = 0$, then $[h, b] = h^{-1}byb = h^{-1}h^b = h^b h^{-1}$. Assume the assertion true for $m \geq 0$. Then we have

$$\begin{aligned}
[h,{}_{m+2}b] &= [[h,{}_{m+1}b], b] \\
&= [h^{(-1)^m 2^m b} h^{(-1)^{m+1} 2^m}, b] \\
&= h^{(-1)^{m+2} 2^m} h^{(-1)^{m+1} 2^m b} b h^{(-1)^m 2^m b} h^{(-1)^{m+1} 2^m} b \\
&= h^{(-1)^{m+2} 2^m} h^{(-1)^{m+1} 2^m b} h^{(-1)^m 2^m} h^{(-1)^{m+1} 2^m b} \\
&= h^{(-1)^{m+2} 2^{m+1}} h^{(-1)^{m+1} 2^{m+1} b}
\end{aligned}$$

□

Proposition 3.1.9. *There exist a group-word w and a* BFC(w)*-group G such that*

$$[w(G), \underbrace{G, \ldots, G}_{m \text{ times}}]$$

is not FC*-embedded in G for every positive integer m.*

Proof. Let G and w be as in (3.1) and (3.2), respectively. Then, by [9, Proposition 4.1], G is a BFC(w)-group. Denote by $K = A \times A^b$ the base group of G. For any odd integer $t \geq 1$, let $N = \langle v_0^t, (v_0^b)^t \rangle$, where $v_0 \in A$ is the non-trivial value of $v(x, y)$ in A. Notice that N is central in K and closed under conjugation by b so that N is a normal subgroup of G. Also, since K/N has odd exponent p^2tn and $|G/K| = 2$, we have

$$K/N = \{g^2N | g \in G\}$$

and consequently $(K/N)_v = (G/N)_w$. Hence $v_0N \in (G/N)_w$, and therefore $v_0^kN \in w(G/N)$ for any integer k. It follows that

$$b^{[v_0^k,{}_m b]}N \in (bN)^{[w(G/N),G/N,\ldots,G/N]},$$

where G/N is repeated exactly m times. Now

$$b^{[v_0^k,{}_m b]}N = b[b,[v_0^k,{}_m b]]N = b[v_0^k,{}_{m+1} b]^{-1}N.$$

By Lemma 3.1.8 we have

$$\begin{aligned} b[v_0^k,{}_{m+1} b]^{-1}N &= bv_0^{k(-1)^m 2^m} v_0^{k(-1)^{m+1}2^m b}N \\ &= b(v_0^{(-1)^m 2^m} v_0^{(-1)^{m+1}2^m b})^k N, \end{aligned}$$

where $v_0^{(-1)^m 2^m} v_0^{(-1)^{m+1}2^m b}N$ has order t in G/N. Thus

$$|\{b^{[v_0^k,{}_m b]}N | k \in \mathbb{Z}\}| = t$$

and so

$$|(bN)^{[w(G/N),G/N,\ldots,G/N]}| \geq t.$$

In particular $|b^{[w(G),G,\ldots,G]}| \geq t$. Since t is an arbitrary odd positive integer, we conclude that $b^{[w(G),G,\ldots,G]}$ is infinite. Therefore

$$[w(G),\underbrace{G,\ldots,G}_{m \text{ times}}]$$

is not FC-embedded in G for every positive integer m. □

Corollary 3.1.10. *There exist a group-word w which is not $\frac{1}{m}$-concise for every positive integer m, and neither 0-concise.*

3.2 Perfectly embedded subgroups

Here we introduce the concept of a perfectly embedded subgroup of a group. Let H and K be subgroups of a group G with $H \leq K$. We say that H is *perfectly embedded* in K if $[H,K] = H$ that is, if H is generated by all commutators of elements of H with elements of K. When $K = G$, we simply refer to H as a perfectly embedded subgroup of G. In this section, we study groups with some restrictions on perfectly embedded subgroups, highlighting structural properties and interactions with commutator subgroups.

We first focus on some simple examples. The trivial subgroup is always perfectly embedded, and every perfectly embedded subgroup is clearly normal.

Every non-central minimal normal subgroup H of a group G is perfectly embedded in G. Indeed, $[H, G]$ is a non-trivial normal subgroup of G contained in H, so $[H, G] = H$ by minimality of H.

If G is a perfect group, then $[H, G]$ is perfectly embedded in G for every subgroup H of G. Indeed, since $[H, G]$ is normal in G, we have $[H, G, G] \leq [H, G]$. On the other hand, $[H, G, G] = [G, H, G]$, and by the Three Subgroups Lemma it follows that $[G, G, H] \leq [H, G, G]$. As G is perfect, we get $[G, G, H] = [G, H] = [H, G]$.

Throughout the section, we exploit concrete examples constructed using GAP, in particular making use of the `PerfectGroup` library. The computational verification of several examples was also carried out in GAP; the corresponding procedure is reported in Section 4.5 of the appendix.

3.2.1 Groups in which every normal subgroup is perfectly embedded

We start our analysis considering the class of all groups G in which every normal subgroup H is perfectly embedded in K, in the extremal cases $K = G$ and $K = H$. We also show that the case $K = N_G(H)$, which would seem equally natural to be studied, is actually too restrictive, because every non-trivial group contains a subgroup that is not perfectly embedded in its normalizer.

Let $\mathcal{X}$ denote the class of groups in which all normal subgroups are perfectly embedded. In the sequel, sometimes we will say that G is an $\mathcal{X}$-group meaning that the group G belongs to the class $\mathcal{X}$.

The class $\mathcal{X}$ contains all non-abelian simple groups, and is contained within the class of perfect groups. Both inclusions are strict: for instance, $\mathrm{PerfectGroup}(960, 1)$ is a non-simple $\mathcal{X}$-group, whereas $\mathrm{SL}(2, 5)$ is perfect but does not belong to $\mathcal{X}$. Of course, the class $\mathcal{X}$ does not contain any non-trivial soluble group.

Using simple groups, one can construct numerous examples of groups in the class $\mathcal{X}$. For instance, if A and H are simple groups then the restricted wreath product $G = A \mathrm{wr} H$ is a group belonging to the class $\mathcal{X}$. This is true since the normal subgroups of G are only $\{1\}$, the base group $A^{(H)}$ and G itself. In particular, we point out that if A is infinite, then G is an infinite non-simple group belonging to $\mathcal{X}$.

Every $\mathcal{X}$-group necessarily has trivial center, since $[Z(G), G] = \{1\}$. On the other side, there exist perfect groups with trivial center that do not belong to $\mathcal{X}$, such as $\mathrm{PerfectGroup}(1920, 4)$. Obviously every non-trivial $\mathcal{X}$-group has subgroups which do not belong to $\mathcal{X}$. Also, the class $\mathcal{X}$ is not closed under taking normal subgroups. For example, $\mathrm{PerfectGroup}(960, 1)$ has a non-trivial abelian normal subgroup. Nevertheless, the class $\mathcal{X}$ has remarkable closure properties.

Proposition 3.2.1. *Every homomorphic image of an $\mathcal{X}$-group is an $\mathcal{X}$-group.*

Proof. Let G be an $\mathcal{X}$-group, and let H be a normal subgroup of G. If K/H is a normal subgroup of G/H, then

$$[K/H, G/H] = [K, G]/H = K/H,$$

showing that G/H is an $\mathcal{X}$-group. $\square$

From Proposition 3.2.1 we derive the following criterion.

Proposition 3.2.2. *A group G is an $\mathcal{X}$-group if and only if $Z(G/N)$ is trivial for every normal subgroup N of G.*

Proof. If G is an $\mathcal{X}$-group, then G/N is an $\mathcal{X}$-group by Proposition 3.2.1, and hence $Z(G/N)$ is trivial. Conversely, assume that G does not belong to the class $\mathcal{X}$, so there exists a normal subgroup H of G such that $[H, G] < H$. Then $[H, G]$ is normal in G, hence $Z(G/[H, G])$ has to be trivial. This is a contradiction, as $H/[H, G] \leq Z(G/[H, G])$. $\square$

Proposition 3.2.3. *The class $\mathcal{X}$ is closed under taking direct products of its members.*

Proof. Let G be a group and let G_1, G_2 be $\mathcal{X}$-subgroups of G such that $G = G_1 \times G_2$. Let H be a normal subgroup of G, and denote by π_1 and π_2 the canonical projections of H onto G_1 and G_2, respectively. Clearly,

$$H \leq \pi_1(H) \times \pi_2(H).$$

Let $h_1 \in \pi_1(H)$ and $g_1 \in G_1$. Then there exists $h_2 \in \pi_2(H)$ such that $h_1h_2 \in H$. Thus $[h_1h_2, g_1] \in H$. Since $[h_1, g_1] = [h_1h_2, g_1]$, we get $[\pi_1(H), G_1] \leq H$. Similarly, $[\pi_2(H), G_2] \leq H$, and therefore

$$[\pi_1(H), G_1] \times [\pi_2(H), G_2] \leq H.$$

As G_1 and G_2 are $\mathcal{X}$-groups, we have $[\pi_1(H), G_1] = \pi_1(H)$ and $[\pi_2(H), G_2] = \pi_2(H)$. Hence

$$\pi_1(H) \times \pi_2(H) \leq H,$$

and equality follows. Finally,

$$H = [\pi_1(H), G_1] \times [\pi_2(H), G_2] = [\pi_1(H) \times \pi_2(H), G_1 \times G_2] = [H, G].$$

This shows that G is an $\mathcal{X}$-group, as required. $\square$

It is well-known that a normal subgroup of the direct product of two groups need not coincide with the direct product of its canonical projections, as shown by Goursat's Lemma (see, for instance, [44, Proposition 5.2.5]). However, Proposition 3.2.3 shows that this is the case when considering any normal subgroup of a direct product of $\mathcal{X}$-groups.

Also notice that, by Propositions 3.2.1 and 3.2.3, the study of $\mathcal{X}$-groups reduces to the indecomposable case.

Now we point out some restriction on normal subgroups of an $\mathcal{X}$-group.

Proposition 3.2.4. *Let G be a finite non-simple $\mathcal{X}$-group. If a Sylow 2-subgroup of G is abelian, then $O_2(G)$ is trivial.*

Proof. Assume for a contradiction that $O_2(G)$ is non-trivial, and set $C = C_G(O_2(G))$. Then C contains a Sylow 2-subgroup of G. By Proposition 3.2.1, the quotient group G/C is an $\mathcal{X}$-group. Since G/C has odd order, it is soluble. Hence $G = C$, so $O_2(G) \leq Z(G) = \{1\}$. □

Proposition 3.2.5. *Let G be an $\mathcal{X}$-group, and let N be a cyclic normal subgroup of G. Then $N = \{1\}$.*

Proof. By Proposition 3.2.1, $G/C_G(N)$ is an $\mathcal{X}$-group which embeds into the abelian group $\mathrm{Aut}(N)$. Hence $G/C_G(N) = \{1\}$, so $G = C_G(N)$, giving $N \leq Z(G) = \{1\}$. □

Next we describe the structure of $\mathcal{T}$-groups belonging to the class $\mathcal{X}$. Recall that a group G is said to be a $\mathcal{T}$-group if normality is transitive in G; that is, if $K \trianglelefteq H \trianglelefteq G$ implies $K \trianglelefteq G$ for all normal subgroups H and K of G.

Lemma 3.2.6. *Let G be a finite perfect group, N a non-abelian simple normal subgroup of G, and $C = C_G(N)$. Then $G = N \times C$.*

Proof. By the Schreier Conjecture, now proved, $\mathrm{Out}(N)$ is soluble. Since G/C embeds into $\mathrm{Aut}(N)$, the quotient G/CN is also soluble. As G is perfect, it follows that $CN = G$, and hence $G = N \times C$. □

Proposition 3.2.7. *A finite $\mathcal{T}$-group in $\mathcal{X}$ is a direct product of non-abelian simple groups.*

Proof. Let N be a minimal normal subgroup of G. By transitivity, N is simple, so it is non-abelian by Proposition 3.2.5. Then $G = N \times C$ by Lemma 3.2.6, and $C = C_G(N)$ is an $\mathcal{X}$-group by Proposition 3.2.1. Since normal subgroups of $\mathcal{T}$-groups are again $\mathcal{T}$-groups, an inductive argument yields the result. □

We now turn to the action of the normalizer, considering groups in which every subgroup is perfectly embedded in its normalizer. This condition is clearly stronger than the one defining the class $\mathcal{X}$. We show that it is in fact too restrictive: the trivial group is the only group satisfying it. Equivalently, every non-trivial group admits a subgroup that is not perfectly embedded in its normalizer.

Proposition 3.2.8. *Let G be a group such that $[H, N_G(H)] = H$ for all $H \leq G$. Then G is trivial.*

Proof. First we will show that G is a periodic group. Indeed, assume for a contradiction that G contains an element x of infinite order, and set $H = \langle x \rangle$. The normalizer $N_G(H)$ acts on H by conjugation, which must be non-trivial by hypothesis. Hence, for all $g \in N_G(H)$, we have $x^g = x^{-1}$. For all $n \in \mathbb{Z}$ and $g \in N_G(H)$ we get

$$[x^n, g] = x^{-n} g^{-1} x^n g = x^{-2n}.$$

Thus $[H, \langle g \rangle] = \langle x^2 \rangle$, a proper subgroup of H, and therefore $[H, N_G(H)] = \langle x^2 \rangle \neq H$, contradicting the hypothesis. Then G is periodic, as claimed.

Now we will prove that G is trivial. Indeed, arguing by contradiction suppose that G is non-trivial. As G is periodic, let p be the smallest prime dividing the order of an element of G, and let $x \in G$ have order p. Then $N_G(\langle x \rangle)/C_G(x)$ embeds into $\mathrm{Aut}(\langle x \rangle)$, whose order is $p - 1$. Hence, for every $g \in N_G(\langle x \rangle)$, we have $g^{p-1} \in C_G(x)$. By minimality of p, this implies $g \in C_G(x)$, so $N_G(\langle x \rangle) = C_G(x)$. Consequently,

$$[\langle x \rangle, N_G(\langle x \rangle)] = 1,$$

contradicting the hypothesis. □

Finally, it is worth mentioning that groups in which every normal subgroup is perfectly embedded in itself are known as *extra-perfect groups*. Finite extra-perfect groups are characterized as those groups in which every composition factor is a non-abelian simple group (see [14]).

3.2.2 Groups without non-trivial perfectly embedded normal subgroups

We now investigate groups in which no non-trivial normal subgroup is perfectly embedded. It comes out that the groups satisfying this property are precisely the hypocentral groups, that are those groups whose transfinite lower central series terminates at the trivial subgroup. This class of groups also coincides with the class of groups in which no non-trivial subgroup is perfectly embedded in its normalizer.

Theorem 3.2.9. *Let G be a group. The following statements are equivalent:*

(i) $[H, G] < H$ for every non-trivial normal subgroup H of G;

(ii) G is hypocentral;

(iii) $[H, N_G(H)] < H$ for every non-trivial subgroup H of G.

Proof. $(i) \Rightarrow (ii)$. Let G_ω denote the hypocenter of G, that is the intersection of all members of the transfinite lower central series of G. By definition, $[G_\omega, G] = G_\omega$. So $G_\omega = \{1\}$ by hypothesis. Hence G is hypocentral.

$(ii) \Rightarrow (i)$. Assume by contradiction that there exists a non-trivial normal subgroup H of G such that $[H, G] = H$. Arguing by transfinite induction, we will show that H is contained in every term of the transfinite lower central series of G.

The base step is immediate, since $H = [H, G] \leq [G, G]$. Assume that $H \leq \gamma_\alpha(G)$ for some non-limit ordinal α. Then

$$H = [H, G] \leq [\gamma_\alpha(G), G] = \gamma_{\alpha+1}(G).$$

Now let α be a limit ordinal, and assume that $H \leq \gamma_\beta(G)$ for all $\beta < \alpha$. Then

$$H \leq \bigcap_{\beta<\alpha} \gamma_\beta(G) = \gamma_\alpha(G).$$

Thus H is contained in the hypocenter of G, contradicting the assumption that G is hypocentral.

$(iii) \Rightarrow (i)$. This is obvious.

$(i) \Rightarrow (iii)$. Since (i) is equivalent to (ii), the group G is hypocentral. As hypocentrality is inherited by subgroups, $N_G(H)$ is also hypocentral. Since H is normal in $N_G(H)$, it follows that $[H, N_G(H)] < H$. □

We conclude this section by considering groups whose non-trivial normal subgroups are not perfect. Recall that a group is called *hypoabelian* if its perfect core is trivial, or equivalently if its transfinite derived series reaches the trivial subgroup.

Theorem 3.2.10. *A group G is hypoabelian if and only if $[H, H] < H$ for every non-trivial normal subgroup H of G.*

Proof. Assume that G is hypoabelian and suppose, by contradiction, that there exists a non-trivial normal subgroup H of G such that $[H, H] = H$. Then H is contained in every term of the derived series of G, and hence lies in the perfect core of G, contradicting hypoabelianity.

Conversely, assume that $[H, H] < H$ for every non-trivial normal subgroup H of G. Let G_α denote the perfect core of G. Then $[G_\alpha, G_\alpha] = G_\alpha$, which forces $G_\alpha = \{1\}$. Hence G is hypoabelian. □

3.2.3 Perfect embeddings of abelian normal subgroups

In this section we restrict our attention to abelian normal subgroups and consider the class $\mathcal{Z}$ of groups in which every abelian normal subgroup is perfectly embedded. Clearly, this class contains the above class $\mathcal{X}$, and the inclusion is strict: for instance, any group without non-trivial abelian normal subgroups is a $\mathcal{Z}$-group. It follows that all simple groups are $\mathcal{Z}$-groups, as well as all symmetric groups of degree $n \geq 3$.

Every $\mathcal{Z}$-group has trivial center, since $[Z(G), G] = 1$. Notice, however, that there exist soluble $\mathcal{Z}$-groups: the smaller example is the symmetric group of degree 3.

Lemma 3.2.11. *Let $G = A \rtimes H$ be a finite group with A abelian. If there exists $h \in H$ such that $C_A(h) = \{1\}$, then $[A, H] = A$. In particular, every element of A can be expressed as a commutator $[a, h]$ for some $a \in A$.*

Proof. Set $C = \{[a, h] \mid a \in A\}$ and consider the map $f : A \to C$ defined by $f(a) = [a, h]$. This map is injective: if $[a_1, h] = [a_2, h]$, then $a_1 a_2^{-1} \in C_A(h) = \{1\}$, giving $a_1 = a_2$. Since $C \subseteq A$, we get $A = C$ and the result follows. ☐

Proposition 3.2.12. *Let $G = K \rtimes H$ be a finite group such that every abelian normal subgroup of G is contained in K. If there exists an element $h \in H$ with $C_K(h) = \{1\}$, then G is a $\mathcal{Z}$-group.*

Proof. Let A be an abelian normal subgroup of G. By hypothesis $A \leq K$, and thus $C_A(h) = \{1\}$. By Lemma 3.2.11, $[A, H] = A$. Since $[A, H] \leq [A, G] \leq A$, we obtain $[A, G] = A$. Therefore G is a $\mathcal{Z}$-group. ☐

Corollary 3.2.13. *Every finite Frobenius group is a $\mathcal{Z}$-group.*

Proof. In a Frobenius group, every abelian normal subgroup lies in the kernel. Now the result follows by Proposition 3.2.12. ☐

By contrast, one may also consider the class $\mathcal{W}$ of groups in which no non-trivial abelian normal subgroup is perfectly embedded. Examples of $\mathcal{W}$-groups are all groups without non-trivial abelian normal subgroups, as well as all hypocentral groups (by Theorem 3.2.9).

Proposition 3.2.14. *Every abelian minimal normal subgroup of a $\mathcal{W}$-group is central.*

Proof. Let G be a $\mathcal{W}$-group, and let H be an abelian minimal normal subgroup of G. By hypothesis, $[H, G] < H$. Since $[H, G]$ is a normal subgroup of G, minimality of H implies $[H, G] = \{1\}$, so $H \leq Z(G)$. ☐

The class $\mathcal{W}$ is not closed under quotients, as shown by $\mathrm{SL}(2,3)$. However, we can prove the following result.

Proposition 3.2.15. *Let G be a finite soluble group such that every factor group of G is a $\mathcal{W}$-group. Then G is nilpotent.*

Proof. Let N be an abelian minimal normal subgroup of G. By Proposition 3.2.14, $N \leq Z(G)$, so $Z(G)$ is non-trivial. Applying the same argument iteratively to $G/Z(G)$, we obtain that the upper central series reaches G, showing that G is nilpotent. $\square$

Chapter 4

Appendix: GAP functions

The computation done in the thesis was performed using GAP [34]. We now describe the main functions used as support throughout the work.

4.1 Classes of groups

We start by introducing some functions that allow us to work just with a specific class of groups.

To justify the first function we present, we refer to the characterization of Frobenius groups given by Bertram (see [7, Lemma 1]). The second function is justified by well-known facts about 2-Frobenius groups, that we recalled in subsection 1.1.2.

```
## This function checks if a group is a Frobenius group.
IsFrobeniusGroup := function(G)
    local N;

    N := FittingSubgroup(G);

    if Order(N)=1 then
        return false;
    fi;

    if Order(N)=Order(G) then
        return false;
    fi;

    return NrConjugacyClasses(G)=
           NrConjugacyClasses(G/N)+(NrConjugacyClasses(N)-1)/Index(G, N);
```

```
end;

## This function checks if a group is a 2-Frobenius group.
IsTwoFrobeniusGroup := function(G)
    local N,K;
    N := FittingSubgroup(G);

    if Order(N)=1 then
        return false;
    fi;

    if IsFrobeniusGroup(G/N)=false then
        return false;
    fi;

    K:=FittingSubgroup(G/N);

    if IsFrobeniusGroup(PreImage(NaturalHomomorphismByNormalSubgroup(G,N),K))
    =false then
        return false;
    fi;

    return
    FittingSubgroup(PreImage(NaturalHomomorphismByNormalSubgroup(G,N),K))
            =N;

end;

## This function checks if a group is a Dedekind group.
IsDedekind := function(G)
    if IsAbelian(G) then
        return true;
    else;
        return Size(AllSubgroups(G))=Size(NormalSubgroups(G));
    fi;
end;

## This fuction checks if a group is an AC-Group.
IsACGroup:=function(G)
    local non_central;
```

```
    non_central := Filtered(G, x -> not x in Center(G));;

    return ForAll(non_central, x-> IsAbelian(Centralizer(G,x)));

end;

## This function checks if a group is cyclic-by abelian group.
IsCyclicByAbelian:=function(G)
    return ForAny(NormalSubgroups(G), N-> IsCyclic(N) and IsAbelian(G/N));
end;
```

4.2 The commuting and the nilpotent graphs

In this section, we show the functions used as computational support for the commuting and the nilpotent graph. In particular, for a group G, we show how to construct $\Delta_{\mathrm{comm}}(G)$ and $\Delta_{\mathrm{nil}}(G)$ on GAP; however, we point out that for the construction of $\Gamma_{\mathrm{comm}}(G)$ and $\Gamma_{\mathrm{nil}}(G)$ it is simply needed a change in the vertex set of the graph.

```
## It is necessary to load the package ‘‘Grape’’ to work on graphs.
LoadPackage("Grape");

## This function constructs the commuting graph of a group.
CommGraph:=function(G)
    local act, rel, non_central, gamma;

    act := {g, alpha} -> Image(alpha, g);
    rel := {x, y} -> x <> y and x*y = y*x;
    non_central := Filtered(G, x -> not x in Center(G));;
    gamma := Graph(AutomorphismGroup(G), non_central, act, rel, true);;
    return gamma;
end;

## This function calculates the diameter of any graph. If a graph is
## disconnected it returns -1.
Diameter(gamma);

## This function constructs the nilpotent graph of a group.
NilGraph := function(G)
```

```
    local act, rel, non_nilp, gamma;

    act := {g, alpha} -> Image(alpha, g);
    rel := {x, y} -> x <> y and IsNilpotent(Group(x,y));
    non_nilp := Difference(G,Union(UpperCentralSeries(G)));
    gamma := Graph(AutomorphismGroup(G), non_nilp, act, rel, true);
    return gamma;
end;


## This function calculates the maximum of the diameters of the connected
## components of a graph.
DiamConnComp:=function(G, gamma)
    local act, x, CC, rel, non_central, diam, d;

    CC:=ConnectedComponents(gamma);
    diam:=0;

    for x in CC do
        d:=Diameter(InducedSubgraph(gamma,x));
        if d>diam then
            diam:=d;
        fi;
    od;

    return diam;
end;

## This function calculates which elements of a group give rise to a specific
## connected component of a graph.
ConnectedComponentAsGroup:=function(gamma,ConnectedComponent)
    local x, grp, V;

    grp:=[];
    V:=Vertices(ConnectedComponent);

    for x in V do
        Add(grp,gamma.names[x]);
    od;
```

```
    return grp;
end;
```

4.3 The normalizing graph

In this section, we focus on the functions that helped us with the directed normalizing graph and the undirected normalizing graph associated with a group. In particular, in addition to the construction of the graphs, we show the functions that calculate the universal vertices of these graphs.

```
## This function calculates the universal bididirectional vertices of the
## directed normalizing graph.
Univ:=function(G)
    local x, y, H, K, el_def, el_one, l;

    el_one:=[];
    el_def:=[];

    for x in G do
        if IsNormal(G, Subgroup(G, [x]))=true then
            Add(el_one, x);
        fi;
    od;

    for y in el_one do
        l:=1;
        for x in G do
            H:=Subgroup(G, [x]);
            K:=Subgroup(G, [x,y]);
            if IsNormal(K,H)=false then
                l:=0;
                break;
            fi;
        od;
        if l=1 then
            Add(el_def, y);
        fi;
    od;
```

```
    return el_def;
end;

## This function constructs the directed normalizing graph of a group.
NormGraph:=function(G)
    local act, rel, univ, non_univ, gamma;

    act := {g, alpha} -> Image(alpha, g);
    rel := {x, y} -> x <> y and IsNormal(Subgroup(G,[x,y]),Subgroup(G,[x]));
    univ:=Univ(G);
    non_univ := Filtered(G, x -> not x in univ);;
    gamma := Graph(G, non_univ, function(x,g) return x; end, rel, true);;
    return gamma;
end;


## This function calculates the Baer Norm of a group.
BaerNorm:=function(G)
    local x, H;

    H:=G;

    for x in G do
        H:=Intersection(H,Normalizer(G,Subgroup(G,[x])));
    od;

    return H;
end;


## This function calculates the universal vertices of the undirected
## normalizing graph.
UnivUn:=function(G)
    local x, H, el_def, el_one;

    el_one:=[];
    el_def:=[];
    H:=BaerNorm(G);

    for x in G do
        if IsNormal(G, Subgroup(G, [x]))=true then
```

```
            Add(el_one, x);
        fi;
    od;

    el_def:=Union(List(H),el_one);
    return el_def;
end;

## This function constructs the undirected normalizing graph of a group.
NormGraphUn:=function(G)
    local rel, gamma;

    rel := {x, y} -> x <> y and (IsNormal(Subgroup(G,[x,y]),Subgroup(G,[x]))
                                     or IsNormal(Subgroup(G,[x,y]),Subgroup(G,[y])));
    gamma := Graph(G, AsList(G), function(x,g) return x; end, rel, true);;
    return gamma;
end;
```

4.4 The verbal graph

In this section, we focus on the useful functions we have created to investigate the verbal graph associated with a group.

```
## The following function allows to define any 2-variables word.
## As an example, the commutator word [x,y] is used.
word:=function(x,y)
    return Comm(x,y);
end;

## This function calculates the set of bidirectional universal vertices of
## the verbal graph.
Univword:=function(G)
    local x, y, el_def, el,l;

    el:=[];
    el_def:=[];

    for x in G do
        l:=1;
```

```
        for y in G do
            if Order(word(x,y))<>1 then
                l:=0;
                break;
            fi;
        od;
        if l=1 then
            Add(el, x);
        fi;
    od;

    for x in el do
        l:=1;
        for y in G do
            if Order(word(y,x))<>1 then
                l:=0;
                break;
            fi;
        od;
        if l=1 then
            Add(el_def, x);
        fi;
    od;

    return el_def;
end;

## This function calculates the marginal subgroup of a group given a
## 2-variables group word.
Marginal:=function(G)
    local x, y, h, mar, k;

    mar:=[];
    k:=1;

    for x in G do
        for y in G do
            for h in G do
                if word(x*h,y)<>word(h,y) then
                    k:=0;
```

```
                fi;
                if word(h*x,y)<>word(h,y) then
                    k:=0;
                fi;
                if word(h,y*x)<>word(h,y) then
                    k:=0;
                fi;
                if word(h,x*y)<>word(h,y) then
                    k:=0;
                fi;
            od;
        od;
        if k=1 then
            Add(mar,x);
        fi;
    od;

    return mar;
end;

## This function constructs the directed verbal graph associated with a group.
VerbalGraph:=function(G)
    local act, rel, univ, non_univ, gamma;

    act := {g, alpha} -> Image(alpha, g);
    rel := {x, y} -> x <> y and word(x,y)=Identity(G);
    univ:=Univword(G);
    non_univ := Filtered(G, x -> not x in univ);;
    gamma := Graph(G, non_univ, function(x,g) return x; end, rel, true);;
    return gamma;
end;

## This function constructs the undirected verbal graph associated with a group.
VerbalGraph:=function(G)
    local act, rel, univ, non_univ, gamma;

    act := {g, alpha} -> Image(alpha, g);
    rel := {x, y} -> x <> y and (word(x,y)=Identity(G) or word(y,x)=Identity(G));
    univ:=Univword(G);
    non_univ := Filtered(G, x -> not x in univ);;
```

```
    gamma := Graph(G, non_univ, function(x,g) return x; end, rel, true);;
    return gamma;
end;
```

4.5 Perfectly embedded subgroups

In this last section, we show the functions that helped us understand which groups belonged to which classes considered in Section 3.2.

```
## This function checks if a group belongs to the class of groups in which
## all normal subgroups are perfectly embedded.
IsClassX:=function(G)
    local x, flag, l;

    flag:=true;
    l:=NormalSubgroups(G);

    for x in l do
        if CommutatorSubgroup(x,G)<>x then
            flag:=false;
            break;
        fi;
    od;

    return flag;
end;

## This function checks if a group belongs to the class of groups in which
## all abelian normal subgroups are perfectly emebedded.
IsClassZ:=function(G)
    local x, flag, l;

    flag:=true;
    l:=Filtered(NormalSubgroups(G),IsAbelian);

    for x in l do
        if CommutatorSubgroup(x,G)<>x then
            flag:=false;
            break;
```

```
        fi;
    od;

    return flag;
end;

## This function checks if a group belongs to the class of groups in which
## all non-trivial abelian normal subgroups are not perfectly emebedded.
IsClassW:=function(G)
    local x, flag, l;

    flag:=true;
    l:=Filtered(NormalSubgroups(G), g -> IsAbelian(g)=true and Size(g)>1);

    for x in l do
        if CommutatorSubgroup(x,G)=x then
            flag:=false;
            break;
        fi;
    od;

    return flag;
end;
```

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